Pure and Applied Mathematics Quarterly
Volume 18, Number 6, 2379–2411, 2022

The Lagrangian density of the disjoint union of a 3-uniform tight path and a matching and the Turán number of its extension

Pingge Chen*, Jinhua Liang, and Yuejian Peng†

Abstract: Given a positive integer n and an r-uniform hypergraph F, the *Turán number* $ex(n, F)$ is the maximum number of edges in an F-free r-uniform hypergraph on n vertices. The *Turán density* of F is defined as $\pi(F) = \lim_{n\to\infty} ex(n, F)/\binom{n}{r}$. The *Lagrangian density* of an r-uniform graph F is $\pi_\lambda(F) = \sup\{r!\lambda(G) : G \text{ is } F\text{-free}\}$, where $\lambda(G)$ is the Lagrangian of G. In 1989, Sidorenko [20] showed that the Lagrangian density of a hypergraph F is the same as the Turán density of its extension. For an r-uniform graph F on t vertices, it is clear that $\pi_\lambda(F) \geq r!\lambda(K^r_{t-1})$, where K^r_{t-1} is the complete r-uniform graph on $t-1$ vertices. We say that an r-uniform hypergraph F on t vertices is λ-perfect if $\pi_\lambda(F) = r!\lambda(K^r_{t-1})$. A result of Motzkin and Straus implies that all graphs are λ-perfect. A conjecture proposed in [23] states that for $r \geq 3$, there exists an integer n such that if F and H are λ-perfect r-uniform graphs on at least n vertices, then the disjoint union of F and H is λ-perfect. The conjecture has been verified in [23] for a 3-uniform tight star $T_t = \{123, 124, \ldots, 12(t+2)\}$ and a λ-perfect 3-uniform graph for $t \geq 3$ (Sidorenko [20] showed that T_t is λ-perfect). The case $t = 2$ remains unsolved. In this paper, we shall show that the disjoint union of $T_2 \cong \{123, 234\}$ and a 3-uniform matching is λ-perfect(Jiang-Peng-Wu [9] showed that a 3-uniform matching is λ-perfect). Moreover, using a stability argument of Pikhurko [16], together with a transference technique between the Lagrangian density of an r-uniform graph and the Turán density of its extension, we also obtain the Turán numbers of their extensions.

Keywords: Hypergraph Lagrangian, Lagrangian density, Turán number.

Received May 31, 2021.

2010 Mathematics Subject Classification: 05C35, 05C65.

*The author is supported by National Natural Science Foundation of China (No. 12101221), National Natural Science Foundation of Hunan Province, China (No. 2021JJ30208).

†The author is supported in part by National Natural Science Foundation of China (No. 11931002).

1. Notations and definitions

For a set V and a positive integer r, let $V^{(r)}$ denote the family of all r-subsets of V. An *r-uniform graph* or *r-graph* G consists of a set $V(G)$ of vertices and a set $E(G) \subseteq V(G)^{(r)}$ of edges. Let $|G|$ denote the number of edges of G. An edge $e = \{a_1, a_2, \ldots, a_r\}$ will be simply denoted by $a_1 a_2 \ldots a_r$. An r-graph H is a *subgraph* of an r-graph G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In particular, a subgraph H is *spanning* if $V(H) = V(G)$. A subgraph of G *induced* by $V' \subseteq V$, denoted as $G[V']$, is the r-graph with vertex set V' and edge set $E' = \{e \in E(G) : e \subseteq V'\}$. Let K_t^r denote the *complete r-graph* on t vertices, that is, the r-graph on t vertices containing all r-subsets of the vertex set as edges.

The r-uniform t-matching, denoted by M_t^r, is the r-graph with t pairwise disjoint edges. For a positive integer n, let $[n]$ denote $\{1, 2, 3, \ldots, n\}$.

Given an r-graph F, an r-graph G is called *F-free* if it does not contain a copy of F as a subgraph. For a fixed positive integer n and an r-graph F, the *Turán number* of F, denoted by $ex(n, F)$, is the maximum number of edges in an F-free r-graph with n vertices. An averaging argument of Katona-Nemetz-Simonovits [11] showed that the sequence $\frac{ex(n,F)}{\binom{n}{r}}$ is a non-increasing sequence. Hence, $\lim_{n\to\infty} \frac{ex(n,F)}{\binom{n}{r}}$ exists. The *Turán density* of F is defined as

$$\pi(F) = \lim_{n\to\infty} \frac{ex(n,F)}{\binom{n}{r}}.$$

For 2-graphs, Erdős-Stone-Simonovits determined the asymptotic values of Turán numbers of all non-bipartite graphs. However, very few results are known for hypergraphs. For example, the well known conjecture of Turán that $\pi(K_4^{(3)}) = 5/9$ is not completely solved although the upper bounds given in [3] and [18] are close to the conjectured value, where $K_4^{(3)}$ is the complete 3-graph with 4 vertices. A recent survey on Turán numbers of r-uniform hypergraphs can be found in [12]. Johnston and Lu introduced the Turán density of non-uniform hypergraphs in [10].

Lagrangian has been a useful tool in estimating the Turán density of a hypergraph.

Definition 1.1. Let G be an r-graph on $[n]$ and let $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Define the Lagrange function of G as

$$\lambda(G, \vec{x}) = \sum_{e \in E(G)} \prod_{i \in e} x_i.$$

The *Lagrangian* of G, denoted by $\lambda(G)$, is defined as

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in \Delta\},$$

where

$$\Delta = \{\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \; for\ every\ i \in [n]\}.$$

The value x_i is called the *weight* of the vertex i and a vector $\vec{x} \in \Delta$ is called a *feasible weighting* on G. A feasible weighting $\vec{x}$ is called an *optimum weighting* on G if $\lambda(G, \vec{x}) = \lambda(G)$.

Given an r-graph F, the *Lagrangian density* $\pi_\lambda(F)$ of F is

$$\pi_\lambda(F) = \sup\{r!\lambda(G) : G\ is\ F\text{-}free\}.$$

The Lagrangian density of an r-graph is closely related to its Turán density. We say that a pair of vertices $\{i, j\}$ is *covered* in a hypergraph H if there exists $e \in H$ such that $\{i, j\} \subseteq e$. We say that a hypergraph H covers pairs if every pair of vertices is covered in H. The *extension* of an r-graph F, denoted by H^F, is defined as follows. For each pair of vertices $v_i, v_j \in V(F)$ not covered in F, we add a set B_{ij} of $r-2$ new vertices and the edge $\{v_i, v_j\} \cup B_{ij}$, where all B_{ij} are pairwise disjoint over all such pairs $\{i, j\}$.

Proposition 1.1 ([19, 16]). *Let F be an r-graph. Then*
(i) $\pi(F) \leq \pi_\lambda(F)$*;*
(ii) $\pi(H^F) = \pi_\lambda(F)$*. In particular, if F covers pairs, then $\pi(F) = \pi_\lambda(F)$.*

For an r-graph H on t vertices, it is clear that $\pi_\lambda(H) \geq r!\lambda(K^r_{t-1})$. We say that an r-uniform hypergraph H on t vertices is λ-*perfect* if $\pi_\lambda(H) = r!\lambda(K^r_{t-1})$. Theorem 2.1 implies that all 2-graphs are λ-perfect. It is interesting to explore what kind of hypergraphs are λ-perfect. Sidorenko [20] showed that the r-fold enlargement of a tree with order greater than some number A_r is λ-perfect. Hefetz and Keevash [6] showed that a 3-uniform matching of size 2 is λ-perfect. Jiang-Peng-Wu [9] extended to that any 3-uniform matching is λ-perfect. Pikhurko [16] and Norin-Yepremyan [15] showed that an r-uniform tight path of length 2 is λ-perfect for $r = 4$ and $r = 5$ or 6 respectively. Bene Watts, Norin and Yepremyan [1] showed that an r-uniform matching of size 2 is not λ-perfect (by determining its Lagrangian density) for $r \geq 4$ confirming a conjecture of Hefetz and Keevash [6]. Wu-Peng-Chen [22] showed the same result for $r = 4$ independently. Jenssen [8] showed that a path of length 2 formed by two edges intersecting at $r-2$ vertices is λ-perfect for $r = 3, 4, 5, 6, 7$. An r-uniform hypergraph is *linear* if any two edges have at most 1 vertex in common. Wu-Peng [21] showed that a 3-uniform linear path of length 3 or 4 is λ-perfect. Hu-Peng-Wu [7] showed that the disjoint union of a 3-uniform linear path of length 2 or 3 and a 3-uniform matching is λ-perfect. Yan-Peng [23] showed that the 3-uniform linear cycle of length 3 ({123, 345, 561}) is λ-perfect, and F_5 ({123, 124, 345}) is not λ-perfect (by determining its Lagrangian density). In [23], the following conjecture is proposed.

Conjecture 1.1 ([23])**.** *(1) For $r \geq 3$, there exists n such that a linear r-graph with at least n vertices is λ-perfect.*

(2) For $r \geq 3$, there exists n such that if G, H are λ-perfect r-graphs with at least n vertices, then the disjoint union of G and H, denoted by $G \uplus H$, is λ-perfect.

Yan-Peng [23] also verified the conjecture for a 3-uniform tight star $T_t = \{123, 124, 125, 126, ..., 12(t+2)\}$ and a λ-perfect 3-uniform graph for $t \geq 3$. The case that $t = 2$ is unsolved.

In this paper, we show that the disjoint union of T_2 and a 3-uniform t-matching (denoted by M_t^3) is λ-perfect. Precisely, let Q_{t+2} be the 3-graph with vertex set $[3t+4]$ and edge set $\{123, 234\} \uplus M_t^3$. We show that the Lagrangian density of Q_{t+2} is $3!\lambda(K_{3t+3}^3)$. We also give the Turán numbers of their extensions by using a similar stability argument for larger enough n as in [16] and several other papers.

2. Preliminaries

In this section, we give some useful properties of the Lagrange function. The following fact follows immediately from the definition of the Lagrangian.

Fact 2.1. *Let G_1, G_2 be r-graphs and $G_1 \subseteq G_2$. Then $\lambda(G_1) \leq \lambda(G_2)$.*

Given an r-graph G and a set S of vertices, the *link* of S in G, denoted by $L_G(S)$, is the hypergraph with edge set $\{e \subset V(G) \setminus S : e \cup S \in E(G)\}$. In particular, $S = \{i\}$, we write $L_G(\{i\})$ as $L_G(i)$. The *degree* of i is $d_G(i) = |L_G(i)|$, the number of edges containing i. Given $i, j \in V(G)$, define

$$L_G(j \setminus i) = \{e \in \binom{V(G)}{r-1} : i \notin e, e \cup \{j\} \in E(G) \text{ and } e \cup \{i\} \notin E(G)\}.$$

In other words, $L_G(j \setminus i)$ is the set of $(r-1)$-tuples in the neighborhood of j but not in the neighborhood of i. We say that an $(r-1)$-tuple e is in the neighborhood of a vertex u if $\{u\} \cup e$ is an edge. When there is no confusion, we will drop the subscript G in $L_G(j \setminus i)$. We say G on vertex set $[n]$ is *left-compressed* if for every i, j, $1 \le i < j \le n$, $L_G(j \setminus i) = \emptyset$. Equivalently, G on $[n]$ is *left-compressed* if $j_1 j_2 \cdots j_r \in E(G)$ implies $i_1 i_2 \cdots i_r \in E(G)$, wherever $i_p \le j_p$ for $1 \le p \le r$. Let $i, j \in V(G)$, define

$$\pi_{ij}(G) = (E(G) \setminus \{e \cup \{j\} : e \in L_G(j \setminus i)\}) \bigcup \{e \cup \{i\} : e \in L_G(j \setminus i)\}.$$

In other words, $\pi_{ij}(G)$ is an r-graph obtained from G by replacing an edge f containing j but not i by $(f \setminus \{j\}) \cup \{i\}$ if $(f \setminus \{j\}) \cup \{i\}$ is not an edge in G. We say that $\pi_{ij}(G)$ is obtained from G by compressing j to i. By the definition of $\pi_{ij}(G)$, it's straightforward to verify the following fact.

Fact 2.2. *Let G be an r-graph on $[n]$. Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be a feasible weighting on G. If $x_i \ge x_j$, then $\lambda(\pi_{ij}(G), \vec{x}) \ge \lambda(G, \vec{x})$.*

An r-graph G is *dense* if for every subgraph G' of G with $|V(G')| < |V(G)|$ we have $\lambda(G') < \lambda(G)$. This is equivalent to that no weight in an optimum weighting on G is zero.

Fact 2.3 ([5]). *Let $G = (V, E)$ be a dense r-graph. Then G covers pairs.*

In [13], Motzkin and Straus determined the Lagrangian of any given 2-graph.

Theorem 2.1 (Motzkin and Straus [13]). *If G is a 2-graph in which a maximum complete subgraph has t vertices, then $\lambda(G) = \lambda(K_t^2) = \frac{1}{2}(1 - \frac{1}{t})$.* □

The *support* of a vector $\vec{x}$ is $\sigma(\vec{x}) = \{i : x_i \neq 0 \; for \; i \in [n]\}$.

Fact 2.4 ([5]). *Let G be an r-graph on $[n]$. Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be an optimum weighting on G. Then*

$$\frac{\partial \lambda(G, \vec{x})}{\partial x_i} = r\lambda(G)$$

for every $i \in \sigma(\vec{x})$.

Fact 2.5 ([21]). *If G is a T_2-free 3-graph on $[n]$ $(n \ge 4)$. Then $\lambda(G) \le \frac{1}{24}$.*

Proof. Since G is T_2-free, then every pair is covered by at most one edge. Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be an optimum weighting on G. By Fact 2.4, $\frac{\partial \lambda(G,\vec{x})}{\partial x_i} = 3\lambda(G)$ for all $i \in \sigma(\vec{x})$. Summing over $i \in \sigma(\vec{x})$ we obtain $3|\sigma(\vec{x})|\lambda(G) = \sum_{i \in \sigma(\vec{x})} \frac{\partial \lambda(G,\vec{x})}{\partial x_i} \le \sum_{1 \le i < j \le n} x_i x_j \le \frac{1}{2}$. Note that $|\sigma(\vec{x})| \ge 4$ (otherwise $\lambda(G) \le \frac{1}{27}$). So $\lambda(G) \le \frac{1}{6|\sigma(\vec{x})|} \le \frac{1}{24}$. □

Theorem 2.2 ([9]). *Let $t \geq 2$ be an integer. Let G be an M_t^3-free 3-graph. Then $\lambda(G) \leq \lambda(K_{3t-1}^3)$.*

Fact 2.6. *Let G be an r-graph on $[n]$. Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be a feasible weighting on G. Let $i, j \in [n]$, $i \neq j$. Suppose that $L_G(i \setminus j) = L_G(j \setminus i) = \emptyset$. Let $\vec{y} = (y_1, y_2, \ldots, y_n)$ be defined by letting $y_\ell = x_\ell$ for every $\ell \in [n] \setminus \{i, j\}$ and letting $y_i = y_j = \frac{1}{2}(x_i + x_j)$, then $\lambda(G, \vec{y}) \geq \lambda(G, \vec{x})$.*

Proof. Since $L_G(i \setminus j) = L_G(j \setminus i) = \emptyset$, we have

$$\lambda(G, \vec{y}) - \lambda(G, \vec{x}) = \sum_{\{i,j\} \subseteq e \in G} \left[\frac{(x_i + x_j)^2}{4} - x_i x_j \right] \prod_{k \in e \setminus \{i,j\}} x_k \geq 0. \qquad \square$$

Let K_{3t+3}^{3-} be the 3-graph obtained by removing one edge from K_{3t+3}^3.

Fact 2.7. *Let $t \geq 1$ be an integer. Let G be a 3-graph on $[3t{+}3]$. If $G \neq K_{3t+3}^3$, then there exists a positive real $c_1 = c_1(t)$ such that $\lambda(G) \leq \lambda(K_{3t+3}^{3-}) \leq \lambda(K_{3t+3}^3) - c_1$,*

If $V_1, \ldots, V_s$ are disjoint sets of vertices, let $\Pi_{i=1}^s V_i = V_1 \times V_2 \times \ldots \times V_s = \{(x_1, x_2, \ldots, x_s) : \forall i \in [s], x_i \in V_i\}$. We will use $\Pi_{i=1}^s V_i$ to also denote the set of the corresponding unordered s-sets. If L is a hypergraph on $[m]$, then a *blowup* of L is a hypergraph G whose vertex set can be partitioned into $V_1, \ldots, V_m$ such that $E(G) = \bigcup_{e \in L} \prod_{i \in e} V_i$. The following proposition follows immediately from the definition and is implicit in many papers (see [12] for instance).

Proposition 2.1. *Let $r \geq 2$. Let L be an r-graph and G be a blowup of L. Suppose $|V(G)| = n$. Then $|G| \leq \lambda(L) n^r$.* $\square$

3. Lagrangian density of Q_{t+2}

Clearly, K_{3t+3}^3 is Q_{t+2}-free. In this section, we will show that the maximum possible Lagrangian among all Q_{t+2}-free 3-graphs is uniquely achieved by K_{3t+3}^3. Our main results are as follows.

