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Span of restriction of Hilbert theta functions

Gabriele Bogo and Yingkun Li

Abstract: In this paper, we study the diagonal restrictions of certain Hilbert theta series for a totally real field F, and prove that they span the corresponding space of elliptic modular forms when the F is quadratic or cubic. Furthermore, we give evidence of this phenomenon when F is quartic, quintic and sextic.

1. Introduction

Theta functions are classical examples of holomorphic modular forms. Given a positive definite, unimodular $\mathbb{Z}$-lattice L of rank $8m$ with $m \in \mathbb{N}$, the associated theta function

$$\theta_L(\tau) := \sum_{\lambda \in L} q^{Q(\lambda)}, \ q := \mathbf{e}(\tau) := e^{2\pi i \tau}, \tag{1}$$

is in M_{4m}, the space of elliptic modular forms of weight $4m$ on $\mathrm{SL}_2(\mathbb{Z})$. For example, the theta functions associated to the E_8 lattice and Leech lattice Λ_{24} are explicitly given as

$$\theta_{E_8}(\tau) = E_4(\tau), \ \theta_{\Lambda_{24}}(\tau) = E_4(\tau)^3 - 720\Delta(\tau), \tag{2}$$

where $E_{2k}(\tau)$ is the Eisenstein series of weight $2k$ and $\Delta(\tau)$ is the Ramanujan Δ-function.

For $N \in \mathbb{N}$, we denote

$$\mathcal{M}_{\mathbb{Q}}^{(N)} := \bigoplus_{k \in \mathbb{N}} M_{Nk} \tag{3}$$

the finitely generated graded algebra of elliptic modular forms with weights divisible by N, and would like to consider the subalgebra $\mathcal{M}_{\mathbb{Q}}^{\theta} \subset \mathcal{M}_{\mathbb{Q}}^{(4)}$ generated by theta functions of unimodular lattices. Using the relation

$$\theta_{L_1 \oplus L_2}(\tau) = \theta_{L_1}(\tau)\theta_{L_2}(\tau). \tag{4}$$

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for any two unimodular lattices L_1, L_2, we see that $\mathcal{M}_{\mathbb{Q}}^{\theta}$ is simply the span of such theta functions. Equation (2) and the fact $\mathcal{M}_{\mathbb{Q}}^{(4)} = \mathbb{C}[E_4, \Delta]$ imply that

$$\mathcal{M}_{\mathbb{Q}}^{\theta} = \mathcal{M}_{\mathbb{Q}}^{(4)}. \tag{5}$$

The construction of theta functions also extends to the case of Hilbert modular forms. Let F be a totally real field of degree d with ring of integers $\mathcal{O}_F$, and denote $\alpha_j \in \mathbb{R}$ the real embeddings of $\alpha \in F$ for $1 \le j \le d$. For $N \in \mathbb{N}$, denote $\mathcal{M}_F^{(N)}$ the algebra of holomorphic Hilbert modular forms of parallel weight Nk for $k \in \mathbb{N}$. Given a totally positive definite, $\mathbb{Z}$-unimodular $\mathcal{O}_F$-lattice L of rank $8m$ (see Definition 1), the associated theta function

$$\theta_L(\tau) := \sum_{\lambda \in L} \prod_{j=1}^{d} q_j^{Q(\lambda)_j}, \ \tau = (\tau_1, \dots, \tau_d) \in \mathbb{H}^d, \ q_j := \mathbf{e}(\tau_j), \tag{6}$$

is a Hilbert modular form of parallel weight $4m$ on $\mathrm{SL}_2(\mathcal{O}_F)$. It is well-known that such lattice exists precisely when

$$m \in \frac{1}{d_2}\mathbb{N}, \ d_2 := \gcd(2, d) \tag{7}$$

(see Prop. 2.1). As a result, the relationship between $\mathcal{M}_F^{(4/d_2)}$ and the subalgebra $\mathcal{M}_F^{\theta}$ generated by such θ_L is not clear.

On the other hand, we have the following diagonal restriction map

$$\begin{aligned} \mathcal{M}_F^{(N)} &\to \mathcal{M}_{\mathbb{Q}}^{(Nd)} \\ f &\mapsto f^{\Delta}(\tau) := f(\tau^{\Delta}), \end{aligned}$$

where $\tau^{\Delta} = (\tau, \dots, \tau) \in \mathbb{H}^d$. In this note, we will investigate the question about the image of $\mathcal{M}_F^{\theta}$ under this map, which is denoted by $(\mathcal{M}_F^{\theta})^{\Delta}$ and contained in $\mathcal{M}_{\mathbb{Q}}^{(4d/d_2)}$. The main result is as follows.

Theorem 1.1. *For a totally real field F of degree $d = 2, 3$, we have*

$$(\mathcal{M}_F^{\theta})^{\Delta} = \mathcal{M}_{\mathbb{Q}}^{(4d/d_2)}. \tag{8}$$

Based on this, it is then natural to make the following conjecture.

Conjecture 1. Equation (8) holds for any totally real field F of degree d.

To prove Theorem 1.1, we apply an instance of the Siegel-Weil formula to see that the Hecke Eisenstein series $E_{F,k}$ defined in (13) is contained in $\mathcal{M}_F^\theta$ for all $k \in (4/d_2)\mathbb{N}$. Then we calculate the Petersson inner product between the diagonal restriction of $E_{F,k}$ and an elliptic cusp form. For $d = 2$, this inner product is related to Fourier coefficients of half-integral weight modular forms by a result of Kohnen-Zagier [7]. For $d \geq 3$, we give an expression for this inner product in terms of a sum over the double coset $\Gamma_{F,\infty}\backslash\Gamma_F/\Gamma_\mathbb{Q}$ (see Prop. 3.1). When $d = 3$, we related this double coset to orders in a cubic field F (see Section 4). Using these results, we show that when $d = 2, 3$, $\mathcal{M}_\mathbb{Q}^{(4d/d_2)}$ can be generated by $E_{F,k}^\Delta$ and θ_L^Δ for a $\mathbb{Z}$-unimodular $\mathcal{O}_F$-lattice L.

The same approach can be used to check conjecture 1 numerically when $d \in \{4, 5, 6, 8, 10\}$. We list some results for $d = 4, 5, 6$ and F has small discriminants in the last section (see Theorem 6.1).

2. Preliminary

Let F be a totally real field of degree d with ring of integers $\mathcal{O}_F$ and different $\mathfrak{d}_F$. Denote $\mathrm{Cl}(F)$ the (wide) class group of F. Let (V, Q) be an F-quadratic space of dimension n. We say that V is *totally positive* if $V \otimes_{\iota(F)} \mathbb{R}$ is totally positive for every real embedding $\iota : F \hookrightarrow \mathbb{R}$. In that case, $\mathrm{SO}_V(\mathbb{R})$ is compact and the double quotient $\mathrm{SO}_V(F)\backslash\mathrm{SO}_V(\hat{F})/K$ is a finite set for any open compact subgroup $K \subset \mathrm{SO}_V(\hat{F})$. Here $\mathbb{A}_F$ and $\hat{F}$ are the adele and finite adele of F.

A finitely generated $\mathcal{O}_F$-module $L \subset V$ is called a ($\mathcal{O}_F$-)lattice if $L \otimes_{\mathcal{O}_F} F = V$. We denote $\hat{L} := L \otimes \hat{\mathbb{Z}} \subset \hat{V} = V \otimes \hat{\mathbb{Q}}$. If $Q(L) \subset \mathfrak{d}_F^{-1}$, we say that L is $\mathbb{Z}$-*even integral* and call the lattice

$$L' := \{y \in V : (y, L) \subset \mathfrak{d}_F^{-1}\} \tag{9}$$

its $\mathbb{Z}$-*dual*. Viewed as a $\mathbb{Z}$-lattice with respect to $Q_\mathbb{Q}(x) := \mathrm{tr}_{F/\mathbb{Q}} Q(x)$, such L is even integral with dual L'.

Definition 1. An $\mathcal{O}_F$-lattice L is said to be $\mathbb{Z}$-*unimodular* if $L' = L$.

