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An alternative proof of general factor structure theorem*

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Abstract: Let G be a graph, and $H : V(G) \to 2^{\mathbb{N}}$ a set function associated with G. A spanning subgraph F of graph G is called a *general factor* or an *H-factor* of G if $d_F(x) \in H(x)$ for every vertex $x \in V(G)$. The existence of H-factors is, in general, an NP-complete problem. H-factor problems are considered as one of most general factor problem because many well-studied factors (e.g., perfect matchings, f-factor problems and (g, f)-factor problems) are special cases of H-factors. Lovász [The factorization of graphs (II), Acta Math. Hungar., 23 (1972), 223–246] gave a structure description of H-optimal subgraphs and obtained a deficiency formula. In this paper, we introduce a new type of alternating path to study Lovász's canonical structural partition of graphs and consequently obtain an alternative and shorter proof of Lovász's deficiency formula for H-factors. Moreover, we also obtain new properties regarding Lovász's canonical structural partition of H-factors.

Keywords: degree constrained factor, alternating path, changeable trail.

1. Introduction

In this paper, we consider finite undirected graphs without loops or multiple edges. For a graph $G = (V, E)$, the degree of x in G is denoted by $d_G(x)$, and the set of vertices adjacent to x in G is denoted by $N_G(x)$. For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$ and $G - S = G[V(G) - S]$. For disjoint vertex subsets S and T, $E_G(S, T)$ is the set of edges between S and T in G. Given a trail P and a spanning subgraph R, let $R\Delta P$ denote the spanning subgraph with vertex set $V(R)$ and edge set $(E(R) - E(P)) \cup$

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$(E(P) - E(R))$. A trail of a graph G is called *Eulerian trail* if it contains all edges of G. A graph is *Eulerian* if it has a closed Eulerian trail. For two positive integers a, b with $a < b$, we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$. Let P be a trail. For $x \in V(P)$, let x^i denote the i-th appearance of x on the trail and let x^e denote the last appearance of x. For $u, v \in V(P)$, let $P^{u^i v^j}$ denote the subtrail of P from u^i to v^j. We use P^{uv} for $P^{u^i v^j}$ when there is no confusion arisen. Notations and terminologies are not defined here may be found in [5].

For a given graph G, we associate an integer set $H(x)$ with each vertex $x \in V(G)$ (i.e., H is a set mapping from $V(G)$ to $2^{\mathbb{N}}$). Given a spanning subgraph F of G, F is a *general factor* or an H-*factor* of G if $d_F(x) \in H(x)$ for every vertex $x \in V(G)$. By specifying $H(x)$ to be an interval or a special set, an H-factor would become an f-factor, an $[a, b]$-factor or a (g, f)-factor, respectively. For a general mapping H, the decision problem of determining whether a graph has an H-factor is known to be NP-complete. In fact, when $H(x)$ contains a "gap" with more than one element, H-factor problem is an NP-complete problem. Interestingly, Lovász [4] showed that Four-Colors Problem is reducible to H-factor problem with $H(x) = \{1\}$ or $\{0, 3\}$ for each $x \in V$. So it is reasonable to conclude that finding a characterization for H-factors in general is a challenging problem and hence it is natural to turn our attention to H-factor problems in which $H(x)$ contains only one-element gaps. Furthermore, Lovász also conjectured that the general factor problem with one-element gaps could be solved in polynomial time and Cornuéjols [1] confirmed this conjecture. For $H(x)$ being intervals, H-factor becomes (g, f)-factor. By using alternative trails, the authors [6] gave a structure characterization of Lovasz's (g, f)-Factor Theorem [2], which also implied a simple proof of (g, f)-Factor Theorem.

Assume that H satisfies the property:

$$(*) \quad \text{if } i \notin H(x), \text{ then } i+1 \in H(x), \quad \text{for } mH(x) \leq i \leq MH(x),$$

where $mH(x) = \min\{r \mid r \in H(x)\}$ and $MH(x) = \max\{r \mid r \in H(x)\}$. Let $MH(S) = \sum_{u \in S} MH(u)$, $mH(S) = \sum_{v \in S} mH(v)$. Given an integer set X and an integer a, let $X \pm a = \{i \pm a \mid i \in X\}$. Let $f : V(G) \to \mathbb{N}$ and let $H \pm f : V(G) \to 2^{\mathbb{N}}$ be a set function such that $(H \pm f)(x) = H(x) \pm f(x)$ for all $x \in V(G)$. Lovász [3] obtained a sufficient and necessary condition for the existence of H-factors with the property $(*)$ and a deficiency formula for H-optimal subgraphs. In this paper, we use the traditional matching theory technique − alternative path − which has dealt effectively with other factor problems to prove Lovász's deficiency formula. However, we need to modify the usual alternative paths to *changeable trails* to handle the more complicated structures in this case.

Let F be a spanning subgraph of G such that $d_F(v) \leq MH(v)$ for all $v \in V(G)$. A trail $P = v_0v_1 \dots v_k$ with $d_F(x) \in H(x)$ for every $x \in V(P) - \{v_0, v_k\}$ is called a F_H*-changeable trail* (or *changeable trail* for simplicity when there is no confusion arisen) if it satisfies the following conditions:

(a) $v_0v_1 \notin E(F)$ and $d_F(v_0) < mH(v_0)$;
(b) $d_{F\triangle P}(x) \in H(x)$, for every $x \in V(P) - \{v_0, v_k\}$;
(c) for all $1 \leq l \leq k$, sub-trail $P' = v_0v_1 \dots v_l$ satisfies condition (b) as well.

Remarks:

(1) In fact, we could restrict the vertex degree of trail P to $d_P(x) \leq 4$, because that the number of times of a vertex is repeated in a trail two times is enough for us to construct augmenting subgraph to improve the deficiency.
(2) Condition (c) is necessary, because a repeated vertex in a sub-trail of P may not satisfy the condition (b). For instance, let $P = v_0v_1v_2v_3v_1v_4$ and $P' = v_0v_1v_2v_3$, where $|H(v_1)| = 1$, $v_0v_1, v_1v_2 \notin E(F)$ and $v_1v_3, v_1v_4 \in E(F)$. One may see that if $d_{F\triangle P}(v_1) \in H(v_1)$, then $d_{F\triangle P'}(v_1) \notin H(v_1)$. Hence the condition (c) is independent of the condition (b).
(3) For any $x \in V(P) - \{v_0, v_k\}$, $d_{F\triangle P}(x) \equiv d_F(x) \pmod 2$ and if $v_0 = v_k$, then $d_{F\triangle P}(v_0) \equiv d_F(v_k)$.

