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Electrotechnics, quantum modularity and CFT

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Abstract: Based on older and recent work of Don Zagier and collaborators, a relation between algebraic K-theory and modularity is investigated. It arises from the study of the behaviour of certain q-hypergeometric functions near the roots of unity. In typical cases, one finds intricate phenomena described by "quantum modularity". Here it is investigated when they reduce to ordinary modular invariance. For the q-hypergeometric functions under study ("Nahm sums") this leads to systems of algebraic equations. I had conjectured an equivalence between modular invariance of the q-hypergeometric functions and a K-theoretic torsion property of all solutions of the algebraic equations. This rough and ready conjecture was not quite correct, though it pointed in a rewarding direction. In particular, Zagier's recent work with Calegari and Garoufalidis proved that modular invariance implies the torsion property for a special solution of the algebraic system. Here their argument is generalised. The q-hypergeometric series should be understood as convolutions of Jacobi forms (as defined and explored by Eichler and Zagier). The Jacobi forms are vector valued, with components described by some finite set $\mathcal{M}$. For each element of $\mathcal{M}$ one has a specific system of algebraic equations and for all of them at least one solution must have a modified K-theoretic torsion property. A related inversion property conjectured twenty years ago is proven. At greater depth new structures do appear. There is a relation to algebraic geometry, namely to vector bundles over tori. Particularly intriguing is a possible relation between CFT (mathematics) and CFT (physics).

Keywords: CFT, Nahm sums, K3 group.

1. Introduction

In a well-known poem, Schiller attributed an interpretation of the three space dimensions to Confucius. He concluded by: *Nur Beharrung führt zum Ziel,*

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Nur die Fülle führt zur Klarheit, Und im Abgrund wohnt die Wahrheit. In view of our limited time, most mathematicians concentrate on parts one and three, and only some of the greatest have the leisure to explore the breadth of the world to gain greater clarity. Don Zagier sometimes regrets that his very nature leads him to explore such broad and clear vistas instead of opening the deepest mines, but mathematical nature has a tendency to twist dimensions. From time to time he enjoys the experience to find himself at an unexpected depth while following his curiosity in a lateral direction. For me it was a pleasure to find half-hidden passages between all those gardens he has brought into fruit and flower. Unfortunately politics and nature limited our interactions and we even did not manage to write a paper together. Our contributions to the meeting "Frontiers in Number Theory, Physics and Geometry" in Les Houches ([11], [7]) should be read as a dialogue, however. If a reader of the present article is not yet immersed in modularity and K_3, (s)he might better look at them first. Still, the following two opening sections of the present article are self-contained. The next two sections provide the connection to another domain of mathematics and some calculations. The final outlook is speculative. Unavoidably, an anniversary celebration has a time limit, so that many paths could not be pursued with sufficient leisure and persistence. I hope that this can be forgiven by the readers. As far as Don Zagier himself is concerned, I regard a small aspect of his recent lecture series in Bonn and Trieste as a continuation of our dialogue and the present article as my first response.

2. Electrotechnics and the extension of the Rogers dilogarithm

One focus of my Les Houches contribution was the theory of integrable perturbations of conformal quantum field theory. I did not (nor do I now) have a good mathematical understanding of it and had to speculate a lot. Thus I was reluctant to bring in even more tenuous links outside mathematics proper, though they had given me much inspiration. A contribution to Zagier's anniversary may be a legitimate place to correct the omission of such a reference. At the time it seemed likely that the gap between the Bloch group and full K_3 could be bridged by a proper consideration of the domain of the Rogers dilogarithm, and indeed Zickert later proved that this is true [12]. One needs the fully extended Bloch group instead of partial extensions that occur in other contexts. I had found it in an advertisement, but learned later that it was known, though with a description that I cannot remember [8]. That advertisement is the only one in all my life that left a long-lasting positive impression. I read it when I came to Bonn to get my PhD and spent much time browsing *Scientific American* in the physics library. At the time I found it too cumbersome to get its source [9].

The first half of the advertisement is worth citing: "Ohm's Law – 1964. It's called van der Pauw's theorem. And in my opinion, it is a most striking example of how a rather sophisticated piece of mathematics led to an extremely practical result. Why? Because of its underlying elegance and simplicity. Before van der Pauw came along, the determination of specitic resistivity [later called ρ, W.N.] – especially in the case of semi-conductors – while not exactly difficult, was a time-consuming and lengthy procedure. The best procedure available was to grind cylindrical or prismatic rods and to determine accurately both length and cross section.

Van der Pauw, on the other hand, showed that it is possible to measure the specific resistivity of a flat sample of constant thickness but otherwise arbitrary shape merely by placing four small contacts (1,2,3,4) on the circumference. Then if we define R_{12} as the ratio of the voltage between 1 and 2, and the current between 3 and 4, and similarly R_{23}, we have:

$$e^{-\pi d R_{12}/\rho} + e^{-\pi d R_{23}/\rho} = 1,$$

where d is the thickness of the sample.

Let me indicate the proof. First take an infinite half plane. Then all potentials are logarithmic and the proof is elementary. Next map conformly to obtain the detailed contour and remember that resistance is a conformal invariant. That's all."

The text was written by Casimir in his role as Director of Philips Research Laboratories, obviously with an audience of mathematical physicists in mind. For mathematicians one should add that the shape must be contractible and that the two-dimensional electric potential is governed by a Green's function of $\partial\bar{\partial}$. Thus for the infinite upper half plane, R_{12} becomes proportional to the logarithm u of the cross ratio of the four points in given order and similarly for v and a cyclically permuted order. One obtains

$$e^u + e^v = 1,$$

which is essentially the equation written down by van der Pauw.

Definition 2.1. Let F be a subfield of $\mathbb{C}$. Then $P(F)$ is the set of pairs (u, v) in $(\mathbb{C} \cup \{-\infty\})^2$, so that e^u and e^v lie in F and satisfy van der Pauw's equation $e^u + e^v = 1$. Here the exponential denotes the standard map from the semigroup $\mathbb{C} \cup \{-\infty\}$ to $\mathbb{C}$, with $e^{-\infty} = 0$.

In the following we will use the notation $x = e^u$ (and $x_i = e^{u_i}$). For $F = \mathbb{C}$, we have differentials $dLi_2(x) = -vdu$ and $dL(u, v) = (udv - vdu)/2$. When one cuts the complex x-plane by removing the interval $(1, \infty)$ on the real axis, the dilogarithm $Li_2(x)$ is a function taking values in $\mathbb{C}$, defined by integration of $-vdu$ and $Li_2(0) = 0$. The Rogers dilogarithm is given by $L(u, v) = Li_2(x) + uv/2$. The inverse image of the cut x-plane under the map $(u, v) \mapsto e^u$ has components labeled by integers n, with $\mathcal{I}m(v) \in (-\pi + n, \pi + n)$. The component with $n = 0$ will be called principal. The points $(-\infty, 0)$ and $(0, -\infty)$ are assigned to the principal component. By continuity, $L(-\infty, 0) = 0$ and $L(0, -\infty) = \pi^2/6$. Since $vdu = ve^v dv/(e^v - 1)$, the residues of $dL(u, v)$ in $P(\mathbb{C})$ are integral multiples of $2\pi i$. Thus $L(u, v)$ is determined up to integral multiples of $(2\pi i)^2$, when it is defined by integration of $dL(u, v)$ along arbitrary paths. Both the multivaluedness and the unique principal value will be relevant later.

One mainly is interested in the case that F is a field of algebraic numbers. From our first childhood experiences with integration the amalgam of algebra and logarithms should not be surprising, and the central role of the logarithmic embedding in number theory shows that it goes deep. Thus we will use the additive group of logarithms of algebraic numbers and the corresponding $\mathbb{Z}$-linear wedge product, with the elementary notation $(u_1+u_2)\wedge v = u_1\wedge v+u_2\wedge v$ instead of the customary $\exp(u_1)\exp(u_2) \wedge \exp(v) = \exp(u_1) \wedge \exp(v) + \exp(u_2) \wedge \exp(v)$. There are various versions of the K_3-group of F, with definitions of different depth and accessibility. In this article, depth is not aimed for. The main disadvantage is that any equivalence with more abstract and better generalisable definitions of K_3 is conjectural. In particular, the work of Garoufalidis and Zagier [4] uses a deeper definition. Thus their results go beyond what can be explained in this article. Nevertheless, it seems clear that there must be easily accessible and easily usable definitions of all relevant K_3 groups for number fields. We shall need two versions, here called K_3 and K_3^m.

Definition 2.2. The abelian group $K_3(F)$ consists of the finite formal sums $\sum_i a_i[u_i, v_i]$ with $a_i \in \mathbb{Z}$ and $(u_i, v_i) \in P(F)$ for which $\sum_i a_i u_i \wedge v_i = 0$, modulo sums for which $\sum_i a_i L(u_i, v_i)$ is an integral multiple of $(2\pi i)^2$.

Torsion elements of $K_3(F)$ are those for which $\sum_i a_i L(u_i, v_i)$ is a rational multiple of $(2\pi i)^2$. Sorry, I should have said $(2\pi i)^2/24$ instead of $(2\pi i)^2$ to stay close to the usual definition. The devil is in the details, but for the recognition of torsion elements it makes no difference. The reason for the denominator 24 will become clear in a moment.

Using logarithms helps to understand the relation to the Bloch-Wigner (-Lobachevsky) function $D(x)$, which is defined by

$$D(x) = \mathcal{I}m(L(u,v) - \bar{u}v/2).$$

Now $\mathcal{I}m(\bar{u}v/2)$ is $\mathbb{Z}$-linear and anti-symmetric. Thus it only depends on $u \wedge v$, so that $\sum_i u_i \wedge v_i = 0$ implies

$$\sum_i D(x_i) = \sum_i \mathcal{I}m(L(u_i, v_i)).$$

Obviously, $\sum_i L(u_i, v_i)$ contains more information than its imaginary part. On the other hand, the logarithmic definition yields some redundancy compared to the purely algebraic formulation, since one has to distinguish between $[u, v]$ and $[u + 2\pi i, v]$ or $[u, v + 2\pi i]$. Since one has $L(u + 2\pi i, v) - L(u, v) = \pi i v$ and $L(u, v + 2\pi i) - L(u, v) = -\pi i u$ this suggests the definition of a modified group K_3^m. Here $P(F)$ is replaced by F itself. To have a unified notation we represent elements x of F by pairs u, v as before, but with u and v considered as logarithms of algebraic numbers modulo $2\pi i$.

