Pure and Applied Mathematics Quarterly
Volume 19, Number 1, 85–93, 2023

Functional equations of polygonal type for multiple polylogarithms in weights 5, 6 and 7*

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Abstract: We present new functional equations of multiple polylogarithms in weights 5, 6 and 7 and use them for explicit depth reduction. These identities generalize the crucial identity $\mathbf{Q}_4$ from the recent work of Goncharov and Rudenko that was used in their proof of the weight 4 case of Zagier's Polylogarithm Conjecture.

Keywords: Polylogarithms, functional equations, cluster relations, Zagier's Polylogarithm Conjecture.

1. Introduction

In their recent breakthrough paper [5] Goncharov and Rudenko envisaged a very promising new strategy to prove Zagier's Polylogarithm Conjecture (ZPC) by relating it to cluster algebra complexes. A crucial ingredient in their proof of the case of the conjecture in weight 4 was a new functional equation relating the multiple polylogarithms $I_{3,1}(x, y)$ and $\mathrm{Li}_4(z)$, which they denoted by $\mathbf{Q}_4$.

This result prompted our experimental search for higher analogues, and our computer implementation allowed us to find analogues, with a combinatorial structure inspired by and quite reminiscent of their $\mathbf{Q}_4$, for higher weights.

These findings date back to 2018 at the MPI Bonn and were both communicated to Don Zagier and subsequently presented at a workshop on cluster algebras and the geometry of scattering amplitudes at the Higgs Centre in Edinburgh in March 2020. Our more ambitious goal of finding a bootstrapping procedure that would produce analogous results for general weight has now apparently been superseded by Rudenko's beautiful new preprint [6] pertaining to Goncharov's depth conjecture. As our approach does not seem to

arXiv: 2012.09840
Received November 4, 2021.
2010 Mathematics Subject Classification: Primary 11G55; secondary 33E20, 39B32.

*Dwelling on numbers: Zetas are great in every respect.

take the exact same symmetries into account, it has the big disadvantage of being harder to generalize but on the other hand it may have produced identities that are of a slightly different nature—both potentially in the use of symmetries and of the choice of functions—than the ones that we anticipate to appear eventually in his already announced 'cluster polylogarithm' preprint (with Matveiakin).

We provide functional equations of type $\mathbf{Q}_n$ up to weight $n = 7$. Our formulas, found with the help of intensive computer calculations, will likely differ from the ones Rudenko derives in that we use a potentially different set of functions and impose cyclic symmetry. We anticipate that they could still be beneficial, in particular in view of the remaining task of 'conditional' further depth reduction which may well result from combining suitable specializations of the functional equations we give.

2. Analogues of the functional equations $\mathbf{Q}_n$ for $n = 5, 6, 7$

In [4], having successfully solved the weight 3 case of ZPC, Goncharov reduced the weight 4 case of his more encompassing Freeness Conjecture to an explicit calculation which would express $I_{3,1}(V(x,y),z)$, with the five term relation $V(x,y)$ in one slot, in terms of Li_4. This was indeed shown to hold with 122 rather non-obvious terms (concocted as products of up to four cross ratios in 6 variables) in [3].

In [5] Goncharov and Rudenko found an alternative and more conceptual way, introducing complexes of cluster algebras, to derive an equation that solves the same question without the need of giving those 122 terms explicitly, and which furthermore has the important property of suggesting generalizations to higher weight. The ensuing connection to moduli spaces $\mathfrak{M}_{0,k}$ (of genus zero curves with k marked points) suggests considering polyangulations of convex $2N$-gons for suitable integers N, resulting in pictures of the type given in the figures below.