Theorem 3.1. *Let G be a Q_{t+2}-free 3-graph. Then $\lambda(G) \leq \lambda(K_{3t+3}^3) = \frac{(3t+1)(3t+2)}{6(3t+3)^2}$. Furthermore, there exists a positive real $c = c(t)$ such that $\lambda(G) \leq \lambda(K_{3t+3}^3) - c$ for any K_{3t+3}^3-free 3-graph G.*

Corollary 3.1. $\pi_\lambda(Q_{t+2}) = 3!\lambda(K_{3t+3}^3)$.

Proof. Since K_{3t+3}^3 is Q_{t+2}-free, then $\pi_\lambda(Q_{t+2}) \geq 3!\lambda(K_{3t+3}^3)$. On the other hand, by Theorem 3.1, $\pi_\lambda(Q_{t+2}) \leq 3!\lambda(K_{3t+3}^3)$. Therefore, $\pi_\lambda(Q_{t+2}) = 3!\lambda(K_{3t+3}^3)$. $\square$

3.1. Left-compressing a Q_{t+2}-free 3-graph

Let

$$Q'_{t+2} = \{a_1a_2c, b_1b_2c\} \uplus M_t^3,$$

and

$$Q''_{t+3} = \{a_1b_1b_2, b_1b_2a_2, a_1cd_2, a_2cd_1\} \uplus M_{t-1}^3.$$

To prove Theorem 3.1, we will prove the following crucial results.

Lemma 3.1. *Let $t \geq 1$ be an integer. Then there exists a positive real c such that the following holds. Let G be a 3-graph on $[n]$ and let $1 \leq i < j \leq n$. If G is Q_{t+2}-free, then*

(1) either $\lambda(G) \leq \lambda(K_{3t+3}^3) - c$, or $\pi_{ij}(G)$ is Q_{t+2}-free.

(2) Furthermore, if G is K_{3t+3}^3-free and the pair $\{i,j\}$ is covered by an edge of G, then $\pi_{ij}(G)$ is K_{3t+3}^3-free.

Proof. (1) Suppose that $\pi_{ij}(G)$ contains a copy of Q_{t+2}, denoted by Q. There is $e \in Q$ such that $i \in e \in \pi_{ij}(G)$, $j \notin e$ and $e' = e \setminus \{i\} \cup \{j\} \in G$. Otherwise, Q is also a copy of Q_{t+2} in G, it is a contradiction. There are two cases in terms of the degree of i in Q.

Case 1: $d_Q(i) = 1$. If there exists no $f \in Q$ such that $j \in f$, then $Q \setminus \{e\} \cup \{e'\}$ forms a copy of Q_{t+2} in G. If there exists one edge f such that $j \in f \in Q$, then f is an independent edge in Q and $f' = f \setminus \{j\} \cup \{i\} \in G$. So $Q \setminus \{e, f\} \cup \{e', f'\}$ forms a copy of Q_{t+2} in G.

Case 2: $d_Q(i) = 2$. Let $Q = \{e_1, e_2, e_3, \cdots, e_{t+2}\}$ and $|e_1 \cap e_2| = 2$.

If $e'_1 = e_1 \setminus \{i\} \cup \{j\} \in G$, $e'_2 = e_2 \setminus \{i\} \cup \{j\} \in G$ and $j \in e_3$, then $Q \setminus \{e_1, e_2, e_3\} \cup \{e'_1, e'_2, e_3 \setminus \{j\} \cup \{i\}\}$ forms a copy of Q_{t+2}. Otherwise, without loss of generality, we assume that $e'_1 = e_1 \setminus \{i\} \cup \{j\} \in G$ but $e'_2 = e_2 \setminus \{i\} \cup \{j\} \notin G$. If $j \in e_2$ with $d_Q(j) = 1$, then $\{e'_1, e_2, e_3, \cdots, e_{t+2}\}$ forms a copy of Q_{t+2}. If $d_Q(j) = 0$, we get a new subgraph $\{e'_1, e_2, e_3, \cdots, e_{t+2}\}$ isomorphic to $Q'_{t+2} = \{a_1a_2c, b_1b_2c\} \uplus M_t^3$ in G. In Section 3.2, we will show the following lemma indicating that $\lambda(G) \leq \lambda(K_{3t+3}^3) - c$ in this case.

Lemma 3.2. *Let $t \geq 1$ be an integer. Then there exists a positive real c such that $\lambda(G) \leq \lambda(K_{3t+3}^3) - c$ for any dense Q_{t+2}-free 3-graph with $Q'_{t+2} \subseteq G$.*

If $j \in e_3$, we have $e_3, e'_3 = e_3 \setminus \{j\} \cup \{i\} \in G$, then we get $\{e'_1, e_2, e_3, e'_3, e_4, \cdots, e_{t+2}\}$ isomorphic to $Q''_{t+3} = \{a_1b_1b_2, b_1b_2a_2, a_1cd_2, a_2cd_1\} \uplus M_{t-1}^3$ in G. In Section 3.3, we will show the following lemma indicating that $\lambda(G) \leq \lambda(K_{3t+3}^3) - c$ in this case.

Lemma 3.3. *Let $t \geq 1$ be an integer. Then there exists a positive real c such that $\lambda(G) \leq \lambda(K_{3t+3}^3) - c$ for any Q_{t+2}-free 3-graph with $Q''_{t+3} \subseteq G$.*

(2) We assume that $\{i, j\}$ is covered by an edge g of G. Suppose for contradiction that $\pi_{ij}(G)$ contains a copy K of K^3_{3t+3}. Clearly, $V(K)$ must contain i. If $j \in V(K)$, then it is easy to see that K is also in G, contradicting G being K^3_{3t+3}-free. By the definition of $\pi_{ij}(G)$, all the edges in K not containing i are also in G. If $j \notin V(K)$, $V(K)$ contains at least $3t+1$ vertices outside g by our assumption. So K contains a copy of $Q_{(t-1)+2}$ disjoint from g, which lies in G. Now, $Q_{(t-1)+2} \uplus \{g\}$ is a copy of Q_{t+2} in G, a contradiction. □

Next, we perform the following algorithm.

Algorithm 3.1.
Input: *An r-graph G on $[n]$.*
Output: *A dense and left-compressed r-graph G'.*
Step 1. *If G is dense, then go to step 2. Otherwise, replace G by a dense subgraph G' with the same Lagrangian, and relabel the vertices of G' if necessary such that an optimum weighting $\vec{y}$ of G' satisfying $y_i \geq y_j$ if $i < j$. Then go to step 2.*
Step 2. *If G is left-compressed, then terminate. Otherwise, let $\vec{y}$ be an optimum weighting of G such that there exist vertices i, j satisfying $i < j$, $y_i \geq y_j$ and $L_G(j \setminus i) \neq \emptyset$. Replace G by $\pi_{ij}(G)$ and go to step 1.*

Note that the algorithm terminates after finite many steps since Step 2 reduces the parameter $s(G) = \sum_{e \in G} \sum_{i \in e} i$ by at least 1 each time and Step 1 reduces the number of vertices by at least 1 each time.

Lemma 3.4. *There exists a positive real c such that the following holds. Let G be a Q_{t+2}-free (and K^3_{3t+3}-free) 3-graph. Then either $\lambda(G) \leq \lambda(K^3_{3t+3}) - c$ or there exists a dense and left-compressed Q_{t+2}-free (and K^3_{3t+3}-free) 3-graph G' with $|V(G')| \leq |V(G)|$ and $\lambda(G') \geq \lambda(G)$.*

Proof. If for any c, we have $\lambda(G) > \lambda(K^3_{3t+3}) - c$, then we apply Algorithm 3.1 to G and let G' be the final graph. Then G' is dense and left-compressed. By Fact 2.2, $\lambda(G') \geq \lambda(G)$. By Lemma 3.1, G' is Q_{t+2}-free (and K^3_{3t+3}-free). □

Proof of Theorem 3.1. By Lemma 3.4, we may assume that G is dense and left-compressed. Suppose $V(G) = [n]$. If $n \leq 3t+3$, then by Fact 2.1, we have $\lambda(G) \leq \lambda(K^3_{3t+3})$. Furthermore, if G is K^3_{3t+3}-free, then by Fact 2.7, $\lambda(G) \leq \lambda(K^{3-}_{3t+3}) \leq \lambda(K^3_{3t+3}) - c_1$ for some positive c_1 (independent of G). Hence, we may assume that $n \geq 3t+4$. Let $\vec{x} = (x_1, x_2, \cdots, x_n)$ be an optimum weighting of G. Since G is left-compressed, then it is clear that $x_1 \geq x_2 \geq \cdots \geq x_n$. By Fact 2.3, G covers pairs. So $i(n-1)n \in G$, for some $i < n-1$. Since G is left-compressed, we have $1(n-1)n \in G$, this implies that $\forall\ i, j$, where $2 \leq i < j \leq n$, $1ij \in G$ and furthermore $L_G(1) = K^2_{n-1}$.

Suppose $x_1 = a$. Since $\vec{y} = (\frac{x_2}{1-a}, \dots, \frac{x_n}{1-a})$ is a feasible weighting on $L_G(1)$, then by Theorem 2.1, we have

$$\lambda(L_G(1), \{x_2, x_3, \cdots, x_n\}) = \sum_{2 \le i < j \le n} x_i x_j = (1-a)^2 \lambda(L_G(1), \vec{y}) < \frac{1}{2}(1-a)^2.$$

Let $F = G[\{2, 3, \cdots, n\}]$. For $t = 1$. Suppose F contains a copy of $T_2^{(3)}$. Since $n \ge 7, \exists\, i, j \in \{2, 3, \cdots, n\}$, such that $i, j \notin V(T_2^{(3)})$. Now, $\{1ij, T_2^{(3)}\}$ forms a copy of Q_3 in G, contradicting G being Q_3-free. Hence F must be $T_2^{(3)}$-free. Note that $\vec{y}$ is a feasible weighting on F. By Fact 2.5, we have $\lambda(F, \vec{y}) \le \frac{1}{24}$. Thus,

$$\begin{aligned}
\lambda(G) = \lambda(G, \vec{x}) &= a\lambda(L_G(1), \{x_2, x_3, \cdots, x_n\}) + \lambda(F, \{x_2, x_3, \cdots, x_n\}) \\
&< \frac{1}{2}a(1-a)^2 + \frac{1}{24}(1-a)^3 \\
&= \frac{1}{2}(1-a)^2 \left[a + \frac{1}{12}(1-a)\right] \\
&\le \frac{1}{2}\left(\frac{24}{11}\right)^2 \cdot \frac{1}{27} = \frac{32}{363} \le \lambda(K_6^3) - 10^{-3}.
\end{aligned}$$

For $t \ge 2$. Suppose F contains a copy of M_t^3. Since $n \ge 3t + 4, \exists\, i, j, k \in \{2, 3, \cdots, n\}$, such that $i, j, k \notin M_t^3$. Now, $\{1ij, 1jk\} \uplus M_t^3$ forms a copy of Q_{t+2} in G, contradicting G being Q_{t+2}-free. Hence F must be M_t^3-free. Note that $\vec{y}$ is a feasible weighting on F. By Theorem 2.2, we have $\lambda(F, \vec{y}) \le \lambda(K_{3t-1}^3)$. Let $s = 3t - 1$ and $\mu = \frac{s^2 - 3s + 2}{6s^2}$. Thus,

$$\begin{aligned}
\lambda(G) = \lambda(G, \vec{x}) &= a\lambda(L_G(1), \{x_2, x_3, \cdots, x_n\}) + \lambda(F, \{x_2, x_3, \cdots, x_n\}) \\
&< \frac{1}{2}a(1-a)^2 + \lambda(K_{3t-1}^3)(1-a)^3 \\
&= (1-a)^2 \left[\frac{1}{2}a + \frac{(3t-2)(3t-3)}{6(3t-1)^2}(1-a)\right] \\
&= (1-a)^2 \left[\frac{1}{2}a + \mu(1-a)\right] \\
&= (1-a)^2 \left[(\frac{1}{2} - \mu)a + \mu\right] \\
&= (1-a)(1-a)(2a + \frac{\mu}{\frac{1}{4} - \frac{1}{2}\mu})(\frac{1}{4} - \frac{1}{2}\mu) \\
&\le \left[\frac{1}{3}\left((1-a) + (1-a) + (2a + \frac{\mu}{\frac{1}{4} - \frac{1}{2}\mu})\right)\right]^3 (\frac{1}{4} - \frac{1}{2}\mu)
\end{aligned}$$

$$= \frac{1}{54(\frac{1}{2}-\mu)^2}$$
$$= \frac{2s^4}{3(2s^2+3s-2)^2}.$$

Since $s = 3t-1$, we have

$$\lambda(K^3_{3t+3}) = \binom{3t+3}{3}(\frac{1}{3t+3})^3 = \frac{s^2+5s+6}{6(s+4)^2}.$$

Hence,

$$\begin{aligned}\lambda(G) - \lambda(K^3_{3t+3}) &\leq \frac{2s^4}{3(2s^2+3s-2)^2} - \frac{s^2+5s+6}{6(s+4)^2} \\ &= \frac{4s^4(s+4)^2 - (s^2+5s+6)(2s^2+3s-2)^2}{6(2s^2+3s-2)^2(s+4)^2} \\ &= -\frac{21s^4+65s^3-50s^2-52s+24}{6(2s^2+3s-2)^2(s+4)^2},\end{aligned}$$

which is negative for every $s \geq 1$. Let

$$c = \min\left\{10^{-3}, c_1, \frac{21s^4+65s^3-50s^2-52s+24}{6(2s^2+3s-2)^2(s+4)^2}\right\}.$$

Then $\lambda(G) \leq \lambda(K^3_{3t+3}) - c$ and the proof is completed. □

We owe the proof of Lemma 3.2 and Lemma 3.3.

3.2. Proof of Lemma 3.2

Let

$$Q'_{t+2} = \{a_1a_2c, b_1b_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t,1}d_{t,2}d_{t,3}\}.$$

Lemma 3.5. *Let G be a dense Q_3-free 3-graph. If G contains a spanning subgraph Q'_3, then there exists a vertex v in $V(G)$ such that the link $L(v)$ contains no K_3.*

Proof. If $L(a_1)$ contains no K_3, then we are done. Otherwise, we show that the only possible sets forming a copy of K_3 in $L(a_1)$ are $\{a_2, c, d_k\}$ $(k = 1, 2, 3)$. If any of the triples in $\{a_2, c, b_1, b_2\}$ forms a copy of K_3 in $L(a_1)$, for example, if $\{a_2, c, b_1\}$ forms a copy of K_3 in $L(a_1)$, that is, $a_1a_2c, a_1cb_1, a_1a_2b_1 \in G$, then any two edges of those and the independent edge $d_1d_2d_3$ forms a copy of Q_3 in G. Similarly, other cases can not happen. If any of the triples with one vertex in $\{a_2, c, b_1, b_2\}$ and two vertices in $\{d_1, d_2, d_3\}$ forms a copy of K_3 in $L(a_1)$, if $\{x, y, z\}$ forms a copy of K_3 in $L(a_1)$, where $x \in \{a_2, c, b_1, b_2\}$, $y, z \in \{d_1, d_2, d_3\}$, then $\{yza_1, d_1d_2d_3, b_1b_2c\}$ forms a copy of Q_3 in G. If $\{d_1, d_2, d_3\}$ forms a copy of K_3 in $L(a_1)$, then $\{d_1d_2a_1, d_1d_2d_3, b_1b_2c\}$ is a copy of Q_3 in G.

Next, we consider the triples with two vertices in $\{a_2, c, b_1, b_2\}$ and one vertex in $\{d_1, d_2, d_3\}$. If $\{a_2, b_i, d_k\}$ $(i = 1, 2; k = 1, 2, 3)$ forms a copy of K_3 in $L(a_1)$, then $\{a_1a_2b_i, a_1a_2c, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{c, b_i, d_k\}$ $(i = 1, 2; k = 1, 2, 3)$ forms a copy of K_3 in $L(a_1)$, then $\{a_1cb_i, b_1b_2c, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{b_1, b_2, d_k\}$ $(k = 1, 2, 3)$ forms a copy of K_3 in $L(a_1)$, then $\{a_1b_1b_2, b_1b_2c, d_1d_2d_3\}$ is a copy of Q_3 in G.

Therefore, the only possible sets forming a copy of K_3 in $L(a_1)$ are $\{a_2, c, d_k\}$ $(k = 1, 2, 3)$. Switching a_1 and b_1, we can show identically that the only possible sets forming a copy of K_3 in $L(b_1)$ are $\{b_2, c, d_k\}$ $(k = 1, 2, 3)$. Without loss of generality, we may assume that $\{a_2, c, d_1\}$ forms a copy of K_3 in $L(a_1)$, that is, $a_1a_2c, a_1a_2d_1, a_1cd_1 \in G$. We have that $\{b_2, c, d_k\}$ $(k = 2, 3)$ can not form a copy of K_3 in $L(b_1)$, otherwise, $\{b_1b_2d_k, b_1b_2c, a_1a_2d_1\}$ $(k = 2, 3)$ is a copy of Q_3 in G. If $L(b_1)$ contains no K_3, then we are done. Otherwise, $\{b_2, c, d_1\}$ forms a copy of K_3 in $L(b_1)$, that is, $b_1b_2c, b_1b_2d_1, b_1cd_1 \in G$. We will show that $L(d_2)$ contains no K_3.

