Pure and Applied Mathematics Quarterly
Volume 18, Number 6, 2461–2478, 2022

A greedy algorithm for the connected positive influence dominating set in k-regular graphs*

Mengmeng He, Bo Hou, Wen Liu, Weili Wu,
Ding-Zhu Du, and Suogang Gao†

Abstract: For a graph $G = (V, E)$, a vertex subset $C \subseteq V$ is a connected positive influence dominating set of G if every node v in $V \setminus C$ has at least a fraction ρ $(0 < \rho < 1)$ of its neighbors in C and the subgraph of G induced by C is connected. In this paper, let G be a regular graph with degree k. We present a greedy algorithm to compute a connected positive influence dominating set in G, and it is proved that the approximation ratio of the algorithm is $2 + \ln(k^2 + 2k)$.

Keywords: Connected positive influence dominating set, greedy algorithm, potential function.

1. Introduction

Online social network is a network composed of individuals who share the same interest and purpose which provides a powerful medium of communicating, sharing and disseminating information, and spreading influence beyond the traditional social interactions within a traditional social network setting. Online social network has developed significantly in recent years. For example, online social network sites like Facebook and MySpace are among the most popular sites on the Internet; online social networks have also raised special interest among commercial businesses, medical and pharmaceutical companies as a channel to influence the opinion of their customers; even police have utilized the information in online social network sites to track down criminals. A great number of research has been done to understand the properties of

Received June 25, 2021.
2010 Mathematics Subject Classification: Primary 05C69; secondary 68W25, 68W40.
*This work was supported by the NSF of China (No. 11971146), the NSF of Hebei Province (Nos. A2017403010, A2019205089 and A2019205092), the Graduate Innovation Grant Program of Hebei Normal University (No. CXZZSS2021045).
†Corresponding author. E-mail: sggaomail@163.com.

social networks and how to effectively utilize social networks to spread information and influence. In social networks, both positive and negative influence can spread. In this paper, we study how to optimize the global positive influence in the k-regular networks by identifying a small group of dominating nodes which are positively influential.

In 2009, Wang et al. [9] first gave the concept of positive influence dominating set (PIDS) in online social networks and in order to find the positive influence dominating set they proposed a greedy algorithm. Furthermore, Wang et al. [10] proved that finding the positive influence dominating set (PIDS) with minimum cardinality is an APX-hard problem. They described a greedy algorithm with an approximation ratio $H(\delta)$, where $H(\cdot)$ is the harmonic function and δ represents the maximum node degree of the graph representing a social network. In 2012, Zhang et al. [13] focused on the PIDS problem in power-law graphs and proved that the greedy algorithm had a constant approximation ratio, and simulation results also demonstrated that greedy algorithm can effectively select a small scale PIDS. In 2014, Dinh et al. [2] studied the positive influence dominating set (PIDS) problem that seeks for a minimal set of nodes P such that all other nodes in the network had at least a fraction ρ of their neighbors in P. They also studied a different formulation, called total positive influence dominating set (TPIDS), in which even nodes in PIDS were required to have a fraction ρ of neighbors inside PIDS. In 2017, Ran et al. [8] presented an approximation algorithm for minimum partial positive influence dominating set (MPPIDS) and gave an approximation algorithm with performance ratio $\gamma H(\Delta)$, where $\gamma = 1/(1-(1-\rho)\eta)$, $\eta \approx \Delta^2/\delta$ and Δ, δ are respectively the maximum and minimum node degrees of the graph. In 2020, Yao et al. [12] studied the connected positive influence dominating set (CPIDS) problem and partial positive influence dominating set (PPIDS) problem with $\rho = \frac{1}{2}$ in the k-regular graphs. When $k = 3$, they proposed an algorithm with an approximation ratio of $H(12)$ for CPIDS and an algorithm with an approximation ratio of $H(9)$ for PPIDS.

In this paper, we mainly focus on finding the minimum connected positive influence dominating set with $0 < \rho < 1$ in the k-regular graph. We design a greedy algorithm for the connected positive influence dominating set (CPIDS) problem in the k-regular graph. The approximation ratio of the greedy algorithm is $2 + \ln(k^2 + 2k)$, where k is the degree of the k-regular graph.

2. Preliminaries

In this section, let $G = (V, E)$ be an undirected graph and denote by $N_G(u)$ the set of neighbors of a node $u \in V$ and $d(u) = |N_G(u)|$ the degree of u.

Definition 2.1 ([2]). *Given an undirected graph $G = (V, E)$, a subset $C \subseteq V$ is a positive influence dominating set (PIDS) of G, if for each $u \in V \setminus C$, we have $|N_G(u) \cap C| \geq \rho d(u)$ for some constant $0 < \rho < 1$, where $d(u)$ is the degree of the node u. In the PIDS problem, our goal is to find a PIDS of minimum cardinality.*

Definition 2.2. *Given a graph $G = (V, E)$, if a PIDS induces a connected subgraph, it is called a connected positive influence dominating set (CPIDS). In the CPIDS problem, our goal is to find a CPIDS of minimum cardinality.*

Definition 2.3. *A k-regular graph is a simple, undirected, connected graph $G = (V, E)$ with every node degree of k.*

In this article, we mainly focus on designing an approximation algorithm to solve the minimum connected positive influence dominating set problem in k-regular graphs.