Further to the notation used in [5] we introduce the following shorthand: Any subpolygon comes equipped with a partition of its internal angles into two subsets (these often correspond in [6] to 'even' and 'odd' polytopes, but we note that our conventions allow successive even or odd indices for lower depth terms). We equip the angles of one of the two sets with a slice of pi, indicating that the associated argument is given as the cyclic ratio (already used extensively in [5])

$$[x_{i_1}, \dots, x_{i_{2m}}] := (-1)^m \frac{(x_{i_1} - x_{i_2})(x_{i_3} - x_{i_4}) \dots (x_{i_{2m-1}} - x_{i_{2m}})}{(x_{i_2} - x_{i_3})(x_{i_4} - x_{i_5}) \dots (x_{i_{2m}} - x_{i_1})},$$

where i_1 corresponds to any one of the m labeled angles. Moreover, we indicate the order in which the arguments are to be taken by indices 1, 2, etc. inside the given subpolygons. Finally, each picture stands for the sum over all cyclic permutations of the indices (corresponding to the rotations of the polygon).

Our functions are slight variants of the standard polylogarithm functions, indicated by an 'N' in the notation, e.g. $I^N_{3,1,1}$ stands for

$$I^N_{3,1,1}(x,y,z) = -\operatorname{Li}_{3,1,1}((xyz)^{-1}, z, y)$$

and more generally

$$\begin{aligned} I^N_{n_1,\ldots,n_d}(a_1,\ldots,a_d) &= I_{n_1,\ldots,n_d}(a_1,(a_2\ldots a_d)^{-1},(a_2\ldots a_{d-1})^{-1},\ldots,a_2^{-1}) \\ &= (-1)^d \operatorname{Li}_{n_1,\ldots,n_d}((a_1\ldots a_d)^{-1}, a_d, a_{d-1},\ldots,a_2) \end{aligned} \tag{1}$$

(these variants have been chosen as they satisfy simpler coproduct expressions than the underlying iterated integrals, see e.g. [2, p. 248]).

With the above explanations, we can now give the functional equations for weights 5, 6 and 7 that should play the role of the crucial equation $\mathbf{Q}_4$ in [5]. The main characterizing feature of these identities (and the reason why we propose them as generalizations of $\mathbf{Q}_4$) is that the arguments of the iterated integrals are cross-ratios and higher cyclic ratios attached to $2N$-gons that form a complete polyangulation of a subpolygon by even polygons. Moreover, the highest depth terms in these identities consist of a single depth d term of the form $I_{n-d+1,\{1\}^{d-1}}$, evaluated at all possible full quadrangulations of a $2N$-gon, up to cyclic symmetry.

3. Identity in weight 5

Our candidate for the functional equation $\mathbf{Q}_5$ is shown in Figure 1. It involves four different types of iterated integrals: $I^N_{3,1,1}$, $I^N_{3,2}$, $I^N_{4,1}$, and I^N_5.

Theorem 1. *The weight 5 multiple polylogarithm identity shown in Figure 1 holds modulo products for generic $x_1,\ldots,x_8$.*

As explained in Section 2 each term in this figure stands for cyclically symmetric sum of eight terms with indices taken modulo 8, e.g. the first term stands for the following expression (where we employ the shorthand $[r,s,t,u]^{(j)} := [x_{r+j}, x_{s+j}, x_{t+j}, x_{u+j}]$)

$$-4\sum_{j=1}^{8} I^N_{3,1,1}\Big([1,2,3,4]^{(j)},[1,4,5,6]^{(j)},[1,6,7,8]^{(j)}\Big)\,.$$

