Pure and Applied Mathematics Quarterly
Volume 18, Number 6, 2599–2618, 2022

Singular Turán numbers of stars

Gaoxing Sun, Heng Li*, Qinghou Zeng, and Jianfeng Hou

Abstract: Suppose that G is a graph and H is a subgraph of G. We call H singular if the vertices of H either have the same degree in G or have pairwise distinct degrees in G. Let $T_S(n, H)$ be the largest number of edges of a graph with n vertices that does not contain a singular copy of H. The problem of determining $T_S(n, H)$ was studied initially by Caro and Tuza, who obtained an asymptotic bound for each H. In this paper, we consider the case that H is a star, and determine the exact values of $T_S(n, K_{1,2})$ for all n, $T_S(n, K_{1,4})$ and $T_S(n, K_{1,2s+1})$ for sufficiently large n.

Keywords: Singular, Turán number, star, H-free.

1. Introduction

The classical *Turán number* of a graph H, denoted by $\text{ex}(n, H)$, is the maximum number of edges in an n-vertex graph not containing H as a subgraph. For $H = K_{r+1}$, the only extremal graph is the so-called *Turán graph*, denoted by $T_r(n)$, which is the balanced complete r-partite graph with each part of size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$. For general graph H with chromatic number $p + 1$, Erdős-Stone-Simonovits Theorem [6] shows that

$$\text{ex}(n, H) = \left(1 - \frac{1}{p} + o(1)\right) \binom{n}{2}.$$

Turán type problem has been studied widely, see [15, 9, 12, 13, 14, 5], especially for bipartite graphs [3, 11, 7, 10].

Albertson [1] considered the maximum number of edges in graphs that have no copy of K_p with all degrees equal, and gave an exact bound. It was extended by Caro and Tuza [4] who initiated the so-called *singular Turán number*. Let G be a graph, and let H be a subgraph of G. We say H is *singular* in G if the vertices of H either have the same degree in G, or have

Received June 30, 2021.
2010 Mathematics Subject Classification: 05C35, 05C07.
*Corresponding author.

pairwise distinct degrees in G. If G does not contain a singular H, then G is singular H-free. The singular Turán number, denoted by $T_S(n,H)$, is the largest number of edges of a singular H-free graph with n vertices. For general graph H with $r+1$ vertices and chromatic number $p+1$, Caro and Tuza [4] obtained an asymptotic bound by proving that

$$T_S(n,H) = \left(1 - \frac{1}{pr} + o(1)\right)\binom{n}{2}.$$

Determining the exact value of $T_S(n,H)$ seems not easy, even for special H, and there are few theoretical results. For $H = K_3$, Caro and Tuza showed that $T_S(4k+2, K_3) = 6k^2 + 6k + 1$, and gave lower and upper bounds of $T_S(n, K_3)$ for other cases of n. This was further improved by Gerbner, Patkós, Vizer and Tuza [8], who showed the following theorem.

Theorem 1.1 (Gerbner, Patkós, Vizer and Tuza [8])**.** *Let k be a nonnegative integer. Then*

(1) $T_S(4, K_3) = 5$, *and* $T_S(4k, K_3) = 6k^2 - 2$ *if* $k \geq 2$,
(2) $T_S(4k+1, K_3) = 6k^2 + 2k$, *and*
(3) $6k^2 + 8k + 1 \leq T_S(4k+3, K_3) \leq 6k^2 + 8k + 3$.

In this paper, we focus on the case that H is a star. Let $t_r(n)$ denote the number of edges in the Turán graph $T_r(n)$. We first consider $T_S(n, K_{1,2s+1})$ and establish the following theorem.

Theorem 1.2. *For any integer $s \geq 1$ and sufficiently large n, there is an absolute constant $C(s)$ such that*

$$T_S(n, K_{1,2s+1}) = t_{2s+1}(n) + sn - C(s).$$

We also determine the exact value of $T_S(n, K_{1,2})$ for all n, and $T_S(n, K_{1,4})$ for sufficiently large n as follows.

Theorem 1.3. *Let k be any nonnegative integer. Then*

(1) $T_S(4k, K_{1,2}) = 4k^2 + k$ *if* $0 \leq k \leq 3$, *and* $T_S(4k, K_{1,2}) = 4k^2 + 2k - 4$ *if* $k \geq 4$,
(2) $T_S(4k+1, K_{1,2}) = 4k^2 + 3k$,
(3) $T_S(4k+2, K_{1,2}) = 4k^2 + 6k + 1$, *and*
(4) $T_S(4k+3, K_{1,2}) = 4k^2 + 6k + 2$ *if* $0 \leq k \leq 1$, *and* $T_S(4k+3, K_{1,2}) = 4k^2 + 7k$ *if* $k \geq 2$.

Theorem 1.4. *For sufficiently large $n \equiv \ell$ (mod 8),we have*

$$T_S(n, K_{1,4}) = \begin{cases} t_4(n) + \frac{3}{2}n - 20 & \text{if } \ell = 0, \\ t_4(n) + \frac{11}{8}n - \frac{123}{8} & \text{if } \ell = 1, \\ t_4(n) + \frac{3}{2}n - 17 & \text{if } \ell = 2, \\ t_4(n) + \frac{11}{8}n - \frac{65}{8} & \text{if } \ell = 3, \\ t_4(n) + \frac{3}{2}n - 10 & \text{if } \ell = 4, \\ t_4(n) + \frac{11}{8}n - \frac{95}{8} & \text{if } \ell = 5, \\ t_4(n) + \frac{3}{2}n - 17 & \text{if } \ell = 6, \\ t_4(n) + \frac{11}{8}n - \frac{141}{8} & \text{if } \ell = 7. \end{cases}$$

This paper is organized as follows. In Section 2, we determine the exact values of the singular Turán number of stars with even number of vertices. In Sections 3 and 4, we prove Theorems 1.3 and 1.4.

Notation. Let $G = (V(G), E(G))$ be a graph. For any $v \in V(G)$, denote $N_G(v)$ the set of *neighbors* of v in G and $d_G(v)$ the *degree* of v in G. Denote by $\Delta(G)$ the maximum degree of G. For any $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S. Denote by $e(G)$ the number of edges in G. Usually, we write $[k] := \{1, \ldots, k\}$.

2. Singular Turán numbers of the star $K_{1,2s+1}$

In this section, we prove Theorem 1.2. We first present a useful lemma given by Brouwer [2].

Lemma 2.1 (Brouwer [2])**.** *If H is a K_{r+1}-free graph on n vertices which is not r-partite, then H has at most $t_r(n) - \lfloor n/r \rfloor + 1$ edges, assuming $n \geq 2r+1$.*

We also give the following lemma, whose simple proof is left to the reader.

Lemma 2.2. *Let G be an r-partite graph with $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r(G)$ and $e(G) = t_r(n) - f(G)$. Then, for each $i \in [r]$, there exists some constant $C(r) \geq 0$ such that*

$$\left| |V_i| - \frac{n}{r} \right| \leq C(r)\sqrt{f(G)}.$$

Gerbner, Patkós, Vizer and Tuza [8] established a general upper bound of $T_S(n, H)$ in concluding remarks. We give a proof here for complement.