Definition 2.3. The abelian group $K_3^m(F)$ consists of equivalence classes of finite formal sums $\sum_i a_i[u_i, v_i]$ as for $K_3(F)$, but with u_i and v_i defined up to addition of integral multiples of $2\pi i$. The sums have to satisfy the condition that $\sum_i a_i u_i \wedge v_i = 2\pi i \wedge w$, where w is the logarithm of an element of F. Two such sums are equivalent, iff their difference is the image of a vanishing sum in $K_3(F)$. Torsion elements of K_3^m in the extended sense are those sums for which $\sum_i a_i L(u_i, v_i)$ is of the form $2\pi i w$, where w is the logarithm of an element of F.

By definition, one has an injection of $K_3(F)$ into $K_3^m(F)$. The definition of extended torsion expresses the hope that the concept will turn out to be sufficiently close to torsion in the common sense, but this question will not be investigated. The fundamental example of a vanishing element in $K_3(F)$ is $\sum_{i=1}^5 [u_i, v_i]$, where u_i and v_i have period 5 as functions of i and satisfy the five-term relation $u_i = v_{i-1} + v_{i+1}$. This yields $\sum_{i=1}^5 L(u_i, v_i) = 4\pi^2/8$ and $\sum_{i=1}^5 L(v_i, u_i) = 4\pi^2/12$. One may put $(u_0, v_0) = (0, -\infty)$ and $(u_1, v_1) = (u_{-1}, v_{-1}) = (-\infty, 0)$. This yields $(u_2, v_2) = (v_{-2}, u_{-2})$ and $L(u, v) + L(v, u) = 4\pi^2/24$ for $(u, v) \in P(F)$. In the commonly used algebraic notation this means that $[x] + [1-x] = 0$. On the other hand, the translation of the equally common relation $[x] + [1/x] = 0$ needs more care. The logarithms of $1/x$ and $1 - 1/x$ yield the pairs $(-u, v - u \pm \pi i)$. In $K_3(F)$ one has $2[u, v] + [-u, v - u + \pi i] + [-u, v - u - \pi i] = 0$, but $[u, v] + [-u, v - u + \pi i]$ alone is not even an element of $K_3(F)$. It is extended torsion in K_3^m, because $L(u,v)+L(-u,v-u+\pi i) = \pi i u/2 + \pi^2/3$.

When one uses vectors $u = (u_1,\dots,u_r)$, $v = (v_1,\dots,v_r)$ in $\mathbb{C}^r$ and a symmetric $r \times r$ matrix A with entries A_{ij} in $\mathbb{Q}$, then $v \wedge Av = 0$. Thus the equation $u = Av$ yields elements $\sum_{i=1}^r [u_i, v_i]$ in K_3. We also will have to consider equations $u - 2\pi i B = Av$ with $B \in \mathbb{Q}^r$. Solutions yield elements in K_3^m and one can ask if they are extended torsion. The following conventions concerning matrices and vectors will be used. They are integral, iff all their entries are integral. The notation of scalar products is suppressed. The components of x^A are $\prod_j x_j^{A_{ij}}$. A always will be symmetric, with entries in $\mathbb{Q}$ and rank denoted by r. A is even, iff $m = An$ with m, n both in $\mathbb{Z}^r$ implies that mn is even. For integral A this property agrees with the usual definition, but for invertible A it implies that A and A^{-1} are both even or both odd. As an aside, in conformal field theory the odd case implies the presence of fermions. Fractions are odd, if they can be written as p/q with p, q both odd. In this case A and pA/q are both even or both odd. When all entries of A have odd denominators and the diagonal entries have even numerators, then A is even. Thus there exist integers s_+, s_- such that $2^s A$ is even for integers s with $s \geq s_+$ or $s \leq s_-$.

It is more common to transform the equation $u = Av$ to the algebraic system $x = (1-x)^A$, as it was done by Meinardus in the case $r = 1$ [6]. All references to Meinardus in the present article refer to that paper, which contains several important ideas. They are developed for $r = 1$ only, but generalisation to arbitrary r is immediate in each case. Meinardus made the substitution $Q = 1 - x$ (with different letters) and wrote $1 - Q = Q^A$. This form was used at Les Houches, too, but will be used with some reluctance, since Q also will denote a quadratic form. When A is not integral, one has to worry about the choice of roots. Only when x and $1 - x$ are restricted to positive real values, rational powers are defined without ambiguity. The unique solution of $x = (1-x)^A$ with this property will be called the positive solution. The corresponding values of u_i, v_i are taken to be real, but they are or course negative. Since these values belong to the principal domain, $\sum_i L(u_i, v_i)$ takes values in $\mathbb{C}$, indeed in $\mathbb{R}$.

In general, the definition of Q^A needs choices of roots Q_i^{1/a_i}, $i \in \{1,\dots,r\}$, where a_i is the lowest common denominator of A_{ij}, $j \in \{1,\dots,r\}$. A solution of $1 - Q = Q^A$ must include such a choice. The equation $u = Av$ is more convenient than its exponential form, since it avoids any ambiguity. We shall say that a solution x of $x = (1-x)^A$ is torsion, iff the (rather a) corresponding sum $\sum_{i=1}^r [u_i, v_i]$ is a torsion element of K_3. Changing the logarithms $\log x_i$ and $\log(1 - x_i)$ by multiples of $2\pi i$ yields the equation $u - \lambda = Av$, where

$\lambda \in 2\pi i\Lambda$, $\Lambda = \mathbb{Z}^r + A\mathbb{Z}^r$. Changing the logarithms while maintaining $u = Av$ changes $L(u, v)$ by multiples of $2\pi^2$, but only by irrelevant multiples of $4\pi^2$ when A is even. For a given Q_i one can choose a unique logarithm v_i with $\mathcal{I}m(v_i) \in (-\pi, \pi]$. For integral A this yields a unique representative (Av, v) of any solution of $1 - Q = Q^A$ in the principal domain or on its boundary and a corresponding value of $\sum_i L(u_i, v_i)$ in $\mathbb{C}$. There should be a generalisation of this principal value for non-integral A, but this needs further thought.

A simple example for a 5-torsion element comes from the five-term relation when all pairs (u_i, v_i) are equal. This yields $u = 2v$, thus the logarithm of the golden ratio. Special solutions of the five-term relation arise, when one puts $(u_{-i}, v_{-i}) = (u_i, v_i)$. This yields $v_1 = u_0/2$, $v_2 = u_1 - v_0$, $u_2 = v_1 + v_2$, with arbitrarily chosen u_0. Then $[u_0, v_0] + 2([u_1, v_1] + [u_2, v_2]) = 0$ in K_3. In particular, if $[u_0, v_0]$ is torsion, then $[u_1, v_1] + [u_2, v_2]$ is torsion, too. Note that one has to include the squareroot of x_0 in the relevant number field for this construction to make sense. The argument also applies if one starts with several values, say u_0 and u_0', so that $[u_0, v_0] + [u_0', v_0']$ can be written as $-2([u_1, v_1] + [u_2, v_2] + [u_1', v_1'] + [u_2', v_2'])$. We will refer to this procedure as the doubling construction (or should it be halving?).

3. Examples and counterexamples for a conjecture

Consider an affine quadratic form $Q(n) = nAn/2 + Bn + C$, $n \in \mathbb{C}^r$, A a positive definite symmetric $r \times r$ matrix with entries in $\mathbb{Q}$, $B \in \mathbb{C}^r$, $C \in \mathbb{C}$. The notation of scalar products in terms like nAn and Bn will be suppressed. Though for $r = 1$ they have a long history, q-hypergeometric series

$$F_Q = \sum_{n \in \mathbb{N}^r} q^{Q(n)}/(q)_n$$

have been called Nahm sums. Here $(q)_n = \prod_{i=1}^r (q)_{n_i}$, with Pochhammer's notation. F_Q will be understood as a function defined on the upper complex half-plane with coordinate τ, using $q = \exp(2\pi i\tau)$ and $q^a = \exp(2\pi i a\tau)$. Sometimes we will write $F_{A,B,C}$ instead of F_Q. When F_Q is modular, the triple (A, B, C) also is called modular. I had conjectured that this happens for suitable B and C, iff all solutions of $u = Av$ yield torsion elements $\sum_i [u_i, v_i]$ in K_3. Indeed the latter condition appears to be sufficient. Moreover, Calegari, Garoufalidis and Zagier now have proven that the positive solution indeed must be torsion, in a deeper sense than what will be explained here [2]. A straightforward example has been given above, namely $r = 1$, $A = (2)$, where the positive solution of $x = (1 - x)^2$ is the golden ratio, which is 5-torsion. The only other solution of this equation is the Galois conjugate of the golden ratio, which of course is 5-torsion, too. The corresponding modular Nahm sums are the two Rogers-Ramanujan functions, for $B = 0$ and $B = 1$ respectively. For $r = 1$ Zagier proved that there are exactly seven modular Nahm sums, for which A must take one of the values 1, 2 or 1/2 ([11], p. 56). This agrees with the list of torsion elements satifsying $u = Av$. Thus for $r = 1$ the original rough conjecture is true. The invariance of the list under the inversion $A \mapsto A^{-1}$ is natural, since it just exchanges u and v, which yields an involution in the set of torsion elements. For $r = 2$, a search for candidate matrices A yielded three series and 22 individual cases, consisting of 11 inversion pairs. Among the latter. there are four pairs where all solutions of $u = Av$ are torsion, and these are exactly those for which modular F_Q exist. Two of the pairs come from the doubling construction described above. Let $1 - Q = Q^A$ with A of rank r. Doubling yields $X^2 = Q$ and the system $1 - X = X^{2A}X'$, $1 - X' = XX'$. Thus one finds a new matrix $D(A) = \begin{pmatrix} 2A & I \\ I & I \end{pmatrix}$ of rank $2r$, with completely analogous behaviour concerning torsion in K_3. Modular Nahm sums for A yield modular Nahm sums for $D(A)$, by rescaling of τ. The most remarkable of the seven remaining pairs has $A = \begin{pmatrix} 8 & 5 \\ 5 & 4 \end{pmatrix}$ and was worked out in some detail by Zagier. The equation $1 - Q = Q^A$ yields unit algebraic integers Q_1, Q_2 that are rational functions of each other and separately satisfy equations of order 8. For Q_1 the corresponding polynomial factorises as

$$(Q_1^4 + Q_1^3 + 3Q_1^2 - 3Q_1 - 1)(Q_1^4 - Q_1^3 + 3Q_1^2 - 3Q_1 + 1).$$

The first factor yields the positive solution and factors into quadratic polynomials, when $\mathbb{Q}$ is extended by the golden ratio. Thus it has abelian Galois group $\mathbb{Z}_4$ and a numerical check shows that the corresponding pair Q_1, Q_2 is torsion. The discriminant of the first factor is $-5^4 \cdot 19$, which yields class number 1. The Galois group of the second factor is generic, so one hardly expects that the roots yield torsion elements $[Q_1] + [Q_2]$. Indeed it is easy to check that the imaginary part of $L(u_1, v_1) + L(u_2, v_2)$ does not vanish. Nevertheless this second factor also has its peculiar properties. Its discriminant is 229 and $\mathbb{Q}(\sqrt{229})$ is the first example for primes with $p = 1$ modulo 4 for which the class number is different from 1, namely 3. Some connection between the factors seems to come from $229 = 12 \cdot 19 + 1$. More than for any other major branch of mathematics, learning number theory feels like the exploration of physical nature. Any botanist would be happy to discover an intricate little flower like Zagier's example. Perhaps it corresponds to a nice hyperbolic knot. The example shows that not only the behaviour of the positive solution is relevant.

