

Indefinite theta series: the case of an N -gon

JENS FUNKE AND STEPHEN KUDLA*

To Don Zagier, with our admiration

Abstract: In this note, we provide a construction of the indefinite theta series attached to N -gons in the symmetric space of an indefinite inner product space of signature $(m - 2, 2)$ following the suggestions of Section C in the recent paper of Alexandrov, Banerjee, Manschot, and Pioline, [2]. We prove the termwise absolute convergence of the holomorphic mock modular part of these series and also obtain an interpretation of the coefficients of this part as linking numbers. Thus, we prove the convergence conjecture of [2] provided none of the vectors in the collection $\mathcal{C} = \{C_1, \dots, C_N\}$ is a null vector. It should be noted that the use of linking numbers and a homotopy argument eliminates the need for an explicit parametrization of a surface S spanning the N -gon that was used in an essential way in our previous work [4]. We indicate how our method could be carried over to a more general situation for signature $(m - q, q)$ where higher homotopy groups are now involved. In the last section, we apply the method to the case of a dodecahedral cell in the symmetric space of a quadratic form of signature $(m - 3, 3)$.

Keywords: Indefinite theta series, mock modular forms.

1. Introduction

1.1. Indefinite theta series

The construction of theta series for lattices with indefinite quadratic forms continues to be a topic of considerable interest. If (L, Q) is a lattice of rank m with integer valued positive definite quadratic form, the classical theta series

$$(1.1) \quad \theta_\mu(\tau) = \sum_{x \in \mu + L} \mathbf{q}^{Q(x)}, \quad \mathbf{q} = e(\tau) = e^{2\pi i \tau},$$

Received September 22, 2021.

2010 Mathematics Subject Classification: 11F27, 11F37.

*The author is supported by an NSERC Grant.

is termwise absolutely convergent and defines a holomorphic modular form of weight $\frac{1}{2}m$, by Poisson summation. Here $\tau = u + iv \in \mathfrak{H}$, the upper half plane, and $\mu \in L^\vee$, the dual lattice of L . For a quadratic lattice (L, Q) of signature (p, q) with $pq > 0$, the series (1.1) no longer converges in general due to (i) the presence of lattice vectors x with $Q(x) < 0$ and (ii) the existence infinitely many lattice vectors x with a given value $Q(x) = n > 0$, due, for example, for the existence of an infinite group Γ_L of isometries of L . To handle this case, the classic construction due to Siegel introduces an auxiliary variable

$$z \in D(V) := \{ z \in \text{Gr}_q^o(V) \mid Q|_z < 0 \},$$

where $\text{Gr}_q^o(V)$ is the Grassmannian of oriented q -planes in $V = L \otimes_{\mathbb{Z}} \mathbb{R}$. The variable z determines a positive definite inner product

$$(x, x)_z = (x, x) + 2R(x, z), \quad R(x, z) = -(\text{pr}_z(x), \text{pr}_z(x)),$$

which coincides with $(\ , \)$ on z^\perp and with $-(\ , \)$ on z . Here pr_z denotes the orthogonal projection to z . Siegel's theta series

$$\theta_\mu(\tau, z) = \sum_{x \in \mu + L} e^{-2\pi R(x, z)v} \mathbf{q}^{Q(x)}$$

is a non-holomorphic modular form of weight $\frac{1}{2}(p - q)$ in τ and a smooth function of $z \in D$. There are also 'geometric' theta series introduced in [9] and [10], which are valued in $A^q(D)$, the space of smooth q -forms on D , and are of weight $\frac{1}{2}(p + q)$ in τ . Both these series and Siegel's are invariant under the action of an arithmetic subgroup $\Gamma_L \subset O(V)$ on D . As a consequence one can use the Siegel-type theta series and the geometric theta series as integral kernel to lift (automorphic) forms for one group to the other. See Section 2 for a more detailed review of these theta series.

Another approach to the construction of indefinite theta series¹ involves restricting the summation to lattice vectors lying in suitable cones on which the quadratic form takes positive values. When they are termwise absolutely convergent, such sums define holomorphic functions of τ . The standard holomorphic Eisenstein series E_k series can be viewed as such an indefinite theta series of signature $(1, 1)$ but in general they are rarely modular due to the lack of a suitable Poisson summation formula. Another non-modular example for signature $(1, 1)$ was constructed by Zagier in Section 2.3 in [7] in his cel-

¹This was suggested by Zagier in conversations with the second author at Maryland in the early 1980s.

brated joint work with Hirzebruch where the holomorphic generating series was completed by a non-holomorphic part to obtain a modular object.

A more systematic understanding of the nature of such series emerged from the seminal work of Zwegers [16]; for an overview see [15]. In the case of a quadratic lattice of signature $(m-1, 1)$, suppose that C and C' are negative vectors in $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ lying in the same component of the light cone, i.e., such that $(C, C') < 0$. Then Zwegers considered the series

$$(1.2) \quad \vartheta_{\mu}(\tau, \mathcal{C}) = \sum_{x \in \mu + L} (\operatorname{sgn}((x, C)) - \operatorname{sgn}((x, C'))) \mathbf{q}^{Q(x)},$$

and showed that it is termwise absolutely convergent and defines a mock modular form, i.e., it has a non-holomorphic modular completion of weight $\frac{1}{2}m$

$$\hat{\vartheta}_{\mu}(\tau, \mathcal{C}) = \vartheta_{\mu}(\tau, \mathcal{C}) - \sum_{x \in \mu + L} (E^c((x, \underline{C})\sqrt{2v}) - E^c((x, \underline{C}')\sqrt{2v})) \mathbf{q}^{Q(x)},$$

involving the complementary error function $E^c(u) = 2 \operatorname{sgn}(u) \int_{|u|}^{\infty} e^{-\pi t^2} dt$.

Alexandrov, Banerjee, Manschot, and Pioline, [1], established the analog of Zwegers' result for signature $(m-2, 2)$. For a collection $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$ of four negative vectors in V satisfying certain conditions, they showed that

$$\hat{\vartheta}_{\mu}(\tau, \mathcal{C}) = \sum_{x \in \mu + L} \sum_{j=1}^4 E_2(C_j, C_{j+1}, x\sqrt{2}) \mathbf{q}^{Q(x)}$$

is the non-holomorphic modular completion of the holomorphic generating series

$$(1.3) \quad \vartheta_{\mu}(\tau, \mathcal{C}) = \sum_{x \in \mu + L} \sum_{j=1}^4 \operatorname{sgn}(x, C_j) \operatorname{sgn}(x, C_{j+1}) \mathbf{q}^{Q(x)}.$$

Here

$$E_2(x; c, c') = \int_z e^{\pi(y - \operatorname{pr}_z(x), y - \operatorname{pr}_z(x))} \operatorname{sgn}(y, c) \operatorname{sgn}(y, c') dy,$$

is the higher error function defined in [1] with z the negative 2-plane spanned by c and c' with projection $\operatorname{pr}_z(x)$ onto z and an appropriately normalized measure dy . In a subsequent paper [2], the authors associate similar generating series to a collection $\mathcal{C} = \{C_1, \dots, C_N\}$ of N negative vectors and conjecture their modularity. Nazaroglu [11] discussed the generalization to

arbitrary signature $(m - q, q)$, while Raum [13] considered more general positive polyhedral cones. All these results employ a modularity criterion for indefinite theta series obtained by Vignéras [12].

It is important to note that the sign functions occurring in the summations (1.2) and (1.3) actually imply that only terms with $Q(x) > 0$ contribute. While this is easy to see in (1.3), it is not at all evident in general.

In [8, 4, 5] we developed a different approach based on the geometric theta series of Kudla-Millson [9, 10]. Recall that the Kudla-Millson theta series

$$\theta_\mu(\tau, \varphi_{KM}) = \sum_{x \in L + \mu} \varphi_{KM}(x, \tau)$$

is a closed differential q -form on the symmetric space D associated to V which in τ transforms like a modular form of weight $m/2$. Then for certain collections \mathcal{C} of negative vectors we associate a (compact) q -cell $S(\mathcal{C})$ in D and consider the integral

$$\int_{S(\mathcal{C})} \theta_\mu(\tau, \varphi_{KM}),$$

which inherits the modularity from $\theta_\mu(\tau, \varphi_{KM})$. On the other hand, under certain conditions, we can compute the theta integral explicitly as a combination of generalized error functions and give a geometric interpretation of the holomorphic generating series. In particular, we obtained new proofs for the results obtained in [16], [1], and [11] for ‘cubical’ input data, but we also established a new result when \mathcal{C} gives rise to geodesic q -simplices.

1.2. The case of the N -gon

In the present paper, we consider another type of sign function introduced in [2]. Suppose that $\text{sig}(V) = (m - 2, 2)$ and that L is a lattice in V such that with $L \subset L^\vee$. Let $\mathcal{C} = \{C_1, \dots, C_N\}$ be a collection of N negative vectors in V such that the oriented spans $z_j = [C_j, C_{j+1}]$ are negative 2-planes all lying in the same connected component of D , say D^+ . This amounts to the set of conditions

$$(1.4) \quad (C_j, C_j) < 0,$$

$$(1.5) \quad (C_j, C_j)(C_{j+1}, C_{j+1}) - (C_j, C_{j+1})^2 > 0,$$

$$(1.6) \quad (C_j, C_j)(C_{j-1}, C_{j+1}) - (C_j, C_{j-1})(C_j, C_{j+1}) < 0,$$

on inner products that we will sometimes refer to as the **N -gon conditions**. Here the first two conditions are equivalent to z_j being a negative 2-plane,

while the third ensure that all z_j are indeed in the same component of D . Note that the last of these conditions is equivalent to $(-C_{j-1\perp j}, C_{j+1\perp j}) < 0$, where for example, adopting the notation of [1],

$$C_{j-1\perp j} = C_{j-1} - \frac{(C_{j-1}, C_j)}{(C_j, C_j)} C_j$$

is the projection of C_{j-1} to C_j^\perp . In particular, the projections $-C_{j-1\perp j}$ and $C_{j+1\perp j}$ lie in the same nap of the light cone in the space $V_j := C_j^\perp$ of signature $(m-2, 1)$. Thus

$$(1.7) \quad \gamma_j(s) := [C_j, (s-1)C_{j-1} + sC_{j+1}], \quad s \in [0, 1],$$

defines a geodesic curve in $D'_{C_j} \cap D^+$ joining $z_{j-1} = [C_{j-1}, C_j]$ and $z_j = [C_j, C_{j+1}]$. Here, for a negative vector $C \in V$, $D'_C \subset D$ is the set of oriented negative 2-planes containing C . Then

$$(1.8) \quad \gamma(\mathcal{C}) = \sum_j \gamma_j$$

is a closed and piecewise smooth (not necessarily simple) curve in D^+ with the z_j 's as its 'vertices'. This is the N -gon of the title. Let $S = S(\mathcal{C})$ be any oriented 2-cell in D^+ with boundary $\gamma(\mathcal{C})$.

For $x \in V$, define the quantities

$$(1.9) \quad \varepsilon(x; \mathcal{C}) := \sum_{j=1}^N \operatorname{sgn}(x, C_j) \operatorname{sgn}(x, C_{j+1}) - \operatorname{sgn}(\mathbf{v}, C_j) \operatorname{sgn}(\mathbf{v}, C_{j+1}),$$

and

$$(1.10) \quad \mathbf{w}(\mathcal{C}) := - \sum_{j=1}^N \operatorname{sgn}(\mathbf{v}, C_j) \operatorname{sgn}(\mathbf{v}, C_{j+1}),$$

where \mathbf{v} is any negative vector in V . Here $\operatorname{sgn}(0) = 0$. We will see that $\mathbf{w}(\mathcal{C})$ is independent of the choice of \mathbf{v} . Note that $|\varepsilon(x; \mathcal{C})| \leq 2N$.

Our main result concerning the N -gon is the following.

Theorem 1.1. *Let $\mathcal{C} = \{C_1, \dots, C_N\}$ be a collection of vectors satisfying the N -gon conditions. For $\mu \in L^\vee/L$ the series*

$$\vartheta_\mu(\tau, \mathcal{C}) = \sum_{x \in \mu + L} \varepsilon(x; \mathcal{C}) \mathbf{q}^{Q(x)}$$

is termwise absolutely convergent. Its modular completion given by

$$\hat{\vartheta}_\mu(\tau, \mathcal{C}) = \sum_{x \in \mu+L} \left(\mathbf{w}(\mathcal{C}) + \sum_{j=1}^N E_2(C_j, C_{j+1}, x\sqrt{2}) \right) \mathbf{q}^{Q(x)}$$

is a non-holomorphic modular form of weight $\frac{m}{2}$. Further,

$$\hat{\vartheta}_\mu(\tau, \mathcal{C}) = 4 \int_S \theta_\mu(\tau, \varphi_{KM}).$$

For $N = 4$ we will see below that one always has $\mathbf{w}(\mathcal{C}) = 0$, and one recovers the main result of [1] which was also considered in [8]. For triangles ($N = 3$) one has $\mathbf{w}(\mathcal{C}) = 1$, and this is a very special case of the results in [5]. In general, we show $-N + 4 \leq \mathbf{w}(\mathcal{C}) \leq N - 2$ and $\mathbf{w}(\mathcal{C}) \equiv -N \pmod{4}$. The invariant $\mathbf{w}(\mathcal{C})$ is somewhat curious, and we discuss some of its properties in section 6.

For general N , this establishes the convergence conjecture made in [2] while also clarifying the role of the constant $\mathbf{w}(\mathcal{C})$. Finally, our proof will show that, as before, in the generating series $\vartheta_\mu(\tau, \mathcal{C})$ for the signs $\varepsilon(x; \mathcal{C})$ only terms with $Q(x) > 0$ contribute.