As a convention, the trivial lattice in the trivial F-vector space is totally positive and $\mathbb{Z}$-unimodular. Consider the monoid

$$\mathcal{U}_F^+ := \{(L, Q) : L \text{ is an even } \mathbb{Z}\text{-unimodular } \mathcal{O}_F\text{-lattice and totally positive}\} \tag{10}$$

with respect to $\oplus$, and denote $\mathcal{U}_F^{+,n} \subset \mathcal{U}_F^+$ the subset of lattices of rank n. We first have the following result.

Proposition 2.1. *The set* $\mathcal{U}_F^{+,n}$ *is non-empty precisely when* $(8/d_2) \mid n$.

Proof. Satz 1 in [1] implies that there exists definite, unimodular $\mathcal{O}_F$-lattices in the sense loc. cit. if and only if $(8/d_2) \mid n$. Furthermore since n is even, all of the 2^d possible definite signatures will appear in the set of definite, unimodular $\mathcal{O}_F$-lattices of rank n. One can then use the fact that the class $\mathfrak{d}_F$ in the class group is a square to translate this result to the existence $\mathbb{Z}$-unimodular lattices. (see the proof of Prop. 2.5 in [10] for details). ☐

Remark 2.2. For $L \in \mathcal{U}_F^{+,n}$ and $h \in \mathrm{SO}_V(\hat{\mathbb{Q}})$ with $V = L \otimes_{\mathcal{O}_F} F$, the lattice

$$h \cdot L := (h \cdot \hat{L}) \cap V \subset V \tag{11}$$

is also in $\mathcal{U}_F^{+,n}$.

For each $L \in \mathcal{U}_F^{+,n}$, let $\theta_L(\tau)$ be the associated theta function defined in (6). It is a Hilbert modular form of parallel weight $n/2$ for $\mathrm{SL}_2(\mathcal{O}_F)$. Now, the Siegel-Weil formula [14, 17] gives us the following result.

Proposition 2.3. *Let* F *be a totally real field of degree* d. *Then*

$$\int_{\mathrm{SO}_V(F)\backslash \mathrm{SO}_V(\mathbb{A}_F)/\mathrm{SO}_V(\mathbb{R})} \theta_{h\cdot L}(\tau) dh = \kappa E_{F,n/2}(\tau), \tag{12}$$

for some positive constant κ, *where* $E_{F,k}$ *is the Hecke Eisenstein series of parallel weight* k *defined by*

(13)

$$E_{F,k}(\tau) := 1 + \zeta_F(k)^{-1} \sum_{\mathcal{A}=[\mathfrak{a}]\in \mathrm{Cl}(F)} \mathrm{Nm}(\mathfrak{a})^k \sum_{(c,d)\in \mathfrak{a}^2/\mathcal{O}_F^\times,\ c\neq 0} \prod_{j=1}^{d} (c_j\tau_j + d_j)^{-k}$$

In particular, $E_{F,k} \in \mathcal{M}_F^\theta$ *for all* $k \in (4/d_2)\mathbb{N}$.

Remark 2.4. The Hecke Eisenstein series have the well-known Fourier expansion (see [15, 19])

$$E_{F,k}(\tau) = 1 + \frac{2^d}{\zeta_F(1-k)} \sum_{t\in \mathfrak{d}_F^{-1},\ t\gg 0} \sigma_{k-1}(t\mathfrak{d}_F) \prod_{j=1}^{d} q_j^{t_j\tau_j} \tag{14}$$

with $\sigma_r(\mathfrak{a}) := \sum_{\mathfrak{b}|\mathfrak{a},\ \mathfrak{b}\subset \mathcal{O}_F} \mathrm{Nm}(\mathfrak{b})^r$ for any integral ideal $\mathfrak{a}$ and $r \in \mathbb{N}$.

Proof. By the Siegel-Weil formula, the left hand side of (12) equals to the Eisenstein series

$$E_L(\tau) = v^{-n/4} \sum_{\gamma \in B(F)\backslash \mathrm{SL}_2(F)} \Phi_L(\gamma g_\tau, n/2 - 1),$$

where $B \subset \mathrm{SL}_2$ is the standard Borel subgroup, and Φ_L is the Siegel-Weil section associated to the lattice L (see e.g. [8, section I.3]). For $t \in F^\times$, the t-th Fourier coefficient of E_L is given by

$$\prod_{\mathfrak{p} < \infty} W_{t,\mathfrak{p}}(1, n/2 - 1, \Phi_{L,\mathfrak{p}})$$

up to constant independent of t. Here $W_{t,\mathfrak{p}}(g, s, \phi)$ is the local Whittaker function (see e.g. [18]). Since L is $\mathbb{Z}$-unimodular, the local lattice $L \otimes \mathcal{O}_{F,\mathfrak{p}}$ in $V \otimes F_\mathfrak{p}$ is self-dual for every finite place $\mathfrak{p}$. Standard calculations (see e.g. [9]) then gives us

$$W_{t,\mathfrak{p}}(1, s, \Phi_{L,\mathfrak{p}}) = \sum_{m=0}^{\mathrm{ord}_\mathfrak{p}(t\mathfrak{d}_{F_\mathfrak{p}})} \mathrm{Nm}(\mathfrak{p})^s$$

when $t \in \mathfrak{d}_{F_\mathfrak{p}}^{-1}$, and zero otherwise. So up to a constant, the Eisenstein series E_L and $E_{F,n/2}$ have the same non-constant term Fourier coefficients, hence agree. Now the left hand side of (12) is just a sum of θ_{L_j} over certain $L_j \in \mathcal{U}_F^{+,n}$ by Remark 2.2. Combining this with Prop. 2.1 finishes the proof. □

We can rewrite the Hecke-Eisenstein series $E_{F,k}$ as

$$E_{F,k}(\tau) := 1 + \sum_{\substack{\mathcal{A}=[\mathfrak{a}]\in \mathrm{Cl}(F) \\ (c,d)\in \mathfrak{a}^2/\mathcal{O}_F^\times \\ c\neq 0 \\ \mathcal{O}_F c + \mathcal{O}_F d = \mathfrak{a}}} \left(\frac{\mathrm{Nm}(\mathfrak{a})}{\mathrm{Nm}(c)}\right)^k \prod_{j=1}^d (\tau_j + d_j/c_j)^{-k}$$

For any $\beta \in F$, there is unique $\mathcal{A} = [\mathfrak{a}]$ and $(c, d) \in \mathfrak{a}^2/\mathcal{O}_F^\times$ with $c \neq 0$ such that $\mathfrak{a} = \mathcal{O}_F c + \mathcal{O}_F d$ and $\beta = d/c$. Therefore, we denote

$$A_\beta := \frac{\mathrm{Nm}(c)}{\mathrm{Nm}(\mathfrak{a})} \in \mathbb{Z} - \{0\}. \tag{15}$$

It is easy to check this definition does not depend on the choice of the representative $\mathfrak{a}$, and

$$A_{\beta+a,k} = A_{\beta,k} \tag{16}$$

for all $a \in \mathbb{Z}$. Then we have

$$E_{F,k}(\tau) = 1 + \sum_{\beta \in F} A_\beta^{-k} \prod_{j=1}^{d} (\tau_j + \beta_j)^{-k}. \tag{17}$$

3. Petersson inner product calculations

In this section, let $F/\mathbb{Q}$ be totally real with degree $d \geq 3$. We will give an expression for the Petersson inner product between the diagonal restriction of the Hecke Eisenstein series $E_{F,k}$ and an elliptic cusp form f of weight dk.