So far the rough conjecture looked good, but even for $r = 2$ one still has to study the three candidate series. They are based on the three $r = 1$ cases $A = 1$ or 2 or $1/2$ by a copycat construction. Let u', v' be any solution of $u = Av$ for the rank 1 matrix (A). One wants to have a rank 2 matrix $C(A)$, so that $(u', u'), (v', v')$ is a solution of $u = C(A)v$. In particular, this implies that the positive solution for $C(A)$ arises by copying the positive solution for A and inherits its torsion property. The obvious choice is $C(A, \alpha) = \begin{pmatrix} A - \alpha & \alpha \\ \alpha & A - \alpha \end{pmatrix}$, with $\alpha < A/2$ in order to have a positive definite $C(A, \alpha)$. Indeed, Vlasenko and Zwegers showed that for suitable α the modular invariant Nahm sums for A lift to modular invariant sums for $C(A, \alpha)$ [10]. In general, the equation $u = C(A, \alpha)v$ will have accessory solutions that do not come from $u = Av$ by copying, and there is no particular reason why they should yield torsion elements. In detail, the picture has some complications, because the series have individual features. Still, those for $A = 2$ and $A = 1/2$ are related by inversion and behave analogously. Only one of them will be discussed.

For $C(1, \alpha)$, the rough conjecture turned out to be true again. Indeed, adding the two components in $u = C(1, \alpha)v$ yields $u_1 + u_2 = v_1 + v_2$, thus $(u_1, v_1) = (v_2, u_2)$, up to a multiple of $2\pi i$. The formal sum $[v_2, u_2] + [u_2, v_2]$ always is torsion, since $L(u_2, v_2) + L(v_2, u_2) = \pi^2/6$. The case $C(2, \alpha)$ is more interesting. The only case where all solutions of $u = C(2, \alpha)v$ are torsion is $\alpha = 2/3$, and indeed one finds corresponding modular functions F_Q. But Vlasenko and Zwegers observed that for $\alpha = 1/2$ and suitable B, C the corresponding F_Q are modular, too, namely the Rogers-Ramanujan functions with τ rescaled by 2. The positive solution is torsion, but the equation $u = C(2, 1/2)v$ has accessory solutions (given by sixth roots of unity) that do not yield torsion elements. For higher r Vlasenko and Zwegers found even more cases where F_Q is modular but some solutions of $u = Av$ are not torsion. They also used the doubling construction described above. In particular, $A = D(C(2, 1/2))$ is an integral $r = 4$ matrix for which the rough conjecture does not hold. All counterexamples to my original conjecture found by these authors come from copycat constructions. The basic solution agrees with the rough conjecture and provides the modular functions, but the construction introduces accessory solutions without the torsion property. In that sense the extended A may be called reducible. No counterexample with irreducible A appears to have turned up anywhere. It might be worthwhile to pursue this line of reasoning.

An interesting feature of the doubling construction is the fact that the doubling of an invertable matrix A sometimes yields a non-invertable one. In particular $D((1/2)) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. The equation $u = Av$ continues to make good sense, indeed the new system is essentially equivalent to the one given by $C(2,0)$. The case of non-invertable A has been treated by Garoufalidis and Zagier in the context of knot theory ([5], p. 48). One can generalise $u = Av$ to $Bu + Av = 0$, iff there are $r \times r$ matrices C, D so that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a symplectic matrix. For $B = -I$ and invertible A one can take $(C, D) = (0, A^{-1})$, for $B = -I$ and $A = D((1/2))$ one choice is $(C, D) = (I, 0)$. This possibility is important, but will not be considered further. In the following, A^{-1} will be used without comment.

Altogether examples and counterexamples show that my rough conjecture was somewhat superficial. To dig deeper one needs a more refined study of F_Q. One natural path is the study of F_Q when τ approaches 0. This is the one Zagier and my group followed about twenty years ago, but he already contemplated a study of F_Q close to other cusps. Recently, he and his collaborators studied an infinite number of points of $\mathbb{Q}$ simultaneously. Here we will calculate the behaviour of F_Q at all cusps.

4. Recursion formulas and vector bundles

In his Les Houches contribution, Zagier presented three ways to study F_Q close to $\tau = 0$. This paper will follow his example, but generalised to arbitrary $\tau \in \mathbb{Q}$. We want to find the modular triples A, B, C. The dependence of F_Q on C is just given by a global factor q^C. When A, B are given, there is at most one value of C for which $F_{A,B,C}$ can be modular. This is the value of a certain quadratic polynomial in B, with coefficients that are rational functions of A and x, x being the positive solution of $x = (1-x)^A$ ([11], p. 50). The dependence on B is rather subtle. For a given A there often are only a few isolated values of B that yield modular triples A, B, C, but sometimes there are families given by an affine linear function on $\mathbb{Q}$. The isolated values all appear to lie in $\Lambda/2$, where $\Lambda = \mathbb{Z}^r + A\mathbb{Z}^r$. The appearance of Λ is not surprising, because the translations of $Q(n)$ by integral shifts of n are equivalent to translations of B by elements of $A\mathbb{Z}^r$. This suggests that B should first be studied modulo shifts by vectors in Λ.

There is a corresponding functional equation used by Meinardus and by Zagier as his first approach to the calculation of F_Q. Here it will be interpreted first as a recursion formula and then as the definition of a vector bundle (up to unique isomorphism). For the recursion, both A and τ are fixed and one considers a system of equations

$$F_A(B) - F_A(B + e_i) = q^{A_{ii}/2 + B_i} F_A(B + Ae_i) \tag{1}$$

with $i = 1, \ldots, r$ and $e_1 = (1, 0, 0, \ldots)$, *etc.* For an individual i, we will refer to (1) as the i-th recursion formula. It is easy to check that the recursion formulas are satisfied by $F_{A,B,C}$, but in the following the system will be investigated all by itself. In parentheses, note that the semi-symplectic matrix pair (A, I) appears again, because one can write $F_A(B) = q^{A_{ii}/2+B_i} F_A(B+Ae_i)+F_A(B+Ie_i)$. Here the obvious generalisation will not be investigated.

For a few choices of A, the recursion has been well studied, in particular solutions with $F_A(+\infty) = 1$. For $A = 0$ and $A = 1$ the equation yields product formulas for $F_A(B)$ found already by Euler. For $A = 0$,

$$\sum_{n\in\mathbb{N}} x^n/(q)_n = (x; q)_\infty^{-1},$$

a formula that calculates the Fourier transform of $1/(q)_n$ and will be crucial in the next section. For $A = 2$ the recursion implies

$$F_2(B)/F_2(B + 1) = 1 + q^{B+1} F_2(B + 2)/F_2(B + 1),$$

which for $B = 0$ yields the continued fraction expansion of the ratio of the two Rogers-Ramanujan functions. At the start of his Les Houches article, Zagier refers to it as "surely among the most beautiful formulas in mathematics".

For $A = 1$ the recursion yields Euler's formula

$$F_1(B, \tau) = \prod_{n\in\mathbb{N}} \left(1 + q^{B+n+1/2}\right).$$

The Jacobi triple product allows to check which values of B yield modular functions $F_{1,B,C}$. Jacobi's formula implies that $F_1(B)F_1(-B)$ is modular up to a factor q^C. For $(F_{1,B,C})^2$ to be modular, one needs $F_1(B)$ and $F_1(-B)$ that are equal up to a multiplicative constant. This is true, iff $B \in \{0, 1/2, -1/2\}$. Indeed, $F_1(1/2)$ and $F_1(-1/2)$ differ just by a factor $1 + q^0$ equal to 2. In detail, the Jacobi triple product yields the modular triples $(1, 0, -1/48)$ and $(1, \pm 1/2, 1/24)$. The sign change from B to $-B$ can be understood as yielding the simplest Coxeter group, namely the Weyl group of $SU(2)$. There are analogues for Lie groups of higher rank, with applications in conformal quantum field theory [7]. For general A much more needs to be done, we only can make a start.