Let H be a set function satisfying the property (*) and F any *spanning* subgraph of G. Given a subset $S \subseteq V(G)$, the *deficiency of subgraph* $G[S]$ in F is defined as

$$def_H[F; S] = \sum_{x \in S} \min\{|d_F(x) - r| \mid r \in H(x)\}.$$

In particular, $def_H[F; x] = \min\{|d_F(x) - r| \mid r \in H(x)\}$ is the deficiency of vertex x in F. We can measure F's "deviation" from H-factors by defining the *deficiency* of F with respect to H as

$$def_H[F] = \sum_{x \in V(F)} \min\{|d_F(x) - r| \mid r \in H(x)\}.$$

The *total deficiency* of G with respect to H is

$$def_H(G) = \min\{def_H[F] \mid F \text{ is a spanning subgraph of } G\}.$$

Note that $def_H[G] \neq def_H(G)$. Clearly, $def_H(G) = 0$ if and only if there exists an H-factor. A subgraph F is called *H-optimal*, if F is a spanning subgraph of G and $def_H[F] = def_H(G)$. Of course, any H-factor is H-optimal.

Let $I_H(x) = \{d_F(x) \mid F \text{ is any } H\text{-optimal subgraph}\}$. Lovász [3] studied the structure of H-factors by introducing a Gallai-Edmonds type canonical partition of $V(G)$ as follows:

$$\begin{aligned} C_H(G) &= \{x \mid I_H(x) \subseteq H(x)\}, \\ A_H(G) &= \{x \mid \min I_H(x) \geq MH(x)\}, \\ B_H(G) &= \{x \mid \max I_H(x) \leq mH(x)\}, \\ D_H(G) &= V(G) - A_H(G) - B_H(G) - C_H(G). \end{aligned}$$

Based on this canonical partition, Lovász obtained the deficiency formula for H-optimal subgraphs, which implied a sufficient and necessary condition of H-factors, where H satisfies the property (*).

Theorem 1.1 (Lovász, [5]). *The deficiency formula is*

(1)
$$def_H(G) = mH(B_H(G)) + c(D_H(G)) - MH(A_H(G)) - \sum_{x \in B_H(G)} d_{G-A_H(G)}(x),$$

where $c(D_H(G))$ denotes the number of components of $G[D_H(G)]$.

In this paper, we firstly give an alternative description of the canonical partition (A, B, C, D) by deploying changeable trails and then provide a new proof of Theorem 1.1.

A changeable trail P is *odd* if the last edge doesn't belong to F; otherwise, P is *even*. We call a changeable trail P an *augmenting changeable trail* if $def_H[F \triangle P] < def_H[F]$. Moreover, the trails of length zero are considered as even changeable trails.

Suppose that G does not have H-factors. Let F be an H-optimal spanning subgraph of G such that $E(F)$ is minimal. Clearly, there is a vertex $v \in V$ such that $d_F(v) \notin H(v)$, so the deficiency of F is positive. Let

$$B_0 = \{x \in V(G) \mid d_F(x) \notin H(x)\}.$$

Since $E(F)$ is minimal and H satisfies (*), we have $d_F(v) \leq MH(v)$ for all $v \in V(G)$ and

$$B_0 = \{x \in V(G) \mid d_F(x) < mH(x)\}.$$

We define $D(G)$ to be a vertex set consisting of three types of vertices as follows:

(i) $\{v \mid \exists$ both of an even changeable trail and an odd changeable trail from B_0 to $v\}$;
(ii) $\{v \mid mH(v) < d_F(v) \leq MH(v)$ and $\exists$ an even changeable trail from B_0 to $v\}$;
(iii) $\{v \mid mH(v) \leq d_F(v) < MH(v)$ and $\exists$ an odd changeable trail from B_0 to $v\}$.

The sets $A(G)$ and $B(G)$ are defined as follows:

$$A(G) = \{v \mid d_F(v) = MH(v) \text{ and } \exists \text{ an odd but not even changeable trail from } B_0 \text{ to } v\}$$

$$B(G) = \{v \mid d_F(v) \leq mH(v) \text{ and } \exists \text{ an even but not odd changeable trail from } B_0 \text{ to } v\},$$

and $C(G) = V(G) - A(G) - B(G) - D(G)$. We abbreviate $D(G), A(G), B(G)$ and $C(G)$ by D, A, B and C, respectively.

We claim that A, B, C, D is indeed a partition of $V(G)$. From the definitions of A and B, clearly $A \cap B = \emptyset$. Let $x \in A$. Since $d_F(x) = MH(x)$, it does not satisfy condition (iii) in the definition of D. Note that there does not exist an even changeable trail from B_0 to x. So x can not satisfy (i) or (ii) of D. Hence we infer that $A \cap D = \emptyset$. With similar discussion, one can see that $B \cap D = \emptyset$. Hence they form a partition of $V(G)$.

Since every trail of length zero is considered as an even trail, by the definitions of B and D, $B_0 \subseteq B \cup D$. So for $v \in C$, we have $d_F(v) \in H(v)$. Following the above discussion, if $H(v)$ is an integer interval with more than one element, then $v \notin D$.

Though the definition of (A, B, C, D) depends on the subgraph F, the partition (A, B, C, D) of graph G is independent of the choice of F. The proof of independence will be given at the very end of this paper (Theorem 3.13).

To illustrate the new concepts introduced in this section, here is an example (Fig. 1). Let $H : V(G) \to 2^{\mathbb{N}}$ be a set function defined as follows: $H(u_i) = \{0, 2\}$ for $i = 1, \ldots, 8$; $H(v_j) = \{1\}$ for $j = 1, \ldots, 10$; and $H(x_k) = H(w_k) = \{1\}$ and $H(y_k) = \{3\}$ for $k = 1, 2$. Let F be the spanning subgraphs with bold edges in Figure 1. It is not hard to verify that $def_H(G) = 2$ and F is an H-optimal subgraph. Clearly, $B_0 = \{y_1, y_2\}$. Then $P_1 = y_1u_5v_5u_4v_4x_2v_8u_8v_9u_9u_8v_6$ is an odd changeable trail and $P_2 = y_1u_5v_4u_4v_5x_1v_1$ is an even changeable trail. By definitions, the partition is $A = \{x_1, x_2\}$, $B = \{y_1, y_2\}$, $C = \{w_1, w_2\}$ and D consists of the remaining vertices.

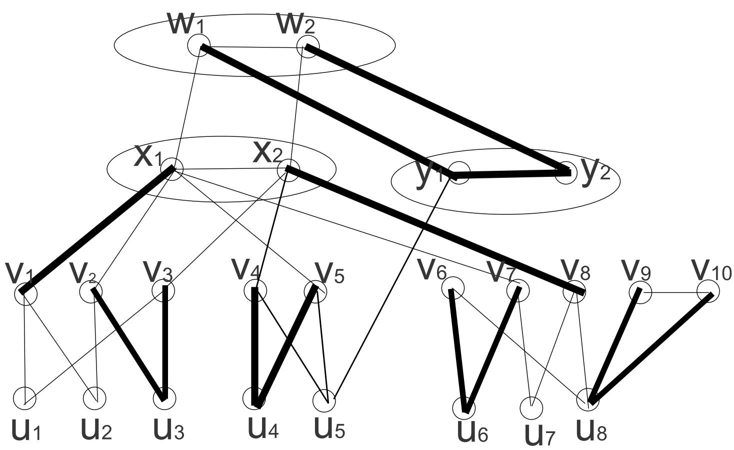

Figure 1: Changeable trails and partition (A, B, C, D).