Remark 1.2. *One can also consider a collection $\mathcal{C} = \{C_1, \dots, C_N\}$ of N vectors in V satisfying (1.4) and (1.5), omitting the condition (1.6). In this case, the oriented negative 2-planes $z_j = [C_j, C_{j+1}]$ need not all lie in the same component of D . The analogue of Theorem 1.1 for such more general collections is Theorem 3.4 where the jumps between components are taken into account.*

As in our prior work, this result is obtained by computing the integral of the geometric theta kernel over the oriented 2-cell S with boundary $\gamma(\mathcal{C})$. Since S is compact, this integral can be computed termwise, so the key is to compute the local theta integral

$$I^0(x, \mathcal{C}) = \int_S \varphi_{KM}^0(x)$$

for $x \in V$ and the Schwartz 2-form used to construct the theta kernel. Further note that since φ_{KM} is closed the integral does not depend on the choice of S .

The result given in Theorem 3.1 involves some new ideas. In our previous work, we needed to impose some additional conditions on \mathcal{C} which ensured that we could explicitly parameterized the cell S . In this paper, we introduce a very general homotopy argument which allows to compute the local

theta integral. Furthermore, the argument also shows that Fourier coefficients $\varepsilon(x; \mathcal{C})$ of the holomorphic generating series $\vartheta_\mu(\tau, \mathcal{C})$ can be interpreted as the linking number between the curve $\gamma(\mathcal{C})$ and a divisor D_x in D .

The conjecture made in [2] allows for the C 's to be rationally null. While we do not consider this case in this paper, we give an example, Theorem 6.6, where we explain that previous work of the first author [3] realizes Zagier's Eisenstein series [14, 7] as indefinite theta series associated to signature $(1, 2)$! For an outline approach to the general null vector case, see Remark 6.7.

In Section 7, we outline how the results of this paper can be extended to general signature $(m - q, q)$, and in Section 8 we carry this out explicitly for $q = 3$ for a geodesic dodecahedron. The cases of a cube and a tetrahedron were discussed in [5], while other solids can be covered by our methods as well.

2. Some machinery

In this section, we review some of the basic machinery involved in the construction of indefinite theta series.

2.1. Non-holomorphic theta series

Let $(V, (\cdot, \cdot))$ be an inner product space over \mathbb{R} of signature (p, q) with $pq > 0$, and let

$$D = D(V) := \{ z \in \text{Gr}_q^o(V) \mid Q|_z < 0 \}$$

be the space of oriented negative q -planes in V . Note that D has two components D^\pm . Here $Q(x) = \frac{1}{2}(x, x)$. For $z \in D$ and $x \in V$, the associated majorant is

$$(x, x)_z = (x, x) + 2R(x, z), \quad R(x, z) = -(\text{pr}_z(x), \text{pr}_z(x)).$$

The Siegel Gaussian is

$$\varphi_0(x) = \varphi_0(x, z) = e^{-\pi(x, x)_z} = e^{-\pi(x, x)} \varphi_0^0(x), \quad \varphi_0^0(x) = e^{-2\pi R(x, z)}.$$

There is also a Schwartz form, see [9],

$$\varphi_{KM}(x) = e^{-\pi(x, x)} \varphi_{KM}^0(x)$$

valued in $A^q(D)$, the space of smooth q -forms on D . These forms are equivariant with respect to the natural action of the orthogonal group $G = O(V)$:

$$\varphi_0 \in [S(V) \otimes A^0(D)]^G, \quad \varphi_{KM} \in [S(V) \otimes A^q(D)]^G.$$

Moreover $\varphi_{KM}(x)$ is closed, $d\varphi_{KM}(x) = 0$. With respect to the Weil representation action of the metaplectic group $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ on the Schwartz space $S(V)$, the functions φ_0 and φ_{KM} are eigenfunctions of weights $\frac{1}{2}(p-q)$ and $\frac{1}{2}(p+q)$ respectively for the subgroup $\widetilde{\mathrm{SO}}(2)$.

If $V = L \otimes \mathbb{R}$ for a quadratic lattice L with dual lattice $L^\vee \supset L$, then the (non-holomorphic) theta series for $\mu \in L^\vee$ and $\tau = u + iv \in \mathfrak{H}$,

$$\theta_\mu(\tau, \varphi_0) = \sum_{x \in L + \mu} \mathbf{q}^{Q(x)} \varphi_0^0(v^{\frac{1}{2}}x), \quad \mathbf{q} = e(\tau),$$

and

$$\theta_\mu(\tau, \varphi_{KM}) = \sum_{x \in L + \mu} \mathbf{q}^{Q(x)} \varphi_{KM}^0(v^{\frac{1}{2}}x),$$

have modular transformations of weights $\frac{1}{2}(p-q)$ and $\frac{1}{2}(p+q)$ respectively. The second of these defines a closed q -form on D , invariant with respect to the discriminant group

$$\Gamma_L = \{ \gamma \in \mathrm{O}(V) \mid \gamma L = L, \gamma|_{L^\vee/L} = \mathrm{id} \}.$$

More details can be found in Section 2 of [5].

2.2. Subspaces of D

There are sub-symmetric spaces of D attached non-null vectors in V as follows.

If $x \in V$ is a nonzero vector, let

$$(2.1) \quad D_x = \{ z \in D \mid x \in z^\perp \} = \{ z \in D \mid R(x, z) = 0 \}.$$

This space is only non-empty if $(x, x) > 0$, and in that case can be identified with the the space $D(V_x)$ of oriented negative q -planes in $V_x = x^\perp$. Note that $\mathrm{sig}(V_x) = (p-1, q)$ so that D_x has codimension q in D .

Similarly, for a non-zero vector x , let

$$(2.2) \quad D'_x = \{ z \in D \mid x \in z \} = \{ z \in D \mid (x, x)_z = (x, x) \}.$$

This space is non-empty if and only if $(x, x) < 0$, in which case $\mathrm{sig}(V_x) = (p, q-1)$. Then there is an isomorphism

$$D(V_x) \xrightarrow{\sim} D'_x, \quad \zeta \mapsto [x, \zeta],$$

where, for an oriented $q - 1$ -plane ζ in V_x , $[x, \zeta]$ is the oriented q -plane in V spanned by x and ζ with the orientation induced by prepending x to a properly oriented basis for ζ . Note that D'_x has codimension p in D .

In the case of main interest to us where $\text{sig}(V) = (m - 2, 2)$, the D_x 's for positive vectors x will have complex codimension 1 in the hermitian symmetric space D and the cycles D'_x for negative vectors x will be hyperbolic spaces of real dimension $m - 2$, fixed points of anti-holomorphic involutions of D .

3. The main result

We now consider the construction proposed in [2], and suppose that $\text{sig}(V) = (m - 2, 2)$ and that L is a lattice in V such that with $L \subset L^\vee$. Our version of the ABMP data is a collection $\mathcal{C} = \{C_1, \dots, C_N\}$ of N vectors in V satisfying the N -gon conditions (1.4), (1.5), and (1.6) and hence defining an N -gon $\gamma(\mathcal{C})$ in D^+ . Let $S = S(\mathcal{C})$ be an oriented 2-chain in D^+ with boundary $\gamma(\mathcal{C})$ and let

$$I^0(x, \mathcal{C}) = \int_S \varphi_{KM}^0(x)$$

be the integral of the 2-form $\varphi_{KM}^0(x)$ over S . Since $\varphi_{KM}^0(x)$ is closed, this integral does not depend on the choice of S . We will refer to it as the local theta integral. Our main result is the following.

Theorem 3.1. (i) *The local theta integral is given explicitly by*

$$4 I^0(x; \mathcal{C}) = \mathbf{w}(\mathcal{C}) + \sum_{j=1}^N E_2(C_j, C_{j+1}, x\sqrt{2}),$$

where, for a pair of negative vectors c_1 and c_2 , $E_2(c_1, c_2; x)$ is the generalized error function defined in [1] and $\mathbf{w}(\mathcal{C})$ is defined by (1.10).

(ii) *Let*

$$\Phi(x; \mathcal{C}) := \lim_{r \rightarrow \infty} I^0(rx, \mathcal{C}).$$

Then

$$4 \Phi(x; \mathcal{C}) = \mathbf{w}(\mathcal{C}) + \sum_{j=1}^N \text{sgn}(x, C_j) \text{sgn}(x, C_{j+1}) = \varepsilon(x; \mathcal{C}).$$

(iii) *For $x \in V$, recall that $D_x = \{ z \in D \mid x \in z^\perp \}$. Then*

$$\Phi(x; \mathcal{C}) \neq 0 \implies D_x \cap S \neq \emptyset.$$

In particular, for $x \neq 0$, $\Phi(x; \mathcal{C}) \neq 0$ implies that $(x, x) > 0$, since D_x is non-empty.

(iv) Suppose that $(x, x) > 0$. The cycle $D_x \subset D$ has complex codimension 1 and the space $D^+ - D_x \cap D^+$ is a punctured disk bundle over the contractible space D_x . In particular,

$$(3.1) \quad \pi_1(D^+ - D_x \cap D^+) = \mathbb{Z},$$

canonically. Suppose that x is regular with respect to \mathcal{C} , i.e., $(x, C_j) \neq 0$ for all j . Then,

- (a) $4\Phi(x; \mathcal{C}) \in \mathbb{Z}$,
- (b) the closed loop $\gamma(\mathcal{C})$ lies in $D^+ - D_x \cap D^+$, and,
- (c) under the identification (3.1), its homotopy class is given by $4\Phi(x; \mathcal{C})$.

(v) The constant $\mathbf{w}(\mathcal{C})$ is independent of the choice of negative vector \mathbf{v} . Moreover, $|\mathbf{w}(\mathcal{C})| \leq N - 2$ and $\mathbf{w}(\mathcal{C}) \equiv -N \pmod{4}$.

Remark 3.2. (a) For x regular with respect to \mathcal{C} , we can view $4\Phi(x; \mathcal{C})$ as the linking number of the cycles $\gamma(\mathcal{C})$ and D_x .

(b) The fact that $\mathbf{w}(\mathcal{C})$ does not depend on the choice of the negative vector seems to be a deeper result. At least we do not know a direct geometric proof.

(c) Once the independence of the choice of negative vector is known, the other conditions on $\mathbf{w}(\mathcal{C})$ given in (v) are immediate. Indeed, for a negative vector \mathbf{v} with $(\mathbf{v}, C_1) = 0$, at least 2 terms in (1.10) vanish so that $|\mathbf{w}(\mathcal{C})| \leq N - 2$. On the other hand, we can always take \mathbf{v} so that all the inner products $(\mathbf{v}, C_j) \neq 0$ for all j , and so $-\mathbf{w}(\mathcal{C})$ is the sum of successive products of a collection of N non-zero signs and hence $-\mathbf{w}(\mathcal{C}) \equiv N \pmod{4}$.

(d) In Section 6, we will show by examples that all values of $\mathbf{w}(\mathcal{C})$ subject to the conditions in (v) occur for some choice of \mathcal{C} , cf. (6.9).

Corollary 3.3. For $\mu \in L^\vee/L$ the series

$$\vartheta_\mu(\tau, \mathcal{C}) = \sum_{x \in \mu + L} \varepsilon(x; \mathcal{C}) \mathbf{q}^{Q(x)}$$

is termwise absolutely convergent. Its modular completion is given by

$$\hat{\vartheta}_\mu(\tau, \mathcal{C}) = \sum_{x \in \mu + L} \left(\mathbf{w}(\mathcal{C}) + \sum_{j=1}^N E_2(C_j, C_{j+1}, x\sqrt{2}) \right) \mathbf{q}^{Q(x)}$$

and

$$\hat{\vartheta}_\mu(\tau, \mathcal{C}) = 4 \int_S \theta_\mu(\tau, \varphi_{KM}).$$

The weight $\frac{m}{2}$ modular transformation behavior of $\hat{\vartheta}_\mu(\tau, \mathcal{C})$ follows immediately from that of the theta form.

Proof. The termwise absolute convergence follows from the argument in section 4.6 of [8]. Recall that the key point is that there is a positive definite inner product $(\ , \)_S$ on V such that $(x, x)_z \geq (x, x)_S$ for all $z \in S$ and all $x \in V$, cf. Lemma 4.11 of [8]. On the other hand, if $x \in V$ with $\Phi(x; \mathcal{C}) \neq 0$, there exists a point $z_0 \in S \cap D_x$, and at such a point $(x, x) = (x, x)_{z_0} \geq (x, x)_S$. Thus

$$\sum_{x \in \mu+L} |\Phi(x; \mathcal{C})| e^{-\pi v(x,x)} \leq \frac{1}{2} N \sum_{x \in \mu+L} e^{-\pi v(x,x)_S}$$

converges, since $|\Phi(x; \mathcal{C})|$ is bounded by $\frac{1}{2} N$.

