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Ricci-flat 5-regular graphs

Heidi Lei and Shuliang Bai

Abstract: The notion of Ricci curvature of Riemannian manifolds in differential geometry has been extended to other metric spaces such as graphs. The Ollivier-Ricci curvature between two vertices of a graph can be seen as a measure of how closely connected the neighbors of the vertices are compared to the distance between them. A Ricci-flat graph is then a graph in which each edge has curvature 0. There has been previous work in classifying Ricci-flat graphs under different definitions of Ricci curvature, notably graphs with large girth and small degrees under the definition of Lin-Lu-Yau, which is a modification of Ollivier's definition of Ricci curvature. In this paper, we continue the effort of classifying Ricci-flat graphs and study specifically Ricci-flat 5-regular graphs under the definition of Lin-Lu-Yau, we prove that a Ricci-flat 5-regular symmetric graph must be isomorphic to a graph of 72 vertices called RF^5_{72}.

1. Introduction

Ricci curvature is an important concept in differential geometry with wide applications in theoretical physics, such as general relativity and superstring theory. Essentially, Ricci curvature measures the amount of deviation in the volume of a section of a geodesic ball in a Riemannian manifold compared to its counterpart in Euclidean space. Naturally, a Ricci-flat manifold is a Riemannian manifold in which the Ricci curvature vanishes everywhere. They hold significance in physics as they represent vacuum solutions to the analogues of Einstein's equations generalized to Riemannian manifolds. One special class of Ricci-flat manifolds is Calabi-Yau manifolds, whose existence was conjectured by E. Calabi and proved by S.-T. Yau. There has been ongoing research in determining and analyzing the structures of Ricci-flat manifolds. One branch of such studies attempt to generalize the notion of Ricci curvature to other metric spaces, including discrete settings, such that analogues of important results in Riemannian manifolds such as Bonnet-Myers theorem hold.

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Bakry-Emery-Ricci curvature generalizes Ricci curvature by defining a diffusion process on the manifold, and it has been studied on graphs in [6] and [11]. Y. Ollivier defines a sense of Ricci curvature using transportation distance and Markov chains on metric spaces including graphs in [13] and [12]. Ollivier-Ricci curvature on graph captures the idea that curvature describes the average distance between points inside small balls compared to the distance between their centers by distributing masses on a vertex and its neighbors, transferring the mass to another vertex and its neighbors, and calculating the transportation distance between the two vertices using an optimal transport plan. Ollivier-Ricci curvature is parametrized by its idleness, the amount of mass placed on the vertex themselves. The rest of the mass is distributed evenly among its neighbors. The Ollivier-Ricci curvature that is most studied is when the idleness is O. Y. Lin, L. Lu, and S.-T. Yau modified Ollivier's definition of Ricci curvature to be the negative derivative when the idleness approaches 1 in Ollivier's definition, thus eliminating the idleness parameter [10]. With the modified Lin-Lu-Yau-Ricci curvature, they were able to study the Ricci curvature of Cartesian product graphs, random graphs, and other special classes of graphs.

[2] studied the Ollivier-Ricci curvature of graphs as a function of the chosen idleness parameter and showed that this idleness function is concave and piece-wise linear with at most 3 linear parts on its domain [0,1], with at most 2 linear parts in the case of a regular graph. Therefore, the Lin-Lu-Yau-Ricci curvature is equivalent to the negative of the slope of the last linear piece of the idleness function.

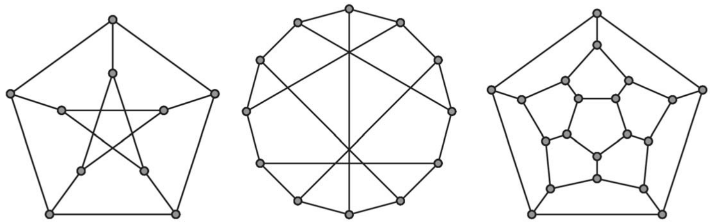
A graph is Ricci-flat if the Ricci curvature vanishes on all edges. It is noticeable that the Ricci flat graphs under Lin-Lu-Yau's definition are different from the Ricci flat graphs defined in 1996 by Chung and Yau [3], in which the definition is given by making connection with the algorithmic Harnack inequality. In [3], a graph G is said to be Ricci-flat at vertex x if there is a local k-frame in a neighborhood of x so that for all i,

$$\bigcup_j (\eta_i \eta_j)x = \bigcup_j (\eta_j \eta_i)x,$$

where $\eta_1, \ldots, \eta_k$ are injective mappings from a neighborhood of x into V so that (1) x is adjacent to $\eta_i x$ for $1 \leq i \leq k$ and (2) $\eta_i x \neq \eta_j x$ if $i \neq j$.

An example of Ricci flat graph using above definition is the lattice type graph, which is also Ricci flat under Lin-Lu-Yau's definition.

Another related work by Hua and Lin in 2019 [8], they classified unweighted graphs satisfying the curvature dimension condition $CD(0, \infty)$



(A) Petersen graph (B) Triplex (C) Dodecahedral graph

Figure 1.1: Ricci-flat 3-regular graphs.

whose girth are at least five, in which the infinite path P_k and cycle graphs C_n $(n \geq 6)$ are also Ricci flat under Lin-Lu-Yau's definition.

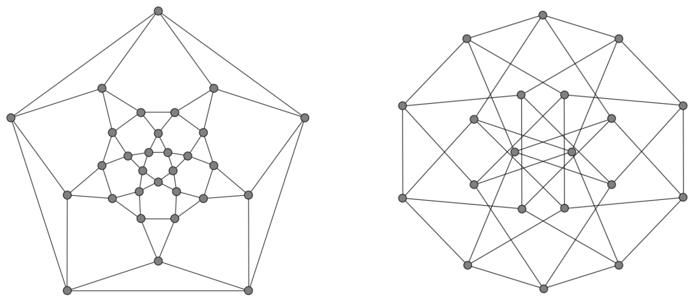
We will focus on the Ricci flat graphs under the Lin-Lu-Yau's definition. The problem of classifying Ricci-flat graphs under Lin-Lu-Yau's definition has been tackled through different angles and additional constraints. [9] classified Ricci-flat graphs with girth at least 5. [4] classified Ricci-flat cubic graphs of girth 5. [7] constructed an infinite family of distinct Ricci-flat graphs of girth four with edge-disjoint 4-cycles and completely characterize all Ricci-flat graphs of girth four with vertex-disjoint 4-cycles. [1] classified Ricci-flat graphs with maximum degree at most 4. The previous results on the classification of Ricci-flat regular graphs of small degree under Lin-Lu-Yau's definition is summarized below:

1. The Ricci flat 2-regular graphs are isomorphic to the infinite path and the cycle graph C_n with $n \geq 6$.
2. The Ricci flat 3-regular graphs are isomorphic to the Petersen graph, the Triplex graph and the dodecahedral graph.
3. The Ricci flat 4-regular graphs are isomorphic to one of two finite graphs: the icosidodecahedral graph and G_{20}; or are isomorphic to infinitely many lattice-type graphs in the terms of [1] in which each graph is locally a 4-regular grid.

[9] showed that Cartesian products of Ricci-flat regular graphs are Ricci-flat with the following theorem.