2. Technical preparations

As technical preparations, we present a lemma and an observation about changable trails.

Lemma 2.1. *Let G be Eulerian. Let R be a subgraph of G. Then G has a closed Eulerian trail C such that for every $v \in V(G)$, the number of pairs of consecutive edges through vertices v and alternatively in $E(R)$ and $E(G) - E(R)$ is equal to* $\min\{d_R(v), d_{G-E(R)}(v)\}$.

Proof. Let $V_1 = \{v \in V(G) \mid d_R(v) < d_G(v)/2\}$ and $V_2 = V(G) - V_1$. Let $U = \cup_{v \in V(G)}\{v_{i1}, v_{i2} \mid 1 \le i \le l_v\}$ and $M = \cup_{v \in V(G)}\{v_{i1}v_{i2}, vv_{i1}, vv_{i2} \mid 1 \le i \le l_v\}$, where $l_v = |d_R(v) - d_{G-E(R)}(v)|/2$. Let $M' = \cup_{v \in V_1}\{vv_{i1}, vv_{i2} \mid 1 \le i \le l_v\}$ and $M'' = \cup_{v \in V_2}\{v_{i1}v_{i2} \mid 1 \le i \le l_v\}$. Let G' be a graph with vertex set $V(G) \cup U$ and edge set $E(G) \cup M$. Let R' be a subgraph with vertex set $V(G) \cup U$ and edge set $E(R) \cup M' \cup M''$. One can see that $d_{R'}(x) = d_{G'}(x)/2$ for all $x \in V(G')$.

Claim 1. Let $\mathcal{G}$ be an Eulerian graph and let $\mathcal{M} \subseteq E(\mathcal{G})$ such that $d_{\mathcal{G}}(x) = 2d_{\mathcal{G}-\mathcal{M}}(x)$ for all $x \in V(\mathcal{G})$. Then $\mathcal{G}$ has an Eulerian trail that is an $\mathcal{M}$-alternative trail.

By induction on $|E(\mathcal{G})|$. Note that $|\mathcal{M}| = |E(\mathcal{G})|/2$ and $|E(\mathcal{G})| \equiv 0 \pmod 2$. The result is true for $|E(\mathcal{G})| \in \{2, 4\}$, because $\mathcal{G}$ is a cycle of length two or four in this case. Suppose that the result holds for any $|E(\mathcal{G})| \le m$, where $m \ge 4$. Let $\mathcal{G}$ be an Eulerian graph with $|E(\mathcal{G})| = m+2$, where $m \ge 4$.

Let $\mathcal{C}_0$ be a maximal $\mathcal{M}$-alternative trail of $\mathcal{G}$. Since $d_{\mathcal{G}}(x) = 2d_{\mathcal{G}-\mathcal{M}}(x)$ for all $x \in V(\mathcal{G})$, then $\mathcal{C}_0$ is a closed trail. Suppose that $E(\mathcal{G}) \setminus E(\mathcal{C}_0) \neq \emptyset$. Let $\mathcal{G}_1, \dots, \mathcal{G}_r$ be non-trivial components of $\mathcal{G} - E(\mathcal{C}_0)$. Note that for every $x \in V(\mathcal{G})$, $d_{\mathcal{G}-E(C_0)}(x) = 2d_{\mathcal{G}-E(C_0)-\mathcal{M}}(x)$. So for every $x \in V(\mathcal{G}_i)$, $d_{\mathcal{G}_i}(x) = 2d_{\mathcal{G}_i-\mathcal{M}}(x)$ and $|E(\mathcal{G}_i)| \equiv 0 \pmod 2$. By induction hypothesis, $\mathcal{G}_i$ has an $\mathcal{M}$-alternative trail $\mathcal{C}_i$ for $1 \leq i \leq r$. Since $\mathcal{G}$ is connected, we may obtain an $\mathcal{M}$-alternative trail of $\mathcal{G}$ from $\mathcal{C}_0, \dots, \mathcal{C}_r$. This completes the proof of Claim 1.

By Claim 1, G' has an R'-alternative Eulerian trail C'. By deleting the edge set M and vertex set U from $\mathcal{M}$, the resulted trail is the desired trail. This completes the proof. □

Eulerian trails in Lemma 2.1 are closed. Clearly, we have a similar result for open Eulerian trails. We state it as corollary below.

Corollary 2.2. *Let G be a connected graph with Eulerian trails and let $x, y \in V(G)$ such that $d_G(x)$ and $d_G(y)$ are odd. Let R be a subgraph of G. Then G has an Eulerian trail P from x to y such that for every $v \in V(G)$, the number of pairs of consecutive edges through vertices v and alternatively in $E(R)$ and $E(G) - E(R)$ is equal to $\min\{d_R(v), d_{G-E(R)}(v)\}$.*

Let G be a graph and let R be a subgraph of G. By Lemma 2.1 and Corollary 2.2, for every changeable trail P with starting vertex v_0 and $x \in V(P)$, we may make the following assumptions.

Observation 2.3. *(i) If $d_P(x) = 2d_{P-E(R)}(x)$, then the two consecutive edges on P incident to x are alternatively in $E(R)$ and $E(G) - E(R)$.*

(ii) If $d_P(x) > 2d_{P-E(R)}(x)$, then the two consecutive edges on P incident to x belong to $E(R)$ or are alternatively in $E(R)$ and $E(G) - E(R)$.

(iii) If $d_P(x) < 2d_{P-E(R)}(x)$, then the two consecutive edges on P incident to x belong to $E(G)-E(R)$ or are alternatively in $E(R)$ and $E(G)-E(R)$.

(iv) For every subtrail P' of P and $x \in V(P')$, if $d_{R\triangle E(P)}(x) > d_R(x)$ (or $d_{R\triangle E(P)}(x) < d_R(x)$), then $d_{R\triangle E(P')}(x) \geq d_R(x) - 1$ (or $d_{R\triangle E(P')}(x) \leq d_R(x) + 1$, resp.), with equality if and only if P' is an even (odd, resp.) changeable trail and $d_{P'}(x) = 1$.

(v) If P is a closed odd trail, then every subtrail back along opposite direction of P starting vertex v_0 is a changeable trail.

