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Quasi-random graphs of given density and Ramsey numbers*

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Abstract: A cornerstone contribution due to Chung, Graham and Wilson (1989) implies that many graph properties of different nature are equivalent. Graphs that satisfy any (and thus all) of the properties are called quasi-random graphs. In this paper, we construct families of quasi-random graphs for any given edge density, which are regular but not strongly regular. Moreover, we obtain a lower bound for Ramsey number $r(K_1 + G)$ in which the graph G contains no isolated vertex, which extends a classical result by Shearer (1986) and independently Mathon (1987).

Keywords: Ramsey number, quasi-randomness, Weil bound.

1. Introduction

Random graphs have been proven to be one of the most important tools in modern graph theory. How can we tell when a given graph behaves like a random graph and how does one construct such graphs? This leads us to a concept of *quasi-random graphs* (or pseudo-random graphs).

It was Thomason [33, 34] who introduced the notation of jumbled graphs in order to measure the similarity between the edge distribution of quasi-random graphs and random graphs. Another cornerstone contribution on this topic due to Chung, Graham and Wilson [11] revealed the equivalence of a number of disparate graph properties with respect to a constant $p \in (0, 1)$, all possessed asymptotically almost surely by random graph $G(n, p)$ in which each edge appears randomly with probability p. This fundamental result opened many new horizons.

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Let $G = (V, E)$ be a graph. Write $e(G)$ for the number of edges of G and $p = e(G)/\binom{n}{2}$ for the edge density, or density simply, of G. For a subset U of V, write $e(U)$ for the number of edges in the subgraph induced by U. Let (G_n) be a sequence of graphs, where G_n has n vertices, and let $p = p(n)$ be a parameter. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be eigenvalues of G with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. For fixed $0 < p < 1$, two of these equivalent properties are as follows.

$\mathcal{P}(\lambda)$: $e(G) \geq \frac{pn^2}{2} + o(n^2)$, $\lambda_1 \sim pn$ and $\lambda_2 = o(\lambda_1)$.

$\mathcal{P}(U)$: For each $U \subseteq V(G)$, $e(U) = p\binom{|U|}{2} + o(n^2)$.

Graphs that satisfy any (and thus all) of the properties are called quasi-random graphs, see [8, 9, 10] and other related references. Quasi-randomness is a limit property, and quasi-random graphs are refereed to a sequence of graphs (G_n). For convenience, we also simply call G_n p-quasi-random if the sequence of graphs (G_n) is p-quasi-random when there is no confusion. A survey on this topic is that of Krivelevich and Sudakov [23].

The following fact follows easily from the property $\mathcal{P}(U)$.

Proposition 1.1. *For a sequence of graphs (G_n) of order n, let $0 < p < 1$ be fixed and (p_n) be a sequence of numbers with $0 < p_n < 1$. If G_n is p_n-quasi-random, and $p_n \to p$ as $n \to \infty$, then G_n is p-quasi-random as $n \to \infty$.*

We also have the following simple property.

Proposition 1.2. *Consider two sequences of quasi-random graphs (G_n) and (G'_n) with densities p and p', respectively, where G_n and G'_n have the same vertex set V_n of size n, whose edge sets are disjoint. Let G''_n be the graph on vertex set V_n whose edge set consists of that of G_n and G'_n. Then G''_n is $(p+p')$-quasi-random.*

Proof. Let U be a subset of V_n. Then the edges of G''_n in U consists of that of G_n and G'_n. As G_n and G'_n satisfies the property $\mathcal{P}(U)$, and thus the number of edges of G''_n in U is

$$p\binom{|U|}{2} + p'\binom{|U|}{2} + o(n^2) = (p+p')\binom{|U|}{2} + o(n^2).$$

So G''_n satisfies the property $\mathcal{P}(U)$ for the edge density $p + p'$. □

We call a graph G *strongly regular* with parameters n, d, γ_1, γ_2, denoted by $srg(n, d, \gamma_1, \gamma_2)$, if it has order n, and it is d-regular, and any pair of vertices have γ_1 common neighbors if they are adjacent, and γ_2 common neighbors otherwise. For a strongly regular graph G, the following well-known result gives the spectrum of G, whose proof can be found in [19] and from which it is easy to verify whether or not a sequence of strongly regular graphs satisfies $\mathcal{P}(\lambda)$.