Theorem 1.1 ([9])**.** *Suppose that G is d_G-regular and H is d_H-regular. Then the Ricci curvature of $G \,\square\, H$ is given by*

$$\kappa^{G\square H}((u_1,v),(u_2,v)) = \frac{d_G}{d_G+d_H}\kappa^{G}(u,u_2),$$



(a) Icosidodecahedral graph (b) G_{20}

Figure 1.2: Ricci-flat 4-regular graphs.

$$\kappa^{G\square H}((u,v_1),(u,v_2)) = \frac{d_H}{d_G + d_H}\kappa^H(v,v_2)$$

where $u \in V(G), v \in V(H), u_1u_2 \in E(G)$*, and* $v_1v_2 \in E(H)$*.*

Corollary 1.1.1. *If both* G *and* H *are Ricci-flat regular graphs, so is the Cartesian product graph* $G \square H$*.*

Therefore, one class of Ricci-flat 5-regular graphs is the Cartesian product of a Ricci-flat 3-regular graph and a Ricci-flat 2-regular graph. As shown by [9] and [1], the Ricci-flat 3-regular graph has girth 5 and is either the Petersen graph, the Triplex graph, or the dodecahedral graph. The Ricci-flat 2-regular graph is either the cycle of length at least six or the infinite path.

In this paper, we study Ricci-flat 5-regular graphs that are not of the Cartesian product type. The paper is organised as follows: In Section 2, we formalize the definition of Ricci curvature on graphs outlined in the introduction following the notations of Lin-Lu-Yau in [10].

In Section 3, we analyze the local structure of a 5-regular graph by proving a more general result concerning regular graphs. Deferring the definition of local characteristics to Section 3, Lemma 3.1 essentially determines the Ricci curvature of an edge in a regular graph given its local environment. As a straightforward corollary, the local structure of any Ricci-flat regular graph can be determined by setting $\kappa(x,y) = 0$. There are five possible sets of local characteristics for a Ricci-flat 5-regular graph, and refer to them by type-A to type-E. See Fig. 3.1 for a schematic representation of the local structure of the edges.

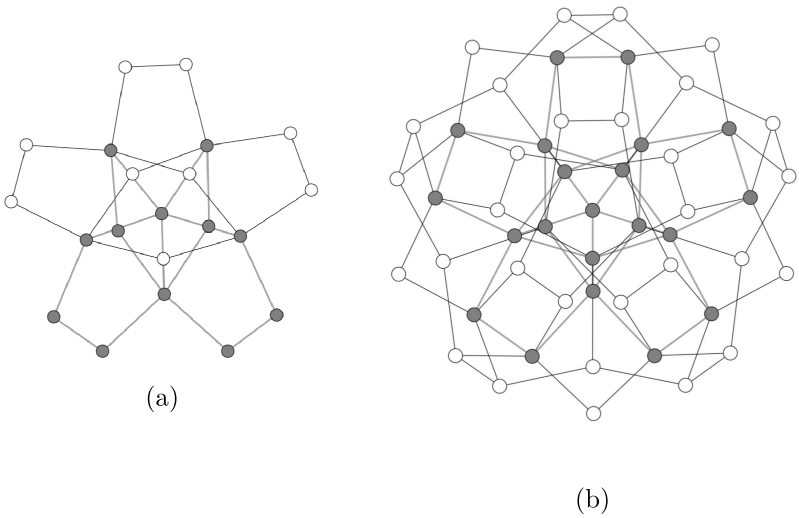


Figure 1.3: Ricci-flat 5-regular symmetric graph of order 72.

Lemma 3.1. Let xy be an edge in a d-regular graph G with local characteristics (N_0, N_1, N_2). Then the Ricci-curvature of the edge xy is given by

$$\kappa(x,y) = -2 + \frac{4 + 3N_0 + 2N_1 + N_2}{d}.$$

Corollary 3.1.1. Let xy be an edge in a Ricci-flat 5-regular graph G. Then the local characteristics (N_0, N_1, N_2) of edge xy must be one of the following five types listed in Table 3.1.

When the symmetry condition for Ricci-flat graphs is not imposed, the possible cases for the construction of the graph grow enormously. The main difficulty of such a classification lies in the lack of leverageable symmetries. In Section 4, we restrict our attention to symmetric graphs and found that Ricci-flat 5-regular symmetric graph must be isomorphic to a 5-regular symmetric graph of order 72, which we denote RF^5_{72}. Fig. 1.3 shows the subgraph induced by 2-neighborhood and 3-neighborhood of a vertex in RF^5_{72}, i.e. the subgraph induced by all vertices within a distance of 2 and 3, respectively, from the central vertex. The type-E local structure of an edge is highlighted in (a) and the the 2-neighborhood graph of RF^5_{72} shown in (a) is highlighted in (b). An adjacency list for RF^5_{72} can be found in the appendix.

Theorem 4.1. If G is a Ricci-flat 5-regular symmetric graph, then G is isomorphic to RF^5_{72}.

2. Notations and definitions

Let $G = (V, E)$ represent an undirected connected graph with vertex set V and edge set E without multiple edges or self loops. A vertex y is a neighbor of x if $xy \in E$. For a vertex $x \in V$, we denote the neighbors of x as $\Gamma(x)$ and the degree of x, i.e. the number of its neighbors, as d_x. If two vertices x, y are neighbors, we use $x \sim y$ to represent this relation. Let C_n represent a cycle of length n.

Definition 2.1. A probability distribution over the vertex set $V(G)$ is a mapping $\mu : V \to [0, 1]$ satisfying $\sum_{x \in V} \mu(x) = 1$. Suppose that two probability distributions μ_1 and μ_2 have finite support. A *coupling* between μ_1 and μ_2 is a mapping $A : V \times V \to [0, 1]$ with finite support such that for any $x, y \in V$

$$\sum_{y \in V} A(x, y) = \mu_1(x) \text{ and } \sum_{x \in V} A(x, y) = \mu_2(y).$$

Definition 2.2. The *transportation distance* between two probability distributions μ_1 and μ_2 is defined as follows:

$$W(\mu_1, \mu_2) = \inf_A \sum_{x,y \in V} A(x, y) d(x, y),$$

where the infimum is taken over all coupling A between μ_1 and μ_2.

By the theory of linear programming, the transportation distance is also equal to the optimal solution of its dual problem. Thus, we also have

$$W(\mu_1, \mu_2) = \sup_f \sum_{x \in V} f(x)[\mu_1(x) - \mu_2(x)]$$

where f is a Lipschitz function satisfying

$$|f(x) - f(y)| \le d(x, y).$$

Definition 2.3 ([13])**.** Let $G = (V, E)$ be a simple graph, for any $x, y \in V$ and $\alpha \in [0, 1]$, the *α-Ricci curvature* κ_α is defined to be

$$\kappa_\alpha(x, y) = 1 - \frac{W(\mu_x^\alpha, \mu_y^\alpha)}{d(x, y)},$$

where the probability distribution μ_x^α is defined as:

$$\mu_x^\alpha(z) = \begin{cases} \alpha, & \text{if } z = x, \\ \dfrac{1-\alpha}{d_x}, & \text{if } z \sim x, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.4 ([10])**.** Let $G = (V, E)$ be a simple graph, for any $x, y \in V$, the Lin-Lu-Yau *Ricci curvature* $\kappa(x, y)$ is defined as

$$\kappa(x, y) = \lim_{\alpha \to 1} \frac{\kappa_\alpha(x, y)}{1-\alpha},$$

where $\kappa_\alpha(x, y)$ is the α-Ricci curvature defined in above definition.