Proposition 4.1. *Let A be a rational and symmetric (not necessarily positive definite) $r \times r$ matrix and $\Lambda = \mathbb{Z}^r + A\mathbb{Z}^r$. Then there exists a finite set of vectors V_k in Λ, $k \in \{1, \ldots, \rho(A)\}$, so that for generic B, linear combinations of the recursion formulas (1) allow to calculate $F_A(B + V)$ for any $V \in \Lambda$ in terms of $F_A(B + V_k)$, $k \in \{1, \ldots, \rho(A)\}$.*

Proof: We show that for sufficiently large R the set of vectors in Λ of length less that R is sufficient. It will suffice to find a positive number d, so that if B_0 lies on the boundary of a ball of radius $R+d$ the value of $F_A(B_0)$ can be calculated as a linear combination of values $F_A(B_m)$ with B_m in the concentric ball of radius R. Let P be the tangent plane through B_0, oriented so that the ball lies to the left. In the i-th recursion formula one can substitute $B = B_0$ or $B = B_0 - e_i$ or $B = B_0 - Ae_i$, so that one of the three points $B, B+e_i, B+Ae_i$ is equal to B_0 and none lies to the right of P. If for some i among these three points only B_0 lies on P, then the i-th recursion formula allows to calculate $F_A(B_0)$ in terms of the remaining two values among $F_A(B)$, $F_A(B+e_i)$, $F_A(B+Ae_i)$, for which the argument of F_A lies to the left of P. Planes P for which such an i does not exist will be called special. The recursion is translationally invariant, so we can assume that $B_0 = 0$. Then P can be identified with a hyperplane in $\mathbb{R}^r$. For special hyperplanes P the set $\{1,\dots,r\}$ can be written as the disjoint union of two sets $E(P)$, $S(P)$, and $S(P)$ can be written as the disjoint union of three sets $S_a(P), S_b(P), S_c(P)$ as follows. We use angular brackets for the span of vectors. We put $i \in E(P)$, iff $\langle e_i, Ae_i\rangle \subseteq P$ and neither $Ae_i = 0$ nor $Ae_i = e_i$. For $i \in S(P)$ we put $i \in S_a(P)$, iff $\langle e_i, Ae_i\rangle \cap P = \langle Ae_i\rangle$, $i \in S_b(P)$, iff $\langle e_i, Ae_i\rangle \cap P = \langle Ae_i - e_i\rangle$ and $i \in S_c(P)$, iff neither of these two statements is true. Note that $i \in S_c(P)$ implies $\langle e_i, Ae_i\rangle \cap P = \langle e_i\rangle$. Let

$$\tilde{A}(P) = A - \sum_{i\in S_b(P)} e_i e_i^T .$$

For $i \in S(P)$ we use the notations $\beta_i = \tilde{A}e_i$ for $i \in S_a(P) \cup S_b(P)$ and $\beta_i = e_i$ for $i \in S_c(P)$. Let $n(P)$ be the normal of P. We have $n(P)\tilde{A}(P)e_i = 0$ for $i \in E(P) \cup S_a(P) \cup S_b(P)$. Because A is symmetric this implies $\tilde{A}n(P) \in \langle e_i \mid i \in S_c(P)\rangle$. Since $n(P) \in \langle e_i \mid i \in S_a(P) \cup S_b(P)\rangle$, this inclusion can be written in the form

$$\sum_{i\in S(P)} a_i\beta_i = 0.$$

The β_i are points of the lattice Λ, so that the coefficients a_i can be taken to be integral. Using a linear combination of $\sum_{i\in S(P)} |a_i|$ recursion formulas, the i-th one taken $|a_i|$ times with suitably shifted B, one obtains a multiple of $F_A(0)$ as a linear combination of $F_A(B_m)$, $m = 1,\dots,M$, where $M = \sum_{i\in S(P)} |a_i|$ and all B_m are $\mathbb{Z}$-linear combinations of the β_i and lie to the left of P. As an example, let $a_1 = 2$, $a_2 = 1$. Then on starts with $F_A(0)$ and $F_A(\beta_1)$, adds a linear combination of $F_A(\beta_1)$ and $F_A(2\beta_1)$ given by the first recursion formula to eliminate $F_A(\beta_1)$, then continues with $F_A(2\beta_1)$, $F_A(2\beta_1+\beta_2)$ and the second recursion formula, then with $F_A(2\beta_1 + \beta_2)$ and some $F_A(2\beta_1 + \beta_2 + \beta_i)$ and so on. Shifting all points by B, one obtains a formula expressing a multiple of $F_A(B)$ as a linear combination of $F_A(B + B_m)$, $m = 1, \ldots, M$. If $\tilde{A}e_i \neq 0$ for all i the coefficient of $F_A(B)$ has the form $1 - (-)^{N_b} q^{W\tilde{A}W/2+BW}$, where $N_b = \sum_{i \in S_b(P)} a_i$ and

$$W = \sum_{i \in S_a(P) \cup S_b(P)} a_i e_i.$$

Generically, that coefficient is invertible.

Let $\mathcal{B}(P) = \{B_m \mid m = 1, \ldots, M\}$. Let $U(\mathcal{B}(P))$ be the open set of hyperplanes P' so that all points in $\mathcal{B}(P)$ lie to the left of P'. A finite number of those sets, say $U(\mathcal{B}(P_k))$, $k \in K$, covers the compact space of all hyperplanes. Let

$$\delta(P) = max_{k \in K}\ min\{d(B, P_k) \mid B \in \mathcal{B}(P_k)\},$$

where $d(B, P)$ is the distance of B from P when B lies to the left of P and 0 otherwise. This defines a continuous positive function on the same compact space, with a positive minimum δ_0. For any d with $d < \delta_0$ and sufficiently large R the local difference between the surface of a ball of radius R through 0 and its tangent plane at 0 is small. Thus $d(B, P_k) \geq \delta_0$ with $B \in \mathcal{B}(P_k)$ implies that B lies in the interior of the ball.

By induction on the distance from the origin it follows that the set of all vectors $V_k \in \Lambda$ contained in a ball of radius $R + d$ around 0 satisfies the statement of the theorem. Indeed, for any V with $|V| > R + d$ consider the sphere of radius $|V|$ and the ball of radius $R + d$ on the inside of this sphere that has the same tangent plane at V. Then $F_A(V)$ can be calculated in terms of $F_A(V')$ with $|V'| < |V| - d$. By translational invariance of the recursion equation, the result remains true for $B + V, B + V_k$. ■

When one replaces B by AB in the recursion formula for A one obtains an equivalent formula for A^{-1}. This inversion duality has already been discussed extensively in the Les Houches lectures ([11]. p. 52; ([7], p. 114). For the appropriate value of C mentioned above it reads

$$(A, B, C) \mapsto \left(A^{-1}, A^{-1}B, \frac{r}{24} + \frac{1}{2}BA^{-1}B - C\right).$$

In particular, it yields

Lemma 4.2.

$$\rho(A^{-1}) = \rho(A).$$

Conjecturally, if one triple A, B, C is modular, the dual triple is modular, too. One aspect of this duality will be proved in the next section.

A lower bound for $\rho(A)$ can be established by finding a sufficiently large vector space of functions on Λ that satisfy the recursion for $q = 1$. An upper bound can be established as follows. Let λ_i, $i = 1, \ldots, r$ be a basis of Λ. If one can guess convenient vectors V_k, $k = 1, \ldots, \rho'(A)$ and show that the values of $F_A(B + V_k \pm \lambda_i)$ can all be calculated from those of $F_A(B + V_k)$ by the recursion, then it is obvious that $\rho'(A) \geq \rho(A)$. When A is simple enough, one can determine $\rho(A)$ by getting coinciding lower and upper bounds. But already for Zagier's example $A = \begin{pmatrix} 8 & 5 \\ 5 & 4 \end{pmatrix}$ such a proof might better be left to AI.

The following theorem is essentially a corollary to Proposition 4.1.

Theorem 4.3. *Let $\Lambda^{\mathbb{C}} = (2\pi i/\tau)\mathbb{Z})^r \oplus \Lambda$. A vector bundle $E(A)$ of rank $\rho(A)$ over the torus $\mathbb{C}^r/\Lambda^{\mathbb{C}}$ can be defined as follows. Let V_k with $k \in \{1, \ldots, \rho(A)\}$ satisfy the statement in Proposition 4.1. For B in a fundamental domain D_A of $\mathbb{C}^r/\Lambda^{\mathbb{C}}$ let the fibre of $E(A)$ over B be the direct sum of the fibres over $B+V_k$ of the trivial complex line bundle over $\mathbb{C}^r$. At the boundary of D_A let the transition functions for shifts by $2\pi i e_i/\tau$ be given by the unit of $GL(\rho(A), \mathbb{C})$ and those for shifts by vectors V in Λ by the linear transformation that expresses $F_A(B + V_k + V)$ in terms of $F_A(B + V_l)$, k, l in $\{1, \ldots, \rho(A)\}$. The vector bundle $E(A)$ is defined up to unique isomorphisms.*

Proof: Shifts of B by multiples of $2\pi i/\tau$ leave q^{Be_i} invariant for all i. Thus unit transition matrices for these shifts are compatible with those describing shifts by vectors V in Λ. The latter are compatible with each other, since for positive definite A they are simultaneously satisfied by $F_{A,B,0}$. The transition functions are rational functions of q^B, with coefficients in $\mathbb{Z}[q^{1+D}]$, D a common denominator of the matrix elements of $A/2$. Thus for a generic choice of D_A they are regular in a neighbourhood of the boundary of D_A. They are invertible since by analytic continuation one obtains the values of $F_A(B+V_k)$ for all k in terms of $F_A(B + V_k - V)$. The choice of a generic D_A can be avoided by using limits in the space of bundles over a torus. This space is a smooth orbifold [1], so one can take limits and transfer result from generic to arbitrary q with $|q| < 1$. Different choices of the V_k yield isomorphic bundles, with isomorphisms again given by the recursion relations. ■

As an example we first treat the case $r = 1$, $A > 0$. A fundamental domain of the recursion over $\mathbb{R}$ is the interval from 0 to A. For $A = m/n$ with $gcd(m, n) = 1$ one finds $\rho(A) = max\{m, n\}$. The transition matrix for $A = 1$ is already given by the recursion equation itself. For $A = m/n$ one can take $V_k = k/n$ with $k = 1, \ldots, m$ for $m > n$ and $k = 1, \ldots, n$ for $m < n$. For integral A with $A > 1$ the corresponding transition matrix for a shift by 1 is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 0 & 1 \\ q^{-A/2-B} & -q^{-A/2-B} & 0 & \ldots & 0 \end{pmatrix}.$$

For $A = 2$ one has $\rho(A) = 2$. The pair $F_{2,B,0}, F_{2,B+1,0}$ yields a trivial subbundle of $E(A)$. The quotient bundle is isomorphic to the determinant bundle and has Chern number -1. Thus there are no further global sections. In general, the dimension of the space of sections is finite, as for all holomorphic vector bundles on compact manifolds. Thus it certainly will be possible to rephrase the study of Nahm sums in the language of algebraic geometry and to use its powerful machinery. In particular, $F_{A,B,0}$ can be characterised as the global section all components of which approach 1 at infinity.