3. Main theorem

In the following lemmas, let F be an H-optimal subgraph of graph G with minimal $E(F)$. Without loss generality, we may assume that G contains no

H-factors, i.e., $B_0 \neq \emptyset$. Let τ denote the number of components of graph $G[D]$ and $D_1, \ldots, D_\tau$ be the components of the subgraph induced by D.

From the definition of augmenting changeable trail, it is easy to see the next lemma.

Lemma 3.1. *An H-optimal subgraph F does not contain an augmenting changeable trail.*

For changeable trails, we have the following lemma.

Lemma 3.2. *Let P_1 be a changeable trail from x to y such that $d_{P_1}(y) = 1$ and let P_2 be a changeable trail from v to y. Suppose that $V(P_1) \cap V(P_2) = \{y\}$ and $d_F(y) \in H(y)$. Then $P_1 \cup P_2$ is a changeable trail if one of the following three conditions holds.*

- *(i) One of P_1 and P_2 is an odd changeable trail and other is an even changeable trail.*
- *(ii) Both P_1 and P_2 are odd changeable trails, and $d_{F \triangle E(P_2 \cup P_1)}(y) \in H(y)$.*
- *(iii) Both P_1 and P_2 are even changeable trails, and $d_{F \triangle E(P_2 \cup P_1)}(y) \in H(y)$.*

Proof. We may assume that both P_1 and P_2 are changeable trails satisfying the result in Corollary 2.2. Let $P := P_1 \cup P_2$ be a trail from x to v along P_1 and reverse order of P_2. Now we show that P is a changeable trail from x to v. For any $u \in V(P_2)$, since F is H-optimal and P_2 is a changeable trail, we may assume that one of the following results holds.

(1) If $d_{F \triangle E(P_2)}(u) = d_F(u)$, then the two consecutive edges on P_2 incident to u are alternatively in $E(F)$ and $E(G) - E(F)$;

(2) If $d_{F \triangle E(P_2)}(u) > d_F(u)$ and $u \notin \{v, x\}$, then $d_F(u) \equiv d_{F \triangle E(P_2)}(u) \pmod 2$, and
$[d_F(u), d_{F \triangle E(P_2)}(u)] \cap H(u) = \{d_F(u), d_F(u) + 2, \ldots, d_{F \triangle E(P_2)}(u)\}$;

(3) If $d_{F \triangle E(P_2)}(u) < d_F(u)$ and $u \notin \{v, x\}$, then $d_F(u) \equiv d_{F \triangle E(P_2)}(u) \pmod 2$, and
$[d_{F \triangle E(P_2)}(u), d_F(u)] \cap H(u) = \{d_{F \triangle E(P_2)}(u), d_{F \triangle E(P_2)}(u) + 2, \ldots, d_F(u)\}$.

Let $z \in V(P_2) - y$ and $w \in V(P_2^{yz}) - z - y$. So by (1), (2) and (3), we have $d_{F \triangle E(P_1 \cup P_2^{yz})}(w) \in H(w)$ and $dist(H(z), d_{F \triangle (E(P_1 \cup P_2^{yz})}(z)) = 1$.

For any subtrail P' of P started from x to t such that $y \in V(P') - x - t$, we have $d_{F \triangle E(P')}(y) \equiv d_F(y) \pmod 2$ and

$$\min\{d_{F \triangle E(P_2)}(y), d_F(y)\} - 1 \leq d_{F \triangle E(P')}(y) \leq \max\{d_{F \triangle E(P_2)}(y), d_F(y)\} + 1.$$

Note that $d_{F \triangle E(P_2)}(y) \neq d_F(y) \pmod 2$

Consider (i). Then we have

$$\min\{d_{F\triangle E(P_2)}(y), d_F(y)\} \leq d_{F\triangle E(P')}(y) \leq \max\{d_{F\triangle E(P_2)}(y), d_F(y)\}.$$

By parity, one can see that

$$\min\{d_{F\triangle E(P_2)}(y)+1, d_F(y)\} \leq d_{F\triangle E(P')}(y) \leq \max\{d_{F\triangle E(P_2)}(y)-1, d_F(y)\}.$$

Hence if $d_{F\triangle E(P_2)}(y) + 1 \leq d_F(y)$, then $d_{F\triangle E(P')}(y) \in \{d_{F\triangle E(P_2)}(y)+1, d_{F\triangle E(P_2)}(y)+3, \ldots, d_F(y)\}$. Else if $d_{F\triangle E(P_2)}(y) - 1 > d_F(y)$, then $d_{F\triangle E(P')}(y) \in \{d_F(y), d_F(y)+2, \ldots, d_{F\triangle E(P_2)}(y)-1\}$. In both of cases, we have $d_{F\triangle E(P')}(y) \in H(y)$.

By symmetry, it is sufficient for us to consider (ii). If $d_{F\triangle E(P)}(y) \leq d_F(y)$, then $d_{F\triangle E(P_2)}(y) < d_F(y)$. Since P_2 is an odd changeable trail, $d_{F\triangle E(P')}(y) \in \{d_{F\triangle E(P_2)}+1, \ldots, d_F(y)-2, d_F(y)\} \subseteq H(y)$. So we may assume that $d_{F\triangle E(P)}(y) > d_F(y)$, then $d_{F\triangle E(P')}(y) \in \{d_F(y), d_F(y)+2, \ldots, d_{F\triangle E(P)}(y)\} \subseteq H(y)$. This completes the proof. □

With similar discussion, we have the following result.

Lemma 3.3. *Let P_1 be a changeable trail from x to y and let P_2 be a changeable trail from v to u such that $d_{P_1}(y) = 1$, $V(P_1) \cap V(P_2) = \{y\}$ and $y \in V(P_2) - v$. Then $P_1 \cup P_2^{y^e u}$ is a changeable trail if one of the following three conditions hold.*

(i) Both P_1 and $P_2^{vy^e}$ are odd (or even) changeable trails.

(ii) P_1 is an odd changeable trail, $P_2^{vy^e}$ is an even changeable trail and $d_F(w)+2 \in H(w)$.

(iii) P_1 is an even changeable trail, $P_2^{vy^e}$ is an odd changeable trail and $d_F(w)-2 \in H(w)$.

Next we show that each D_j can only possess a small deficiency.

Lemma 3.4. *$def_H[F; V(D_j)] \leq 1$ for $j = 1, \ldots, \tau$.*

Proof. Suppose, to the contrary, that $def_H[F; V(D_i)] > 1$. Let $v_0 \in V(D_i)$ with $def_H[F; v_0] \geq 1$. Then $v_0 \in B_0$ and thus $d_F(v_0) < mH(v_0)$. So v_0 is of type (i) and there exists an odd changeable trail P from a vertex s of B_0 to v_0. We claim $s = v_0$. Otherwise, $def_H[F] > def_H[F\Delta P]$, a contradiction since F is H-optimal. Furthermore, if $def_H[F; v_0] \geq 2$, then $s = v_0$ and so $def_H[F] > def_H[F\Delta P]$, a contradiction again. So we have $def_H[F; v_0] = 1$ and $def_H[F; u] \leq 1$ for any $u \in V(D_i) - v_0$. Moreover, $d_F(v_0)+1 \in H(v_0)$ and $d_F(v_0)+2 \notin H(v_0)$.