For $\alpha \in M_{m,n}(F)$ and $1 \leq j \leq d$, we write $\alpha_j \in M_{m,n}(\mathbb{R})$ with $1 \leq j \leq d$ for the real embeddings of α. We identify $\mathbb{P}^1(F) \cong B(F)\backslash \mathrm{SL}_2(F)$ via

$$\beta \mapsto \begin{cases} \left(\begin{smallmatrix} * & * \\ 1 & \beta \end{smallmatrix}\right) & \beta \in F, \\ \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) & \beta = \infty. \end{cases} \tag{18}$$

Let $S_0 \cup \{\infty\} \subset \mathbb{P}^1(F)$ be a set of representatives of the double coset space

$$B(F)\backslash \mathrm{SL}_2(F)/\mathrm{SL}_2(\mathbb{Z})\,.$$

Then $S_0 \subset F - \mathbb{Q}$ and we can use (17) to express the diagonal restriction of $E_{F,k}$ as

$$E^{\Delta}_{F,k}(\tau) = E_{dk} + \sum_{\beta \in S_0} E_{F,k,\beta}(\tau), \tag{19}$$

where

$$E_{F,k,\beta}(\tau) := \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} A^{-k}_{-\gamma^{-1}\cdot(-\beta_j)} \prod_{j=1}^{d} (\tau - \gamma^{-1} \cdot (-\beta_j))^{-k}$$

with $\tau \in \mathbb{H}$. Note that E_{dk} is just the elliptic Eisenstein series of weight dk.

Let $f(\tau) = \sum_{n \geq 1} c_n q^n \in S_{dk}$ be a cusp form. We are interested in estimating its inner product with $E^{\Delta}_{F,k}$. By the usual unfolding process, we obtain

$$\begin{aligned} \langle E^{\Delta}_{F,k}, f \rangle &= \sum_{\beta \in S_0} \int_{\Gamma_\infty \backslash \mathbb{H}} E^{\infty}_{F,k,\beta}(\tau) \overline{f(\tau)} v^{dk} \frac{du dv}{v^2} \\ &= \sum_{\beta \in S_0} \int_0^\infty \sum_{n \geq 1} \overline{c_n} a_{F,k,\beta}(n, v) e^{-2\pi n v} v^{dk-1} \frac{dv}{v}, \end{aligned}$$

where $\Gamma_\infty := B(\mathbb{Q}) \cap \mathrm{SL}_2(\mathbb{Z})$ and

$$\begin{aligned} E^{\infty}_{F,k,\beta}(\tau) &:= \sum_{\gamma\in\Gamma_\infty} A^{-k}_{-\gamma^{-1}\cdot(-\beta)} \prod_{j=1}^{d} (\tau - \gamma^{-1}\cdot(-\beta_j))^{-k} \\ &= 2A_\beta^{-k} \prod_{j=1}^{d} (\tau+\beta_j+b)^{-k} = \sum_{n\in\mathbb{Z}} a_{F,k,\beta}(n,v)\mathbf{e}(nu). \end{aligned} \tag{20}$$

for $\beta = d/c \in S_0$. Here we have $r_{-\gamma\cdot(-\beta)} = r_\beta$ for all $\gamma \in \Gamma_\infty$ by (16). It is easy to see that

$$\begin{aligned} a_{F,k,\beta}(n,v) &= 2A_\beta^{-k} \int_{\mathbb{R}} \prod_{j=1}^{d} (u+iv+\beta_j)^{-k} \mathbf{e}(-nu)du \\ &= 4\pi i(-A_\beta)^{-k} \sum_{z\in Z(\beta)} \mathrm{Res}_{x=z} \left(\mathbf{e}(nx) \prod_{j=1}^{d} (x-(\beta_j+iv))^{-k} \right), \end{aligned} \tag{21}$$

where $Z(\beta) := \{\beta_j + iv : 1 \le j \le d\} \subset \mathbb{H}$ since

$$\sum_{z\in Z(\beta)} \mathrm{Res}_{x=z} \left(\mathbf{e}(nx) \prod_{j=1}^{d} (x-z_j)^{-k} \right) = \frac{1}{2\pi i} \int_{\mathbb{R}} \mathbf{e}(nx) \prod_{j=1}^{d} (x-z_j)^{-k} dx. \tag{22}$$

Suppose β_j's are all distinct. Then

$$\begin{aligned} &\sum_{z\in Z(\beta)} \mathrm{Res}_{x=z} \left(\mathbf{e}(nx) \prod_{j=1}^{d} (x-(\beta_j+iv))^{-k} \right) \\ &= \frac{1}{\Gamma(k)} \sum_{j=1}^{d} \left(\frac{d}{dx}\right)^{k-1} \left(\frac{\mathbf{e}(nx)}{\prod_{j'=1,\ j'\neq j}^{d} (x-(\beta_{j'}+iv))^k} \right) |_{x=\beta_{j'}+iv} \\ &= \frac{\mathbf{e}(niv)}{\Gamma(k)} \sum_{j=1}^{d} \sum_{\ell=0}^{k-1} \frac{\mathbf{e}(n\beta_j)e^{-2\pi nv}}{(2\pi i n)^{k-1-\ell}} \binom{k-1}{\ell} \left(\frac{P_{d-1,k,\ell}}{Q_{d-1,k+\ell}} \right) (\beta_j-\beta_1,\dots,\beta_j-\beta_d), \end{aligned}$$

where $P_{m,k,\ell}, Q_{m,r} \in \mathbb{Q}[x_1,\dots,x_m]$ are symmetric polynomials of degrees $(m-1)\ell$ and mr defined by

$$\begin{aligned} P_{m,k,\ell}(x_1,\dots,x_m) &:= (x_1\dots x_m)^{k+\ell} (\partial_{x_1}+\dots+\partial_{x_m})^{\ell} (x_1\dots x_m)^{-k}, \\ Q_{m,r}(x_1,\dots,x_m) &:= (x_1\dots x_m)^r. \end{aligned} \tag{23}$$

Note that

$$\frac{P_{m,k,\ell}}{Q_{m,k+\ell}}(x_1,\dots,x_m) = (-1)^\ell \ell! \sum_{r=(r_j)\in\mathbb{N}^m,\ \sum_j r_j=\ell} \left(\!\!\binom{k}{r}\!\!\right) \prod_{j=1}^m x_j^{-k-r_j}, \tag{24}$$

where $\left(\!\binom{k}{r}\!\right) := \frac{k^{(r_1)}\dots k^{(r_m)}}{r_1!\dots r_m!}$ for $r = (r_1,\dots,r_m) \in \mathbb{N}^m$ with $k^{(n)} := k(k+1)\dots(k+n-1)$. Substituting this into the unfolding gives us the following result.

Proposition 3.1. *Suppose F is a totally real field of degree $d \geq 3$ and there is no intermediate field between F and $\mathbb{Q}$. For any $k \in 2\mathbb{N}$ and $f(\tau) = \sum_{n\geq 1} c(n)q^n \in S_{dk}$, we have*

$$\begin{aligned} \langle E^{\Delta}_{F,k}, f\rangle = {} & \frac{i\Gamma(dk-1)}{(4\pi)^{dk-2}\Gamma(k)} \sum_{\ell=0}^{k-1} (2\pi i)^{k-1-\ell} \sum_{\beta\in S_0} A_\beta^{-k} \\ & \times \sum_{j=1}^{d} \left(\frac{P_{d-1,k,\ell}}{Q_{d-1,k+\ell}}\right)(\beta_j-\beta_1,\dots,\beta_j-\beta_{j-1},\beta_j-\beta_{j+1},\dots,\beta_j-\beta_d) \\ & \times \sum_{n\geq 1} \frac{\mathbf{e}(n\beta_j)\overline{c_n}}{n^{(d-1)k+\ell}}, \end{aligned} \tag{25}$$

where the polynomials $P_{m,k,\ell}$ and $Q_{m,r}$ are defined in (23).

Remark 3.2. The condition that there is no intermediate field between F and $\mathbb{Q}$ implies that $\beta_i = \beta_j$ if and only if $i = j$ for all $\beta \in F - \mathbb{Q}$. A similar but more complicated formula for the inner product can be derived without this condition.