Lemma 2.3. *Let n be a positive integer, and let H be a graph with r vertices. Then*

$$T_S(n, H) \leq \text{ex}(n, K_r) + \text{ex}(n, H).$$

Proof. Suppose that G is a singular H-free graph with n vertices. Let G_1 be the spanning subgraph induced by the edges that connect vertices of the same degree, and let $G_2 = G - E(G_1)$. Since G_1 is H-free, we have

$$e(G_1) \leq \mathrm{ex}(n, H). \tag{1}$$

Note that G_2 is K_r-free; otherwise G_2 has a copy of H with all degree distinct. This implies that $e(G_2) \leq \mathrm{ex}(n, K_r)$, which together with (1) yields that

$$e(G) = e(G_1) + e(G_2) \leq \mathrm{ex}(n, K_r) + \mathrm{ex}(n, H),$$

completing the proof of Lemma 2.3. □

Now, we define a kind of regular graph. The *kth power* of a graph G, denoted by G^k, has vertex set $V(G)$ in which vertices are adjacent if the distance between them in G is at most k. Note that C_n^k is a $2k$-regular graph if $n \geq 2k+1$.

Proof of Theorem 1.2. For the lower bound, we construct a graph G as follows. Let $a_1 < a_2 < \cdots < a_{2s+1}$ be positive integers such that $\sum_{i=1}^{2s+1} a_i = n$ and $\sum_{1\leq i<j\leq 2s+1} a_i a_j$ is maximum. Let $K_{a_1,\ldots,a_{2s+1}}$ be a complete $(2s+1)$-partite graph with $V = V_1 \cup V_2 \cup \cdots \cup V_{2s+1}$ and $|V_i| = a_i$ for $i \in [2s+1]$. Then, for each $i \in [2s+1]$, we embed a $2s$-regular graph $C_{a_i}^s$ in V_i, which means that $G[V_i]$ is $K_{1,2s+1}$-free. Moreover, G has no $K_{1,2s+1}$ with all degree distinct as G only has $2s+1$ distinct degrees. Thus, G is singular $K_{1,2s+1}$-free with

$$e(G) = t_{2s+1}(n) + sn - C(s),$$

where $C(s) = t_{2s+1}(n) - \sum_{1\leq i<j\leq 2s+1} a_i a_j > 0$.

We claim that there exists some constant $C_s \geq 0$ such that $C(s) \leq C_s s^3$. Let $b_i = \left\lfloor \frac{n}{2s+1} \right\rfloor - s + i - 1$ for $i \in [2s]$ and $b_{2s+1} = n - \sum_{i=1}^{2s} b_i$. Then, we have $b_1 < b_2 < \cdots < b_{2s+1}$ and $\sum_{i=1}^{2s+1} b_i = n$. Recall that $\sum_{1\leq i<j\leq 2s+1} a_i a_j$ is maximum. Clearly, we have

$$\begin{aligned}\sum_{1\leq i<j\leq 2s+1} a_i a_j \geq \sum_{1\leq i<j\leq 2s+1} b_i b_j &= \sum_{1\leq i<j\leq 2s} b_i b_j + b_{2s+1} \sum_{1\leq i\leq 2s} b_i \\ &= t_{2s+1}(n) - O(s^3).\end{aligned}$$

Thus, $C(s) = t_{2s+1}(n) - \sum_{1\leq i<j\leq 2s+1} a_i a_j \leq t_{2s+1}(n) - (t_{2s+1}(n) - O(s^3)) = O(s^3)$.

Now we prove the upper bound. Suppose that n is sufficiently large. Let G be a singular $K_{1,2s+1}$-free graph with n vertices. Let G_1 be the spanning subgraph induced by the edges that connect vertices of the same degree, and let $G_2 = G - E(G_1)$. Then G_1 is $K_{1,2s+1}$-free, which implies that

$$e(G_1) \leq \frac{2sn}{2} = sn. \tag{2}$$

Clearly, G_2 is K_{2s+2}-free. If $e(G_2) \leq t_{2s+1}(n) - \left\lfloor \frac{n}{2s+1} \right\rfloor + 1$, then

$$e(G) = e(G_1) + e(G_2) \leq sn + t_{2s+1}(n) - \left\lfloor \frac{n}{2s+1} \right\rfloor + 1 \leq t_{2s+1}(n) + sn - C(s),$$

as desired. Suppose that

$$e(G_2) \geq t_{2s+1}(n) - \left\lfloor \frac{n}{2s+1} \right\rfloor + 2. \tag{3}$$

By Lemma 2.1, G_2 is $(2s+1)$-partite with $V(G_2) = A_1 \cup A_2 \cup \cdots \cup A_{2s+1}$. For each $i \in [2s+1]$, it follows from Lemma 2.2 and (3) that

$$|A_i| \geq \frac{n}{2s+1} - O(\sqrt{n}). \tag{4}$$

Next, we show that all vertices in A_i have the same degree in G.

Claim 2.4. *For each $i \in [2s+1]$, we have $d_G(u) = d_G(v)$ for all $u, v \in A_i$.*

Proof. Without loss of generality, we may assume that there exist $x_1, x_2 \in A_1$ such that $d_G(x_1) \neq d_G(x_2)$. Let $B_i = \{v \in A_i \mid d_{G_2}(v) \leq n - |A_i| - 1\}$, then $e(G_2) \leq t_{2s+1}(n) - |B_i|$ for $i \in [2s+1]$. If there exists some i such that $|B_i| \geq |A_i|/2$, then this together with (4) implies that

$$\begin{aligned} e(G) =& e(G_1) + e(G_2) \\ &\leq sn + t_{2s+1}(n) - |B_i| \\ &\leq sn + t_{2s+1}(n) - \frac{n}{2(2s+1)} + O(\sqrt{n}) \\ &\leq t_{2s+1}(n) + sn - \Omega(n), \end{aligned}$$

as desired. Otherwise, $|B_i| < |A_i|/2$ for each $i \in [2s+1]$. Then there exists a vertex $y \in A_2$ such that $d_{G_2}(y) = n - |A_2|$. Let $\mathcal{F}$ be the family of all $(2s+1)$-sets $\{x_1, x_2, x_3, \ldots, x_{2s+1}\}$ with $x_i \in A_i$ for $3 \leq i \leq 2s+1$. Then $|\mathcal{F}| = \prod_{k=3}^{2s+1} |A_k|$. Since $d_{G_2}(y) = n - |A_2|$ and G has no $K_{1,2s+1}$ with all degree distinct, there exist $1 \le i \ne j \le 2s+1$ such that $d_G(x_i) = d_G(x_j)$. This means that the edge $x_i x_j \notin E(G_2)$. Sum over the sets in $\mathcal{F}$ gives that $e(G_2)$ has at least $\prod_{k=3}^{2s+1} |A_k|$ missing edges in total. For a missing edge $x_i x_j$, if $i, j \ge 3$, then $x_i x_j$ is counted at most $\frac{\prod_{k=3}^{2s+1} |A_k|}{|A_i||A_j|}$ times. Otherwise, if $i \le 2$ and $j \ge 3$, then $x_i x_j$ is counted at most $\frac{\prod_{k=3}^{2s+1} |A_k|}{|A_j|}$ times. Thus, the number of missing edges in G_2 is at least $\min\{|A_i| \,|\, 3 \le i \le 2s+1\}$. This together with (4) implies that

$$e(G_2) \le t_{2s+1}(n) - \frac{n}{2s+1} + O(\sqrt{n}),$$

and then

$$e(G) = e(G_1) + e(G_2) \le sn + t_{2s+1}(n) - \frac{n}{2s+1} + O(\sqrt{n}) \le t_{2s+1}(n) + sn - C(s).$$