Lemma 1.1. *Let G be a connected $srg(n,d,\gamma_1,\gamma_2)$ with $n \geq 3$. Then $\lambda_1 = d$ is an eigenvalue with multiplicity $m_1 = 1$, and any eigenvalue $\lambda \neq \lambda_1$ satisfies*

$$\lambda^2 + (\gamma_2 - \gamma_1)\lambda + (\gamma_2 - d) = 0.$$

The equation has two distinct solutions λ_2 and λ_3 with $\lambda_2 > 0 > \lambda_3$, and λ_3 is an eigenvalue. If $d + (n-1)\lambda_3 \neq 0$, then λ_2 is also an eigenvalue. The multiplicities of m_2 and m_3 can be determined by $m_2 + m_3 = n - 1$ and $d + m_2\lambda_2 + m_3\lambda_3 = 0$.

The existence of quasi-random graphs for any given density p is ensured by random graph $G(n,p)$. However, only a few explicit constructions for quasi-random graphs are known. The most mentioned quasi-random graphs are the Paley graphs with edge density $p = 1/2$. For a prime power q and an integer $k \geq 2$, Delsarte and Goethals and to Turyn (unpublished, reported in [30]) defined $srg\left(q^2, k(q-1), q+k^2-3k, k(k-1)\right)$, which is quasi-random by checking $\mathcal{P}(\lambda)$ as its spectrum can be determined by Lemma 1.1.

The edge density of the above $srg\left(q^2, k(q-1), q+k^2-3k, k(k-1)\right)$ is $\frac{k}{q+1}$. Hence, for any fixed $p \in (0,1)$, if we take q and $k = k(q)$ such that $k/q \to p$ as $q \to \infty$, then this strongly regular graph is p-quasi-random as $q \to \infty$.

2. Quasi-random graphs with any given edge density

In this section, we shall construct new families of quasi-random graphs for any given edge density. The idea of the following construction comes from Shearer [31], Mathon [22], Füredi [17], and Axenovich, Füredi and Mubayi [4].

Let $k \geq 3$ and $q \equiv 1 \pmod{2k}$ be a prime power. Denote $q = sk+1$. Let β be a primitive element of F_q,

(1)
$$F_0 = \{0\}, \quad F_1 = \left\{1, \beta^k, \ldots, \beta^{(s-1)k}\right\}, \quad \text{and} \quad F_i = \beta^{i-1}F_1 \quad \text{for} \quad i = 1,2,\ldots,k.$$

The *cyclotomic association scheme* with k classes corresponds to the edge coloring of K_q on F_q: an edge $\{x,y\}$ of K_q is assigned color i if and only if $x - y \in F_i$, $1 \leq i \leq k$. $H_i(q,k)$ are the *cyclotomic graphs* generated by edges in color i for $1 \leq i \leq k$.

One can find the following result in [5, p. 66], and we include a proof for completeness.

Lemma 2.1. *Let $q \equiv 1 \pmod{2k}$ be a prime power and $H_i(q,k)$, $1 \le i \le k$, be defined as above. They are isomorphic to each other, and form an edge partition of K_q. Furthermore, each of them is $(q-1)/k$-regular, and the number of common neighbors of each pair of adjacent vertices is the same.*

Proof. We shall verify that $H_1(q,k)$ is isomorphic to $H_i(q,k)$. Define a bijection $\phi(z) = \beta^{i-1}z$ on $F(q)$. Then $\phi(x) - \phi(y) = \beta^{i-1}(x-y)$, and thus x and y are adjacent in $H_1(q,k)$ if and only if $\phi(x)$ and $\phi(y)$ are adjacent in $H_i(q,k)$.

Note that $F_1, F_2, \ldots, F_k$ form a partition of F_q^*, where $F_q^* = F_q \setminus \{0\}$. Thus any edge of K_q is colored by one from $\{1,2,\ldots,k\}$.