Definition 2.4 ([1]). *Consider a finite set E and a function $f : 2^E \to \mathbb{Z}$, where 2^E denotes the power set of E (i.e. the family of all subsets of E). The function f is said to be submodular if for any two elements A and B in 2^E,*

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B).$$

Definition 2.5 ([1]). *Assume that $f(\cdot)$ is a submodular function on subsets of E. Define*

$$\Delta_B f(A) \equiv f(A \cup B) - f(A)$$

for any subsets A and B of E; that is, $\Delta_B f(A)$ is the extra amount of $f(\cdot)$ value we gain by adding B to A. When $B = \{x\}$ is a singleton, we simply write

$$\Delta_x f(A) = f(A \cup \{x\}) - f(A) \tag{1}$$

instead of $\Delta_{\{x\}} f(A)$.

Definition 2.6 ([1]). *A function f on 2^E is said to be monotone increasing if, for all $A, B \subseteq E$,*

$$A \subseteq B \Rightarrow f(A) \leq f(B).$$

Lemma 2.7 ([1, Lemma 2.24]). *Let f be a function on all subsets of a set E. Then f is submodular if and only if, for any two subsets $A \subseteq B$ of E and any element $x \notin B$,*

$$\Delta_x f(A) \geq \Delta_x f(B).$$

Definition 2.8 ([1]). *Let $G=(V,E)$ be a graph. For a node subset $C\subseteq V$, denote by $G\langle C\rangle$ the subgraph with node set V and edge set $D(C)$, where $D(C)$ is the set of all edges incident on some vertices in C.*

Definition 2.9 ([1]). *With reference to Definition 2.8, define $q(C)$ to be the number of connected components of the subgraph $G\langle C\rangle$.*

Lemma 2.10 ([1, Lemma 2.44]). *Given a graph $G=(V,E)$ and two node subsets $A,B\subseteq V$. If $A\subseteq B$, then $\Delta_y q(A)\le\Delta_y q(B)$ for any $y\in V$.*

3. Potential function

In this section, we construct a potential function $g(\cdot)$ and give the properties of the potential function. In the rest of this paper the graphs we mentioned are k-regular graphs.

We now introduce the construction of the potential function $g(\cdot)$.

For a graph $G=(V,E)$ and a node subset $C\subseteq V$, denote by $N_C(u)$ the set of nodes in C which are adjacent with u in G for every node $u\in V$. In particular, we use $N_G(u)$ to denote the node set adjacent with u in G. For a given constant ρ $(0<\rho<1)$, define

$$S_i^C=\{u\in V\setminus C \mid |N_C(u)|=i\}\ (i=0,...,\lceil\rho k\rceil-1),$$
$$S_{\lceil\rho k\rceil}^C=\{u\in V\setminus C \mid |N_C(u)|\ge\lceil\rho k\rceil\}.$$

For a node subset C, we say that a node u is black if $u\in C$, gray if $u\in S_{\lceil\rho k\rceil}^C$, red if $u\in\cup_{i=1}^{\lceil\rho k\rceil-1}S_i^C$, and white if $u\in S_0^C$. Denote by $\mathcal{B}_C$, $\mathcal{G}_C$, $\mathcal{R}_C$, and $\mathcal{W}_C$ the set of black, gray, red, and white nodes with respect to C, respectively.

We first define two functions $n(\cdot)$, $p(\cdot)$ on 2^V. For a subset $C\subseteq V$, define

$$n(C)=\sum_{u\in V}n_C(u), \tag{2}$$

where

$$n_C(u)=\begin{cases}0, & u\in C,\\ \lceil\rho k\rceil-i, & u\in S_i^C\quad i=0,...,\lceil\rho k\rceil.\end{cases} \tag{3}$$

Define $p(C)$ to be the number of connected components of the subgraph $G[C]$ induced by C.

Let

$$m(\cdot)=p(\cdot)+q(\cdot)$$

and

$$g = n(\cdot) + p(\cdot) + q(\cdot),$$

where $q(\cdot)$ is from Definition 2.9.

Suppose that f is a function on 2^V. For a node subset $C \subseteq V$ and a node $x \in V$, define

$$\Delta'_x f(C) = f(C) - f(C \cup \{x\}).$$

Observe that

$$\Delta'_x f(C) = -\Delta_x f(C),$$
$$\Delta'_x m(C) = \Delta'_x p(C) + \Delta'_x q(C),$$

and

$$\Delta'_x g(C) = \Delta'_x n(C) + \Delta'_x p(C) + \Delta'_x q(C).$$

In the following, we give the properties of the functions above.

