

Hilbert reciprocity using K -theory localization

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Abstract: Usually the boundary map in K -theory localization only gives the tame symbol at K_2 . It sees the tamely ramified part of the Hilbert symbol, but no wild ramification. Gillet has shown how to prove Weil reciprocity using such boundary maps. This implies Hilbert reciprocity for curves over finite fields. However, phrasing Hilbert reciprocity for number fields in a similar way fails because it crucially hinges on wild ramification effects. We resolve this issue, except at $p = 2$. Our idea is to pinch singularities near the ramification locus. This fattens up K -theory and makes the wild symbol visible as a boundary map.

Keywords: Hilbert reciprocity law, Moore sequence, localization sequence, Hilbert symbol, tame symbol.

1. Introduction

In the discipline of K -theory the term *tame symbol* is a special name for the boundary map $\partial_v : K_2(F) \rightarrow \kappa(v)^\times$ for a field F with a discrete valuation v which has residue field $\kappa(v)$. This map comes from the localization sequence, but it is also very easy to give an explicit formula:

$$(1.1) \quad \partial_v\{f, g\} := (-1)^{v(f)v(g)} \overline{f^{v(g)}g^{-v(f)}} \in \kappa(v)^\times,$$

where $\overline{(-)}$ is the reduction modulo the valuation ideal.

In number theory the meaning of the term tame symbol is more restrictive. Every finite extension F/\mathbb{Q}_p with its usual valuation v and residue field $\kappa(v)$ is equipped, through a construction coming from local class field theory, with its Hilbert symbol

$$h_v : K_2(F) \longrightarrow \mu(F),$$

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where $\mu(F)$ is the group of roots of unity. We shall recall its construction and further background in §2.1. The residue field satisfies $\kappa(v)^\times \cong \mathbb{F}_q^\times$ and using the Teichmüller lift these roots of unity lift injectively to elements of $\mu(F)$. The image of this lift agrees precisely with the direct summand of those roots in $\mu(F)$ of order prime to p . Consider only the output of h_v on this summand,

$$K_2(F) \longrightarrow \mu(F)[\text{prime-to-}p\text{-part}] \cong \kappa(v)^\times.$$

Then this map agrees up to an isomorphism with the tame symbol. The complementary p -torsion part $\mu(F)[p^\infty]$ has an interpretation in terms of wild ramification. Seen from this viewpoint, the tame symbol takes its name from encoding the tamely ramified information in the Hilbert symbol. Unlike the tame symbol, the full Hilbert symbol has no simple formula as in Equation 1.1. Formulas for the p -part of h_v are known as *explicit reciprocity laws* and this is a whole area of research in itself. Many explicit formulas exist, of varying generality, and all of them are very complicated (as a very incomplete list, we name Artin–Hasse [AH28], Shafarevich [Š50], Wiles [Wil78], Brückner [Bru79], Vostokov [Vos78], Henniart [Hen81], de Shalit [dS86], ...¹).

Taking this into account, it is almost surprising that the localization sequence in K -theory, a result which works in broad generality and has nothing to do with the specifics of number theory, captures a part of the Hilbert symbol at all.

It has been a long-standing folklore question whether it would also be possible to express the full Hilbert symbol, including the wild part, as a boundary map in a localization sequence. We answer this affirmatively in this paper, except at places over the prime $p = 2$ and real places. In other words: We cannot capture 2-torsion phenomena with what we do.

Whenever we work with a number field F , we have the instinct to work with its ring of integers \mathcal{O}_F as an integral model. Recall that an *order* \mathcal{R} in F is any subring $\mathcal{R} \subseteq F$ such that $\mathbb{Q} \cdot \mathcal{R} = F$ and which is finitely generated as an abelian group. The ring of integers \mathcal{O}_F agrees with the unique maximal order in F . This intuition is informed from ring-theoretic properties: The maximal order is regular, it is our only chance to get a Dedekind domain, to get unique factorization in prime ideals. However, addressing our problem, it is best to shrink the order. First, proceed locally.

¹There are many more works and I apologize to those I have not explicitly listed.

Theorem 1.1. *For any finite extension F/\mathbb{Q}_p with p odd, there exists a well-defined local subring $(\mathcal{R}, \tilde{\mathfrak{m}})$ of the valuation ring such that the localization sequence for the closed-open complement*

$$\text{Spec } \mathcal{R}/\tilde{\mathfrak{m}} \hookrightarrow \text{Spec } \mathcal{R} \hookrightarrow \text{Spec } F$$

outputs

$$K_2(\mathcal{R}) \longrightarrow K_2(F) \xrightarrow{\partial} K_{\tilde{\mathfrak{m}},1}(\mathcal{R}) \longrightarrow 1$$

and such that the boundary map is the full Hilbert symbol. In particular, $K_{\tilde{\mathfrak{m}},1}(\mathcal{R}) \cong \mu(F)$. If F/\mathbb{Q}_p is unramified, \mathcal{R} is the usual ring of integers.