In the span of $F_{A,B,C}$ with A fixed, B in a fixed equivalence class modulo Λ, C arbitrary, the dimension of the subspace of modular functions is bounded by $\rho(A)$, since linear combinations with nontrivial powers of q as coefficients destroy modularity. The same modular function up to a multiplicative constant may appear for several values of B, however. For $r = 1$ we have seen one example, namely $F_{1,-1/2,1/24} = 2F_{1,1/2,1/24}$. The case $A = D((1))$ appears to inherit its special behaviour from (1), since $F_{A,(-1,1/2),0} = 2F_{A,(1,1/2),0}$. These equations are easily derived from the recursion system. The fact that $F_{A,(0,1/2),0}$ also can be made modular by multiplication with some q^C shows that $\rho(D((1))) \geq 2$ without further work. For $A = D((1))^{-1}$ inversion duality yields $F_{A,(-3/2,2),1} = 2F_{A,(1/2,0),0}$. Note the factor of q. The case $A = 2$ is of a different type. Here again $\rho(A) = 2$ and there are two modular functions for the equivalence class of $(0, 0)$. On the other hand, $F_{2,-1,0} = F_{2,1,0} + F_{2.0.0}$. Both summands on the right hand side become modular for a suitable choice of C. The two values differ, however, so that $F_{2,-1,0}$ cannot be made modular by multiplication with any q^C. The same argument applies to any other B in Λ different from 0,1.

Whenever $\rho(A) = 2$, projectivisation of the vector bundle transforms the recursion to an expansion by continued fraction, as for the ratio of the Rogers-Ramanujan functions considered above. For $A = D((1))$,

$$F_A(b, b) = \left(1 + q^{b+1/2}\right) F_A(b + 1, b + 1) + q^{b+1} F_A(b + 2, b + 2),$$

with the corresponding continued fraction expansion of $F_A(1,1)/F_A(0,0)$. For suitable values of C both $A,(1,1)$ and $A,(0,0)$ yields modular triples. From the point of view of physics, the Rogers-Ramanujan functions yield partition functions of the (2,5)-minimal model, whereas $A,(1,1)$ and $A,(0,0)$ for $A = D((1))$ yield partition functions of the (3,8)-minimal model. What happens for the (5,13)-minimal model is anyone's guess.

The vector bundle changes smoothly for τ in the upper or in the lower half plane. Over $\tau \in \mathbb{Q}$ it degenerates to a vector bundle over $\mathbb{R}^r/\Lambda$, with fibre $\mathbb{C}(\tau)^{\rho(A)}$. The work of Garoufalidis, Zagier and Scholze discussed in Zagier's recent Bonn/Trieste lectures certainly means that a better description should use the Habiro ring. Most of the content of these lectures is now available in print [5], but I have not fully understood it yet. Inversion duality implies that the bundle for A, q is isomorphic to the one for A^{-1}, q^{-1}. Thus one can glue the bundle space for A over the upper half plane to the complex conjugate of the one for A^{-1} over the lower half plane, very much in the spirit of "quantum modularity" [5].

For $\tau = 0$ the transition matrices for shifts by e_i do not depend on B. Their simultaneous diagonalization yields a split of $E(A)$ into line bundles. Those can be trivialised by sections that transform linearly under shifts of B, in other words, by exponential functions of B. Insertion of $F_{A,B} = Q^B$ in the recursion relation yields the familiar algebraic equations $1 - Q = Q^A$. I expect that the following statements will be easy to prove for mathematicians with the necessary background.

Conjecture 4.4. *$\rho(A)$ is the degree of the system $1 - Q = Q^A$ of algebraic equations.*

Conjecture 4.5. *When q is a primitive c-th root of unity, the recursion system (1) has a solution basis of the form $f_{A,B} = \phi(B)Q^{B/c}$. Here ϕ is a periodic function with period lattice $c\Lambda$ that takes values in a number field that is a solvable extension of the field of solutions of $1 - Q = Q^A$.*

Indeed, a formula for ϕ will be derived in the next section from the limiting behaviour of $F_{A,B}$ close to the cusps.

5. Inversion and the neighbourhood of cusps

By construction, q-hypergeometric series have a simple behaviour near the cusp at infinity. The intricate behaviour at other cusps, when q approaches a root of unity is studied in the context of quantum modularity. One possible perspective on modular Nahm sums is the question how ordinary modular behaviour arises out of the more general one for special values of the parameters. For this purpose one has to study the sums close to arbitrary cusps in $\mathbb{Q}$. For their evaluation one can use Poisson summation in two different, and it turns out dual ways. The direct approach was developed by Zagier and used to calculate the behaviour close to $\tau = 0$. In recent work with Garoufalidis it was extended to other cusps. This extension yielded a firm connection between modularity and K_3.

To apply Poisson summation directly to $q^{Q(n)}/(q)_n$ one needs a smooth interpolation for the denominator, including an extension to negative real values of n. Pochhammer's $(q)_n$ is a q-deformed factorial, defined by $(q)_0 = 1$ and $(q)_n = (1-q^n)(q)_{n-1}$ for $n \in \mathbb{N}$. The ordinary factorial $n!$ arises for q close to 1. An easy interpolation for $|q| < 1$ follows Gauss' definition of the Gamma function. On first defines $(x;q)_0 = 1$, $(x;q)_{n+1} = (1-xq^n)(x;q)_n$ for $n \in \mathbb{N}$. In the limit $n \to \infty$ this yields $(x;q)_\infty$ for arbitrary x. Then $1/(q)_n = (q^{n+1};q)_\infty/(q)_\infty$ is the wanted interpolation. Since the first aim of Garoufalidis and Zagier was the calculation of F_Q up to arbitrary polynomially suppressed corrections, only an integral over a neighbourhood of the maximum of $q^{Q(n)}/(q)_n$ had to be evaluated. Negative n was irrelevant at this stage, but might be important. When one continues $1/(q)_n$ to negative integers n, one finds 0, since the factor $(1-q^{n+1})$ vanishes for $n=-1$. The derivative with respect to n looks interesting, however. Up to a simple factor one finds $\sum q^{Q(-n)}(q^{-1})_{n-1}$. This sum belongs to the Habiro ring and might be called the Habiro shadow of the Nahm sum.

When one uses the modularity of $q^{1/24}(q)_\infty$ and the standard notation $\tilde{q} = \exp(-2\pi i/\tau)$, Zagier's approach yields

$$F_Q(\tau) = q^{r/24}\tilde{q}^{-r/24}(-i\tau)^{r/2}\int q^{Q(n)}(q^n q;q)_\infty dn(1+\mathcal{O}(|\tau|^N)). \tag{2}$$

for any $N \in \mathbb{N}$. Zagier then showed that for τ close to 0 one can use the Euler-Maclaurin formula for $\log(x;q)_\infty$. The leading term is $Li_2(x)/log\ q$. Thus the maximum of $q^{Q(n)}(q^n q;q)_\infty$ is given by the equation $1-Q = Q^A$ for $Q = q^n$. Integrating around the maximun yields an asymptotic expansion

$$\begin{aligned} \log F_Q(\tau) = \quad & L(u,v)/(-2\pi i\tau) + B\log Q - \tfrac{1}{2}\log(Q + A(1-Q)) \\ & + \textstyle\sum_{k=1}^{N-1} c_k(A,B,Q)\tau^k + \mathcal{O}(|\tau|^N). \end{aligned} \tag{3}$$

Here (u,v) is the positive solution of $u = Av$, $Q = e^v$ and the c_k are fairly complicated rational functions. For neighbourhoods of other rational values of τ this approach was developed in [4]. An asymptotic expansion is an element of $\mathbb{C}(\tau)$. Substitution of τ by $-\tau$ yields another element of $\mathbb{C}(\tau)$, though $F_Q(\tau)$

is not defined for values of τ in the lower complex half-plane. This fact will be used below.

The dual use of Poisson summation goes back to Meinardus and had been used by my group. The approach may be motivated by the well-known fact that the Fourier transform of a product is the convolution of the Fourier transform of the factors. Moreover, we have seen that the Fourier transform of $1/(q)_n$ is $(x;q)_\infty^{-1}$. For $r=1$ this yields Meinardus' formula

$$F_Q(\tau) = \oint \Theta_Q(\tau,z)(x;q)_\infty^{-1}\, dx/2\pi i x, \tag{4}$$

where we put $x = \exp(2\pi i z)$,

$$\Theta_Q(\tau,z) = \sum_{n\in\mathbb{Z}} q^{Q(n)} x^{-n},$$

and where the integral is taken along a small circle around 0. As usual, estimation of F_Q can be done by expanding the integration circle to let it pass through a point where the integrand is stationary.

Now one can evaluate $\Theta_Q(\tau,z)$ by Poisson summation. The resulting sum can be understood as yielding an integral over the universal cover of the original integration circle. These two steps taken by Meinardus can be replaced by direct application of

$$\sum_{n\in\mathbb{N}} f_n g_n = \int \tilde{f}(-z) g(z) dz. \tag{5}$$

Here $g(z) = \sum_n g_n \exp(2\pi i n z)$ and $\tilde{f}$ denotes the Fourier transform, $\tilde{f}(z) = \int f(n)\exp(2\pi i n z) dn$, where $f(n)$ is a smooth interpolation of f_n. In our case, $f(n) = q^{Q(n)}$, $g_n = 1/(q)_n$. We first consider the case $Q(n) = nAn/2$. This yields

$$F_Q(\tau) = (-i\tau)^{-r/2} det(A)^{-1/2} \int \exp\left(-\pi i z A^{-1} z/\tau\right)(x;q)_\infty^{-1} dz, \tag{6}$$

where the integral is taken along some parallel to the real z-plane. When one makes the substitution $x = q^n$, one sees that the latter formula is close to (2), but with A replaced by A^{-1} and n substituted by n/τ and a corresponding integration over an imaginary plane. This yields the following theorem.

