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On squares of Hecke eigenforms

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1. Introduction

In a recent paper D. Bao [1] studied certain linear relations between Hecke eigenforms of weight $2k$ and squares of Hecke eigenforms of weight k. In particular, for a positive even integer k let S_k be the space of cusp forms of weight k for $\Gamma_1 := SL_2(\mathbf{Z})$. According to a widely believed conjecture of Maeda [4] the Hecke algebra over $\mathbf{Q}$ of S_k is simple (i.e. is a single number field) whose Galois closure over $\mathbf{Q}$ has Galois group isomorphic to the full symmetric group in d letters where $d = dim\ S_k$. Granting this conjecture both in weight k and weight $2k$ it was shown in [1] that the Petersson scalar products $\langle f^2, F\rangle$ are non-zero for all normalized Hecke eigenforms f in S_k and F in S_{2k}. Note that the numbers

$$\frac{\langle F, f^2\rangle}{\langle F, F\rangle}$$

are the coefficients of f^2 when written as a linear combination of an orthogonal basis of Hecke eigenforms F in S_{2k}. The above non-vanishing statement can be put more neatly in saying that f^2 for all f is a generator of S_{2k} viewed as a module over the Hecke algebra.

The question arises if one can get some simple explicit formulas for the quantities $\langle f^2, F\rangle$. As one naturally would expect such formulas should be related to modular forms of half-integral weight. Indeed, this is predicted by the identity

$$\mathcal{S}_1(f(4z)\theta(z)) = f^2(z) \tag{1}$$

for any normalized Hecke eigenform f in S_k. Here

$$\theta(z) = \sum_{n\in\mathbf{Z}} q^{n^2}$$

is the standard theta function of weight $\frac{1}{2}$ where as throughout z is in the complex upper half-plane $\mathcal{H}$ and $q = e^{2\pi i z}$ for $z \in \mathcal{H}$. Furthermore $\mathcal{S}_1$ is the

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first Shimura lift from the plus space $S^+_{k+1/2}$ of weight $k+\frac{1}{2}$ and level 4 to S_{2k} (see Sect. 2).

Identity (1) or variants of it follow by explicit calculations from the multiplicative properties of the Fourier coefficients of f. They seem to have been discovered first by Selberg and were later re-discovered by several people including H. Cohen and the author. In particular, with f replaced by the Eisenstein series

$$G_k(z) = \frac{1}{2}\zeta(1-k) + \sum_{n\geq 1} \sigma_{k-1}(n)q^n$$

of weight k for Γ_1 and properly added constant term, (1) was used in [7] in connection with Waldspurger's theorem on central critical L-values.

In this paper, we in particular will give an explicit identity for $\langle f^2, F\rangle$ which involves special values of partial L-functions attached to Hecke eigenforms of weight $k+\frac{1}{2}$ corresponding to Hecke eigenforms of weight $2k$ under the Shimura lift. Indeed, using (1) this will follow from a slightly more general similar statement in the context of modular forms of half-integral weight. The proof of the latter uses Rankin's method as given in [6] (with the Eisenstein series replaced by a Poincaré series) and the Petersson formulas for Poincaré series both in the integral and the half-integral weight case.

Note that our results do not depend on any unproved conjectures. Moreover, they seem to imply some curious identities involving Hecke eigenforms. We shall give two explicit examples, one relating to cubes of Hecke eigenforms and one other which gives an algebraicity result for a special value of partial L-functions of a Hecke eigenform of half-integral weight.

A precise formulation of our results and proofs will be given in the next sections. At the end of the paper, we will discuss some open questions and speculations.

2. Preliminaries on modular forms

Throughout k denotes a positive even integer. We will suppose that $k \geq 6$ since otherwise $S_{2k} = \{0\}$.

If f_1 and f_2 are modular forms of weight k on a congruence subgroup $\Gamma \subset \Gamma_1$ one of which is a cusp form we will denote by

$$\langle f_1, f_2\rangle = \frac{1}{[\Gamma_1 : \Gamma]} \int_{\Gamma\backslash\mathcal{H}} f_1(z)\overline{f_2(z)}y^k d\mu \quad (z = x+iy, d\mu = \frac{dxdy}{y^2})$$

their Petersson scalar product.