Firstly, we consider the triples in $\{a_1, a_2, d_1, d_3\}$. If any of the triples in $\{a_1, a_2, d_1, d_3\}$ forms a copy of K_3 in $L(d_2)$, for example, if $\{a_1, a_2, d_1\}$ forms a copy of K_3 in $L(d_2)$, that is, $a_1a_2d_2, a_1d_1d_2, a_2d_1d_2 \in G$, then any two edges of those and the edge b_1b_2c forms a copy of Q_3 in G. Similarly, other cases can not happen.

Secondly, we consider the triples with one vertex in $\{a_1, a_2, d_1, d_3\}$ and two vertices in $\{b_1, b_2, c\}$. If any of $\{b_1, b_2, a_i\}$, $\{b_1, b_2, d_k\}$ $(i = 1, 2; k = 1, 3)$ forms a copy of K_3 in $L(d_2)$, then $\{b_1b_2d_2, b_1b_2c, a_1a_2d_1\}$ is a copy of Q_3 in G. If $\{b_j, c, a_i\}$, $\{b_j, c, d_k\}$ $(i, j = 1, 2; k = 1, 3)$ forms a copy of K_3 in $L(d_2)$, then $\{b_jcd_2, b_1b_2c, a_1a_2d_1\}$ is a copy of Q_3 in G.

Thirdly, if $\{b_1, b_2, c\}$ forms a copy of K_3 in $L(d_2)$, then $\{b_1b_2d_2, b_1b_2c, a_1a_2d_1\}$ is a copy of Q_3 in G.

Finally, we consider the triples with two vertices in $\{a_1, a_2, d_1, d_3\}$ and one vertex in $\{b_1, b_2, c\}$. If any of $\{a_1, a_2, b_j\}$, $\{a_1, a_2, c\}$ $(j = 1, 2)$ forms a copy of K_3 in $L(d_2)$, then $\{a_1a_2d_2, a_1a_2d_1, b_1b_2c\}$ is a copy of Q_3 in G. If any of $\{a_i, d_k, b_j\}$, $\{a_i, d_k, c\}$ $(i, j = 1, 2; k = 1, 3)$ forms a copy of K_3 in $L(d_2)$, then $\{a_id_kd_2, d_1d_2d_3, b_1b_2c\}$ is a copy of Q_3 in G. If $\{d_1, d_3, b_j\}$ $(j = 1, 2)$ forms a copy of K_3 in $L(d_2)$, then $\{d_1b_jd_2, d_1d_2d_3, a_1a_2c\}$ is a copy of Q_3 in G.

If $\{d_1, d_3, c\}$ forms a copy of K_3 in $L(d_2)$, i.e., $d_1d_2d_3, d_1d_2c, d_3d_2c \in G$. Let's consider the pair $\{a_2, b_2\}$. If $a_2b_2a_1 \in G$, then $\{a_1a_2b_2, a_1a_2c, d_1d_2d_3\}$ forms a copy of Q_3 in G. If $a_2b_2b_1 \in G$, then $\{a_2b_1b_2, b_1b_2c, d_1d_2d_3\}$ forms a copy of Q_3 in G. If $a_2b_2c \in G$, then $\{b_2a_2c, a_1a_2c, d_1d_2d_3\}$ forms a copy of Q_3 in G. If $a_2b_2d_1 \in G$, then $\{b_2a_2d_1, a_1a_2d_1, d_2d_3c\}$ forms a copy of Q_3 in G. If $a_2b_2d_k \in G$ $(k = 2, 3)$, then $\{a_2b_2d_k, a_1cd_1, b_1cd_1\}$ forms a copy of Q_3 in G. So the pair $\{a_2, b_2\}$ can not be covered by any edge of G, by Fact 2.3, it is a contradiction. The proof is complete. □

Lemma 3.6. *Let G be a dense Q_{t+2}-free 3-graph. If G contains a spanning subgraph Q'_{t+2}, then there exists a vertex v in $V(G)$ such that the link $L(v)$ contains no K_{t+2}.*

Proof. Note that $V(G) = V(Q'_{t+2})$. If $L(a_1)$ contains no K_{t+2}, then we are done. Otherwise, we will show that the only possible sets forming a copy of K_{t+2} in $L(a_1)$ are $\{a_2, c, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ ($k_i = 1$ *or* 2 *or* $3; i = 1, 2, \cdots, t$).

We apply induction on t. By the proof of Lemma 3.5, the conclusion holds for $t = 1$. Suppose that the conclusion holds for $t - 1$ $(t \geq 2)$. We will show that the conclusion holds for t. Let G' be the subgraph of G induced on $V(G) \setminus \{d_{t,1}, d_{t,2}, d_{t,3}\}$. Then G' is $Q_{(t-1)+2}$-free 3-graph and G' contains a spanning subgraph $Q'_{(t-1)+2}$.

We consider the $(t + 2)$-sets of vertices with at least two vertices in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. If $\{x_1, x_2, \cdots, x_t, y_1, y_2\}$ forms a copy of K_{t+2} in $L(a_1)$, where $y_1, y_2 \in \{d_{t,1}, d_{t,2}, d_{t,3}\}$, $x_1, x_2, \cdots, x_t \in V(G) \setminus \{a_1, y_1, y_2\}$, then $\{y_1y_2a_1, d_{t,1}d_{t,2}d_{t,3}, b_1b_2c, d_{1,1}d_{1,2}d_{1,3}, \cdots, d_{t-1,1}d_{t-1,2}d_{t-1,3}\}$ forms a copy of Q_{t+2} in G.

Next, we consider the $(t + 2)$-sets of vertices with at most one vertex in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. By the induction hypothesis, the vertices forming a copy of K_{t+1} in $L_{G'}(a_1)$ must be of the form $\{a_2, c, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t-1,k_{t-1}}\}$ ($k_i = 1$ *or* 2 *or* $3; i = 1, 2, \cdots, t-1$). Thus, the only possible sets forming a copy of K_{t+2} in $L(a_1)$ are $\{a_2, c, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ ($k_i = 1$ *or* 2 *or* $3; i = 1, 2, \cdots, t$). Switching a_1 and b_1, we can show identically that the only possible sets forming a copy of K_{t+2} in $L(b_1)$ are $\{b_2, c, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ ($k_i = 1$ *or* 2 *or* $3; i = 1, 2, \cdots, t$).

Without loss of generality, we may assume that $\{a_2, c, d_{1,1}, d_{2,1}, \cdots, d_{t,1}\}$ forms a copy of K_{t+2} in $L(a_1)$, that is, $xya_1 \in G$, where $x, y \in \{a_2, c, d_{1,1}, d_{2,1}, \cdots, d_{t,1}\}$. In particular, $a_1a_2d_{i,1}, a_1cd_{i,1} \in G$ $(i \in [t])$.

If $L(b_1)$ contains no K_{t+2}, then we are done. Otherwise, we can obtain that the only possible set forming a copy of K_{t+2} in $L(b_1)$ is $\{b_2, c, d_{1,1}, d_{2,1}, \cdots, d_{t,1}\}$. Indeed, if $\{b_2, c, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ ($k_i = 2$ *or* $3, i = 1, 2, \cdots, t$) forms a copy of K_{t+2} in $L(b_1)$, then $\{b_1b_2c, b_1b_2d_{i,k_i}, a_1a_2d_{i,1}, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3},$

$\cdots, d_{i-1,1}d_{i-1,2}d_{i-1,3}, d_{i+1,1}d_{i+1,2}d_{i+1,3}, \cdots, d_{t,1}d_{t,2}d_{t,3}\}$ forms a copy of Q_{t+2} in G. So the only possible set forming a copy of K_{t+2} in $L(b_1)$ is $\{b_2, c, d_{1,1}, d_{2,1}, \cdots, d_{t,1}\}$, that is, $xyb_1 \in G$, where $x, y \in \{b_2, c, d_{1,1}, d_{2,1}, \cdots, d_{t,1}\}$. In particular, $b_1cd_{i,1} \in G$ $(i \in [t])$. We will show that $\exists\ i \in [t]$, such that $L(d_{i,2})$ contains no K_{t+2}.

We claim that the only possible sets forming a copy of K_{t+2} in $L(d_{1,2})$ are $\{d_{1,1}, d_{1,3}, c, d_{2,k_2}, d_{3,k_3}, \cdots, d_{t,k_t}\}$ $(k_i = 1$ *or* 2 *or* $3; i = 1, 2, \cdots, t)$. Applying induction on t. By the proof of Lemma 3.5, the conclusion holds for $t = 1$. For $t = 2$. We consider the 4-sets of vertices with at least two vertices in $\{d_{2,1}, d_{2,2}, d_{2,3}\}$. If $\{x_1, x_2, y_1, y_2\}$ forms a copy of K_4 in $L(d_{1,2})$, where $y_1, y_2 \in \{d_{2,1}, d_{2,2}, d_{2,3}\}$, $x_1, x_2 \in V(G)\backslash\{d_{1,2}, y_1, y_2\}$, then $\{y_1y_2d_{1,2}, d_{2,1}d_{2,2}d_{2,3}, b_1b_2c, a_1a_2d_{1,1}\}$ is a copy of Q_4 in G. In addition, we consider the 4-sets of vertices with at most one vertex in $\{d_{2,1}, d_{2,2}, d_{2,3}\}$. Let $G^0 = G[\{a_1, a_2, c, b_1, b_2, d_{1,1}, d_{1,2}, d_{1,3}\}]$. Since the vertices forming a copy of K_3 in $L_{G^0}(d_{1,2})$ must be of the form $\{d_{1,1}, d_{1,3}, c\}$, then the only possible sets forming a copy of K_4 in $L(d_{1,2})$ are $\{d_{1,1}, d_{1,3}, c, d_{2,k_2}\}$ $(k_2 = 1, 2, 3)$.

Suppose that the conclusion holds for $t-1$ $(t \geq 3)$, that is, if $\{a_2, c, d_{1,1}, d_{2,1}, \cdots, d_{t-1,1}\}$ forms a copy of K_{t+1} in $L_{G'}(a_1)$ and $\{b_2, c, d_{1,1}, d_{2,1}, \cdots, d_{t-1,1}\}$ forms a copy of K_{t+1} in $L_{G'}(b_1)$, then we can obtain that the only possible sets forming a copy of K_{t+1} in $L_{G'}(d_{1,2})$ are $\{d_{1,1}, d_{1,3}, c, d_{2,k_2}, d_{3,k_3}, \cdots, d_{t-1,k_{t-1}}\}$ $(k_i = 1$ *or* 2 *or* $3; i = 1, 2, \cdots, t-1)$. We will show that the conclusion holds for t.

Firstly, we consider the $(t+2)$-sets of vertices with at least two vertices in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. If $\{x_1, x_2, \cdots, x_t, y_1, y_2\}$ forms a copy of K_{t+2} in $L(d_{1,2})$, where $y_1, y_2 \in \{d_{t,1}, d_{t,2}, d_{t,3}\}$, $x_1, x_2, \cdots, x_t \in V(G) \setminus \{d_{1,2}, y_1, y_2\}$, then $\{y_1y_2d_{1,2}, d_{t,1}d_{t,2}d_{t,3}, a_1a_2d_{1,1}, b_1b_2c, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t-1,1}d_{t-1,2}d_{t-1,3}\}$ forms a copy of Q_{t+2} in G.

Next, we consider the $(t+2)$-sets of vertices with at most one vertex in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. By the induction hypothesis, the only possible sets forming a copy of K_{t+1} in $L_{G'}(d_{1,2})$ are $\{d_{1,1}, d_{1,3}, c, d_{2,k_2}, d_{3,k_3}, \cdots, d_{t-1,k_{t-1}}\}$.

Thus the only possible sets forming a copy of K_{t+2} in $L(d_{1,2})$ are $\{d_{1,1}, d_{1,3}, c, d_{2,k_2}, \cdots, d_{t,k_t}\}$. Similarly, we have that the only possible sets forming a copy of K_{t+2} in $L(d_{i,2})$ $(i = 2, 3, \cdots, t)$ are $\{d_{i,1}, d_{i,3}, c, d_{1,k_1}, d_{2,k_2}, \cdots, d_{i-1,k_{i-1}}, d_{i+1,k_{i+1}}, \cdots, d_{t,k_t}\}$.

If $\exists\ i \in [t]$, such that $L(d_{i,2})$ contains no K_{t+2}, then we are done. Otherwise, $\{d_{i,1}, d_{i,3}, c, d_{1,k_1}, d_{2,k_2}, \cdots, d_{i-1,k_{i-1}}, d_{i+1,k_{i+1}}, \cdots, d_{t,k_t}\}$ forms a copy of K_{t+2} in $L(d_{i,2})$ $(i \in [t])$, that is, $xyd_{i,2} \in G$, where $x, y \in \{d_{i,1}, d_{i,3}, c, d_{1,k_1}, d_{2,k_2}, \cdots, d_{i-1,k_{i-1}}, d_{i+1,k_{i+1}}, \cdots, d_{t,k_t}\}$. In particular, $d_{i,2}d_{i,3}c \in G$. Next, we consider the pair $\{a_2, b_2\}$. If $a_2b_2a_1 \in G$, then $\{a_1a_2b_2, a_1a_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. If $a_2b_2b_1 \in G$, then

$\{b_1b_2a_2, b_1b_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. If $a_2b_2c \in G$, then $\{b_2a_2c, a_1a_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. If $a_2b_2d_{i,1} \in G$ ($i \in [t]$), then $\{a_2b_2d_{i,1}, a_1a_2d_{i,1}, d_{i,2}d_{i,3}c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{i-1,1}d_{i-1,2}d_{i-1,3}, d_{i+1,1}d_{i+1,2}d_{i+1,3}, \cdots, d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. If $a_2b_2d_{i,k_i} \in G$ ($k_i = 2,3; i \in [t]$), then $\{a_2b_2d_{i,k_i}, a_1cd_{i,1}, b_1cd_{i,1}, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{i-1,1}d_{i-1,2}d_{i-1,3}, d_{i+1,1}d_{i+1,2}d_{i+1,3}, \cdots, d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. So the pair $\{a_2, b_2\}$ can not be covered by any edge of G, which contradicting Fact 2.3. □

Lemma 3.7. *Let G be a Q_3-free 3-graph. If G contains a subgraph Q_3' and $|V(G)| = |V(Q_3')| + 1$, then there exists a vertex v in $V(G)$ such that the link $L(v)$ contains no K_3.*

Proof. Let $u \in V(G) \setminus V(Q_3')$. If $L(u)$ contains no K_3, then we are done. Otherwise, we show that the only possible sets forming a copy of K_3 in $L(u)$ are $\{a_i, b_j, d_k\}$ ($i, j = 1, 2; k = 1, 2, 3$). If any of the triples in $\{a_1, a_2, c, b_1, b_2\}$ forms a copy of K_3 in $L(u)$, for example, if $\{a_1, a_2, c\}$ forms a copy of K_3 in $L(u)$, that is, $a_1a_2u, a_1cu, a_2cu \in G$, then any two edges of those and the independent edge $d_1d_2d_3$ forms a copy of Q_3 in G. Similarly, other cases can not happen. If any of the triples with one vertex in $\{a_1, a_2, c, b_1, b_2\}$ and two vertices in $\{d_1, d_2, d_3\}$ forms a copy of K_3 in $L(u)$, for example, if $\{d_1, d_2, c\}$ forms a copy of K_3 in $L(u)$, then $\{d_1d_2u, d_1d_2d_3, a_1a_2c\}$ forms a copy of Q_3 in G. Similarly, other cases can not happen. If $\{d_1, d_2, d_3\}$ forms a copy of K_3 in $L(u)$, then $\{d_1d_2u, d_1d_2d_3, a_1a_2c\}$ is a copy of Q_3 in G.

Next, we consider the triples with two vertices in $\{a_1, a_2, c, b_1, b_2\}$ and one vertex in $\{d_1, d_2, d_3\}$. If $\{a_1, a_2, d_k\}$ ($k = 1, 2, 3$) forms a copy of K_3 in $L(u)$, then $\{a_1a_2u, a_1a_2c, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{b_1, b_2, d_k\}$ ($k = 1, 2, 3$) forms a copy of K_3 in $L(u)$, then $\{b_1b_2u, b_1b_2c, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{a_i, c, d_k\}$ ($i = 1, 2; k = 1, 2, 3$) forms a copy of K_3 in $L(u)$, then $\{a_icu, a_1a_2c, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{b_j, c, d_k\}$ ($j = 1, 2; k = 1, 2, 3$) forms a copy of K_3 in $L(u)$, then $\{b_jcu, b_1b_2c, d_1d_2d_3\}$ is a copy of Q_3 in G.

Therefore, the only possible sets forming a copy of K_3 in $L(u)$ are $\{a_i, b_j, d_k\}$ ($i, j = 1, 2; k = 1, 2, 3$). Without loss of generality, we may assume that $\{a_1, b_1, d_1\}$ forms a copy of K_3 in $L(u)$, that is $a_1b_1u, a_1d_1u, b_1d_1u \in G$. We will show that $L(b_2)$ contains no K_3.

Firstly, we consider the triples in $\{a_1, a_2, c, b_1, u\}$. For example, if $\{a_1, a_2, c\}$ forms a copy of K_3 in $L(b_2)$, that is, $a_1a_2b_2, a_1cb_2, a_2cb_2 \in G$, then any two edges of those and the independent edge $d_1d_2d_3$ forms a copy of Q_3 in G. Similarly, other cases can not happen.