This completes the proof of Claim 2.4. □

If there do not exist $i, j \in [2s+1]$ and $i \ne j$ such that $|A_s| = |A_t|$, then we have that

$$e(G) = e(G_1) + e(G_2) \le sn + \sum_{1 \le i < j \le 2s+1} |A_i||A_j| \le t_{2s+1}(n) + sn - C(s),$$

as required. So, without loss of generality, we may suppose that $|A_1| = |A_2|$. Let $D_i = A_i \backslash B_i$ for each $i \in [2]$. Recall that $|B_i| \le |A_i|/2$. So, by (4), we have

$$|D_i| \ge |A_i|/2 \ge \frac{n}{4s+2} - O(\sqrt{n}). \tag{5}$$

Choose $x \in D_1$ and $y \in D_2$, we obtain that $d_{G_2}(x) = d_{G_2}(y) = n - |A_1|$. Since $d_G(x) \ne d_G(y)$, we have $d_{G_1}(x) \ne d_{G_1}(y)$. It follows from G_1 is $K_{1,2s+1}$-free that either $d_{G_1}(x) < 2s$ or $d_{G_1}(y) < 2s$. If $d_{G_1}(x) < 2s$, then $d_{G_1}(z) < 2s$ for all $z \in D_1$ by Claim 2.4. Thus,

$$e(G_1) \le sn - \frac{|D_1|}{2} = sn - \Omega(n). \tag{6}$$

Otherwise, $d_{G_1}(z) < 2s$ for all $z \in D_2$, implying that

$$e(G_1) \le sn - \frac{|D_2|}{2} = sn - \Omega(n). \tag{7}$$

In view of (6) and (7), we have

$$e(G) = e(G_1) + e(G_2) \leq sn - \Omega(n) + t_{2s+1}(n) \leq t_{2s+1}(n) + sn - C(s).$$

Thus, for any singular $K_{1,2s+1}$-free graph G with n vertices, $e(G) \leq t_{2s+1}(n) + sn - C(s)$ for sufficiently large n. This completes the proof of Theorem 1.2. □

3. Singular Turán number of $K_{1,2}$

In this section, we consider the singular Turán number of $K_{1,2}$ and give a proof of Theorem 1.3.

Proof of Theorem 1.3. Let G be a singular $K_{1,2}$-free graph with n vertices. Let G_1 be the spanning subgraph induced by the edges that connect vertices of the same degree, and let $G_2 = G - E(G_1)$. Note that $\Delta(G_1) \leq 1$. So,

$$e(G_1) \leq \left\lfloor \frac{n}{2} \right\rfloor. \tag{8}$$

Note that G has no $K_{1,2}$ with all degree distinct. This implies that, for each $uv \in E(G_2)$, we have $d_G(x) = d_G(v)$ for $x \in N_{G_2}(u)$ and $d_G(y) = d_G(u)$ for $y \in N_{G_2}(v)$. It follows that $G_2[N_{G_2}(u) \cup N_{G_2}(v)]$ is a bipartite graph. In a similar flavor for other edges, we conclude that G_2 is a bipartite graph with parts A and B.

Suppose that $n = 4k + i$ for $0 \leq i \leq 3$.

Case 1. $i = 0$. First, we give the lower bound of $T_S(4k, K_{1,2})$ by showing the following constructions: if $k \geq 4$, then let F be a graph by adding perfect matchings in both parts of $K_{2k-2,2k+2}$; if $0 \leq k \leq 3$, then let F be a graph by adding a perfect matching in one part of $K_{2k,2k}$. Then F is singular $K_{1,2}$-free with

$$e(F) = \begin{cases} 4k^2 + 2k - 4 & \text{if } k \geq 4, \\ 4k^2 + k & \text{if } 0 \leq k \leq 3. \end{cases}$$

Thus,

$$T_S(4k, K_{1,2}) \geq \begin{cases} 4k^2 + 2k - 4 & \text{if } k \geq 4, \\ 4k^2 + k & \text{if } 0 \leq k \leq 3. \end{cases}$$

Next, we give the upper bound of G. Recall that G_2 is a bipartite graph with $V(G_2) = A \cup B$. If $|A| \leq 2k - 2$, then this together with (8) yields that

$$e(G) = e(G_1) + e(G_2) \leq \left\lfloor \frac{n}{2} \right\rfloor + |A|(n - |A|) \leq 2k + (4k^2 - 4) = 4k^2 + 2k - 4. \tag{9}$$

Suppose $|A| = 2k-1$. If there exist vertices $u \in A$, $v \in B$ satisfying $d_{G_2}(u) = 2k+1$ and $d_{G_2}(v) = 2k-1$, then the vertices in A (or B) have the same degree in G. Since A is odd, $d_G(x) = 2k+1$ for all $x \in A$. Similarly, $d_G(y) = 2k-1$ for all $y \in B$. Thus,

$$e(G) = (2k-1)(2k+1) = 4k^2 - 1. \tag{10}$$

Otherwise, every vertex in A (or B) misses at least one edge to the other part in G_2. This together with (8) yields that

$$e(G) = e(G_1) + e(G_2) \leq 2k + ((2k+1)(2k-1) - (2k-1)) = 4k^2 - 1. \tag{11}$$

Suppose $|A| = 2k$. Again, if there exist vertices $u \in A$, $v \in B$ satisfying $d_{G_2}(u) = 2k$ and $d_{G_2}(v) = 2k$, then $d_G(x) \leq 2k+1$ for all $x \in V(G)$, and either $d_G(x) = 2k$ for all $x \in A$ or $d_G(y) = 2k$ for all $y \in B$. Thus,

$$e(G) = (2k \cdot 2k + 2k(2k+1))/2 = 4k^2 + k. \tag{12}$$

Otherwise, there are at least $2k$ edges missing in G_2. This together with (8) yields that

$$e(G) = e(G_1) + e(G_2) \leq 2k + (2k \cdot 2k - 2k) = 4k^2. \tag{13}$$

Combining (9), (10), (11), (12) and (13), we have

$$T_S(4k, K_{1,2}) \leq \begin{cases} 4k^2 + 2k - 4 & \text{if } k \geq 4, \\ 4k^2 + k & \text{if } 0 \leq k \leq 3. \end{cases}$$

Case 2. $i = 1$. First, we give the lower bound of $T_S(4k+1, K_{1,2})$. Let F be a graph by adding a perfect matching in the part with $2k$ vertices of $K_{2k,2k+1}$. Thus,

$$T_S(4k+1, K_{1,2}) \geq e(F) = 4k^2 + 3k$$

as F is singular $K_{1,2}$-free.