If the two forms are of half-integral weight $k+\frac{1}{2}$ on a congruence subgroup $\Gamma \subset \Gamma_0(4)$ (where $\Gamma_0(4)$ consists of matrices in Γ_1 with lower left component divisible by 4), we define their scalar product in the same way, mutatis mutandis and with the pre-factor $\frac{1}{[\Gamma_1:\Gamma]}$ replaced by $\frac{1}{[\Gamma_0(4):\Gamma]}$. The values of the scalar products then are independent of the choice of Γ and the usual computation rules are valid. We continue to use the above notation also in the case of non-holomorphic modular forms provided the integrals are convergent.

We let $\{f_1, ..., f_d\}$ (resp. $\{F_1, ..., F_e\}$) be the orthogonal basis of normalized Hecke eigenforms of weight k (resp. weight $2k$) for S_k (resp. S_{2k}). Note that these bases are uniquely determined up to permutation. We shall write $a_\mu(n)\,(n \geq 1)$ for the n-th Fourier coefficient of f_μ. Note that the $a_\mu(n)$ are real.

We denote by $S_{k+1/2}$ the space of cusp forms of weight $k+\frac{1}{2}$ and level 4 and by $S^+_{k+1/2}$ the subspace of those forms whose n-th Fourier coefficients vanish until $n \equiv 0, 1 \pmod 4$ [5, 9].

The space $S^+_{k+1/2}$ is Hecke isomorphic to S_{2k}. We let $\{g_1, ..., g_e\}$ be an orthogonal basis of Hecke eigenforms of $S^+_{k+1/2}$ with g_ν corresponding to F_ν for all ν, i.e. g_ν and F_ν have the same Hecke eigenvalues. This basis is uniquely determined up to permutation and multiplication with non-zero scalars. We may and will assume that the Fourier coefficients $c_\nu(n)\,(n \geq 1)$ of g_ν are real.

The map $\mathcal{S}_1$ defined by

$$\sum_{n\geq 1} c(n)q^n \mapsto \sum_{n\geq 1}\Big(\sum_{d|n} d^{k-1}c(\frac{n^2}{d^2})\Big)q^n$$

maps $S^+_{k+1/2}$ to S_{2k} and commutes with all Hecke operators (first Shimura lift). One has

$$\mathcal{S}_1(g_\nu) = c_\nu(1)F_\nu \tag{2}$$

for all ν [5, 9].

3. Statement of main result

For $g = \sum_{n\geq 1} c(n)q^n \in S^+_{k+1/2}$ and $n \in \mathbf{N}$ we put

$$\ell(g,n) := \sum_{r\in\mathbf{Z}} \frac{c(4n+r^2)}{(4n+r^2)^{k-1/2}}. \tag{3}$$

Since $c(n) = \mathcal{O}(n^{k/2+1/4})$ by the usual Hecke bound we see that the series in (3) is absolutely convergent.

We set

$$\kappa_k := \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-3)}{2^k (k-2)!}. \tag{4}$$

Note that κ_k is a rational number.

Theorem. *For each $n \in \mathbf{N}$ one has*

$$\sum_{\mu=1}^{d} \frac{a_\mu(n)}{n^{k-1}\langle f_\mu, f_\mu\rangle} f_\mu(4z)\theta(z) = \kappa_k \sum_{\nu=1}^{e} \frac{\ell(g_\nu, n)}{\langle g_\nu, g_\nu\rangle} g_\nu(z) \tag{5}$$

where κ_k is given by (4).

The proof will be given in Sect. 5. Before we will state and prove some Corollaries in the next section.

4. Discussion of corollaries

If we apply $\mathcal{S}_1$ on both sides of (5) and observe (1) and (2) we obtain

Corollary 1. *For each $n \in \mathbf{N}$ one has*

$$\sum_{\mu=1}^{d} \frac{a_\mu(n)}{n^{k-1}\langle f_\mu, f_\mu\rangle} f_\mu^2(z) = \kappa_k \sum_{\nu=1}^{e} \frac{\ell(g_\nu, n) c_\nu(1)}{\langle g_\nu, g_\nu\rangle} F_\nu(z) \tag{6}$$

with κ_k given by (4).