The theta form

$$\theta_\mu(\tau, \varphi_{KM}) = \sum_{x \in \mu+L} \varphi_{KM}^0(v^{\frac{1}{2}}x) q^{Q(x)}$$

is termwise integrable over the compact 2-chain S . By (i) of Theorem 3.1, this integral is the given expression $\hat{\vartheta}_\mu(\tau, \mathcal{C})$. □

3.1. A slight extension

One can also consider a collection $\mathcal{C} = \{C_1, \dots, C_N\}$ of N vectors in V satisfying (1.4) and (1.5), but omitting (1.6). In this case $z_j = [C_{j-1}, C_j]$ is a negative 2-plane, but the z_j 's need not lie in the same component of D . Let $z_j^* = [C_j, -C_{j+1}]$, so that the negative 2-planes z_j and z_j^* coincide but have opposite orientations and hence lie on different components of D . Let

$$\nu_j := \begin{cases} +1 & \text{if } z_j \text{ and } z_{j+1} \text{ lie in the same component of } D, \\ -1 & \text{if } z_j \text{ and } z_{j+1} \text{ lie in different components of } D. \end{cases}$$

Then

$$-\nu_{j-1} = \operatorname{sgn} \left((C_j, C_j)(C_{j-1}, C_{j+1}) - (C_j, C_{j-1})(C_j, C_{j+1}) \right),$$

so that condition (1.6) on \mathcal{C} is equivalent to the condition that $\nu_j = +1$ for all j . To obtain an N -gon, we follow the path connecting successive points in the same component as z_1 , using z_j or z_j^* as needed. Note that

$$(3.2) \quad 1 = \prod_{j=1}^N \nu_j.$$

It is easily checked that the resulting path $\gamma(\mathcal{C})$ is given by

$$(3.3) \quad [C_1, C_2] \longrightarrow [C_2, \nu_1 C_3] \longrightarrow [C_3, \nu_1 \nu_2 C_4] \longrightarrow \dots \\ \dots \longrightarrow [C_{N-1}, \nu_1 \dots \nu_{N-2} C_N] \longrightarrow [C_N, \nu_1 \dots \nu_{N-1} C_1] \longrightarrow [C_1, \nu_1 \dots \nu_N C_2] \\ = [C_1, C_2].$$

Of course, if condition (1.6) is satisfied so that $\nu_j = 1$ for all j , then this is the N -gon considered before. Note that now

$$(3.4) \quad \gamma_j(s) = [C_j, (s-1)\nu_1 \dots \nu_{j-2} C_{j-1} + s\nu_1 \dots \nu_{j-1} C_{j+1}].$$

Let $S = S(\mathcal{C})$ be an oriented 2-chain in D^+ with boundary $\gamma(\mathcal{C})$.

We let

$$\boldsymbol{\nu}_j = \nu_1 \dots \nu_{j-1} \\ \varepsilon(x, \mathcal{C}) = \sum_{j=1}^N \boldsymbol{\nu}_j \left(\operatorname{sgn}(x, C_j) \operatorname{sgn}(x, C_{j+1}) - \operatorname{sgn}(\boldsymbol{\nu}, C_j) \operatorname{sgn}(\boldsymbol{\nu}, C_{j+1}) \right),$$

and

$$w(\mathcal{C}) = - \sum_{j=1}^N \boldsymbol{\nu}_j \operatorname{sgn}(\boldsymbol{\nu}, C_j) \operatorname{sgn}(\boldsymbol{\nu}, C_{j+1}).$$

The following result extends Theorem 3.1 and Corollary 3.3 when condition (1.6) is dropped. Its proof, which we omit, is the same, using the paths (3.4) and keeping track of the extra signs $\boldsymbol{\nu}_j$ in the calculation.

Theorem 3.4. *Suppose that $\mathcal{C} = \{C_1, \dots, C_N\}$ is a collection of vectors satisfying (1.4) and (1.5). Then the series*

$$\vartheta_\mu(\tau, \mathcal{C}) = \sum_{x \in \mu+L} \varepsilon(x, \mathcal{C}) \mathbf{q}^{Q(x)}$$

is termwise absolutely convergent. Its modular completion given by

$$\hat{\vartheta}_\mu(\tau, \mathcal{C}) = \sum_{x \in \mu+L} \left(w(\mathcal{C}) + \sum_{j=1}^N \boldsymbol{\nu}_j E_2(C_j, C_{j+1}, x\sqrt{2}) \right) \mathbf{q}^{Q(x)}$$

has weight $\frac{m}{2}$. Further,

$$\hat{\vartheta}_\mu(\tau, \mathcal{C}) = 4 \int_S \theta_\mu(\tau, \varphi_{KM}).$$

3.2. Comparison with [2]

We finish this section by explaining the relation to the setup of [2]. As in section C of [2], suppose that N is even and consider a collection of vectors $\mathcal{C}' = \{C'_1, \dots, C'_N\}$ satisfying the conditions

$$(3.5) \quad (C'_j, C'_j) < 0$$

$$(3.6) \quad (C'_j, C'_j)(C'_{j+1}, C'_{j+1}) - (C'_j, C'_{j+1})^2 > 0$$

$$(3.7) \quad (C'_j, C'_j)(C'_{j-1}, C'_{j+1}) - (C'_j, C'_{j-1})(C'_j, C'_{j+1}) > 0.$$

Here we have excluded the case of null vectors and have kept in mind the fact that our inner product is the negative of the one in [2]. Note that the left side of the third condition is $(C'_j, C'_j)(C'_{j-1\perp j}, C'_{j+1\perp j})$, so that this condition is equivalent to (C.10) in [2]. The first two conditions are unchanged if the C'_j 's are changed by signs. If we let

$$(3.8) \quad \mathcal{C} = \{C'_1, C'_2, -C'_3, -C'_4, C'_5, \dots, (-1)^{N/2-1}C'_N\},$$

then the collection $\mathcal{C} = \{C_1, \dots, C_N\}$ satisfies our conditions above, i.e., the sign in the third condition is reversed. Of course, the transformation (3.8) can be reversed to yield ABMP type collections from ours. Thus, our results can be applied to prove the conjecture in section C of [2], provided all of the vectors C'_j are timelike, i.e., non-null. We have kernel

$$4\Phi(x; \mathcal{C}) = w(\mathcal{C}) + \sum_{j=1}^N (-1)^{j-1} \operatorname{sgn}(x, C'_j) \operatorname{sgn}(x, C'_{j+1})$$

and

$$w(\mathcal{C}) = - \sum_{j=1}^N (-1)^{j-1} \operatorname{sgn}(\mathbf{v}, C'_j) \operatorname{sgn}(\mathbf{v}, C'_{j+1})$$

for any negative vector \mathbf{v} .

Note that our answer differs slightly from, and corrects, what was proposed in [2].

The termwise absolute convergence goes over immediately to the case where the kernel is

$$\Phi(x, P; \mathcal{C}) = P(x)\Phi(x; \mathcal{C})$$

where $P(x)$ is a function of at most polynomial growth. Also recall that it is shown in [2] that the case of odd N follows from the case of N -even. It remains to remove the restriction that all the C'_j 's are non-null.

4. Proofs

We begin by considering what happens on the regular set

$$\text{Reg}(\mathcal{C}) = V \setminus \bigsqcup_j V_j, \quad V_j = C_j^\perp,$$

and write $\gamma = \gamma(\mathcal{C})$. If x is regular with respect to \mathcal{C} , then $D_x \cap \gamma$ is empty, because every point of γ has the form $[C_j, \star]$ and hence is not in D_x since $(x, C_j) \neq 0$.

Recall from Lemma 5.1 (ii) of [5], that on the set $D - D_x$, we have

$$\varphi_{KM}^0(x) = d\Psi_{KM}^0(x),$$

for an explicit 1-form $\Psi_{KM}^0(x)$. For x regular with respect to \mathcal{C} , so that γ is contained in $D - D_x$, let

$$J^0(x; \mathcal{C}) := \int_\gamma \Psi_{KM}^0(x).$$

Note that, from the formula for $\Psi_{KM}^0(x)$ on the set $D - D_x$, cf. (5.9) and (5.6) in Section 5 of [5],

$$(4.1) \quad \lim_{r \rightarrow \infty} \Psi_{KM}^0(rx) = 0.$$

and hence

$$(4.2) \quad \lim_{r \rightarrow \infty} J^0(rx, \mathcal{C}) = 0.$$

For any x (not necessarily regular), recall that we defined

$$(4.3) \quad \Phi(x, \mathcal{C}) = \lim_{r \rightarrow \infty} I^0(rx, \mathcal{C}).$$

Lemma 4.1. *Suppose that $D_x \cap S = \emptyset$. Then the limit (4.3) exists and is equal to 0.*

Proof. If $D_x \cap S$ is empty, $\Psi_{KM}^0(x)$ is smooth on γ ,

$$I^0(x, \mathcal{C}) = J^0(x, \mathcal{C}),$$

by Stokes, and

$$\lim_{r \rightarrow \infty} I^0(rx, \mathcal{C}) = \lim_{r \rightarrow \infty} J^0(rx, \mathcal{C}) = 0. \quad \square$$

Corollary 4.2. *If $\Phi(x, \mathcal{C}) \neq 0$, then $D_x \cap S \neq \emptyset$.*

Here the condition $\Phi(x, \mathcal{C}) \neq 0$ includes the possibility that the limit does not exist.

Now the analogue of the computations in [8] and [5] yields the following.

Proposition 4.3. *For $x \in \text{Reg}(\mathcal{C})$,*

$$J^0(x, \mathcal{C}) = \int_{\gamma} \Psi_{KM}^0(x) = \frac{1}{4} \sum_{j=1}^N \left(E_2(C_j, C_{j+1}, x\sqrt{2}) - \text{sgn}(x, C_j) \text{sgn}(x, C_{j+1}) \right).$$

Proof. Using the formulas from [5], we have

$$\begin{aligned} & \int_{\gamma} \Psi_{KM}^0(x) \\ &= \sum_j \int_{\gamma_j} \Psi_{KM}^0(x) \\ &= - \sum_j \sqrt{2}(x, \underline{C}_j) \int_1^{\infty} e^{-2\pi t^2(x, \underline{C}_j)^2} \int_{\gamma_j} \varphi_{KM}^{V_j, 0}(tx_{\perp C_j}) dt \\ &= - \sum_j \sqrt{2}(x, \underline{C}_j) \int_1^{\infty} e^{-2\pi t^2(x, \underline{C}_j)^2} I^0(-C_{j-1\perp j}, C_{j+1\perp j}; tx_{\perp j}) dt \\ &= -\frac{1}{2} \sum_j (\sqrt{2}x, \underline{C}_j) \\ &\quad \times \int_1^{\infty} e^{-2\pi t^2(x, \underline{C}_j)^2} [E_1(C_{j+1\perp j}, tx_{\perp j}\sqrt{2}) + E_1(C_{j-1\perp j}, tx_{\perp j}\sqrt{2})] dt \\ &= \frac{1}{4} \left[\sum_j E_2(C_j, C_{j+1}; x\sqrt{2}) - \sum_j \text{sgn}(C_j, x) \text{sgn}(C_{j+1}, x) \right]. \end{aligned}$$

Here in the first step we use the relation of Corollary 6.3,

$$\kappa_j^* \Psi_{KM}^0(x) = -\sqrt{2}(x, \underline{C}_j) \int_1^{\infty} e^{-2\pi t^2(x, \underline{C}_j)^2} \varphi_{KM}^{V_j, 0}(tx_{\perp C_j}) dt,$$

where we write $\kappa_j : D(V_j) \rightarrow D(V)$ for the standard embedding. This reduces the integral along γ_j in D'_{C_j} to the integral yielding Zwegers' identity. In the last step, we use the recursion formula of Proposition 7.3 of [5]. Recall, that $\underline{C} = |(C, C)|^{-\frac{1}{2}}C$. □

Proposition 4.4. *For $x \in \text{Reg}(\mathcal{C})$, the limit (4.3) defining $\Phi(x; \mathcal{C})$ exists and*

$$I^0(x; \mathcal{C}) = \Phi(x; \mathcal{C}) + J^0(x; \mathcal{C})$$

$$= \Phi(x; \mathcal{C}) + \frac{1}{4} \sum_{j=1}^N \left(E_2(C_j, C_{j+1}, x\sqrt{2}) - \operatorname{sgn}(x, C_j) \operatorname{sgn}(x, C_{j+1}) \right).$$

Corollary 4.5. *On the regular set $\operatorname{Reg}(\mathcal{C})$, the quantity*

$$\Phi(x; \mathcal{C}) - \frac{1}{4} \sum_{j=1}^N \operatorname{sgn}(x, C_j) \operatorname{sgn}(x, C_{j+1})$$

is the restriction of the continuous function

$$I^0(x; \mathcal{C}) - \frac{1}{4} \sum_{j=1}^N E_2(C_j, C_{j+1}, x\sqrt{2})$$

on V .

Proof of Proposition 4.4. The point is the following result, which reinvents the wheel.

Lemma 4.6. *For x regular with respect to \mathcal{C} and for $r > 0$, the difference*

$$I^0(rx; \mathcal{C}) - J^0(rx; \mathcal{C}), \quad r > 0,$$

is independent of r . In particular, the limit (4.3) exists due to (4.2).

Proof. From the definition, cf. Section 5 of [5] and with the scaling by $r^{\frac{1}{2}}$ used there, we have

$$\begin{aligned} \Psi_{KM}^0(r^{\frac{1}{2}}x) &= - \int_1^\infty \psi_{KM}^0(t^{\frac{1}{2}}r^{\frac{1}{2}}x) t^{-1} dt \\ &= - \int_r^\infty \psi_{KM}^0(t^{\frac{1}{2}}x) t^{-1} dt, \end{aligned}$$

where $\psi_{KM}^0(x)$ is the Schwartz 1-form defined in (5.6) of [5]. Then

$$\frac{\partial}{\partial r} \Psi_{KM}^0(r^{\frac{1}{2}}x) = r^{-1} \psi_{KM}^0(r^{\frac{1}{2}}x).$$

Recall that, for $R(x, z) = |(\operatorname{pr}_z(x), \operatorname{pr}_z(x))|$,

$$\psi^0(t^{\frac{1}{2}}x) = (\text{form valued polynomial in } t^{\frac{1}{2}}x) \cdot e^{-2\pi t R(x,z)},$$

so that the integral only makes sense when $R(x, z) > 0$, i.e., on $D - D_x$, and the exponential decay (4.1) holds on this set. On the other hand, by (i) of

Lemma 5.1, we have

$$r \frac{\partial}{\partial r} \varphi_{KM}^0(r^{\frac{1}{2}}x) = d\psi_{KM}^0(r^{\frac{1}{2}}x).$$

Thus,

$$\begin{aligned} & r \frac{\partial}{\partial r} (I^0(r^{\frac{1}{2}}x; S) - J^0(r^{\frac{1}{2}}x; S)) \\ &= \int_S r \frac{\partial}{\partial r} \varphi_{KM}^0(r^{\frac{1}{2}}x) - \int_{\partial S} r \frac{\partial}{\partial r} \Psi_{KM}^0(r^{\frac{1}{2}}x) \\ &= \int_S d\psi_{KM}^0(r^{\frac{1}{2}}x) - \int_{\partial S} \psi^0(r^{\frac{1}{2}}x) \\ &= 0. \end{aligned} \quad \square$$

This proves the proposition. □

A crucial property of the form $\varphi_{KM}^0(x)$ is the following, whose proof will be given in the next section.