Naturally, a Ricci-flat graph is defined to be a graph in which the Ricci curvature of each edge is zero.

Definition 2.5 ([10])**.** A graph G is *Ricci-flat* if $\kappa(x, y) = 0$ for all edges $xy \in E$.

Next, we provide definitions for some properties of a graph that concern its symmetries, more precisely, its automorphism group, which is the group of permutations of vertices preserving edge connectivity.

Definition 2.6. A graph G is *edge-transitive* if its automorphism group acts transitively on its edges, i.e., for all pairs of edges $e_1, e_2 \in E$ there exists an automorphism $\varphi : e_1 \mapsto e_2$.

Definition 2.7. A graph G is *vertex-transitive* if its automorphism group acts transitively on its vertices, i.e., for all pairs of vertices $v_1, v_2 \in V$ there exists an automorphism $\varphi : v_1 \mapsto v_2$.

Definition 2.8. A graph G is *symmetric* if it is both edge-transitive and vertex-transitive.

Definition 2.9. A graph G is *arc-transitive* (also called *symmetric* by some authors) if its automorphism group acts transitively on ordered pairs of adjacent vertices, i.e., for all ordered pairs of adjacent vertices $(u_1, v_1), (u_2, v_2)$, there exists an automorphism $\varphi : u_1 \mapsto u_2, v_1 \mapsto v_2$.

Although in general symmetric graphs are not necessarily arc-transitive, for graphs of odd degree, the two notions are equivalent. Thus in our case of 5-regular symmetric graphs, we can simply consider arc-transitive graphs.

Lemma. *If a graph G is of odd degree, then it is arc-transitive if and only if it is symmetric.*

Proof. Let G be a symmetric graph of odd degree. Suppose to the contrary G is not arc-transitive. Then, there are two distinct orbits of the arcs of G under the automorphism group. Since G is vertex transitive, then the directed graph induced by one orbit of the arcs has the same indegree and outdegree for each vertex. However, in a directed graph, the sums of outdegree and indegree over all vertices must be equal, so the degree of G must be even. $\square$

3. Local structures with zero curvature

The Ricci-curvature of an edge xy describes roughly the "closeness" of the neighbors of vertices x and y. In order to formulate how close the two sets of neighbors $\Gamma(x)$ and $\Gamma(y)$ are, we define the local characteristics of edge xy as follows.

Consider all possible bijective pairings $p : \Gamma(x)\setminus\{y\} \to \Gamma(y)\setminus\{x\}$ between neighbors of x and y excluding themselves such that each neighbor of x is paired uniquely with a neighbor of y. Sort all the distances between paired vertices $d(x_i, p(x_i))$ into a non-decreasing sequence $S(p)$. Let $S(p')$ be the least sequence by lexicographic order taken from all possible pairings p between the neighbor sets. The *local characteristics* (N_0, N_1, N_2) of edge xy is defined such that N_i is the number of occurrences of i in the sequence $S(p')$. In other words, N_i describes the number of $(i+3)$-cycles C_{i+3} supporting edge xy with disjoint pairs of neighbors of x and y.

The curvature of an edge in a regular graph is then completely determined by its local characteristics.

Lemma 3.1. *Let xy be an edge in a d-regular graph G with local characteristics (N_0, N_1, N_2). Then the Ricci-curvature of the edge xy is given by*

$$\kappa(x, y) = -2 + \frac{4 + 3N_0 + 2N_1 + N_2}{d}.$$

Proof. Since G is d-regular, we have $\mu_x^\alpha(x) = \mu_y^\alpha(y) = \alpha$, $\mu_x^\alpha(y) = \mu_y(x) = \frac{1-\alpha}{d}$, and $\mu_x^\alpha(v_x) = \mu_y(v_y) = \frac{1-\alpha}{d}$ for $v_x \in \Gamma(x) - \{y\}$ and $v_y \in \Gamma(y) - \{x\}$. The main idea of the proof is to show that the optimal transport plan is to transfer $\alpha - \frac{1-\alpha}{d}$ from vertex x to y, and $\frac{1-\alpha}{d}$ from vertices in $\Gamma(x) - \{y\}$ to their paired vertex in $\Gamma(y) - \{x\}$ in the distance-minimizing pairing p'.

Let $S(p')$ be the least sequence associated with the pairing p' used in the above definition of the local characteristics of edge xy. Let $A(u, v) : V \times V \to$

$[1, 0]$ be a coupling function such that

$$A(u,v) = \begin{cases} \alpha - \dfrac{1-\alpha}{d}, & \text{if } u = x,\ v = y, \\ \dfrac{1-\alpha}{d}, & \text{if } v = p'(u), \\ 0, & \text{otherwise.} \end{cases}$$

Since we'll be taking the limit as $\alpha \to 1$, assume that $\alpha > \frac{1-\alpha}{d}$. Then the transportation distance is bounded above by

$$\begin{aligned} W(\mu_x^\alpha, \mu_y^\alpha) &\leq \sum_{u,v \in V} A(u,v) d(u,v) \\ &= A(x,y)d(x,y) + \sum_{d(u,p'(u))=1,2,3} A(u,p'(u))d(u,p'(u)) \\ &= \left(\alpha - \frac{1-\alpha}{d}\right) \cdot 1 + \frac{1-\alpha}{d} \cdot (N_1 + 2N_2 + 3(d-1-N_0-N_1-N_2)) \\ &= 3 - 2\alpha - \frac{1-\alpha}{d}(4 + 3N_0 + 2N_1 + N_2). \end{aligned}$$

In order to differentiate between the paired neighbors of x and y, define the following sets of vertices:

$$\begin{aligned} V_0 &= \{v \in \Gamma(x) - \{y\} \mid d(v, p'(v)) = 0\}, \\ X_1 &= \{v \in \Gamma(x) - \{y\} \mid d(v, p'(v)) = 1\}, \\ X_2 &= \{v \in \Gamma(x) - \{y\} \mid d(v, p'(v)) = 2\}, \\ X_3 &= \{v \in \Gamma(x) - \{y\} \mid d(v, p'(v)) = 3\}, \\ Y_3 &= \{v \in \Gamma(y) - \{x\} \mid d(p'^{-1}(v), v) = 3\}. \end{aligned}$$

We define a Lipschitz function $f : V \to \mathbb{R}$ by the following procedure:

1. $f(x) = 2$, $f(y) = 1$, $f(x_2) = 3$ for $x_3 \in X_3$, and $f(y_3) = 0$ for $y_3 \in Y_3$.
2. For $v_0 \in V_0$, if $v_0 \in \Gamma(X_3)$, then $f(v_0) = 2$; otherwise $f(v_0) = 1$. For $x_1 \in X_1$, if $x_1 \in \Gamma(X_3)$, then $f(x_1) = 2$ and $f(p'(x_1)) = 1$; otherwise $f(x_1) = 3$ and $f(p'(x_1)) = 2$. For $x_2 \in X_2$, if $x_2 \in \Gamma(X_3)$, then $f(x_2) = 2$, $f(p'(x_2)) = 0$, and $f(v_2) = 1$ for all $v_2 \in \Gamma(x_2) \cup \Gamma(p'(x_2))$; otherwise $f(x_1) = 3$, $f(p'(x_2)) = 1$, and $f(v_2) = 2$.
3. For the remaining vertices v, if $v \in \Gamma(x)$ for $f(X) = 3$, then $f(v) = 2$; otherwise $f(v) = 1$.