Secondly, we consider the triples with two vertices in $\{a_1, a_2, c, b_1, u\}$ and one vertex in $\{d_1, d_2, d_3\}$. If $\{a_1, a_2, d_k\}$ ($k = 1, 2, 3$) forms a copy of K_3 in

$L(b_2)$, then $\{a_1a_2b_2, a_1a_2c, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{a_i, c, d_k\}$ ($i = 1,2; k = 1,2,3$) forms a copy of K_3 in $L(b_2)$, then $\{a_icb_2, a_1a_2c, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{a_i, b_1, d_k\}$ ($i = 1,2; k = 1,2,3$) forms a copy of K_3 in $L(b_2)$, then $\{a_ib_1b_2, b_1b_2c, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{a_1, u, d_k\}$ ($k = 1,2,3$) forms a copy of K_3 in $L(b_2)$, then $\{ua_1b_2, ua_1b_1, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{a_2, u, d_1\}$ forms a copy of K_3 in $L(b_2)$, then $\{ud_1b_2, ud_1b_1, a_1a_2c\}$ is a copy of Q_3 in G. If $\{a_2, u, d_k\}$ ($k = 2,3$) forms a copy of K_3 in $L(b_2)$, then $\{a_2d_kb_2, a_1b_1u, b_1ud_1\}$ is a copy of Q_3 in G. If $\{c, b_1, d_1\}$ forms a copy of K_3 in $L(b_2)$, then $\{b_1d_1b_2, b_1d_1u, a_1a_2c\}$ is a copy of Q_3 in G. If $\{c, b_1, d_k\}$ ($k = 2,3$) forms a copy of K_3 in $L(b_2)$, then $\{b_1d_kb_2, cb_1b_2, a_1ud_1\}$ is a copy of Q_3 in G. If $\{c, u, d_k\}$ ($k = 1,2,3$) forms a copy of K_3 in $L(b_2)$, then $\{cub_2, b_1b_2c, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{b_1, u, d_k\}$ ($k = 1,2,3$) forms a copy of K_3 in $L(b_2)$, then $\{b_1ub_2, b_1b_2c, d_1d_2d_3\}$ is a copy of Q_3 in G.

Thirdly, we consider the triples with one vertex in $\{a_1, a_2, c, b_1, u\}$ and two vertices in $\{d_1, d_2, d_3\}$. If $\{d_k, d_t, a_i\}$ or $\{d_k, d_t, b_1\}$ or $\{d_k, d_t, c\}$ or $\{d_k, d_t, u\}$ ($1 \le k < t \le 3; i = 1,2$) forms a copy of K_3 in $L(b_2)$, then $\{d_kd_tb_2, d_1d_2d_3, a_1a_2c\}$ is a copy of Q_3 in G.

Finally, if $\{d_1, d_2, d_3\}$ forms a copy of K_3 in $L(b_2)$, then $\{d_1d_2b_2, d_1d_2d_3, a_1a_2c\}$ is a copy of Q_3 in G. The proof is complete. □

Lemma 3.8. *Let G be a Q_{t+2}-free 3-graph. If G contains a subgraph Q'_{t+2} and $|V(G)| = |V(Q'_{t+2})| + 1$, then there exists a vertex v in $V(G)$ such that the link $L(v)$ contains no K_{t+2}.*

Proof. Let $u \in V(G) \setminus V(Q'_{t+2})$. If $L(u)$ contains no K_{t+2}, then we are done. Otherwise, we show that the only possible sets forming a copy of K_{t+2} in $L(u)$ are $\{a_i, b_j, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ ($i, j = 1,2; k_s = 1$ *or* 2 *or* $3; s = 1,2,\cdots,t$). We apply induction on t. By the proof of Lemma 3.7, the conclusion holds for $t = 1$. Suppose that the conclusion holds for $t-1$ ($t \ge 2$). We will show that the conclusion holds for t.

We consider the $(t+2)$-sets of vertices with at least two vertices in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. If $\{x_1, x_2, \cdots, x_t, y_1, y_2\}$ forms a copy of K_{t+2} in $L(u)$, where $y_1, y_2 \in \{d_{t,1}, d_{t,2}, d_{t,3}\}$, $x_1, x_2, \cdots, x_t \in V(G) \setminus \{u, y_1, y_2\}$, then $\{y_1y_2u, d_{t,1}d_{t,2}d_{t,3}, a_1a_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t-1,1}d_{t-1,2}d_{t-1,3}\}$ is a copy of Q_{t+2} in G.

Next, we consider the $(t+2)$-sets of vertices with at most one vertex in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. Let $G' = G[V(G) \setminus \{d_{t,1}, d_{t,2}, d_{t,3}\}]$. Then G' is $Q_{(t-1)+2}$-free 3-graph and it contains a subgraph $Q'_{(t-1)+2}$. By the induction hypothesis, the vertices forming a copy of K_{t+1} in $L_{G'}(u)$ must be of the form $\{a_i, b_j, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t-1,k_{t-1}}\}$ ($i, j = 1,2; k_s = 1$ *or* 2 *or* 3; $s = 1,2,\cdots,t-1$). Thus the only possible sets forming a copy of K_{t+2}

in $L(u)$ are $\{a_i, b_j, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ $(i,j = 1,2; k_s = 1$ *or* 2 *or* $3; s = 1,2,\cdots,t)$.

Without loss of generality, we may assume that $\{a_1, b_1, d_{1,1}, d_{2,1}, \cdots, d_{t,1}\}$ forms a copy of K_{t+2} in $L(u)$. In this case, $\{a_1, b_1, d_{1,1}, d_{2,1}, \cdots, d_{t-1,1}\}$ forms a copy of K_{t+1} in $L(u)$, we will show that $L(b_2)$ contains no K_{t+2}. We apply induction on t. By the proof of Lemma 3.7, the conclusion holds for $t = 1$. Suppose that the conclusion holds for $t-1$ $(t \geq 2)$, that is, if $\{a_1, b_1, d_{1,1}, d_{2,1}, \cdots, d_{t-1,1}\}$ forms a copy of K_{t+1} in $L_{G'}(u)$, then we have $L_{G'}(b_2)$ contains no K_{t+1}. We will show that the conclusion holds for t.

We consider the $(t+2)$-sets of vertices with at least two vertices in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. If $\{x_1, x_2, \cdots, x_t, y_1, y_2\}$ forms a copy of K_{t+2} in $L(b_2)$, where $y_1, y_2 \in \{d_{t,1}, d_{t,2}, d_{t,3}\}$, $x_1, x_2, \cdots, x_t \in V(G) \setminus \{b_2, y_1, y_2\}$, then $\{y_1y_2b_2, d_{t,1}d_{t,2}d_{t,3}, a_1a_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t-1,1}d_{t-1,2}d_{t-1,3}\}$ is a copy of Q_{t+2} in G.

We consider the $(t+2)$-sets of vertices with at most one vertex in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. By the induction hypothesis, the vertices can not form a copy of K_{t+1} or K_{t+2} in $L_{G'}(b_2)$. Thus $L(b_2)$ contains no K_{t+2} in G. □

Lemma 3.9. *Let G be a dense Q_3-free 3-graph. If G contains a subgraph Q_3' and $|V(G)| = |V(Q_3')| + 2$, then there exists a vertex v in $V(G)$ such that the link $L(v)$ contains no K_3.*

Proof. Let $u_1, u_2 \in V(G) \setminus V(Q_3')$. If $L(u_1)$ contains no K_3, then we are done. Otherwise, we show that the only possible sets forming a copy of K_3 in $L(u_1)$ are $\{a_i, b_j, d_k\}$ or $\{c, u_2, d_k\}$ $(i,j = 1,2; k = 1,2,3)$. If any of the triples in $\{a_1, a_2, c, b_1, b_2, u_2\}$ forms a copy of K_3 in $L(u_1)$, for example, if $\{a_1, a_2, c\}$ forms a copy of K_3 in $L(u_1)$, that is, $a_1a_2u_1, a_1cu_1, a_2cu_1 \in G$, then any two edges of those and the independent edge $d_1d_2d_3$ forms a copy of Q_3 in G. Similarly, other cases can not happen. If any of the triples with one vertex in $\{a_1, a_2, c, b_1, b_2, u_2\}$ and two vertices in $\{d_1, d_2, d_3\}$ forms a copy of K_3 in $L(u_1)$, for example, if $\{d_1, d_2, u_2\}$ forms a copy of K_3 in $L(u_1)$, then $\{d_1d_2u_1, d_1d_2d_3, a_1a_2c\}$ forms a copy of Q_3 in G. Similarly, other cases can not happen. If $\{d_1, d_2, d_3\}$ forms a copy of K_3 in $L(u_1)$, then $\{d_1d_2u_1, d_1d_2d_3, a_1a_2c\}$ is a copy of Q_3 in G.

Next, we consider the triples with two vertices in $\{a_1, a_2, c, b_1, b_2, u_2\}$ and one vertex in $\{d_1, d_2, d_3\}$.

If $\{a_1, a_2, d_k\}$ $(k=1,2,3)$ forms a copy of K_3 in $L(u_1)$, then $\{a_1a_2u_1, a_1a_2c, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{b_1, b_2, d_k\}$ $(k = 1,2,3)$ forms a copy of K_3 in $L(u_1)$, then $\{b_1b_2u_1, b_1b_2c, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{a_i, c, d_k\}$ $(i = 1,2; k = 1,2,3)$ forms a copy of K_3 in $L(u_1)$, then $\{a_icu_1, a_1a_2c, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{b_j, c, d_k\}$ $(j = 1,2; k = 1,2,3)$ forms a copy of

K_3 in $L(u_1)$, then $\{b_j c u_1, b_1 b_2 c, d_1 d_2 d_3\}$ is a copy of Q_3 in G. If $\{a_i, u_2, d_k\}$ $(i = 1, 2; k = 1, 2, 3)$ forms a copy of K_3 in $L(u_1)$, then $\{a_i u_2 u_1, d_k u_2 u_1, b_1 b_2 c\}$ forms a copy of Q_3. If $\{b_j, u_2, d_k\}$ $(j = 1, 2; k = 1, 2, 3)$ forms a copy of K_3 in $L(u_1)$, then $\{b_j u_2 u_1, d_k u_2 u_1, a_1 a_2 c\}$ forms a copy of Q_3. Therefore, the only possible sets forming a copy of K_3 in $L(u_1)$ are $\{a_i, b_j, d_k\}$ or $\{c, u_2, d_k\}$ $(i, j = 1, 2; k = 1, 2, 3)$. Switching u_1 and u_2, we can show identically that the only possible sets forming a copy of K_3 in $L(u_2)$ are $\{a_i, b_j, d_k\}$ or $\{c, u_1, d_k\}$ $(i, j = 1, 2; k = 1, 2, 3)$.

Case 1: A set in the form of $\{a_i, b_j, d_k\}$ $(i, j = 1, 2; k = 1, 2, 3)$ forms a copy of K_3 in $L(u_1)$.

Without loss of generality, we assume that $\{a_1, b_1, d_1\}$ forms a copy of K_3 in $L(u_1)$, that is, $a_1 b_1 u_1, a_1 d_1 u_1, b_1 d_1 u_1 \in G$. We will show that $L(u_2)$ contains no K_3. Recall that the only possible sets forming a copy of K_3 in $L(u_2)$ are $\{a_i, b_j, d_k\}$ or $\{c, u_1, d_k\}$ $(i, j = 1, 2; k = 1, 2, 3)$.

If $\{a_1, b_i, d_1\}$ $(i = 1, 2)$ forms a copy of K_3 in $L(u_2)$, then $\{a_1 d_1 u_2, a_1 d_1 u_1, b_1 b_2 c\}$ is a copy of Q_3 in G. If $\{a_1, b_1, d_i\}$ $(i = 2, 3)$ forms a copy of K_3 in $L(u_2)$, then $\{a_1 b_1 u_2, a_1 b_1 u_1, d_1 d_2 d_3\}$ is a copy of Q_3 in G. If $\{a_1, b_2, d_i\}$ $(i = 2, 3)$ forms a copy of K_3 in $L(u_2)$, then $\{b_2 d_i u_2, a_1 b_1 u_1, b_1 d_1 u_1\}$ is a copy of Q_3 in G. If $\{a_2, b_1, d_1\}$ forms a copy of K_3 in $L(u_2)$, then $\{b_1 d_1 u_2, b_1 d_1 u_1, a_1 a_2 c\}$ is a copy of Q_3 in G. If $\{a_2, b_1, d_i\}$ $(i = 2, 3)$ forms a copy of K_3 in $L(u_2)$, then $\{a_2 d_i u_2, a_1 b_1 u_1, b_1 d_1 u_1\}$ is a copy of Q_3 in G. If $\{a_2, b_2, d_i\}$ $(i = 1, 2, 3)$ forms a copy of K_3 in $L(u_2)$, then $\{a_2 b_2 u_2, b_1 d_1 u_1, a_1 b_1 u_1\}$ is a copy of Q_3 in G. If $\{c, u_1, d_1\}$ forms a copy of K_3 in $L(u_2)$, then $\{u_1 d_1 u_2, a_1 u_1 d_1, b_1 b_2 c\}$ is a copy of Q_3 in G. If $\{c, u_1, d_i\}$ $(i = 2, 3)$ forms a copy of K_3 in $L(u_2)$, then $\{c d_i u_2, a_1 b_1 u_1, b_1 d_1 u_1\}$ is a copy of Q_3 in G.

From the above, we have $L(u_2)$ contains no K_3.

Case 2: A set in the form of $\{c, u_2, d_i\}$ $(i = 1, 2, 3)$ forms a copy of K_3 in $L(u_1)$.

Without loss of generality, we assume that $\{c, u_2, d_1\}$ forms a copy of K_3 in $L(u_1)$, that is, $c u_2 u_1, c d_1 u_1, u_2 d_1 u_1 \in G$. We claim that either $L(u_2)$ contains no K_3 or $\{c, u_1, d_1\}$ forms a copy of K_3 in $L(u_2)$. Recall that the only possible sets forming a copy of K_3 in $L(u_2)$ are $\{a_i, b_j, d_k\}$ or $\{c, u_1, d_k\}$ $(i, j = 1, 2; k = 1, 2, 3)$.

If $\{a_i, b_j, d_1\}$ $(i, j = 1, 2)$ forms a copy of K_3 in $L(u_2)$, then $\{b_j d_1 u_2, u_2 d_1 u_1, a_1 a_2 c\}$ is a copy of Q_3 in G. If $\{a_i, b_j, d_k\}$ $(i, j = 1, 2, k = 2, 3)$ forms a copy of K_3 in $L(u_2)$, then $\{a_i b_j u_2, b_j d_k u_2, c u_1 d_1\}$ is a copy of Q_3 in G. If $\{c, u_1, d_i\}$ $(i = 2, 3)$ forms a copy of K_3 in $L(u_2)$, then $\{u_1 d_i u_2, u_1 d_1 u_2, a_1 a_2 c\}$ is a copy of Q_3 in G.

Therefore, if $L(u_2)$ contains no K_3, then we are done. Otherwise, $\{c, u_1, d_1\}$ forms a copy of K_3 in $L(u_2)$, that is, $c u_1 u_2, u_1 d_1 u_2, c d_1 u_2 \in G$, then we will show that $L(d_2)$ contains no K_3.

Firstly, we consider the triples in $\{a_1, a_2, u_1, u_2, d_1, d_3\}$. If any of the triples in $\{a_1, a_2, u_1, u_2, d_1, d_3\}$ forms a copy of K_3 in $L(d_2)$, for example, if $\{a_1, u_1, d_1\}$ forms a copy of K_3 in $L(d_2)$, that is, $a_1u_1d_2, a_1d_1d_2$, $u_1d_1d_2 \in G$, then any two edges of those and the edge b_1b_2c forms a copy of Q_3 in G. Similarly, other cases can not happen.

Secondly, we consider the triples with two vertices in $\{a_1, a_2, u_1, u_2, d_1, d_3\}$ and one vertex in $\{b_1, b_2, c\}$.

If any of $\{a_1, a_2, b_i\}$, $\{a_1, a_2, c\}$ $(i = 1, 2)$ forms a copy of K_3 in $L(d_2)$, then $\{a_1a_2d_2, a_1a_2c, u_1u_2d_1\}$ is a copy of Q_3 in G. If $\{a_i, u_j, b_k\}$ $(i, j, k = 1, 2)$ forms a copy of K_3 in $L(d_2)$, then $\{a_ib_kd_2, u_1d_1u_2, u_1d_1c\}$ is a copy of Q_3 in G. If $\{a_i, u_j, c\}$ $(i, j = 1, 2)$ forms a copy of K_3 in $L(d_2)$, then $\{a_icd_2, a_1a_2c, u_1u_2d_1\}$ is a copy of Q_3 in G. If any of $\{a_i, d_j, b_k\}$, $\{a_i, d_j, c\}$ $(i, k = 1, 2; j = 1, 3)$ forms a copy of K_3 in $L(d_2)$, then $\{a_id_jd_2, d_1d_2d_3, b_1b_2c\}$ is a copy of Q_3 in G. If any of $\{u_1, u_2, b_i\}$ $(i = 1, 2)$, $\{u_1, u_2, c\}$ forms a copy of K_3 in $L(d_2)$, then $\{u_1u_2d_2, u_1u_2d_1, a_1a_2c\}$ is a copy of Q_3 in G. If any of $\{u_i, d_j, b_k\}$, $\{u_i, d_j, c\}$ $(i, k = 1, 2; j = 1, 3)$ forms a copy of K_3 in $L(d_2)$, then $\{u_id_jd_2, d_1d_2d_3, a_1a_2c\}$ is a copy of Q_3 in G. If $\{d_1, d_3, b_i\}$ $(i = 1, 2)$ forms a copy of K_3 in $L(d_2)$, then $\{d_1b_id_2, d_1d_2d_3, a_1a_2c\}$ is a copy of Q_3 in G.

