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On finite multiple zeta values of level two

Masanobu Kaneko, Takuya Murakami, and Amane Yoshihara

Dedicated to Professor Don Zagier on the occasion of his 70th birthday

Abstract: We introduce and study a "level two" analogue of finite multiple zeta values. We give conjectural bases of the space of finite Euler sums as well as that of usual finite multiple zeta values in terms of these newly defined elements. A kind of "parity result" and certain sum formulas are also presented.

Keywords: Multiple zeta values, Finite multiple zeta values, Finite Euler sums.

1. Definitions and conjectures

The finite multiple zeta value $\zeta_{\mathcal{A}}(k_1, \ldots, k_r)$ is an element in the $\mathbb{Q}$-algebra $\mathcal{A}$ defined by

$$\mathcal{A} := \prod_p \mathbb{Z}/p\mathbb{Z} \Big/ \bigoplus_p \mathbb{Z}/p\mathbb{Z} = \{(a_{(p)})_p \,|\, a_{(p)} \in \mathbb{Z}/p\mathbb{Z}\}/\sim .$$

Here, p runs over all prime numbers, and the relation $(a_{(p)})_p \sim (b_{(p)})_p$ means that the equality $a_{(p)} = b_{(p)}$ holds in $\mathbb{Z}/p\mathbb{Z}$ for all but a finite number of p. We often identify a representative $a = (a_{(p)})_p$ with the element in $\mathcal{A}$ that it defines. Precisely, $\zeta_{\mathcal{A}}(k_1, \ldots, k_r)$ is defined as follows.

Definition 1.1. For a tuple of positive integers $(k_1, \ldots, k_r)$ (called an index), define the ($\mathcal{A}$-) finite multiple zeta value $\zeta_{\mathcal{A}}(k_1, \ldots, k_r) \in \mathcal{A}$ by

$$\zeta_{\mathcal{A}}(k_1, \ldots, k_r)_{(p)} = \sum_{0<m_1<\cdots<m_r<p} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \mod p. \tag{1}$$

This is a finite analogue of the usual multiple zeta value in $\mathbb{R}$:

$$\zeta(k_1, \ldots, k_r) = \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}. \tag{2}$$

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To ensure the convergence, we need the condition $k_r > 1$ here, but for $\zeta_{\mathcal{A}}(k_1,\ldots,k_r)$, obviously we do not need such a restriction.

In recent years, a vast amount of work has been done on the classical multiple zeta value (2) and its numerous variants and generalizations including the finite multiple zeta value (1). A central conjecture concerning finite multiple zeta values predicts a deep connection between finite and classical multiple zeta values (see [7, 8] for the precise statement). For references to the extensive literature on the subject, one may refer to the book [16] by Zhao and the website [5] maintained by Hoffman.

In this paper, we consider the following "level two" variant $\zeta_{\mathcal{A}}^{(2)}(k_1,\ldots,k_r)$ of $\zeta_{\mathcal{A}}(k_1,\ldots,k_r)$.

Definition 1.2. For an index $(k_1,\ldots,k_r)$, define the finite multiple zeta value of level two $\zeta_{\mathcal{A}}^{(2)}(k_1,\ldots,k_r)$ in $\mathcal{A}$ by

$$\zeta_{\mathcal{A}}^{(2)}(k_1,\ldots,k_r)_{(p)} = \sum_{0<m_1<\cdots<m_r<p/2} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \mod p. \tag{3}$$

The difference is that the summation extends up to $p/2$ instead of p. We mention that this sum for special indices was already considered in several literatures, for instance in Pilehrood-Pilehrood-Tauraso [11]. See also [16].

For later use, we note here that, by putting $n_i = 2m_i$ first and then changing n_i with $p - n_{r+1-i}$, we have two expressions

$$\zeta_{\mathcal{A}}^{(2)}(k_1,\ldots,k_r)_{(p)} = 2^{k_1+\cdots+k_r} \sum_{\substack{0<n_1<\cdots<n_r<p \\ n_i:\,\mathrm{even}}} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}} \mod p \tag{4}$$

$$= (-2)^{k_1+\cdots+k_r} \sum_{\substack{0<n_1<\cdots<n_r<p \\ n_i:\,\mathrm{odd}}} \frac{1}{n_1^{k_r}\cdots n_r^{k_1}} \mod p. \tag{5}$$