Since the neighborhood of a vertex x in $H_1(q,k)$ is $x + F_1$, which contains $(q-1)/k$ vertices, so $H_1(q,k)$ is $(q-1)/k$-regular.

Now, let $\{u,v\}$ be an edge of $H_1(q,k)$ with $u - v = \beta^{tk}$. Define a bijection ψ on F_q as $\psi(z) = \frac{z-u}{v-u}$. For any distinct vertices x and y, as

$$\psi(x) - \psi(y) = \frac{x-y}{u-v} = (x-y)\beta^{(q-1-t)k} = (x-y)\beta^{t_1 k},$$

where $0 \le t_1 \le s-1$ with $q-1-t \equiv t_1 \pmod{s}$, we have that x and y are adjacent in $H_1(q,k)$ if and only if $\psi(x)$ and $\psi(y)$ are adjacent in $H_1(q,k)$, and thus ψ is an automorphism of $H_1(q,k)$. Since $\psi(u) = 0$ and $\psi(v) = 1$, we have the number of common neighbors of u and v in $H_1(q,k)$ is as same as that of 0 and 1. □

Definition 2.1. *Let A be an additive group, and let $S \subseteq A \setminus \{0\}$ be an inverse-closed subset. The* Cayley graph *$\Gamma(A,S)$ is defined on vertex set A, in which a pair of vertices u and v are adjacent if $u - v \in S$.*

Remark. If we set $S_i = \{\beta^{i-1}, \beta^{i-1+k}, \beta^{i-1+2k}, \ldots\}$, then $H_i(q,k)$ is just the Cayley graph $\Gamma(F_q, S_i)$.

The following well-known result establishes a simple and elegant relation between eigenvalues of the Cayley graph $\Gamma(A,S)$ and the character on S, see e.g. Alon [1].

Lemma 2.2. *Let A be an additive group, and let $S \subseteq A \setminus \{0\}$ be an inverse-closed subset. Then each eigenvalue λ of the Cayley graph $\Gamma(A,S)$ has the form*

$$\lambda = \sum_{s \in S} \psi(s),$$

where ψ is a character of A.

Now, we construct new families of quasi-random graphs for any given edge density as follows.

Theorem 2.1. *Let* $q_n \equiv 1 \pmod{2k_n}$, *and let the Cayley graph* $\Gamma(F_{q_n}, S_i)$ *be defined as above. For any fixed real number* $0 < p < 1$, *there exists a rational sequence* (r_n) *such that*

$$G_n = \Gamma(F_{q_n}, T_n),$$

where $T_n = \bigcup_{i=0}^{r_n-1} S_i$, *is* p*-quasi-random as* $n \to \infty$.

In order to prove Theorem 2.1, we also need the following Weil bound. The characters of a finite field F_q are group homomorphisms from F_q or $F_q^* = F_q \setminus \{0\}$ to

$$S^1 = \{z : |z| = 1\} = \{e^{i\theta} : 0 \le \theta < 2\pi\}$$

respectively, where S^1 is a multiplicative group of complex numbers.

An *additive character* of F_q is a function $\psi : F_q \to S^1$ such that for any $x, y \in F_q$,

$$\psi(x+y) = \psi(x)\psi(y).$$

Clearly $\psi(0) = 1$ and $\psi(-x) = \overline{\psi(x)}$. The trivial function ψ_0 with $\psi_0(x) \equiv 1$ is also called the *principle additive character* of $F(q)$.