If $\{d_1, d_3, c\}$ forms a copy of K_3 in $L(d_2)$, i.e., $d_1d_2d_3, d_1d_2c, d_3d_2c \in G$. Let's consider the pairs $\{a_i, b_j\}$ $(i, j = 1, 2)$. If $a_ib_ju_k \in G$ $(k = 1, 2)$, then $\{a_ib_ju_k, d_1d_2d_3, d_1d_2c\}$ forms a copy of Q_3 in G. If $a_ib_jc \in G$, then $\{a_ib_jc, b_1b_2c, d_1d_2d_3\}$ forms a copy of Q_3 in G. If $a_1b_ja_2 \in G$, then $\{a_1b_ja_2, a_1a_2c, d_1d_2d_3\}$ forms a copy of Q_3 in G. If $a_ib_1b_2 \in G$, then $\{a_ib_1b_2, b_1b_2c, d_1d_2d_3\}$ forms a copy of Q_3 in G. Since G is dense, by Fact 2.3, the pairs $\{a_i, b_j\}$ must be covered by an edge in the form of $a_ib_jd_k$. If $a_ib_jd_k \in G$ $(k = 2, 3)$, recall that $cu_1u_2, u_1u_2d_1 \in G$, then $\{cu_1u_2, u_1u_2d_1, a_ib_jd_2\}$ forms a copy of Q_3 in G. So $a_1b_1d_1, a_1b_2d_1, a_2b_1d_1, a_2b_2d_1 \in G$. Then we have $\{cu_1u_2, a_1b_1d_1, a_1b_2d_1\}$ forms a copy of Q_3 in G.

Thirdly, we consider the triples with one vertex in $\{a_1, a_2, u_1, u_2, d_1, d_3\}$ and two vertices in $\{b_1, b_2, c\}$. If any of $\{b_1, b_2, a_i\}$, $\{b_1, b_2, u_i\}$, $\{b_1, b_2, d_j\}$ $(i = 1, 2; j = 1, 3)$ forms a copy of K_3 in $L(d_2)$, then $\{b_1b_2d_2, b_1b_2c, u_1u_2d_1\}$ is a copy of Q_3 in G. If $\{b_i, c, a_j\}$, $\{b_i, c, u_j\}$, $\{b_i, c, d_k\}$ $(i, j = 1, 2; k = 1, 3)$ forms a copy of K_3 in $L(d_2)$, then $\{b_icd_2, b_1b_2c, u_1u_2d_1\}$ is a copy of Q_3 in G.

Finally, if $\{b_1, b_2, c\}$ forms a copy of K_3 in $L(d_2)$, then $\{b_1b_2d_2, b_1b_2c, u_1u_2d_1\}$ is a copy of Q_3 in G. The proof is complete. ☐

Lemma 3.10. *Let G be a dense Q_{t+2}-free 3-graph. If G contains a subgraph Q'_{t+2} and $|V(G)| = |V(Q'_{t+2})| + 2$, then there exists a vertex v in $V(G)$ such that the link $L(v)$ contains no K_{t+2}.*

Proof. Let $u_1, u_2 \in V(G) \setminus V(Q'_{t+2})$. If $L(u_1)$ contains no K_{t+2}, then we are done. Otherwise, we show that the only possible sets forming a copy of

K_{t+2} in $L(u_1)$ are $\{a_i, b_j, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ or $\{c, u_2, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ $(i, j = 1, 2; k_s = 1$ *or* 2 *or* $3; s = 1, 2, \cdots, t)$. We apply induction on t. By the proof of Lemma 3.9, the conclusion holds for $t = 1$. Suppose that the conclusion holds for $t-1$ $(t \geq 2)$. We will show that the conclusion holds for t. Let $G' = G[V(G) \setminus \{d_{t,1}, d_{t,2}, d_{t,3}\}]$.

Consider the $(t+2)$-sets of vertices with at least two vertices in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. If $\{x_1, x_2, \cdots, x_t, y_1, y_2\}$ forms a copy of K_{t+2} in $L(u_1)$, where $y_1, y_2 \in \{d_{t,1}, d_{t,2}, d_{t,3}\}$, $x_1, x_2, \cdots, x_t \in V(G) \setminus \{u_1, y_1, y_2\}$, then $\{y_1y_2u_1, d_{t,1}d_{t,2}d_{t,3}, a_1a_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t-1,1}d_{t-1,2}d_{t-1,3}\}$ is a copy of Q_{t+2} in G.

Next, consider the $(t+2)$-sets of vertices with at most one vertex in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. By the induction hypothesis, the vertices forming a copy of K_{t+1} in $L_{G'}(u_1)$ must be of the form $\{a_i, b_j, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t-1,k_{t-1}}\}$ or $\{c, u_2, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t-1,k_{t-1}}\}$ $(i, j=1, 2; k_s=1$ *or* 2 *or* $3; s=1, 2, \cdots, t-1)$. Thus the only possible sets forming a copy of K_{t+2} in $L(u_1)$ are $\{a_i, b_j, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ or $\{c, u_2, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ $(i, j = 1, 2; k_s = 1$ *or* 2 *or* $3;$ $s = 1, 2, \cdots, t)$. Switching u_1 and u_2, we can show identically that the only possible sets forming a copy of K_{t+2} in $L(u_2)$ are $\{a_i, b_j, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ or $\{c, u_1, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ $(i, j = 1, 2; k_s = 1$ *or* 2 *or* $3; s = 1, 2, \cdots, t)$.

Case 1: A set in the form of $\{a_i, b_j, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ $(i, j = 1, 2; k_s = 1$ *or* 2 *or* $3; s = 1, 2, \cdots, t)$ forms a copy of K_{t+2} in $L(u_1)$.

We will show that $L(u_2)$ contains no K_{t+2}. Applying induction on t. By the proof of Lemma 3.9, the conclusion holds for $t = 1$. Suppose that the conclusion holds for $t-1$ $(t \geq 2)$. We will show that the conclusion holds for t.

Consider the $(t+2)$-sets of vertices with at least two vertices in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. If $\{x_1, x_2, \cdots, x_t, y_1, y_2\}$ forms a copy of K_{t+2} in $L(u_2)$, where $y_1, y_2 \in \{d_{t,1}, d_{t,2}, d_{t,3}\}$, $x_1, x_2, \cdots, x_t \in V(G) \setminus \{u_2, y_1, y_2\}$, then $\{y_1y_2u_2, d_{t,1}d_{t,2}d_{t,3}, a_1a_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t-1,1}d_{t-1,2}d_{t-1,3}\}$ is a copy of Q_{t+2} in G.

Next, consider the $(t+2)$-sets of vertices with at most one vertex in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. By the induction hypothesis, the vertices can not form a copy of K_{t+1} or K_{t+2} in $L_{G'}(u_2)$. Thus $L(u_2)$ contains no K_{t+2}.

Case 2: A set in the form of $\{c, u_2, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ $(k_s = 1$ *or* 2 *or* $3;$ $s = 1, 2, \cdots, t)$ forms a copy of K_{t+2} in $L(u_1)$.

Without loss of generality, we assume that $\{c, u_2, d_{1,1}, d_{2,1}, \cdots, d_{t,1}\}$ forms a copy of K_{t+2} in $L(u_1)$, that is, $xyu_1 \in G$, where $x, y \in \{c, u_2, d_{1,1}, d_{2,1}, \cdots, d_{t,1}\}$. In particular, $u_1cd_{i,1}, u_1u_2d_{i,1} \in G$ $(i \in [t])$. In this case, $\{c, u_2, d_{1,1}, d_{2,1}, \cdots, d_{t-1,1}\}$ forms a copy of K_{t+1} in $L(u_1)$. We claim that the only possible set forming a copy of K_{t+2} in $L(u_2)$ is $\{c, u_1, d_{1,1}, d_{2,1}, \cdots, d_{t,1}\}$. Applying induction on t. By the proof of Lemma 3.9, the conclusion holds for $t = 1$.

Suppose that the conclusion holds for $t-1$ ($t \geq 2$). We will show that the conclusion holds for t.

Consider the $(t+2)$-sets of vertices with at least two vertices in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. If $\{x_1, x_2, \cdots, x_t, y_1, y_2\}$ forms a copy of K_{t+2} in $L(u_2)$, where $y_1, y_2 \in \{d_{t,1}, d_{t,2}, d_{t,3}\}$, $x_1, x_2, \cdots, x_t \in V(G) \setminus \{u_2, y_1, y_2\}$, then $\{y_1y_2u_2, d_{t,1}d_{t,2}d_{t,3}, a_1a_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t-1,1}d_{t-1,2}d_{t-1,3}\}$ is a copy of Q_{t+2} in G.

Next, consider the $(t+2)$-sets of vertices with at most one vertex in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. By the induction hypothesis, the vertices forming a copy of K_{t+1} in $L_{G'}(u_2)$ must be of the form $\{c, u_1, d_{1,1}, d_{2,1}, \cdots, d_{t-1,1}\}$. If $\{c, u_1, d_{1,1}, d_{2,1}, \cdots, d_{t-1,1}, d_{t,k_t}\}$ ($k_t = 2, 3$) forms a copy of K_{t+2} in $L(u_2)$, then $\{u_1u_2d_{t,k_t}, u_1u_2d_{t,1}, a_1a_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t-1,1}d_{t-1,2}d_{t-1,3}\}$ is a copy of Q_{t+2} in G. From the above, if $L(u_2)$ contains no K_{t+2}, then we are done. Otherwise, we have that $\{c, u_1, d_{1,1}, d_{2,1}, \cdots, d_{t,1}\}$ forms a copy of K_{t+2} in $L(u_2)$. We claim that the only possible set forming a copy of K_{t+2} in $L(a_1)$ is $\{a_2, c, d_{1,1}, d_{2,1}, \cdots, d_{t,1}\}$. We will apply induction on t. Let's first show for $t = 1$. Suppose that $\{c, u_2, d_{1,1}\}$ forms a copy of K_3 in $L(u_1)$ and $\{c, u_1, d_{1,1}\}$ forms a copy of K_3 in $L(u_2)$. We show that the only possible set forming a copy of K_3 in $L(a_1)$ is $\{a_2, c, d_{1,1}\}$.

Firstly, we consider the triples in $\{a_2, c, b_1, b_2, u_1, u_2\}$. If any of the triples in $\{a_2, c, b_1, b_2, u_1, u_2\}$ forms a copy of K_3 in $L(a_1)$, for example, if $\{a_2, c, u_1\}$ forms a copy of K_3 in $L(a_1)$, that is, $a_1a_2c, a_1a_2u_1, a_1cu_1 \in G$, then any two edges of those and the edge $d_{1,1}d_{1,2}d_{1,3}$ forms a copy of Q_3 in G. Similarly, other cases can not happen.

Secondly, we consider the triples with one vertex in $\{a_2, c, b_1, b_2, u_1, u_2\}$ and two vertices in $\{d_{1,1}, d_{1,2}, d_{1,3}\}$. If $\{x, y, z\}$ forms a copy of K_3 in $L(a_1)$, where $x \in \{a_2, c, b_1, b_2, u_1, u_2\}$, $y, z \in \{d_{1,1}, d_{1,2}, d_{1,3}\}$, then $\{yza_1, d_{1,1}d_{1,2}d_{1,3}, b_1b_2c\}$ is a copy of Q_3 in G.

Thirdly, if $\{d_{1,1}, d_{1,2}, d_{1,3}\}$ forms a copy of K_3 in $L(a_1)$, then $\{d_{1,1}d_{1,2}a_1, d_{1,1}d_{1,2}d_{1,3}, b_1b_2c\}$ is a copy of Q_3 in G.

Finally, we consider the triples with two vertices in $\{a_2, c, b_1, b_2, u_1, u_2\}$ and one vertex in $\{d_{1,1}, d_{1,2}, d_{1,3}\}$. If $\{a_2, b_i, d_{1,k}\}$ ($i = 1, 2; k = 1, 2, 3$) forms a copy of K_3 in $L(a_1)$, then $\{a_1a_2b_i, a_1a_2c, d_{1,1}d_{1,2}d_{1,3}\}$ is a copy of Q_3 in G. If $\{a_2, u_i, d_{1,k}\}$ ($i = 1, 2; k = 1, 2, 3$) forms a copy of K_3 in $L(a_1)$, then $\{a_1a_2u_i, a_1a_2c, d_{1,1}d_{1,2}d_{1,3}\}$ is a copy of Q_3 in G. If $\{c, b_i, d_{1,k}\}$ ($i = 1, 2; k = 1, 2, 3$) forms a copy of K_3 in $L(a_1)$, then $\{a_1cb_i, a_1a_2c, d_{1,1}d_{1,2}d_{1,3}\}$ is a copy of Q_3 in G. If $\{c, u_i, d_{1,k}\}$ ($i = 1, 2; k = 1, 2, 3$) forms a copy of K_3 in $L(a_1)$, then $\{a_1cu_i, cu_1u_2, d_{1,1}d_{1,2}d_{1,3}\}$ is a copy of Q_3 in G. If $\{b_1, b_2, d_{1,k}\}$ ($k = 1, 2, 3$) forms a copy of K_3 in $L(a_1)$, then $\{a_1b_1b_2, b_1b_2c, d_{1,1}d_{1,2}d_{1,3}\}$ is a copy of Q_3 in G. If $\{b_i, u_j, d_{1,1}\}$ ($i, j = 1, 2$) forms a copy of K_3 in $L(a_1)$, then $\{a_1u_jd_{1,1}, u_1u_2d_{1,1}, b_1b_2c\}$ is a copy of Q_3 in G. If $\{b_i, u_j, d_{1,k}\}$ ($i, j =$

$1,2;k=2,3)$ forms a copy of K_3 in $L(a_1)$, then $\{a_1b_id_{1,k}, cu_1u_2, u_1u_2d_{1,1}\}$ is a copy of Q_3 in G. If $\{u_1,u_2,d_{1,k}\}$ $(k=1,2,3)$ forms a copy of K_3 in $L(a_1)$, then $\{u_1u_2a_1, u_1u_2c, d_{1,1}d_{1,2}d_{1,3}\}$ is a copy of Q_3 in G. If $\{a_2,c,d_{1,k}\}$ $(k=2,3)$ forms a copy of K_3 in $L(a_1)$, then $\{a_1a_2d_{1,k}, a_1a_2c, u_1u_2d_{1,1}\}$ is a copy of Q_3 in G. So the only possible set forming a copy of K_3 in $L(a_1)$ is $\{a_2,c,d_{1,1}\}$.

Suppose that it holds for $t-1$ $(t\geq 2)$, that is, if $\{c,u_2,d_{1,1},d_{2,1},\cdots,d_{t-1,1}\}$ forms a copy of K_{t+1} in $L_{G'}(u_1)$, and $\{c,u_1,d_{1,1},d_{2,1},\cdots,d_{t-1,1}\}$ forms a copy of K_{t+1} in $L_{G'}(u_2)$. Then the only possible set forming a copy of K_{t+1} in $L_{G'}(a_1)$ is $\{a_2,c,d_{1,1},d_{2,1},\cdots,d_{t-1,1}\}$. We will show that the conclusion holds for t.

Consider the $(t+2)$-sets of vertices with at least two vertices in $\{d_{t,1},d_{t,2},d_{t,3}\}$. If $\{x_1,x_2,\cdots,x_t,y_1,y_2\}$ forms a copy of K_{t+2} in $L(a_1)$, where $y_1,y_2\in\{d_{t,1},d_{t,2},d_{t,3}\}$, $x_1,x_2,\cdots,x_t\in V(G)\setminus\{a_1,y_1,y_2\}$, then $\{y_1y_2a_1, d_{t,1}d_{t,2}d_{t,3}, b_1b_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3},\cdots,d_{t-1,1}d_{t-1,2}d_{t-1,3}\}$ is a copy of Q_{t+2} in G.

Next, consider the $(t+2)$-sets of vertices with at most one vertex in $\{d_{t,1},d_{t,2},d_{t,3}\}$. By the induction hypothesis, the vertices forming a copy of K_{t+1} in $L_{G'}(a_1)$ must be of the form $\{a_2,c,d_{1,1},d_{2,1},\cdots,d_{t-1,1}\}$. If $\{a_2,c,d_{1,1},d_{2,1},\cdots,d_{t-1,1},d_{t,k}\}$ $(k=2,3)$ forms a copy of K_{t+2} in $L(a_1)$, then $\{a_1a_2d_{t,k}, a_1a_2c, u_1u_2d_{t,1}, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3},\cdots,d_{t-1,1}d_{t-1,2}d_{t-1,3}\}$ is a copy of Q_{t+2} in G.