As is well-known the matrix

$$(a_\mu(n))_{1\leq n\leq d, 1\leq \mu\leq d}$$

is invertible. (Indeed, S_k has a basis whose (d,d)-matrix of first Fourier coefficients is the unit matrix.) Therefore

$$A := \Big(\frac{a_\mu(n)}{n^{k-1}\langle f_\mu, f_\mu\rangle}\Big)_{1\leq n\leq d, 1\leq \mu\leq d}$$

is invertible. We put

$$B := \kappa_k \Big(\frac{\ell(g_\nu, n) c_\nu(1)}{\langle g_\nu, g_\nu \rangle}\Big)_{1\le n\le d, 1\le \nu\le e}.$$

We then find from (6)

Corollary 2. *With the above notation one has*

$$\begin{pmatrix} f_1^2 \\ . \\ . \\ . \\ f_d^2 \end{pmatrix} = A^{-1} B \begin{pmatrix} F_1 \\ . \\ . \\ . \\ F_e \end{pmatrix}.$$

Corollary 2 gives an explicit expression for f_ν^2 $(1 \le \nu \le d)$ in terms of the Hecke basis $\{F_1, ..., F_e\}$.

Next, we shall prove

Corollary 3. *i) For any $g \in S^+_{k+1/2}$ the function*

$$\sum_{n\ge 1} n^{k-1} \ell(g, n) q^n$$

is in S_k.

ii) One has

$$\sum_{\mu=1}^{d} \frac{f_\mu(z') f_\mu^2(z)}{\langle f_\mu, f_\mu \rangle} = \kappa_k \sum_{\nu=1}^{e} \frac{c_\nu(1)}{\langle g_\nu, g_\nu \rangle} \Big(\sum_{n\ge 1} n^{k-1} \ell(g_\nu, n) e^{2\pi i n z'}\Big) F_\nu(z).$$

In particular

$$\sum_{\mu=1}^{d} \frac{f_\mu^3(z)}{\langle f_\mu, f_\mu \rangle} = \kappa_k \sum_{\nu=1}^{e} \frac{c_\nu(1)}{\langle g_\nu, g_\nu \rangle} \Big(\sum_{n\ge 1} n^{k-1} \ell(g_\nu, n) q^n\Big) F_\nu(z).$$

Indeed, if we multiply both sides of (5) with $n^{k-1} e^{2\pi i n z'}$ $(z' \in \mathcal{H})$ and then sum up over n we obtain

$$\sum_{\mu=1}^{d} \frac{f_\mu(z') f_\mu(4z) \theta(z)}{\langle f_\mu, f_\mu \rangle} = \kappa_k \sum_{\nu=1}^{e} \Big(\sum_{n\ge 1} n^{k-1} \ell(g_\nu, n) e^{2\pi i n z'}\Big) \frac{g_\nu(z)}{\langle g_\nu, g_\nu \rangle}. \qquad (7)$$

Taking the scalar product with a fixed $g_\rho(z)$ ($\rho = 1, ..., e$) reveals that

$$\sum_{n\geq 1} n^{k-1}\ell(g_\rho, n)e^{2\pi i n z'}$$

is in S_k. This proves *i)*.

To prove *ii)* we apply $\mathcal{S}_1$ on both sides of (7) and then use (1) and (2).

Remark. Using the calculations in the next section and an obvious slight modification of the arguments given in [6], one sees that the linear map given in *i)* above is the adjoint map of $S_k \to S^+_{k+1/2}$, $f(z) \mapsto f(4z)\theta(z)$, up to some non-zero constant.