If an additive character ψ on F_q is nontrivial ($\psi \neq \psi_0$) and $S = \{x^k : x \in F_q\}$ for $k \ge 1$, then

$$\left| \sum_{y \in S} \psi(y) \right| \le (k-1)\sqrt{q},$$

which is the Weil bound [35]. In particular, if S is a multiplicative subgroup of F_q, then we have a slightly better bound for this special case which is due to Alon and Bourgain [2, Lemma 2.7] as follows,

$$\left| \sum_{y \in S} \psi(y) \right| \le \sqrt{q}. \tag{2}$$

Proof of Theorem 2.1. Let (p_n) be a sequence of positive rational numbers such that $p_n \to p$ as $n \to \infty$. Suppose $p_n = r_n/k_n$, where r_n and k_n are positive integers with $k_n \ge 2$ and $(r_n, k_n) = 1$. Note that there are infinitely many primes $q \equiv 1 \pmod{2k_n}$, so we can take a large prime $q_n \equiv 1 \pmod{2k_n}$ such that $q_1 < q_2 < \cdots$ and

$$r_n k_n \sqrt{q_n} = o(q_n), \quad (n \to \infty).$$

Let β_n be a primitive element of F_{q_n}, $S_i = \{\beta_n^{i-1}, \beta_n^{i-1+k_n}, \cdots\}$ and $T_n = \cup_{i=0}^{r_n-1} S_i$. Let

$$G_n = \Gamma(F_{q_n}, T_n).$$

The vertex set of G_n is F_{q_n}, whose edge set consists of all edges of $H_i(q_n, k_n)$ for $0 \le i \le r_n - 1$.

By Lemma 2.2, each eigenvalue λ of $H_1(q_n, k_n)$ has the form

$$\lambda = \sum_{s \in S_1} \psi(s),$$

where ψ is an additive character of F_{q_n}. The trivial character ψ_0 determines the largest eigenvalue $\lambda_1 = |S_1| = (q_n - 1)/k$. For each other eigenvalue λ of $H_1(q_n, k_n)$, we have an additive character $\psi \ne \psi_0$ such that

$$|\lambda| = \left| \sum_{s \in S_1} \psi(s) \right| \le \sqrt{q_n} = o(\lambda_1),$$

in which the inequality follows from (2) since $S_1 = \{1, \beta^k, \ldots, \beta^{q-1}\}$ is a multiplicative group. Thus $H_1(q_n, k_n)$ and hence any $H_i(q_n, k_n)$ is $1/k_n$-quasi-random by noting that $H_i(q_n, k_n)$ is $(q_n - 1)/k_n$ regular, which implies that the density of G_n is $p_n = r_n/k_n$ by Lemma 1.2. Therefore, from Lemma 1.1, G_n is p-quasi-random as $n \to \infty$. This completes the proof of Theorem 2.1. $\square$

Remarks. Lemma 2.1 tells us that $H_i(q, k)$ is "half" strongly regular. However, $H_i(q, k)$ is not strongly regular generally. For example, consider the graph $H_1(13, 3)$ with respect to primitive element $\beta = 2$ of F_{13}. Let us list neighborhoods of some vertices as follows.

$$N(0) = \{1, 5, 8, 12\}, \; N(2) = \{1, 3, 7, 10\}, \; N(4) = \{3, 5, 9, 12\}.$$

For non-adjacent pairs $\{0, 2\}$ and $\{0, 4\}$, we have $|N(0) \cap N(2)| \ne |N(0) \cap N(4)|$.

For sparse graphs with density $p = o(1)$, the situation is significantly more complicated as revealed by Chung and Graham [10]. The first remarked fact is that the properties defined for quasi-random graphs with fixed density may not equivalent. They found some equivalent properties under certain conditions. One of the properties is that $\lambda_1 \sim pn$ and $\lambda_2 = o(pn)$. For a given positive function $f(n)$ with $f(n) = o(1)$, it is similar to construct $G_n = \Gamma(F_{q_n}, S)$ of density p_n such that $p_n \sim f(n)$. Naturally, we may ask which properties (in which form) of quasi-randomness for fixed density can be preserved for sparse quasi-random graphs.

3. Lower bounds for Ramsey numbers $r(K_1 + G)$

For a graph G, the Ramsey number $r(G)$ is the minimum integer N such that any red/blue edge coloring of K_N contains a monochromatic copy of G.