In particular, $\zeta_{\mathcal{A}}^{(2)}(k_1,\ldots,k_r)$ may be viewed as a finite analogue of Hoffman's "t-value" [4], up to a constant multiple. We further note that, if we write the sum on the right as

$$\sum_{\substack{0<n_1<\cdots<n_r<p \\ n_i:\,\mathrm{odd}}} \frac{1}{n_1^{k_r}\cdots n_r^{k_1}} = 2^{-r} \sum_{0<n_1<\cdots<n_r<p} \frac{(1-(-1)^{n_1})\cdots(1-(-1)^{n_r})}{n_1^{k_r}\cdots n_r^{k_1}},$$

we see that $\zeta_{\mathcal{A}}^{(2)}(k_1,\ldots,k_r)$ can be written as a $\mathbb{Q}$-linear combination of "finite Euler sums," as studied for instance in Zhao [15, 16].

We introduce three $\mathbb{Q}$-subspaces of $\mathcal{A}$ spanned by the usual finite multiple zeta values, our level-two analogues, and the finite Euler sums.

Definition 1.3. For each integer $k \geq 0$, define the $\mathbb{Q}$-vector spaces $\mathcal{Z}^{(1)}_{\mathcal{A},k}$, $\mathcal{Z}^{(2)}_{\mathcal{A},k}$, and $\mathcal{E}_k$ in $\mathcal{A}$ by $\mathcal{Z}^{(1)}_{\mathcal{A},0} = \mathcal{Z}^{(2)}_{\mathcal{A},0} = \mathcal{E}_0 = \mathbb{Q}$ and

$$\mathcal{Z}^{(1)}_{\mathcal{A},k} := \sum_{\substack{k_1+\cdots+k_r=k \\ r\geq 1,\, \forall k_i \geq 1}} \mathbb{Q}\cdot \zeta_{\mathcal{A}}(k_1,\ldots,k_r) \quad (k\geq 1),$$

$$\mathcal{Z}^{(2)}_{\mathcal{A},k} := \sum_{\substack{k_1+\cdots+k_r=k \\ r\geq 1,\, \forall k_i \geq 1}} \mathbb{Q}\cdot \zeta^{(2)}_{\mathcal{A}}(k_1,\ldots,k_r) \quad (k\geq 1),$$

and

$$\mathcal{E}_k := \mathbb{Q}\text{-span of all finite Euler sums of weight } k,$$

namely, all elements in $\mathcal{A}$ of the form

$$\left(\sum_{0<m_1<\cdots<m_r<p} \frac{(\pm 1)^{m_1}\cdots(\pm 1)^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}} \mod p \right)_p$$

with $k_1+\cdots+k_r = k$ $(r\geq 1, \forall k_i \geq 1)$ and with all possible signs in the numerator. Further, we set

$$\mathcal{Z}^{(1)}_{\mathcal{A}} := \sum_{k=0}^{\infty} \mathcal{Z}^{(1)}_{\mathcal{A},k}, \quad \mathcal{Z}^{(2)}_{\mathcal{A}} := \sum_{k=0}^{\infty} \mathcal{Z}^{(2)}_{\mathcal{A},k}, \quad \mathcal{E} := \sum_{k=0}^{\infty} \mathcal{E}_k.$$

Proposition 1.4. *The space $\mathcal{Z}^{(2)}_{\mathcal{A}}$ is a $\mathbb{Q}$-subalgebra of $\mathcal{E}$.*

Proof. That the space $\mathcal{Z}^{(2)}_{\mathcal{A}}$ is contained in $\mathcal{E}$ has already been remarked above. And that $\mathcal{Z}^{(2)}_{\mathcal{A}}$ is closed under multiplication is seen by the fact that the standard harmonic (or stuffle) product rule applies also to the defining sum of $\zeta^{(2)}_{\mathcal{A}}(k_1,\ldots,k_r)$ in (3). ($\mathcal{E}$ is a $\mathbb{Q}$-algebra by the same reasoning.) $\square$

Based on an evidence supported by numerical experiments, we propose the following conjecture.

Conjecture 1.5. *i)* $\mathcal{Z}^{(2)}_{\mathcal{A}} = \mathcal{E}$.

ii) The set $\{\zeta^{(2)}_{\mathcal{A}}(k_1,\ldots,k_r) \mid r\geq 1, \forall k_i : \text{odd} \geq 1\}$ *forms a linear basis of* $\mathcal{Z}^{(2)}_{\mathcal{A}}$.