Next, we consider the upper bound. Note that exactly one of $|A|$ and $|B|$ is odd. If there exist vertices $u \in A$, $v \in B$ satisfying $d_{G_2}(u) = |B|$ and $d_{G_2}(v) = |A|$, then the vertices in A (or B) have the same degree in G. Then $|B| \leq d_G(x) \leq |B| + 1$ for all $x \in |A|$. Moreover, if $d_G(x) = |B| + 1$, then $|A|$ is even, and $G[B]$ is empty. The similar statements also hold for the vertices in $|B|$. Without loss of generality, we assume that $|A|$ is even. Thus,

$$e(G) = |A|(n - |A|) + \frac{|A|}{2} \leq 4k^2 + 3k. \tag{14}$$

Otherwise, every vertex in A (or B) misses an edge to the other part in G_2. This together with (8) yields that

$$e(G) = e(G_1) + e(G_2) \leq \left\lfloor \frac{n}{2} \right\rfloor + (|A|(n - |A|) - |A|) \leq 4k^2 + 2k. \tag{15}$$

Thus, by (14) and (15), we have

$$T_S(4k+1, K_{1,2}) \leq 4k^2 + 3k.$$

Case 3. $i = 2$. Again, we give the lower bound of $T_S(4k+2, K_{1,2})$ firstly. Let F be a graph by adding perfect matchings in both parts of $K_{2k,2k+2}$. Thus,

$$T_S(4k+2, K_{1,2}) \geq e(F) = 4k^2 + 6k + 1$$

as F is singular $K_{1,2}$-free.

Next, we consider the upper bound. If $|A| \leq 2k$, then by (8),

$$e(G) = e(G_1) + e(G_2) \leq \left\lfloor \frac{n}{2} \right\rfloor + |A|(n - |A|) \leq 4k^2 + 6k + 1. \tag{16}$$

Suppose $|A| = 2k+1$. We claim that there do not exist vertices $u \in A$ and $v \in B$ satisfying $d_{G_2}(u) = 2k+1$ and $d_{G_2}(v) = 2k+1$. Since otherwise $d_G(x) = 2k+1$ for all $x \in V(G)$ as $|A|$ and $|B|$ are odd. Thus, every vertex in A (or B) misses at least one edge to the other part in G_2. This together with (8) yields that

$$e(G) = e(G_1) + e(G_2) \leq 2k+1 + ((2k+1)^2 - (2k+1)) = 4k^2 + 4k + 1. \tag{17}$$

Combining (16) and (17), we have

$$T_S(4k+2, K_{1,2}) \leq 4k^2 + 6k + 1.$$

Case 4. $i = 3$. First, we give the lower bound of $T_S(4k+3, K_{1,2})$ by showing the following constructions: if $k \geq 2$, then let F be a graph by adding a perfect matching in the part with $2k$ vertices of $K_{2k,2k+3}$; if $0 \leq k \leq 1$, then let F be the graph $K_{2k+1,2k+2}$. Thus, we have

$$T_S(4k+3, K_{1,2}) \geq e(F) = \begin{cases} 4k^2 + 7k & \text{if } k \geq 2, \\ 4k^2 + 6k + 2 & \text{if } 0 \leq k \leq 1. \end{cases}$$

as F is singular $K_{1,2}$-free.

Next, we consider the upper bound. By the same argument as in Case 2, we have either

$$e(G) = |A|(n - |A|) + \frac{|A|}{2} \le 4k^2 + 7k,$$

or

$$e(G) = e(G_1) + e(G_2) \le \left\lfloor \frac{n}{2} \right\rfloor + (|A|(n - |A|) - |A|) \le 4k^2 + 6k + 2.$$

Thus, we have

$$T_S(4k+3, K_{1,2}) \le \begin{cases} 4k^2 + 7k & \text{if } k \ge 2, \\ 4k^2 + 6k + 2 & \text{if } 0 \le k \le 1. \end{cases}$$

This completes the proof of Theorem 1.3. □

4. Singular Turán number of $K_{1,4}$

In this section, we consider the singular Turán number of $K_{1,4}$ and prove Theorem 1.4.

Firstly, we give the lower bound by the following construction: Let a_1, a_2, a_3, a_4 be positive integers with $\sum_{i=1}^{4} a_i = n$. The choices of a_i can be found in Table 1. Let $K^*_{a_1,a_2,a_3,a_4}$ be the graph obtained from the complete 4-partite graph K_{a_1,a_2,a_3,a_4} by adding a 3-regular graph to the parts with even vertices and a 2-regular graph to the part with odd vertices. Then $K^*_{a_1,a_2,a_3,a_4}$ has 4 distinct degrees, which means $K^*_{a_1,a_2,a_3,a_4}$ has no $K_{1,4}$ with all degree distinct. Moreover, $G[V_i]$ is $K_{1,4}$-free for $i \in [4]$. So, G is singular $K_{1,4}$-free.

Table 1: The graph $K^*_{a_1,a_2,a_3,a_4}$

n	a_1	a_2	a_3	a_4	$e(K^*_{a_1,a_2,a_3,a_4})$	$C(\ell)$
$8k$	$2k-4$	$2k-2$	$2k+2$	$2k+4$	$t_4(n) + \frac{3}{2}n - 20$	20
$8k+1$	$2k-4$	$2k$	$2k+2$	$2k+3$	$t_4(n) + \frac{11}{8}n - \frac{123}{8}$	$\frac{123}{8}$
$8k+2$	$2k-4$	$2k$	$2k+2$	$2k+4$	$t_4(n) + \frac{3}{2}n - 17$	17
$8k+3$	$2k-2$	$2k$	$2k+2$	$2k+3$	$t_4(n) + \frac{11}{8}n - \frac{65}{8}$	$\frac{65}{8}$
$8k+4$	$2k-2$	$2k$	$2k+2$	$2k+4$	$t_4(n) + \frac{3}{2}n - 10$	10
$8k+5$	$2k-2$	$2k$	$2k+2$	$2k+5$	$t_4(n) + \frac{11}{8}n - \frac{95}{8}$	$\frac{95}{8}$
$8k+6$	$2k-2$	$2k$	$2k+2$	$2k+6$	$t_4(n) + \frac{3}{2}n - 17$	17
$8k+7$	$2k-2$	$2k$	$2k+4$	$2k+5$	$t_4(n) + \frac{11}{8}n - \frac{141}{8}$	$\frac{141}{8}$

Thus, for $n = 8k + \ell$ with $0 \le \ell \le 7$,

$$T_S(n, K_{1,4}) \ge \begin{cases} t_4(n) + \frac{11}{8}n - C(\ell) & n \text{ is odd} \\ t_4(n) + \frac{3}{2}n - C(\ell) & n \text{ is even} \end{cases},$$

where $C(\ell)$ is the constant given in Table 1.