Proposition 4.7. *For x regular with respect to \mathcal{C} , $4\Phi(x, \mathcal{C})$ is in \mathbb{Z} .*

On the regular set, by Proposition 4.4, we have the identity

$$(4.4) \quad I^0(x; \mathcal{C}) - \frac{1}{4} \sum_{j=1}^N E_2(C_j, C_{j+1}, x\sqrt{2}) = \Phi(x; \mathcal{C}) - \frac{1}{4} \sum_{j=1}^N \operatorname{sgn}(x, C_j) \operatorname{sgn}(x, C_{j+1}).$$

The left hand side here is a continuous function on all of V , while, by Proposition 4.7, the right side lies in $\frac{1}{4}\mathbb{Z}$ and hence is locally constant on the regular set. Thus, the left hand side of (4.4) is constant on V . Computing the right hand side on any negative vector \mathbf{v} in $\operatorname{Reg}(\mathcal{C})$ and noting that $D_{\mathbf{v}}$ is empty so that $\Phi(\mathbf{v}; \mathcal{C}) = 0$, we obtain the constant

$$-\frac{1}{4} \sum_{j=1}^N \operatorname{sgn}(\mathbf{v}, C_j) \operatorname{sgn}(\mathbf{v}, C_{j+1}) = \frac{1}{4} \mathbf{w}(\mathcal{C}).$$

In particular, this value is independent of the choice of \mathbf{v} , as claimed.

Thus, we have

$$4I^0(x; \mathcal{C}) = \mathbf{w}(\mathcal{C}) + \sum_{j=1}^N E_2(C_j, C_{j+1}, x\sqrt{2}),$$

valid for all x in V , as claimed in (i) of Theorem 3.1. Using this expression, we see that

$$4\Phi(x; \mathcal{C}) = \mathbf{w}(\mathcal{C}) + \sum_{j=1}^N \operatorname{sgn}(x, C_j) \operatorname{sgn}(x, C_{j+1})$$

for all x in V . This completes the proof of parts (i), (ii) and (iii) of Theorem 3.1. Part (iv) will be proved in the next section.

5. Proof of Proposition 4.7

Let D^+ be the component of D containing $\gamma = \gamma(\mathcal{C})$ and let $D_x^+ = D_x \cap D^+$. Let \mathcal{T}_ν be the tube of radius ν around the cycle D_x^+ in D^+ , and take ν small enough so that \mathcal{T}_ν is bounded away from $\gamma(\mathcal{C})$. In particular, γ lies in $D^+ - \mathcal{T}_\nu$. Note that, for ν sufficiently small, $\mathcal{T}_\nu - D_x^+$ is a punctured disk bundle over the contractible space D_x^+ . Taking a point $\zeta_0 \in \gamma$ as base point, we have

$$\pi_1(D^+ - \mathcal{T}_\nu, \zeta_0) = \pi_1(D^+ - D_x^+, \zeta_0) \simeq \mathbb{Z}.$$

More explicitly, let D_{x,z_0}^\vee be the complementary cycle to D_x^+ through a point $z_0 \in D_x^+$. Here recall that D_x^+ is the fixed point locus of the involution $\sigma_x : D^+ \rightarrow D^+$ induced by reflection in the subspace $V_x = x^\perp$. Then D_{x,z_0}^\vee is the fixed point locus of the involution $\sigma_x \circ \theta_{z_0}$ where θ_{z_0} is the Cartan involution at z_0 . Alternatively, D_{x,z_0}^\vee is one component of the cycle associated to the subspace

$$V_{x,z_0}^\vee = z_0 + \mathbb{R}x$$

of V of signature $(1, 2)$. Since D^+ is the union of the cycles D_{x,z_0}^\vee as z_0 runs over D_x^+ , we choose z_0 so that our base point ζ_0 lies in D_{x,z_0}^\vee . We choose an isomorphism $D_{x,z_0}^\vee \simeq \mathfrak{H}$ with $z_0 \mapsto i$ so that $\mathcal{T}_\nu \cap D_{x,z_0}^\vee$ maps to a small disk about i . The image in \mathfrak{H} of the base point ζ_0 lies outside of this disk and we choose a disk Δ_i in \mathfrak{H} centered at i and passing through this image. A generator of $\pi_1(D^+ - \mathcal{T}_\nu, \zeta_0)$ is then given by the image γ_{x,z_0,ζ_0} in $D^+ - \mathcal{T}_\nu$ of the counterclockwise boundary of Δ_i .

Our piecewise smooth loop $\gamma(\mathcal{C})$ in $D^+ - \mathcal{T}_\nu$ passes through ζ_0 and is homotopic to an integer multiple k of γ_{x,z_0,ζ_0} . Moreover, the homotopy can be realized as a piecewise smooth oriented 2-chain \mathbf{h} in $D - \mathcal{T}_\nu$, i.e.,

$$\partial \mathbf{h} = \gamma(\mathcal{C}) - k \cdot \gamma_{x,z_0,\zeta_0}.$$

Let S_{x,z_0} be the image of Δ_i in $D_{x,z_0}^\vee \subset D^+$. We may then take

$$S = \mathbf{h} + k \cdot S_{x,z_0},$$

so that

$$I^0(x; \mathcal{C}) = \int_{\mathbf{h}} \varphi_{KM}^0(x) + k \cdot \int_{S_{x,z_0}} \varphi_{KM}^0(x).$$

Now $R(x, z)$ is uniformly bounded away from 0 on $D^+ - \mathcal{T}_\nu$, and so

$$\lim_{r \rightarrow \infty} \int_{\mathbf{h}} \varphi_{KM}^0(rx) = 0.$$

Claim. On the other hand,

$$\lim_{r \rightarrow \infty} \int_{S_{x,z_0}} \varphi_{KM}^0(rx) = \int_{D_{x,z_0}^\vee} \varphi_{KM}^0(x) = \frac{1}{4},$$

by the Thom form property of $\varphi_{KM}^0(x)$, [9], [10]. Thus

$$(5.1) \quad 4 \Phi(x; \mathcal{C}) = k.$$

To justify the limit, note that

$$\int_{S_{x,z_0}} \varphi_{KM}^0(rx) = \int_{D_{x,z_0}^\vee} \varphi_{KM}^0(rx) - \int_{D_{x,z_0}^\vee - S_{x,z_0}} \varphi_{KM}^0(rx).$$

But $R(x, z)$ is bounded below by some R_0 , uniformly for $z \in D^+ - \mathcal{T}_\nu$ and the second integral here decays at least like $C \exp(-\pi r^2 \frac{1}{2} R_0)$ as r goes to infinity. On the other hand, by Proposition 6.2 of [10], we have

$$\int_{D_{x,z_0}^\vee} \varphi_{KM}^0(rx) = \frac{1}{4}.$$

Note that the integral in [10] is taken over both connected components, whereas our domain of integration D_{x,z_0}^\vee is connected.

Remark. The formula (5.1) gives a nice explanation of the integer $4 \Phi(x; \mathcal{C})$.

6. Examples: signature (1, 2)

We now consider the case where (V, Q) is the quadratic space of traceless 2×2 matrices with the quadratic form $Q(X) = \det(X)$. The corresponding bilinear form $(X, Y) = -\text{tr}(XY)$ has signature is $(1, 2)$. There are identifications

$$\mathfrak{H}^\pm \xrightarrow{\sim} \mathbb{H}^\pm \xrightarrow{\sim} D = D(V)$$

where $\mathfrak{H}^\pm = \mathbb{C} - \mathbb{R}$ and $\mathbb{H}^\pm = \{X \in V \mid Q(X) = 1\}$ given by

$$z = x + iy \mapsto X(z) = y^{-1} \begin{pmatrix} -x & |z|^2 \\ -1 & x \end{pmatrix}, \quad X \mapsto X^\perp = \mathbf{z},$$

where a basis u_0, u_1 for the negative 2-plane X^\perp is properly oriented if $X \wedge u_0 \wedge u_1 \in \Lambda^3(V)^+$. Here we orient V by choosing the ordered orthogonal basis

$$e_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} & -1 \\ -1 & \end{pmatrix}$$

and taking $\Lambda^3(V)^+ = \mathbb{R}_{>0} \cdot e_1 \wedge e_2 \wedge e_3$. The components of \mathbb{H}^\pm are determined by $X \in \mathbb{H}^\pm$ if $\text{sgn}(X, e_1) = \pm 1$. The inverse of the second isomorphism is given as follows. If u_0, u_1 is an orthogonal basis for the negative 2-plane \mathbf{z} , take a vector $X \in \mathbf{z}^\perp$ with $Q(X) = 1$ and such that $X \wedge u_0 \wedge u_1 \in \Lambda^3(V)^+$. Explicitly, if $u_0 = u_{10}e_1 + u_{20}e_2 + u_{30}e_3$ and $u_1 = u_{11}e_1 + u_{21}e_2 + u_{31}e_3$, the vector X is given by the hyperbolic cross product

$$X = u_0 \times u_1 = (u_{20}u_{31} - u_{30}u_{21})e_1 + (u_{10}u_{31} - u_{30}u_{11})e_2 - (u_{10}u_{21} - u_{20}u_{11})e_3.$$

(Note the signs for the second and third component).

Vectors $X \in V$ with $Q(X) \neq 0$ define cycles in D by

$$D_X = \{\mathbf{z} \in D \mid X \in \mathbf{z}^\perp\}, \quad \text{if } Q(X) > 0,$$

and

$$c_X = \{\mathbf{z} \in D \mid X \in \mathbf{z}\}, \quad \text{if } Q(X) < 0,$$

so that D_X is a union of two points D_X^\pm and c_X is a union of two geodesic c_X^\pm , one in each component of D . Note that D_X determines the positive line $\mathbb{R} \cdot X$ when $Q(X) > 0$, and c_X determines the negative line $\mathbb{R} \cdot X$ when $Q(X) < 0$. In the later case $\mathbb{R} \cdot X$ is the intersection of any two distinct points on c_X , viewed as negative 2-planes.

These cycles can be described in \mathfrak{H}^\pm in a familiar classical way. For

$$X = \begin{pmatrix} b & 2c \\ -2a & -b \end{pmatrix} =: [a, b, c], \quad -Q(X) = b^2 - 4ac,$$

we have

$$(X, X(z)) = \frac{2}{y}(a|z|^2 + bx + c)$$

and

$$R(X, z) = -(\text{pr}_z(X), \text{pr}_z(X)) = \frac{2}{y^2}|az^2 + bz + c|^2.$$

Then

$$D_X = \frac{-b}{2a} \pm \frac{i\sqrt{d}}{2a}, \quad \text{if } Q(X) > 0$$

and

$$c_X = \{ z \in \mathfrak{H}^\pm; a|z|^2 + b\text{Re}(z) + c = 0 \}, \quad \text{if } Q(X) < 0.$$

Note that c_X^+ separates $D^+ \simeq \mathfrak{H}^+$ into two components. We orient the cycle by asserting that the component given by $(X(z), X) > 0$ lies to the left of the geodesic and the one given by $(X(z), X) < 0$ to the right. Hence for $X = [a, b, c]$ with $a \neq 0$ as above, c_X^+ is a semi-circle in D^+ oriented clockwise when $a > 0$ and the vertical geodesic arising from $X = [0, b, c]$ gives for $b > 0$ the oriented half line $(-\frac{c}{b} + i\infty, -\frac{c}{b})$.

Given two points z_1, z_2 in the same component of \mathfrak{H}^\pm , say \mathfrak{H}^+ , the unique geodesic c_{12} containing both points arises from the negative vector

$$Y(z_1, z_2) := X(z_1) \times X(z_2) = \frac{1}{y_1 y_2} \begin{pmatrix} \frac{1}{2}(|z_1|^2 - |z_2|^2) & x_1|z_2|^2 - x_2|z_1|^2 \\ x_1 - x_2 & \frac{1}{2}(|z_2|^2 - |z_1|^2) \end{pmatrix}.$$

Indeed, this negative vector is orthogonal to both $X(z_1)$ and $X(z_2)$ and hence lies in \mathfrak{z}_1 and in \mathfrak{z}_2 , so that \mathfrak{z}_1 and \mathfrak{z}_2 both lie on c_{12} . Moreover, c_{12} is oriented so that it runs from z_1 to z_2 . By $\text{SL}_2(\mathbb{R})$ equivariance, it suffices to check for $z_1 = i$ and $z_2 = ir, r > 0$ and this case is immediate.

Now suppose that we have three distinct points z_1, z_2 and z_3 in \mathfrak{H}^+ and hence geodesic arcs c_{12} and c_{23} associated to $Y_{12} = X(z_1) \times X(z_2)$ and $Y_{23} = X(z_2) \times X(z_3)$. The negative vectors Y_{12} and Y_{23} both lie in $\mathfrak{z}_2 \in D^+$ so that it is natural to ask if $[Y_{12}, Y_{23}]$ is a properly oriented basis.

Lemma 6.1. *The cross product of Y_{12} and Y_{23} is given by*

$$(6.1) \quad (X(z_1) \times X(z_2)) \times (X(z_2) \times X(z_3)) = \alpha(z_1, z_2, z_3) X(z_2).$$

where

$$(6.2) \quad \alpha(z_1, z_2, z_3) = \frac{x_1(|z_2|^2 - |z_3|^2) + x_2(|z_3|^2 - |z_1|^2) + x_3(|z_1|^2 - |z_2|^2)}{2y_1 y_2 y_3}.$$

In particular,

$$(6.3) \quad \mathbf{z}_2 = [X(z_1) \times X(z_2), X(z_2) \times X(z_3)] \iff \alpha(z_1, z_2, z_3) > 0.$$

Moreover, $\alpha(z_1, z_2, z_3) > 0$ if and only if the geodesics associated to $X(z_1) \times X(z_2)$ and $X(z_2) \times X(z_3)$ perform a left turn at z_2 .