Lemma 3.1. *If $C \subseteq V$, then $\Delta'_x m(C) \geq 0$ for every $x \in V$.*

Proof. Note that $\Delta'_x m(C) = \Delta'_x p(C) + \Delta'_x q(C)$. Let x be any node of V. We consider two increments $\Delta'_x p(C)$ and $\Delta'_x q(C)$. We divide possibilities into the following three cases according with the color of x.

(i) Suppose that x is black. Then we have $C \cup \{x\} = C$. Therefore,

$$\Delta'_x p(C) = \Delta'_x q(C) = 0,$$
$$\Delta'_x m(C) = \Delta'_x p(C) + \Delta'_x q(C) = 0.$$

(ii) Suppose that x is white. This means that x has no neighbors in C.

We first consider $\Delta'_x p(C)$. Since x is white, the number of connected components in $G[C]$ will increase by one after we put the node x into set C. Therefore,

$$\Delta'_x p(C) = p(C) - p(C \cup \{x\}) = p(C) - (p(C) + 1) = -1.$$

Next, we consider $\Delta'_x q(C)$. If x is adjacent to other white nodes, then at least two white connected components in $G\langle C\rangle$ are merged into one after we put the node x into set C; if x is not adjacent to other white nodes, then the connected components adjacent to x are merged into one with x. Therefore,

$$\Delta'_x q(C) = q(C) - q(C \cup \{x\}) \geq q(C) - (q(C) - 1) = 1.$$

Combining the above discussions of the two increments, we have

$$\Delta'_x m(C) = \Delta'_x p(C) + \Delta'_x q(C) \geq (-1) + 1 = 0.$$

(iii) Suppose that x is red or gray. This means that x has neighbors in C.

We first consider $\Delta'_x p(C)$. Since x has neighbors in C, the number of connected components in $G[C]$ doesn't increase after we put x in C. Therefore,

$$\Delta'_x p(C) = p(C) - p(C \cup \{x\}) \geq p(C) - p(C) = 0.$$

Next, we consider $\Delta'_x q(C)$. If x is adjacent to other white nodes, the number of white connected components in $G\langle C\rangle$ will be reduced by at least one; if x is not adjacent to other white nodes, the number of connected components in $G\langle C\rangle$ doesn't change. Therefore,

$$\Delta'_x q(C) = q(C) - q(C \cup x) \geq 0.$$

Combining the above discussions of the two increments, we have

$$\Delta'_x m(C) = \Delta'_x p(C) + \Delta'_x q(C) \geq 0 + 0 = 0.$$

Combining the above discussions of the three cases, we have $\Delta'_x m(C) \geq 0$ for every $x \in V$. □

Lemma 3.2. *If $C \subseteq V$, then $\Delta'_x g(C) \geq 0$ for every $x \in V$.*

Proof. Note that

$$\begin{aligned} \Delta'_x g(C) &= \Delta'_x n(C) + \Delta'_x p(C) + \Delta'_x q(C) \\ &= \Delta'_x n(C) + \Delta'_x m(C). \end{aligned}$$

We consider two increments $\Delta'_x n(C)$ and $\Delta'_x m(C)$. Let x be any node of V. We divide possibilities into the following four cases according with the color of x.

(i) Suppose that x is black. Then we have $C \cup \{x\} = C$. Therefore,

$$\Delta'_x n(C) = \Delta'_x p(C) = \Delta'_x q(C) = 0,$$
$$\Delta'_x g(C) = \Delta'_x n(C) + \Delta'_x p(C) + \Delta'_x q(C) = 0.$$

(ii) Suppose that x is white. Then we have $n_C(x) = \lceil \rho k \rceil$.

We consider $\Delta'_x n(C)$. By the definition of $\Delta'_x n(C)$, we have

$$\Delta'_x n(C) = \sum_{u \in V} (n_C(u) - n_{C\cup\{x\}}(u)).$$

We consider each of the terms in this summation. We divide possibilities into the following three cases:

Case 1. $u \in C$. In this case, we have

$$n_C(u) - n_{C\cup\{x\}}(u) = 0.$$

Case 2. $u = x$. In this case, we have

$$n_C(u) - n_{C\cup\{x\}}(u) = n_C(x) - n_{C\cup\{x\}}(x) = \lceil \rho k \rceil - 0 = \lceil \rho k \rceil > 0.$$

Case 3. $u \in S_i^C$ $(i = 0, ..., \lceil \rho k \rceil - 1)$ and $u \neq x$. In this case, we have

$$n_C(u) - n_{C\cup\{x\}}(u) = \lceil \rho k \rceil - i - (\lceil \rho k \rceil - i) = 0,$$

or

$$n_C(u) - n_{C\cup\{x\}}(u) = \lceil \rho k \rceil - i - (\lceil \rho k \rceil - i - 1) = 1.$$

Therefore,

$$\Delta'_x n(C) = \sum_{u \in V} (n_C(u) - n_{C\cup\{x\}}(u)) > 0.$$

By Lemma 3.1, we have

$$\Delta'_x g(C) = \Delta'_x n(C) + \Delta'_x m(C) > 0 + 0 = 0.$$

(iii) Suppose that x is gray. Then we have $n_C(x) = 0$.