Remark 1.6. The conjectural dimension (as a $\mathbb{Q}$-vector space) of $\mathcal{E}_k$ $(k \geq 1)$ is given by the Fibonacci number F_k $(= F_{k-1} + F_{k-2},\ F_1 = F_2 = 1)$ (cf. [16, §8.6.3]). The cardinality of the set in ii) above is easily seen to be equal to F_k. Also note that the number of $\zeta_{\mathcal{A}}^{(2)}(k_1, \ldots, k_r)$ of weight k is 2^{k-1} which is much smaller than that of finite Euler sums of weight k, namely $2 \cdot 3^{k-1}$.

Proposition 1.7. *The space $\mathcal{Z}_{\mathcal{A}}^{(1)}$ of ordinary finite multiple zeta values is contained in $\mathcal{Z}_{\mathcal{A}}^{(2)}$; we have the inclusions $\mathcal{Z}_{\mathcal{A}}^{(1)} \subset \mathcal{Z}_{\mathcal{A}}^{(2)} \subset \mathcal{E}$.*

Proof. This can be seen from the identity

$$\zeta_{\mathcal{A}}(k_1, \ldots, k_r) = \sum_{i=0}^{r} (-1)^{k_{i+1}+\cdots+k_r} \zeta_{\mathcal{A}}^{(2)}(k_1, \ldots, k_i) \zeta_{\mathcal{A}}^{(2)}(k_r, \ldots, k_{i+1}), \tag{6}$$

where we set $\zeta_{\mathcal{A}}^{(2)}(\emptyset) = 1$, together with Proposition 1.4 (that $\mathcal{Z}_{\mathcal{A}}^{(2)}$ is closed under multiplication). This identity, which is useful in several places in the rest of the paper, is a consequence of the following division of the sum

$$\sum_{0<m_1<\cdots<m_r<p} = \sum_{i=0}^{r} \sum_{0<m_1<\cdots<m_i<p/2<m_{i+1}<\cdots<m_r<p}$$

in the definition and the change $m_j \to p - m_j$ for $j = i+1, \ldots, r$. □

Remark 1.8. This is reminiscent of the definition of the "symmetric multiple zeta values," a conjectural real counterpart of finite multiple zeta values. See [7, 8] for the details on this.

Also from the numerical experiments, we surmise

Conjecture 1.9. *i) If all k_i are greater than 1, each $\zeta_{\mathcal{A}}^{(2)}(k_1, \ldots, k_r)$ is in $\mathcal{Z}_{\mathcal{A}}^{(1)}$.*

ii) The set $\{\zeta_{\mathcal{A}}^{(2)}(k_1, \ldots, k_r) \mid r \geq 1, \forall k_i : odd \geq 3\}$ constitutes a basis of $\mathcal{Z}_{\mathcal{A}}^{(1)}$.

Remark 1.10. The conjectural dimension of $\mathcal{Z}_{\mathcal{A}}^{(1)}$ is given by the sequence d_{k-3} defined recursively by $d_k = d_{k-2} + d_{k-3},\ d_0 = 1, d_1 = 0, d_2 = 1$. (cf. [7, 16]). The cardinality of the set in ii) above equals d_{k-3}.

2. Examples in low depths and a parity result

First, we define two specific elements $L(2)$ and $Z(k)$ $(k \geq 2)$ in $\mathcal{A}$ as

$$L(2)_{(p)} := \frac{2^{p-1} - 1}{p} \bmod p \quad \text{and} \quad Z(k)_{(p)} := \frac{B_{p-k}}{k} \bmod p,$$

where B_n denotes the nth Bernoulli number. Note that, by the definition of $\mathcal{A}$, we may ignore possible (finitely many) p's such that the right-hand sides are not well defined. These elements are respectively a natural analogue of $\log 2$ and the conjectural "true" analogue of $\zeta(k) \bmod \pi^2$ in $\mathcal{A}$. We refer the reader to [8] for more details on these. We first recall known formulas for depth (the length of the index) less than or equal to 2 ([12, Th. 5.2], [10, Lem. 1]). We give proofs for the convenience of the reader.

Proposition 2.1 ([12], [10]). *i)* $\zeta_{\mathcal{A}}^{(2)}(1) = -2L(2)$ *and* $\zeta_{\mathcal{A}}^{(2)}(k) = (2-2^k)Z(k)$ *for* $k \geq 2$.