Let $q \equiv 1 \pmod 4$ be a prime power, and F_q the finite field of order q. Define a function $\chi : F_q \to \mathbb{Z}$ by $\chi(x) = x^{(q-1)/2}$, called the *quadratic residue character*, namely

$$\chi(x) = \begin{cases} 1 & x \text{ is quadratic, } x \neq 0, \\ 0 & x = 0, \\ -1 & x \text{ is non-quadratic.} \end{cases} \tag{3}$$

The Paley graph P_q is defined on vertex set F_q, in which a pair of vertices $u, v \in F_q$ are adjacent if and only if $\chi(u - v) = 1$. Let us point out that the cyclotomic graphs $H_1(q, 2)$ and $H_2(q, 2)$ are exactly the Paley graph P_q.

A result of Shearer [31] and independently Mathon [22] was that if the Paley graph P_q contains no K_t, then

$$r(K_{t+1}) \geq 2(q + 1) + 1. \tag{4}$$

This can give the currently best lower bounds of $r(K_t)$ for small t, except for $t = 4, 5, 6, 8$, see [18, 15, 21, 6]. It is clear that $r(K_2) = 2$ and $r(K_3) = 6$. In [18], Greenwood and Gleason proved $r(K_4) = 18$, and there is no other known exact values $r(K_t)$ for $t \geq 5$. For $r(K_5)$, we know that $43 \leq r(K_5) \leq 48$ in Exoo [15] and Angeltveit and McKay [3] for the lower and upper bounds respectively.

For vertex disjoint graphs G and H, let $H + G$ be a graph obtained from H and G by adding new edges to connect H and G completely. The family of graphs in the form $K_1 + G$ contains graphs in the form of $K_m + H$ such as the book graph $B_n^{(m)} = K_m + \overline{K_n}$, and we shall give a constructive lower bound for $r(K_1 + G)$ as follows.

Theorem 3.1. *Let $q \equiv 1 \pmod 4$ be a prime power. If the Paley graph P_q contains no G with minimum degree $\delta(G) \geq 1$, then $r(K_1 + G) \geq 2(q + 1) + 1$.*

Remark. Since $K_{t+1} = K_1 + K_t$, Theorem 3.1 generalizes the lower bound (4).

Let $B_n^{(m)}$ be the book graph that consists of n copies of K_{m+1} sharing a common K_m. The study of Ramsey numbers of books goes back to [14, 29]. Erdős, Faudree, Rousseau and Schelp [14] using the random graph $G(N, 1/2)$, and later Thomason [32] using the Paley graph obtained that $r(B_n^{(k)}, B_n^{(k)}) \geq (2^k + o_k(1))n$. Recently, Conlon [12] establishes that $r(B_n^{(k)}, B_n^{(k)}) = (2^k +$

$o_k(1))n$, which confirms a conjecture of Thomason [32] asymptotically and also gives an answer to a problem proposed by Erdős [14]. Using a different method, the upper bound has been improved slightly to that $r(B_n^{(k)}, B_n^{(k)}) \leq 2^k n + O_k(\frac{n}{(\log\log\log n)^{1/25}})$, see Conlon, Fox and Wigderson [13]. For more Ramsey numbers on books, we refer the reader to [7, 16, 26, 27, 28] and other related references.

When $m = 2$, it is shown that if $4n+1$ is a prime power, then $r(B_n^{(2)}) = 4n+2$; see Rousseau and Sheehan [29]. As a corollary, the following result improves the lower bound in [14, 32, 24, 25] when $m = 3$.

Corollary 3.1. *If $4n+1$ is a prime power, then $r(B_n^{(3)}) \geq 8n+5$.*

Proof. As $B_n^{(3)} = K_1 + B_n^{(2)}$, the assertion follows by considering the largest $B_n^{(2)}$ in P_q when $q = 4n+1$. □

Remark. From Corollary 3.1 and the upper bound due to Conlon, Fox and Wigderson [13], we know that if $4n+1$ is a prime power, then $8n+5 \leq r(B_n^{(3)}) \leq 8n + O(\frac{n}{(\log\log\log n)^{1/25}})$. It is natural to ask that whether $r(B_n^{(3)}) \leq 8n + c$ and $2^m n + c_m \leq r(B_n^{(m)}) \leq 2^m n + d_m$ for infinitely many n, where c, c_m and d_m are constants. For the upper bound, Thomason [32] indeed conjectured that $r(B_n^{(m)}) \leq 2^m(m+n-2)+2$ for all $m \geq 1$. This conjecture is trivially true when $m = 1$ and its truth for $m = 2$ due to Rousseau and Sheehan [29] follows from Goodman's theorem [20]. If Thomason's conjecture holds for $m = 3$, then we can get $8n+5 \leq r(B_n^{(3)}) \leq 8n+10$ if $4n+1$ is a prime power.