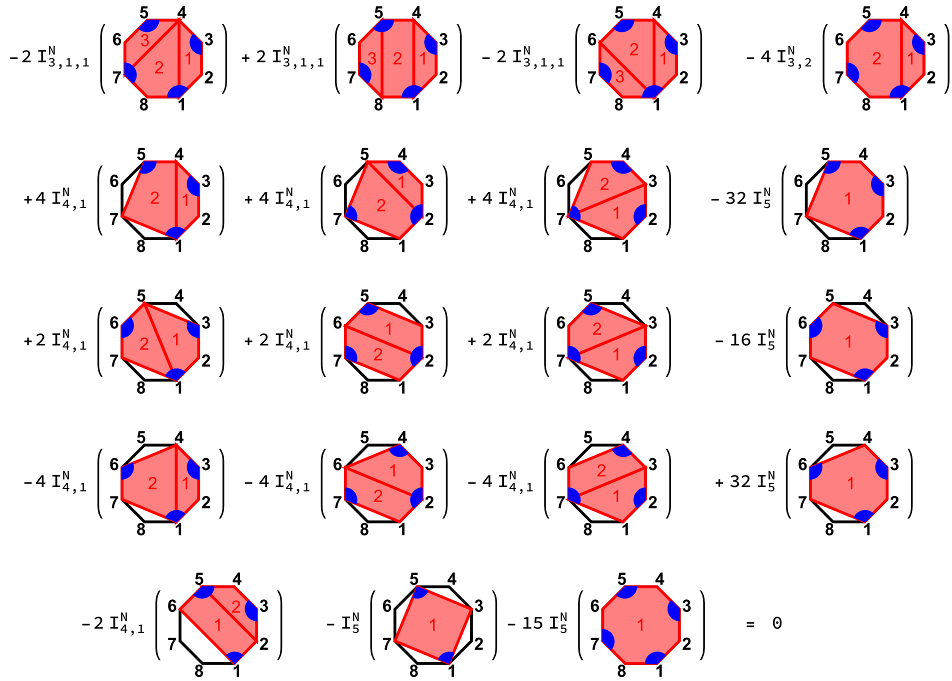

Figure 1: A version of $\mathbf{Q}_5$.

By a suitable specialization (or, more precisely, degeneration, since one needs to take limiting values) of the arguments we obtain an explicit reduction of a single term $I_{3,1,1}(x,y,z)$ to multiple polylogarithms of depth at most 2. When speaking of depth reduction we always work modulo products.

Proposition 2. *The function $I_{3,1,1}(x,y,z)$ can be written as a linear combination of at most 47 terms, in terms of the functions $I_{3,2}$, $I_{4,1}$, and I_5 (containing 1, 23 and 23 terms, respectively).*

Proof. Specializing the identity of Theorem 1 to $x_5 = x_3 = x_1$ produces a combination of one generic $I^N_{3,1,1}$ term and several degenerate ones, up to the inversion and reversion of $I^N_{3,1,1}$. The generic term can then be isolated by subtracting the specialization of Theorem 1 to $x_2 = x_1$. □

Proposition 3. *The function $I_{3,2}(x,y)$ can be written as a linear combination of at most 24 terms, via the functions $I_{4,1}$ and I_5 (containing 12 terms each).*

Proof. Specializing the above identity for $I^N_{3,1,1}$ to $x_7 = x_2$ reduces the $I^N_{3,1,1}$ term to $I^N_{3,2}$. (The $I^N_{3,2}$ term that occurs in the formula of Proposition 2 goes away under this degeneration.) □

A different reduction of $I_{3,2}$ to $I_{4,1}$ and Li_5, involving a lot more complicated Li_5 terms, but fewer and simpler $I_{4,1}$ terms was given in [2].

Corollary 4. *Every weight* 5 *multiple polylogarithm can be expressed in terms of* $I_{4,1}$ *and* I_5*.*

Proof. This follows from the two preceding propositions together with an explicit reduction of any iterated integral in weight 5 to $I_{3,1,1}$ that was given in [2]. □

4. Identities in weight 6

Our candidate for the identity $\mathbf{Q}_6$ is given in Figure 2 below. It involves $I^N_{4,1,1}$, $I^N_{4,2}$, $I^N_{5,1}$, and I^N_6.