See Theorem 4.17. The ring \mathcal{R} , which we shall call the *optimal order*, will usually be singular. We explain its construction later in the paper. Recall that $K_{\mathfrak{m},1}(\mathcal{O}_F)$ gives only the prime-to- p order roots of unity when using the valuation ring \mathcal{O}_F instead of \mathcal{R} . The localization sequence in K -theory is compatible under switching from working locally to globally. Running the same idea globally, one gets the full Hilbert reciprocity law – except that since the above only works for odd p , we must invert 2. This also washes away the contribution from all real places.

Theorem 1.2. *Suppose F is any number field. Then we construct a well-defined order $\mathcal{R} \subseteq \mathcal{O}_F$ such that we get a commutative diagram with exact rows*

$$\begin{array}{ccccccc} K_2(\mathcal{R}) & \longrightarrow & K_2(F) & \xrightarrow{\partial} & \bigoplus_{v \text{ finite}} K_{\tilde{\mathfrak{m}},1}(\mathcal{R}_v) & \longrightarrow & SK_1(\mathcal{R}) \longrightarrow 0 \\ & & \parallel & & \downarrow \oplus_v \phi & & \downarrow \\ & & K_2(F) & \xrightarrow{h_v} & \bigoplus_{v \text{ noncomplex}} \mu(F_v) & \xrightarrow{\cdot \frac{m_v}{m}} & \mu(F) \longrightarrow 0 \end{array}$$

in the category of abelian groups up to 2-primary torsion. The top row is the localization sequence with respect to the maximal ideals of \mathcal{R} . The bottom row is the Moore sequence (a way to phrase Hilbert reciprocity). The downward arrows are isomorphisms (up to 2-primary torsion).

See Theorem 5.8, where we also elaborate on the formulation “up to 2-primary torsion”. One can deduce statements from this with a more classical flavour.

Corollary. *Let F be a number field with $\sqrt{-1} \notin F$. Then the statement of Hilbert reciprocity up to sign, i.e.*

$$\prod_{v \text{ noncomplex}} h_v(\alpha, \beta)^{\frac{m_v}{m}} = \pm 1 \quad \text{for all } \alpha, \beta \in F^\times,$$

can be phrased as the property of being a complex ($d^2 = 0$) for a localization sequence in K -theory.

As we will explain in §3, Gillet proved Hilbert reciprocity for function fields over finite fields using K -theory localization by proving the more general Weil reciprocity theorem over all base fields. Under the number field/function field dictionary one would hope that this proof generalizes to give Hilbert reciprocity for number fields. This fails for various reasons, one being that one needs the *full* Hilbert symbol and not just the tame symbol. Our method solves this. However, the version of Hilbert reciprocity it proves – if we only use K -theory localization and nothing else – then takes values in the group SK_1 of the global (singular) order we refer to in Theorem 1.2.

It seems difficult to compute this group without using tools which would also go into conventional proofs of Hilbert reciprocity.

It is a common phenomenon in Algebraic K -theory that adding nil-thickenings or singularities makes K -groups more complicated. For example, the Contou-Carrère symbol is a fattening up of the tame symbol for nil-thickenings and in its tangent functor one can find the residue symbol for log differential forms. The residue symbol only becomes visible thanks to nil-thickenings. The Contou-Carrère symbol is a different kind of generalization of the tame symbol than the Hilbert symbol. Nonetheless, our main idea is similar. If we think in terms of the function field analogy, the orders \mathcal{R} correspond to (affine models of) singular algebraic curves and $\text{Spec } \mathcal{O}_F \rightarrow \text{Spec } \mathcal{R}$ is the normalization, the resolution of singularities. Using this analogy, working with \mathcal{R} is like pinching additional singularities at the branching locus of F over \mathbb{Q} . This then has the effect to fatten up the K -theory with support at these singularities. As it turns out, this additional complexity is precisely the wild part of the Hilbert symbol. Under the normalization, going to \mathcal{O}_F , K -theory loses the wild data and keeps only the tame symbol in the boundary map.

Digging a littler deeper, there is a hierarchy of rings \mathcal{R}_i between the optimal and the maximal order,

$$\mathcal{R} \subseteq \mathcal{R}_i \subseteq \mathcal{O}_F$$

which “can see”, as i increases, less and less of the wild part of the Hilbert symbol. We shall study these in more detail in a future text.

Considering local metaplectic extensions, Toshiaki Suzuki has worked with rings similar to what we do in §4 with the purpose to formulate modified Gauss sums [Suz02, Suz06, Suz12]. He does not use K -theory, but probably our local optimal order agrees with his subrings from [Suz02].

Let me note that the recent papers [AB19, BHv21] also produce, without relying on global class field theory, a sequence

$$K_2(F) \xrightarrow{\partial} \bigoplus_v \mu(F_v) \longrightarrow K_2(\text{LCA}_F)_{/div} \longrightarrow 0,$$

which is then shown to be equivalent to Moore’s sequence. The methods in these papers are completely different from what we do in this paper.

2. What is the Hilbert symbol?

2.1. Local theory

In this section, we shall recall the definition and purpose of the Hilbert symbol and the statement of the Hilbert reciprocity law. Readers familiar with these concepts are invited to skip ahead.