We prove the upper bound of $T_s(n, K_{1,4})$. Suppose that n is sufficiently large. Let G^* be a singular $K_{1,4}$-free graph with

$$e(G^*) = T_s(n, K_{1,4}) \geq \begin{cases} t_4(n) + \frac{11}{8}n - C(\ell) & n \text{ is odd} \\ t_4(n) + \frac{3}{2}n - C(\ell) & n \text{ is even} \end{cases}, \tag{18}$$

where $C(\ell)$ is the constant given in Table 1.

Let G_1^* be the spanning subgraph induced by the edges that connect vertices of the same degree, and let $G_2^* = G^* - E(G_1^*)$. Since G_1^* is $K_{1,4}$-free, we have $d_{G_1^*}(v) \leq 3$ for $v \in V(G^*)$. Thus,

$$e(G_1^*) \leq \frac{3n}{2}. \tag{19}$$

By a similar argument as that in the proof of Theorem 1.2, we have the following properties of G_2^*, whose proof details are omitted.

Lemma 4.1. *G_2^* is 4-partite, and the vertices in each part have the same degree in G^*.*

By Lemma 4.1, let G_2^* be a 4-partite graph with $V(G_2^*) = V_1 \cup V_2 \cup V_3 \cup V_4$. For $i \in [4]$, it follows from $e(G_2^*) \geq e(G^*) - \frac{3}{2}n \geq t_4(n) - \Omega(n)$ and Lemma 2.2 that

$$|V_i| \geq n/4 - O(\sqrt{n}). \tag{20}$$

Choose $v_i \in V_i$ such that $d_{G_1^*}(v_i) = \min\{d_{G_1^*}(v) \mid v \in V_i\}$. Then we have the follow property.

Lemma 4.2. *For each $i \in [4]$, we have $d_{G_2^*}(v_i) = n - |V_i|$.*

Proof. By contradiction, we may assume that $d_{G_2^*}(v_1) \leq n - |V_1| - 1$. For $i \in [4]$ and $v \in V_i$, we have $d_{G^*}(v) = d_{G^*}(v_i)$ by Lemma 4.1, which implies that $d_{G_2^*}(v) \leq d_{G_2^*}(v_i)$. This together with (19) and (20) yields that

$$\begin{aligned} e(G^*) = e(G_1^*) + e(G_2^*) &\leq \frac{3}{2}n + \frac{1}{2}\sum_{i=1}^{4}\sum_{v \in V_i} d_{G_2^*(v)} \\ &\leq \frac{3}{2}n + \frac{1}{2}\sum_{i=1}^{4}|V_i|(n - |V_i|) - \frac{1}{2}|V_1| \\ &\leq t_4(n) + \frac{11}{8}n + O(\sqrt{n}), \end{aligned}$$

a contradiction to (18) when n is even.

Suppose that $|V_j|$ is odd for some $j \in [4]$. Since $d_{G_1^*}(v_j) \leq d_{G_1^*}(v) \leq 3$ for $v \in V_j$, we have $d_{G_1^*}(v_j) \leq 2$. If $j = 1$, we have $d_{G^*}(v_1) = d_{G_1^*}(v_1) + d_{G_2^*}(v_1) \leq n - |V_1| + 1$. Thus,

$$\begin{aligned} e(G^*) &= \frac{1}{2} \sum_{v \in V(G^*)} d_{G^*}(v) \\ &\leq \frac{1}{2} \left(|V_1|(n - |V_1| + 1) + \sum_{i=2}^{4} |V_i|(n - |V_i| + 3) \right) \\ &\leq t_4(n) + \frac{1}{2} \left(|V_1| + 3(n - |V_1|) \right) \\ &\leq t_4(n) + \frac{10}{8} n + O(\sqrt{n}), \end{aligned}$$

a contradiction to (18). If $j \neq 1$, we have $d_{G^*}(v_1) \leq n - |V_1| + 2$ and $d_{G^*}(v_j) \leq n - |V_j| + 2$. Thus,

$$\begin{aligned} e(G^*) &= \frac{1}{2} \sum_{v \in V(G^*)} d_{G^*}(v) \\ &\leq \frac{1}{2} \left(\sum_{i \in \{1, j\}} |V_i|(n - |V_i| + 2) + \sum_{i \in [4] \setminus \{1, j\}} |V_i|(n - |V_i| + 3) \right) \\ &\leq t_4(n) + \frac{1}{2} \left(2(|V_1| + |V_j|) + 3(n - |V_1| - |V_j|) \right) \\ &\leq t_4(n) + \frac{10}{8} n + O(\sqrt{n}), \end{aligned}$$

a contradiction to (18). □

Next, we consider the structure of G_1^* as follows.

Lemma 4.3. *For $i \in [4]$, if n is even, then all $G_1^*[V_i]$ are 3-regular; otherwise, all $G_1^*[V_i]$ are 3-regular, expect one of $G_1^*[V_i]$ is 2-regular.*

Proof. According to the parity of n, we divide our proof into the following two cases.

Case 1. n is even. For $i \in [4]$, it suffices to prove that $d_{G_1^*}(v_i) \geq 3$. Indeed, we have $d_{G_1^*}(v) = 3$ for $v \in V(G^*)$ by the definition of v_i and $\Delta(G_1^*) \leq 3$, which implies that all $G_1^*[V_i]$ are 3-regular.

By contradiction, we may assume that $d_{G_1^*}(v_1) \leq 2$. Then $d_{G^*}(v_1) \leq n - |V_1| + 2$ by Lemma 4.2. Recall that $d_{G^*}(v) = d_{G^*}(v_i)$ for each $v \in V_i$ and each $i \in [4]$ by Lemma 4.1. We have $d_{G^*}(v) \leq n - |V_1| + 2$ for each $v \in V_1$. For $2 \leq i \leq 4$, $d_{G^*}(v) \leq n - |V_i| + 3$ for each $v \in V_i$. Thus,

$$\begin{aligned} e(G^*) &= \frac{1}{2} \sum_{v \in V(G^*)} d_{G^*}(v) \\ &\leq \frac{1}{2} \left(|V_1|(n - |V_1| + 2) + \sum_{i=2}^{4} |V_i|(n - |V_i| + 3) \right) \\ &\leq t_4(n) + \frac{1}{2} \left(2|V_1| + 3(n - |V_1|)\right) \\ &\leq t_4(n) + \frac{11}{8} n + O(\sqrt{n}), \end{aligned}$$

where the last inequality follows from (20), a contradiction to (18).

Case 2. n is odd. Again, without loss of generality, we aim to prove that

(i) $|V_1|$ is odd and $d_{G_1^*}(v_1) = 2$;
(ii) $|V_i|$ is even and $d_{G_1^*}(v_i) = 3$ for $2 \leq i \leq 4$.