We now turn to special values. Writing $A_\nu(n)$ ($n \geq 1$) for the n-th Fourier coefficient of F_ν ($1 \leq \nu \leq e$) the matrix

$$(A_\nu(n))_{1\leq n\leq e, 1\leq \nu\leq e}$$

is invertible as already stated above. We therefore obtain from (6) by equating first Fourier coefficients an identity

$$Cp = r,$$

where p is a d-column, r is an e-column with

$$p_\mu = \frac{a_\mu(n)}{n^{k-1}\langle f_\mu, f_\mu\rangle} \quad (\mu = 1, ..., d),$$
$$r_\nu = \kappa_k \frac{\ell(g_\nu, n)c_\nu(1)}{\langle g_\nu, g_\nu\rangle} \quad (\nu = 1, ..., e)$$

and where C is an (e, d)-matrix with entries in the field of algebraic numbers $\overline{\mathcal{Q}}$. In particular, we obtain

Corollary 4. *Let* $\nu \in \{1, ..., e\}$. *Then for all* $n \in \mathbf{N}$ *the numbers*

$$\frac{\ell(g_\nu, n)c_\nu(1)}{\langle g_\nu, g_\nu\rangle}$$

lie in the finite-dimensional $\overline{\mathcal{Q}}$*- vector space*

$$\overline{\mathcal{Q}}\frac{1}{\langle f_1, f_1\rangle} + ... + \overline{\mathcal{Q}}\frac{1}{\langle f_d, f_d\rangle}.$$

5. Proof of main result

We shall now give the proof of the Theorem stated in Sect. 3.

For each $n \in \mathbf{N}$ we let $P_{k,n} \in S_k$ be the n-th Poincaré series characterized by the condition

$$\langle f, P_{k,n}\rangle = \frac{(k-2)!}{(4\pi n)^{k-1}} a_f(n) \tag{8}$$

for all $f \in S_k$, where $a_f(n)$ denotes the n-th Fourier coefficient of f. Explicitly, one has

$$P_{k,n}(z) = \frac{1}{2} \sum_{\gamma \in \Gamma_{1,\infty}\backslash\Gamma_1} e^{2\pi i n z}|_k\gamma, \tag{9}$$

where $\Gamma_{1,\infty} = \{\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) | m \in \mathbf{Z}\}$ and as usual for any function $h : \mathcal{H} \to \mathbf{C}$ and for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbf{R})$ we put

$$(h|_k\gamma)(z) := (ad-bc)^{k/2}(cz+d)^{-k} h(\frac{az+b}{cz+d}). \tag{10}$$

From (8) using the Petersson formula and multiplying with θ we find that

$$P_{k,n}(4z)\theta(z) = \frac{(k-2)!}{(4\pi)^{k-1}} \sum_{\mu=1}^{d} \frac{a_\mu(n)}{n^{k-1}\langle f_\mu, f_\mu\rangle} f_\mu(4z)\theta(z). \tag{11}$$

On the other hand, we note that $P_{k,n}(4z)\theta(z) \in S^+_{k+1/2}$. Hence by Petersson's formula again we have

$$P_{k,n}(4z)\theta(z) = \sum_{\nu=1}^{e} \frac{\overline{\langle g_\nu(z), P_{k,n}(4z)\theta(z)\rangle}}{\langle g_\nu, g_\nu\rangle} g_\nu(z). \tag{12}$$

We shall prove

Proposition. *For any $g \in S^+_{k+1/2}$ one has*

$$\langle g(z), P_{k,n}(4z)\theta(z)\rangle = 2^{-2k+1}\frac{\Gamma(k-1/2)}{\pi^{k-1/2}}\ell(g,n). \tag{13}$$

The assertion of the Theorem immediately follows from the Proposition and by comparing (11) and (12). We also observe our assumption that $c_\nu(n)$

is real and the identity

$$\Gamma(k-1/2)=\frac{1\cdot 3\cdot 5\cdot\dots\cdot(2k-3)}{2^{k-1}}\sqrt{\pi}.$$

We will now give the *proof of the Proposition* which technically is somewhat involved. For the reader's convenience we shall be rather detailed.

We shall start with some definitions.