Suppose F is a local field of characteristic zero, i.e. a finite extension of \mathbb{Q}_p or \mathbb{R} . In this section, let us exclude the case $F = \mathbb{C}$. Write $m := \#\mu(F) < \infty$ for the number of roots of unity in F .² Then the *local reciprocity map* (also known as *local Artin map*) is a group homomorphism

$$\text{Art}_F: F^\times \longrightarrow \text{Gal}(F^{\text{ab}}/F),$$

where F^{ab}/F is the maximal abelian extension. It is easy to describe for $F = \mathbb{Q}_p$ or \mathbb{R} , but for a general local field setting up Art_F is too complicated than we would wish to recall here. The Artin map becomes an isomorphism after profinite completion³, so it gives tight control over the abelian field extensions of F . The *Hilbert symbol* is the map

$$(2.1) \quad \begin{aligned} h: F^\times \times F^\times &\longrightarrow \mu(F) \\ (x, y) &\longmapsto \frac{\text{Art}_F(x) \sqrt[m]{y}}{\sqrt[m]{y}}. \end{aligned}$$

We need to explain this: Since F contains a primitive m -th root of unity, the extension $F(\sqrt[m]{y})/F$ is Kummer, and in particular an abelian extension. Only

²We have excluded \mathbb{C} because it is the only local field with infinitely many roots of unity. However, it also has trivial Hilbert symbol, so we do not miss out on anything by skipping this case.

³The Galois group is of course already profinite. However, the left side changes. For p -adic fields the completion injects into a strictly bigger group, while for the reals the completion map is surjective. The kernel is the subgroup of positive reals.

because of this, $\text{Art}_F(x)$ acts on it. All Galois conjugates of $\sqrt[m]{y}$ differ by a root of unity, so the map takes values in $\mu(F)$. It is well-defined because if we pick a different root for $\sqrt[m]{y}$, then (as long as we use the same one in the numerator and denominator of Equation 2.1) they both differ by the same root of unity from our previous choices, which then cancels out in the fraction.

One checks that h is bilinear and respects the Steinberg relation, so we obtain a factorization

$$h: K_2(F) \longrightarrow \mu(F).$$

In a nutshell, one could say that h sets up a link between the local reciprocity map and Kummer theory. Suppose F is a finite extension of \mathbb{Q}_p . As $\mu(F)$ is a torsion group, it splits canonically into its p -primary part and a prime-to- p part,

$$\mu(F) \cong \mu(F)[p^\infty] \oplus \bigoplus_{\ell \neq p} \mu(F)[\ell^\infty].$$

We have no use for further distinguishing the primes in the prime-to- p summand, we just write $\mu(F)[\text{prime-to-}p]$ for it all.

Explicit formulas for the p -part of the Hilbert symbol, i.e. its values in the summand $\mu(F)[p^\infty]$, are very complicated. If we write $m = p^k m_0$ with $(p, m_0) = 1$, then $F(\sqrt[m]{y p^k}) = F(\sqrt[m_0]{y})$ is at worst tamely ramified over F , and in this case the value of the Hilbert symbol $(x, y p^k)$ in the summand $\mu(F)[p^\infty]$ is trivial. Hence, such a case corresponds to (at worst) tame ramification. In this situation, the Hilbert symbol agrees with the tame symbol, see Lemma 4.11 below. The values in the complementary summand $\mu(F)[p^\infty]$ are therefore also sometimes called the wild part of the Hilbert symbol, or even a *wild symbol* formula.

2.2. Global theory

Suppose F is a number field. For any noncomplex place write F_v for its completion at v . Define $m := \#\mu(F) < \infty$ and $m_v := \#\mu(F_v) < \infty$.

Theorem 2.1 (Moore sequence). *Let F be a number field. Then the sequence*

$$(2.2) \quad K_2(F) \longrightarrow \bigoplus_{v \text{ noncomplex}} \mu(F_v) \xrightarrow{\cdot \frac{m_v}{m}} \mu(F) \longrightarrow 0$$

is exact, where the first arrow is the Hilbert symbol, and the second is taking the $\frac{m_v}{m}$ -th power.

The statement that this is a complex ($d^2 = 0$) amounts to the classical Hilbert reciprocity law. Moore provided the slight strengthening that the sequence is exact. We refer to [CW72] for a nice proof.

Example 2.2. For $F = \mathbb{Q}$ we have $\mu(F) := \{\pm 1\}$. The finite places are just the ordinary prime numbers. If p is odd, $\frac{m_p}{m} = \frac{p-1}{2}$ is half the order of \mathbb{F}_p^\times and coprime to p , so the Hilbert symbol is just the tame symbol, but effectively only contributes values in $\mathbb{F}_p^\times / \mathbb{F}_p^{\times 2} \cong \{\pm 1\}$ thanks to taking the $\frac{m_p}{m}$ -th power. One finds that this composition $K_2(F) \rightarrow \{\pm 1\}$ is given in terms of the Legendre symbol for all odd p . Only for $p = 2, \infty$ the Hilbert symbol differs from the tame symbol. The fact that $d^2 = 0$ in Equation 2.2 then becomes the Gauss quadratic reciprocity law.