Indeed, as in the proof of Case 1, if $d_{G_1^*}(v_i) = 3$, then $G_1^*(V_i)$ is 3-regular and $d_{G_2^*}(v) = n - |V_i|$ for all $v \in V_i$. If all but one of $G_1^*[V_i]$ are 3-regular, then G_2^* is a complete 4-partite graph. Consider the part V_1 with $d_{G_1^*}(v_1) \leq 2$. All the vertices in V_1 have the same degree both in G^* and in G_2^*, so do in G_1^*. Thus, $G_1^*[V_1]$ is regular. Since $|V_1|$ is odd, $G_1^*[V_1]$ is not 3-regular. To maximum $e(G^*)$, $G_1^*[V_1]$ should be 2-regular.

If there exists $k \in [4]$ such that $d_{G_1^*}(v_k) \leq 1$, then $d_{G^*}(v) \leq n - |V_i| + 1$ for each $v \in V_k$. For $i \in [4] \setminus \{k\}$, $d_{G^*}(v) \leq n - |V_i| + 3$ for each $v \in V_i$. Thus,

$$\begin{aligned} e(G^*) &= \frac{1}{2} \sum_{v \in V(G^*)} d_{G^*}(v) \\ &\leq \frac{1}{2} \left(|V_k|(n - |V_k| + 1) + \sum_{i \in [4],\ i \neq k} |V_i|(n - |V_i| + 3) \right) \\ &\leq t_4(n) + \frac{10}{8} n + O(\sqrt{n}), \end{aligned}$$

a contradiction to (18).

Thus, we have $d_{G_1^*}(v_i) \geq 2$ for $i \in [4]$. Suppose that there exist s and t with $1 \leq s < t \leq 4$ such that $d_{G_1^*}(v_s) = d_{G_1^*}(v_t) = 2$. Then $d_{G^*}(v) \leq n - |V_i| + 2$ for each $v \in V_i$ and every $i \in \{s, t\}$. Moreover, if $i \in [4] \setminus \{s, t\}$, then $d_{G^*}(v) \le n - |V_i| + 3$ for $v \in V_i$. Again,

$$\begin{aligned} e(G^*) &= \frac{1}{2} \sum_{v \in V(G^*)} d_{G^*}(v) \\ &\le \frac{1}{2} \left(\sum_{i \in \{s,t\}} |V_i|(n - |V_i| + 2) + \sum_{i \in [4] \setminus \{s,t\}} |V_i|(n - |V_i| + 3) \right) \\ &\le t_4(n) + \frac{10}{8} n + O(\sqrt{n}), \end{aligned}$$

a contradiction to (18). □

Next, we may assume that $|V_i| < |V_{i+1}|$ for $i \in [3]$. If n is even, then each $|V_i|$ is even and every $G_1^*[V_i]$ is 3-regular by Lemma 4.3. Thus,

$$e(G_1^*) = \frac{3}{2} n. \tag{21}$$

Note that $|V_{i+1}| \ge |V_i| + 2$ for $i \in [3]$. Thus,

$$|V_1| \le n/4 - 3. \tag{22}$$

For the case $n = 8k$, we have $|V_1| \le 2k - 3$ by (22), and then $|V_1| \le 2k - 4$ as $|V_1|$ is even. We claim that $|V_1| = 2k - 4$. Since otherwise, $|V_1| \le 2k - 6$ as $|V_1|$ is even. It follows from (21) that

$$\begin{aligned} e(G^*) = e(G_1^*) + e(G_2^*) &= \frac{3}{2} n + \sum_{1 \le i < j \le 4} |V_i||V_j| \\ &\le \frac{3}{2} n + |V_1|(n - |V_1|) + t_3(n - |V_1|) \\ &\le \frac{3}{2} n + |V_1|(n - |V_1|) + \frac{(n - |V_1|)^2}{3} \\ &\le \frac{3}{2} n + t_4(n) - 24, \end{aligned}$$

a contradiction to (18). Thus, $|V_2| + |V_3| + |V_4| = 6k + 4$. It follows from $|V_{i+1}| \ge |V_i| + 2$ that $|V_2| = 2k - 2$. Note that

$$e(G_2^*) = \sum_{1 \le i < j \le 4} |V_i||V_j| = 20k^2 - 12k - 28 + |V_3|(4k + 6 - |V_3|).$$

To maximum $e(G_2^*)$, we have $|V_3| = 2k+2$ and $|V_4| = 2k+4$ as $|V_3|$ and $|V_4|$ are even. Thus, G^* is exactly the graph $K^*_{2k-4,2k-2,2k+2,2k+4}$ given in Table 1. For the remaining cases of even n, analogous arguments are given in Appendix.

If n is odd, then exactly one part V_t has odd vertices for some $t \in [4]$ by Lemma 4.3. Thus,

$$e(G_1^*) = \frac{2|V_t|}{2} + \frac{3(n-|V_t|)}{2} = \frac{3n-|V_t|}{2}. \tag{23}$$

For the case $n = 8k+1$, we consider $t = 1$ firstly. Then

$$|V_2| - |V_1| \geq 3 \tag{24}$$

by Lemmas 4.2 and 4.3. Note that $|V_{i+1}| \geq |V_i| + 2$ for $i = 2, 3$. This together with (24) implies $|V_1| \leq 2k-5$. We claim that $|V_1| = 2k-5$. Since otherwise, $|V_1| \leq 2k-7$ as $|V_1|$ is odd. It follows from (23) that

$$\begin{aligned} e(G^*) &= e(G_1^*) + e(G_2^*) \\ &= \frac{3n-|V_1|}{2} + \sum_{1\leq i<j\leq 4} |V_i||V_j| \\ &\leq \frac{3n-|V_1|}{2} + |V_1|(n-|V_1|) + t_3(n-|V_1|) \\ &\leq \frac{11}{8}n + t_4(n) - \frac{251}{8}, \end{aligned}$$

a contradiction to (18). Thus, $|V_2| + |V_3| + |V_4| = 6k+6$. It follows from $|V_{i+1}| \geq |V_i| + 2$ for $i = 2, 3$ and (24) that $2k-1 \leq |V_2| \leq 2k$. If $|V_2| = 2k-2$, then

$$e(G_2^*) = \sum_{1\leq i<j\leq 4} |V_i||V_j| = 20k^2 - 10k + 10 + |V_3|(4k+8-|V_3|).$$

To maximum $e(G_2^*)$, we have $|V_3| = 2k+2$ and $|V_4| = 2k+6$. Thus, $e(G_2^*) = t_4(n) - 34$. This together with (23) implies that

$$\begin{aligned} e(G^*) &= e(G_1^*) + e(G_2^*) \\ &= \frac{3n-|V_1|}{2} + t_4(n) - 34 \\ &= \frac{11}{8}n + t_4(n) - \frac{251}{8}, \end{aligned}$$

a contradiction to (18). If $|V_2| = 2k$, then

$$e(G_2^*) = \sum_{1\le i<j\le 4} |V_i||V_j| = 20k^2 - 6k - 30 + |V_3|(4k+6-|V_3|).$$