It is easy to check that f is indeed 1-Lipschitz, and as a result the transportation distance is bounded below by

$$\begin{aligned}
&W(\mu_x^\alpha, \mu_y^\alpha) \\
&\quad \geq \sum_{v\in V} f(v)[\mu_x^\alpha(v) - \mu_y^\alpha(v)] \\
&\quad = f(x)\left(\alpha - \frac{1-\alpha}{d}\right) + f(y)\left(\frac{1-\alpha}{d} - \alpha\right) + \sum_{v\in V_0}\left(\frac{1-\alpha}{d} - \frac{1-\alpha}{d}\right) \\
&\qquad + \sum_{v\in\Gamma(x)-\{y\}-V_0} f(v)\left(\frac{1-\alpha}{d} - 0\right) + \sum_{v\in\Gamma(y)-\{x\}-V_0} f(v)\left(0 - \frac{1-\alpha}{d}\right) \\
&\quad = (f(x) - f(y))\left(\alpha - \frac{1-\alpha}{d}\right) + \frac{1-\alpha}{d}\left(\sum_{i=1}^{3}\sum_{x_i\in X_i}(f(x_i) - f(p'(x_i)))\right) \\
&\quad = \frac{1-\alpha}{d}\cdot(N_1 + 2N_2 + 3(d-1-N_0-N_1-N_2)) \\
&\quad = 3 - 2\alpha - \frac{1-\alpha}{d}(4 + 3N_0 + 2N_1 + N_2).
\end{aligned}$$

Since the two bounds are equal, we have

$$W(\mu_x^\alpha, \mu_y^\alpha) = 3 - 2\alpha - \frac{1-\alpha}{d}(4 + 3N_0 + 2N_1 + N_2).$$

Therefore, the Ricci curvature of edge xy is

$$\kappa(x,y) = \lim_{\alpha\to 1}\frac{1 - W(\mu_x^\alpha, \mu_y^\alpha)}{1-\alpha} = -2 + \frac{4 + 3N_0 + 2N_1 + N_2}{d}.$$

□

Corollary 3.1.1. *Let xy be an edge in a Ricci-flat 5-regular graph G. Then the local characteristics (N_0, N_1, N_2) of edge xy must be one of the following five types listed in Table 3.1.*

Proof. With $\kappa = 0$ and $d_x = 5$, Lemma 3.1 gives

$$3N_0 + 2N_1 + N_2 = 6.$$

Table 3.1: Local characteristics for edges in Ricci-flat 5-regular graphs

Type-A	$(2,0,0)$
Type-B	$(1,1,1)$
Type-C	$(1,0,3)$
Type-D	$(0,3,0)$
Type-E	$(0,2,2)$

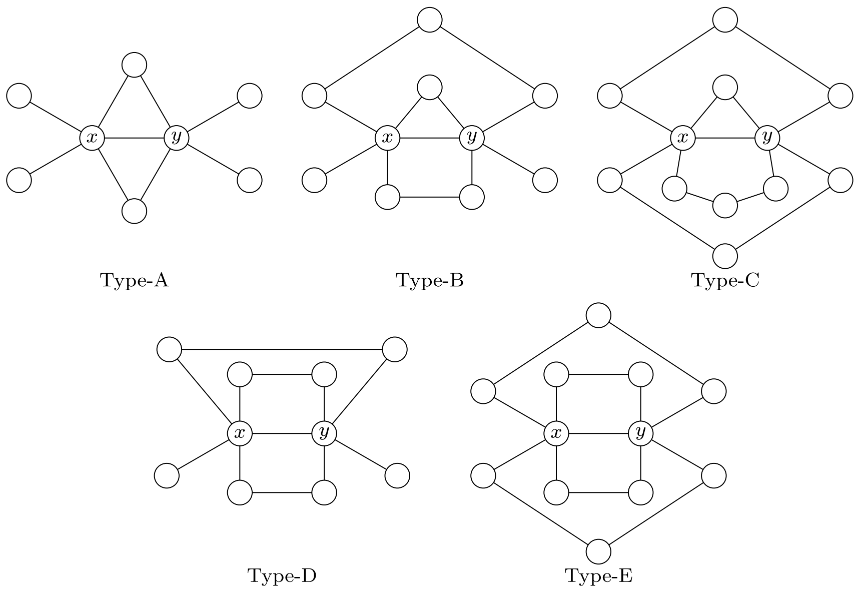


Figure 3.1: Local structures of edge xy in Ricci-flat 5-regular graphs.

Since there are only 4 vertices in $\Gamma(x) - y$, we have $N_0 + N_1 + N_2 \leq 4$. All solutions of the above are given in Table 3.1. A schematic drawing of each local structure is shown in Fig. 3.1. Note that the vertices that are not neighbors of x and y (in this case the middle vertex in a 5-cycle) may be the same vertex as other vertices in the graph as long as the local characteristic of xy remains the same. For example, in Type-B, the top vertex in the 5-cycle can coincide with the vertex in the 3-cycle without changing the local characteristic of xy. □

It is worth noting that in each type of local structure, at least two pairs of vertices given by the pairing p' have to have distance less than 3. Moreover, excluding type-A, each type requires at least three pairs of vertices with distance less than 3.

4. Ricci-flat 5-regular symmetric graphs

In this section, we classify Ricci-flat 5-regular graphs G that are symmetric.

Theorem 4.1. *If G is a Ricci-flat 5-regular symmetric graph, then G is isomorphic to RF^5_{72}.*

For a symmetric graph G, every edge in G must have the same local structure. Therefore, we classify G based on the local structure of its edges.

4.1. Ricci-flat 5-regular symmetric graphs of girth 3

We show that Ricci-flat 5-regular graphs cannot contain 3-cycles and hence cannot have girth 3.

Lemma 4.2. *If G is a Ricci-flat 5-regular symmetric graph, then the edges in G are not type-A.*

Proof. Let xy be an edge in G, v_1, v_2 be common vertices of x and y, and x_1, x_2, y_1, y_2 be the neighbors of x and y respectively, as shown in Fig. 4.1. Consider edge xx_1, which needs to be in two C_3 for it to be type-A. Clearly, $x_1 \nsim y$ considering edge xy, so x_1 must be connected to two of the vertices in the set $\{v_1, v_2, x_2\}$. Since v_1 and v_2 are interchangeable, i.e., there exists an automorphism $\varphi : v_1 \mapsto v_2$, we have wlog $x_1 \sim v_1$. However, this is a contradiction since connecting x_1v_1 forms a C_5 on edge v_1y, which makes v_1y no longer type-A. □

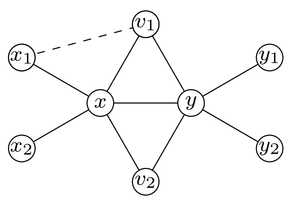


Figure 4.1: Type-A.