We consider $\Delta'_x n(C)$. By the definition of $\Delta'_x n(C)$, we have

$$\Delta'_x n(C) = \sum_{u \in V} (n_C(u) - n_{C\cup\{x\}}(u)).$$

We consider each of the terms in this summation. We divide possibilities into the following three cases:

Case 1. $u \in C$. In this case, we have

$$n_C(u) - n_{C\cup\{x\}}(u) = 0.$$

Case 2. $u = x$. In this case, we have

$$n_C(u) - n_{C\cup\{x\}}(u) = n_C(x) - n_{C\cup\{x\}}(x) = 0 - 0 = 0.$$

Case 3. $u \in S_i^C$ $(i = 0, ..., \lceil \rho k \rceil - 1)$ and $u \neq x$. In this case, we have

$$n_C(u) - n_{C\cup\{x\}}(u) = \lceil \rho k \rceil - i - (\lceil \rho k \rceil - i) = 0,$$

or

$$n_C(u) - n_{C\cup\{x\}}(u) = \lceil \rho k \rceil - i - (\lceil \rho k \rceil - i - 1) = 1.$$

Therefore,

$$\Delta'_x n(C) = \sum_{u\in V} (n_C(u) - n_{C\cup\{x\}}(u)) \geq 0.$$

By Lemma 3.1, we have

$$\Delta'_x g(C) = \Delta'_x n(C) + \Delta'_x m(C) \geq 0 + 0 = 0.$$

(iv) Suppose that x is red. Then we have $n_C(x) = \lceil \rho k \rceil - i$.
We consider $\Delta'_x n(C)$. By the definition of $\Delta'_x n(C)$, we have

$$\Delta'_x n(C) = \sum_{u\in V} (n_C(u) - n_{C\cup\{x\}}(u)).$$

We consider each of the terms in this summation. We divide possibilities into the following three cases:

Case 1. $u \in C$. In this case, we have

$$n_C(u) - n_{C\cup\{x\}}(u) = 0.$$

Case 2. $u = x$. In this case, we have

$$n_C(u) - n_{C\cup\{x\}}(u) = n_C(x) - n_{C\cup\{x\}}(x) = \lceil \rho k \rceil - i - 0 = \lceil \rho k \rceil - i > 0.$$

Case 3. $u \in S_i^C$ $(i = 0, ..., \lceil \rho k \rceil - 1)$ and $u \neq x$. In this case, we have

$$n_C(u) - n_{C\cup\{x\}}(u) = \lceil \rho k \rceil - i - (\lceil \rho k \rceil - i) = 0,$$

or

$$n_C(u) - n_{C\cup\{x\}}(u) = \lceil \rho k \rceil - i - (\lceil \rho k \rceil - i - 1) = 1.$$

Therefore,

$$\Delta'_x n(C) = \sum_{u\in V} (n_C(u) - n_{C\cup\{x\}}(u)) > 0.$$

By Lemma 3.1, we have

$$\Delta'_x g(C) = \Delta'_x n(C) + \Delta'_x m(C) > 0 + 0 = 0.$$

Combining the above discussions of the four cases, we have

$$\Delta'_x g(C) \geq 0$$

for every $x \in V$. □

4. Approximation algorithm

In this section, we give a greedy algorithm for the connected positive influence dominating set problem.

Lemma 4.1. *Suppose $C \subseteq V$. Then C is a connected positive influence dominating set if and only if $\Delta'_x g(C) = 0$ for every $x \in V$.*

Proof. Suppose C is a connected positive influence dominating set. Then $p(C) = q(C) = 1$. By (2), (3) we have

$$n(C) = \sum_{u \in V} n_C(u) = \sum_{u \in V} 0 = 0.$$

Hence,

$$g(C) = n(C) + p(C) + q(C) = 2.$$

Since V is a connected positive influence dominating set, we have $g(V) = 2$. By Lemma 3.2, we have $g(C) \geq g(C \cup \{x\})$ for every $x \in V$. Then

$$g(C) \geq g(C \cup \{x\}) \geq g(V).$$

Therefore, $g(C \cup \{x\}) = g(C)$ for every $x \in V$, that is, $\Delta'_x g(C) = 0$ for every $x \in V$. The necessity is proved.