To maximum $e(G_2^*)$, we have $|V_3| = 2k+2$ and $|V_4| = 2k+4$. Thus, $e(G_2^*) = t_4(n) - 30$. This together with (23) implies that

$$\begin{aligned} e(G^*) &= e(G_1^*) + e(G_2^*) \\ &= \frac{3n-|V_1|}{2} + t_4(n) - 30 \\ &= \frac{11}{8}n + t_4(n) - \frac{155}{8}, \end{aligned}$$

a contradiction to (18). By similar arguments given in Appendix, we have

$$e(G^*) = \begin{cases} t_4(n) + \frac{11}{8}n - \frac{139}{8} & \text{if } t = 2, \\ t_4(n) + \frac{11}{8}n - \frac{131}{8} & \text{if } t = 3, \\ t_4(n) + \frac{11}{8}n - \frac{123}{8} & \text{if } t = 4. \end{cases}$$

If $t = 2, 3$, this leads to a contradiction to (18); if $t = 4$, G^* is exactly the graph $K^*_{2k-4,2k,2k+2,2k+3}$ given in Table 1. The remainder cases of Theorem 1.4 are analogous and we prove them in Appendix.

Appendix

In this section, we give the details of other cases of Theorem 1.4.

Case 1. n is even. It suffices to give upper bound of $e(G_2^*)$ as $e(G_1^*) = 3n/2$. Note that $|V_{i+1}| \ge |V_i| + 2$ for $i \in [3]$. Thus, we have $|V_1| \le n/4 - 3$.

Subcase 1.1. $n = 8k+2$. In this case, we have $|V_1| \le 2k-4$. If $|V_1| \le 2k-6$, then

$$e(G_2^*) = \sum_{1\le i<j\le 4} |V_i||V_j| \le |V_1|(n-|V_1|) + \frac{(n-|V_1|)^2}{3} \le t_4(n) - 28.$$

Suppose $|V_1| = 2k-4$. It follows from $|V_{i+1}| \ge |V_i| + 2$ that $|V_2| \le 2k$. If $|V_2| = 2k-2$, then $e(G_2^*) = e(K_{2k-4,2k-2,2k+4,2k+4}) = t_4(n) - 25$. If $|V_2| = 2k$, then $e(G_2^*) = e(K_{2k-4,2k,2k+2,2k+4}) = t_4(n) - 17$.

Subcase 1.2. $n = 8k+4$. In this case, we have $|V_1| \le 2k-2$. If $|V_1| \le 2k-4$, then

$$e(G_2^*) \le |V_1|(n-|V_1|) + \frac{(n-|V_1|)^2}{3} \le t_4(n) - 18.$$

Suppose $|V_1| = 2k-2$. It follows from $|V_{i+1}| \geq |V_i| + 2$ that $|V_2| = 2k$. Thus, $e(G_2^*) = e(K_{2k-2,2k,2k+2,2k+4}) = t_4(n) - 10$.

Subcase 1.3. $n = 8k+6$. In this case, we have $|V_1| \leq 2k-2$. If $|V_1| \leq 2k-4$, then

$$e(G_2^*) \leq |V_1|(n - |V_1|) + \frac{(n-|V_1|)^2}{3} \leq t_4(n) - 20.$$

Suppose $|V_1| = 2k-2$. It follows from $|V_{i+1}| \geq |V_i| + 2$ that $|V_2| = 2k$. Thus, $e(G_2^*) = e(K_{2k-2,2k,2k+2,2k+6}) = t_4(n) - 17$.

Case 2. n is odd. Since there is exactly one part V_t with odd number of vertices for some $t \in [4]$, we have

$$e(G_1^*) = \frac{2|V_t|}{2} + \frac{3(n-|V_t|)}{2} = \frac{3n-|V_t|}{2}.$$

For $t \in [3]$, we have $|V_{t+1}| - |V_t| \geq 3$, and then

$$|V_4| \geq |V_3| + 2 \geq |V_2| + 4 \geq |V_1| + 7, \text{ for } t = 1; \tag{25}$$
$$|V_4| \geq |V_3| + 2 \geq |V_2| + 5 \geq |V_1| + 6, \text{ for } t = 2; \tag{26}$$
$$|V_4| \geq |V_3| + 3 \geq |V_2| + 4 \geq |V_1| + 6, \text{ for } t = 3; \tag{27}$$
$$|V_4| \geq |V_3| + 1 \geq |V_2| + 3 \geq |V_1| + 5, \text{ for } t = 4. \tag{28}$$

Subcase 2.1. $n = 8k+1$ and $t \in \{2,3,4\}$. In this case, we have $|V_1| \leq 2k-4$ by (26), (27) and (28). If $|V_1| \leq 2k-6$, then

$$e(G^*) \leq \frac{3n-|V_t|}{2} + |V_1|(n-|V_1|) + t_3(n-|V_1|) \leq \frac{11}{8}n + t_4(n) - \frac{187}{8},$$

where the last inequality follows from $|V_t| > |V_1|$. Thus, $|V_1| = 2k-4$.

For $t = 2$, it follows from (26) that $|V_2| \leq 2k-1$. Then

$$e(G^*) \leq \begin{cases} \frac{3n-|V_2|}{2} + e(K_{2k-4,2k-3,2k+2,2k+6}) = \frac{11}{8}n + t_4(n) - \frac{243}{8}, \\ \frac{3n-|V_2|}{2} + e(K_{2k-4,2k-1,2k+2,2k+4}) = \frac{11}{8}n + t_4(n) - \frac{139}{8}. \end{cases}$$

For $t = 3$, it follows from (27) that $|V_2| \leq 2k$. Then

$$e(G^*) \leq \begin{cases} \frac{3n-|V_3|}{2} + e(K_{2k-4,2k-2,2k+1,2k+6}) = \frac{11}{8}n + t_4(n) - \frac{227}{8}, \\ \frac{3n-|V_3|}{2} + e(K_{2k-4,2k,2k+1,2k+4}) = \frac{11}{8}n + t_4(n) - \frac{131}{8}. \end{cases}$$

For $t=4$, it follows from (28) that $|V_2| \leq 2k$. Then

$$e(G^*) \leq \begin{cases} \frac{3n-|V_4|}{2} + e(K_{2k-4,2k-2,2k+2,2k+5}) = \frac{11}{8}n + t_4(n) - \frac{211}{8}, \\ \frac{3n-|V_4|}{2} + e(K_{2k-4,2k,2k+2,2k+3}) = \frac{11}{8}n + t_4(n) - \frac{123}{8}. \end{cases}$$

Subcase 2.2. $n \neq 8k+1$ and $t=1$. In this case, we have $|V_1| \leq 2k-3$ by (25). If $|V_1| \leq 2k-5$, then

$$e(G^*) \leq \frac{3n-|V_1|}{2} + |V_1|(n-|V_1|) + \frac{(n-|V_1|)^2}{3} \leq \frac{11}{8}n + t_4(n) - \frac{153}{8}.$$