We define D_i^1 to be a vertex set consisting of three types of vertices as follows:

(1) $\{w \in V(D_i) \mid \exists$ an even changeable trail R_1 and an odd changeable trail R_2 from v_0 to w such that $V(R_1) \cup V(R_2) \subseteq V(D_i)\}$;
(2) $\{w \in V(D_i) \mid mH(w) < d_F(w) \leq MH(w)$ and $\exists$ an even changeable trail R from v_0 to w such that $V(R) \subseteq V(D_i)\}$;
(3) $\{w \in V(D_i) \mid mH(w) \leq d_F(w) < MH(w)$ and $\exists$ an odd changeable trail$R$ from v_0 to w such that $V(R) \subseteq V(D_i)\}$.

Let P be a closed odd changeable trail with started vertex v_0 and let $S = \{v \in V(P) \mid d_P(v) = 2d_{P-E(F)}(v)\}$. By Observation 2.3, every subtrail along P with one end v_0 (clockwise or anticlockwise) is a changeable trail. Hence for $v \in S$, v belongs to type (1). Consider $u \in V(P) - S$. If $d_P(u)/2 < d_{P-E(F)}(u)$, then $d_F(u) \leq d_{F\Delta E(P)}(u) - 2$, $d_F(u) < MH(u)$ and there exists an odd changeable trail from v_0 to u. So u is type (3). Else if $d_P(u)/2 > d_{P-E(F)}(u)$, with similar discussion, u is type (2). Hence we have $V(P) \subseteq D_i^1$.

One can see that $D_i^1 \neq \emptyset$ and D_i^1 is well-defined.

Claim 1. $D_i^1 = V(D_i)$.

Suppose that the claim does not hold. Since D_i is connected, there exists an edge $xy \in E(G)$ such that $x \in V(D_i) - D_i^1$ and $y \in D_i^1$.

We assume $xy \in E(F)$ (resp. $xy \notin E(F)$) and show that there exists an *even* (resp. odd) changeable trail P_1 from v_0 to x such that $yx \in E(P_1)$ and $V(P_1) - x \subseteq D_i^1$. If y is type (1) or (3), then there exists an odd changeable trail R_1 from v_0 to y such that $V(R_1) \subseteq D_i^1$ and $R_1 \cup xy$ is an even changeable trail from v_0 to x. Else, i.e., y is type (2), then there exists an even changeable trail R_2 from v_0 to y such that $V(R_2) \subseteq D_i^1$ and we have $mH(y) < d_F(y) \leq MH(y)$. Since F is H-optimal and H satisfies property (*), so $d_F(y) - 1 \notin H(y)$ and $d_F(y), d_F(y) - 2 \in H(y)$. Hence $R_2 \cup xy$ is the required trail P_1.

Since $x \notin D_i^1$, x can not be of type (ii). So there exists an odd (resp. even) changeable trail P_2 from a vertex t of B_0 to x. We claim $t = v_0$. Otherwise, suppose that $t \neq v_0$. If $V(P_1) \cap V(P_2) = \{x\}$, then $def_H[F] > def_H[F \triangle (P_1 \cup P_2)]$, a contradiction since F is H-optimal. Let z be the first vertex along P_1 appearing in P_2. Note that $V(P_1^{v_0z}) \cap V(P_2^{tz}) = \{z\}$. If $P_1^{v_0z}$ is an odd (or even) changeable trail and P_2^{tz} is an even (odd, resp.) changeable trail, then $P_1^{v_0z} \cup P_2^{tz}$ is an augmenting changeable trail, a contradiction to that F is H-optimal. So we may assume that both $P_1^{v_0z}$ and P_2^{tz} are odd (or even). By Lemma 3.3, $P_1^{v_0z} \cup P_2^{zx}$ is an odd changeable trail from v_0 to x. Thus we have $t = v_0$.

Let $w \in P_2$ be the last vertex of P_2 which belongs to D_i^1. Let w' be the successor vertex in P_2. By symmetry, we may assume that $ww' \in E(F)$. If

$w \in D_i^1$ is of type (1) or (3), by the definition of D_i^1, there exists an odd changeable trail P_3 from v_0 to w such that $V(P_3) \subseteq D_i^1$. One can see that $P_3 \cup P_2^{wx}$ is an odd changeable trail. If w is type (2), then there exists an even changeable trail P_4 from v_0 to w such that $V(P_4) \subseteq D_i^1$. Moreover, since F is H-optimal and H satisfies property (*), we have $d_F(w), d_F(w)-2 \in H(w)$ and $d_F(w)-1 \notin H(w)$. Thus one can see that $P_4 \cup P_2^{wx}$ is an odd changeable trail. Hence by the definition of D_i^1, we have $V(P_2^{wx}) \subseteq D_i^1$, a contradiction. This completes the proof of Claim.

Let $u \in V(D_i) - v_0$ and $def_H[F;u] = 1$. Note that $u \in B_0$ and so $d_F(u) < mH(u)$. One can see that u is not type (2). Thus there exists an odd changeable trail R from v_0 to u. Hence we have $def_H[F] > def_H[F \triangle R]$, a contradiction again. So we show that $def_H[F; V(D_i)] \leq 1$. □

Using the above lemma, we have the following result.

Lemma 3.5. *For $i = 1, \ldots, \tau$, if $def_H[F; V(D_i)] = 1$, then*

(a) $E_G(D_i, B) \subseteq E(F)$;
(b) $E_G(D_i, A) \cap E(F) = \emptyset$.

Proof. Let $def_H[F; x] = 1$, where $x \in V(D_i)$. Suppose the lemma does not hold.

To show (a), let $uv \notin E(F)$, where $u \in V(D_i)$ and $v \in V(B)$. If u is of type (i) or type (ii), from the proof of Lemma 3.4, then there exists an even changeable trail P of D_i from x to u. Hence $P \cup uv$ is an odd changeable trail from x to v, a contradiction to $v \in B$. Else, u is of type (iii), then there exists an odd changeable trail $P \subseteq D_i$ from x to u. Since F is H-optimal and H satisfies the property (*), we have $d_F(u) \in H(u)$, $d_F(u)+1 \notin H(u)$ and $d_F(u)+2 \in H(u)$. Hence $P \cup uv$ is an odd changeable trail from x to v, a contradiction to $v \in B$ again.