Example 3.3. Let $d = 3$ and $k = 2$. Then

$$\frac{P_{d-1,k,\ell}}{Q_{d-1,k+\ell}}(x,y) = \begin{cases} 1/(xy)^2, & \ell = 0, \\ -2(x+y)/(xy)^3, & \ell = 1. \end{cases}$$

Set $\gamma_1 := \beta_2 - \beta_3, \gamma_2 := \beta_3 - \beta_1, \gamma_3 := \beta_1 - \beta_2$, we have

$$\begin{aligned} & \sum_{\ell=0}^{k-1} (2\pi i n)^{k-1-\ell} \sum_{j=1}^{d} \frac{P_{d-1,k,\ell}}{Q_{d-1,k+\ell}}(\beta_j-\beta_1,\dots,\beta_j-\beta_d)\mathbf{e}(n\beta_j) \\ & = \left(\frac{2\pi i n}{(\gamma_2\gamma_3)^2} + \frac{2(\gamma_3-\gamma_2)}{(\gamma_2\gamma_3)^3}\right)\mathbf{e}(n\beta_1) + \left(\frac{2\pi i n}{(\gamma_1\gamma_3)^2} + \frac{2(\gamma_1-\gamma_3)}{(\gamma_1\gamma_3)^3}\right)\mathbf{e}(n\beta_2) \end{aligned}$$

$$+\left(\frac{2\pi in}{(\gamma_1\gamma_2)^2}+\frac{2(\gamma_2-\gamma_1)}{(\gamma_1\gamma_2)^3}\right)\mathbf{e}(n\beta_3).$$

For $d=3$ and $k-1\geq\ell\geq 0$, we can write explicitly

$$\begin{aligned}&\sum_{j=1}^{d}\frac{P_{d-1,k,\ell}}{Q_{d-1,k+\ell}}(\beta_j-\beta_1,\ldots,\beta_j-\beta_d)\mathbf{e}(n\beta_j)\\&=\frac{P_{2,k,\ell}}{Q_{2,k+\ell}}(\gamma_3,-\gamma_2)\mathbf{e}(n\beta_1)+\frac{P_{2,k,\ell}}{Q_{2,k+\ell}}(-\gamma_3,\gamma_1)\mathbf{e}(n\beta_2)\\&+\frac{P_{2,k,\ell}}{Q_{2,k+\ell}}(-\gamma_2,-\gamma_1)\mathbf{e}(n\beta_3).\end{aligned}$$

Using the inequalities $k^{(a)}k^{(b)}\leq k^{(a+b)}$, $(x_1+x_2+x_3)^2\leq 3(x_1^2+x_2^2+x_3^2)$,

$$\sum_{\sigma\in S_3}x_{\sigma(1)}^a x_{\sigma(2)}^b x_{\sigma(3)}^c\leq\frac{a!b!c!}{(a+b+c)!}(x_1+x_2+x_3)^{a+b+c},\ x_i,a,b,c\geq 0\tag{26}$$

and Equation (24), we obtain the bound

$$\begin{aligned}&\left|\sum_{j=1}^{d}\frac{P_{d-1,k,\ell}}{Q_{d-1,k+\ell}}(\beta_j-\beta_1,\ldots,\beta_j-\beta_d)\mathbf{e}(n\beta_j)\right|\\&\leq\left|\frac{P_{2,k,\ell}}{Q_{2,k+\ell}}(\gamma_3,-\gamma_2)\right|+\left|\frac{P_{2,k,\ell}}{Q_{2,k+\ell}}(-\gamma_3,\gamma_1)\right|+\left|\frac{P_{2,k,\ell}}{Q_{2,k+\ell}}(-\gamma_2,-\gamma_1)\right|\\&\leq\frac{\ell!}{|\gamma_1\gamma_2\gamma_3|^{k+\ell}}\sum_{a+b=\ell}\frac{k^{(a)}k^{(b)}}{a!b!}\left(|\gamma_1^b\gamma_2^a\gamma_3^{k+\ell}|+|\gamma_2^b\gamma_3^a\gamma_1^{k+\ell}|+|\gamma_3^b\gamma_1^a\gamma_2^{k+\ell}|\right)\\&\leq\ell!\frac{(|\gamma_1|+|\gamma_2|+|\gamma_3|)^{k+2\ell}}{|\gamma_1\gamma_2\gamma_3|^{k+\ell}}\frac{(k+\ell)!}{(k+2\ell)!}\frac{\ell+1}{2}k^{(\ell)}\\&\leq\frac{\binom{k-1+\ell}{\ell}}{\binom{k+2\ell}{\ell}}(\ell+1)!\frac{3^{k/2+\ell}}{2}\frac{(\gamma_1^2+\gamma_2^2+\gamma_3^2)^{k/2+\ell}}{|\gamma_1\gamma_2\gamma_3|^{k+\ell}}.\end{aligned}$$

4. Double coset and binary cubic forms

When $d=3$, we can identify the double coset $B(F)\backslash\mathrm{SL}_2(F)/\mathrm{SL}_2(\mathbb{Z})-\{\infty\}$ with orders in $\mathcal{O}_F$ in the following way. Let $f(X,Y)=AX^3+BX^2Y+CXY^2+DY^3$ and

$$\mathcal{Q}_F:=\{f(X,Y)\in\mathbb{Z}[X,Y]:f(\beta,1)=0\text{ for some }\beta\in F\backslash\mathbb{Q}\}$$

be the set of integral binary cubic forms with a root in $F-\mathbb{Q}$. A form is primitive if its coefficients have no common factor. There is a natural action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{Q}_F$ that preserves the discriminant

$$\begin{aligned}\Delta(f) &:= A^6((\beta_1-\beta_2)(\beta_1-\beta_3)(\beta_2-\beta_3))^2 \\ &= 18ABCD + B^2C^2 - 4AC^3 - 4B^3D - 27A^2D^2,\end{aligned} \tag{27}$$

and the subset of primitive forms. The quantity

$$P(f) := B^2 - 3AC > 0 \tag{28}$$

is the leading coefficient of the Hessian of f, which is a positive definite quadratic form and a coinvariant of f. For every $f \in \mathcal{Q}_F$, Prop. 2 in [4] gives us $f' \sim_{\mathrm{SL}_2(\mathbb{Z})} f$ satisfying

$$P(f') \le \sqrt{\Delta(f')} = \sqrt{\Delta(f)}. \tag{29}$$

Given $\beta \in F-\mathbb{Q}$, we can associate to it a primitive element $f_\beta \in \mathcal{Q}_F$ defined by

$$f_\beta(X,Y) := A_\beta \prod_{j=1}^{3}(X-\beta_j Y) = A_\beta X^3 + B_\beta X^2Y + C_\beta XY^2 + D_\beta Y^3 \in \mathcal{Q}_F. \tag{30}$$

Note that $f_\beta(\beta,1)=0$ and the right action of $\mathrm{SL}_2(\mathbb{Z})$ on $B(F)\backslash\mathrm{SL}_2(F)$ corresponds to its natural action on $\mathcal{Q}_F$.

To any binary cubic form f with non-zero discriminant and $f(\beta,1)=0$ we can associate the free $\mathbb{Z}$-module of rank 3

$$\mathcal{O}_f := \mathbb{Z} + \mathbb{Z}A\beta + \mathbb{Z}(A\beta^2 + B\beta + C) \subset \mathbb{Q}(\beta), \tag{31}$$

which is also a commutative ring. A classical result of Delone and Faddeev tells us that this gives a bijection between $\mathrm{GL}_2(\mathbb{Z})$-classes of binary cubic forms with non-zero discriminants and isomorphism classes of commutative rings that are free $\mathbb{Z}$-modules of rank 3 [6]. If we restrict β to be in a fixed field F, then $\mathcal{O}_f$ is an order in $\mathcal{O}_F$, and $\mathcal{O}_{f_1}, \mathcal{O}_{f_2} \subset \mathcal{O}_F$ are the same if and only if $f_1, f_2 \in \mathcal{Q}_F$ are $\mathrm{GL}_2(\mathbb{Z})$-equivalent (see e.g. [12, Lemma 3.1]). Furthermore, we have

$$\Delta(f) = \Delta(\mathcal{O}_f) = D_F[\mathcal{O}_F : \mathcal{O}_f]^2 \tag{32}$$

with $\Delta(\cdot)$ the discriminant. For $s = [\beta] \in \mathbb{P}^1(F)/\mathrm{SL}_2(\mathbb{Z}) - \{\infty\}$, we then denote

$$\mathcal{O}_s := \mathcal{O}_{f_\beta}, \Delta(s) := \Delta(\mathcal{O}_s). \tag{33}$$

The discussions above lead to the following result.