Lemma 4.3. *If G is a Ricci-flat 5-regular symmetric graph, then the edges in G are not type-B or type-C.*

Proof. Consider a vertex x_0 in G and its neighbors $x_i, 1 \leq i \leq 5$. Since every edge is type-B or type-C, it is in a C_3. For edge x_0x_1, wlog $x_1 \sim x_2$. For edge x_0x_3, wlog $x_3 \sim x_4$. Then, edge x_0x_5 cannot be in a C_3 since connecting x_5 with any other vertex will result in two C_3 on an edge, which is a contradiction since none of the edges are type-A. □

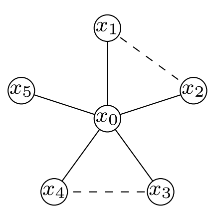


Figure 4.2: Type-B or Type-C.

4.2. Ricci-flat 5-regular symmetric graphs of girth 4

Before proving that Ricci-flat 5-regular symmetric graphs with type-D edges do not exist, we prove a short lemma using the technique of double counting to show that a Ricci-flat graph containing only type-D edges must contain two 4-cycles sharing two edges, that is, the subgraph shown in Fig. 4.3 which is isomorphic to the complete bipartite graph $K_{3,2}$.

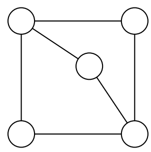


Figure 4.3: Complete bitartite graph $K_{3,2}$.

Lemma 4.4. *If G is a Ricci-flat 5-regular graph containing only type-D edges, then it contains $K_{3,2}$ as a subgraph.*

Proof. We show by contradiction that there doesn't exist a Ricci-flat 5-regular graph with only type-D edges that does not contain $K_{2,3}$, i.e., in which all 4-cycles share at most one edge. Suppose such a graph G exists. Consider a vertex x_0 in G and its neighbors $x_i, 1 \leq i \leq 5$. Since all C_4 share at most one edge, each one of the five edges x_0x_i is in exactly three C_4. Thus, the number of ordered pair (x_0x_i, C_4^*) where $x_0x_i \in C_4^*$ should be 15. On the other hand, each C_4 through vertex x_0 contains two edges x_i and x_ix_j. Thus, the number of ordered pairs (x_0x_i, C_4^*) should be even, and we have reached a contradiction. ☐

Lemma 4.5. *If G is a Ricci-flat 5-regular symmetric graph, then the edges in G are not type-D.*

Proof. Since G is symmetric and of odd degree, it must be arc-transitive. As a result, the neighborhood of an edge $u_1v_1 \in G$ denoted by $\Gamma(u_1v_1)$, i.e., the subgraph induced by $\Gamma(u_1) \cup \Gamma(v_1)$ must be isomorphic to the neighborhood of any other edge $\Gamma(u_2v_2)$. Since by Lemma 4.4, G must contain $K_{3,2}$ as a subgraph. Symmetry implies that each edge of G is contained in a $K_{3,2}$. We classify all possible neighborhoods of an edge xy such that xy is in a $K_{3,2}$ and there is an automorphism $\varphi : \Gamma(xy) \to \Gamma(xy)$ mapping xy to yx. Let x_i and y_i be the neighbors of x and y excluding themselves, and wlog $x_i \sim y_i$ for $i = 1, 2, 3$ and $d(x_4, y_4) = 3$. In order to form a $K_{3,2}$ on xy, we have wlog either $x_1 \sim y_2$ or $x_1 \sim y_4$.

Assume that $x_1 \sim x_2$ and $x_1 \nsim y_4$, we break into two cases based on the number of connections between x_i and y_j.

1. Suppose each x_i, $i = 1, 2, 3$ is connected to at most one $y_j, j \neq i$.

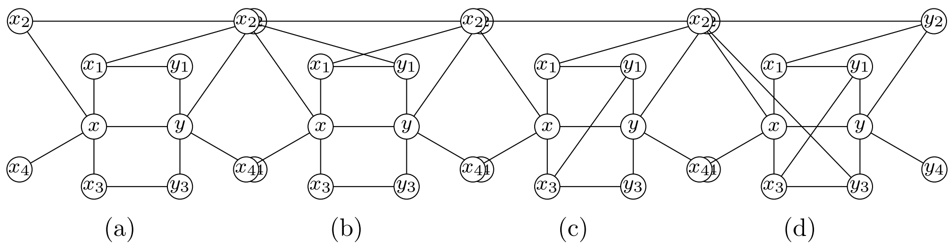


Figure 4.4

(a) Assume that xy has the neighborhood shown in Fig. 4.4(a). Consider edge xx_4, which cannot form a C_4 through xy as it has distance 3 to all the non-adjacent vertices. Thus, it must form a C_4 with each x_i, $i = 1, 2, 3$ by connecting x_4 to a new neighbor of x_i namely z_i. For the neighborhood of xx_4, we need to connect one of z_i to x_j, $i, j \in \{1, 2, 3\}$. Note that xx_1 is already in three C_4, namely x_1y_1yx, $x_1y_2x_2x$ and $x_1z_1x_4x$. Since we have $y_2 \sim y$, its neighborhood including the fifth neighbor of x_1 is isomorphic to $\Gamma(xy)$. Thus, the neighbors of x_1 and x are not further connected, and we have $x_1 \nsim z_2, z_3$ and $z_1 \nsim x_2, x_3$. Therefore, we must have either $x_2 \sim z_3$ or $z_2 \sim x_3$.

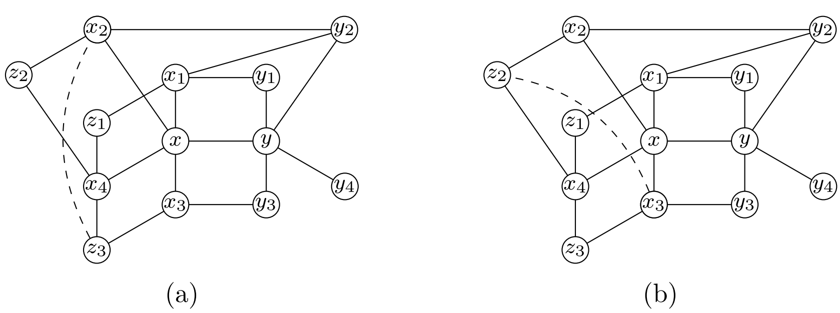


Figure 4.5: Type-D.

If $x_2 \sim z_3$ as in Fig. 4.5(a), consider edge xx_2, which is already in three C_4. Let v be the fifth neighbor of x_2, we have $d(v, x_1) = 3$.

However, as x_1 is connected to y_2, a neighbor of x_2, the neighborhood of xx_2 is not isomorphic to $\Gamma(xy)$, contradiction.