Theorem 5. *The weight* 6 *multiple polylogarithm identity shown in Figure 2 holds modulo products for generic* $x_1, \dots, x_9$*.*

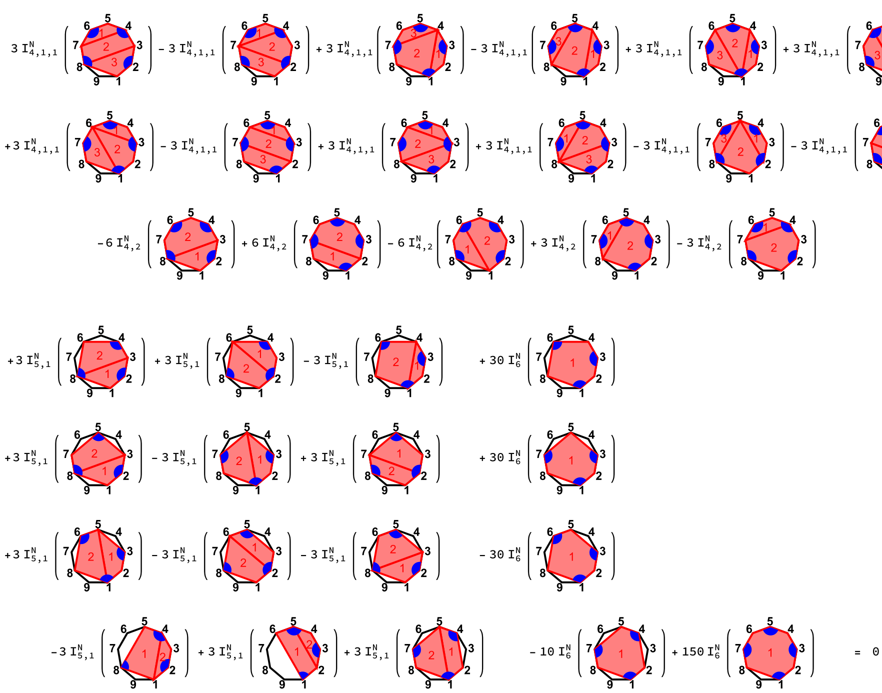

Figure 2: A version of $\mathbf{Q}_6$.

4.1. Reduction from depth 4 to depth 3 and identities of type $\mathbf{Q}_n^d$

To perform the depth reduction from 4 to 3 in the weight 6 case, by analogy to what was done in the previous section, we need a slightly different, 'off-diagonal' identity of type $\mathbf{Q}_n$. The 'diagonal' identities $\mathbf{Q}_6$ and $\mathbf{Q}_5$ presented above, as well as the original identities $\mathbf{Q}_4$ and $\mathbf{Q}_3$ of Goncharov and Rudenko, have the property that the highest depth of the iterated integrals that occur in $\mathbf{Q}_n$ is $\lceil n/2 \rceil$. The weight 6 identity whose top layer structure is indicated in Figure 3 below, however, has highest depth 4, and we will call it $\mathbf{Q}_6^4$, indicating the highest depth in the superscript. (In this notation the candidate for $\mathbf{Q}_6$ presented above would be called $\mathbf{Q}_6^3$.)

The specific identity that we give involves the functions $I^N_{3,1,1,1}$, $I^N_{4,1,1}$, $I^N_{3,2,1}$, $I^N_{5,1}$, $I^N_{4,2}$, $I^N_{3,3}$, and I^N_6. We note that, unlike the identities shown in Figures 1, 2, and 4 (and breaking the convention that was set up in Section 2), in this identity each term represents a *signed* cyclic symmetrization, introducing a sign $(-1)^j$ after cyclically shifting by j steps, i.e. we replace each term by $\sum_{j=1}^{10}(-1)^j F(x_{1+j},\dots,x_{10+j})$ instead of $\sum_{j=1}^{10} F(x_{1+j},\dots,x_{10+j})$.

Theorem 6. *There exists a cyclically symmetric multiple polylogarithm identity in weight* 6*, depth* 4 *with* 168 *cyclic orbits of terms whose top layer structure is shown in Figure 3 that holds modulo products for generic* $x_1,\dots,x_{10}$.