3. Gillet’s proof of Weil reciprocity

To motivate this paper, we must recall Gillet’s very elegant proof of Hilbert reciprocity for curves over finite fields. This is the counterpart, under the number field/function field dictionary, of the Hilbert reciprocity law which we discuss in the present article. Gillet first proves something more general.

Theorem 3.1 (Weil reciprocity). *Let k be a field. Let X/k be an integral smooth⁴ proper curve with function field $F := k(X)$. Then the composition*

$$K_n(F) \xrightarrow{\partial} \bigoplus_v K_{n-1}(\kappa(v)) \xrightarrow{N_{\kappa(v)/k}} K_{n-1}(k)$$

is zero, where ∂ denotes the boundary map of the Algebraic K -theory localization sequence with respect to zero-dimensional support, v runs through the places of X which are trivial over k , and $N_{\kappa(v)/k}$ is the norm map.

The set of places v can be identified with the set of codimension one irreducible closed subschemes, and equivalently with the set of closed points of X . The local rings $(\mathcal{O}_{X,P}, \mathfrak{m}_P)$ are DVRs and the resulting valuation defines a place v on the function field, and conversely these valuations pin down a local ring and therefore a closed point.

Proof (Gillet). We shall freely use results from Algebraic K -theory. The reader will find in Appendix §A a summary, in blackbox format, of what we need.

⁴Gillet’s proof actually does not require smoothness if one works with G -theory localization rather than K -theory. But discussing this detour would distract us too much from our main story. We favour K -theory for good reasons over G -theory in this paper.

Following Gillet, we use the localization sequence of Algebraic K -theory applied to X and Z being any zero-dimensional integral closed subscheme. We get the fiber sequence

$$(3.1) \quad K_Z(X) \longrightarrow K(X) \longrightarrow K(X - Z).$$

Note that we may split Z into its disjoint connected components and get $K_Z(X) = \bigoplus_P K_P(X)$, where P runs through the finitely many points of Z . Finally, since X is smooth, it is regular, and by devissage we obtain $K_P(X) \cong K(P)$ (i.e. regard P as a zero-dimensional closed subscheme). Note that this means that $K(P) = K(\kappa(v))$, where $\kappa(v) = \mathcal{O}_{X,P}/\mathfrak{m}_P$ is the residue field of the DVR belonging to the valuation v . Next, we arrange this in a direct system running over $Z \subseteq Z'$, partially ordered under containment. As K -theory commutes with filtering colimits, we obtain the top row in

$$(3.2) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & K_n(F) & \xrightarrow{\partial} & \bigoplus_v K_{n-1}(\kappa(v)) & \longrightarrow & K_{n-1}(X) \longrightarrow \cdots \\ & & & \searrow & \downarrow N_{\kappa(v)/k} & \swarrow R\pi_* & \\ & & & & K_{n-1}(k) & & \end{array}$$

Since the structure map $\pi : X \rightarrow k$ is proper, the derived pushforward $R\pi_*$ induces a map $K(X) \rightarrow K(k)$. Since the closed embeddings $i_P : P \hookrightarrow X$ are also proper, $\pi \circ i_P$ induces maps $K(P) \rightarrow K(k)$. Since the composition of two consecutive arrows in a complex is zero ($d^2 = 0$), the existence of the right diagonal arrow implies that $\sum_P \pi_* i_{P*} = 0$. Upon unravelling this, the $\pi \circ i_P$ correspond to the pushforward along the finite morphisms $\text{Spec } \kappa(v) \rightarrow \text{Spec } k$, which is nothing but the norm map on K -theory. \square

The above proof, which uses only fairly general tools from K -theory and the existence of $R\pi_*$ now gives a complete proof of Hilbert reciprocity for curves over finite fields.

Theorem 3.2 (Hilbert reciprocity for function fields). *Suppose X/\mathbb{F}_q is a geometrically integral smooth proper curve with function field $F := \mathbb{F}_q(X)$. Then Hilbert reciprocity holds: The composition of*

$$(3.3) \quad K_2(F) \xrightarrow{h_v} \bigoplus_v \mu(F_v) \xrightarrow{\frac{m_v}{m}} \mu(F)$$

is zero, where $m_v := \#\mu(F_v)$ and $m := \mu(F)$.

We have only discussed the Hilbert symbol in characteristic zero in §2.1, but the same definitions go through in general. We will not work in positive characteristic anywhere else in this paper.