Theorem 5.1. *As asymptotic expansions around $\tau = 0$,*

$$(\log F_{A^{-1},0,0})(\tau) = \log\left(det(A)^{1/2} q^{r/24}\tilde{q}^{-r/24}\right) + (\log F_{A.0.0})(-\tau).$$

Proof: This follows from three simple facts. First, the Poisson and dual Poisson evaluations of F_Q must coincide. Second, let $f(x)$ be a real analytic function with maximum at 0. Then for t positive and close to 0 the expansions for $\int \exp(-f(x)/t)dx$, with an integral along the real axis and $\int \exp(f(x)/t)dx$, with an integral along the imaginary axis, transform into each other by a sign change of t. Third, denote by $EM(f,\tau)$ the boundary sum in the Euler-Maclaurin expansion for $\sum_{n\in\mathbb{N}} f(n,\tau)$. Then

$$EM(\log(xq;q)_\infty,\tau) + EM(\log(x;q))_\infty, -\tau) = 0.$$

This follows most easily from $(xq;q)_n(x;q^{-1})_m = (xq^n;q^{-1})_{m+n}$. ■

An inversion property of this kind was conjectured by Zagier ([11], p. 52). It is just what one expects when $F_{A,0,0}$ and $F_{A^{-1},0,0}$ both can be made modular by multiplication with some q^C, since then $c_k(A,0,e^v)$ and $c_k(A^{-1},0,e^u)$ must both vanish for $k>1$. Note, however, that the result cannot be generalised to the case when B and its dual $A^{-1}B$ are different from zero. since $q^{nA^{-1}n/2+BA^{-1}n}$ cannot be related to $q^{nAn/2+Bn}$ by a Fourier transformation. Instead, one obtains $q^{nAn/2}$ multiplied by a character $\exp(2\pi iBn)$. Obviously, Nahm sums should be generalised to include such characters. More precisely, $\Theta_Q(\tau,z)$ should be treated as one component of a vector valued Jacobi form in the sense of Eichler and Zagier [3]. The fact that z takes values in $\mathbb{C}^r$ instead of $\mathbb{C}$ does not make much of a difference. The weight of Θ_Q is $r/2$, the index $A^{-1}/2$. It is natural to extend my conjecture to convolutions of other Jacobi forms with analogues of $(x;q)_\infty^{-1}$. Indeed, some analogues of the latter function have appeared in knot theory, but no systematic investigation has been done yet. The vector space spanned by the components of Θ will be called Jacobi space. For $A=2$ it is spanned by $\sum q^{n^2}x^n$ and $\sum q^{(n+1/2)^2}x^{n+1/2}$. No character needs to be considered. For $A=1$ the Jacobi space is spanned by the three basic odd theta series introduced by Jacobi himself. One of them includes a character $(-)^n$. One way to treat Nahm sums with characters arises from a remark concerning the perturbation series (3) in ([11], p. 51). Formally, one gets $\exp(2\pi iBn)$ from q^{Bn} when one replaces $B\tau$ by B. This replacement can be done in (3), because the expansion in terms of B,τ,Q can be rearranged to yield an asymptotic expansion in terms of $B\tau,\tau$ and a modified Q. For the latter, Zagier obtained the x-deformed equation $1-Q=e^xQ^A$, where $e^x=q^B$. This topic needs a separate investigation, but is mentioned here, because just such a modification of $1-Q=Q^A$ will come up soon.

Procedures analogous to the direct and inversion dual Poisson summations also can be used to evaluate the behaviour of F_Q close to other cusps a/c in $\mathbb{Q}$. For the direct approach, see [4]. Here the dual one will be presented. The most straightforward procedure would be to perform modular transformations $\tau \mapsto (a\tau + b)/(c\tau + d)$ for $\tilde{f}$ and g in eq. (5) and to evaluate the result close to $\tau = i\infty$. For $\tilde{f}$ this yields a sum over the components of the Jacobi space. To facilitate comparison with [4] this will be done somewhat implicitly, by direct evaluation of F_Q close to a/c. We use the notation $\exp(2\pi i(a/c + \tau)) = \zeta q$, where $\zeta = \exp(2\pi i a/c)$ is a root of unity. Now $\zeta^{Q(n)}$ is periodic, so that we have to consider sums $\sum p_n f_n g_n$ with periodic p. Let M be the sublattice of $\mathbb{Z}^r$ consisting of the elements m satisfying $p_{m+n} = p_n$ for $m \in M$, and M^{v} its dual. In analogy to (5) one obtains

$$\sum_{n\in\mathbb{Z}^r} p_n f_n g_n = |\mathbb{Z}/M|^{-1} \sum_{\nu\in M^{\mathrm{v}}/\mathbb{Z}^r} P(\nu) \int \tilde{f}(\nu - z) g(z) dz, \tag{7}$$

where $P(v) = \sum_{n\in\mathbb{Z}^r/M} p_n \exp(-2\pi i n\nu)$. We need $p_n = \zeta^{Q(n)}$. The sublattice of $\mathbb{Z}^r$ given by $\zeta^{Q(n+m)} = \zeta^{Q(n)}$ will be called M_Q^ζ, or just M, when Q and ζ are obvious from the context. With $x = \exp(2\pi i z)$ and $f(n) = q^{Q(n)}$ as before,

$$F_Q\left(\frac{a}{c} + \tau\right) = |\mathbb{Z}/M|^{-1} \sum_{\nu\in M^{\mathrm{v}}/\mathbb{Z}^r} G(Q, \zeta, \nu) \int \tilde{f}(\nu - z)(x; \zeta q)_\infty^{-1} dz,$$

where G is a Gauss sum

$$G(Q, \zeta, \nu) = \sum_{n\in\mathbb{Z}^r/M} \zeta^{Q(n)} \exp(-2\pi i n\nu).$$

The Fourier transform of f is standard. It remains to evaluate $(x; \zeta q)_\infty^{-1}$ for small τ. A result in terms of an asymptotic series was given in [2], but we need an integral representation with manageable kernel. For this purpose Jacobi's triple product identity can be written in the form

$$q^{1/12}(x; q)_\infty (x^{-1}q; q)_\infty = (-x)^{1/2}\theta_1(\tau, z)/\eta(\tau).$$

Since the right hand side has simple Jacobi transformations, it is easy to evaluate close to any cusp. The factorisation on the left hand side can be characterised by the fact that $(x; q)_\infty$ is holomorphic, non-vanishing for $|x| < 1$ and equal to 1 for $x = 0$, whereas $(x^{-1}q; q)_\infty$ is holomorphic and non-vanishing for $|x| > |q|$. In the range $(|q|, 1)$ neither $(x; q)_\infty$ nor $(x; q)_\infty$ has winding, so that the logarithm of $x^{1/2}\theta_1(\tau, z)$ is defined. As in the construction of the Laurent expansion, this logarithm can be split by an application of Cauchy's theorem into one part that extends to $x = 0$ and another one that extends to $x^{-1} = 0$. Thus for $|x| < r$, $r \in (|q|, 1)$ the logarithm of $(x;q)_\infty$ can be calculated by a Cauchy integral over a circle with radius r. We shall do it in the limit $r \to 1$. Thus

$$\log\left((x;q)_\infty)\right) = \oint \log\left(e^{\pi i w}\theta_1(\tau, w)\right)\omega,$$

where

$$\omega = \frac{dw}{\exp(2\pi i(w-z)) - 1}.$$

With a slight abuse of notation, we regard the circle integral as a w-integral over $\mathbb{R}/\mathbb{Z}$, for definiteness over $(0,1)$. To avoid an overabundance of $2\pi i$ in the following calculations, we use Zagier's notation $\mathbf{e}(z) = e^{2\pi i z}$. When we evaluate $\log\theta_1(\tau, w)$, additive constants can be neglected, since $\oint \omega = 0$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. The modular transformation $\tau' \mapsto (a\tau' + b)/(c\tau' + d)$ maps $-1/(c^2\tau) - d/c$ to $a/c + \tau$. Thus up to a multiplicative constant

$$\theta_1\left(\tau + \frac{a}{c}, w\right) \sim \mathbf{e}\left(-\frac{w^2}{2\tau}\right)\theta_1\left(-\frac{1}{c^2\tau} - \frac{d}{c}, -\frac{w}{c\tau}\right).$$

Moreover

$$\theta_1\left(-\frac{1}{c^2\tau} - \frac{d}{c}, -\frac{w}{c\tau}\right) = \mathbf{e}\left(\frac{w}{2c\tau}\right)\Psi_+(w)\Psi_-(w),$$

with

$$\Psi_+(w) = \prod_{n=0}^{\infty}\left(1 - \mathbf{e}\left(-\frac{w}{c\tau} - \frac{dn}{c}\right)\tilde{q}^{n/c^2}\right)$$

and

$$\Psi_-(w) = \prod_{n=-\infty}^{c-1}\left(1 - \mathbf{e}\left(\frac{w-1}{c\tau} + \frac{dn}{c}\right)\tilde{q}^{-n/c^2}\right).$$

For τ close to 0 in the upper half-plane and $w \in (0,1)$ all factors of Ψ_+ and almost all factors of Ψ_- are close to 1. The exceptional factors of Ψ_- are those for which $c - n < cw$. With $n' = c - n$ we have in this case

$$\begin{aligned}
&\log\left(1 - \mathbf{e}\left(\tfrac{w-1}{c\tau} + \tfrac{dn}{c}\right)\tilde{q}^{-n/c^2}\right) = 2\pi i\tfrac{w-n'/c}{c\tau} \\
&+2\pi i\left(\tfrac{1}{2} - \tfrac{dn'}{c} + \left[\tfrac{dn'}{c}\right]\right) + \log\left(1 - \mathbf{e}\left(-\tfrac{w-1}{c\tau} - \tfrac{dn}{c}\right)\tilde{q}^{n/c^2}\right).
\end{aligned}$$

The integral part $[dn'/c]$ occurs, because the phase of θ_1 can only change in the range $(-\pi, \pi)$, when w moves past a pole. Summing up these contributions yields up to an additive constant

$$\frac{1}{2\pi i}\log\theta_1\left(\tau, w+\frac{a}{c}\right) = -\frac{\left(cw-[cw]-\frac{1}{2}\right)^2}{2c^2\tau} + \mathcal{D}(w) + \frac{1}{2\pi i}\log\Psi(w),$$

where

$$\mathcal{D}(w) = \sum_{n=1}^{[cw]}\left(\frac{1}{2}-\frac{dn}{c}+\left[\frac{dn}{c}\right]\right),$$

$\Psi(w) = \hat{\Pi}_+(w)\hat{\Pi}_-(w)$,

$$\hat{\Pi}_+(w) = \prod_{n=-[cw]}^{\infty}\left(1-\mathbf{e}\left(-\frac{w}{c\tau}-\frac{dn}{c}\right)\tilde{q}^{n/c^2}\right)$$

and

$$\hat{\Pi}_-(w) = \prod_{n=-\infty}^{c-[cw]-1}\left(1-\mathbf{e}\left(\frac{w-1}{c\tau}+\frac{dn}{c}\right)\tilde{q}^{-n/c^2}\right).$$

We want to evaluate $(x;\zeta q)_\infty$ with $\zeta = \mathbf{e}(a/c)$, using

$$\log(x;\zeta q)_\infty = \oint \log\left(\mathbf{e}\left(\frac{w}{2}\right)\theta_1\left(\tau, w+\frac{a}{c}\right)\right)\omega.$$