(We have also found a similar identity $\mathbf{Q}_3^3$ in weight 3 involving depth 3 functions and $\mathbf{Q}_4^4$ in weight 4 involving depth 4 functions and also analogous identities $\mathbf{Q}_5^3$ and $\mathbf{Q}_5^4$ in weight 5.)

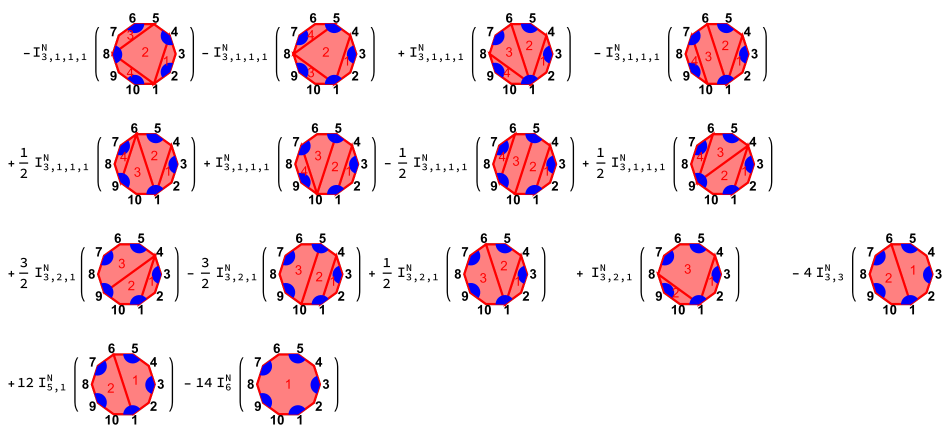

Figure 3: Top layer of a weight 6 depth 4 identity.

Using the identity $\mathbf{Q}_6^4$, similarly to the case of weight 5 outlined in the previous section, one can obtain depth reduction of certain iterated integrals.

Proposition 7. *The function $I_{3,1,1,1}(w,x,y,z)$ can be written as a linear combination of the functions $I_{3,2,1}$, $I_{4,1,1}$, $I_{4,2}$, $I_{5,1}$, and I_6.*

Proof. Specializing to $x_7 = x_5 = x_3 = x_1$ produces a combination consisting of one generic $I^N_{3,1,1,1}$ term and several degenerate ones, up to the inversion and reversion relations for $I^N_{3,1,1,1}$. The generic term can then again be isolated by subtracting the specialization to $x_4 = x_1$. □

Proposition 8. *The function $I_{3,2,1}(x,y,z)$ can be expressed as a linear combination of $I_{4,1,1}$, $I_{4,2}$, $I_{5,1}$ and I_6.*

Proof. The proof goes along the same lines as the proof of Proposition 3 Specializing the above reduction for $I^N_{3,1,1,1}$ to $x_7 = x_2$ reduces the $I^N_{3,1,1,1}$ term to $I^N_{3,2,1}$. (The original $I^N_{3,2,1}$ term goes away.) □

In general, using the dihedral symmetries of $I_{n_1,n_2,\ldots,n_k}$ allows one to express any iterated integral in terms $I_{3,\{1\}^a}$ or $I_{2,\{1\}^b,2,\{1\}^c}$. By considering the stuffle product of $\mathrm{Li}_{2,2,\{1\}^b}$ and $\mathrm{Li}_{\{1\}^c}$ one can express $I_{2,\{1\}^b,2,\{1\}^c}$ in terms of $I_{2,\{1\}^{<b},2,\{1\}^{c'}}$ and lower depth. Iteratively, this shows $I_{2,2,\{1\}^{a-1}}$ suffices amongst integrals with indices involving only 1's apart from two 2's. Finally the shuffle product of $I_{3,\{1\}^a}$ and I_1 expresses $I_{2,2,\{1\}^{a-1}}$ in terms of integrals with indices involving only 1's apart from a single 3, meaning $I_{3,\{1\}^a}$ alone is sufficient. Combining this with Proposition 8 gives us the following.