Proof. Write $k := \mathbb{F}_q$ for the base field. We use Weil reciprocity for $n = 2$ and obtain

$$(3.4) \quad K_2(F) \xrightarrow{\partial} \bigoplus_v K_1(\kappa(v)) \xrightarrow{N_{\kappa(v)/k}} K_1(k).$$

Since F is a function field of a geometrically integral curve, we have the sequence $k^\times \hookrightarrow F^\times \xrightarrow{v} \bigoplus \mathbb{Z}$ sending a function to its Weil divisor, and since \mathbb{Z} has no non-trivial torsion, it follows that all roots of unity must come from k^\times , but since $k = \mathbb{F}_q$, all its non-zero elements are roots of unity. Hence, $K_1(k) = \mu(F)$. Similarly, each $\kappa(v)/\mathbb{F}_q$ is a finite field extension, and thus $\kappa(v)$ is itself a finite field and therefore $K_1(\kappa(v)) = \mu(F_v)$. Note that since k has characteristic $p > 0$ and we have $\mu(F_v) = \kappa(v)^\times \oplus \mu(F_v)[p^\infty]$ by the Teichmüller lift, we must have $\mu(F_v)[p^\infty] = 1$ as there cannot be non-trivial p -power roots of unity in a char $p > 0$ field. This also implies that the Hilbert symbol agrees with the tame symbol. By now, Equation 3.4 has almost completely transformed into Equation 3.3. It remains to show that $N_{\kappa(v)/k}$ agrees with multiplication by $\frac{m_v}{m}$, but

$$\frac{m_v}{m} = \frac{q^{[\kappa(v):k]} - 1}{q - 1} = 1 + q + q^2 + \dots + q^{[\kappa(v):k]-1},$$

and the extension $\kappa(v)/k$ is Galois and its Galois group generated by the Frobenius $x \mapsto x^q$. Thus, taking the $\frac{m_v}{m}$ -th power does the same as taking the norm. □

Inspired by this beautifully short proof, people have been wondering about the possibility to prove Hilbert reciprocity for number fields using the same idea.

For the sake of completeness, let us quickly go through the same steps in the number field setting and see how and why they fail.

Let F be a number field and take $X := \text{Spec } \mathcal{O}_F$, where \mathcal{O}_F denotes the ring of integers. The localization sequence of Equation 3.1 works fine, where again Z is running through the direct system of zero-dimensional closed subschemes partially ordered by inclusion. We obtain the exact sequence

$$\dots \longrightarrow K_n(F) \xrightarrow{\partial} \bigoplus_v K_{n-1}(\kappa(v)) \xrightarrow{\beta_n} K_{n-1}(\mathcal{O}_F) \longrightarrow \dots,$$

where v runs through the finite places. What should the replacement for π in Diagram 3.2 be? We might speculate about $\pi : \text{Spec } \mathcal{O}_F \rightarrow \text{Spec } \mathbb{Z}$ or about some map to a conjectural \mathbb{F}_1 , but there is a problem we cannot avoid: Soulé has proven that for all even $n \geq 2$ the maps β_n are zero [Wei13, Chapter V, Theorem 6.8]. So even if we can set up some magical diagram (with $n \geq 2$ even)

$$\begin{array}{ccccccc} \cdots & \longrightarrow & K_n(F) & \xrightarrow{\partial} & \bigoplus_v K_{n-1}(\kappa(v)) & \xrightarrow{0} & K_{n-1}(\mathcal{O}_F) \longrightarrow \cdots \\ & & \searrow & & \downarrow & & \swarrow \cdots \\ & & & & K_{n-1}(?) & & \end{array}$$

for some carefully chosen “?”, we see that for $n = 2$ the statement of our theorem is void because already the individual maps $\pi_* \circ i_{P*}$ are necessarily zero. So clearly their sum is zero. We learn nothing new.

Is this because we have no infinite places here and some Arakelov theory would solve the problem? At first, one could say yes. The cited result of Soulé also tells us that for affine curves over finite fields, e.g. $\text{Spec } \mathcal{O}_X(X - pt)$, the same vanishing holds. But we could take a number field without real places. Since Hilbert reciprocity among the infinite places only sees the real ones, this leads us to situations where Hilbert reciprocity is exclusively a statement about a cancellation of values at *finite* places, but we still have the same issue. So this is not the core of the problem.

More crucially, we only see the *tame* symbol here. Consider for example the extensions $\mathbb{Q}_p(\zeta_{p^m})/\mathbb{Q}_p$. These are totally ramified. As $m \rightarrow +\infty$, the group $\mu(\mathbb{Q}_p(\zeta_{p^m})) = \langle \zeta_{p^m} \rangle \times \mathbb{F}_q^\times$ grows arbitrarily large, yet the receptacle of the tame symbol

$$K_2(\mathbb{Q}_p(\zeta_{p^m})) \xrightarrow{\partial} K_1(\mathbb{F}_p) = \mathbb{F}_p^\times$$

remains unchanged. This shows how the discrepancy between the Hilbert symbol and the tame symbol can grow arbitrarily large.

Example 3.3 (continues Example 2.2). Another good example is the quadratic reciprocity law at $p = 2$. The tame symbol here is a map $K_2(\mathbb{Q}_2) \rightarrow \mathbb{F}_2^\times = \{1\}$, and contains no information. The Hilbert symbol of \mathbb{Q}_2 however is non-trivial.

Even once we know that the Hilbert reciprocity law holds, it does not really imply anything about a reciprocity law for tame symbols, so it is not too shocking that Soulé’s result reveals that considering the tame symbol yields no interesting invariant whatsoever.

4. Local theory

Let F/\mathbb{Q}_p be a finite extension. We write F_0/\mathbb{Q}_p for its maximal unramified subextension. Denote by $(\mathcal{O}_F, \mathfrak{m})$ the local ring which is the ring of integers of F . Analogously, write (\mathcal{O}_0, p) for the ring of integers of F_0 .