Remark 6.2. For example, for $z_1 = i$, $z_2 = ir$ and $z_3 = x + iy$, the expression in the lemma is $x(1 - r^2)/2ry$ so that, if $r > 1$, the expression is positive for $x < 0$ (left turn) and negative for $x > 0$ (right turn).

Here is a recipe for recovering N -gon data $\mathcal{C} = \{C_1, \dots, C_N\}$ from a geodesic N -gon in $\mathfrak{H} \simeq D^+$ with vertices $\mathcal{Z} = \{z_1, \dots, z_N\}$. The geodesic arc in D^+ containing $\mathbf{z}_{j-1} = [C_{j-1}, C_j]$ and $\mathbf{z}_j = [C_j, C_{j+1}]$ is $c_{C_j}^+$, while that containing \mathbf{z}_j and $\mathbf{z}_{j+1} = [C_{j+1}, C_{j+2}]$ is $c_{C_{j+1}}^+$. These arcs must be distinct since otherwise the vectors C_j and C_{j+1} would be collinear, whereas they span the 2-plane \mathbf{z}_j . Also, if $z_j \in \mathfrak{H}$ corresponds to $\mathbf{z}_j \in D^+$, then C_j must be proportional to $Y_{j-1j} = X(z_{j-1}) \times X(z_j)$ since they determine the same geodesic arc. As observed earlier, the geometry does not change if the vectors in \mathcal{C} are scaled independently by positive constants. Up to such a scaling, we may then write $C_j = \epsilon_j Y_{j-1j}$ for some signs $\epsilon_j = \pm 1$. Also notice that at each vertex z_j , either a left or a right turn is made, since successive arcs are distinct. Let $\tau_j = +1$ for a left turn and -1 for a right turn. It then follows from Lemma 6.1 that

$$\mathbf{z}_j = [Y_{j-1j}, \tau_j Y_{jj+1}] = [\epsilon_j C_j, \tau_j \epsilon_{j+1} C_{j+1}] = [C_j, C_{j+1}],$$

so that

$$(6.4) \quad \epsilon_{j+1} = \epsilon_j \tau_j.$$

Also note that the geometry does not change if all C_j are scaled by -1 . Thus, we may assume that $\epsilon_1 = 1$ and conclude that, for $j \geq 2$,

$$(6.5) \quad \epsilon_j = \prod_{1 \leq i \leq j-1} \tau_i.$$

Note that the crucial fact is that the chain of geodesics must close up, so that

$$\mathbf{z}_N = [C_N, C_1] = [Y_{N-1N}, \tau_N Y_{N,1}] = [\epsilon_N C_N, \tau_N \epsilon_1 C_1].$$

Since $\epsilon_1 = 1$, this amounts to the total turning condition

$$(6.6) \quad \prod_{j=1}^N \tau_j = 1.$$

Thus, we have the following recovery recipe.

Proposition 6.3. (i) Let $\mathcal{Z} = \{z_1, \dots, z_N\}$ be the set of vertices of an N -gon in $\mathfrak{H} \simeq D^+$ defined by a collection $\mathcal{C} = \{C_1, \dots, C_N\}$. Then the geodesic arcs joining successive vertices are distinct and, up to admissible scaling, $C_j = \epsilon_j X(z_{j-1}) \times X(z_j)$ where ϵ_j is determined by the turning signs τ_j via (6.5).
(ii) Suppose that $\mathcal{Z} = \{z_1, \dots, z_N\}$ is an ordered set of N points in \mathfrak{H} with no three consecutive points collinear. Let

$$C_j = \epsilon_j X(z_{j-1}) \times X(z_j),$$

where ϵ_j is defined by (6.5) such that the total turning condition (6.6) is satisfied. Then the collection

$$\mathcal{C} = \{C_1, \dots, C_N\}$$

satisfies the N -gon conditions and the associated N -gon has vertices z_j .

Proof. It remains to check that the collection \mathcal{C} in part (ii) satisfies the N -gon conditions. The cross product is equivariant for the action of $SL_2(\mathbb{R})$, i.e., $(g \cdot A) \times (g \cdot B) = g \cdot (A \times B)$. Since

$$\alpha(i, ir, x + iy) = (2ry)^{-1} x(1 - r^2),$$

the requirements that the z_j 's give rise to distinct successive geodesics imply that $\alpha(z_{j-1}, z_j, z_{j+1}) \neq 0$ for all j and hence $C_j \times C_{j+1}$ is a nonzero multiple of $X(z_j)$. Thus, $[C_j, C_{j+1}]$ is a negative 2-plane. We must still check the condition

$$(6.7) \quad (C_j, C_j)(C_{j-1}, C_{j+1}) - (C_j, C_{j-1})(C_j, C_{j+1}) < 0,$$

where we note that

$$\epsilon_{j-1}\epsilon_{j+1} = \tau_{j-1}\tau_j.$$

By equivariance, it suffices to consider the sequence of 4 points

$$\{z_{j-2}, z_{j-1}, z_j, z_{j+1}\} = \{i, ir, x + iy, x' + iy'\}$$

with $r > 1$. The turning signs are then

$$\tau_{j-1} = \text{sgn}(\alpha(i, ir, x + iy)) = \text{sgn}(x(1 - r^2))$$

$$\tau_j = \operatorname{sgn}(\alpha(ir, x + iy, x' + iy')) = \operatorname{sgn}(r^2(x' - x) + x|z'|^2 - x'|z|^2).$$

The quantity on the left side of (6.7) is then $\tau_{j-1}\tau_j$ times

$$(2r^3y^3y')^{-1}(1 - r^2)x(r^2 + |z|^2)(r^2(x - x') + x'|z|^2 - x|z'|^2),$$

so that condition (6.7) holds! □

Remark 6.4. *Note that the condition (6.6) in the characterization of the N -gons arising from collections \mathcal{C} precludes configurations with an odd number of right turns. For example, a triangle with edges oriented clockwise cannot arise. However, if the N -gon defined by $\mathcal{Z} = \{z_1, \dots, z_N\}$ has an even number of left turns one can simply swap the orientation of the N -gon to obtain an ‘admissible’ collection \mathcal{C} , and the indefinite theta series to the surface S associated to \mathcal{Z} will be the negative of the one associated to \mathcal{C} . This takes care of N odd. On the other hand, for N even it is easy to construct an N -gon with $(N - 1)$ left turns and one right turn. In that case Theorem 3.4 applies, whose formula also can be easily shown directly as follows. The right turn occurs at z_N then use another negative vector C'_N to geodesically cut-off the right turn close to Z_N to bisect S into a (clockwise oriented) triangle formed by $\{C_N, C_1, C'_N\}$ and an ‘admissible’ $(N + 1)$ -gon obtained from $\{C_1, \dots, C_N, C'_N\}$. Then combining the two indefinite theta series gives the one given in Theorem 3.4.*

Next, we consider the invariant

$$w(\mathcal{C}) = - \sum_j \operatorname{sgn}((e_2, C_j)) \operatorname{sgn}((e_2, C_{j+1})).$$

Note that

$$(e_2, Y(z_1, z_2)) = (y_1y_2)^{-1}(|z_2|^2 - |z_1|^2),$$

so that

$$\operatorname{sgn}((e_2, C_j)) = \epsilon_j \operatorname{sgn}(|z_j|^2 - |z_{j-1}|^2).$$

Since $\epsilon_j\epsilon_{j+1} = \tau_j$, see (6.4),

$$(6.8) \quad \operatorname{sgn}((e_2, C_j)) \operatorname{sgn}((e_2, C_{j+1})) = \tau_j \operatorname{sgn}(|z_j|^2 - |z_{j-1}|^2) \operatorname{sgn}(|z_{j+1}|^2 - |z_j|^2).$$

Thus, the contribution of this term is determined by the turn made at z_j and whether the incoming and outgoing edges at this point are inward or outward bound with respect to the origin in \mathbb{C} , information that can be read off immediately from the configuration $\mathcal{Z} = \{z_1, \dots, z_N\}$.

For the chain

$$\mathcal{Z} = \{ z_0, z_1, \dots, z_{N-1} \},$$

only left turns occur, i.e., $\tau_j = 1$ for all j , and for $1 \leq j \leq N - 2$ the incident edges are both outbound, so these vertices contribute $2 - N$ to the sum. At the vertices z_0 and z_{N-1} the incident edges are have opposite directions and so these two vertices contribute an additional $+2$, and $w(\mathcal{C}) = 4 - N$.

Next let $z'_r = r + iT$ for $T \gg 1$ and $r \in \mathbb{Z}$, and consider the chain

$$\mathcal{Z} = \{ z_0, \dots, z_{k-1}, z'_{\ell-1}, z'_{\ell-2}, \dots, z'_0 \}, \quad N = k + \ell.$$

Left turns are made at the vertices $z_0, \dots, z_{k-1}, z'_{\ell-1}$ and z'_0 , and right turns are made at the remaining vertices $z'_1, \dots, z'_{\ell-2}$. Thus, we require² that $\ell \geq 2$ is even. At the vertices z_0 and $z'_{\ell-1}$, the incident edges are have opposite directions, so these contribute $+2$. At the vertices z_1, \dots, z_{k-1} , the incident edges are both outbound resulting in a net contribution of $1 - k$ to $w(\mathcal{C})$. At the vertex z'_0 the incident edges are both inbound yielding a contribution of -1 . Finally, at the remaining vertices, $z'_1, \dots, z'_{\ell-2}$ where right turns occur, the incident edges are both inbound, so these contribute $\ell - 2$. Hence $w(\mathcal{C}) = \ell - k = N - 2k$. Since $k \geq 1$ and $\ell \geq 2$ resulting range of values for $w(\mathcal{C})$ is

$$(6.9) \quad \begin{aligned} &\{4 - N, 8 - N, \dots, N - 8, N - 4\} && \text{for } N \text{ even,} \\ &\{4 - N, 8 - N, \dots, N - 6, N - 2\} && \text{for } N \text{ odd.} \end{aligned}$$

Remark 6.5. Any quadratic space V of signature $(m - 2, 2)$ for $m \geq 3$ contains a subspace of signature $(1, 2)$ and the N -gons just described can be realized in $D(V)^+$. Thus, all of the values in (6.9) occur in the general case.

We close the section with some additional explicit examples.

Consider for $T > 1$

$$\begin{aligned} C_1 &= \begin{pmatrix} -1 & 1 \\ & 1 \end{pmatrix} = \frac{1}{2}e_1 - e_2 - \frac{1}{2}e_3, & C_3 &= \begin{pmatrix} 1 & 1 \\ & -1 \end{pmatrix} = \frac{1}{2}e_1 + e_2 - \frac{1}{2}e_3, \\ C_2 &= \begin{pmatrix} & T^2 + 1/4 \\ 1 & \end{pmatrix} = \frac{T^2 - 3/4}{2}e_1 - \frac{T^2 + 5/4}{2}e_3, & C_4 &= \begin{pmatrix} & -1 \\ -1 & \end{pmatrix} = e_3. \end{aligned}$$

These vectors satisfy the N -gon condition, and the corresponding vertices in \mathfrak{H}^+ are

$$z_1 = \frac{1}{2} + Ti, \quad z_2 = -\frac{1}{2} + Ti, \quad z_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad z_4 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

²Formally, for $\ell = 0$ we obtain the previous example.

Note that $c_{C_1}^+$ is the vertical geodesic $x = \frac{1}{2}$ going up, $c_{X_3}^+$ the one $x = -\frac{1}{2}$ going down, $c_{C_2}^+$ is the half circle $|z|^2 = T^2 + 1/4$ going anti-clockwise, and c_4^+ the half circle $|z| = 1$ going clockwise. Hence $\gamma(\mathcal{C})$ is the boundary of the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ cut-off at height T (and connected by the geodesic arc segment), and we can take $S_T(\mathcal{C})$ to be region enclosed by $\gamma(\mathcal{C})$. Further, note $w(\mathcal{C}) = 0$ and hence

$$I(D_X, S_T(\mathcal{C})) := \frac{1}{4}\varepsilon(X; \mathcal{C}) = \frac{1}{4} \sum_{j=1}^4 \mathrm{sgn}(X, C_j)\mathrm{sgn}(X, C_{j+1})$$

gives the intersection number $I(D_X, S_T(\mathcal{C}))$ of the 0-cycle D_X with the surface $S_T(\mathcal{C})$, that is, $I(D_X, S_T(\mathcal{C})) = 1$ if D_X lies in the interior of $S_T(\mathcal{C})$ and $I(D_X, S_T(\mathcal{C})) = 0$ if D_X lies outside $S_T(\mathcal{C})$.

We now fix the lattice $L = \{[a, b, c]; a, b, c \in \mathbb{Z}\}$ which is stabilized by $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Then

$$\int_{S_T(\mathcal{C})} \theta(\tau, \varphi_{KM}, L) = \frac{1}{4} \sum_{j=1}^4 \sum_{X \in L} E_2(C_j, C_{j+1}, X\sqrt{2}) \mathbf{q}^{\mathbf{Q}(X)}$$

is the modular completion of the

$$\sum_{X \in L; \mathbf{Q}(X) \geq 0} I(D_X, S_T(\mathcal{C})) \mathbf{q}^{\mathbf{Q}(X)}.$$

Note for fixed $n > 0$, the quantity $\sum_{\substack{X \in L \\ \mathbf{Q}(X) = n}} I(D_X, S_T(\mathcal{C}))$ exactly counts twice the number of CM points of discriminant $-n$ in the standard fundamental domain \mathcal{F} for $\mathrm{SL}_2(\mathbb{Z})$ below the geodesic defined by C_2 (the factor 2 arises from x and $-x$ giving rise to the same CM point.) Thus, we can view the theta integral as the modular completion for the generating series of “truncated” class numbers.