We let

$$W_4:=\begin{pmatrix}0&-1\\4&0\end{pmatrix}.$$

Then according to (10), W_4 acts on functions $h:\mathcal{H}\to\mathbf{C}$ in weight k by

$$h(z)\mapsto (h|_kW_4)(z)=(2z)^{-k}h(-\frac{1}{4z}).$$

We also define an operation of W_4 on functions h as above in weight $k+\frac{1}{2}$ by

$$h(z)\mapsto (h|_{k+\frac{1}{2}}W_4)(z):=(-2iz)^{-k-1/2}h(-\frac{1}{4z}).$$

We also set

$$(h|V_4)(z):=h(4z).$$

Furthermore, if

$$H(z)=\sum_{m\in\mathbf{Z}}a(m,y)e^{2\pi imx}\qquad(z=x+iy\in\mathcal{H})$$

is a convergent Fourier series we define

$$(H|U_4)(z):=\sum_{m\in\mathbf{Z}}a(4m,\frac{y}{4})e^{2\pi imx}.$$

If $a(m,y)=a(m)e^{-2\pi my}$ with $a(m)\in\mathbf{C}$ and so H is a holomorphic Fourier series, then U_4 restricts to the usual Hecke operator of degree 4 sending modular forms to modular forms.

We recall that the subspace $S^+_{k+1/2}\subset S_{k+1/2}$ can be characterized as the subspace of functions g which satisfy

$$g|U_4=(-1)^{k/2}2^k(g|_{k+1/2}W_4)\tag{14}$$

[3].

We will now start to evaluate the scalar product on the left of (13) which in the following will be abbreviated as I.

Since

$$(P_{k,n}|_k W_4)(z) = 2^k P_{k,n}(4z)$$

and

$$(\theta|_{1/2} W_4)(z) = \theta(z)$$

we see that

$$\begin{aligned}(-1)^{k/2} 2^{-k} I &= (-1)^{k/2} \langle g, (P_{k,n}|_k W_4)\theta \rangle \\ &= \langle g, (P_{k,n}\theta)|_{k+1/2} W_4 \rangle \\ &= \langle g|_{k+1/2} W_4, P_{k,n}\theta \rangle\end{aligned}$$

where in the last line we have used that W_4 in weight $k + \frac{1}{2}$ is an hermitian involution.

Let us put

$$G(z) := (g|_{k+1/2} W_4)(z) \cdot \overline{\theta(z)} \cdot \sqrt{y}.$$

Then G transforms like a modular form of weight k on $\Gamma_0(4)$ and

$$(-1)^{k/2} 2^{-k} I = \langle G, P_{k,n} \rangle.$$

Let

$$P_{k,n,4}(z) = \frac{1}{2} \sum_{\gamma \in \Gamma_0(4)_\infty \backslash \Gamma_0(4)} e^{2\pi i n z}|_k \gamma \tag{15}$$

be the n-th Poincaré series of weight k on $\Gamma_0(4)$. It follows from the definitions (9) and (15) that

$$P_{k,n} = P_{k,n,4}|tr$$

where

$$P_{k,n,4}|tr := \sum_{\gamma \in \Gamma_0(4) \backslash \Gamma_1} P_{k,n,4}|_k \gamma$$

("trace map"). Hence by elementary properties of the scalar product

$$\begin{aligned}(-1)^{k/2} 2^{-k} I &= \langle G, P_{k,n,4}|tr \rangle \\ &= \sum_{\gamma \in \Gamma_0(4) \backslash \Gamma_1} \langle G, P_{k,n,4}|_k \gamma \rangle\end{aligned}$$

$$= \sum_{\gamma\in\Gamma_0(4)\backslash\Gamma_1} \langle G|_k\gamma^{-1}, P_{k,n,4}\rangle.$$

The group $\Gamma_0(4)$ has index 6 in Γ_1 and we take as a set of representatives for $\Gamma_0(4)\backslash\Gamma_1$ the matrices

$$\begin{pmatrix}1&0\\4&1\end{pmatrix}, \begin{pmatrix}1&0\\2&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&\mu\\0&1\end{pmatrix} \quad (\mu \pmod 4).$$

Since G is invariant in weight k under $\Gamma_0(4)$ we thus conclude that
(16)

$$(-1)^{k/2}2^{-k}I = \langle G, P_{k,n,4}\rangle + \langle G|_k \begin{pmatrix}1&0\\-2&1\end{pmatrix}, P_{k,n,4}\rangle + 4\cdot\langle G|_k \begin{pmatrix}1&0\\-1&1\end{pmatrix} G, P_{k,n,4}\rangle.$$

Recall that $\Gamma_0(4)$ has three inequivalent cusps (usually represented by $i\infty, 0$ and $\frac{1}{2}$) and that the functions

$$G, G|_k \begin{pmatrix}1&0\\-2&1\end{pmatrix}, G|_k \begin{pmatrix}1&0\\-1&1\end{pmatrix},$$

respectively are "the" expansions of G at $i\infty, -\frac{1}{2}$ (equivalent to $\frac{1}{2}$) and -1 (equivalent to 0), respectively.