We recall that we may realize $\mathcal{O}_F = \mathcal{O}_0[T]/(f)$, where f is an Eisenstein polynomial for the prime p , [Rei03, (5.6)]. The image of the variable, $\pi := \overline{T}$, is a uniformizer of F , so that $\mathfrak{m} = (\pi)\mathcal{O}_F$. Conversely, the minimal polynomial of any uniformizer of F over \mathcal{O}_0 will be a possible choice for the Eisenstein polynomial f , [Rei03, (5.12)].

As a result of this presentation, \mathcal{O}_F is a finitely generated free \mathcal{O}_0 -module, and we may take

$$(4.1) \quad \mathcal{O}_F = \mathcal{O}_0 \langle 1, \pi, \dots, \pi^{e-1} \rangle$$

as a basis, where $e := \deg f$ is the ramification index of F/\mathbb{Q}_p . Moreover, recall that there is a non-canonical decomposition

$$(4.2) \quad F^\times \cong \langle \pi^{\mathbb{Z}} \rangle \times \langle \zeta_{q-1} \rangle \times U_F^1,$$

where ζ_{q-1} is a primitive $(q - 1)$ -st root of unity, and

$$U_F^r := 1 + \mathfrak{m}^r \mathcal{O}_F \quad (\text{for } r \geq 1)$$

denotes the r -th higher unit group.

Let us consider the subrings⁵

$$(4.3) \quad R_m := \mathcal{O}_0 + \mathfrak{m}^m \mathcal{O}_F \quad (\text{for } m \geq 0)$$

inside \mathcal{O}_F .

Lemma 4.1. *The ring R_m is a Noetherian local domain with field of fractions F . Its maximal ideal is*

$$(4.4) \quad \tilde{\mathfrak{m}}_m = \begin{cases} p\mathcal{O}_0 + \mathfrak{m}^m \mathcal{O}_F & \text{for } m \geq 1 \\ \mathfrak{m} \mathcal{O}_F & \text{for } m = 0. \end{cases}$$

The ring R_m is a module-finite \mathcal{O}_0 -algebra of finite index $[\mathcal{O}_F : R_m] < \infty$. Moreover, R_m is a compact subring of \mathcal{O}_F (in the valuation topology). The

⁵This is a special case of a general observation: If $R \subseteq S$ is a subring of any ring S , and I any ideal of S , then $R + I$ is a ring.

unit group is non-canonically isomorphic to

$$(4.5) \quad R_m^\times \cong \langle \zeta_{q-1} \rangle \times (1 + \tilde{\mathfrak{m}}_m).$$

If $1 \leq m \leq e$ (with $e := e_{F/\mathbb{Q}_p}$ the absolute ramification index), $\tilde{\mathfrak{m}}_m = \mathfrak{m}^m \mathcal{O}_F$.

Proof. Since $R_0 = \mathcal{O}_F$, the case $m = 0$ is clear. From now on assume $m \geq 1$. It is clear that the $R_m = \mathcal{O}_0 + \mathfrak{m}^m \mathcal{O}_F$ are subrings of \mathcal{O}_F . It follows that R_m is a domain. Next, it is immediate to check that $\tilde{\mathfrak{m}}_m \subseteq R_m$ is an ideal. Since $\tilde{\mathfrak{m}}_m \subseteq \mathfrak{m}$, it does not contain 1, so it is a proper ideal in R_m . If we can show that all elements in $R_m \setminus \tilde{\mathfrak{m}}_m$ are units, it will follow that $\tilde{\mathfrak{m}}_m$ is the unique maximal ideal. Before we do this, note that the decomposition of Equation 4.2 applied to F_0 shows that $\zeta_{q-1} \in \mathcal{O}_0$ (since both F and F_0 have the same residue field; alternatively retrieve ζ_{q-1} as the Teichmüller lift of a primitive $(q - 1)$ -st root of unity of the finite residue field). Suppose $x \in R_m \setminus \tilde{\mathfrak{m}}_m$. Using Equation 4.2 for F we write it as

$$(4.6) \quad x = \pi^n \cdot \zeta_{q-1}^i \cdot u \quad \text{with} \quad u \in U_F^1.$$

Since $x \in \mathcal{O}_F$, we have $n \geq 0$. Assume $n \geq 1$. Then

$$x \in R_m \cap \mathfrak{m} \mathcal{O}_F = (\mathcal{O}_0 + \mathfrak{m}^m \mathcal{O}_F) \cap \mathfrak{m} \mathcal{O}_F = (\mathcal{O}_0 \cap \mathfrak{m} \mathcal{O}_F) + \mathfrak{m}^m \mathcal{O}_F = \tilde{\mathfrak{m}}_m$$

since $m \geq 1$ and $\mathcal{O}_0 \cap \mathfrak{m} \mathcal{O}_F = p \mathcal{O}_0$. Contradiction. Hence, we must have $n = 0$. As $\zeta_{q-1} \in \mathcal{O}_0$, we deduce