Proposition 4.1. *The map*

$$\begin{gathered}\mathbb{P}^1(F)/\mathrm{SL}_2(\mathbb{Z}) - \{\infty\} \to \{\mathcal{O} : \mathcal{O} \subset \mathcal{O}_F \textit{ is an order}\}/\cong \\ s \mapsto \mathcal{O}_s\end{gathered}$$

is well-defined and $(2|\mathrm{Aut}(\mathcal{O}_F)|)$*-to-1.*

Remark 4.2. The quantity $|\mathrm{Aut}(\mathcal{O}_F)|$ is either 3 or 1 depending on $F/\mathbb{Q}$ is Galois or not.

Finally, the following Dirichlet series

$$\eta_F(s) := \sum_{\mathcal{O}\subset\mathcal{O}_F \text{ order}} [\mathcal{O}_F : \mathcal{O}]^{-s} = \sum_{\mathcal{O}\subset\mathcal{O}_F \text{ order}} \frac{D_F^{s/2}}{\Delta(\mathcal{O})^{s/2}}. \tag{34}$$

can be factorized in the following way by a result of Datskovsky and Wright [5] (see [12, Lemma 3.2])

$$\eta_F(s) = \frac{\zeta_F(s)}{\zeta_F(2s)}\zeta(2s)\zeta(3s-1). \tag{35}$$

5. Proof of Theorem 1.1

We are now ready to prove Theorem 1.1. The cases of $d = 2, 3$ are proved separately.

Proof of Theorem 1.1 for $d = 2$. For $k = 2, 4$, the space M_{2k} is 1-dimensional and spanned by the Eisenstein series E_{2k}. Since θ_L^Δ is non-trivial for any $L \in \mathcal{U}_F^+$, the claim follows in these two base cases as $M^\theta_{F,k}$ is non-trivial by Prop. 2.1 (see also [13] for an explicit construction). More generally, we know that $\mathcal{M}^{(4)}_{\mathbb{Q}} = \mathbb{Q}[E_4, \Delta]$. Therefore, it suffices to show that $\Delta \in S_{12}$ is in $(M^\theta_{F,6})^\Delta$. As M_{12} is 2-dimensional and

$$E_4^3 = E_{12} + \frac{432000}{691}\Delta \in (M^\theta_{F,6})^\Delta, \tag{36}$$

we just need to produce a form $f \in (M^{\theta}_{F,6})^{\Delta}$ linearly independent from E_4^3. For this purpose, we apply Prop. 2.3 with $k = 6$ to get

$$f(\tau) := (E^{\Delta}_{F,6})(\tau) = 1 + \frac{4}{\zeta_F(-5)} \sum_{m \geq 1} q^m \sum_{\nu \in \mathfrak{d}_F^{-1},\ \nu \gg 0,\ \mathrm{tr}(\nu) = m} \sigma_5((\nu)\mathfrak{d}_F).$$

By Theorem 6 in [7], we know that

$$f = E_{12} - \frac{12}{691} \frac{c(D)}{\zeta_F(-5)} \Delta, \tag{37}$$

where $c(D)$ is the D-th Fourier coefficient of the half-integral weight form

$$g(\tau) = \sum_{D \in \mathbb{N}} c(D) q^D := \frac{1}{8\pi i}(2E_4(4\tau)\theta'(\tau) - E_4'(4\tau)\theta(\tau))$$

spanning the Kohnen plus space $S^+_{13/2}$. Now using the estimate $L(k, \chi_D) > 2 - \zeta(k)$ for $k \geq 2$ (see e.g. Equation (3) in [2]) we know that $\zeta_F(1-k) = D^{k-1/2} \frac{4\Gamma(k)^2}{(-4\pi)^k} \zeta_F(k)$ satisfies

$$|\zeta_F(-5)| > 0.01 \cdot D^{11/2}.$$

On the other hand, the Hecke bound for $c(D)$ yields

$$|c(D)| \leq c \cdot D^{13/4}, \ c := e^{2\pi} \max_{\tau \in \mathbb{H}} |g(\tau)| v^{13/4} < 10$$

Comparing with (36), it is clear that f and E_4^3 are linearly independent for all fundamental discriminant $D > 0$. This finishes the proof of Theorem 1.1 for $d = 2$. □

Using the calculation in Section 3 and the correspondence in Section 4, we can prove the following lemma.

Lemma 5.1. *For $d = 3, k \geq 3$ and $f(\tau) = \sum_{n \geq 1} c_f(n) q^n \in S_{3k}$, let $c_f > 0$ be a constant such that*

$$|c_f(n)| \leq c_f \cdot n^{3k/2}$$

for all $n \geq 1$. Then we have the bound

$$|\langle E^{\Delta}_{F,k}, f \rangle| \leq C_k c_f D_F^{-k/4} \tag{38}$$

for all cubic field F, with $C_k := 6c_k \frac{\zeta(k/2)^3}{\zeta(k)^2} \zeta(3k/2 - 1)$ and the constant c_k given in (39).

Proof. Let $a_k := \frac{\Gamma(3k-1)}{\Gamma(k)}(4\pi)^{2-3k}$. For $\beta \in S_0 \subset F$, recall that f_β is the binary cubic form associated to it in (30), which has coefficients $A_\beta, B_\beta, C_\beta, D_\beta$. Using (25), the estimate in Example 3.3 and (29), we obtain the bound

$$\begin{aligned}
|\langle E^{\Delta}_{F,k}, f\rangle| &\le a_k \sum_{\ell=0}^{k-1}(2\pi)^{k-1-\ell}\sum_{n\ge 1}\frac{|c_f(n)|}{n^{2k+\ell}} \\
&\quad \times \sum_{\beta\in S_0} A_\beta^{-k}\left|\sum_{j'=1}^{d}\frac{P_{d-1,k,\ell}}{Q_{d-1,k+\ell}}(\beta_{j'}-\beta_1,\dots,\beta_{j'}-\beta_d)\mathbf{e}(n\beta_{j'})\right| \\
&\le c_f\cdot a_k\sum_{\ell=0}^{k-1}(2\pi)^{k-1-\ell}\zeta(k/2+\ell)\frac{\binom{k-1+\ell}{\ell}}{\binom{k+2\ell}{\ell}}(\ell+1)!\frac{3^{k/2+\ell}}{2} \\
&\quad \times \sum_{\beta\in S_0} A_\beta^{-k}\frac{((\beta_1-\beta_2)^2+(\beta_2-\beta_3)^2+(\beta_3-\beta_1)^2)^{k/2+\ell}}{((\beta_1-\beta_2)^2(\beta_2-\beta_3)^2(\beta_3-\beta_1)^2)^{(k+\ell)/2}} \\
&\le 2^{-1}c_f\cdot a_k\sum_{\ell=0}^{k-1}(2\pi)^{k-1-\ell}\zeta(k/2+\ell)\frac{\binom{k-1+\ell}{\ell}}{\binom{k+2\ell}{\ell}}(\ell+1)!6^{k/2+\ell} \\
&\quad \times \sum_{\beta\in S_0}\frac{P(f_\beta)^{k/2+\ell}}{\Delta(f_\beta)^{(k+\ell)/2}} \\
&\le c_f\cdot c_k\sum_{\beta\in S_0}\Delta(f_\beta)^{-k/4} \le c_f\cdot c_k\cdot 2|\mathrm{Aut}(\mathcal{O}_F)|\cdot D_F^{-k/4}\eta_F\left(\tfrac{k}{2}\right).
\end{aligned}$$

Here the constant c_k is defined by

$$c_k := \frac{\Gamma(3k-1)}{2\Gamma(k)}(4\pi)^{2-3k}\sum_{\ell=0}^{k-1}(2\pi)^{k-1-\ell}\zeta(k/2+\ell)\frac{\binom{k-1+\ell}{\ell}}{\binom{k+2\ell}{\ell}}(\ell+1)!6^{k/2+\ell}. \tag{39}$$

For the last steps, we used Prop. 4.1. Combining this with (35) and applying $\zeta_F(s) \le \zeta(s)^3$ for $s > 1$, we have

$$\begin{aligned}
|\langle E^{\Delta}_{F,k}, f\rangle| &\le c_f c_k 2|\mathrm{Aut}(\mathcal{O}_F)|\frac{\zeta_F(\frac{k}{2})}{\zeta_F(k)}\zeta(k)\zeta(\tfrac{3k}{2}-1)D_F^{-k/4} \\
&\le 6c_f c_k\frac{\zeta(\frac{k}{2})^3}{\zeta(k)^2}\zeta(\tfrac{3k}{2}-1)D_F^{-k/4}
\end{aligned}$$

for $k \ge 3$. This finishes the proof. ∎

Remark 5.2. For $k = 4$, the bound above gives $C_4 < 5.79$. We can obtain a better bound by estimating the second to the last line in Example 3.3 case by case for each $\ell = 0, 1, 2, 3$, instead of using (26). The improved bound is

$$|\langle E^{\Delta}_{F,4}, f\rangle| \leq 0.067 c_f D_F^{-1}$$

for all totally real cubic field F.