So the only possible set forming a copy of K_{t+2} in $L(a_1)$ is $\{a_2,c,d_{1,1},d_{2,1},\cdots,d_{t,1}\}$. If $L(a_1)$ contains no K_{t+2}, then we are done. Otherwise, we assume that $\{a_2,c,d_{1,1},d_{2,1},\cdots,d_{t,1}\}$ forms a copy of K_{t+2} in $L(a_1)$. Let's consider the pair $\{a_2,b_2\}$. If $a_2b_2a_1\in G$, then $\{a_1a_2b_2, a_1a_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3},\cdots,d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. If $a_2b_2b_1\in G$, then $\{b_1b_2a_2, b_1b_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3},\cdots,d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. If $a_2b_2c\in G$, then $\{a_2b_2c, a_1a_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3},\cdots,d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. If $a_2b_2u_i\in G$ $(i=1,2)$, then $\{a_2b_2u_i, a_1cd_{1,1}, u_{3-i}cd_{1,1}, d_{2,1}d_{2,2}d_{2,3},\cdots,d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. If $a_2b_2d_{i,1}\in G$ $(i\in[t])$, then $\{a_2b_2d_{i,1}, a_1a_2d_{i,1}, cu_1u_2, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3},\cdots, d_{i-1,1}d_{i-1,2}d_{i-1,3}, d_{i+1,1}d_{i+1,2}d_{i+1,3},\cdots,d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. If $a_2b_2d_{i,k_i}\in G$ $(k_i=2,3;i\in[t])$, then $\{a_2b_2d_{i,k_i}, a_1cd_{i,1}, u_1cd_{i,1}, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3},\cdots,d_{i-1,1}d_{i-1,2}d_{i-1,3}, d_{i+1,1}d_{i+1,2}d_{i+1,3},\cdots,d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. Thus we obtain that the pair $\{a_2,b_2\}$ can not be covered by any edge of G, which contradicting Fact 2.3. So we have $L(a_1)$ contains no K_{t+2}. This completes the proof. $\square$

Lemma 3.11. *Let G be a dense Q_3-free 3-graph. If G contains a subgraph Q_3' and $|V(G)|>|V(Q_3')|+2$, then there exists a vertex v in $V(G)$ such that the link $L(v)$ contains no K_3.*

Proof. Let $u_1, u_2, \cdots, u_p \in V(G) \setminus V(Q_3')$ $(p \geq 3)$. If $L(u_1)$ contains no K_3, then we are done. Otherwise, we show that the only possible sets forming a copy of K_3 in $L(u_1)$ are $\{a_i, b_j, d_k\}$ or $\{c, u_l, d_k\}$, $(i, j = 1, 2; k = 1, 2, 3; 2 \leq l \leq p)$. If any of the triples in $\{a_1, a_2, c, b_1, b_2, u_2, u_3, \cdots, u_p\}$ forms a copy of K_3 in $L(u_1)$, for example, if $\{a_1, c, u_2\}$ forms a copy of K_3 in $L(u_1)$, that is, $a_1cu_1, cu_2u_1, a_1u_2u_1 \in G$, then any two edges of those and the independent edge $d_1d_2d_3$ forms a copy of Q_3 in G. Similarly, other cases can not happen. If any of the triples with one vertex in $\{a_1, a_2, c, b_1, b_2, u_2, u_3, \cdots, u_p\}$ and two vertices in $\{d_1, d_2, d_3\}$ forms a copy of K_3 in $L(u_1)$, for example, if $\{d_1, d_2, u_2\}$ forms a copy of K_3 in $L(u_1)$, then $\{d_1d_2u_1, d_1d_2d_3, a_1a_2c\}$ forms a copy of Q_3 in G. Similarly, other cases can not happen. If $\{d_1, d_2, d_3\}$ forms a copy of K_3 in $L(u_1)$, then $\{d_1d_2u_1, d_1d_2d_3, a_1a_2c\}$ is a copy of Q_3 in G.

Next, we consider the triples with two vertices in $\{a_1, a_2, c, b_1, b_2, u_2, u_3, \cdots, u_p\}$ and one vertex in $\{d_1, d_2, d_3\}$. If $\{a_1, a_2, d_k\}$ $(k = 1, 2, 3)$ forms a copy of K_3 in $L(u_1)$, then $\{a_1a_2u_1, a_1a_2c, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{b_1, b_2, d_k\}$ $(k = 1, 2, 3)$ forms a copy of K_3 in $L(u_1)$, then $\{b_1b_2u_1, b_1b_2c, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{a_i, c, d_k\}$ $(i = 1, 2; k = 1, 2, 3)$ forms a copy of K_3 in $L(u_1)$, then $\{a_icu_1, a_1a_2c, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{b_j, c, d_k\}$ $(j = 1, 2; k = 1, 2, 3)$ forms a copy of K_3 in $L(u_1)$, then $\{b_jcu_1, b_1b_2c, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{a_i, u_l, d_k\}$ $(i = 1, 2; k = 1, 2, 3; 2 \leq l \leq p)$ forms a copy of K_3 in $L(u_1)$, then $\{a_iu_lu_1, d_ku_lu_1, b_1b_2c\}$ forms a copy of Q_3. If $\{b_i, u_l, d_k\}$ $(i = 1, 2; k = 1, 2, 3; 2 \leq l \leq p)$ forms a copy of K_3 in $L(u_1)$, then $\{b_iu_lu_1, d_ku_lu_1, a_1a_2c\}$ forms a copy of Q_3. If $\{u_l, u_t, d_k\}$ $(k = 1, 2, 3; 2 \leq l < t \leq p)$ forms a copy of K_3 in $L(u_1)$, then $\{u_lu_tu_1, d_ku_tu_1, a_1a_2c\}$ forms a copy of Q_3.

Therefore, the only possible sets forming a copy of K_3 in $L(u_1)$ are $\{a_i, b_j, d_k\}$ or $\{c, u_l, d_k\}$ $(i, j = 1, 2; k = 1, 2, 3; 2 \leq l \leq p)$.

Case 1: A set in the form of $\{a_i, b_j, d_k\}$ $(i, j = 1, 2; k = 1, 2, 3)$ forms a copy of K_3 in $L(u_1)$.

Without loss of generality, we may assume that $\{a_1, b_1, d_1\}$ forms a copy of K_3 in $L(u_1)$, that is, $a_1b_1u_1, a_1d_1u_1, b_1d_1u_1 \in G$. We will show that $L(u_3)$ contains no K_3. Note that the only possible sets forming a copy of K_3 in $L(u_3)$ are $\{a_i, b_j, d_k\}$ or $\{c, u_l, d_k\}$ $(i, j = 1, 2; k = 1, 2, 3; l = 1, 2, 4, \cdots, p)$.

If $\{a_1, b_1, d_k\}$ $(k = 1, 2, 3)$ forms a copy of K_3 in $L(u_3)$, then $\{a_1b_1u_3, a_1b_1u_1, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{a_1, b_2, d_1\}$ forms a copy of K_3 in $L(u_3)$, then $\{a_1d_1u_3, a_1d_1u_1, b_1b_2c\}$ is a copy of Q_3 in G. If $\{a_1, b_2, d_k\}$ $(k = 2, 3)$ forms a copy of K_3 in $L(u_3)$, then $\{b_2d_ku_3, a_1b_1u_1, a_1d_1u_1\}$ is a copy of Q_3 in G. If $\{a_2, b_1, d_1\}$ forms a copy of K_3 in $L(u_3)$, then $\{b_1d_1u_3, b_1d_1u_1, a_1a_2c\}$ is a copy of Q_3 in G. If $\{a_2, b_1, d_k\}$ $(k = 2, 3)$ forms a copy of K_3 in $L(u_3)$, then $\{a_2d_ku_3, a_1b_1u_1, a_1d_1u_1\}$ is a copy of Q_3 in G. If $\{a_2, b_2, d_k\}$ $(k = 1, 2, 3)$ forms a copy of K_3 in $L(u_3)$, then $\{a_2b_2u_3, a_1b_1u_1, a_1d_1u_1\}$ is a copy of Q_3 in G.

If $\{c, u_1, d_1\}$ forms a copy of K_3 in $L(u_3)$, then $\{u_1d_1u_3, b_1d_1u_1, a_1a_2c\}$ is a copy of Q_3 in G. If $\{c, u_1, d_k\}$ $(k = 2, 3)$ forms a copy of K_3 in $L(u_3)$, then $\{cd_ku_3, a_1b_1u_1, a_1d_1u_1\}$ is a copy of Q_3 in G. If $\{c, u_l, d_k\}$ $(l = 2, 4, \cdots, p; k = 1, 2, 3)$ forms a copy of K_3 in $L(u_3)$, then $\{cu_lu_3, a_1b_1u_1, a_1d_1u_1\}$ is a copy of Q_3 in G.

From the above, we have $L(u_3)$ contains no K_3.

Case 2: A set in the form of $\{c, u_l, d_k\}$ $(2 \le l \le p; k = 1, 2, 3)$ forms a copy of K_3 in $L(u_1)$.

Without loss of generality, we may assume that $\{c, u_2, d_1\}$ forms a copy of K_3 in $L(u_1)$, that is, $cu_2u_1, cd_1u_1, u_2d_1u_1 \in G$. In this case, we will also show that $L(u_3)$ contains no K_3. Note that the only possible sets forming a copy of K_3 in $L(u_3)$ are $\{a_i, b_j, d_k\}$ or $\{c, u_l, d_k\}$ $(i, j = 1, 2; k = 1, 2, 3; l = 1, 2, 4, \cdots, p)$.

If $\{a_i, b_j, d_k\}$ $(i, j = 1, 2, k = 1, 2, 3)$ forms a copy of K_3 in $L(u_3)$, then $\{a_ib_ju_3, cu_1u_2, d_1u_1u_2\}$ is a copy of Q_3 in G. If $\{c, u_l, d_k\}$ $(l = 1, 2; k = 1, 2, 3)$ forms a copy of K_3 in $L(u_3)$, then $\{cu_lu_3, cu_1u_2, d_1d_2d_3\}$ is a copy of Q_3 in G. If $\{c, u_l, d_k\}$ $(l = 4, \cdots, p; k = 2, 3)$ forms a copy of K_3 in $L(u_3)$, then $\{cd_1u_1, u_2d_1u_1, u_ld_ku_3\}$ is a copy of Q_3 in G.

Therefore, the only possible sets forming a copy of K_3 in $L(u_3)$ are $\{c, u_l, d_1\}$ $(l = 4, \cdots, p)$. In this case, $cu_lu_3, cd_1u_3, u_ld_1u_3 \in G$. We consider the pairs $\{b_1, u_l\}$ $(l = 4, \cdots, p)$. If $b_1u_la_i \in G$ $(i = 1, 2)$, then $\{b_1u_la_i, cd_1u_1, cd_1u_3\}$ is a copy of Q_3 in G. If $b_1u_lc \in G$, then $\{b_1u_lc, cb_1b_2, d_1d_2d_3\}$ is a copy of Q_3 in G. If $b_1u_lb_2 \in G$, then $\{b_1u_lb_2, cb_1b_2, d_1d_2d_3\}$ is a copy of Q_3 in G. If $b_1u_ld_1 \in G$, then $\{b_1u_ld_1, u_ld_1u_3, ca_1a_2\}$ is a copy of Q_3 in G. If $b_1u_ld_k \in G$ $(k = 2, 3)$, then $\{b_1u_ld_k, cd_1u_1, cd_1u_3\}$ is a copy of Q_3 in G. If $b_1u_lu_2 \in G$, then $\{b_1u_lu_2, cd_1u_1, cd_1u_3\}$ is a copy of Q_3 in G. If $b_1u_lu_k \in G$ $(k = 3, \cdots, p, k \ne l)$, then $\{b_1u_lu_k, cd_1u_1, cu_1u_2\}$ is a copy of Q_3 in G. Since G is dense, the pairs $\{b_1, u_l\}$ must be covered by an edge in the form of $b_1u_lu_1 \in G$. Next, we consider the pairs $\{b_2, u_l\}$ $(l = 4, \cdots, p)$. Switching b_1 and b_2, we have $b_2u_lu_1 \in G$, then $\{b_2u_lu_1, b_1u_lu_1, ca_1a_2\}$ is a copy of Q_3 in G. It is a contradiction.

From the above, we have $L(u_3)$ contains no K_3. □

Lemma 3.12. *Let G be a dense Q_{t+2}-free 3-graph. If G contains a subgraph Q'_{t+2} and $|V(G)| > |V(Q'_{t+2})| + 2$, then there exists a vertex v in $V(G)$ such that the link $L(v)$ contains no K_{t+2}.*

Proof. Let $u_1, u_2, \cdots, u_p \in V(G)\backslash V(Q'_{t+2})$ $(p \ge 3)$. If $L(u_1)$ contains no K_{t+2}, then we are done. Otherwise, we show that the only possible sets forming a copy of K_{t+2} in $L(u_1)$ are $\{a_i, b_j, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ or $\{c, u_l, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ $(i, j = 1, 2; l = 2, 3, \cdots, p; k_s = 1 \textit{ or } 2 \textit{ or } 3; s = 1, 2, \cdots, t)$. We apply induction on t. By the proof of Lemma 3.11, the conclusion holds for

$t = 1$. Suppose that the conclusion holds for $t-1$ ($t \geq 2$). We show that the conclusion holds for t. Let $G' = G[V(G) \setminus \{d_{t,1}, d_{t,2}, d_{t,3}\}]$.

Consider the $(t+2)$-sets of vertices with at least two vertices in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. If $\{x_1, x_2, \cdots, x_t, y_1, y_2\}$ forms a copy of K_{t+2} in $L(u_1)$, where $y_1, y_2 \in \{d_{t,1}, d_{t,2}, d_{t,3}\}$, $x_1, x_2, \cdots, x_t \in V(G) \setminus \{u_1, y_1, y_2\}$, then $\{y_1y_2u_1, d_{t,1}d_{t,2}d_{t,3}, a_1a_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t-1,1}d_{t-1,2}d_{t-1,3}\}$ is a copy of Q_{t+2} in G.

Next, consider the $(t+2)$-sets of vertices with at most one vertex in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. By the induction hypothesis, the vertices forming a copy of K_{t+1} in $L_{G'}(u_1)$ must be of the form $\{a_i, b_j, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t-1,k_{t-1}}\}$ or $\{c, u_l, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t-1,k_{t-1}}\}$ ($i, j = 1, 2; l = 2, 3, \cdots, p; k_s = 1$ *or* 2 *or* 3; $s = 1, 2, \cdots, t-1$). Thus the only possible sets forming a copy of K_{t+2} in $L(u_1)$ are $\{a_i, b_j, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ or $\{c, u_l, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ ($i, j = 1, 2; l = 2, 3, \cdots, p; k_s = 1$ *or* 2 *or* $3; s = 1, 2, \cdots, t$).

Case 1: A set in the form of $\{a_i, b_j, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ ($i, j = 1, 2; k_s = 1$ *or* 2 *or* $3; s = 1, 2, \cdots, t$) forms a copy of K_{t+2} in $L(u_1)$.

We will show that $L(u_3)$ contains no K_{t+2}. Applying induction on t. By the proof of Lemma 3.11, the result holds for $t = 1$. Suppose that the conclusion holds for $t-1$ ($t \geq 2$), that is, if $\{a_i, b_j, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t-1,k_{t-1}}\}$ forms a copy of K_{t+1} in $L_{G'}(u_1)$. Then $L_{G'}(u_3)$ contains no K_{t+1}. We will show that the conclusion holds for t.

Consider the $(t+2)$-sets of vertices with at least two vertices in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. If $\{x_1, x_2, \cdots, x_t, y_1, y_2\}$ forms a copy of K_{t+2} in $L(u_3)$, where $y_1, y_2 \in \{d_{t,1}, d_{t,2}, d_{t,3}\}$, $x_1, x_2, \cdots, x_t \in V(G) \setminus \{u_3, y_1, y_2\}$, then $\{y_1y_2u_3, d_{t,1}d_{t,2}d_{t,3}, a_1a_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t-1,1}d_{t-1,2}d_{t-1,3}\}$ is a copy of Q_{t+2} in G.

Next, we consider the $(t+2)$-sets of vertices with at most one vertex in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. By the induction hypothesis, the vertices can not form a copy of K_{t+1} or K_{t+2} in $L_{G'}(u_3)$. Thus $L(u_3)$ contains no K_{t+2}.

Case 2: A set in the form of $\{c, u_l, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t,k_t}\}$ ($l = 2, 3, \cdots, p$; $k_s = 1$ *or* 2 *or* $3; s = 1, 2, \cdots, t$) forms a copy of K_{t+2} in $L(u_1)$.