We will now compute these expansions and then the expressions on the right of (16).

We denote by $b(m)\,(m \geq 1)$ the Fourier coefficients of $g|_{k+1/2}W_4$. Then we find from the definitions that

$$G(z) = \sqrt{y} \sum_{m\geq 1, r\in\mathbf{Z}} b(m)e^{-2\pi(m+r^2)y}e^{2\pi i(m-r^2)x}$$

and hence the n-th Fourier coefficient of G equals

$$a(G; n, y) := \sqrt{y}\sum_{r\in\mathbf{Z}} b(n+r^2)e^{-2\pi(n+2r^2)y}.$$

Therefore, by the standard unfolding argument and the usual integral representation of the Γ-function, we infer that

$$\begin{aligned}\langle G, P_{k,n,4}\rangle &= \int_0^\infty a(G; n, y)e^{-2\pi ny}y^{k-2}dy\\ &= \sum_{r\in\mathbf{Z}} b(n+r^2)\int_0^\infty e^{-4\pi(n+r^2)y}y^{k-3/2}dy\end{aligned}$$

$$= \frac{\Gamma(k-1/2)}{(4\pi)^{k-1/2}} \sum_{r\in\mathbf{Z}} \frac{b(n+r^2)}{(n+r^2)^{k-1/2}}.$$

By (14), one has

$$b(n) = (-1)^{k/2} 2^{-k} c(4n)$$

and hence we find that

$$\langle G, P_{k,n,4}\rangle = (-1)^{k/2} 2^{-3k+1} \frac{\Gamma(k-1/2)}{\pi^{k-1/2}} \sum_{r\in\mathbf{Z}, r\ even} \frac{c(4n+r^2)}{(4n+r^2)^{k-1/2}}. \tag{17}$$

Next, we turn to

$$G_1 := G|_k \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}^{-1} \Gamma_0(4) \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \Gamma_0(4)$$

and hence G_1 is modular on $\Gamma_0(4)$.

Since $\theta|_{1/2} W_4 = \theta$ and

$$Im(-\frac{1}{4z})^{1/2} = \frac{\sqrt{y}}{2|z|} \tag{18}$$

we can write

$$G = (-1)^{k/2} \Big(g\overline{\theta} \cdot Im^{1/2}\Big)|_k W_4.$$

Therefore, as

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = W_4^{-1} \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} W_4$$

we find that

$$G_1 = (-1)^{k/2} \Big(g\overline{\theta} \cdot Im^{1/2}\Big)|_k \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} |_k W_4. \tag{19}$$

Let us compute the action of $\left(\begin{smallmatrix} 1 & 1/2 \\ 0 & 1 \end{smallmatrix}\right)$ in weight k in (19). We have

$$g(z+\frac{1}{2}) = \sum_{m\geq 1} (-1)^m c(m) q^m$$

$$= 2 \cdot \sum_{m \geq 1, m\, even} c(m)q^m - g(z).$$

Since g is in the plus space, we infer that

$$\sum_{m \geq 1, m\, even} c(m)q^m = \sum_{m \geq 1, m \equiv 0 \pmod 4} c(m)q^m$$
$$= (g|U_4V_4)(z)$$
$$= (-1)^{k/2}2^k(g|_{k+1/2}W_4)|V_4$$

(use (14)).