If $z_2 \sim x_3$ as in Fig. 4.5(b), then edge xx_3 is in three C_4 and has isomorphic neighborhood to xy. Consider edge xx_2, which needs another C_4 formed through a new neighbor of x_2 namely v since x_2 cannot connect to any of the existing vertices. However, $v \nsim x_1$ considering the neighborhood of xx_1, $v \nsim x_3$ considering the neighborhood of xx_3. Thus, the third C_4 on edge xx_2 cannot be formed, a contradiction.

(b) Assume that xy has the neighborhood shown in Fig. 4.4(b). Similar to Case 1(a), we have $z_i \sim x_i$ for $i = 1, 2, 3$ where z_i are neighbors of x_4 as in Fig. 4.6(a). Since the neighborhood of edge xx_4 needs to be isomorphic to $\Gamma(xy)$, we must have wlog either $z_1 \sim x_2$ or $z_1 \sim x_3$. However, $d(z_1, x_3) = 3$ considering edge xx_1, which is already in three C_4, so we must have $z_1 \sim x_2$ and also $z_2 \sim x_1$. Next, we consider edge xx_3, which is in two C_4 and needs to form a C_4 through either xx_1 or xx_2. Since xx_1 and xx_2 are equivalent edges under an automorphism, let the C_4 pass through xx_1. Since x_1 is at maximum degree, x_3 must be connected to one of the neighbors of x_1. However, none of the neighbors of x_1 can be connected to x_3 given the neighborhood structure of edges xy and xx_4, and we have reached a contradiction.

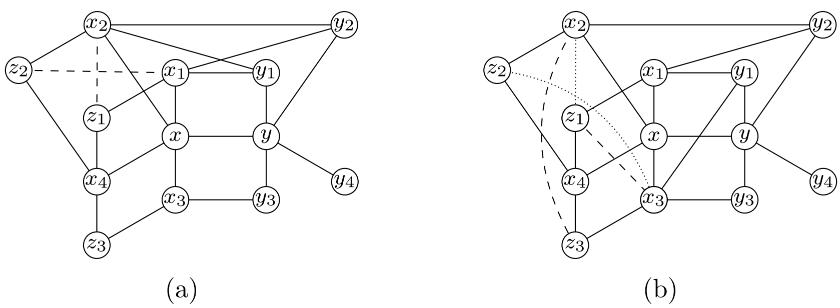


Figure 4.6: Type-D.

(c) Assume that xy has the neighborhood shown in Fig. 4.4(c). Similar to Case 1(a), we have $z_i \sim x_i$ for $i = 1, 2, 3$ where z_i are neighbors of x_4 as in Fig. 4.6(b), and we need to connect neighbors of x_4 and x so that the neighborhood of xx_4 is isomorphic to $\Gamma(xy)$. Since xx_1 is already in three C_4, we have $x_1 \nsim z_2, z_3$. Thus, we have either $x_2 \sim z_3$ and $x_3 \sim z_1$ (shown with dashed lines), or $x_2 \sim z_1$ and $x_3 \sim z_2$ (shown with dotted lines). However, in each case four C_4 are created on edge xx_1 and xx_3 respectively, a contradiction.

(d) Assume that xy has the neighborhood shown in Fig. 4.4(d) respectively. The argument for Case 1(c) applies similarly.

2. Suppose each x_i, $i = 1, 2, 3$ is connected to at most two $y_j, j \neq i$. Wlog, let $x_1 \sim y_2, y_3$, and by the automorphism φ there needs to be a vertex y_j such that it is connected to two $x_i, i \neq j$. By casework, we have the following potential neighborhoods of xy shown in Fig. 4.7 in which there exists an automorphism φ.

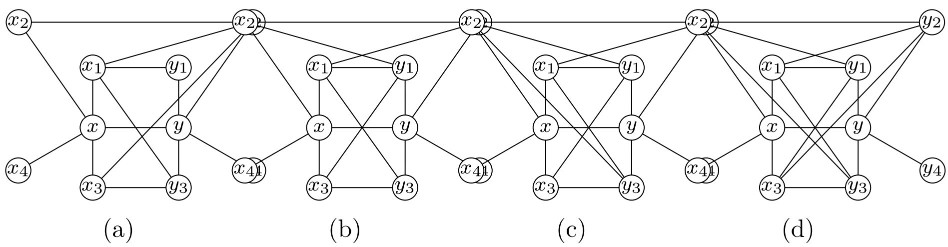


Figure 4.7: Type-D.

(a) Assume that xy has the neighborhood shown in Fig. 4.7(b). We relabel the vertices by interchanging y_1 and y_2 and redraw the graph in Fig. 4.8 to highlight the symmetry and its similarity to the following cases.

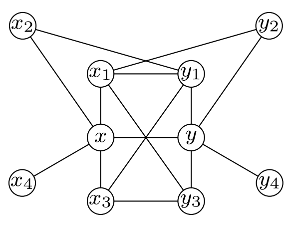


Figure 4.8: Type-D.

Consider edge xx_4 and note that x_4 has distance 3 to y_4 and the fifth neighbor of x_1. Since it does not connect to any existing vertex either, it cannot form a C_4 through both xx_1 and xy, a contradiction.

(b–d) Assume that xy has the neighborhood shown in Fig. 4.7(b)–(d) respectively. Consider edge xx_4 and a similar contradiction arises as in Case 2(a).

Therefore, we proved that we must have wlog $x_1 \sim y_4$. Then, x_4 must also be connected to a neighbor of y given the automorphism φ. Note that $x_4 \nsim y_1$ or else $d(x_4, y_4) = 1$, so we have wlog $x_4 \sim y_3$. Moreover, note that $x_4 \nsim y_2$, since if X_4 is connected to two neighbors of y, y_4 must be connected two neighbors of x as well, resulting in $d(x_4, y_4) = 1$. Similarly, $y_4 \nsim x_2, x_3$. Therefore, we have $x_1 \sim y_4$ and $x_4 \sim y_3$. We combine this with the discussion of whether x_i and y_j where $i, j \in \{1, 2, 3\}, i \neq j$ is connected above, and arrive at several possibility for $\Gamma(xy)$.

1. Suppose $x_i \nsim y_j$ for all $i, j \in \{1, 2, 3\}$ and $i \neq j$ as in Fig. 4.9.

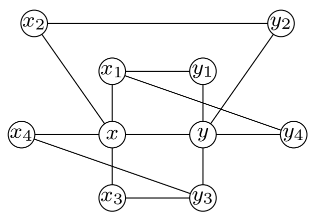


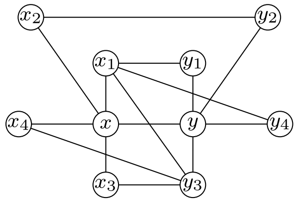
Figure 4.9: Type-D.

Consider edge xx_1. Let z_1, z_2 be the two new neighbors of x_1. Then, for $\Gamma(xx_1)$ to be isomorphic to $\Gamma(xy)$, wlog we have z_1 connected to two neighbors of x excluding x_1, and z_2 connected to the remaining neighbor of x. Since x_3 and x_4 are interchangeable, there are two cases.