Without loss of generality, we assume that $\{c, u_2, d_{1,1}, d_{2,1}, \cdots, d_{t,1}\}$ forms a copy of K_{t+2} in $L(u_1)$, that is, $xyu_1 \in G$, where $x, y \in \{c, u_2, d_{1,1}, d_{2,1}, \cdots, d_{t,1}\}$. In particular, $u_1cd_{i,1}, u_1u_2d_{i,1} \in G$ ($i \in [t]$). In this case, $\{c, u_2, d_{1,1}, d_{2,1}, \cdots, d_{t-1,1}\}$ forms a copy of K_{t+1} in $L(u_1)$. We claim that the only possible sets forming a copy of K_{t+2} in $L(u_3)$ are $\{c, u_l, d_{1,1}, d_{2,1}, \cdots, d_{t,1}\}$ ($l = 4, \cdots, p$). Applying induction on t. By the proof of Lemma 3.11, the result holds for $t = 1$. Suppose that the result holds for $t-1$ ($t \geq 2$), that is, if $\{c, u_2, d_{1,1}, d_{2,1}, \cdots, d_{t-1,1}\}$ forms a copy of K_{t+1} in $L_{G'}(u_1)$. Then the only possible sets forming a copy of K_{t+1} in $L_{G'}(u_3)$ are $\{c, u_l, d_{1,1}, d_{2,1}, \cdots, d_{t-1,1}\}$ ($l = 4, \cdots, p$). We will show that the conclusion holds for t.

Firstly, we consider the $(t+2)$-sets of vertices with at least two vertices in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. If $\{x_1, x_2, \cdots, x_t, y_1, y_2\}$ forms a copy of K_{t+2} in $L(u_3)$, where $y_1, y_2 \in \{d_{t,1}, d_{t,2}, d_{t,3}\}$, $x_1, x_2, \cdots, x_t \in V(G) \setminus \{u_3, y_1, y_2\}$, then $\{y_1y_2u_3, d_{t,1}d_{t,2}d_{t,3}, a_1a_2c, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t-1,1}d_{t-1,2}d_{t-1,3}\}$ is a copy of Q_{t+2} in G.

Secondly, we consider the $(t+2)$-sets of vertices with at most one vertex in $\{d_{t,1}, d_{t,2}, d_{t,3}\}$. By the induction hypothesis, the vertices forming a copy of K_{t+1} in $L_{G'}(u_3)$ must be of the form $\{c, u_l, d_{1,1}, d_{2,1}, \cdots, d_{t-1,1}\}$ $(l = 4, \cdots, p)$. If $\{c, u_l, d_{1,1}, d_{2,1}, \cdots, d_{t-1,1}, d_{t,k}\}$ $(k = 2, 3)$ forms a copy of K_{t+2} in $L(u_3)$, then $\{u_1cd_{t,1}, u_1u_2d_{t,1}, u_ld_{t,k}u_3, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t-1,1}d_{t-1,2}d_{t-1,3}\}$ is a copy of Q_{t+2} in G. Thus the only possible sets forming a copy of K_{t+2} in $L(u_3)$ are $\{c, u_l, d_{1,1}, d_{2,1}, \cdots, d_{t,1}\}$ $(l = 4, \cdots, p)$.

If $L(u_3)$ contains no K_{t+2}, then we are done. Otherwise, we assume that $\{c, u_l, d_{1,1}, d_{2,1}, \cdots, d_{t,1}\}$ forms a copy of K_{t+2} in $L(u_3)$. In this case, $cd_{i,1}u_3, u_ld_{i,1}u_3 \in G$ $(i \in [t])$. We consider the pairs $\{b_1, u_l\}$ $(l = 4, \cdots, p)$. If $b_1u_la_i \in G$ $(i = 1, 2)$, then $\{b_1u_la_i, cd_{1,1}u_1, cd_{1,1}u_3, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. If $b_1u_lc \in G$, then $\{b_1u_lc, cb_1b_2, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. If $b_1u_lb_2 \in G$, then $\{b_1u_lb_2, cb_1b_2, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. If $b_1u_ld_{i,1} \in G$ $(i \in [t])$, then $\{b_1u_ld_{i,1}, u_ld_{i,1}u_3, ca_1a_2, d_{1,1}d_{1,2}d_{1,3}, \cdots, d_{i-1,1}d_{i-1,2}d_{i-1,3}, d_{i+1,1}d_{i+1,2}d_{i+1,3}, \cdots, d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. If $b_1u_ld_{i,k_i} \in G$ $(k_i = 2, 3; i \in [t])$, then $\{b_1u_ld_{i,k_i}, cd_{i,1}u_1, cd_{i,1}u_3, d_{1,1}d_{1,2}d_{1,3}, \cdots, d_{i-1,1}d_{i-1,2}d_{i-1,3}, d_{i+1,1}d_{i+1,2}d_{i+1,3}, \cdots, d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. If $b_1u_lu_2 \in G$, then $\{b_1u_lu_2, cd_{1,1}u_1, cd_{1,1}u_3, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{k+2} in G. If $b_1u_lu_k \in G$ $(k = 3, \cdots, n, k \neq l)$, then $\{b_1u_lu_k, cd_{1,1}u_1, cu_1u_2, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. Since G is dense, then the pair $\{b_1, u_l\}$ must be covered by an edge in the form of $b_1u_lu_1$. Next, we consider the pairs $\{b_2, u_l\}$ $(l = 4, \cdots, p)$. Switching b_1 and b_2, we have $b_2u_lu_1 \in G$, then $\{b_2u_lu_1, b_1u_lu_1, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t,1}d_{t,2}d_{t,3}\}$ is a copy of Q_{t+2} in G. It is a contradiction.

From the above, we have $L(u_3)$ contains no K_{t+2}. $\square$

Proof of Lemma 3.2. Let $\vec{x}$ be an optimum weighting of G. By Lemmas 3.6, 3.8, 3.10, and 3.12, there exists a vertex v in $V(G)$ such that $L(v)$ contains no K_{t+2}. By Fact 2.4, we have

$$3\lambda(G) = \lambda(L(v), \vec{x}) \leq \binom{t+1}{2}(\frac{1}{t+1})^2 = \frac{t}{2(t+1)}.$$

Since

$$\lambda(K^3_{3t+3}) = \binom{3t+3}{3}(\frac{1}{3t+3})^3 = \frac{(3t+2)(3t+1)}{6(3t+3)^2}.$$

Hence

$$\lambda(G) - \lambda(K^3_{3t+3}) \leq \frac{t}{6(t+1)} - \frac{(3t+2)(3t+1)}{6(3t+3)^2} = -\frac{1}{3(3t+3)^2}.$$

Let $c = \frac{1}{3(3t+3)^2}$.

Then $\lambda(G) \leq \lambda(K^3_{3t+3}) - c$. □

3.3. Proof of Lemma 3.3

Let

$$Q''_{t+3} = \{a_1b_1b_2, b_1b_2a_2, a_1cd_2, a_2cd_1, d_{1,1}d_{1,2}d_{1,3}, d_{2,1}d_{2,2}d_{2,3}, \cdots, d_{t-1,1}d_{t-1,2}d_{t-1,3}\}.$$

Lemma 3.13. *Let G be a Q_3-free 3-graph. If G contains a spanning subgraph Q''_4, then there exists a vertex v in $V(G)$ such that the link $L(v)$ contains no K_3.*

Proof. Since G is Q_3-free, we will show that $L(d_1)$ contains no K_3.

If any of $\{b_1, b_2, a_i\}$ $(i = 1, 2)$, $\{b_1, b_2, c\}$, $\{b_1, b_2, d_2\}$ forms a copy of K_3 in $L(d_1)$, then $\{b_1b_2d_1, b_1b_2a_2, a_1cd_2\}$ is a copy of Q_3 in G. If any of $\{a_1, c, b_i\}$ $(i = 1, 2)$, $\{a_1, c, a_2\}$, $\{a_1, c, d_2\}$ forms a copy of K_3 in $L(d_1)$, then $\{a_1cd_1, a_1cd_2, a_2b_1b_2\}$ is a copy of Q_3 in G. If any of $\{a_2, b_i, a_1\}$, $\{a_2, b_i, c\}$, $\{a_2, b_i, d_2\}$ $(i = 1, 2)$ forms a copy of K_3 in $L(d_1)$, then $\{a_2b_id_1, a_2b_2b_1, a_1cd_2\}$ is a copy of Q_3 in G. If any of $\{a_1, d_2, a_2\}$, $\{a_1, d_2, b_i\}$ $(i = 1, 2)$ forms a copy of K_3 in $L(d_1)$, then $\{a_1d_2d_1, a_1cd_2, a_2b_1b_2\}$ is a copy of Q_3 in G. If any of $\{c, d_2, a_2\}$ $\{c, d_2, b_i\}$ $(i = 1, 2)$ forms a copy of K_3 in $L(d_1)$, then $\{cd_2d_1, cd_2a_1, a_2b_1b_2\}$ is a copy of Q_3 in G, it is a contradiction.

From the above, we have $L(d_1)$ contains no K_3. □

Lemma 3.14. *Let G be a Q_{t+2}-free 3-graph. If G contains a spanning subgraph Q''_{t+3}, then there exists a vertex v in $V(G)$ such that the link $L(v)$ contains no K_{t+2}.*

Proof. We claim that $L(d_1)$ contains no K_{t+2}. Applying induction on t. By the proof of Lemma 3.13, the conclusion holds for $t = 1$. Suppose that the conclusion holds for $t - 1$ $(t \geq 2)$.

We will show that the conclusion holds for t.

Let $G' = G[V(G) \setminus \{d_{t-1,1}, d_{t-1,2}, d_{t-1,3}\}]$. We consider the $(t+2)$-sets of vertices with at least two vertices in $\{d_{t-1,1}, d_{t-1,2}, d_{t-1,3}\}$. If $\{x_1, x_2, \cdots, x_t, y_1, y_2\}$ forms a copy of K_{t+2} in $L(d_1)$, where $y_1, y_2 \in \{d_{t-1,1}, d_{t-1,2}, d_{t-1,3}\}$,

$x_1, x_2, \cdots, x_t \in V(G) \setminus \{d_1, y_1, y_2\}$, then $\{y_1y_2d_1, d_{t-1,1}d_{t-1,2}d_{t-1,3}, a_2b_1b_2, a_1cd_2, d_{1,1}d_{1,2}d_{1,3}, \cdots, d_{t-2,1}d_{t-2,2}d_{t-2,3}\}$ is a copy of Q_{t+2} in G.

Now consider the $(t+2)$-sets of vertices with at most one vertex in $\{d_{t-1,1}, d_{t-1,2}, d_{t-1,3}\}$. By the induction hypothesis, the vertices can not form a copy of K_{t+1} or K_{t+2} in $L_{G'}(d_1)$. Thus $L(d_1)$ contains no K_{t+2}. □

Lemma 3.15. *Let G be a Q_3-free 3-graph. If G contains a subgraph Q_4'' and $|V(G)| \geq |V(Q_4'')| + 1$, then there exists a vertex v in $V(G)$ such that the link $L(v)$ contains no K_3.*

Proof. Let $u_1, u_2, \cdots, u_p \in V(G) \setminus V(Q_4'')$. If $L(d_1)$ contains no K_3, then we are done. Otherwise, we show that the only possible sets forming a copy of K_3 in $L(d_1)$ are $\{a_1, b_i, u_j\}$ $(i = 1, 2; j = 1, 2, \cdots, p)$.

Firstly, we consider the triples in $\{a_1, a_2, b_1, b_2\}$. If $\{a_1, a_2, b_i\}$ $(i = 1, 2)$ forms a copy of K_3 in $L(d_1)$, then $\{a_2b_id_1, a_2b_1b_2, a_1cd_2\}$ is a copy of Q_3 in G. If $\{a_i, b_1, b_2\}$ $(i = 1, 2)$ forms a copy of K_3 in $L(d_1)$, then $\{b_1b_2d_1, b_1b_2a_2, a_1cd_2\}$ is a copy of Q_3 in G.

Secondly, we consider the triples with one vertex in $\{a_1, a_2, b_1, b_2\}$ and two vertices in $\{c, d_2, u_1, u_2, \cdots, u_p\}$. If $\{x, y, z\}$ forms a copy of K_3 in $L(d_1)$, where $x, y \in \{c, d_2, u_1, u_2, \cdots, u_p\}, z \in \{a_1, a_2, b_1, b_2\}$, then $\{xyd_1, a_1b_1b_2, a_2b_1b_2\}$ forms a copy of Q_3 in G.

Thirdly, we consider the triples in $\{c, d_2, u_1, u_2, \cdots, u_p\}$. If $\{c, d_2, u_1\}$ forms a copy of K_3 in $L(d_1)$, that is, $cd_2d_1, cu_1d_1, d_2u_1d_1 \in G$, then any two edges of those and the edge $a_1b_1b_2$ make a copy of Q_3 in G. Similarly, other cases can not happen.

Finally, we consider the triples with two vertices in $\{a_1, a_2, b_1, b_2\}$ and one vertex in $\{c, d_2, u_1, u_2, \cdots, u_p\}$. If any of $\{a_1, a_2, c\}, \{a_1, b_i, c\}$ $(i = 1, 2)$ forms a copy of K_3 in $L(d_1)$, then $\{a_1cd_1, a_1cd_2, a_2b_1b_2\}$ is a copy of Q_3 in G. If any of $\{a_1, a_2, d_2\}, \{a_1, b_i, d_2\}$ $(i = 1, 2)$ forms a copy of K_3 in $L(d_1)$, then $\{a_1d_2d_1, a_1cd_2, a_2b_1b_2\}$ is a copy of Q_3 in G. If $\{a_1, a_2, u_j\}$ $(j = 1, 2, \cdots, p)$ forms a copy of K_3 in $L(d_1)$, then $\{a_2u_jd_1, a_2cd_1, a_1b_1b_2\}$ is a copy of Q_3 in G. If any of $\{b_1, b_2, c\}, \{b_1, b_2, d_2\}, \{b_1, b_2, u_j\}$ $(j = 1, 2, \cdots, p)$ forms a copy of K_3 in $L(d_1)$, then $\{b_1b_2d_1, b_1b_2a_2, a_1cd_2\}$ is a copy of Q_3 in G. If any of $\{a_2, b_i, c\}, \{a_2, b_i, d_2\}, \{a_2, b_i, u_j\}$ $(i = 1, 2; j = 1, 2, \cdots, p)$ forms a copy of K_3 in $L(d_1)$, then $\{a_2b_id_1, a_2b_1b_2, a_1cd_2\}$ is a copy of Q_3 in G.

Therefore, we obtain the only possible sets forming a copy of K_3 in $L(d_1)$ are $\{a_1, b_i, u_j\}$ $(i = 1, 2; j = 1, 2, \cdots, p)$. Without loss of generality, we assume that $\{a_1, b_1, u_1\}$ forms a copy of K_3 in $L(d_1)$, that is, $a_1b_1d_1, a_1u_1d_1, b_1u_1d_1 \in G$. We will show that $L(d_2)$ contains no K_3.

Firstly, we consider the triples in $\{a_1, a_2, b_1, b_2\}$. If $\{a_1, a_2, b_i\}$ $(i = 1, 2)$ forms a copy of K_3 in $L(d_2)$, then $\{a_1b_id_2, a_1b_1b_2, a_2cd_1\}$ is a copy of Q_3 in G. If

$\{a_i, b_1, b_2\}$ $(i = 1, 2)$ forms a copy of K_3 in $L(d_2)$, then $\{b_1b_2d_2, b_1b_2a_1, a_2cd_1\}$ is a copy of Q_3 in G.

Secondly, we consider the triples with two vertices in $\{a_1, a_2, b_1, b_2\}$ and one vertex in $\{c, d_1, u_1, u_2, \cdots, u_p\}$. If $\{a_1, a_2, c\}$ forms a copy of K_3 in $L(d_2)$, then $\{a_2cd_2, a_2cd_1, a_1b_1b_2\}$ is a copy of Q_3 in G. If $\{a_1, a_2, d_1\}$ forms a copy of K_3 in $L(d_2)$, then $\{a_2d_1d_2, a_2cd_1, a_1b_1b_2\}$ is a copy of Q_3 in G. If $\{a_1, a_2, u_j\}$ $(j = 1, 2, \cdots, p)$ forms a copy of K_3 in $L(d_2)$, then $\{a_1u_jd_2, a_1cd_2, a_2b_1b_2\}$ is a copy of Q_3 in G. If any of $\{b_1, b_2, c\}$, $\{b_1, b_2, d_1\}$, $\{b_1, b_2, u_j\}$ $(j = 1, 2, \cdots, p)$ forms a copy of K_3 in $L(d_2)$, then $\{b_1b_2d_2, b_1b_2a_1, a_2cd_1\}$ is a copy of Q_3 in G. If any of $\{a_1, b_i, c\}$, $\{a_1, b_i, d_1\}$, $\{a_1, b_i, u_j\}$ $(i = 1, 2; j = 1, 2, \cdots, p)$ forms a copy of K_3 in $L(d_2)$, then $\{a_1b_id_2, a_1b_1b_2, a_2cd_1\}$ is a copy of Q_3 in G. If any of $\{a_2, b_i, c\}$, $\{a_2, b_i, d_1\}$, $\{a_2, b_i, u_j\}$ $(i = 1, 2; j = 1, 2, \cdots, p)$ forms a copy of K_3 in $L(d_2)$, then $\{a_2b_id_2, a_2b_1b_2, a_1u_1d_1\}$ is a copy of Q_3 in G.

Thirdly, we consider the triples with one vertex in $\{a_1, a_2, b_1, b_2\}$ and two vertices in $\{c, d_1, u_1, u_2, \cdots, u_p\}$. If $\{x, y, z\}$ forms a copy of K_3 in $L(d_2)$, where $x, y \in \{c, d_1, u_1, u_2, \cdots, u_p\}, z \in \{a_1, a_2, b_1, b_2\}$, then $\{xyd_2, a_1b_1b_2, a_2b_1b_2\}$ forms a copy of Q_3 in G.