Corollary 9. *Every weight 6 multiple polylogarithm can be expressed in terms of $I_{4,1,1}$, $I_{4,2}$, $I_{5,1}$ and I_6.*

5. Identity in weight 7

Finally, we indicate the top level structure of our candidate for $\mathbf{Q}_7$ in Figure 4. It involves the iterated integrals $I^N_{4,1,1,1}$, $I^N_{5,1,1}$, $I^N_{4,2,1}$, $I^N_{6,1}$, $I^N_{5,2}$, $I^N_{4,3}$, and I^N_7.

Theorem 10. *There exists a cyclically symmetric multiple polylogarithm identity in weight 7 and depth 4 with 121 cyclic orbits of terms whose top layer structure is shown in Figure 4 that holds modulo products for generic $x_1,\ldots,x_{10}$.*

Similarly to the proof of Proposition 2 we get the following.

Proposition 11. *The function $I_{4,1,1,1}(w,x,y,z)$ can be written as a linear combination of lower depth functions.*

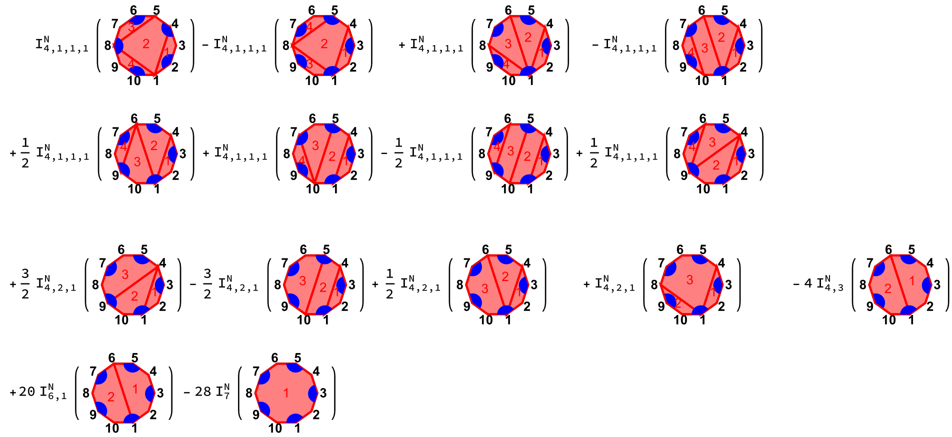

Figure 4: Top layer of a version of $\mathbf{Q}_7$.

Proof. The same specialization to $x_7 = x_5 = x_3 = x_1$ produces a combination of one generic $I^N_{4,1,1,1}$ term and several degenerate ones, up to the inversion and reversion relations for $I^N_{4,1,1,1}$. The generic term is isolated by subtracting the specialization to $x_4 = x_1$. □

Remark 12. It was checked with extensive symbolic calculations that $I_{5,1,1}$ functions of the type appearing in $\mathbf{Q}_7$, along with similar functions of lower depth, were sufficient to span the space of weight 7 iterated integrals. More precisely, we were able to confirm that the dimension of the space of such functions agrees with the dimension computed by Brown in [1] for weight $k = 7$ iterated integrals in $n = 8$ projective variables (the dimension being $\frac{1}{k}\sum_{d|k}\mu(\frac{k}{d})\sum_{i=2}^{n-2} i^d = 53\,820$). By the known reduction of weight 7 integrals to depth 5 (hence involving $5 + 3 = 8$ projective variables), this implies that every weight 7 integral can be expressed in terms of $I_{5,1,1}$, $I_{5,2}$, $I_{6,1}$ and I_7. Given this large dimension, we were unable to extract explicit formulas for this reduction.

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