Conversely, suppose $\Delta'_x g(C) = 0$ for every $x \in V$. By Lemmas 3.1, 3.2 and since $\Delta'_x g(C) = \Delta'_x n(C) + \Delta'_x m(C)$, we have

$$\begin{aligned} \Delta'_x n(C) &= 0, \\ \Delta'_x m(C) &= 0 \end{aligned}$$

for every $x \in V$.

Claim 1. $C \neq \emptyset$.

Assume $C = \emptyset$ for a contradiction. By the definitions of $n(\cdot)$, $p(\cdot)$, $q(\cdot)$, we have $n(\emptyset) = \lceil \rho k \rceil |V|$, $p(\emptyset) = 0$, $q(\emptyset) = |V|$. Let x be a node in G, we have

$$n(\{x\}) = (\lceil \rho k \rceil - 1)|N_G(x)| + \lceil \rho k \rceil (|V| - |N_G(x)| - 1),$$

$$\begin{aligned} p(\{x\}) &= 1, \\ q(\{x\}) &= |V| - |N_G(x)|. \end{aligned}$$

Then

$$\begin{aligned} \Delta'_x g(\emptyset) =& g(\emptyset) - g(\{x\}) \\ =& (n(\emptyset) - n(\{x\})) + (p(\emptyset) - p(\{x\})) + (q(\emptyset) - q(\{x\})) \\ =& (\lceil \rho k \rceil |V| - ((\lceil \rho k \rceil - 1)|N_G(x)| + \lceil \rho k \rceil (|V| - |N_G(x)| - 1))) \\ & + (0 - 1) + (|V| - (|V| - |N_G(x)|)) \\ =& 2|N_G(x)| + \lceil \rho k \rceil - 1. \end{aligned}$$

Since G is a k-regular graph, we have $|N_G(x)| = k \geq 1$. Then

$$\Delta'_x g(\emptyset) = \lceil \rho k \rceil + 2k - 1 \geq \lceil \rho k \rceil + 2 - 1 > \lceil \rho k \rceil > 0.$$

This contradiction establishes Claim 1.

Claim 2. For any node $u \in V$, $n_C(u) = 0$.

Suppose that there exists a node $u_0 \in V$ such that $n_C(u_0) > 0$ for a contradiction. By the definition of $n_C(\cdot)$ and node shading, we have u_0 is a red node or a white node. If u_0 is a red node, we have

$$\begin{aligned} \Delta'_{u_0} n(C) &= \sum_{u \in V} (n_C(u) - n_{C \cup \{u_0\}}(u)) \geq n_C(u_0) - n_{C \cup \{u_0\}}(u_0) \\ &= n_C(u_0) \\ &= \lceil \rho k \rceil - i > 0, \end{aligned}$$

which contradicts $\Delta'_x n(C) = 0$. If u_0 is a white node, by the proof of Lemma 3.2, we have

$$\Delta'_{u_0} n(C) \geq n_C(u_0) - n_{C \cup \{u_0\}}(u_0) = \lceil \rho k \rceil > 0,$$

which contradicts $\Delta'_x n(C) = 0$. Claim 2 holds.

By the definition of $n(\cdot)$ and Claim 2, we have C is a positive influence dominating set.

Claim 3. $G[C]$ is a connected graph.

Let $\mathcal{C}$ be a connected component of the subgraph $G[C]$. Denote by B the node set of $\mathcal{C}$. Let A be a subset of $V \setminus B$ in which the node is adjacent to at least one node of B.

To prove that $G[C]$ is a connected graph, it suffices to prove that $V = A \cup B$. To do this, suppose, by way of contradiction, that $V \neq A \cup B$. Since G is connected, there exists a node $x \notin A \cup B$ that is adjacent to a node

$y \in A \cup B$. Since all nodes adjacent to B are in A, we know that y must be in A. Since C is a positive influence dominating set, then x must be of gray or black. Now, if x is of gray, then we have $p(C \cup \{y\}) \leq p(C)$ and $q(C \cup \{y\}) < q(C)$. If x is of black, then we have $p(C \cup \{y\}) < p(C)$ and $q(C \cup \{y\}) \leq q(C)$. In either case, we get $\Delta'_y m(C) = \Delta'_y p(C) + \Delta'_y q(C) > 0$, which contradicts $\Delta'_x m(C) = 0$. Claim 3 holds.

Combining the above three claims, C is a connected positive influence dominating set. □

Based on Lemma 4.1, we give the following Algorithm 1 for the connected positive influence dominating set problem.

Algorithm 1 Algorithm for the CPIDS

Input: A k-regular graph $G = (V, E)$.
Output: A connected positive influence dominating set of G.