It is natural to take the limit $T \rightarrow \infty$. In fact, the first author considered exactly the resulting theta integral over the non-compact domain \mathcal{F} in his PhD thesis written under the supervision of the second author. We have

Theorem 6.6. [3] [Zagier’s weight 3/2 Eisenstein series.] *The theta series $\theta(\tau, \varphi_{KM}, L)$ is exponentially decaying as a differential form at all rational cusps of \mathfrak{H} (but not term wise), and the theta integral $\int_{\mathcal{F}} \theta(\tau, \varphi_{KM}, L)$ is equal to Zagier’s weight 3/2 Eisenstein series [14, 7], i.e.,*

$$2 \int_{\mathcal{F}} \theta(\tau, \varphi_{KM}, L) = \sum_{n \geq 0} H(n)q^n + \frac{v^{-1/2}}{16\pi} \sum_{n \in \mathbb{Z}} \beta(4\pi n^2 v)q^{-n^2}.$$

Here $H(n)$ denotes the Kronecker Hurwitz class number (with $H(0) = -1/12$) and $\beta(s) = \int_1^\infty e^{-st}t^{-3/2}dt$.

Remark 6.7. Since $\lim_{T \rightarrow \infty} \frac{1}{T^2}C_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ we can view the limit process as replacing C_2 with the null vector $u := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In fact, [2] in their treatment allow the C 's to be rationally null as well. It is hence suggestive to replace C_2 with u and perform the same analysis as before. However, because of convergence issues one needs to proceed carefully. In fact, the non-holomorphic tail in Zagier's series exactly arises from the failure of term wise convergence.

Nevertheless, in [6] it is shown that the theta series $\theta(\tau, \varphi_{KM})$ extends to a differential form on the Borel-Serre enlargement \overline{D} of the symmetric space D (which eg. for the upper half plane adds a real line to each rational cusp). Then a pair of non-zero non-positive vectors C_j, C_{j+1} , rational if null, will define a point z_j in \overline{D} . If the collection \mathcal{C} satisfies the (analogous) N -gon conditions then one can obtain a closed loop $\gamma(\mathcal{C})$ in \overline{D} as before and then a region $S(\mathcal{C})$ over which one can integrate the theta series. We will revisit this case in the near future. Note that for signature $(p, 1)$ we discussed the null case in [4].

Now consider

$$\begin{aligned} C'_1 &= \begin{pmatrix} -3/2 & -5/2 \\ -1 & 3/2 \end{pmatrix} = -\frac{3}{4}e_1 - \frac{3}{2}e_2 - \frac{7}{4}e_3, & C'_2 &= \begin{pmatrix} & 4 \\ 1 & \end{pmatrix} = \frac{3}{2}e_1 - \frac{5}{2}e_3, \\ C'_3 &= \begin{pmatrix} 3/2 & -5/2 \\ -1 & -3/2 \end{pmatrix} = -\frac{3}{4}e_1 + \frac{3}{2}e_2 - \frac{7}{4}e_3, & C'_4 &= \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = -e_3. \end{aligned}$$

The associated geodesics intersect at the same four points as above (with $T = \sqrt{15}/2$), however in a different configuration:

$$z_1 = \frac{1}{2} + \frac{\sqrt{15}}{2}i, \quad z_2 = -\frac{1}{2} + \frac{\sqrt{15}}{2}i, \quad z_3 = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad z_4 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

Now γ_1 is the geodesic “diagonal” connecting the lower left vertex $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ with the upper right one $\frac{1}{2} + \frac{\sqrt{15}}{2}i$ (in this direction) and γ_3 is the other diagonal connecting $-\frac{1}{2} + \frac{\sqrt{15}}{2}i$ with $\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Of course γ_1 and γ_3 intersect in the interior of the square, but this is not an issue. The geodesic γ_2 is the half circle $|z| = 2$ oriented counter-clockwise while γ_4 connects $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ with $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ on the unit circle. Note that while the geodesic associated to C'_4 is oriented *clockwise*, the geodesic arc segment γ_4 corresponding to C'_4 is oriented *counter-clockwise*. The reason is that $[C'_3, C'_4]$ and $[C'_4, C'_1]$ define points on the upper half plane (and not $[C'_3, -C'_4]$ and $[-C'_4, C'_1]$).

The resulting curve $\gamma(\mathcal{C}')$ defines a ‘butterfly’, and the corresponding surface $S(\mathcal{C}')$ is in fact the union of two oriented triangles,

$$S(\mathcal{C}) = S(C'_1, C'_2, C'_3) - S(-C'_3, -C'_1, C'_4).$$

We have as before $w(\mathcal{C}') = 0$ and hence

$$I^0(x, \mathcal{C}') = \frac{1}{4} \sum_{j=1}^4 E_2(C'_j, C'_{j+1}; x).$$

Considering the two triangles separately we obtain (now one has $w = 1$ each) one obtains the same. Note that the formula

$$I(D_x, S(\mathcal{C}')) = \frac{1}{4} \sum_{j=1}^4 \operatorname{sgn}(x, C_j) \operatorname{sgn}(x, C_{j+1})$$

is still valid: It gives 1 in the upper triangle, -1 in the lower (opposite orientation!), and 0 outside both.

7. A general result

The methods developed above can be applied in a rather more general situation.

Suppose that $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ has signature (p, q) with $q \geq 2$ and let D be the space of oriented negative q -planes in V . Fix a base point $z_0 \in D^+$ where D^+ is one component of D . Let $I = [0, 1]$. Suppose that

$$\gamma : (I^{q-1}, \partial I^{q-1}) \longrightarrow (D, z_0)$$

is a piecewise smooth map, where the notation means that $\gamma : I^{q-1} \longrightarrow D^+$ is a piecewise smooth map with $\gamma(\partial I^{q-1}) = z_0$. Since D^+ is contractible, there exists a homotopy

$$\rho : I^q \longrightarrow D^+$$

such that

$$\rho|_{I^{q-1} \times 0} = \gamma, \quad \rho(I^{q-1} \times \{1\}) = z_0.$$

We can assume that ρ is also piecewise smooth and thus, viewing ρ as a q -chain, we can define

$$I^0(x; \gamma) = \int_{I^q} \rho^* \varphi_{KM}^0(x).$$

Since $\varphi_{KM}^0(x)$ is closed, this integral does not depend on the choice of ρ . It defines a continuous function of $x \in V$.

Let D_x be defined as in (2.1). For $x \neq 0$, D_x is empty if $(x, x) \leq 0$ and is a total geodesic subsymmetric space of codimension q if $(x, x) > 0$. When $(x, x) > 0$, there is a fibration $\text{pr} : D \rightarrow D_x$ where, for $z_1 \in D_1$, the fiber $\text{pr}^{-1}(z_1) = D_{x,z_1}^\vee$ is the subsymmetric space associated to the subspace $V_{x,z_1}^\vee = z_1 + \mathbb{R}x$ of V of signature $(1, q)$. The space $D^+ - D_x^+$ is then a fiber bundle over D_x^+ with fibers $D_{x,z_1}^\vee - \{z_1\}$. Also

$$\pi_{q-1}(D^+ - D_x^+, z_0) = \mathbb{Z}$$

with generator given by a (suitably oriented) sphere S_{x,z_1} in $D_{x,z_1}^\vee - \{z_1\}$ for $z_1 = \text{pr}(z_0) \in D_x^+$.

We say that x is regular with respect to γ if $D_x \cap \gamma(I^{q-1})$ is empty. For $x \neq 0$, this is always the case when $(x, x) \leq 0$. If $(x, x) > 0$ and x is regular with respect to γ , then $z_0 \notin D_x$ and

$$\gamma : (I^{q-1}, \partial I^{q-1}) \rightarrow (D^+ - D_x^+, z_0).$$

Then γ is homotopic to a multiple $k \cdot S_{x,z_1}$ of the generator for $\pi_{q-1}(D^+ - D_x^+, z_0)$. We may suppose that there is a piecewise smooth q -chain \mathbf{h} in $D^+ - D_x^+$ such that

$$\partial \mathbf{h} = \gamma - k \cdot S_{x,z_1}.$$

Thus, if B_{x,z_1} is the closed disk in D_{x,z_1}^\vee with boundary S_{x,z_1} , we can take the q -chain

$$S = \mathbf{h} + k \cdot B_{x,z_1}$$

in the definition of $I^0(x; \gamma)$.

Recall that there is a $q - 1$ -form $\Psi_{KM}^0(x)$ such that

$$\varphi_{KM}^0(x) = d\Psi_{KM}^0(x)$$

and the properties of $\Psi_{KM}^0(x)$ used in the arguments above hold in general.

Let

$$\Phi(x; \gamma) = \lim_{r \rightarrow \infty} I^0(rx; \gamma),$$

and if x is regular with respect to γ , let

$$J^0(x; \gamma) = \int_{I^{q-1}} \gamma^* \Psi_{KM}^0(x).$$

Note that

$$\lim_{r \rightarrow \infty} J^0(rx; \gamma) = 0.$$

Also, we have

$$\begin{aligned} \Phi(x; \gamma) &= \lim_{r \rightarrow \infty} I^0(rx; \gamma) \\ &= \lim_{r \rightarrow \infty} \left(\int_{\mathbf{h}} \varphi_{KM}^0(rx) + k \cdot \int_{B_{x, z_1}} \varphi_{KM}^0(rx) \right) \\ &= k \cdot \int_{D_{x, z_1}^\vee} \varphi_{KM}^0(rx) \\ &= k \cdot \varepsilon(x) 2^{-\frac{q}{2}-1} \end{aligned}$$

by Proposition 6.2 of [10], where the sign $\varepsilon(x)$ depends on choices of orientations which we will not make explicit here. When q is even, $\varepsilon(x) = 1$. Also note that the integral in [10] is twice ours, since we are integrating over one of the two connected components.

Arguing as before, we obtain the following.

Proposition 7.1. (i) *If $\Phi(x; \gamma) \neq 0$ then $D_x \cap S$ is non-empty for any choice of S .*

(ii) *If x is regular with respect to γ , then $\Phi(x; \gamma)$ is an integer and*

$$[\gamma] = \varepsilon(X) \Phi(x; \gamma) \cdot [S_{x, z_1}] \in \pi_{q-1}(D^+ - D_x^+, z_0).$$

(iii) *If x is regular with respect to γ , then*

$$I^0(x; \gamma) = \Phi(x; \gamma) + J^0(x; \gamma).$$

At this level of generality, we do not have an explicit expression for $J^0(x; \gamma)$ like that in Proposition 4.3. The following assumption allows us to continue the argument.

Assumption A. Suppose that, for x regular with respect to γ ,

$$(7.1) \quad J^0(x; \gamma) = \mathcal{E}(x; \gamma) - \mathcal{D}(x; \gamma),$$

where $\mathcal{E}(x; \gamma)$ extends to a continuous function of $x \in V$ and $\mathcal{D}(x; \gamma)$ is valued in $\frac{1}{M}\mathbb{Z}$ for some integer $M \geq 1$.

Then, the difference

$$\Phi(x; \gamma) - \mathcal{D}(x; \gamma) = I^0(x; \gamma) - \mathcal{E}(x; \gamma)$$

takes values in a discrete set for x regular, but also extends to a continuous function on V , hence is a constant

$$\mathbf{c}(\gamma) = \Phi(x; \gamma) - \mathcal{D}(x; \gamma) = -\mathcal{D}(\mathbf{v}; \gamma),$$

for any negative vector \mathbf{v} in V . We obtain

$$I^0(x; \gamma) = \mathcal{E}(x; \gamma) + \mathbf{c}(\gamma)$$

and

$$\Phi(x; \gamma) = \mathcal{D}(x; \gamma) - \mathcal{D}(\mathbf{v}; \gamma).$$

One more piece of information is needed.

Assumption B. Suppose that $|\mathcal{D}(x; \gamma)| \leq B$ for some constant B .

Under these assumptions, we obtain convergence and completion.

(1) The series

$$(7.2) \quad \vartheta_\mu(\tau; \gamma) = \sum_{x \in \mu+L} \Phi(x; \gamma) \mathbf{q}^{Q(x)}$$

is termwise absolutely convergent. For terms in the series with x regular with respect to γ , the coefficients are generalized linking numbers.

(2) The modular completion of $\vartheta_\mu(\tau; \gamma)$ is given by

$$(7.3) \quad \begin{aligned} \hat{\vartheta}_\mu(\tau; \gamma) &= \sum_{x \in \mu+L} (\mathcal{E}(v^{\frac{1}{2}}x; \gamma) + \mathbf{c}(\gamma)) \mathbf{q}^{Q(x)} \\ &= \vartheta_\mu(\tau; \gamma) + \sum_{x \in \mu+L} (\mathcal{E}(v^{\frac{1}{2}}x; \gamma) - \mathcal{D}(v^{\frac{1}{2}}x; \gamma)) \mathbf{q}^{Q(x)}. \end{aligned}$$

The modularity of this completion follows from the identity

$$\hat{\vartheta}_\mu(\tau; \gamma) = \int_S \theta_\mu(\tau, \varphi_{KM}).$$

Remark. In all cases handled so far, an explicit formula for $J^0(x; \gamma)$ confirming Assumptions A and B was obtained inductively. The induction depends on the fact that the ‘faces’ of γ lie in subsymmetric spaces of the form

$$D'_y = \{z \in D \mid y \in z\}$$

for negative vectors y associated to the subspaces y^\perp of V of signature $(p, q - 1)$. Examples for this were given in [5] where we discussed the cases when

the collection \mathcal{C} gives to geodesic q -‘cubes’ respectively q -simplices in the symmetric space D . It would be very interesting to see if further examples can be obtained without this inductive structure.

8. The dodecahedron

In this section, we consider an example for which explicit formulas can again be obtained inductively, beginning with the result above on the 5-gon.