$$(4.7) \quad x \zeta_{q-1}^{-i} - 1 \in \mathfrak{m} \mathcal{O}_F \cap R_m$$

(it lies in $\mathfrak{m} \mathcal{O}_F$ by Equation 4.6, and in R_m since x and ζ_{q-1}^{-1} lie in R_m and R_m is closed under ring operations). Then the geometric series

$$(4.8) \quad \frac{1}{x \zeta_{q-1}^{-i}} = \frac{1}{1 - (1 - x \zeta_{q-1}^{-i})} = 1 + \sum_{l \geq 1} (1 - x \zeta_{q-1}^{-i})^l := \tilde{u}$$

is convergent in \mathcal{O}_F . As the residue field of \mathcal{O}_F is finite, \mathcal{O}_F is compact in the valuation topology. Same for \mathcal{O}_0 . As R_m is the image of the compact space $\mathcal{O}_0 \times \mathcal{O}_F$ under an obvious map, R_m is compact. In particular, R_m is necessarily complete as a metric space. As all partial sums in Equation 4.8 lie in R_m , we deduce that the limit \tilde{u} must also lie in R_m . Returning to our original thread of thoughts, Equation 4.8 shows that $x^{-1} = \zeta_{q-1}^{-i} \tilde{u}$ with $\tilde{u} \in R_m$. This completes the proof that $\tilde{\mathfrak{m}}_m$ is the unique maximal ideal. We

conclude that R_m is a local ring. As this also implies that $R_m^\times = R_m \setminus \tilde{\mathfrak{m}}_m$, the computation starting from $x \in R_m \setminus \tilde{\mathfrak{m}}_m$ and leading to Equation 4.7 proves the inclusion $R_m^\times \subseteq \langle \zeta_{q-1} \rangle \times (1 + \tilde{\mathfrak{m}}_m)$ of Equation 4.5; and Equation 4.8 had already shown the reverse inclusion. Next, let us show that $\text{Frac } R_m = F$. This is easy: Since $F = \text{Frac } \mathcal{O}_F$ is a discrete valuation field, for any $x \in F$ we have $\pi^l x \in \mathcal{O}_F$ for $l \geq 0$ sufficiently big. In particular, $\pi^{l+m} x \in R_m$ and $\pi^{l+m} \in R_m$, so $x = \frac{\pi^{l+m} x}{\pi^{l+m}} \in \text{Frac } R_m$. Finally, by Equation 4.1 we obtain

$$R_m = \mathcal{O}_0 + \pi^m \mathcal{O}_0 \langle 1, \pi, \dots, \pi^{e-1} \rangle,$$

so R_m is finitely generated as an \mathcal{O}_0 -module. As \mathcal{O}_0 is a discrete valuation ring, the structure theorem for finitely generated modules over a principal ideal domain shows that R_m is a finite free \mathcal{O}_0 -module. There cannot be any torsion since it is a submodule of F , hence p -torsionfree. This also implies that R_m is Noetherian (which could also be shown in many other ways). If $1 \leq m \leq e$, then $p \in \pi^e \mathcal{O}_F^\times \subseteq \pi^m \mathcal{O}_F$, so $\tilde{\mathfrak{m}}_m = \mathfrak{m}^m \mathcal{O}_F$. As R_m can be sandwiched as $\mathfrak{m}^m \mathcal{O}_F \subseteq R_m \subseteq \mathcal{O}_F$ but the outer two terms have finite index in one another, we also must have $[\mathcal{O}_F : R_m] < \infty$. This also proves $\text{Frac } R_m = F$ a second time. □

Lemma 4.2. *The residue field of R_m is the same as of \mathcal{O}_F , i.e. $\mathcal{O}_F/\mathfrak{m}$. For $m \geq 1$ we have $R_m = \mathcal{O}_0 + \tilde{\mathfrak{m}}_m$.*

Proof. Suppose $m \geq 1$. Then $R_m = \mathcal{O}_0 + \tilde{\mathfrak{m}}_m$ by Equation 4.4. For subgroups A, B contained in some abelian group one generally has

$$\frac{A+B}{B} \cong \frac{A}{A \cap B}.$$

Hence, for $A := \mathcal{O}_0$ and $B := \tilde{\mathfrak{m}}_m$, both lying inside \mathcal{O}_F , we find

$$\frac{R_m}{\tilde{\mathfrak{m}}_m} = \frac{\mathcal{O}_0 + \tilde{\mathfrak{m}}_m}{\tilde{\mathfrak{m}}_m} \cong \frac{\mathcal{O}_0}{\mathcal{O}_0 \cap \tilde{\mathfrak{m}}_m} = \frac{\mathcal{O}_0}{(\mathcal{O}_0 \cap p\mathcal{O}_0) + (\mathcal{O}_0 \cap \pi^m \mathcal{O}_F)}.$$