Thus

$$g(z + \frac{1}{2}) = (-1)^{k/2}2^{k+1}(g|_{k+1/2}W_4)(4z) - g(z). \tag{20}$$

In particular, (with $g = \theta$) we see that

$$\theta(z + \frac{1}{2}) = 2\theta(4z) - \theta(z). \tag{21}$$

Inserting (20) and (21) into (19) we conclude that

$$G_1 = (-1)^{k/2}\Big([(-1)^{k/2}2^{k+1}(g|_{k+1/2}W_4)(4z) - g(z)] \cdot \overline{[2\theta(4z) - \theta(z)]} \cdot \sqrt{y}\Big)|_k W_4.$$

A simple calculation now shows that the first factor in square brackets acted on by W_4 in weight k equals

$$2^{-k+1/2}(-iz)^{1/2}(g(\frac{z}{4}) - (g|U_4)(z))$$

(we have used (14) again for the second term of the sum).

Therefore, using (18) we finally obtain

$$G_1 = (-1)^{k/2}2^{-k}[g(\frac{z}{4}) - (g|U_4)(z)] \cdot \overline{[\theta(\frac{z}{4}) - \theta(z)]} \cdot \sqrt{y}. \tag{22}$$

We now insert the Fourier expansions into (22). We have

$$g(\frac{z}{4}) - (g|U_4)(z) = \sum_{m \geq 1, m \equiv 0,1 \pmod 4} c(m)q^{m/4} - (g|U_4)(z)$$
$$= \sum_{m \geq 1, m \equiv 1 \pmod 4} c(m)q^{m/4}.$$

In particular, with $g = \theta$ we have

$$\theta(\frac{z}{4}) - \theta(z) = \theta(\frac{z}{4}) - (\theta|U_4)(z) = \sum_{r \in \mathbf{Z}, r\, odd} q^{r^2/4}.$$

Hence, we find

$$G_1 = (-1)^{k/2} 2^{-k} \cdot \sqrt{y} \cdot (\sum_{m \geq 1, m \equiv 1 \pmod 4} c(m) q^{\frac{m}{4}}) (\overline{\sum_{r \in \mathbf{Z}, r\, odd} q^{r^2/4}}).$$

Multiplying out we get

$$G_1 = (-1)^{k/2} 2^{-k} \cdot \sqrt{y} \cdot \sum_{m \geq 1, m \equiv 1 \pmod 4, r \in \mathbf{Z}, r\, odd} c(m) e^{-2\pi \frac{m+r^2}{4} y} \cdot e^{-2\pi i \frac{m-r^2}{4} x}.$$

Note that under the given conditions $\frac{m-r^2}{4}$ is an integer. Hence, we see that the n-th Fourier coefficient of G_1 equals

$$(-1)^{k/2} 2^{-k} \cdot \sqrt{y} \cdot \sum_{m \geq 1, m \equiv 1 \pmod 4, r \in \mathbf{Z}, r\, odd} c(4n + r^2) e^{-2\pi (n + \frac{r^2}{2}) y},$$

and in a similar way as before we obtain

$$\langle G_1, P_{k,n,4} \rangle = (-1)^{k/2} 2^{-3k+1} \frac{\Gamma(k - 1/2)}{\pi^{k-1/2}} \sum_{r \in \mathbf{Z}, r\, odd} \frac{c(4n + r^2)}{(4n + r^2)^{k-1/2}}. \tag{23}$$

We will finally treat

$$G_2 := G|_k \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

We denote by $\Gamma^0(4)$ the subgroup of matrices in Γ_1 with upper right entry divisible by 4 and put

$$\Gamma_0^0(4) := \Gamma_0(4) \cap \Gamma^0(4).$$

Then

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \Gamma_0^0(4) \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \Gamma_0^0(4)$$

and G_2 is modular on $\Gamma_0^0(4)$. The group $\Gamma_0^0(4)$ has index 4 in $\Gamma_0(4)$ and a set of representatives for $\Gamma_0^0(4)\backslash\Gamma_0(4)$ is given by the matrices

$$\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \quad (\mu \pmod 4).$$

In particular, it follows that the trace of G_2 from $\Gamma_0^0(4)$ to $\Gamma_0(4)$ given by

$$tr\ G_2 = \sum_{\mu \pmod 4} G_2|_k \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \quad (\mu \pmod 4))$$

is modular on $\Gamma_0(4)$. Since $P_{k,n,4}$ is invariant under translations we therefore find that

$$\begin{aligned} 4 \cdot \langle G_2, P_{k,n,4}\rangle &= \sum_{\mu \pmod 4} \langle G_2, P_{k,n,4}|_k \begin{pmatrix} 1 & -\mu \\ 0 & 1 \end{pmatrix}\rangle \\ &= \sum_{\mu \pmod 4} \langle G_2|_k \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, P_{k,n,4}\rangle \\ &= \langle tr\ G_2, P_{k,n,4}\rangle. \end{aligned}$$