Let V be a quadratic over \mathbb{Q} with $\text{sig}(V) = (m - 3, 3)$. For a negative vector y , the space $V_y = y^\perp$ has $\text{sig}(V_y) = (m - 3, 2)$ and a collection of vectors in V_y satisfying the 5-gon conditions determine a 5-gon in $D(V_y)$. The idea now is to construct a dodecahedron in $D(V)$ with such 5-gons as its faces.

Let

$$\mathcal{C}^{\text{dod}} = \{ C_0, \dots, C_{11} \}$$

be a collection of negative vectors in V . Let $V_j = C_j^\perp$ and $D'_j = D(V_j) \subset D(V)$. The faces of the dodecahedron will be 5-gons in the spaces D'_j determined by the projections to V_j of suitable ordered subsets of \mathcal{C} that we now describe.

We label the faces of a dodecahedron by elements j in $\mathbb{Z}/12\mathbb{Z}$. Choose an initial face F_0 (the ‘top’) labeled 0. Label the 5 adjacent faces F_1, F_2, F_3, F_4, F_5 , clockwise with respect to the outward normal. Define an involution of $\mathbb{Z}/12\mathbb{Z}$ by $a \mapsto \bar{a} = -(a + 1)$. Faces occur in antipodal pairs and we require that the labels of such a pair are a and \bar{a} . The numbering of the faces is then determined. In particular, the ‘bottom’ face, opposite to the ‘top’ F_0 , will be $F_{\bar{0}} = F_{11}$. The cycle of faces adjacent to the top (resp. bottom) face will be numbered

$$\mathcal{F}(0) = (1, 2, 3, 4, 5),$$

and

$$\mathcal{F}(\bar{0}) = (\bar{5}, \bar{4}, \bar{3}, \bar{2}, \bar{1}).$$

Note that the order is clockwise with respect to the outward normal in both cases. The cycles for the remaining faces are

$$\begin{array}{ll} \mathcal{F}(1) = (0, 5, \bar{3}, \bar{4}, 2) & \mathcal{F}(\bar{1}) = (\bar{0}, \bar{2}, 4, 3, \bar{5}) \\ \mathcal{F}(2) = (0, 1, \bar{4}, \bar{5}, 3) & \mathcal{F}(\bar{2}) = (\bar{0}, \bar{3}, 5, 4, \bar{1}) \\ \mathcal{F}(3) = (0, 2, \bar{5}, \bar{1}, 4) & \mathcal{F}(\bar{3}) = (\bar{0}, \bar{4}, 1, 5, \bar{2}) \\ \mathcal{F}(4) = (0, 3, \bar{1}, \bar{2}, 5) & \mathcal{F}(\bar{4}) = (\bar{0}, \bar{5}, 2, 1, \bar{3}) \\ \mathcal{F}(5) = (0, 4, \bar{2}, \bar{3}, 1) & \mathcal{F}(\bar{5}) = (\bar{0}, \bar{1}, 3, 2, \bar{4}). \end{array}$$

Here the cycle $\mathcal{F}(\bar{j})$ is obtained from $\mathcal{F}(j)$ by taking $\bar{}$ and reversing the order. The face F_j is adjacent to F_i if and only if $j \in \mathcal{F}(i)$, and, in that case, the cycle $\mathcal{F}(j)$ is obtained from $\mathcal{F}(i) = (a, j, b, u, v)$ by the recipe:

$$(8.1) \quad \mathcal{F}(i) = (a, j, b, u, v) \mapsto (b, i, a, \bar{u}, \bar{v}) = \mathcal{F}(j).$$

For example,

$$\mathcal{F}(4) = (\bar{2}, 5, 0, 3, \bar{1}) \mapsto (0, 4, \bar{2}, \bar{3}, 1) = \mathcal{F}(5).$$

The recipe (8.1) shows that, if F_i and F_j are adjacent, the sets $\mathcal{F}(i)$ and $\mathcal{F}(j)$ share precisely 2 elements, say a and b , so that $\mathcal{F}(i)$ contains the subsequence (a, j, b) , and $\mathcal{F}(j)$ contains the subsequence (b, i, a) .

Each vertex is incident to a triple of faces, which we list clockwise with respect to the outward normal. The 5 vertices of a face F_i correspond to successive pairs of elements in $\mathcal{F}(i)$, and, if (u, v) is such a pair, the corresponding vertex is labeled $[i, u, v]$. If the faces F_i and F_j are adjacent, then they share vertices $[i, a, j] = [j, i, a]$ and $[j, b, i] = [i, j, b]$.

Recall that an ordered 5-tuple of vectors $(v_1, v_2, v_3, v_4, v_5)$ in an inner product space W of signature $(p, 2)$ is said to satisfy **the 5-gon conditions** if

$$(8.2) \quad \begin{aligned} & (v_j, v_j) < 0 \\ & (v_j, v_j)(v_{j+1}, v_{j+1}) - (v_j, v_{j+1})^2 > 0 \\ & (v_j, v_j)(v_{j-1}, v_{j+1}) - (v_j, v_{j-1})(v_j, v_{j+1}) < 0. \end{aligned}$$

We write

$$P_j : V \longrightarrow V_j, \quad P_j v = v - \frac{(v, C_j)}{(C_j, C_j)} C_j$$

for the orthogonal projection to $V_j = C_j^\perp$.

We say that a collection $\mathcal{C}^{\text{dod}} = \{C_j\}_{j \in \mathbb{Z}/12\mathbb{Z}}$ satisfies the **dodecahedron conditions** if, for each $i \in \mathbb{Z}/12\mathbb{Z}$, the ordered 5-tuple of vectors

$$(8.3) \quad \mathcal{R}(i) = (P_i C_j)_{j \in \mathcal{F}(i)} = (\mathcal{R}(i)_1, \mathcal{R}(i)_2, \mathcal{R}(i)_3, \mathcal{R}(i)_4, \mathcal{R}(i)_5)$$

satisfies the 5-gon conditions (8.2). Here the ordering in (8.3) is given by the ordering in $\mathcal{F}(i)$ so that, for example, for $F_{\bar{2}}$ the 5-tuple in $V_{\bar{2}}$ is

$$\mathcal{R}(\bar{2}) = (P_{\bar{2}} C_0, P_{\bar{2}} C_{\bar{3}}, P_{\bar{2}} C_5, P_{\bar{2}} C_4, P_{\bar{2}} C_{\bar{1}}).$$

If \mathcal{C}^{dod} is such a collection, then each 5-tuple (8.3) defines a 5-gon $\gamma(i)$ in the space $D(V_i)$ where we recall that $\text{sig}(V_i) = (m - 3, 2)$.

More explicitly, suppose that $j \in \mathcal{F}(i)$ and write $\mathcal{F}(i) = (a, j, b, c, d)$. Note that the orthogonal complement in V_i of the vector $P_i C_j$ is precisely³ $V_i \cap V_j$. Let

$$Q_j^i : V_i \longrightarrow V_i \cap V_j \quad Q_j^i v = v - \frac{(v, P_i C_j)}{(P_i C_j, P_i C_j)} P_i C_j.$$

Define the path $\gamma(i)_j$ by $\gamma(i)_j : [0, 1] \longrightarrow D(V)$,

$$s \mapsto [C_i, P_i C_j, (s - 1)Q_j^i(P_i C_a) + s Q_j^i(P_i C_b)] = [C_i, C_j, (s - 1)C_a + s C_b].$$

Here note that, by the 5-gon condition, $[P_i C_j, (s - 1)Q_j^i(P_i C_a) + s Q_j^i(P_i C_b)]$ is an oriented negative 2-plane in V_i and hence $\gamma(i)_j(s)$ is an oriented negative 3-plane in V . The endpoints of this path are

$$\gamma(i)_j(0) = [C_j, C_i, C_a]$$

and

$$\gamma(i)_j(1) = [C_i, C_j, C_b],$$

oriented negative 3-planes in the same component of $D(V)$. Then the 5-gon $\gamma(i)$ is the sum of paths $\gamma(i)_j$ given by this recipe as j runs over $\mathcal{F}(i)$.

Now suppose that $F(i)$ and $F(j)$ are adjacent faces with $\mathcal{F}(i) = (a, j, b, c, d)$ and $\mathcal{F}(j) = (b, i, a, \bar{c}, \bar{d})$. The 5-gon $\gamma(j)$ in $D(V_j)$ includes a segment

$$\gamma(j)_i(s) = [C_j, P_i C_i, (s - 1)Q_i^j(P_j C_b) + s Q_i^j(P_j C_a)] = [C_j, C_i, (s - 1)C_b + s C_a].$$

Writing

$$\gamma(j)_i(s) = [C_i, C_j, -s C_a + (1 - s)C_b]$$

we see that

$$\gamma(j)_i(s) = \gamma(i)_j(1 - s),$$

i.e., these two ‘edges’ coincide, with opposite parametrization.

Thus, we have proved the following result.

Proposition 8.1. *Suppose that a collection \mathcal{C}^{dod} satisfies the dodecahedral conditions.*

(i) *The 5-gons in the $D(V_i)$ ’s associated to the collections $(P_i C_j)_{j \in \mathcal{F}(i)}$ all lie in the same component of $D(V)$. Denote this component by $D(V)^+$.*

³If $v \in V_i$, then $(v, P_i C_j) = (v, C_j)$.

(ii) For each i , let $S(i)$ be an oriented 2-cell in $D(V_i)^+$ with $\partial S(i) = \gamma(i)$. Then the 2-chain

$$S(\mathcal{C}^{\text{dod}}) = \sum_{i=0}^{11} S(i)$$

in $D(V)^+$ has $\partial S(\mathcal{C}^{\text{dod}}) = 0$.

Corollary 8.2. *There is an oriented 3-chain \mathcal{D}^{dod} in $D(V)^+$ with $\partial \mathcal{D}^{\text{dod}} = S(\mathcal{C}^{\text{dod}})$.*

We may then define a function of $x \in V$ by

$$I^0(x; \mathcal{C}^{\text{dod}}) = \int_{\mathcal{D}^{\text{dod}}} \varphi_{KM}^0(x).$$

Once the 2-cells $S(i)$ have been chosen, $I^0(x; \mathcal{C}^{\text{dod}})$ does not depend on the choice of the 3-chain \mathcal{D}^{dod} . In fact, we will see that there is also no dependence on the 2-cells $S(i)$ in $D(V_i)$.

The regular set is

$$\text{Reg}(\mathcal{C}^{\text{dod}}) = \{ x \in V \mid (x, C_j) \neq 0, \forall j \}.$$

If x is regular, the 5-gons $\gamma(i)$ and the 2-cells $S(i)$ lie in $D(V) - D_x$, since they lie in the subsymmetric spaces $D(V_i) \hookrightarrow D(V)$, where the embedding is given by $\zeta \mapsto [C_i, \zeta]$.

Recall that, on the set $D(V) - D_x$,

$$\varphi_{KM}^0(x) = d\Psi_{KM}^0(x),$$

for an explicit 2-form $\Psi_{KM}^0(x)$ on D . For x is regular with respect to \mathcal{C}^{dod} , let

$$J^0(x; \mathcal{C}^{\text{dod}}) := \int_{S(\mathcal{C}^{\text{dod}})} \Psi_{KM}^0(x).$$

As before, an inductive calculation yields the following.

Proposition 8.3. *For x regular with respect to \mathcal{C}^{dod} ,*

$$\begin{aligned} J^0(x; \mathcal{C}^{\text{dod}}) &= \frac{1}{8} \sum_{\nu} E_3(\nu, x\sqrt{2}) - \text{sgn}(x; \nu) \\ &\quad + \frac{1}{8} \sum_{i \in \mathbb{Z}/12\mathbb{Z}} \mathbf{w}(\mathcal{R}(i)) \left(E_1(C_i; x\sqrt{2}) - \text{sgn}((x, C_i)) \right) \end{aligned}$$

where ν runs over the vertices of the dodecahedron and where, with $\mathcal{R}(i)$ given by (8.3),

$$\mathbf{w}(\mathcal{R}(i)) = - \sum_{\ell=1}^5 \operatorname{sgn}((\mathbf{v}_i, \mathcal{R}(i)_\ell)) \operatorname{sgn}((\mathbf{v}_i, \mathcal{R}(i)_{\ell+1}))$$

for any negative vector $\mathbf{v} \in V_i$.