In the following, we shall give a proof for Theorem 3.1. We will apply the construction due to Shearer [31] and Mathon [22] independently. Write (u, v) for an edge that connects vertices u and v. Let P_q and P'_q be two disjoint copies of Paley graphs. Let V, V' and E, E' be their corresponding vertex and edge sets, respectively, and let λ, λ' be two additional vertices. We define a new graph H_q with vertex set $\{\lambda, \lambda'\} \cup V \cup V'$ and containing the edges

$$\begin{array}{ll} (\lambda, x),\ (\lambda', x') & x \in V; \\ (x, y),\ (x', y') & (x, y) \in E; \\ (x, y'),\ (x', y) & (x, y) \in E'. \end{array}$$

Proof of Theorem 3.1. Let H_q be constructed as above with vertex set $\{\lambda, \lambda'\} \cup V \cup V'$. We aim to show that both H_q and $\overline{H_q}$ contain no $K_1 + G$ as a subgraph. The fact that H_q contains no copy of $K_1 + G$ as a subgraph follows from the following claim since the Paley graph P_q contains no G from the assumption.

Claim. The neighborhood of any vertex of H_q induces a subgraph that is isomorphic to the Paley graph P_q.

Proof. The assertion holds clearly if the vertex u is either λ or λ'. Recall the definition of the Paley graph P_q, $V = F_q$, where $q = 4n + 1$ is a prime power. Let β be a primitive element of F_q. Since for any $a, b \in V$, the map $\psi(x) = a + b - x$ is an automorphism mapping a to b, we have that the Paley graph is vertex transitive. Therefore, it suffices to verify the neighborhood of the vertex $0 \in V$ in H_q by symmetry. From the definition of H_q, the neighborhood of 0 is

$$U = \left\{\lambda, 1, \beta^2, \ldots, \beta^{4n-2}; \beta', \beta^{3'}, \ldots, \beta^{4n-1'}\right\}.$$

Denote $H_q[U]$ by the subgraph induced by the vertices of U in H_q. Define an bijection φ from V to U such that

$$\varphi(0) = \lambda, \ \ \varphi(\beta^{2i}) = \frac{1}{\beta^{2i}} \ \text{ and } \ \varphi(\beta^{2i+1}) = \left(\frac{1}{\beta^{2i+1}}\right)', \ \ i = 0, 1, \ldots, 2n-1.$$

Table 1: Four types of edges

P_q	$(0, \beta^{2i})$	(β^{2i}, β^{2j})	$(\beta^{2i+1}, \beta^{2j+1})$	$(\beta^{2i}, \beta^{2j+1})$
$H[U]$	$(\lambda, \frac{1}{\beta^{2i}})$	$(\frac{1}{\beta^{2i}}, \frac{1}{\beta^{2j}})$	$((\frac{1}{\beta^{2i+1}})', (\frac{1}{\beta^{2j+1}})')$	$(\frac{1}{\beta^{2i}}, (\frac{1}{\beta^{2j+1}})')$

Clearly, φ is an isomorphism from the Paley graph P_q to $H_q[U]$ from the definition of H_q. e.g., $\beta^{2i+1} - \beta^{2j+1}$ is quadratic if and only if $\frac{1}{\beta^{2i+1}} - \frac{1}{\beta^{2j+1}} = \frac{\beta^{2j+1} - \beta^{2i+1}}{\beta^{2i+1}\beta^{2j+1}}$ is quadratic by noting $-1 = \beta^{2n}$ is quadratic as $q = 4n + 1$. i.e., $(\beta^{2i+1}, \beta^{2j+1})$ is an edge in P_q if and only if $(\frac{1}{\beta^{2i+1}}, \frac{1}{\beta^{2j+1}})$ is an edge in P_q hence an edge in H_q. This completes the proof of the claim. $\square$