In particular, $\zeta_{\mathcal{A}}^{(2)}(k) = 0$ *if* k *is even.*

ii) If $k_1 + k_2$ *is odd,*

$$\zeta_{\mathcal{A}}^{(2)}(k_1, k_2) = \frac{1}{2}\left\{(-1)^{k_2}\binom{k_1+k_2}{k_2} + 2^{k_1+k_2} - 2\right\} Z\left(k_1+k_2\right).$$

Proof. i) From the computation using the binomial formula, we have

$$2L(2)_{(p)} = \frac{(1+1)^p - 2}{p} \bmod p = \sum_{i=1}^{p-1} \frac{(-1)^{i-1}}{i} \bmod p.$$

This is equal to $-\zeta_{\mathcal{A}}^{(2)}(1)$ because

$$\sum_{\substack{0<i<p \\ i:\text{even}}} \frac{1}{i} \bmod p = \frac{1}{2}\zeta_{\mathcal{A}}^{(2)}(1)_{(p)} \quad \text{and} \quad \sum_{\substack{0<i<p \\ i:\text{odd}}} \frac{1}{i} \bmod p = -\frac{1}{2}\zeta_{\mathcal{A}}^{(2)}(1)_{(p)},$$

as seen from (4) and (5).

For the second equality, we use the Seki-Bernoulli formula for sum of powers (cf. [1]). We start with

$$\zeta_{\mathcal{A}}^{(2)}(k)_{(p)} = \sum_{0<m<p/2} \frac{1}{m^k} \bmod p = \sum_{0<m<p/2} m^{p-1-k} \bmod p.$$

The last sum, for large enough p, is equal to $\big(B_{p-k}(\frac{p+1}{2}) - B_{p-k}(1)\big)/(p-k)$, where $B_n(x)$ denotes the Bernoulli polynomial (cf. [1, Rem. 4.10]). From the formula $B_n(1/2) = (2^{1-n}-1)B_n$ (easily derived from the distribution relation [1, Prop. 4.9 (7)] for the case $k=2$), we see that this quantity is congruent modulo p to $(2-2^k)B_{p-k}/k$, the result follows. When k is even, $B_{p-k} = 0$ for almost all p and thus $\zeta_{\mathcal{A}}^{(2)}(k) = 0$.

We prove ii) in Example 2.4 after we present a general identity (7). □

The equality in ii) above may be viewed as an analogue of the "parity result" ([6, 13]) in the case of depth 2. In the next proposition we present a general identity from which one can obtain (a kind of) a general parity result. To state the proposition, we introduce the "star" variant $\zeta_{\mathcal{A}}^{(2),\star}(k_1,\dots,k_r)$ defined similarly as $\zeta_{\mathcal{A}}^{(2)}(k_1,\dots,k_r)$ but the summation is over $0<m_1\le\cdots\le m_r<p/2$ rather than with the strict inequalities.

Proposition 2.2. *For any $r\ge 1$ and $k_i\ge 1$, we have*
(7)
$$\zeta_{\mathcal{A}}^{(2)}(k_1,\dots,k_r)=(-1)^{r+k_1+\cdots+k_r}\sum_{i=0}^{r}(-1)^i\zeta_{\mathcal{A}}(k_i,\dots,k_1)\zeta_{\mathcal{A}}^{(2),\star}(k_{i+1},\dots,k_r).$$

Proof. We use (6) and the "antipode identity" (a consequence of the harmonic algebra structure, see [3])

$$\sum_{j=0}^{r}(-1)^j\zeta_{\mathcal{A}}^{(2)}(k_j,\dots,k_1)\zeta_{\mathcal{A}}^{(2),\star}(k_{j+1},\dots,k_r)=\delta_{r,0},\tag{8}$$

where $\delta_{r,0}$ is 0 if $r>0$ and 1 if $r=0$. By using these and the reversal formula $\zeta_{\mathcal{A}}(k_i,\dots,k_1)=(-1)^{k_1+\cdots+k_i}\zeta_{\mathcal{A}}(k_1,\dots,k_i)$, we start with the sum on the right and proceed as