Proof. Using the formulas from [5], we have

$$\begin{aligned} \int_{S(\mathcal{C}^{\text{dod}})} \Psi_{KM}^0(x) &= \sum_i \int_{S(i)} \Psi_{KM}^0(x) \\ &= - \sum_i \sqrt{2}(x, \underline{C}_i) \int_1^\infty e^{-2\pi t^2(x, \underline{C}_i)^2} \int_{S(i)} \varphi_{KM}^{V_j, 0}(tx \perp C_i) dt \\ &= - \sum_i \sqrt{2}(x, \underline{C}_i) \int_1^\infty e^{-2\pi t^2(x, \underline{C}_i)^2} I^0(tx \perp C_i; \mathcal{R}(j)) dt \\ &= -\frac{1}{4} \sum_i (\sqrt{2}x, \underline{C}_i) \int_1^\infty e^{-2\pi t^2(x, \underline{C}_i)^2} \\ &\quad \times \left[\mathbf{w}(\mathcal{R}(i)) + \sum_{\ell=1}^5 E_2(\mathcal{R}(i)_\ell, \mathcal{R}(i)_{\ell+1}, tx \perp C_i \sqrt{2}) \right] dt. \end{aligned}$$

Here in the first step we use the relation of Corollary 6.3,

$$\kappa_j^* \Psi_{KM}^0(x) = -\sqrt{2}(x, \underline{C}_j) \int_1^\infty e^{-2\pi t^2(x, \underline{C}_j)^2} \varphi_{KM}^{V_j, 0}(tx \perp C_j) dt,$$

To apply the induction identity, we regroup the 60 terms in the last expression in 3's according to the vertices. For example, the vertex $[0, 1, 2]$ is shared by the faces F_0, F_1 and F_2 . The corresponding 5-gon data are

$$\begin{aligned} \mathcal{R}(0) &= (P_0C_1, P_0C_2, P_0C_3, P_0C_4, P_0C_5) \\ \mathcal{R}(1) &= (P_1C_0, P_1C_5, P_1C_3, P_1C_4, P_1C_2) \\ \mathcal{R}(2) &= (P_2C_0, P_2C_1, P_2C_4, P_2P_5, P_2C_3). \end{aligned}$$

We collect a term from each of the groups $i = 0, 1$ and 2 amounting to the contribution

$$\begin{aligned} &-\frac{1}{4} (\sqrt{2}x, \underline{C}_0) \int_1^\infty e^{-2\pi t^2(x, \underline{C}_0)^2} E_2(P_0C_1, P_0C_2, tx \perp C_0 \sqrt{2}) dt \\ &-\frac{1}{4} (\sqrt{2}x, \underline{C}_1) \int_1^\infty e^{-2\pi t^2(x, \underline{C}_1)^2} E_2(P_1C_0, P_1C_2, tx \perp C_1 \sqrt{2}) dt \\ &-\frac{1}{4} (\sqrt{2}x, \underline{C}_2) \int_1^\infty e^{-2\pi t^2(x, \underline{C}_2)^2} E_2(P_2C_0, P_2C_1, tx \perp C_0 \sqrt{2}) dt, \end{aligned}$$

where, in the second line, we have employed the symmetry $E_2(c_1, c_2; x) = E_2(c_2, c_1; x)$ of the generalized error function. By the recursion formula of Proposition 7.3 of [5], this contribution is

$$\frac{1}{8} \left(E_3(C_0, C_1, C_2; x\sqrt{2}) - \operatorname{sgn}((x, C_0)) \operatorname{sgn}((x, C_1)) \operatorname{sgn}((x, C_2)) \right).$$

Thus, the total contribution of such terms is

$$\frac{1}{8} \sum_{\boldsymbol{\nu}} E_3(\boldsymbol{\nu}, x\sqrt{2}) - \operatorname{sgn}(x; \boldsymbol{\nu}),$$

where the sum runs over the triples of vectors associated to the vertices of \mathcal{D}^{dod} .

The remaining contribution is

$$\begin{aligned} & -\frac{1}{4} \sum_i \mathbf{w}(\mathcal{R}(i)) \sqrt{2} (x, \underline{C}_i) \int_1^\infty e^{-2\pi t^2(x, \underline{C}_i)^2} dt \\ & = \frac{1}{8} \sum_i \mathbf{w}(\mathcal{R}(i)) \left(E_1(C_i; x\sqrt{2}) - \operatorname{sgn}((x, C_i)) \right). \end{aligned}$$

where $\mathbf{w}(\mathcal{R}(i))$ is as in the proposition. □

Now the topological argument of the previous section can be applied. The resulting modularity statement for the dodecahedron is summarized as follows.

Theorem 8.4. *Suppose $\operatorname{sig}(V) = (m - 3, 3)$ and let $\mathcal{C}^{\text{dod}} = \{ C_0, \dots, C_{11} \}$ be a collection of negative vectors in V satisfying the dodecahedron conditions. Let*

$$\Phi(x; \mathcal{C}^{\text{dod}}) = \mathcal{D}(x; \mathcal{C}^{\text{dod}}) - \mathcal{D}(\mathbf{v}; \mathcal{C}^{\text{dod}}),$$

where \mathbf{v} is any negative vector in V and

$$\mathcal{D}(x; \mathcal{C}^{\text{dod}}) = -\frac{1}{8} \sum_{\boldsymbol{\nu}} \operatorname{sgn}(x; \boldsymbol{\nu}) - \frac{1}{8} \sum_{i \in \mathbb{Z}/12\mathbb{Z}} \mathbf{w}(\mathcal{R}(i)) \operatorname{sgn}((x, C_i)).$$

Here $\boldsymbol{\nu}$ runs over the triples of vectors associated to the vertices of \mathcal{D}^{dod} , and for the ‘face cycle’ $\mathcal{F}(i) = \{ j_1, j_2, j_3, j_4, j_5 \}$, and collection

$$\mathcal{R}(i) = \{ P_i C_{j_1}, P_i C_{j_2}, P_i C_{j_3}, P_i C_{j_4}, P_i C_{j_5} \}$$

of negative vectors in $V_i = C_i^\perp$ associated to the i th face,

$$\mathbf{w}(\mathcal{R}(i)) = - \sum_{\ell=1}^5 \operatorname{sgn}(\langle \mathbf{v}_i, C_{j_\ell} \rangle) \operatorname{sgn}(\langle \mathbf{v}_i, C_{j_{\ell+1}} \rangle),$$

where \mathbf{v}_i is any negative vector in V_i . Let

$$\mathcal{E}(x; \mathcal{C}^{\text{dod}}) = \frac{1}{8} \sum_{\mathbf{v}} E_3(\mathbf{v}, x\sqrt{2}) + \frac{1}{8} \sum_{i \in \mathbb{Z}/12\mathbb{Z}} \mathbf{w}(\mathcal{R}(i)) E_1(C_i; x\sqrt{2}).$$

(i) The integral $I^0(x; \mathcal{C}^{\text{dod}})$ is given by

$$I^0(x; \mathcal{C}^{\text{dod}}) = \mathcal{E}(x; \mathcal{C}^{\text{dod}}) - \mathcal{D}(\mathbf{v}; \mathcal{C}^{\text{dod}})$$

for any negative vector \mathbf{v} in V . In particular, the integral depends only on the collection \mathcal{C}^{dod} and not on the choice of the faces surfaces $S(i)$ in D'_{C_i} .

(ii) For a lattice $L \subset L^\vee \subset V$, the series

$$(8.4) \quad \vartheta_\mu(\tau; \mathcal{C}^{\text{dod}}) = \sum_{x \in \mu + L} \Phi(x; \mathcal{C}^{\text{dod}}) \mathbf{q}^{Q(x)}$$

is termwise absolutely convergent and defines a mock modular form of weight $\frac{1}{2}m$ with completion

$$(8.5) \quad \begin{aligned} \hat{\vartheta}_\mu(\tau; \mathcal{C}^{\text{dod}}) &= \sum_{x \in \mu + L} (\mathcal{E}(v^{\frac{1}{2}}x; \mathcal{C}^{\text{dod}}) - \mathcal{D}(\mathbf{v}; \mathcal{C}^{\text{dod}})) \mathbf{q}^{Q(x)} \\ &= \vartheta_\mu(\tau; \mathcal{C}^{\text{dod}}) + \sum_{x \in \mu + L} (\mathcal{E}(v^{\frac{1}{2}}x; \mathcal{C}^{\text{dod}}) - \mathcal{D}(x; \mathcal{C}^{\text{dod}})) \mathbf{q}^{Q(x)}. \end{aligned}$$

The modularity of this completion follows from the identity

$$\hat{\vartheta}_\mu(\tau; \mathcal{C}^{\text{dod}}) = \int_{\mathcal{D}^{\text{dod}}} \theta_\mu(\tau, \varphi_{KM}).$$

Remark 8.5. (i) It is probably difficult to prove the absolute convergence of (8.4) directly from the set of inequalities imposed by the dodecahedral conditions, a collection 5-gon conditions on each of the 12 sets of projections $\mathcal{R}(i)$.

(ii) Of course, we do not know whether or not such dodecahedrons could arise in the context of [2]. For us the main point is to provide an example in which the method of section 7 can be made to yield an explicit formula. The cube and tetrahedron were handled in [5]. The octahedron should be similar but simpler while the icosahedron will require a bit more work.

Remark 8.6. *It is a natural question to consider the action of the lowering operator $\mathbb{L} = -2iv^2 \frac{\partial}{\partial \bar{\tau}}$ on $\hat{\vartheta}_\mu(\tau; \mathcal{C}^{\text{dod}})$. We do not discuss this here in detail but refer to [5], Section 9 for the consideration of very similar cases. Basically, if the collection \mathcal{C}^{dod} is rational then $\mathbb{L}\hat{\vartheta}_\mu(\tau; \mathcal{C}^{\text{dod}})$ splits into a linear combination (of the complex conjugate) of weight $3/2$ unary theta series times the indefinite theta integrals over the faces of \mathcal{D}^{dod} of signature $(m-3, 2)$.*

A supply of collections satisfying the dodecahedron conditions is provided by a ‘seed’ construction. Fix an oriented negative 3-plane z_0 with properly oriented orthonormal basis u_0, u_1, u_2 so that $z_0 \simeq \mathbb{R}^3$ with the negative of the standard Euclidean inner product. For a regular dodecahedron \mathbb{D} in \mathbb{R}^3 , centered at the origin, let $\mathcal{C}_0^{\text{dod}} = \{c_0, \dots, c_{11}\}$ be the images in z_0 of the outward unit normal vectors to the faces, numbered as described above. In particular, $c_{\bar{r}} = -c_r$. It is then easily checked, using $\text{SO}(3)$ equivariance and the transitivity of the symmetry group of the regular dodecahedron on faces, edges and vertices, that the collection $\mathcal{C}_0^{\text{dod}}$ satisfies the dodecahedron conditions. Of course, the associated ‘dodecahedron’ in $D = D(V)$ degenerates to the point z_0 . For a positive vector v_0 in z_0^\perp and $t = (t_0, \dots, t_{11}) \in \mathbb{R}^{12}$, let

$$\mathcal{C}_t^{\text{dod}} = \mathcal{C}_0^{\text{dod}} + tv_0.$$

Since the dodecahedron conditions define an open subset of V^{12} , these conditions are still satisfied by $\mathcal{C}_t^{\text{dod}}$ for t small. It remains to show that we can take t so that the 20 vertices determined by $\mathcal{C}_t^{\text{dod}}$ are distinct. But the coincidence of any two of these 3-planes amounts to non-trivial linear conditions on the components of t , so the vertices will be distinct for an open set of t 's.

We leave aside the following problem. Let V_{dod}^{12} be the set of collection \mathcal{C}^{dod} satisfying the dodecahedral conditions, and let $(V_{\text{dod}}^{12})^0$ be the subset of collection for which the associated 20 vertices are distinct. The problem is to write down additional conditions defining $(V_{\text{dod}}^{12})^0$ and to determine its connected components. Is there a nice characterization of the geometry of the dodecahedra associated to the different⁴ components?

References

- [1] S. ALEXANDROV, S. BANERJEE, J. MANSCHOT, AND B. PIOLINE, *Indefinite theta series and generalized error functions*, *Selecta Mathematica* **24** (5) (2018), 3927–3972. [MR3874688](#)

⁴At the moment we do not know whether or not $(V_{\text{dod}}^{12})^0$ is connected.

- [2] S. ALEXANDROV, S. BANERJEE, J. MANSCHOT, AND B. PIOLINE, *Multiple $D3$ -instantons and mock modular forms II*, Communications in Mathematical Physics **359** (2018), 297–346. [MR3781452](#)
- [3] J. FUNKE, *Heegner divisors and nonholomorphic modular forms*, Compositio Math. **133** (2002), 289–321. [MR1930980](#)
- [4] J. FUNKE AND S. KUDLA, *Mock modular forms and geometric theta functions for indefinite quadratic forms*, J. Phys. A **50** (40) (2017), 404001. [MR3708083](#)
- [5] J. FUNKE AND S. KUDLA, *On some incomplete theta integrals*, Composito Math. **155** (2019), 1711–1746. [MR3993802](#)
- [6] J. FUNKE AND J. MILLSON, *Boundary behavior of special cohomology classes arising from the Weil representation*, Jussieu Math. J. **12** (2013), 571–634. [MR3062872](#)
- [7] F. HIRZEBRUCH AND D. ZAGIER, *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus*, Inv. Math. **36** (1976), 57–113. [MR0453649](#)
- [8] S. KUDLA, *Theta integrals and generalized error functions*, Manuscripta Math. **155** (2018), 303–333. [MR3763408](#)
- [9] S. KUDLA AND J. MILLSON, *The theta correspondence and harmonic forms I*, Math. Annalen **274** (1986), 353–378. [MR0842618](#)
- [10] S. KUDLA AND J. MILLSON, *The theta correspondence and harmonic forms II*, Math. Annalen **277** (1987), 267–314. [MR0886423](#)
- [11] C. NAZAROGLU, *r -tuple error functions and indefinite theta series of higher depth*, Communications in Number Theory and Physics **12** (3) (2018), 581–608. [MR3862073](#)
- [12] M.-F. VIGNÉRAS, *Séries theta des formes quadratiques indéfinies*. In: Modular functions of one variable VI, Springer Lecture Notes **627** (1977), 227–239. [MR0480352](#)
- [13] M. WESTERHOLT-RAUM, *Indefinite theta series on cones*, preprint (2016), [arXiv:1608.08874v2](#).
- [14] D. ZAGIER, *Nombres de classes et formes modulaires de poids $3/2$* , C. R. Acad. Sci. Paris Sér. A-B **281** (1975), 883–886. [MR0429750](#)
- [15] D. ZAGIER, *Ramanujan’s mock theta functions and their applications, (after Zagiers and Ono-Bringmann)*, Astérisque (326): Exp.

No. 986, vii–viii, 143–164 (2010), 2009. Séminaire Bourbaki. Vol. 2007/2008. [MR2605321](#)

- [16] S. P. ZWEGERS, Mock Theta Functions. Thesis, Utrecht, (2002). [http://igiturarchive, library.uu.nl/dissertations/2003-0127-094324/inhoud.htm](http://igiturarchive.library.uu.nl/dissertations/2003-0127-094324/inhoud.htm).

Jens Funke
Department of Mathematical Sciences
Durham University
South Road
Durham, DH1 3LE
UK
E-mail: jens.funke@durham.ac.uk

Stephen Kudla
Department of Mathematics
University of Toronto
40 St. George St.
Toronto, ON M5S 2E4
Canada
E-mail: skudla@math.toronto.edu