Now we are ready to prove Theorem 1.1 in the cubic case.

Proof of Theorem 1.1 for $d = 3$. Since $\mathcal{M}^{(12)}_{\mathbb{Q}} = \mathbb{C}[E_{12}, \Delta]$, we only have to check that $(\mathcal{M}^{\theta}_{F})^{\Delta} \cap \mathcal{M}^{(12)}_{\mathbb{Q}}$ is 2 dimensional. For any $L \in \mathcal{U}^{+}_{F}$, the diagonal restriction θ^{Δ}_{L} is the theta function for a unimodular lattice P over $\mathbb{Z}$. So we know that $\theta_P \in (\mathcal{M}^{\theta}_{F})^{\Delta}$ for some Niemeier lattice P. To see that it is linearly independent from $E^{\Delta}_{F,4} = 1 + c(1)q + O(q^2)$, it suffices to show that $c(1)$ is not integral. We have checked this numerically for any cubic F with $D_F < 70000$.

More generally, we have

$$\theta_P = E_{12} + (N_2(P) - 65520/691)\Delta,$$

with $N_2(P)$ is the number of norm 2 vectors in P. From Table V in [3], we obtain a list of $N_2(P)$ and

$$|\langle \theta_P, \Delta\rangle| = |N_2(P) - 65520/691|\langle \Delta, \Delta\rangle > 1.22 \times 10^{-6}$$

for any Niemeier lattice P. On the other hand by taking $c_{\Delta} = 1$, the upper bound found in Lemma 5.1 and improved in Remark 5.2 gives us

$$|\langle E^{\Delta}_{F,4}, \Delta\rangle| \;<\; \frac{0.067}{D_F}.$$

So $E^{\Delta}_{F,4}$ and θ_P are linearly independent for $D_F \geq 60000$. This finishes the proof. □

6. Numerical evidence for Conjecture 1

In this section, we approach numerically Conjecture 1 in the case F is a totally real field of degree $d \in \{4, 5, 6\}$. For these choices of d the space $\mathcal{M}^{(4d/d_2)}_{\mathbb{Q}}$ can be in principle generated by the restriction of Eisenstein series and of (at most) one theta function θ_L of rank $8/d_2$. Conjecture 1 reduces then to the verification of the linear independence of θ^{Δ}_{L} and $E^{\Delta}_{F,4/d_2}$ for $d = 5, 6$, and of monomials in $\theta^{\Delta}_{L}, E^{\Delta}_{F,4d/d_2}$ and $E^{\Delta}_{F,k}$ in general for suitable weights k. This approach gives data supporting Conjecture 1 in the case $d = 4, 5$, and in the case $d = 6$ except for two fields F. Our result, for which evidence is given in this final section, is the following.

Theorem 6.1. *Conjecture 1 holds for*

1. $d = 4$ *and* $D_F \leq 10^5$;
2. $d = 5$ *and* $D_F \leq 2 \times 10^6$;
3. $d = 6$ *and* $D_F \leq 5 \times 10^6$ *except for the fields of discriminant* 453789 *and* 1397493.

6.1. A note on the computations

For $l, k \in \mathbb{Z}_{\geq 0}$, let σ_{k-1} be as in Remark 2.4 and define

$$s_l^F(k) \ := \sum_{\substack{\nu \in \mathfrak{d}_F^{-1} \\ \nu \gg 0 \\ \mathrm{tr}(\nu) = l}} \sigma_{k-1}\big((\nu)\mathfrak{d}_F\big)\,.$$

Then the diagonal restriction of $E_{F,k}$ has the following q-expansion at ∞ by (14)

$$E^{\Delta}_{F,k}(\tau) \ = \ 1 \ + \ \frac{2^d}{\zeta_K(1-k)} \sum_{l=0}^{\infty} s_l^F(k)\,. \tag{40}$$

We computed the first few coefficients of the above expansion with PARI/GP [16]. As (40) shows, this reduces to the determination of the functions $s_l^F(k)$ for small values of l (up to $l = 5$ in the case $d = 5$) and different values of k. The main difficulty is to find the totally positive $\nu \in \mathfrak{d}_F^{-1}$ of fixed trace l. Let $(\nu_1, \dots, \nu_d)$ be an integral basis for $\mathfrak{d}_F^{-1}$. Then any $\nu \in \mathfrak{d}_F^{-1}$ is of the form $\nu = v_1\nu_1 + \cdots + v_d\nu_d$ for $(v_1, \dots, v_d) \in \mathbb{Z}^d$ and conversely every vector in $\mathbb{Z}^d$ gives an element $\nu \in \mathfrak{d}_F^{-1}$. If $Q(x_1, \dots, x_d)$ denotes the quadratic form $x_1^2 + \dots x_d^2$, we have, for a totally positive $\nu \in \mathfrak{d}_F^{-1}$, that $Q(\sigma_1(\nu), \dots, \sigma_d(\nu)) < \mathrm{tr}(\nu)^2$. This implies that if $A = (\sigma_i(\nu_j))_{i,j}$ denotes the matrix of the real embeddings of the basis of $\mathfrak{d}_F^{-1}$, we can search the totally positive $\nu \in \mathfrak{d}_F^{-1}$ of fixed trace l among of vectors $v = (v_1, \dots, v_d) \in \mathbb{Z}^d$ satisfying

$$v^T(A^TA)v \ = \ Q(\nu) \ < \ l^2\,.$$

This gives a finite (but large as l and D_F grow) set of vectors on which we can perform the final search. Once the suitable $\nu \in \mathfrak{d}_F^{-1}$ have been determined, it is straightforward to compute $\sigma_k\big((\nu)\mathfrak{d}_F\big)$ for every value of k by using the basic PARI functions.

Remark 6.2. It is possible to investigate also the cases $d = 8, 10$ with the method outlined at the beginning of this section. For the case $d = 8$ we need to compute five coefficients of the q-expansion (40), while for $d = 10$ we need to compute six coefficients. This, together with the size of the discriminants of these fields ($D_F \geq 282300416$ for $d = 8$ and $D_F \geq 443952558373$ for $d = 10$), makes it hard to collect significant data in these cases.

6.2. Tables

d=4 Let F be a totally real field with $[F : \mathbb{Q}] = 4$. In this case, the proof of Conjecture 1 reduces to the statement that $\mathcal{M}_{\mathbb{Q}}^{(8)}$ is spanned by restrictions of Hilbert Eisenstein series on Γ_F. It is easy to see that $\{E_4^2, \Delta E_4, \Delta^2\}$ is a generating set for $\mathcal{M}_{\mathbb{Q}}^{(8)}$. By a dimension argument, $E_{F,2}^{\Delta} = E_4^2$. It follows that ΔE_4 and Δ^2 can be obtained by restriction of Eisenstein series on Γ_F respectively if the sets $\{E_{F,4}^{\Delta}, (E_{F,2}^{\Delta})^2\}$, and $\{(E_{F,2}^{\Delta})^3, E_{F,2}^{\Delta}E_{F,4}^{\Delta}, E_{F,6}^{\Delta}\}$ are both linearly independent.