$$\begin{aligned}
&\sum_{i=0}^{r}(-1)^i\zeta_{\mathcal{A}}(k_i,\dots,k_1)\zeta_{\mathcal{A}}^{(2),\star}(k_{i+1},\dots,k_r)\\
=&\sum_{i=0}^{r}(-1)^{i+k_1+\cdots+k_i}\zeta_{\mathcal{A}}(k_1,\dots,k_i)\zeta_{\mathcal{A}}^{(2),\star}(k_{i+1},\dots,k_r)\\
=&\sum_{i=0}^{r}(-1)^{i+k_1+\cdots+k_i}\\
&\times\sum_{j=0}^{i}(-1)^{k_{j+1}+\cdots+k_i}\zeta_{\mathcal{A}}^{(2)}(k_1,\dots,k_j)\zeta_{\mathcal{A}}^{(2)}(k_i,\dots,k_{j+1})\zeta_{\mathcal{A}}^{(2),\star}(k_{i+1},\dots,k_r)\\
=&\sum_{j=0}^{r}\sum_{i=j}^{r}(-1)^{i+k_1+\cdots+k_j}\zeta_{\mathcal{A}}^{(2)}(k_i,\dots,k_{j+1})\zeta_{\mathcal{A}}^{(2),\star}(k_{i+1},\dots,k_r)\zeta_{\mathcal{A}}^{(2)}(k_1,\dots,k_j)\\
=&(-1)^{r+k_1+\cdots+k_r}\zeta_{\mathcal{A}}^{(2)}(k_1,\dots,k_r).
\end{aligned}$$

□

Remark 2.3. If $k_1+\cdots+k_r \not\equiv r \bmod 2$, we obtain from (7)

$$\zeta_{\mathcal{A}}^{(2)}(k_1,\ldots,k_r)+\zeta_{\mathcal{A}}^{(2),\star}(k_1,\ldots,k_r) = -\sum_{i=1}^{r}(-1)^i\zeta_{\mathcal{A}}(k_i,\ldots,k_1)\zeta_{\mathcal{A}}^{(2),\star}(k_{i+1},\ldots,k_r).$$

Noting $\zeta_{\mathcal{A}}^{(2)}(k_1,\ldots,k_r)+\zeta_{\mathcal{A}}^{(2),\star}(k_1,\ldots,k_r) = 2\zeta_{\mathcal{A}}^{(2)}(k_1,\ldots,k_r)+\sum\zeta_{\mathcal{A}}^{(2)}$ (lower depth), we conclude that if the weight and the depth have a different parity, $\zeta_{\mathcal{A}}^{(2)}(k_1,\ldots,k_r)$ is written as a sum of products of $\zeta_{\mathcal{A}}^{(2)}$'s of lower depths and $\zeta_{\mathcal{A}}(k_1,\ldots,k_r)$. If we view the depth of $\zeta_{\mathcal{A}}(k_1,\ldots,k_r)$ as $r-1$ (this is reasonable in light of our "main conjecture" in [8]), this gives a kind of parity result for finite multiple zeta values of level 2, although it has the term $\zeta_{\mathcal{A}}(k_1,\ldots,k_r)$ of level one.

Example 2.4. i) When $r=2$, the identity (7) becomes

$$\zeta_{\mathcal{A}}^{(2)}(k_1,k_2) \;=\; (-1)^{k_1+k_2}\big(\zeta_{\mathcal{A}}^{(2),\star}(k_1,k_2)-\zeta_{\mathcal{A}}(k_1)\zeta_{\mathcal{A}}^{(2),\star}(k_2)+\zeta_{\mathcal{A}}(k_2,k_1)\big),$$

and this is equal to

$$(-1)^{k_1+k_2}\big(\zeta_{\mathcal{A}}^{(2)}(k_1,k_2)+\zeta_{\mathcal{A}}^{(2)}(k_1+k_2)+\zeta_{\mathcal{A}}(k_2,k_1)\big)$$

because $\zeta_{\mathcal{A}}(k_1)=0$ for any k_1 (see [3, 14], also [7, 8]). If k_1+k_2 is odd, we see from this that

$$\zeta_{\mathcal{A}}^{(2)}(k_1,k_2) \;=\; -\frac{1}{2}\big(\zeta_{\mathcal{A}}^{(2)}(k_1+k_2)+\zeta_{\mathcal{A}}(k_2,k_1)\big).$$