Next we consider (b). Let $uv \in F$, where $u \in V(D_i)$ and $v \in A$. If u is of type (i) or type (iii), from the proof of Lemma 3.4, then there exists an odd changeable trail P of D_i from x to u. Then $P \cup uv$ is an even changeable trail from x to v, a contradiction to $v \in A$. Else, u is of type (ii), then there exists an even changeable trail P of D_i from x to u. Since F is H-optimal and H satisfies the property (*), we have $d_F(u) \in H(u)$, $d_F(u)-1 \notin H(u)$ and $d_F(u)-2 \in H(u)$. Hence $P \cup uv$ is an even changeable trail from x to v, a contradiction to $v \in A$ again. □

From the definition of partition (A, B, C, D), it is not hard to see the next lemma.

Lemma 3.6. *$E_G(B, C \cup B) \subseteq E(F)$, $E_G(A, A \cup C) \cap E(F) = \emptyset$ and $E_G(D, C) = \emptyset$.*

Lemma 3.7. (*a*) *F misses at most one edge of $E_G(V(D_i), B)$. Moreover, if F misses one edge of $E_G(V(D_i), B)$, then $E_G(V(D_i), A) \cap E(F) = \emptyset$;*
(*b*) *F contains at most one edge of $E_G(V(D_i), A)$. Moreover, if F contains one edge of $E_G(V(D_i), A)$, then $E_G(V(D_i), B) \subseteq E(F)$.*

Proof. By Lemmas 3.4 and 3.5, we may assume $def_H[F; V(D_i)] = 0$. Let $u \in V(D_i)$, by the definition of D, there exists a changeable trail P from a vertex x of B_0 to u. Denote the first vertex in P belonging to D_i by y, and the sub-trail of P from x to y by P_1. Let $y_1y \in E(P_1)$, where $y_1 \notin V(D_i)$. Without loss of generality, assume that P_1 is an odd changeable trail (when P_1 is an even changeable trail, the argument is similar). Then we have $y_1y \notin E(F)$. We claim that $P_1 - y$ is even changeable trail. Otherwise, by the definition of changeable trail, we have $d_F(y_1) + 1 \notin H(y_1)$ and $d_F(y_1) + 2 \in H(y_1)$, which imply that $y_1 \in D$, a contradiction. Hence we infer that $y_1 \in B$. Because $y \in D$, y is of type (i) or type (iii).

We define the subset $D_i^1 \subseteq V(D_i)$ which consists of the following vertices:

(1) $\{w \in V(D_i) \mid \exists$ an even changeable trail P_1' and an odd changeable trail P_2' along P_1 from x to w such that $V(P_1') \cup V(P_2') - (V(P_1) - y) \subseteq D_i^1\}$;
(2) $\{w \in V(D_i) \mid mH(w) < d_F(w) \leq MH(w)$ and $\exists$ an even changeable trail P_1' along $P_1$from x to w such that $V(P_1') - (V(P_1) - y) \subseteq D_i^1\}$;
(3) $\{w \in V(D_i) \mid mH(w) \leq d_F(w) < MH(w)$ and $\exists$ an odd changeable trail P_1' along $P_1$from x to w such that $V(P_1') - (V(P_1) - y) \subseteq D_i^1\}$.

Firstly, we show that $D_i^1 \neq \emptyset$. If y is type (iii), then $y \in D_i^1$. If y is type (i) or (ii), then there exists an even changeable trail R_1 from a vertex w of B_0 to y. We have $yy_1 \in E(R_1)$; otherwise, $R_1 \cup yy_1$ is an odd changeable trail from w to y_1, contradicting to $y_1 \in B$. Hence, we may assume $w = x$ and P_1 is a subtrail of R_1. Note that $R_1 - (V(P_1) - y)$ is a closed trail. So by Observation 2.3, $V(R_1) - (V(P_1) - y) \subseteq D_i^1$ and $D_i^1 \neq \emptyset$.

Claim 1. $D_i^1 = V(D_i)$.

Suppose that $V(D_i) \neq D_i^1$. Since D_i is connected, we may choose $v_1v_2 \in E(G)$, such that $v_1 \in D_i^1$ and $v_2 \in V(D_i) - D_i^1$.

We may assume $v_1v_2 \in E(F)$ since $v_1v_2 \notin E(F)$ can be discussed similarly. By the definition of D_i^1, there exists an even changeable trail R_2 from x to v_2 such that $V(R_2) - (V(P_1) - y) - v_2 \subseteq D_i^1$ and $v_1v_2 \in E(R_2)$. If v_2 is type (ii), by the definition of D_i^1, then we have $v_2 \in D_i^1$, a contradiction. So v_2 is type (i) or type (iii). By the definition of D, there exists an odd changeable trail R_3 from a vertex w_1 of B_0 to v_2.

Next we show that $yy_1 \in R_3$. Suppose that $yy_1 \notin R_3$. If $V(R_3) \cap D_i^1 \neq \emptyset$, let z be first vertex of R_3 belonging D_i^1; else let $z = v_2$. Without loss of generality, we suppose that the subtrail R_4 from w_1 to z along R_3 is an odd changeable trail and $z \in D_i^1$. If $z \in D_i^1$ is type (1) or (2), then there is an even changeable trail, say R_5, from x to z along P_1 such that $V(R_5) - (V(P_1) - y) \subseteq D_i^1$. Let R_6 is a subtrail from y_1 to z along R_5. Then $R_4 \cup R_6$ is an odd changeable trail from w_1 to y_1, contradicting to $y_1 \in B$. Else if $z \in D_i^1$ is type (3), since F is H-optimal and H satisfies property (*), then $d_F(z) \in H(z)$, $d_F(z)+1 \notin H(z)$, and $d_F(z)+2 \in H(z)$. Moreover, there is an odd changeable trail R_7 along P_1 from x to z. Let R_8 is a subtrail from y_1 to z along R_7. Then $R_4 \cup R_8$ is an odd changeable trail from w_1 to y_1, contradicting to $y_1 \in B$ again. So $y_1 y \in R_3$. Hence we may assume that $x = w_1$.