In order to study this problem, we compute the restriction of $E_{F,k}$ for $k = 4, 6$. As bases for M_{16} and M_{24}, we choose $\{E_4^4, E_4\Delta\}$ and $\{E_4^6, E_4^3\Delta, \Delta^2\}$ respectively. We have

$$E_{F,4}^{\Delta} = E_4^4 \; + \; bE_4\Delta\,, \quad E_{F,6}^{\Delta} = E_4^6 \; - \; c_1E_4^3\Delta \; + \; c_2\Delta^2\,, \tag{41}$$

for some coefficients $b, c_1, c_2 \in \mathbb{Q}$ that depend on F. To prove Conjecture 1, it suffices to check that b and c_2 are both non-zero. We computed the coefficients b, c_1, c_2 for the first 30 totally real quartic fields F. The results are reported in Table 1. For these fields it is enough to specify the discriminant D_F to uniquely identify the field F (check the number field database [11]). This remark applies also for the fields we consider in the cases $d = 5, 6$.

It turns out that the numerical values of b, c_1, and c_2 are very close to $955, 1439$, and -129930 respectively. These numbers are related to the Eisenstein series of weight 16 and 24 since

$$E_{16} \;=\; E_4^4 + b(E_{16})E_4\Delta\,, \quad E_{24} \;=\; E_4^5 + c_1(E_{24})E_4^2\Delta + c_2(E_{24})\Delta^2,$$

with

$$b(E_{16}) = -\tfrac{3456000}{3617} \sim 955\,,\; c_1(E_{24}) = \tfrac{340364160000}{236364091} \sim 1439\,,$$
$$c_2(E_{24}) = -\tfrac{30710845440000}{236364091} \sim 129930\,.$$

In other words, it seems that the diagonal restriction of $E_{F,4}$ and $E_{F,6}$ are close to E_{16} and E_{24} respectively. In analogy with the proof of Theorem 1.1 in the

Table 1: $d = 4$

D_F	$E^{\Delta}_{F,4}$		$E^{\Delta}_{F,6}$	
	$-b$	$\lvert b - b(E_{16})\rvert$	$\lvert c_1 - c_1(E_{24})\rvert$	$\lvert c_2 - c_2(E_{24})\rvert$
725	$\frac{518400}{541}$	2.7375349	0.00050313732	25.886498
1125	$\frac{1209600}{1261}$	3.7507260	0.00054525118	81.739221
1600	$\frac{16588800}{17347}$	0.80418080	0.00021600333	72.207992
1957	$\frac{3379968}{3541}$	0.96439255	0.00038594892	17.453573
2000	$\frac{3628800}{3793}$	1.2217550	0.00025214822	55.822134
2048	$\frac{83358720}{87439}$	2.1522766	0.00086000436	17.157301
2225	$\frac{4406400}{4601}$	2.2168733	0.00044417599	65.944997
2304	$\frac{6996480}{7337}$	1.8993132	0.00078107824	34.635539
2525	$\frac{40953600}{42787}$	1.6625629	0.00038679430	60.388956
2624	$\frac{31242240}{32681}$	0.48766988	0.00016760431	11.280096
2777	$\frac{30326400}{31739}$	0.0052682944	2.49791×10^{-5}	3.1916173
3600	$\frac{3940800}{4117}$	1.7138725	0.00032163274	63.391164
3981	$\frac{22598400}{23651}$	0.0065088042	1.13683×10^{-5}	16.924484
4205	$\frac{81112320}{84937}$	0.51758364	7.99169×10^{-5}	20.235531
4225	$\frac{31168800}{32567}$	1.5789962	0.00036468807	64.303710
4352	$\frac{14613696}{15301}$	0.40686765	0.00042977636	2.3880196
4400	$\frac{287193600}{300017}$	1.7697819	0.00034047188	63.576584
4525	$\frac{315705600}{329717}$	2.0167958	0.00041971303	62.764065
4752	$\frac{94772160}{99107}$	0.77303737	0.00019694052	7.4011277
4913	$\frac{358572096}{375437}$	0.40870317	4.15983×10^{-5}	2.8025931
5125	$\frac{24364800}{25453}$	1.7587165	0.000037508012	63.196773
5225	$\frac{262310400}{273971}$	1.9505876	0.00039428701	63.490510
5725	$\frac{716947200}{748883}$	1.8674479	0.00042362838	63.447454
5744	$\frac{727626240}{761737}$	0.26820601	7.42018×10^{-5}	5.6160966
6125	$\frac{454636800}{474913}$	1.8174699	0.00042240976	63.162128
6224	$\frac{204809472}{214357}$	0.028287048	2.32205×10^{-5}	5.7256495
6809	$\frac{87570720}{91723}$	0.75775312	0.00019944285	7.0978686
7053	$\frac{1504154880}{1573751}$	0.28894424	0.00013645348	2.0848135
7056	$\frac{191034720}{200123}$	0.90144417	0.00037862134	11.709300
7168	$\frac{670104576}{701855}$	0.72584168	0.00033000104	3.6107465

case $d = 3$, Conjecture 1 holds for $D_F \gg 0$ if the Petersson products of $E^{\Delta}_{F,4}$ and $E^{\Delta}_{F,6}$ with all cusp forms of weight 16 and 24 respectively can be bounded by small quantities as $D_F \to \infty$. If F ranges over the totally real quartic fields with no non-trivial subfields, the decay of the Petersson products as $D_F \to \infty$ can be observed from the data. We expect similar strategy for the proof of Theorem 1.1 when $d = 3$ to work in this case. When F ranges instead over extensions of the form $\mathbb{Q} \subset K \subset F$, where K is a fixed real quadratic field, the data suggest that

$$\langle E^{\Delta}_{F,k}, f\rangle \to \langle E^{\Delta}_{K,2k}, f\rangle \quad \text{as } \mathrm{disc}(F) \to \infty\,.$$

The proof of Conjecture 1 may be obtained then in two steps: first proving that $E_{F,k}$ restrict to the Hilbert Eisenstein series $E_{K,2k}$ on Γ_K as $F \to \infty$, and then using Theorem 1.1 for the real quadratic field K.

d=5 Let F be a totally real field of degree 5. The space $\mathcal{M}^{(20)}_{\mathbb{Q}}$ is generated by the set $\{E_{20}, E_8\Delta, E_4\Delta^3, \Delta^5\}$. In order to get this space by restriction of Hilbert theta series (Conjecture 1), we only need to consider a Hilbert theta function θ_L for $L \in \mathcal{U}^{+,8}_F$ and the Eisenstein series $E_{F,4}, E_{F,8}$, and $E_{F,12}$. Fixing basis for M_{20}, M_{40}, and M_{60}, we find the expressions

(42)

$$\begin{aligned} E^{\Delta}_{F,4} &= E_4^5 + bE_4^2\Delta\,,\\ E^{\Delta}_{F,8} &= E_4^{10} + c_1E_4^7\Delta + c_2E_4^4\Delta^2 + c_3E_4\Delta^3\,,\\ E^{\Delta}_{F,12} &= E_4^{15} + d_1E_4^{12}\Delta + d_2E_4^9\Delta^2 + d_3E_4^6\Delta^3 + d_4E_4^3\Delta^4 + d_5\Delta^5\,, \end{aligned}$$

for $b, c_i, d_i \in \mathbb{Q}$ that depends on F. Since $\theta^{\Delta}_L = 1 + \sum_{n\geq 1} a_nq^n$ with $a_n \in \mathbb{Z}$, in order to prove linear independence of θ^{Δ}_L and $E^{\Delta}_{F,4}$, it suffices to show that $b \notin \mathbb{Z}$. If this holds true, we only need that $c_3 \neq 0$ and $d_5 \neq 0$ to prove Conjecture 1. The results of the computation of b, c_3, and d_5, for the first 30 totally real quintic fields F (ordered by discriminant) can be found in Table (2). Similarly to the case $d = 4$, the numerical values of b, c_i, d_i are close to the coefficients appearing in the expression of the Eisenstein series E_{20}, E_{40}, and E_{60} with respect to the bases specified above:

$$E_{20} = E_4^5 + b(E_{20})E_4^2\Delta\,, \quad E_{40} = E_4^{10} + \sum_{i=1}^{3} c_i(E_{40})E_4^{10-3i}\Delta^i\,,$$

$$E_{60} = E_4^{15} + \sum_{i=1}^{5} d_i(E_{60})E_4^{15-3i}\Delta^i\,,$$

Table 2: $d = 5$

D_F	$E^{\Delta}_{F,4}$		$E^{\Delta}_{F,8}$	$E^{\Delta}_{F,12}$
	$-b$	$\lvert b - b(E_{20})\rvert$	$\lvert c_3 - c_3(E_{40})\rvert$	$\lvert d_5 - d_5(E_{60})\rvert$
14641	$\frac{1017360000}{847811}$	0.060027104	20.049846	602.44929
24217	$\frac{539084160}{449263}$	0.0056153314	3.0959986	626.69793
36497	$\frac{228998016}{190847}$	0.020731861	3.7691249	625.79357
38569	$\frac{1372671360}{1144027}$	0.065169297	7.1399961	53.593811
65657	$\frac{17909631360}{14926259}$	0.050318395	1.4956754	21.447253
70601	$\frac{22786945920}{18989939}$	0.023943997	6.3509437	57.580627
81509	$\frac{1255163040}{1046047}$	0.013653680	0.14871660	33.681371
81589	$\frac{157427145}{131198}$	0.0040921029	3.7633773	5.5844793
89417	$\frac{3299933520}{2750093}$	0.010842438	0.88686411	7.1415794
101833	$\frac{27422375040}{22853437}$	0.00095029875	2.8922290	147.56474
106069	$\frac{8416776960}{7014301}$	0.020817098	1.2397693	6.2320866
117688	$\frac{72647616960}{60544963}$	0.029096490	0.024328539	11.486101
122821	$\frac{2646596160}{2205599}$	0.019992669	3.3965204	46.863235
124817	$\frac{169474446720}{141236923}$	0.0057786083	1.2672930	6.7827247
126032	$\frac{186909793920}{155769041}$	0.0082151643	0.36230593	11.688370
135076	$\frac{39368816640}{32809823}$	0.014954319	0.35027520	21.034030
138136	$\frac{42439256640}{35368523}$	0.0084035031	0.15151378	21.202474
138917	$\frac{30923687520}{25771127}$	0.010994082	3.3294478	176.27092
144209	$\frac{35105335200}{29256611}$	0.013203831	1.2246131	5.0019384
147109	$\frac{79422612480}{66189911}$	0.0041847312	1.4237043	8.3262747
149169	$\frac{316249522560}{263551583}$	0.028634117	1.8194260	15.033097
153424	$\frac{24509153664}{20425187}$	0.023174277	3.4016473	25.402792
157457	$\frac{76544072064}{63790577}$	0.0031668548	1.0412335	7.2747406
160801	$\frac{411236196480}{342716341}$	0.0072814568	0.29819082	10.519558
161121	$\frac{6653973120}{5545309}$	0.0038819428	0.12885785	18.108623
170701	$\frac{125695281600}{104754347}$	0.019242412	2.2046583	104.80927
173513	$\frac{530059904640}{441734773}$	0.026193754	0.10356861	42.247614
176281	$\frac{187387136640}{156166489}$	0.0054077065	1.9881737	26.798380
176684	$\frac{60248727936}{50210921}$	0.011580917	1.0387400	12.861239
179024	$\frac{638510843520}{532132229}$	0.014289468	0.34114599	1.2188617

the relevant values being

$$\begin{aligned} b(E_{20}) &= \tfrac{209520000}{174611} \sim 1199\,, \\ c_3(E_{40}) &= \tfrac{27014542428753690624000000000}{2610827184964491 22051} \sim 103471200 \\ d_5(E_{60}) &= \tfrac{1423152253904739393602157818174020937318400000000000000}{12152331404837555720403049940798202460414 91} \sim 1171094011917. \end{aligned}$$

In Table 2 we do not write the numerical values of c_3, d_5, but of their difference with the coefficients $c_3(E_{40})$ and $d_5(E_{60})$ respectively. Analogously to the case $d = 4$, it seems that the diagonal restriction of $E_{F,4}, E_{F,8}$, and $E_{F,12}$ are close to the Eisenstein series E_{20}, E_{40}, and E_{60} respectively. In particular, since $\langle E_{20}, E_4^2\Delta\rangle = 0$, this implies that the Petersson product

$$\langle E_{F,4}^{\Delta}, E_4^2\Delta\rangle \;=|b - b(E_{20})|\langle E_4\Delta^2, E_4^2\Delta\rangle$$

is small for any field F and may decay as $D_F \to \infty$. Similar considerations apply to the cases $E_{F,8}^{\Delta}$ and $E_{F,12}^{\Delta}$.

d=6 We have that $\mathcal{M}_{\mathbb{Q}}^{12} = \mathbb{C}[E_4^3, \Delta]$. We have only to check that

$$E_{F,2}^{\Delta} \;=\; E_4^3 \;+\; b\cdot\Delta$$

is not the restriction of a Hilbert theta function θ_L. We know this is the case if b is not an integer, as explained in the proof of Theorem 1.1 in the case $d = 3$. However, looking at the values of b computed for the first 30 totally real sextic fields F in Table (3), this is not always the case. Since $\theta_L^{\Delta} = 1 + N_2(L)q + \cdots$, we have to compare, for integral values of b, the number $720 + b$ with the possible values of $N_2(L)$ listed in table V of [3] to check whether they differ or not. This happens in all cases but two: the field of discriminant 453789 has $720 + b = 0 = N_2(\Lambda_{24})$, the field of discriminant 1397493 has $720 + b = 72 = N_2(A_2^{12})$. For these fields our argument can not confirm the validity of Conjecture 1. We checked fields up to $D_F = 5 \times 10^6$ (144 fields) and found no other such instances.

As in the cases $d = 4, 5$, in table (3) we also compare the value of b with $b(E_{12}) = -\frac{432000}{691}$ (see (36)).

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Table 3: $d = 6$

D_F	$E^{\Delta}_{F,2}$ $-b$	$\|b - b(E_{12})\|$	D_F	$E^{\Delta}_{F,2}$ $-b$	$\|b - b(E_{12})\|$
300125	$\frac{21600}{37}$	41.397113	1134389	684	58.819103
371293	$\frac{11808}{19}$	3.7072130	1202933	608	17.180897
434581	$\frac{8352}{13}$	17.280641	1229312	$\frac{27264}{43}$	8.8656144
453789	720	94.819103	1241125	$\frac{28800}{47}$	12.414940
485125	$\frac{7200}{11}$	29.364557	1259712	$\frac{17280}{31}$	67.761542
592661	672	46.819103	1279733	$\frac{11736}{17}$	65.172044
703493	$\frac{2048}{3}$	57.485769	1292517	$\frac{16416}{29}$	59.111932
722000	$\frac{4800}{7}$	60.533388	1312625	$\frac{9000}{13}$	67.126795
810448	$\frac{3456}{5}$	66.019103	1387029	696	70.819103
820125	$\frac{43200}{73}$	33.400075	1397493	648	22.819103
905177	$\frac{3348}{5}$	44.419103	1416125	$\frac{12000}{19}$	6.3980501
966125	675	49.819103	1528713	$\frac{12096}{19}$	11.450682
980125	675	49.819103	1541581	$\frac{8352}{13}$	17.280641
1075648	$\frac{8352}{13}$	17.280641	1683101	$\frac{65088}{103}$	6.7414328
1081856	$\frac{3072}{5}$	10.780897	1767625	$\frac{25200}{41}$	10.546751

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Gabriele Bogo
Fachbereich Mathematik
Technische Universität Darmstadt
Schlossgartenstrasse 7, D–64289
Darmstadt
Germany
E-mail: bogo@mathematik.tu-darmstadt.de

Yingkun Li
Fachbereich Mathematik
Technische Universität Darmstadt
Schlossgartenstrasse 7, D–64289
Darmstadt
Germany
E-mail: li@mathematik.tu-darmstadt.de