Then Proposition 2.1 ii) follows from Proposition 2.1 i) and a formula for $\zeta_{\mathcal{A}}(k_2,k_1)$ in [3, 14, 7, 8].

ii) The case $r=3$ of (7) reads (we set $k=k_1+k_2+k_3$ and use $\zeta_{\mathcal{A}}(k_1)=0$)

$$\zeta_{\mathcal{A}}^{(2)}(k_1,k_2,k_3) = (-1)^{k-1}\big(\zeta_{\mathcal{A}}^{(2),\star}(k_1,k_2,k_3)+\zeta_{\mathcal{A}}(k_2,k_1)\zeta_{\mathcal{A}}^{(2),\star}(k_3)-\zeta_{\mathcal{A}}(k_3,k_2,k_1)\big).$$

If k is even, we have, by writing $\zeta_{\mathcal{A}}^{(2),\star}(k_1,k_2,k_3)$ as a sum of $\zeta_{\mathcal{A}}^{(2)}$ in a usual way,

$$\begin{aligned}&\zeta_{\mathcal{A}}^{(2)}(k_1,k_2,k_3)\\ &=\frac{1}{2}\big(\zeta_{\mathcal{A}}(k_1,k_2,k_3)-\zeta_{\mathcal{A}}^{(2)}(k_1+k_2,k_3)-\zeta_{\mathcal{A}}^{(2)}(k_1,k_2+k_3)+\zeta_{\mathcal{A}}(k_1,k_2)\zeta_{\mathcal{A}}^{(2)}(k_3)\big).\end{aligned}$$

3. Sum formulas

In this section, we present various sum formulas. First, we establish formulas for

$$S(k,r) := \sum_{k_1+\cdots+k_r=k} \zeta_{\mathcal{A}}^{(2)}(k_1,\ldots,k_r)$$

and

$$S_1(k,r) := \sum_{\substack{k_1+\cdots+k_r=k \\ \forall k_i \geq 2}} \zeta_{\mathcal{A}}^{(2)}(k_1,\ldots,k_r),$$

writing them as linear combinations of conjectural basis elements given in Conjectures 1.5 and 1.9 respectively. Set

$$B(k,r) := \sum_{\substack{k_1+\cdots+k_r=k \\ \forall k_i:\mathrm{odd}\geq 1}} \zeta_{\mathcal{A}}^{(2)}(k_1,\ldots,k_r)$$

and

$$B_1(k,r) := \sum_{\substack{k_1+\cdots+k_r=k \\ \forall k_i:\mathrm{odd}\geq 3}} \zeta_{\mathcal{A}}^{(2)}(k_1,\ldots,k_r).$$

Theorem 3.1. *For $1 \leq r \leq k$, we have*

i)

$$S(k,r) = (-1)^{k+r} \sum_{\substack{1\leq i\leq r \\ i\equiv k \bmod 2}} \binom{\frac{k-i}{2}}{r-i} B(k,i)$$

and

ii)

$$S_1(k,r) = (-1)^{k+r} \sum_{\substack{1\leq i\leq r \\ i\equiv k \bmod 2}} \binom{\frac{k-3i}{2}}{r-i} B_1(k,i).$$

The following is the key lemma to prove Theorem 3.1. We consider the $\mathbb{Q}$-vector space spanned by formal symbols $[\mathbf{k}]$ for each index $\mathbf{k}$, equipped with the algebra structure given by the harmonic (stuffle) product $*$. For example, $[(2)]*[(3)] = [(2,3)]+[(3,2)]+[(5)]$. This is isomorphic to Hoffman's harmonic algebra $\mathfrak{h}^1$ ([2]). For more details, we refer [2, 7].

Lemma 3.2. *For $1 \leq r \leq k$ and $0 \leq a \leq r$, set*

$$g(k,r,a) := \sum_{\substack{k_1+\cdots+k_r=k \\ \#\ of\ even\ k_i\ =\ a}} [(k_1,\ldots,k_r)]$$

and

$$g_1(k,r,a) := \sum_{\substack{k_1+\cdots+k_r=k \\ \#\ of\ even\ k_i=a \\ \forall k_i\geq 2}} [(k_1,\ldots,k_r)].$$