Finally, we consider the triples in $\{c, d_1, u_1, u_2, \cdots, u_p\}$. If $\{x, y, z\}$ forms a copy of K_3 in $L(d_2)$, where $x, y, z \in \{c, d_1, u_1, u_2, \cdots, u_p\}$, then $\{xyd_2, a_1b_1b_2, a_2b_1b_2\}$ forms a copy of Q_3 in G.

From the above, we have $L(d_2)$ contains no K_3. □

Lemma 3.16. *Let G be a Q_{t+2}-free 3-graph. If G contains a subgraph Q''_{t+3} and $|V(G)| \geq |V(Q''_{t+3})| + 1$, then there exists a vertex v in $V(G)$ such that the link $L(v)$ contains no K_{t+2}.*

Proof. Let $u_1, u_2, \cdots, u_p \in V(G) \setminus V(Q''_{t+3})$. If $L(d_1)$ contains no K_{t+2}, then we are done. Otherwise, we show that the only possible sets forming a copy of K_{t+2} in $L(d_1)$ are $\{a_1, b_i, u_j, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t-1,k_{t-1}}\}$ $(i=1, 2; j=1, 2, \cdots, p;$ $k_s = 1$ *or* 2 *or* $3; s = 1, 2, \cdots, t-1)$.

Applying induction on t. By the proof of Lemma 3.15, the conclusion holds for $t = 1$. For $t = 2$. We consider the 4-sets of vertices with at least two vertices in $\{d_{1,1}, d_{1,2}, d_{1,3}\}$. If $\{x_1, x_2, y_1, y_2\}$ forms a copy of K_4 in $L(d_1)$, where $y_1, y_2 \in \{d_{1,1}, d_{1,2}, d_{1,3}\}$, $x_1, x_2 \in V(G) \setminus \{d_1, y_1, y_2\}$, then $\{y_1y_2d_1, d_{1,1}d_{1,2}d_{1,3}, a_2b_1b_2, a_1cd_2\}$ is a copy of Q_4 in G.

Now consider the 4-sets of vertices with at most one vertex in $\{d_{1,1}, d_{1,2}, d_{1,3}\}$. Let $G^0 = G[\{a_1, a_2, b_1, b_2, c, d_1, d_2\}]$. Since the vertices forming a copy of K_3 in $L_{G^0}(d_1)$ must be of the form $\{a_1, b_i, u_j\}$ $(i = 1, 2; j = 1, 2, \cdots, p)$. Thus the only possible sets forming a copy of K_4 in $L(d_1)$ are $\{a_1, b_i, u_j, d_{1,k_1}\}$ $(i = 1, 2; j = 1, 2, \cdots, p; k_1 = 1, 2, 3)$. Switching d_1 and d_2, we have that the only possible sets forming a copy of K_4 in $L(d_2)$ are $\{a_2, b_i, u_j, d_{1,k_1}\}$

$(i = 1, 2; j = 1, 2, \cdots, p; k_1 = 1, 2, 3)$. Suppose that the conclusion holds for $t-1$ $(t \geq 3)$. We will show that the conclusion holds for t. Let $G' = G[V(G) \setminus \{d_{t-1,1}, d_{t-1,2}, d_{t-1,3}\}]$.

Consider the $(t+2)$-sets of vertices with at least two vertices in $\{d_{t-1,1}, d_{t-1,2}, d_{t-1,3}\}$. If $\{x_1, x_2, \cdots, x_t, y_1, y_2\}$ forms a copy of K_{t+2} in $L(d_1)$, where $y_1, y_2 \in \{d_{t-1,1}, d_{t-1,2}, d_{t-1,3}\}$, $x_1, x_2, \cdots, x_t \in V(G) \setminus \{d_1, y_1, y_2\}$, then $\{y_1y_2d_1, d_{t-1,1}d_{t-1,2}d_{t-1,3}, a_2b_1b_2, a_1cd_2, d_{1,1}d_{1,2}d_{1,3}, \cdots, d_{t-2,1}d_{t-2,2}d_{t-2,3}\}$ is a copy of Q_{t+2} in G.

Now consider the $(t+2)$-sets of vertices with at most one vertex in $\{d_{t-1,1}, d_{t-1,2}, d_{t-1,3}\}$. By the induction hypothesis, the vertices forming a copy of K_{t+1} in $L_{G'}(d_1)$ must be of the form $\{a_1, b_i, u_j, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t-2,k_{t-2}}\}$ $(i = 1, 2; j = 1, 2, \cdots, p; k_s = 1 \textit{ or } 2 \textit{ or } 3; s = 1, 2, \cdots, t-2)$. Thus the only possible sets forming a copy of K_{t+2} in $L(d_1)$ are $\{a_1, b_i, u_j, d_{1,k_1}, d_{2,k_2}, \cdots, d_{t-1,k_{t-1}}\}$ $(i = 1, 2; j = 1, 2, \cdots, p; k_s = 1 \textit{ or } 2 \textit{ or } 3; s = 1, 2, \cdots, t-1)$.

Without loss of generality, we may assume that $\{a_1, b_1, u_1, d_{1,1}, d_{2,1}, \cdots, d_{t-1,1}\}$ forms a copy of K_{t+2} in $L(d_1)$. In particular, $a_1u_1d_1 \in G$. We will show that $L(d_2)$ contains no K_{t+2}.

Applying induction on t. By the proof of Lemma 3.15, the conclusion holds for $t = 1$. For $t = 2$, recall that the only possible sets forming a copy of K_4 in $L(d_2)$ are $\{a_2, b_i, u_j, d_{1,k_1}\}$ $(i = 1, 2; j = 1, 2, \cdots, p; k_1 = 1, 2, 3)$. But $\{a_2, b_i, u_j, d_{1,k_1}\}$ $(i = 1, 2; j = 1, 2, \cdots, p; k_1 = 1, 2, 3)$ can not form a copy of K_4 in $L(d_2)$. Otherwise, $\{a_2b_id_2, a_2b_1b_2, a_1u_1d_1, d_{1,1}d_{1,2}d_{1,3}\}$ $(i = 1, 2)$ forms a copy of Q_4 in G. Then $L(d_2)$ contains no K_4.

Suppose that the conclusion holds for $t-1$ $(t \geq 3)$, that is, if $\{a_1, b_1, u_1, d_{1,1}, d_{2,1}, \cdots, d_{t-2,1}\}$ forms a copy of K_{t+1} in $L_{G'}(d_1)$, then we have that $L_{G'}(d_2)$ contains no K_{t+1}. We will show that the conclusion holds for t.

Consider the $(t+2)$-sets of vertices with at least two vertices in $\{d_{t-1,1}, d_{t-1,2}, d_{t-1,3}\}$. If $\{x_1, x_2, \cdots, x_t, y_1, y_2\}$ forms a copy of K_{t+2} in $L(d_2)$, where $y_1, y_2 \in \{d_{t-1,1}, d_{t-1,2}, d_{t-1,3}\}$, $x_1, x_2, \cdots, x_t \in V(G) \setminus \{d_2, y_1, y_2\}$, then $\{y_1y_2d_2, d_{t-1,1}d_{t-1,2}d_{t-1,3}, a_2b_1b_2, a_1cd_2, d_{1,1}d_{1,2}d_{1,3}, \cdots, d_{t-2,1}d_{t-2,2}d_{t-2,3}\}$ is a copy of Q_{t+2} in G.

Now consider the $(t+2)$-sets of vertices with at most one vertex in $\{d_{t-1,1}, d_{t-1,2}, d_{t-1,3}\}$. By the induction hypothesis, the vertices can not form a copy of K_{t+1} or K_{t+2} in $L_{G'}(d_2)$. Thus $L(d_2)$ contains no K_{t+2}. □

Proof of Lemma 3.3. Let $\vec{x}$ be an optimum weighting of G. By Lemmas 3.14 and 3.16, there exists a vertex v in $V(G)$ such that $L(v)$ contains no K_{t+2}. The rest of the proof is identical to the proof Lemma 3.2. □

4. Turán number of the extension of Q_{t+2}

Let $T^r_m(n)$ be *the balanced complete* m*-partite* r*-uniform graph on* n *vertices*, i.e., $V(T^r_m(n)) = V_1 \cup V_2 \cup \cdots \cup V_m$ such that $V_i \cap V_j = \emptyset$ for every $1 \le i < j \le m$ and $|V_1| \le |V_2| \le \cdots \le |V_m| \le |V_1| + 1$, and $E(T^r_m(n)) = \{e \in \binom{[n]}{r} : \forall i \in [m], |e \cap V_i| \le 1\}$. Let $t^r_m(n) = |T^r_m(n)|$. Given positive integers m and r, let $[m]_r = m(m-1)\dots(m-r+1)$.

For an r-graph F and $p \ge |V(F)|$, let $\mathcal{K}^F_p$ denote the family of r-graphs H that contains a set C of p vertices, called the *core*, such that the subgraph of H induced by C contains a copy of F and such that every pair of vertices in C is covered in H. Let H^F_p be a member of $\mathcal{K}^F_p$ obtained as follows. Label the vertices of F as $v_1, \dots, v_{|V(F)|}$. Add new vertices $v_{|V(F)|+1}, \dots, v_p$. Let $C = \{v_1, \dots, v_p\}$. For each pair of vertices $v_i, v_j \in C$ not covered in F, we add a set B_{ij} of $r-2$ new vertices and the edge $\{v_i, v_j\} \cup B_{ij}$, where the B_{ij}'s are pairwise disjoint over all such pairs $\{i, j\}$. Note that the extension H^F is the case that $p = |V(F)|$.

Using a stability argument of Pikhurko [16] and a transference technique between the Lagrangian density of an r-uniform graph and the Turán density of its extension in several other papers, we obtain the following result.

Theorem 4.1. *For sufficiently large* n*,* $ex(n, H^{Q_{t+2}}) = t^3_{3t+3}(n)$*. Moreover, if* n *is sufficiently large and* G *is an* $H^{Q_{t+2}}$*-free* 3*-graph on* $[n]$ *with* $|G| = t^3_{3t+3}(n)$*, then* $G = T^3_{3t+3}(n)$*.*

To prove the theorem, we need several results from [2]. Similar results are obtained independently in [15].

Definition 4.1 ([2])**.** Let $m, r \ge 2$ be positive integers. Let F be an r-graph that has at most $m+1$ vertices satisfying $\pi_\lambda(F) \le \frac{[m]_r}{m^r}$. We say that $\mathcal{K}^F_{m+1}$ is m*-stable* if for every real $\varepsilon > 0$ there are a real $\delta > 0$ and an integer n_1 such that if G is a $\mathcal{K}^F_{m+1}$-free r-graph with at least $n \ge n_1$ vertices and more than $(\frac{[m]_r}{m^r} - \delta)\binom{n}{r}$ edges, then G can be made m-partite by deleting at most εn vertices.

Theorem 4.2 ([2])**.** *Let* $m, r \ge 2$ *be positive integers. Let* F *be an* r*-graph that either has at most* m *vertices or has* $m+1$ *vertices one of which has degree* 1*. Suppose either* $\pi_\lambda(F) < \frac{[m]_r}{m^r}$ *or* $\pi_\lambda(F) = \frac{[m]_r}{m^r}$ *and* $\mathcal{K}^F_{m+1}$ *is* m*-stable. Then there exists a positive integer* n_2 *such that for all* $n \ge n_2$ *we have* $ex(n, H^F_{m+1}) = t^r_m(n)$ *and the unique extremal* r*-graph is* $T^r_m(n)$*.* $\square$

The following lemma is proved in [21].

Lemma 4.1 ([21])**.** *Let* $m, r \ge 2$ *be positive integers. Let* F *be an* r*-graph that has at most* $m+1$ *vertices with* $r-1$ *vertices of one edge of degree* 1 *and*

$\pi_\lambda(F) \le \frac{[m]_r}{m^r}$. *Suppose there is a constant $c > 0$ such that for every F-free and K^r_m-free r-graph L, $\lambda(L) \le \lambda(K^r_m) - c$ holds. Then $\mathcal{K}^F_{m+1}$ is m-stable.*

Proof of Theorem 4.1. By Theorem 3.1 and Corollary 3.1, Q_{t+2} satisfies the conditions of Lemma 4.1. So $\mathcal{K}^{Q_{t+2}}_{3t+4}$ is $(3t+3)$-stable. The theorem then follows from Theorem 4.2. □

Remark. As mentioned earlier, Conjecture 1.1 has been verified for a 3-uniform tight star $T_t = \{123, 124, 125, 126, ..., 12(t+2)\}$ and a λ-perfect 3-uniform graph for $t \ge 3$ in [23]. Surprisingly, it seems to be much harder to verify for the case $t = 2$. We think that it is interesting to understand for the case $t = 2$.

References

[1] A. Bene Watts, S. Norin, L. Yepremyan. A Turán theorem for extensions via an Erdős-Ko-Rado theorem for Lagrangians. Combinatorica, **39** (2019), 1149–1171. MR4039605

[2] A. Brandt, D. Irwin, T. Jiang. Stability and Turán numbers of a class of hypergraphs via Lagrangians. Combin., Probab. & Comput., **26** (3) (2017) 367–405. MR3628909

[3] F. Chung and L. Lu. An Upper Bound for the Turán Number $t_3(n, 4)$, Journal of Combinatorial Theory, Series A, **87** (1999), 381–389. MR1704268

[4] P. Frankl and Z. Füredi. Extremal problems whose solutions are the blow-ups of the small Witt-designs. Journal of Combinatorial Theory, Series A, **52** (1989), 129–147. MR1008165

[5] P. Frankl, V. Rödl. Hypergraphs do not jump. Combinatorica, **4** (1984), 149–159. MR0771722

[6] D. Hefetz, P. Keevash. A hypergraph Turán theorem via Lagrangians of intersecting families. Journal of Combinatorial Theory, Series A, **120** (2013), 2020–2038. MR3102173

[7] S. Hu, Y. Peng, B. Wu. Lagrangian densities of unions of Linear paths and matchings and Turán numbers of their extensions. Journal of Combinatorial Designs, **28** (2020), 207–223. MR4057897

[8] M. Jenssen. Continous Optimisation in Extremal Combinatorics. Ph.D. dissertation, London School of Economics and Political Science, 2017.

[9] T. Jiang, Y. Peng, B. Wu. Lagrangian densities of some sparse hypergraphs and Turán numbers of their extensions. European Journal of Combinatorics, **73** (2018), 20–36. MR3836731

[10] T. Johnston, L. Lu. Turán Problems on Non-uniform Hypergraphs. Electron. J. Combin., **21** (2014), no. 4, Paper 4.22, 34 pp. MR3292259

[11] G. Katona, T. Nemetz, M. Simonovits. On a problem of Turán in the theory of graphs. Mat. Lapok., **15** (1964), 228–238. MR0172263

[12] P. Keevash. Hypergrah Turán problems. Surveys in Combinatorics. Cambridge University Press, (2011), 83–140. MR2866732

[13] T.S. Motzkin, E.G. Straus. Maxima for graphs and a new proof of a theorem of Turán. Canad. J. Math., **17** (1965), 533–540. MR0175813

[14] S. Norin, L. Yepremyan. Turán number of generalized triangles. Journal of Combinatorial Theory, Series A, **146** (2017), 312–343. MR3574234

[15] S. Norin, L. Yepremyan. Turán numbers of extensions. Journal of Combinatorial Theory, Series A, **155** (2018), 476–492. MR3741438

[16] O.Pikhurko. An exact Turán result for the generalized triangle. Combinatorica, **28** (2008), 187–208. MR2399018

[17] O.Pikhurko. Exact computation of the hypergraph Turán function for expanded complete 2-graphs. Journal of Combinatorial Theory, Series B, **103** (2013), 220–225. MR3018066

[18] A. Razborov. On 3-hypergraphs with forbidden 4-vertex configurations. SIAM. J. Discrete Math., **24** (2010), 946–963. MR2680226

[19] A.F. Sidorenko. On the maximal number of edges in a homogeneous hypergraph that does not contain prohibited subgraphs. Mat. Zametki, **41** (1987), 433–455. MR0893373

[20] A.F. Sidorenko. Asymptotic solution for a new class of forbidden r-graphs. Combinatorica, **9** (1989), 207–215. MR1030374

[21] B. Wu, Y. Peng. Lagrangian densities of short 3-uniform linear paths and Turán numbers of their extensions. Graphs and Combinatorics, **37** (2021), 711–729. MR4249196

[22] B. Wu, Y. Peng, P Chen. On a conjecture of Hefetz and Keevash on Lagrangians of intersecting hypergraphs and Turán numbers. arXiv:1701.06126v3.

[23] Z. Yan, Y. Peng. Lagrangian densities of hypergraph cycles. Discrete Mathematics, **342** (2019), 2048–2059. MR3943373

[24] A. A. Zykov. On some properties of linear complexes (in Russian). Mat. Sbornik. (N. S.), **24** (1949), 163–188. MR0035428

Pingge Chen
Hunan University of Technology
College of Science
China
E-mail: chenpingge@hut.edu.cn

Jinhua Liang
Hunan University
School of Mathematics
China
E-mail: jh_liang@hnu.edu.cn

Yuejian Peng
Hunan University
School of Mathematics
China
E-mail: ypeng1@hnu.edu.cn