So, we have to compute the Fourier expansion of $tr\ G_2$. Since

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and G is invariant under translations we see that

$$\begin{aligned} tr\ G_2 &= \sum_{\mu \pmod 4} G|_k \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \\ &= \sum_{\mu \pmod 4} G|_k \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Now a straightforward calculation similar as above shows that

$$(G|_k \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})(z) = (-1)^{k/2} 2^{-k-1} g(\frac{z}{4}) \overline{\theta(\frac{z}{4})} \cdot \sqrt{y},$$

hence

$$tr\ G_2 = (-1)^{k/2} 2^{-k-1} \Big(\sum_{\mu \pmod 4} g(\frac{z+\mu}{4}) \overline{\theta(\frac{z+\mu}{4}} \cdot Im(\frac{z+\mu}{4})^{1/2} \Big)$$
$$= (-1)^{k/2} 2^{-k-1} \Big(g\overline{\theta} \cdot \sqrt{y} \Big) | U_4.$$

We have

$$g\overline{\theta} \cdot \sqrt{y} = \sqrt{y} \sum_{m \geq 1,\, r \in \mathbf{Z}} c(m) e^{-2\pi(m+r^2)y} e^{2\pi i(m-r^2)x}$$

and the n-th Fourier coefficient of $(g\overline{\theta} \cdot \sqrt{y})|U_4$ is the $4n$-th Fourier coefficient of $g\overline{\theta} \cdot \sqrt{y}$, with y replaced by $\frac{y}{4}$. Thus the n-th Fourier coefficient of $tr\ G_2$ is

$$(-1)^{k/2} 2^{-k+1} \cdot \sqrt{\frac{y}{4}} \cdot \sum_{r \in \mathbf{Z}} c(4n+r^2) e^{-4\pi(4n+2r^2)y/4}.$$

Hence, as before

$$\langle tr\ G_2, P_{k,n,4} \rangle = (-1)^{k/2} 2^{-k+1} \sum_{r \in \mathbf{Z}} c(4n+r^2) \int_0^\infty e^{-2\pi(4n+2r^2)y} y^{k-3/2} dy$$
$$= (-1)^{k/2} 2^{-3k+1} \frac{\Gamma(k-1/2)}{\pi^{k-1/2}} \sum_{r \in \mathbf{Z}} \frac{c(4n+r^2)}{(4n+r^2)^{k-1/2}}. \tag{24}$$

Altogether, we find from (16), (17), (23) and (24) that

$$I = 2^{-2k+1} \frac{\Gamma(k-1/2)}{\pi^{k-1/2}} \ell(g,n)$$

as claimed. This proves the Proposition.

6. Some open questions and speculations

Let us briefly discuss some open problems connected with the main result of the paper.

i) It seems possible to generalize the Theorem to arbitrary level and character (not necessarily restricted to the context of the plus space), using generalizations of Selberg's identity (1) due to B. Cipra [2] and D. Hansen and Y. Naqvi [3].

ii) A more difficult question is if one can prove analogous results when $\mathcal{S}_1$ is replaced by the D-th Shimura lift where $D > 0$ is a fundamental discriminant [5]. Note that in [7], p. 185 for level 1 two functions $\mathcal{G}_D$ and $\mathcal{F}_{\mathcal{D}}$ built out of Eisenstein series were introduced and it was proved in Propos. 3 that $\mathcal{F}_{\mathcal{D}}$ is the image of $\mathcal{G}_D$ under the D-th Shimura lift. The latter can be viewed as a generalization of formula (1). In the above situation, can one replace the Eisenstein series by any normalized cuspidal Hecke eigenform?

We finally remark that M.K. Pandey and B. Ramakrishnan in [8] give certain generalizations of (1) to the case of the t-th Shimura lift, where t is a squarefree number and the authors do not restrict to the plus space.

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