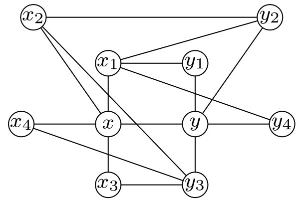
   (a) Assume wlog that $z_1 \sim x_2, x_4$ and $z_2 \sim x_3$. Consider edge xx_4. Note that it cannot have an isomorphic neighborhood to xy because y_3 and z_1, two neighbors of x_4, have degree 3 in the neighborhood of xx_4, a contradiction.

   (b) We have that $z_1 \sim x_3, x_4$ and $z_2 \sim x_2$. Consider edge z_1x_4, which is in two C_4, namely $z_1x_1xx_4$ and $z_1x_3y_3x_4$. However, we also have xx_3, which makes it impossible for $\Gamma(z_1x_4)$ to be isomorphic to $\Gamma(xy)$, a contradiction.

2. Suppose there exists x_i, $i \in \{1, 2, 3\}$ such that $x_i \sim y_j$ for some $j \in \{1, 2, 3\} - \{i\}$, then there are only two non-isomorphic possibilities for $\Gamma(xy)$ by noticing that the automorphism φ sending xy to yx must be $\varphi : x_1 \mapsto y_3, x_2 \mapsto y_2, x_3 \mapsto y_1, x_4 \mapsto y_4$.



(a)



(b)

Figure 4.10: Type-D.

(a) Assume that xy has the neighborhood shown in Fig. 4.10(a). Let the fifth neighbor of x_1 be z_1 and the fifth neighbor of y_3 be z_2. Consider edge x_1y_3, which needs one more C_4 and it must be formed by connecting $z_1 \sim z_2$. This way, edges xx_1 and yy_3 have isomorphic neighborhoods to xy. Next, we consider edge xx_1, which is already in three C_4. To make $\Gamma(xx_1)$ isomorphic to $\Gamma(xy)$, we must have $x_2 \sim z_1$.

Next, we consider edge x_4x. Notice that y_3, a neighbor of x_4, is connected to four neighbors of x, thus it must be mapped to x_1 by the automorphism sending edge x_4x to xy. Thus, x_2, the only neighbor of x not connected to y_3, must be connected to a new neighbor of x_4 namely w_1. Similarly, since vertices x_3 and x_4 are interchangeable, the same analysis applies and x_2 must be connected to a neighbor of x_4. Note that $x_4 \nsim w_1$ since if so, w_1 as a neighbor of x_3 would be connected to two neighbors of x, resulting in $\Gamma(xx_3)$ no longer possible to be isomorphic to $\Gamma(xy)$. Thus, we must have $x_2 \sim w_2 \sim x_4$. However, in this case, xx_2 would be in four C_4, a contradiction.

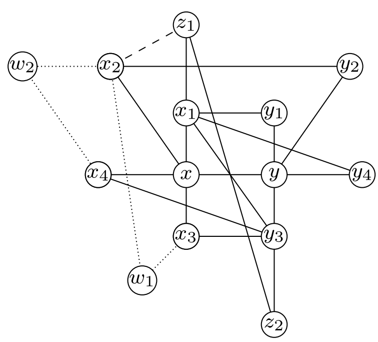


Figure 4.11: Type-D.

(b) Assume that xy has the neighborhood shown in Fig. 4.10(b). Let the fifth neighbor of x_1 be z. Consider edge xx_1, which is in two C_4, so another C_4 needs to be formed through x_1z. Since x_3 and x_4 are interchangeable, let $z \sim x_4$. To make the $\Gamma(xx_1)$ isomorphic to $\Gamma(xy)$, we must have $z \sim x_2, x_3$. However, in this way, $\Gamma(xx_2)$ cannot be isomorphic to $\Gamma(xy)$, a contradiction.

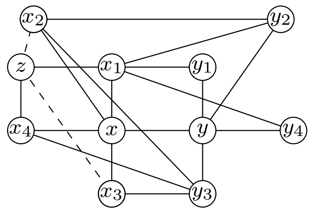


Figure 4.12: Type-D.

□

Next, we move onto symmetric graphs with type-E edges.

Lemma 4.6. *If G is a Ricci-flat 5-regular symmetric graph with type-E edges, then it is isomorphic to RF^5_{72}.*

Proof. We start by considering a C_5 in G and denote its vertices $x_i, 1 \leq i \leq 5$. Since each edge is type-E, it needs to be supported on two C_4. There are only three arrangements of the C_4 on edges in the C_5 under consideration such that each arc in the C_5 are in the same orbit under the automorphism group of this subgraph, since for the two C_4 on an edge x_ix_{i+1}, at least one of them is adjacent to a C_4 on the neighboring edge $x_{i+1}x_{i+2}$. If both C_4 on an edge are adjacent to the two C_4 on neighboring edges of the C_5, we have the first case in Fig. 4.13(A). When only one C_4 is adjacent to a C_4 on the neighboring edges, if there are no three adjacent C_4 in a row, that is, adjacent C_4 on edges $x_ix_{i+1}, x_{i+1}x_{i+2}, x_{i+2}x_{i+3}$, we have the second case shown in Fig. 4.13(B); otherwise, we have the third case shown in Fig. 4.13(C).

All the C_5 in G must have a local structure that is isomorphic to the subgraph shown above in each case. We construct the graph with the aid of a curvature calculator [5]. In the first two cases, contradiction arises in the construction process, while the local structure of a C_5 in the third case can be successfully expanded into a Ricci-flat 5-regular graph on 72 vertices, denoted by RF^5_{72}. The 2-neighborhood and 3-neighborhood of a vertex in RF^5_{72} are shown in Fig. 1.3.

The first case is rejected as the fifth edge adjacent to vertex x_1 cannot be in C_4 while maintaining symmetry and Ricci flat. In fact, the local structure

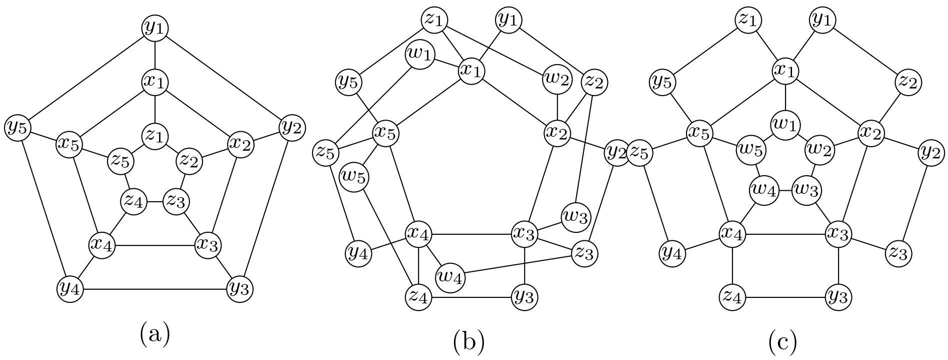


Figure 4.13: Three cases of construction using Type-E edges.

around one vertex must consists of the rotations in C_4 and C_5, together with reflectional symmetry because of arc-transitivity. In Fig. 4.13(B), a common vertex for y_1 and w_1 is needed, similarly to other pairs y_i, w_i for $i = 2, 3, 4, 5$. Thus there can be five C_4s around every vertex, see the rotation order of C_4s from the first two subgraphs of Fig. 4.14. And there is also a unique way to generate C_5 around each vertex according to Fig. 4.13(B).