1: Set $C \leftarrow \emptyset$;
2: **while** there exists a node $x \in V \setminus C$ such that $\Delta'_x g(C) > 0$ **do**
3: Choose a node $x \in V \setminus C$ such that $\Delta'_x g(C)$ is maximized;
4: **end while**
5: $C \leftarrow C \cup \{x\}$
6: **return** $C_g \leftarrow C$

Theorem 4.2. *C_g is a connected positive influence dominating set of G.*

Proof. The terminating condition of Algorithm 1 is $\Delta'_x g(C_g) = 0$ for every node $x \in V$, which implies C_g is a connected positive influence dominating set by Lemma 4.1. □

5. Theoretical analysis

In this section, we first show that the function $-n(\cdot)$ is submodular. Then we show that the approximation ratio of Algorithm 1 is $2 + \ln(k^2 + 2k)$.

We assume that the output of the algorithm is $C_g = \{x_1, x_2, ..., x_g\}$, where $x_1, x_2, ..., x_g$ are selected in order. Set $C_0 = \emptyset$. For $i = 1, 2, ..., g$, denote $C_i = \{x_1, x_2, ..., x_i\}$. Note that $g(C_0) = (\lceil \rho k \rceil + 1)|V|$, $g(C_g) = 2$. In addition, let $C^* = \{y_1, y_2, ..., y_m\}$ be a minimum connected positive influence dominating set of the graph G.

Lemma 5.1. *The function $-n(\cdot)$ is submodular.*

Proof. By Lemma 2.7 and the definitions of $\Delta'_x n(\cdot), \Delta_x n(\cdot)$, we only show that $\Delta'_x n(A) \geq \Delta'_x n(B)$, for any two subsets $A \subseteq B$ of V and any element

$x \notin B$. Observe that

$$\Delta'_x n(A) = \sum_{u \in V} (n_A(u) - n_{A \cup \{x\}}(u))$$

and

$$n_A(u) - n_{A\cup\{x\}}(u) = \begin{cases} n_A(u), & u = x, \\ 0, & u \notin N_G(x) \cup u \in A \cup S^A_{\lceil \rho k \rceil}, \\ 1, & u \in N_G(x) \cap S^A_i \quad (i = 0, ..., \lceil \rho k \rceil - 1). \end{cases}$$

Then

$$\Delta'_x n(A) = n_A(x) + |N_G(x) \cap (\cup_{i=0}^{\lceil \rho k \rceil - 1} S^A_i)| = n_A(x) + |N_G(x) \cap (\mathcal{R}_A \cup \mathcal{W}_A)|.$$

Similarly,

$$\Delta'_x n(B) = n_B(x) + |N_G(x) \cap (\mathcal{R}_B \cup \mathcal{W}_B)|.$$

For simplicity, write $t = \Delta'_x n(A) - \Delta'_x n(B)$. Then

$$t = (n_A(x) - n_B(x)) + (|N_G(x) \cap (\mathcal{R}_A \cup \mathcal{W}_A)| - |N_G(x) \cap (\mathcal{R}_B \cup \mathcal{W}_B)|).$$

By the definition of $n_A(\cdot)$ and since $A \subseteq B$, we have

$$n_A(x) \geq n_B(x).$$

Notice that

$$\mathcal{W}_B \subseteq \mathcal{W}_A$$

and

$$\mathcal{R}_B = \cup_{i=1}^{\lceil \rho k \rceil - 1} S^B_i \subseteq \cup_{i=0}^{\lceil \rho k \rceil - 1} S^A_i = \mathcal{R}_A \cup \mathcal{W}_A.$$

Thus $t \geq 0$, and hence the result follows. □

Lemma 5.2. *For* $i = 1, 2, ..., g$,

$$g(C_i) \leq g(C_{i-1}) - \frac{g(C_{i-1}) - 2}{m} + 1. \tag{4}$$

Proof. Set $C^*_0 = \emptyset$. For $j = 1, 2, ..., m$, denote $C^*_j = \{y_1, y_2, ..., y_j\}$. We note that

$$g(C_{i-1}) - 2 \quad = \quad g(C_{i-1}) - g(C_g)$$

$$\begin{aligned} &= \sum_{j=1}^{m} (\Delta'_{y_j} g(C_{i-1} \cup C^*_{j-1})) \\ &= \Delta'_{C^*} g(C_{i-1}). \end{aligned}$$

By the definition of $\Delta'_x g(C)$, for a node $x \in V$, we have

$$\Delta'_x g(C) = \Delta'_x n(C) + \Delta'_x p(C) + \Delta'_x q(C).$$

For every $y_j \in C^*$, we have

$$\begin{aligned} \Delta'_{y_j} g(C_{i-1} \cup C^*_{j-1}) &= \Delta'_{y_j} n(C_{i-1} \cup C^*_{j-1}) + \Delta'_{y_j} p(C_{i-1} \cup C^*_{j-1}) \\ &+ \Delta'_{y_j} q(C_{i-1} \cup C^*_{j-1}). \end{aligned}$$

By Lemmas 2.10 and 5.1, we have

$$\Delta'_{y_j} n(C_{i-1} \cup C^*_{j-1}) \leq \Delta'_{y_j} n(C_{i-1}) \tag{5}$$

and

$$\Delta'_{y_j} q(C_{i-1} \cup C^*_{j-1}) \leq \Delta'_{y_j} q(C_{i-1}). \tag{6}$$