It remains to verify that $\overline{H_q}$ contains no copy of $K_1 + G$. Suppose to the contrary that $\overline{H_q}$ contains a copy of $K_1 + G$. Let u be the K_1 of the $K_1 + G$, i.e., the center of $K_1 + G$. We claim $u \neq \lambda$. Otherwise, G is contained in $V' \cup \{\lambda'\}$ completely. Note that λ' has no neighbor in V', G must be contained in V' completely as $\delta(G) \geq 1$. However, this will lead to a contradiction since V' induces the Paley graph P_q containing no copy of G in $\overline{H_q}$. Similarly, $u \neq \lambda'$.

Thus, we assume $u \in V$, say $u = 0$ without loss of generality. From the definition of H_q, the neighborhood of 0 in $\overline{H_q}$ is

$$\left\{\beta, \beta^3, \ldots, \beta^{4n-1}; \lambda', 1', \beta^{2'}, \ldots, \beta^{4n-2'}\right\} \cup \{0'\}.$$

Let

$$W = \left\{\beta, \beta^3, \ldots, \beta^{4n-1}; \lambda', 1', \beta^{2'}, \ldots, \beta^{4n-2'}\right\},$$

and denote $\overline{H_q}[W]$ by the subgraph induced by the vertices of W in $\overline{H_q}$. Define an bijection φ from V to W such that

$$\varphi(0) = \lambda', \ \ \varphi(\beta^{2i}) = \left(\frac{1}{\beta^{2i}}\right)' \text{ and } \varphi(\beta^{2i+1}) = \frac{1}{\beta^{2i+1}}, \ \ i = 0, 1, \ldots, 2n-1.$$

Similarly, φ is an isomorphism from the Paley graph $\overline{P_q}$ $(= P_q)$ to $\overline{H_q}[W]$ from the definition of H_q. e.g., $\beta^{2i} - \beta^{2j+1}$ is non-quadratic if and only if $\frac{1}{\beta^{2i}} - \frac{1}{\beta^{2j+1}} = \frac{\beta^{2j+1}-\beta^{2i}}{\beta^{2i}\beta^{2j+1}}$ is quadratic. i.e., $(\beta^{2i}, \beta^{2j+1})$ is an edge in $\overline{P_q}$ if and only if $(\frac{1}{\beta^{2i}}, \frac{1}{\beta^{2j+1}})$ is an edge in H_q, equivalently, $((\frac{1}{\beta^{2i}})', \frac{1}{\beta^{2j+1}})$ is an edge in $\overline{H_q}[W]$.

Now, note that, in $\overline{H_q}$, the neighborhood of the vertex $0'$ is

$$\left\{\lambda, 0, 1, \beta^2, \ldots, \beta^{4n-2}; \beta', \beta^{3'}, \ldots, \beta^{4n-1'}\right\},$$

which is disjoint from W. It follows that G must be contained in W completely as $\delta(G) \geq 1$. However, this is a contradiction since $\overline{H_q}[W]$ is isomorphic to the Paley graph P_q which contains no copy of G. The proof of Theorem 3.1 is completed. $\square$

Remark. For a graph G, the k-color Ramsey number $r_k(G)$ is defined to be the least integer N such that any edge-coloring of K_N with k colors contains a monochromatic copy of G. Applying the cyclotomic graphs defined as in Section 2, we can similarly obtain that for fixed integer $k \geq 3$ and prime power $q \equiv 1 \pmod{2k}$, if G is a graph without isolated vertex and the graph $H_1(q,k)$ contains no G, then $r_k(K_1 + G) \geq k(q+1) + 1$. In particular, Xu and Radziszowski [36] obtained that for fixed $k \geq 3$ and prime power $q \equiv 1 \pmod{2k}$, if the graph $H_1(q,k)$ contains no K_m, then $r_k(K_{m+1}) \geq k(q + (k-1)!) + 1$, which improves that in [31, 22].

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