We write $R_3 = xx_1 \cdots y_1 u_0 u_1 \cdots u_r$, where $y = u_0$ and $v_2 = u_r$. Since R_3 is a trail, some vertices may appear in the sequence more than once. We may choose maximum i such that $u_i \in D_i^1$ and $u_s \notin D_i^1$ for $i+1 \leq s \leq r$. Since $u_0 = y \in D_i^1$ and $v_2 \notin D_i^1$, u_i is well-defined. Denote the sub-trail of R_3 from u_i to v_2 by R_9. Consider $u_i u_{i+1} \in E(F)$ (or $u_i u_{i+1} \notin E(F)$, resp.). We claim that there exists an changeable trail R' from x to u_i along P_1 such that $V(R') - (V(P_1) - y) \subseteq D_i^1$ and $R' \cup R_9$ be a changeable trail. If u_i is type (1) or (3) ((1) or (2), resp.), then there exists an odd (even, resp.) changeable trail R'' such that $V(R'') \subseteq D_i^1 \cup V(P_1)$, which is a desired trail R'. Thus we may assume that u_i is type (2) ((3), resp.). There exists an even (odd, resp.) changeable trail R''' from x to u_i such that $V(R''') \subseteq D_i^1 \cup V(P_1)$ and $mH(u_i) < d_F(u_i) \leq MH(u_i)$ ($mH(u_i) \leq d_F(u_i) < MH(u_i)$, resp.). Since F is H-optimal, we have $d_F(u_i), d_F(u_i) - 2 \in H(u_i)$ and $d_F(u_i) - 1 \notin H(u_i)$ ($d_F(u_i) + 2 \in H(u_i)$ and $d_F(u_i) + 1 \notin H(u_i)$, resp.). It follows that $R'' \cup R_9$ is an odd changeable trail. By observing changeable trails $R'' \cup R_9$ and R_2, one can see that $V(R_9) \subseteq D_i^1$ by the definition of D_i^1, a contradiction.

Claim 2. Let uv be an edge distinct from yy_1, where $u \in V(D_i)$ and $v \in B$. Then $uv \in E(F)$.

Otherwise, suppose that $uv \notin E(F)$. If u is of type (1) or (2), by Claim 1, then there exists an even changeable trail R_{10} from x to u such that $V(R_{10}) - (V(P_1) - y) \subseteq V(D_i)$. Then $R_{10} \cup uv$ is an odd changeable trail from x to v, contradicting to $v \in B$. If u is of type (3), then there exists an odd changeable trail R_{11} from x to u such that $V(R_{11}) - (V(P_1) - y) \subseteq V(D_i)$. Note that $d_F(u) \in H(u)$. Since F is H-optimal, then $d_F(u) + 1 \notin H(u)$ and $d_F(u) + 2 \in H(u)$. Hence $R_{11} \cup uv$ is an odd changeable trail from x to v, contradicting to $v \in B$. So we prove Claim 2.

Claim 3. Let uv be an edge distinct from yy_1, where $u \in V(D_i)$ and $v \in A$. Then $uv \notin E(F)$.

Otherwise, suppose that $uv \in E(F)$. If u is of type (1) or (3), then there exists an odd changeable trail R_{12} from x to u such that $V(R_{12}) - (V(P_1) - y) \subseteq V(D_i)$. Then $R_{12} \cup uv$ is an even changeable trail from x to v, contradicting to $v \in A$. If u is of type (2), then there exists an even changeable trail R_{13} from x to u such that $V(R_{13}) - (V(P_1) - y) \subseteq V(D_i)$. Since F is H-optimal and H satisfies property (*), then $d_F(u) \in H(u)$, $d_F(u)-1 \notin H(u)$ and $d_F(u) - 2 \in H(u)$. So $R_{13} \cup uv$ is an even changeable trail from x to v, contradicting to $v \in A$ again.

By Claim 2, F misses at most one edge of $E_G(V(D_i), B)$. By Claim 3, if $y_1 y \in E_G(V(D_i), B) - E(F)$, then we have $E_G(V(D_i), A) \cap E(F) = \emptyset$. Hence (a) holds. With similar discussion, one can see that (b) holds. This completes the proof. □

Lemma 3.8. *If $def_H[F, V(D_i)] = 0$, then either F misses an edge of $E_G(V(D_i), B)$ or contains an edge of $E_G(V(D_i), A)$.*

Proof. Suppose that $def_H[F, V(D_i)] = 0$. Let $x \in V(D_i)$, we have $x \in D$. Then there exists a changeable trail P from a vertex v_0 of B_0. Clearly, $v_0 \notin V(D_i)$. Let yy' be the first edge of P such that $\{y, y'\} \cap V(D_i) \neq \emptyset$, where $y \in V(D_i)$ and $y' \notin V(D_i)$. By the definitions of A and B, we have $y' \in A \cup B$. If $y' \in A$ (or $y' \in B$), then the subtrail P_1 of P from v_0 to y' is an odd (resp. even) changeable trail. Since $P_1 \cup yy'$ is also changeable trail, then $yy' \in E(F)$ (resp. $yy' \notin E(F)$). Otherwise, we have $d_F(y') + 1 \notin H(y')$ and $d_F(y') + 2 \in H(y')$, which implies $y' \in D$, a contradiction. By Lemma 3.7, F contains (resp. misses) exactly an edge of $E_G(A, V(D_i))$ (resp. $E_G(B, V(D_i))$). This completes the proof. □

With the above lemmas, we have a better picture of the structures of graphs in partition (A, B, C, D) and are ready to prove deficiency formula for H-optimal subgraphs. Recall that τ is the number of components in $G[D]$.

Theorem 3.9. *$def_H(G) = \tau + \sum_{v \in B}(mH(v) - d_{G-A}(v)) - \sum_{v \in A} MH(v)$.*

Proof. Let τ_1 denote the number of components D_i of $G[D]$ with $def_H[F; V(D_i)] = 1$. Let τ_B (or τ_A) be the number of components T of $G[D]$ such that F misses (or contains) one edge from T to B (or A). By Lemmas 3.5, 3.7 and 3.8, we have $\tau = \tau_1 + \tau_A + \tau_B$. Note that $d_F(v) \leq mH(v)$ for all $v \in B$ and $d_F(v) = MH(v)$ for all $v \in A$. So

$$\begin{aligned} def_H(G) &= \tau_1 + mH(B) - \sum_{v \in B} d_F(v) \\ &= \tau_1 + mH(B) - (\sum_{v \in B} d_{G-A}(v) - \tau_B) - (MH(A) - \tau_A) \end{aligned}$$

$$= \tau + mH(B) - \sum_{v \in B} d_{G-A}(v) - MH(A).$$

□

Let X, Y be two disjoint subsets of $V(G)$. Define the modified prescription of H to be

$$H_Y(u) = H(u) - |E_G(u, Y)| \text{ for } u \in V(G) - X - Y.$$

Let K be a component of $G - X - Y$. We defined $H_{Y|K}$ as follows:

$$H_{Y|K}(u) = H_Y(u) \text{ for } u \in V(K).$$

Theorem 3.10. *Let $F_i = F[V(D_i)]$ for $i = 1, \ldots, \tau$. Then $def_{H_{B|D_i}}(F_i) = 1$ and F_i is $H_{B|D_i}$-optimal for $i = 1, \ldots, \tau$.*

Proof. By Lemma 3.4, we have $def_H[F, V(D_i)] \leq 1$. If $def_H[F, V(D_i)] = 1$, by Lemma 3.5, we have $def_{H_{B|D_i}}(F_i) = 1$. If $def_H[F, V(D_i)] = 0$, by Lemma 3.8, then $def_{H_{B|D_i}}(F_i) \leq 1$ holds. Hence we can assume that $def_{H_{B|D_i}}(F_i) \leq 1$.