Thus, $|V_1| = 2k-3$. It follows from (25) that $|V_2| = 2k$. Thus,

$$e(G^*) \leq \begin{cases} \frac{3n-|V_1|}{2} + e(K_{2k-3,2k,2k+2,2k+4}) = \frac{11}{8}n + t_4(n) - \frac{89}{8}, & n = 8k+3, \\ \frac{3n-|V_1|}{2} + e(K_{2k-3,2k,2k+2,2k+6}) = \frac{11}{8}n + t_4(n) - \frac{151}{8}, & n = 8k+5, \\ \frac{3n-|V_1|}{2} + e(K_{2k-3,2k,2k+4,2k+6}) = \frac{11}{8}n + t_4(n) - \frac{165}{8}, & n = 8k+7. \end{cases}$$

Subcase 2.3. $n \neq 8k+1$ and $t \in \{2,3,4\}$. In this case, we have $|V_1| \leq 2k-2$ by (26), (27) and (28). If $|V_1| \leq 2k-4$, then

$$e(G^*) \leq \frac{3n-|V_t|}{2} + |V_1|(n-|V_1|) + \frac{(n-|V_1|)^2}{3} \leq \frac{11}{8}n + t_4(n) - \frac{105}{8}.$$

where the last inequality follows from $|V_t| > |V_1|$. Thus, $|V_1| = 2k-2$.

For $t=2$, it follows from (26) that $|V_2| = 2k-1$. Thus,

$$e(G^*) \leq \begin{cases} \frac{3n-|V_2|}{2} + e(K_{2k-2,2k-1,2k+2,2k+4}) = \frac{11}{8}n + t_4(n) - \frac{81}{8}, & n = 8k+3, \\ \frac{3n-|V_2|}{2} + e(K_{2k-2,2k-1,2k+2,2k+6}) = \frac{11}{8}n + t_4(n) - \frac{143}{8}, & n = 8k+5, \\ \frac{3n-|V_2|}{2} + e(K_{2k-2,2k-1,2k+4,2k+6}) = \frac{11}{8}n + t_4(n) - \frac{165}{8}, & n = 8k+7. \end{cases}$$

For $t=3$, it follows from (27) that $|V_2| = 2k$. Thus,

$$e(G^*) \leq \begin{cases} \frac{3n-|V_3|}{2} + e(K_{2k-2,2k,2k+1,2k+4}) = \frac{11}{8}n + t_4(n) - \frac{73}{8}, & n = 8k+3, \\ \frac{3n-|V_3|}{2} + e(K_{2k-2,2k,2k+1,2k+6}) = \frac{11}{8}n + t_4(n) - \frac{135}{8}, & n = 8k+5, \\ \frac{3n-|V_3|}{2} + e(K_{2k-2,2k,2k+3,2k+6}) = \frac{11}{8}n + t_4(n) - \frac{149}{8}, & n = 8k+7. \end{cases}$$

For $t=4$, it follows from (28) that $|V_2| = 2k$. Thus,

$$e(G^*) \leq \begin{cases} \frac{3n-|V_4|}{2} + e(K_{2k-2,2k,2k+2,2k+3}) = \frac{11}{8}n + t_4(n) - \frac{65}{8}, & n = 8k+3, \\ \frac{3n-|V_4|}{2} + e(K_{2k-2,2k,2k+2,2k+5}) = \frac{11}{8}n + t_4(n) - \frac{119}{8}, & n = 8k+5, \\ \frac{3n-|V_4|}{2} + e(K_{2k-2,2k,2k+4,2k+5}) = \frac{11}{8}n + t_4(n) - \frac{141}{8}, & n = 8k+7. \end{cases}$$

It is easy to check that the maximum value of $e(G^*)$ is achieved when $t = 4$.

Acknowledgements

The authors would like to thank the referees for their careful reading and helpful comments. This work is partially supported by NSFC (Grant No. 12071077, 12001106) and National Natural Science Foundation of Fujian Province (Grant No. 2021J05128).

References

[1] M. Albertson, Turán theorems with repeated degrees, *Discrete Math.* **100** (1992) 235–241. MR1172352

[2] A. Brouwer, Some lotto numbers from an extension of Turán's theorem, in: Afdeling Zuivere Wiskunde 152 (Stichting Mathematish Centrum, Amsterdam, 1981). MR0611510

[3] B. Bukh and M. Tait, Turán number of theta graphs, *Comb. Probab. Comput.* **29** (2020) 495–507. MR4132518

[4] Y. Caro and Zs. Tuza, Singular Ramsey and Turán numbers, *Theory Appl. Graphs* **6** (2019) 1–32. MR3915687

[5] Y. Caro and Zs. Tuza, Regular Turán numbers, *Australas. J. Combin.* **78** (2020) 133–144. MR4148510

[6] P. Erdős and M. Simonovits, A limit theorem in graph theory, *Studia Sci. Math. Hungar.* **1** (1966) 51–57. MR0205876

[7] Z. Füredi and M. Simonovits, The history of the degenerate (bipartite) extremal graph problems, *Bolyai Soc. Math. Stud.* **25** (2013) 169–264. MR3203598

[8] D. Gerbner, B. Patkós, M. Vizer and Zs. Tuza, Singular Turán numbers and WORM-colorings, *Discuss. Math. Graph Theory*, https://doi.org/10.7151/dmgt.2335. MR4445840

[9] W. T. Gowers and B. Janzer, Generalizations of the Ruzsa-Szemerédi and rainbow Turán problems for cliques, *Combin. Probab. Comput.* **30** (2021) 591–608. MR4272854

[10] J. He, J. Ma and T. Yang, Some extremal results on 4-cycles, *J. Combin. Theory Ser. B* **149** (2021) 92–108. MR4208020

[11] D. Y. Kang, J. Kim and H. Liu, On the rational Turán exponents conjecture, *J. Combin. Theory Ser. B* **148** (2021) 149–172. MR4200749

[12] M. Liu, K. Xu and X. Zhang, Extremal graphs for vertex-degree-based invariants with given degree sequences, *Discrete Appl. Math.* **255** (2019) 267–277. MR3926345

[13] J. Ma and Y. Qiu, Some sharp results on the generalized Turán numbers, *European J. Combin.* **84** (2020) 103026. MR4014346

[14] B. Wu and Y. Peng, Two extremal problems related to orders, *J. Comb. Optim.* **35** (2018) 588–612. MR3761255

[15] L. Yuan, Extremal graphs for the k-flower, *J. Graph Theory* **89** (2018) 26–39. MR3828126

Gaoxing Sun
Center for Discrete Mathematics
Fuzhou University
Fuzhou
China
E-mail: gaoxing_sun@sina.com

Heng Li
Center for Discrete Mathematics
Fuzhou University
Fuzhou
China
E-mail: liheng0110@yeah.net

Qinghou Zeng
Center for Discrete Mathematics
Fuzhou University
Fuzhou
China
E-mail: zengqh@fzu.edu.cn

Jianfeng Hou
Center for Discrete Mathematics
Fuzhou University
Fuzhou
China
E-mail: jfhou@fzu.edu.cn