When a summand in the integrand has an explicit splitting into functions of x and x^{-1} that are holomorphic within their unit circles, its integral is given by Cauchy's theorem. First, $x^{1/2} = \sqrt{1-x}\sqrt{x^{-1}-1}$. Second,

$$2\pi^2\left(cz-[cz]-\frac{1}{2c}\right)^2 = Li_2(x^c)+Li_2(x^{-c})+\frac{\pi^2}{6}.$$

Third, $\mathcal{D}(w)$ is a piecewise constant function. When w moves past ka/c, $k\in\{1,\dots,c-1\}$, it increases by $1/2-k/c$. Thus, up to an irrelevant additive constant,

$$2\pi i\mathcal{D}(z) = -\log\left(\frac{\sqrt{1-x^c}}{\sqrt{1-x}}D_{\bar{\zeta}}(x)^{-1/c}\right)+\log\left(\frac{\sqrt{1-x^{-c}}}{\sqrt{1-x^{-1}}}D_\zeta(x^{-1})^{-1/c}\right).$$

Here D_ζ is the cyclic quantum dilogarithm

$$D_\zeta(x) = \prod_{k=1}^{c-1}(1-\zeta^k x)^k,$$

which provides a crucial link to algebraic K-theory, as pointed out in [4]. Note that $D_\zeta(x)^{1/c} D_{\bar\zeta}(x)^{1/c} = (1-x^c)/(1-x)$. Finally, we have

$$\log \hat\Pi_+(w) = \sum_{n=0}^{c-1} \int_{-n/c}^{\infty} \log\left(1 - \mathbf{e}\left(-\frac{w+n/c}{c\tau} - \frac{dn}{c}\right)\right)\omega,$$

$$\log \hat\Pi_-(w) = \sum_{n=0}^{c-1} \int_{\infty}^{n/c} \log\left(1 - \mathbf{e}\left(\frac{w-n/c}{c\tau} - \frac{dn}{c}\right)\right)\omega,$$

Altogether one finds

$$(x;\zeta q)_\infty = \sqrt{1-x^c} D_\zeta(x)^{-1/c} \exp\left(\frac{Li_2(x^c)}{2\pi i c^2 \tau}\right)\Psi(x),$$

where

$$\log \Psi(x) = \\ \textstyle\sum_{n=0}^{c-1} \int_0^\infty \log\left(1 - e^{-2\pi i(\frac{nd}{c}+\frac{w}{c\tau})}\right)\left(\frac{dw}{e^{2\pi i(w-z-n/c)}-1} + \frac{dw}{e^{-2\pi i(w+z-n/c)}-1}\right).$$

For $c = 1$ the result simplifies, since $D_\zeta(x) = 1$ and the sum is restricted to $n = 0$.

We now have

$$F_Q\left(\frac{a}{c}+\tau\right) = |\mathbb{Z}/M|^{-1} \sum_{\nu \in M^{\mathrm{v}}/\mathbb{Z}^r} G(Q,\zeta,\nu) F_Q^\zeta(\nu,\tau),$$

where

$$F_Q^\zeta(\nu,\tau) = \int \tilde f(\nu - z) exp\left(-\frac{Li_2(x^c)}{2\pi i c^2\tau}\right)(1-x^c)^{-1/2} D_\zeta(x)^{1/c}\Psi(x)^{-1} dz.$$

We consider Nahm sums with trivial character, but other elements of the Jacobi space can be treated in the same way. For $Q(n) = nAn/2 + Bn$,

$$\tilde f(z) = (-i\tau)^{-r/2} det(A)^{-1/2} \mathbf{e}\left(-\frac{zA^{-1}z}{2\tau} - BA^{-1}z - \frac{BA^{-1}B\tau}{2}\right).$$

When F_Q is modular, then for small τ the functions $F_Q(a/c+\tau)$ will all be proportional to $\tilde q^h$ with some rational h. Here h can be calculated by deforming the integration domain, following Meinardus' example of $r = 1$ and $\zeta = 1$. For $\zeta = 1$ one has $M^{\mathrm{v}} = \mathbb{Z}^r$, so that one only has to consider the value $\nu = 0$. The leading term of F_Q comes from a stationary point of $\frac{1}{2}uA^{-1}u + Li_2(e^u)$, where $u = 2\pi icz$. The value of this function at the stationary point must be a rational multiple of $(2\pi i)^2$, otherwise F_Q cannot be modular. Because $d(\frac{1}{2}uA^{-1}u + Li_2(e^u)) = 0$ yields $u = Av$, this value is just $L(u,v)$. That was the motivation for my conjecture concerning the relation between modularity and torsion in K_3. At general ζ, we have represented F_Q as a linear combination of certain functions $F_Q^\zeta(\nu)$, $\nu \in M^{\mathrm{v}}$, namely those for which the Gauss sum $G(Q,\zeta,\nu)$ does not vanish. Natural basis elements of the Jacobi space correspond to linear combinations of subsets of these functions that have the same behaviour for small τ.

The leading contribution to $\log F_Q^\zeta(\nu,\tau)$ should come from some stationary point of $\frac{1}{2}(u - 2\pi ic\nu)A^{-1}(u - 2\pi ic\nu) + Li_2(e^u)$, where $u = 2\pi icz$. This yields $u - 2\pi ic\nu = Av$ or

$$1 - Q = \mathbf{e}(c\nu)Q^A.$$

At the stationary point we have

$$\frac{1}{2}(u - 2\pi ic\nu)A^{-1}(u - 2\pi ic\nu) + Li_2(e^u) = L(u,v) - \pi ic\nu v.$$

Let u, v be the stationary point that yields the dominant contribution. Then the integral over z yields

$$F_Q^\zeta(\nu,\tau) = \exp\left(\frac{-2\pi i}{c^2\tau}h\right)\Phi(\nu,\tau)$$

with

$$h = \frac{L(u,v) - \pi ic\nu v}{(2\pi i)^2}$$

and

$$\Phi(\nu,\tau) = \frac{Q^{B/c}}{det(A + Q - AQ)^{1/2}} D_\zeta\left(\mathbf{e}(\nu)Q^{A/c}\right)^{1/c}(1 + \mathcal{O}(|\tau|).$$

For modular F_Q, h must be rational, so that the dominant solution of $1-Q = \mathbf{e}(c\nu)Q^A$ should be extended torsion. For $\nu = 0$ one recovers the torsion property of the positive solution and its corresponding value of h. When $c\nu \in \Lambda$ one obtains the same value, since the equation $1 - Q = \mathbf{e}(c\nu)Q^A$ differs from $1 - Q = Q^A$ at most by a different choice of roots for non-integral A. In general, ν and ν' yield the same h, when $c\nu - c\nu' \in \Lambda$. Let

$M(B) = cM^{\mathrm{v}}/(cM^{\mathrm{v}} \cap \Lambda)$. The value of h corresponding to a class β in $M(B)$ will be called $h(\beta)$. Let $N(\beta)$ be the inverse image of the map $M^{\mathrm{v}}/\mathbb{Z}^r \to M(B)$ given by $\nu \mapsto c\nu$ and

$$\mathbf{F}_Q^\zeta(\beta, \tau) = |\mathbb{Z}/M|^{-1} \sum_{\nu \in N(\beta)} G(Q, \zeta, \nu) F_Q^\zeta(\nu, \tau),$$

so that

$$F_Q\left(\frac{a}{c} + \tau\right) = \sum_{\beta \in M(B)} \mathbf{F}_Q^\zeta(\beta, \tau).$$

We have

$$\mathbf{F}_Q^\zeta(\beta, \tau) = \exp\left(-\frac{\hat{L}(\beta)}{2\pi i c^2 \tau}\right) \mathbf{\Phi}(\beta)(1 + \mathcal{O}(|\tau|),$$

where

$$\mathbf{\Phi}(\beta) = |\mathbb{Z}/M|^{-1} \frac{Q^{B/c}}{det(A + Q - AQ)^{1/2}} \sum_{\nu \in N(\beta)} G(Q, \zeta, \nu) D_\zeta \left(\mathbf{e}(\nu) Q^{A/c}\right)^{1/c}.$$

The splitting of F_Q into a sum over terms with varying dominant exponents corresponds to the splitting of $E(A)$ into line bundles at ζ. The solutions of the recursion equations at ζ should be given by

$$\phi(B) = \sum_{\nu \in N(\beta)} G(Q, \zeta, \nu) D_\zeta \left(\mathbf{e}(\nu) Q^{A/c}\right)^{1/c}.$$

This is the function alluded to in Conjecture 4.5.

The calculation of $M(B, a/c)$ is standard. The lattice M^{v} can be read off from the defining property $\zeta^{Q(n+m)} = \zeta^{Q(n)}$ for $m \in M$, valid for $m, n \in \mathbb{Z}^r$. This yields the two conditions

$$\frac{a}{c} Am \in \mathbb{Z},$$

$$\frac{a}{c}\left(Bm + \frac{1}{2} mAm\right) \in \mathbb{Z}.$$

When $\frac{a}{c}A$ is even, this yields $M(B) = \langle B \rangle$. In any case the map $\left(\mathbb{Z}^r \cap \frac{c}{a}A^{-1}\mathbb{Z}\right) / 2\left(\mathbb{Z}^r \cap \frac{c}{a}A^{-1}\mathbb{Z}\right) \to \mathbb{Z}/2\mathbb{Z}$ given by $m \mapsto \frac{a}{c} mAm$ is linear and can be represented by $m \mapsto m\mu(a/c)$ with $\mu(a/c) \in \mathbb{Z}^r + \frac{a}{c}A\mathbb{Z}^r$. Thus

$$cM^{\mathrm{v}}(a/c) = c\mathbb{Z}^r + aA\mathbb{Z} + \left\langle aB + \frac{1}{2} c\mu \right\rangle.$$

When $\frac{a}{c}A$ is even, we can take $\mu(a/c) = 0$. Let $2^s A$ be even for $s \geq s_+$ or $s \leq s_-$. When p, q are odd, then $q\mu(2^s p/q) = \mu(2^s)$ modulo $2\mathbb{Z}^r + 2^{s+1}\frac{p}{q}A\mathbb{Z}^r$. Thus $\bigcup_{a/c\in\mathbb{Q}} cM^{\mathrm{v}}(\frac{a}{c}) = \mathcal{M}(B)$,

$$\mathcal{M}(B) = \bigcup_{s\in[s_-,0]} \langle B + 2^{-s-1}\mu(2^s)\rangle \cup \bigcup_{s\in[0,s_+]} \langle 2^s B + \mu(2^s)/2\rangle$$

modulo Λ. For any rational B this is a finite set, which means that only a finite set of equations $1-Q = \mathbf{e}(\hat{B})Q^A$, $\hat{B} \in \mathcal{M}(B)$ has to be investigated. For isolated values of B all known examples have $B \in \Lambda/2$, so that $2\mathcal{M}(B) = 0$.