Then we have

i)

$$\sum_{i=1}^{\frac{k-r-a}{2}} [(2i)] * g(k-2i,r,a) = \frac{k-r-a}{2} g(k,r,a) + (a+1)g(k,r+1,a+1)$$

and

ii)

$$\sum_{i=1}^{\frac{k-3r+a}{2}} [(2i)] * g_1(k-2i,r,a) = \frac{k-3r+a}{2} g_1(k,r,a) + (a+1)g_1(k,r+1,a+1).$$

Proof. i) If we compute the harmonic product $[(2i)] * g(k-2i,r,a)$, each resulting term has weight k, depth either r or $r+1$ and the number of even entries a or $a+1$ respectively, i.e., a term appearing in either $g(k,r,a)$ or $g(k,r+1,a+1)$. For a given $[(k_1,\ldots,k_r)]$ in $g(k,r,a)$, the number of possible combinations of i and a term in $g(k-2i,r,a)$ which give $[(k_1,\ldots,k_r)]$ in their harmonic product is $(k-r-a)/2$ because the depth r and the number a of even entries are the same, so the choice is the entry k_j in $[(k_1,\ldots,k_r)]$ which is larger than 2 and the amount $2i$ such that $(k_1,\ldots,k_j-2i,\ldots,k_r)]$ is still an index $(k_j-2i>0)$. Such a pair (k_j,i) is unique, and the total number is $(k-(r-a)-2a)/2 = (k-r-a)/2$ (there are $r-a$ odd entries and a even entries in $(k_1,\ldots,k_r)$, and $k-(r-a)-2a$ is the 'excess' for possible subtraction of $2i$). The term $[(k_1,\ldots,k_r)]$ in $g(k,r+1,a+1)$ comes from $[(2i)] * g(k-2i,r,a)$ by inserting $2i$ to a term in $g(k-2i,r,a)$, and so the choice is $a+1$.

The formula ii) is proved similarly, just by noting the condition that all entries are greater than or equal to 2. □

Proof of Theorem 3.1. i) From Proposition 2.1 i) and i) of the above lemma, one concludes

$$\frac{k-r-a}{2} S(k,r,a) + (a+1)S(k,r+1,a+1) \;=\; 0$$

because $\zeta_{\mathcal{A}}^{(2)}$ obeys the harmonic product rule. From this, and noting $S(k,r,0) = B(k,r)$, we have

$$
\begin{aligned}
&S(k,r,a)\\
&= \left(-\frac{k-r-a+2}{2a}\right) S(k,r-1,a-1)\\
&= \left(-\frac{k-r-a+2}{2a}\right)\left(-\frac{k-r-a+4}{2(a-1)}\right) S(k,r-2,a-2)\\
&= \dots\\
&= \left(-\frac{k-r-a+2}{2a}\right)\left(-\frac{k-r-a+4}{2(a-1)}\right)\cdots\left(-\frac{k-r-a+2a}{2}\right) S(k,r-a,0)\\
&= (-1)^a \binom{\frac{k-r+a}{2}}{a} B(k,r-a).
\end{aligned}
$$

Summing up, we obtain i). The proof of ii) is the same and is omitted. □

In an attempt to find a sum formula which is more close in form to the classical one (cf. [16, Ch. 5]), we discovered experimentally several strange formulas, some we could prove and the other conjectural. Since we feel there still are much to be discovered and our understanding is not mature yet, we mention only several of them, postponing the detailed study in a future publication [9] by the second-named author.

Theorem 3.3. *For $1 \le r \le k$ and a fixed i, we have*

$$\sum_{\substack{k_1+\cdots+k_r=k\\ k_i:odd \text{ and } \forall k_j:even\ (j\neq i)}} \zeta_{\mathcal{A}}^{(2)}(k_1,\dots,k_r) \ = \ c\,\zeta_{\mathcal{A}}^{(2)}(k)$$

for some rational constant c.

Proof. A special case

$$\zeta_{\mathcal{A}}^{(2)}(\underbrace{2,\dots 2}_{i-1},1,\underbrace{2,\dots,2}_{r-i}) = \frac{(-1)^{r-1}}{2^{2r-2}}\binom{2r-1}{2i-1}\zeta_{\mathcal{A}}^{(2)}(2r-1)$$

is proved in [11, Th. 5.4], and we may use this and the harmonic product to establish the theorem. We are not able to obtain a general closed formula of the constant c. □

The result [11, Th. 5.3] is also a special case and there the constant is explicit. The above theorem looks similar to the classical sum formula for multiple zeta values. There are several variants like restricted sum formulas or weighted sum formulas (see for instance [16]). But the next formulas looks rather strange and seems similar to none of these. We introduce one notation.