Next we show the relationship between $\Delta'_{y_j} p(C_{i-1} \cup C^*_{j-1})$ and $\Delta'_{y_j} p(C_{i-1})$. For $y \in V$, $\Delta_y p(C)$ is equal to the number of connected components of $G[C]$ which are adjacent to y minus 1. We consider the number of connected components which are adjacent to any node $y_j \in C^*$ in graph $G[C_{i-1} \cup C^*_{j-1}]$ and $G[C_{i-1}]$. Since C^* is a connected positive influence dominating set, we can always arrange the elements of C^* in an ordering $y_1, y_2, ..., y_m$ such that y_1 is adjacent to a node in C_{i-1} and, for each $j \geq 2$, y_j is adjacent to a node in $\{y_1, y_2, ..., y_{j-1}\}$. For each $j = 1, 2, ..., m$, we note that y_j can dominate at most one additional connected component in the subgraph $G[C_{i-1} \cup C^*_{j-1}]$ than in $G[C_{i-1}]$, which is the one that contains C^*_{j-1}, since all nodes $y_1, y_2, ..., y_{j-1}$ in C^*_{j-1} are connected. Then

$$\Delta'_{y_j} p(C_{i-1} \cup C^*_{j-1}) \leq \Delta'_{y_j} p(C_{i-1}) + 1.$$

Moreover, by inequalities (5) and (6), we have

$$\Delta'_{y_j} g(C_{i-1} \cup C^*_{j-1}) \leq \Delta'_{y_j} g(C_{i-1}) + 1.$$

Then

$$g(C_{i-1}) - 2 = \Delta'_{C^*} g(C_{i-1})$$

$$
\begin{aligned}
&= \sum_{j=1}^{m} \Delta'_{y_j} g(C_{i-1} \cup C^*_{j-1}) \\
&\leq \sum_{j=1}^{m} (\Delta'_{y_j} g(C_{i-1}) + 1).
\end{aligned}
$$

By the pigeonhole principle, there exists an element $y_j \in C^*$ such that

$$\Delta'_{y_j} g(C_{i-1}) + 1 \geq \frac{g(C_{i-1}) - 2}{m}.$$

By the greedy strategy of Algorithm 1,

$$\Delta'_{x_i} g(C_i - 1) \geq \Delta'_{y_j} g(C_i - 1) \geq \frac{g(C_{i-1}) - 2}{m} - 1.$$

Or, equivalently,

$$g(C_i) \leq g(C_{i-1}) - \frac{g(C_{i-1}) - 2}{m} + 1.$$

□

Theorem 5.3. *Algorithm 1 has an approximation ratio* $2 + \ln(k^2 + 2k)$*, where* k *is the degree of the* k*-regular graph.*

Proof. If $g \leq 2m$, then the proof is already done. So we assume that $g > 2m$. Rewrite the inequality (4) as

$$g(C_i) \leq g(C_{i-1}) - \frac{g(C_{i-1}) - 2}{m} + 1.$$

Solving this recurrence relation, we have

$$
\begin{aligned}
g(C_i) - 2 &\leq (g(C_{i-1}) - 2)(1 - \frac{1}{m}) + 1 \\
&\leq (g(C_{i-2} - 2))(1 - \frac{1}{m})^2 + (1 - \frac{1}{m}) + 1 \\
&\leq (g(C_0) - 2)(1 - \frac{1}{m})^i + \sum_{k=0}^{i-1} (1 - \frac{1}{m})^k \\
&= (g(C_0) - 2)(1 - \frac{1}{m})^i + m(1 - (1 - \frac{1}{m})^i) \\
&= (g(C_0) - m - 2)(1 - \frac{1}{m})^i + m
\end{aligned}
$$

$$\begin{aligned} &\leq (g(C_0) - m - 2)e^{-\frac{i}{m}} + m \\ &= ((\lceil \rho k \rceil + 1)|V| - m - 2)e^{-\frac{i}{m}} + m. \end{aligned}$$

From the greedy strategy of Algorithm 1, we reduce the value $g(C_{i-1})$ in each stage i ($i \leq g$). Therefore, $g(C_i) \leq g(C_{i-1}) - 1$. In addition, $g(C_g) = 2$. So we have $2m + 2 \leq g(C_{g-2m})$. Set $i = g - 2m$, and observe that

$$2m \leq g(C_i) - 2 \leq ((\lceil \rho k \rceil + 1)|V| - m - 2)e^{-\frac{i}{m}} + m,$$
$$i \leq m \cdot \ln \frac{(\lceil \rho k \rceil + 1)|V| - m - 2}{m}.$$