When one investigates a given vector valued Jacobi form, the relevant values of B are known and one can put $\mathcal{M} = \bigcup_B \mathcal{M}(B)$. The values $h(\beta)$ with $\beta \in \mathcal{M}$ can be identified with the constants C in the modular triples (A, B, C) coming from the space. The simplest example appears already at $r = 1$, but it is somewhat degenerate. For $A = 1$ the set $\mathcal{M}$ has two elements, represented by $B = 0$ and $B = 1/2$. For $B = 0$ stationarity means $u = v$. This yields $Q = 1/2$ and $L(u, v)/(2\pi i)^2 = -1/48$, thus the modular function $F_{1,0,-1/48}$ considered above. For $B = 1/2$ one has $u - \pi i = v$. This yields $1 - Q = -Q$, thus a stationary point at the boundary of the van der Pauw domain, with $v = +\infty$. Since $(u - \pi i)^2/2 + Li_2(e^u) = -Li_2(e^{-u}) - \pi^2/6$ and $Li_2(0) = 0$, the corresponding value of $(2\pi i)^2 h(\beta)$ is $-\pi^2/6$. It corresponds to the modular function $F_{1,1/2,1/24}$, also considered above.

Starting from the list of modular triples with $r = 2$ given in [11], it is easy to work out further examples. In particular, it has been mentioned that there are cases where A admits an infinity of modular triples (A, B, C). These cases also fit in. The basic example is $A = D(1, \alpha)$. Here the elements of $\mathcal{M}$ have the form $(b, -b)$. Thus one has to consider the system

$$\begin{aligned} 1 - Q_1 &= \mathbf{e}(b)Q_1^{1-\alpha}Q_2^{\alpha}, \\ 1 - Q_2 &= \mathbf{e}(-b)Q_1^{\alpha}Q_2^{1-\alpha}. \end{aligned}$$

Multiplication immediately yields $Q_1 + Q_2 = 1$, so that the solution is torsion for all b.

6. Outlook

The present status can be summarised as follows. The set of matrices A admitting modular triples (A, B, C) is somewhat better understood than twenty years ago. At that time some of the numbers C were understood in terms of the Rogers dilogarithm, namely as values of $L(u, v)/(2\pi i)^2$ with $u = Av$. Now all of these numbers can be interpreted as values of $(L(u,v) - \pi i\hat{B}v)/(2\pi i)^2$, where $u - 2\pi i\hat{B} = Av$. This means that one has to consider more general values of u, v than the torsion elements considered at the time. The concept of extended torsion was introduced to cover this situation. Examples indicate that one still remains within the conventional realm of torsion in K_3, but this needs further study.

Apparently, not all solutions of $u - 2\pi i\hat{B} = Av$ contribute to $F_{A,B,C}$. The relevant solutions should be those that are picked up by the deformation of the integration domain in some integral representation of this function. The result of the deformation should be a representation of F_Q as an inverse Laplace transform, say

$$F_Q(\tau) = (-i\tau)^{-r/2} \sum_{\beta\in\mathcal{M}} \sum_{n\in\mathbb{N}} \int_{h(\beta)}^{\infty} ds f_Q^{\gamma+n}(s) \exp(-2\pi i(s+n)/\tau),$$

with holomorphic functions $f_Q^{\gamma+n}$, modulo details. The integral representations for F_Q derived above involve integrals over functions of the form $\exp(-ik(z)/\tau$. This is close to an inverse Laplace transform, but in general $k(z)$ is complex. The integration domain should be deformed in such a way that $k(z)$ becomes real. At least for $r = 1$ this can be done, though the deformed integration path has singularities. In particular, one finds different branches of the integration path that intersect at points with $v = -\infty$. In the van der Pauw domain this looks like the upper half of the letter X. The apparent singularity can be removed when the path is written as the sum of the two crossing branches of the full X, minus its lower half. The path corresponding to this lower half can be pushed down to a lower smooth path and so on. The procedure will yields stationary points away from the principal part of the van der Pauw domain. As has been explained above, this will yield stationary values of $L(u,v)/(2\pi i)^2$ that differ by integers or half-integers from the principal one. Their constributions should explain the sum over $\mathbb{N}$ in the inverse Laplace transform and eventually the coefficients in F_Q.

This approach may be unnecessarily difficult, however. Study of the vector bundles on the Jacobi torus might yield the result that any F_Q that looks modular at the cusps is indeed modular. Equivalently, any linear combination of Nahm sums that vanishes at all cusps must be zero. Already now, the study of the behaviour of modular F_Q at the cusps yields astonishing results. Due to modularity, their behaviour is simple and periodic. On the other hand, the explicit calculations presented above yield $\mathbf{\Phi}$, a linear combinations of roots of cyclic quantum dilogarithms with Gauss sums as coefficients. The simple values of $\mathbf{\Phi}$ for modular F_Q look rather mysterious. Already for $r = 1$, $A = 1$

and $\zeta = \mathbf{e}(2/3)$ one finds a surprising, though easily checkable result. One finds

$$(8) \qquad \frac{1+2\omega}{3} D_\zeta(x)^{1/3} + \frac{1-\omega}{3}\left(D_\zeta(\omega x)^{1/3} + D_\zeta(\omega^2 x)^{1/3}\right) = e^{2\pi i/9},$$

where $\omega = e^{2\pi i/3}$, $\zeta = \omega^2$ and $x = (1/2)^{1/3}$.

As explained in [7], the physical context of Nahm sums is conformal quantum field theory (CFT). For readers that only need a first impression, it is sufficient to recall the importance of the Jacobi triple product in what has been said above. For physicists, Jacobi's identity follows from the fact that the quantum field theory of a free complex fermion in two spacetime dimensions is isomorphic to the one of a free boson with values on a circle. Specifically, it states the equality of the partition function at genus one. Unsurprisingly, a free complex fermion can be described in terms of two free real fermions. Real fermions are described by the minimal (3,4) CFT and their partition function is given by F_Q with $A = 1$.

Recently, Zagier gave lectures on CFT (physics!) and asked about any relation with class field theory. Now questions about words relate to poetry, not mathematics, but they should not be taken too lightly. Poetry works with accidental relations of words in a particular language. Not all of them are particularly inspiring. At best, the homophony of 'field' in physics and mathematics can illustrate the rift between the two fields of knowledge caused by quantum field theory. Still, some relations are not understood as arbitrary by native speakers. They have come to reflect aspects of the national character, like $\kappa\alpha\lambda o\varsigma\ \kappa'\alpha\gamma\alpha\theta o\varsigma$ in Greek, the two meanings of 'true' in English, the relation of green growth and beauty in old German (grôni enti skôni), or the relation of 'klar' and 'wahr' mentioned in the introduction. For the language of mathematics, Zagier has long been stimulated by the three meanings of q. In this article q-hypergeometric series and the q of modularity have appeared in close linkage. Third, q-deformation comes from quantum mechanics. Indeed, the q-deformed torus embodies Heisenberg's commutator of position and momentum, though it must be complemented by Schrödinger's cat to make real (rather complex) sense.

It seems quite possible that CFT will join this famous company, when the rift between mathematics and physics will be healed. Look again at eq. (8). The extension of $\mathbb{Q}(\omega)$ by $2^{1/3}$ is due to Gauss, who observed that $p = x^2 + 27y^2$ is equivalent to $p \equiv 1 \; mod \; 3$ plus $m^3 \equiv 2 \; mod \; p$ for some integer m. This was the first glimpse of true class field theory. More vaguely, the hidden periodicity of F_Q at cusps, like $2/(2m+1)$ in the example at hand, may recall that the periodicity of Gauss sums prepared for the hidden periodicities in class field theory. Finally, the fact that the number fields occuring in conformal field theory come from abelian extensions has deep roots. It has been seen by all mathematiciens who looked at the Verlinde equations. The devil may not only be in details but also in sounds, but one does not have to be an ancient Greek to be curious about sirenes.

References

[1] M. Atiyah, *Vector Bundles over an Elliptic Curve*, Proc. London Math. Soc. **7** (1957) 414–452. MR0131423

[2] F. Calegari et al., *Bloch groups, algebraic K-Theory, units, and Nahm's conjecture*, (arXiv:1712.04887), to appear in Annales scientifiques de l'École normale supérieure.

[3] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Math. **55**, Birkhäuser, Basel-Boston (1985). MR0781735

[4] S. Garoufalidis and D. Zagier, *Asymptotics of Nahm sums at roots of unity*, The Ramanujan Journal **55** (2021), 219–238. MR4244313

[5] S. Garoufalidis and D. Zagier, *Knots, Perturbative Series and Quantum Modularity*, arXiv:2111.06645 [math.GT].

[6] G. Meinardus, *Asymptotische Aussagen uber Partitionen*, Math. Zeitschr. **59** (1954), 388–398. MR0062781

[7] W. Nahm, *Conformal Field Theory and torsion elements of the Bloch group*, in: Frontiers in Number Theory, Physics and Geometry II, P. Cartier et al. eds., Springer, Berlin-Heidelberg-New York (2006), 67–132. MR2290759

[8] W. Neumann, *Extended Bloch group and the Cheeger–Chern–Simons class*, Geometry & Topology **8** (2004), 413–474. MR2033484

[9] L. van der Pauw, *A method of measuring specific resistivity and Hall effect of discs of arbitrary shape*, Philips Research Reports **13** (1958), 1–9.

[10] M. Vlasenko and S. Zwegers, *Nahm's conjecture: Asymptotic calculations and counterexamples*, Comm. Number Theory and Physics **5** (2011), 617–642. MR2864462

[11] D. Zagier, *The dilogarithm function*, in: Frontiers in Number Theory, Physics and Geometry II, P. Cartier et al. eds., Springer, Berlin-Heidelberg-New York (2006), 3–65. MR2290758

[12] C. Zickert, *The Extended Bloch Group and Algebraic K-Theory*, J. Reine Angew. Math. **704** (2015), 21–54. MR3365773

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