Suppose that the theorem doesn't hold. Let F_i^* be $H_{B|D_i}$-optimal. Then we have $def_{H_{B|D_i}}[F_i^*] = 0$. We consider two cases.

Case 1. $def_H[F; V(D_i)] = 1$.

Then we have $def_H((F - E(F_i)) \cup E(F_i^*)) = def_H(F) - 1$, a contradiction since F is H-optimal.

Case 2. $def_H[F; V(D_i)] = 0$.

By Lemma 3.8, then F contains (or misses) an edge of $E(V(D_i), A)$ (resp. $E(V(D_i), B)$). Let $yy_1 \in E(V(D_i), A) \cap E(F)$, where $y \in V(D_i)$ and $y_1 \in A$ (resp. $yy_1 \in E(V(D_i), B)$ and $yy_1 \notin E(F)$, where $y \in V(D_i)$ and $y_1 \in B$). Since $y \in V(D_i)$, there is an odd (or even) changeable trail P from a vertex x of B_0 to y ($d_F(y), d_F(y) - 2 \in H(y)$ and $d_F(y) - 1 \notin H(y)$, resp.). We claim $yy_1 \in E(P)$, otherwise, $P \cup yy_1$ be an even changeable trail from x to y_1, a contradiction since $y_1 \in A$. Let P_1 be a subtrail of P such that $V(P_1) \cap V(D_i) = \{y\}$. Then we have $def_H[((F - E(F_i)) \triangle P_1) \cup E(F_i^*)] = def_H(F) - 1$, a contradiction again. □

Now we prove Lovász's classic deficiency formula.

Theorem 3.11 (Lovász [3])**.** *The total deficiency of G is*

$$def_H(G) = \max_{\substack{S,T \\ S \cap T = \emptyset}} \tau_H(S, T) - \sum_{x \in T} d_{G-S}(x) - MH(S) + mH(T),$$

where $\tau_H(S,T)$ denotes the number of components K of $G-S-T$ such that K contains no $H_{T|K}$-factors. Moreover, a graph G has an H-factor if and only if for any pair of disjoint sets $S,T \subseteq V(G)$,

$$\tau_H(S,T) - \sum_{x\in T} d_{G-S}(x) - MH(S) + mH(T) \leq 0.$$

Proof. Let F be an arbitrary H-optimal graph of G. Firstly, we show that

$$def_H(G) \geq \max_{\substack{S,T \\ S\cap T=\emptyset}} \tau_H(S,T) - \sum_{x\in T} d_{G-S}(x) - MH(S) + mH(T).$$

Let S and T be two disjoint subsets of $V(G)$, which reach the maximum of the right-hand side. Let $\tau_H(S,T)$ be defined as in the theorem. For $i = 1,, \dots, \tau_H(S,T)$, let C_i denote the component of $G-S-T$ containing no $H_{T|C_i}$-factors. We write $W = C_1 \cup \cdots \cup C_{\tau_H(S,T)}$. If $def_H[F, V(C_i)] = 0$, since C_i contains no $H_{T|C_i}$-factors, then F either misses at least an edge of $E_G(V(C_i),T)$ or contains at least an edges of $E_G(V(C_i),S)$. Let τ_1 denote the number of components of W such that F misses at least an edge of $E_G(V(C_i),T)$ and τ_2 denote the number of the components of W such that F contains at least an edge of $E_G(V(C_i),S)$. Then we have

(2)
$$def_H(G) = def_H[F] \geq \tau_H(S,T) - \tau_1 - \tau_2 + \sum_{x\in S\cup T} \min\{|r - d_F(x)| \mid r \in H(x)\}$$

(3)
$$\geq \tau_H(S,T) - \tau_1 - \tau_2 + \sum_{x\in S}(d_F(x) - MH(x)) + \sum_{x\in T}(mH(x) - d_F(x))$$

(4)
$$\begin{aligned}
&\geq \tau_H(S,T) - \tau_1 - \tau_2 + (e_F(S,T) + \tau_2 - MH(S)) \\
&\quad + \sum_{x\in T}(mH(x) - d_F(x)) \\
&= \tau_H(S,T) - \tau_1 + (e_F(S,T) - MH(S)) + \sum_{x\in T}(mH(x) - d_F(x)) \\
&\geq \tau_H(S,T) - \tau_1 + (e_F(S,T) - MH(S)) \\
&\quad + (mH(T) - (e_F(S,T) + \sum_{x\in T} d_{G-S}(x) - \tau_1)) \\
&= \tau_H(S,T) + mH(T) - MH(S) - \sum_{x\in T} d_{G-S}(x).
\end{aligned}$$

By Theorem 3.9, we have

$$def_H(G) = \tau + \sum_{v\in B}(mH(v) - d_{G-A}(v)) - \sum_{v\in A} MH(v).$$

By Theorem 3.10, D_i contains no $H_{B|D_i}$-factors for $i = 1, \cdots, \tau$. So we have

$$def_H(G) = \tau_H(A, B) + \sum_{v\in B}(mH(v) - d_{G-A}(v)) - \sum_{v\in A} MH(v)$$
$$= \max_{\substack{S,T\\ S\cap T=\emptyset}} \tau_H(S, T) - \sum_{x\in T} d_{G-S}(x) - MH(S) + mH(T).$$

This completes the proof. □

The proof of Theorem 3.11 also imply the following result.

Theorem 3.12. *Let F be an arbitrary H-optimal subgraph of G. Then*

(i) $d_F(v) \in H(v)$ *for all* $v \in C$*;*
(ii) $d_F(v) \geq MH(v)$ *for all* $v \in A$*;*
(iii) $d_F(v) \leq mH(v)$ *for all* $v \in B$*.*

Proof. In the proof of Theorem 3.11, since the equality between (2) and (3) holds, one can see that $d_F(x) \geq MH(x)$ for all $x \in S$ and $d_F(y) \leq mH(y)$ for all $y \in T$. Note that A and B is a special case of S and T. So we have $d_F(x) \geq MH(x)$ for all $x \in A$ and $d_F(y) \leq mH(y)$ for all $y \in B$. □

Theorem 3.12 implies that the partition (A, B, C, D) defined in this paper is equivalent to the original partition (A_H, B_H, C_H, D_H) introduced by Lovász in [3].

Theorem 3.13. $C_H = C$*,* $A_H = A$*,* $B_H = B$ *and* $D_H = D$*.*

Proof. By Theorem 3.12, we have $C \subseteq C_H$, $A \subseteq A_H$, $B \subseteq B_H$. However, the definition of D implies $v \notin C_H \cup A_H \cup B_H$ for any $v \in D$. So we have $D \subseteq D_H$. This completes the proof. □

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