Note that each node has at most k neighbors and so can dominate at most $k+1$ nodes. Hence, $\frac{|V|}{m} \leq k + 1$. It follows that

$$g = 2m + i \leq m(2 + \ln(k(\lceil \rho k \rceil + 1) + \lceil \rho k \rceil)).$$

Since $0 < \rho < 1$, then $\lceil \rho k \rceil \leq k$, we have

$$\begin{aligned} g &\leq m(2 + \ln(k(\lceil \rho k \rceil + 1) + \lceil \rho k \rceil)) \\ &\leq m(2 + \ln(k^2 + 2k)). \end{aligned}$$

□

6. Conclusion

In this paper, we consider the connected positive influence dominating set problem. Firstly, we construct a potential function $g(\cdot)$ and study the properties of $g(\cdot)$. And then, based on the potential function $g(\cdot)$, we present a greedy algorithm for this problem and show the approximation ratio is $(2 + \ln(k^2 + 2k))$. Further, we will study the connected positive influence dominating set problem in general graphs.

Acknowledgement

The authors would like to thank the referees for giving this paper a careful reading and many valuable comments and useful suggestions.

References

[1] D. Z. Du, K. I. Ko, X. D. Hu, Design and Analysis of Approximation Algorithms. *New York: Springer*, 2011. MR2856271

[2] T. N. Dinh, Y. L. Shen, D. T. Nguyen, M. T. Thai, On the approximability of positive influence dominating set in social networks. *J. Comb. Optim.* **27(3)** (2014), 487–503. MR3167714

[3] M. M. D. Khomami, M. A. Haeri, M. R. Meybodi, A. M. Saghiri, An algorithm for weighted positive influence dominating set based on learning automata. *IEEE 4th International Conference on Knowledge-Based Engineering and Innovation. Tehran: IEEE* (2017), 734–740. MR3889003

[4] M. M. D. Khomami, A. Rezvanian, N. Bagherpour, M. R. Meybodi, Minimum positive influence dominating set and its application in influence maximization: a learning automata approach. *Appl. Intell.* **48(3)** (2017), 1–24.

[5] G. Lin, J. Guan, H. B. Feng, An ILP based memetic algorithm for finding minimum positive influence dominating sets in social networks. *Phys. A* **500** (2018), 199–209. MR3777369

[6] J. Pan, T. M. Bu, A fast greedy algorithm for finding minimum positive influence dominating sets in social networks. *IEEE INFOCOM 2019-IEEE Conference on Computer Communications Workshops (INFOCOM WKSHPS 2019). Paris: IEEE* (2019), 360–364.

[7] H. Raei, N. Yazdani, M. Asadpour, A new algorithm for positive influence dominating set in social networks. *IEEE/ACM International Conference on Advances in Social Networks Analysis and Mining. Istanbul: IEEE* (2012), 253–257.

[8] Y. L. Ran, Z. Zhang, H. W. Du, Y. Q. Zhu, Approximation algorithm for partial positive influence problem in social network. *J. Comb. Optim.* **33(2)** (2017), 791–802. MR3604101

[9] F. Wang, E. Camacho, K. Xu, Positive influence dominating set in online social networks. *Combinatorial optimization and applications. Lecture Notes in Comput. Sci.*, 5573. *Berlin: Springer* (2009), 313–321. MR2737120

[10] F. Wang, H. Du, H. W. Du, E. Camacho, K. Xu, W. Lee, S. Yan, S. Shan, On positive influence dominating sets in social networks. *Theoret. Comput. Sci.* **412(3)** (2011), 265–269. MR2789649

[11] X. Z. Wang, X. Y. Li, B. Hou, W. Liu, S. G. Gao, A greedy algorithm for the fault-tolerant outer-connected dominating set problem. *J. Comb. Optim.* **41(1)** (2021), 118–127. MR4204646

[12] X. P. Yao, H. J. Huang, H. W. Du, Connected positive influence dominating set in k-regular graph. *Discrete Appl. Math.* **287** (2020), 65–76. MR4138334

[13] W. Zhang, W. L. Wu, F. Wang, Positive influence dominating sets in power-law graphs. *Soc Netw Anal Min* **2(1)** (2012), 31–37.

Mengmeng He
School of Mathematical Sciences
Hebei Normal University
Shijiazhuang
China
E-mail: 15633606057@163.com

Bo Hou
School of Mathematical Sciences
Hebei Normal University
Shijiazhuang
China
E-mail: houbo1969@163.com

Wen Liu
School of Mathematical Sciences
Hebei Normal University
Shijiazhuang
China
E-mail: liuwen1975@126.com

Weili Wu
Department of Computer Science
University of Texas at Dallas
Richardson
USA
E-mail: weiliwu@utdallas.edu

Ding-Zhu Du
Department of Computer Science
University of Texas at Dallas
Richardson
USA
E-mail: dzdu@utdallas.edu

Suogang Gao
School of Mathematical Sciences
Hebei Normal University
Shijiazhuang
China
E-mail: sggaomail@163.com