Definition 3.4. For an index $(k_1,\dots,k_r)$ of weight $k=k_1+\cdots+k_r$, put

$$C(k_1,\dots,k_r):=\sum_{j=1}^{r-1}(-1)^{k_1+\cdots+k_j}\binom{k}{k_1+\cdots+k_j}.$$

Let Σ_n be the set of permutations of $\{1,\dots,n\}$ (the symmetric group of order n). The following is a theorem for the usual (level one) finite multiple zeta values.

Theorem 3.5. *For a non-empty index $(k_1,\dots,k_r)$ of depth r and weight k, we have*

$$\begin{aligned}&\sum_{\sigma\in\Sigma_r}\Big(r+1-2\sigma^{-1}(r)\Big)\,\zeta_{\mathcal{A}}(k_{\sigma(1)},\dots,k_{\sigma(r)})\\ &=\ (-1)^r\,2\sum_{\tau\in\Sigma_{r-1}}C(k_{\tau(1)},\dots,k_{\tau(r-1)},k_r)\cdot Z(k).\end{aligned}$$

And the next is a level-two counterpart.

Theorem 3.6. *For a non-empty index $(k_1,\dots,k_r)$ of depth r and weight k with k_i even for all $1\le i\le r-1$ and k_r odd, we have*

$$\begin{aligned}&\sum_{\sigma\in\Sigma_r}\Big(r+1-2\sigma^{-1}(r)\Big)\,\zeta^{(2)}_{\mathcal{A}}(k_{\sigma(1)},\dots,k_{\sigma(r)})\\ &=\ (-1)^r\sum_{\tau\in\Sigma_{r-1}}C(k_{\tau(1)},\dots,k_{\tau(r-1)},k_r)\cdot Z(k).\end{aligned}$$

The proofs of both theorems rely on the following lemma.

Lemma 3.7. *For an index $(k_1,\dots,k_r)$, set*

$$R(k_1,\dots,k_r)=\sum_{\sigma\in\Sigma_r}(r+1-2\sigma^{-1}(r))[(k_{\sigma(1)},\dots,k_{\sigma(r)})].$$

Then, we have the identity

$$\begin{aligned}&\sum_{i=1}^{r-1}[(k_i)]*R(k_1,\dots,\check{k}_i,\dots,k_r)\\ &=(r-2)R(k_1,\dots,k_r)+\sum_{1\le i\le r-1}R(k_1,\dots,\check{k}_i,\dots,k_{r-1},k_i+k_r)\end{aligned}$$

$$+\,2\sum_{1\le i<j\le r-1} R(k_i+k_j,k_1,\dots,\check{k}_i,\dots,\check{k}_j\dots,k_{r-1},k_r),$$

where $\check{k}_i$ *means* k_i *is deleted.*

The proof of the lemma is done basically by comparing coefficients of terms on both sides, though this is a bit tedious. And the proofs of theorems are by induction on depths, starting point being explicit formulas in the case of depth 2 (Proposition 2.1 for level 2 and [3, 14], [7, Ex. 7.4] for level 1). The detailed discussion will be given in [9].

We end this paper by a conjecture, which may be viewed as a variant of the weighted sum formula but also strange in form.

Conjecture 3.8. *For* $r\ge 1$ *and* a $(0\le a\le r)$, *one has*

$$\sum_{\substack{\forall k_i\in\{1,2\}\\ \#\{i|k_i=2\}=a}} \left((-1)^{\#\{i|i:odd,k_i=2\}}2^a-1\right)\zeta_{\mathcal{A}}^{(2)}(k_1,\dots,k_r)=0.\quad (\textit{The weight is } r+a.)$$

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Masanobu Kaneko
Faculty of Mathematics
Kyushu University
744, Motooka, Nishi-ku
Fukuoka, 819-0395
Japan
E-mail: mkaneko@math.kyushu-u.ac.jp

Takuya Murakami
Sophia Fukuoka Junior and Senior High School
1-10-10 Terukuni, Chuo-ku
Fukuoka, 810-0032
Japan
E-mail: tak_mrkm@icloud.com

Amane Yoshihara
3-27-14-102, Nishi-Ohmiya, Nishi-ku
Saitama, 331-0078
Japan
E-mail: amanettiyosshy@me.com