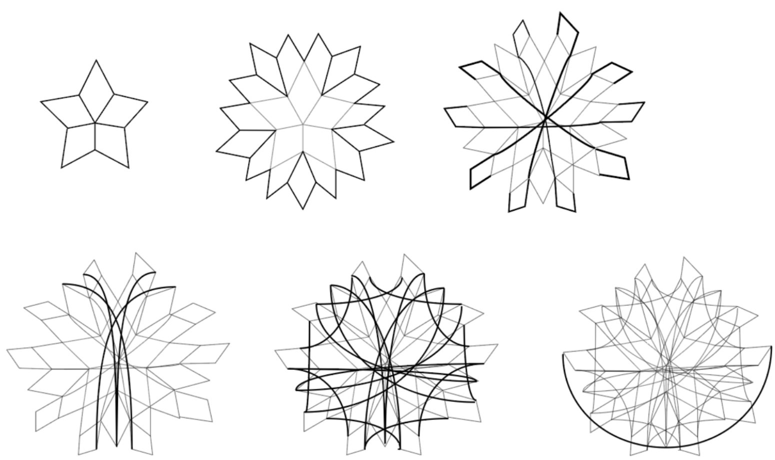
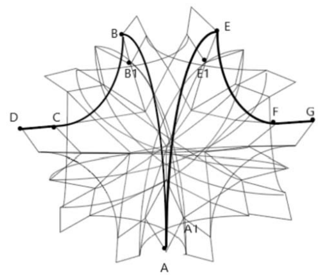


Figure 4.14: Construction Process using (B) of Fig. 4.13, bold edges are newly added at every step.

However, the lack of reflectional symmetry in Fig. 4.13(B) highlights the fact that edges are not equivalent to their inverse, thus it cannot be extended to an arc-transitive graph.



One can see from above construction stage that vertex A reaches degree 6. By the rotations of C_4 around vertices A, B, C, we can determine that edges AB and BC are in a C_4, edges BC and CD are in a C_4. Call the fifth neighbor of A as $A5$, then edges $A5A$ and AB have to be in a C_4. Since $B-B1-A1-A$ is already in one C_4, then $A5$ has to be D. Similarly, A has to be adjacent to vertex G. Thus the degree of A is 6.

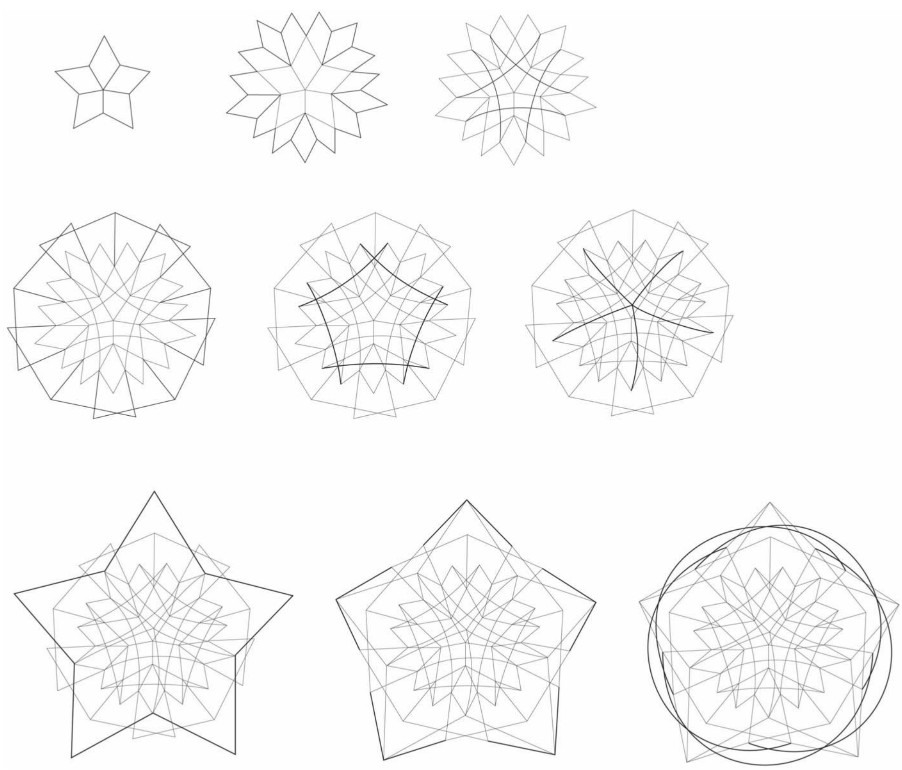


Figure 4.15: Construction Process using (C) of Fig. 4.13.

For the third case, by edge-transitivity, $\{z_1, x_1, y_1\}$ share one C_4, $\{z_1, x_1, x_2\}$, $\{w_1, x_1, z_1\}$, $\{y_1, x_1, x_5\}$ and $\{y_1, x_1, w_1\}$ share one C_5 respectively. That is, there are five 4-cycles and five 5-cycles meeting at each vertex. Thus there is a unique way to generate five C_4s around any vertex. To construct C_5 around one vertex, we have to match the induced subgraph on $\{w_1, \ldots, w_5, x_1, \ldots, x_5\}$ as shown on Fig. 4.13(C), then there is still only one way to generate these C_5s around each vertex.

Fig. 4.15 shows the construction process. We show as many vertices as possible but not all of them, otherwise the picture won't look clear. Eventually Fig. 4.13 (C) will be extended to RF^5_{72}.

This concludes our proof of Theorem 4.1. □

Appendix A. Adjacency list for RF^5_{72}

1:[2,3,4,5,8],
2:[1,12,6,7,9],
3:[11,1,13,7,18],
4:[1,14,6,19,10],
5:[11,1,16,28,20],
6:[2,4,17,29,21],
7:[22,2,3,15,26],
8:[1,23,16,30,10],
9:[2,25,17,30,41],
10:[4,27,8,31,42],
11:[24,3,5,38,32],
12:[33,2,25,15,20],
13:[3,36,26,19,53],
14:[4,37,27,28,54],
15:[55,12,38,39,7],
16:[34,5,40,8,43],
17:[44,35,6,9,31],
18:[23,45,24,3,36],
19:[13,46,4,37,21],
20:[12,5,38,49,62],
21:[26,6,50,19,64],
22:[45,48,39,7,41],
23:[18,40,51,8,65],
24:[11,44,35,18,40],
25:[12,56,47,52,9],

26:[13,57,48,7,21],
27:[55,34,14,39,10],
28:[34,14,58,5,49],
29:[33,35,6,50,54],
30:[59,51,8,9,31],
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32:[11,35,58,61,53],
33:[12,69,29,62,52],
34:[57,48,16,27,28],
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36:[56,13,18,52,63],
37:[56,14,47,60,19],
38:[11,66,15,61,20],
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69:[33,55,70,71,54],
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72:[67,46,70,64,65]

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Heidi Lei
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, 02139
USA
E-mail: hlei@mit.edu

Shuliang Bai
Shing-Tung Yau Center
Southeast University
Nanjing, 210018
China
E-mail: sbai@seu.edu.cn