Since $\pi \mathcal{O}_F \cap \mathcal{O}_0 = p\mathcal{O}_0$, we have $\mathcal{O}_0 \cap \pi^m \mathcal{O}_F = p^m \mathcal{O}_0$, so

$$= \frac{\mathcal{O}_0}{p\mathcal{O}_0 + p^m \mathcal{O}_0} = \frac{\mathcal{O}_0}{p\mathcal{O}_0},$$

which is the residue field of \mathcal{O}_0 , but since F/F_0 is totally ramified, this is the same residue field as of F . Finally, for $m = 0$ the first claim is clearly also true. □

Lemma 4.3. *The group $1 + \tilde{\mathfrak{m}}_m \subseteq R_m^\times$ is a \mathbb{Z}_p -module under exponentiation. Concretely, one may define*

$$(1 + x)^\alpha := \sum_{l \geq 0} \binom{\alpha}{l} x^l$$

for all $\alpha \in \mathbb{Z}_p$, where $\binom{\alpha}{l} = \frac{\alpha(\alpha-1)\dots(\alpha-l+1)}{1 \cdot 2 \dots l}$.

Proof. This result is well-known for U_F^1 , so it is clear that the series converges to an element in U_F^1 . As soon as we show that the limit lies in $1 + \tilde{\mathfrak{m}}_m$, it is then clear that this pairing $\mathbb{Z}_p \times (1 + \tilde{\mathfrak{m}}_m) \rightarrow U_F^1$ restricts to a module structure on $1 + \tilde{\mathfrak{m}}_m$. To see this, first note that $\binom{\alpha}{l} \in \mathbb{Z}_p$ (because for all $\alpha \in \mathbb{Z}$ the value lies in \mathbb{Z} , so by continuity for $\alpha \in \mathbb{Z}_p$ the values must lie in the closure $\overline{\mathbb{Z}} = \mathbb{Z}_p$ with respect to the p -adic topology), so once $l \geq 1$, we have $\binom{\alpha}{l} x^l \in \tilde{\mathfrak{m}}_m$ since $x \in \tilde{\mathfrak{m}}_m$. As $\tilde{\mathfrak{m}}_m$ is a closed subspace of R_m , we deduce that the limit lies in $1 + \tilde{\mathfrak{m}}_m$. This concludes the proof. An alternative approach could be modelled on [FV02, Chapter I, (6.1)]. \square

Lemma 4.4. *R_m is clopen in F and the quotient group F/R_m is discrete.*

Clear.

Lemma 4.5. *The ring \mathcal{O}_F is the integral closure of R_m in F .*

Proof. We use that \mathcal{O}_F is the integral closure of \mathcal{O}_0 in F by [Rei03, (5.6), (i)]. By the transitivity of integral closures, it follows that the intermediate ring $\mathcal{O}_0 \subseteq R_m \subseteq \mathcal{O}_F$ must have the same integral closure. \square

Having shown that R_m is a local ring, we may now apply the following general result.

Proposition 4.6 (Dennis–Stein). *Suppose R is a local ring.*

1. *Then $K_1(R) \cong R^\times$ (via the determinant) and the product map of K -theory,*

$$R^\times \otimes R^\times \longrightarrow K_2(R)$$

is surjective, i.e. every element in K_2 comes from symbols.

2. (Steinberg relation) *If $x, y \in R^\times$ such that $x + y = 1$, then $\{x, y\} = 0$.*

Proof. Since R is a local ring (Lemma 4.1), $R^\times = \mathrm{GL}_1(R) \rightarrow K_1(R)$ is an isomorphism, and split by the determinant $\mathrm{GL}(R) \rightarrow R^\times$. By a result of Dennis–Stein [SD73, Theorem 2.7] for a local ring the map $K_1(R) \otimes K_1(R) \rightarrow K_2(R)$ is surjective. The claim in (2) is standard. \square

Lemma 4.7. *Every element in $K_2(F)$ can be written as $\beta = \{\pi, u_0\} + \sum_{i=1}^r \{u_i, v_i\}$ for some r and $u_i, v_i \in \mathcal{O}_F^\times$.*

Proof. Standard. By linearity it suffices to prove the claim for a pure symbol $\{x, y\}$ with $x, y \in F^\times$. We may write $x = \pi^a u$ and $y = \pi^b v$ for $a, b \in \mathbb{Z}$ and $u, v \in \mathcal{O}_F^\times$, so by linearity

$$\begin{aligned} \{x, y\} &= \{\pi^a, \pi^b\} + \{\pi^a, v\} + \{u, \pi^b\} + \{u, v\} \\ &= ab\{\pi, \pi\} + \{\pi, v^a\} + \{\pi, u^{-b}\} + \{u, v\}. \end{aligned}$$

The Steinberg relation implies that $\{\pi, \pi\} = \{\pi, -1\}$, so this simplifies to $\{\pi, (-1)^{ab} v^a u^{-b}\} + \{u, v\}$, proving the claim. \square

The following is well-known to experts, and in much broader generality. We record it in the version suitable for us.

Lemma 4.8. *Let π be a uniformizer of F . Then for any $u \in U_F^2$ there exist $a, b \in U_F^1$ such that $\{\pi, u\} = \{a, b\}$ holds in $K_2(F)$.*

Proof. Since $u \in U_F^2$, we may write $u = 1 - z$ with $z \in \mathcal{O}_F$ of valuation $v(z) \geq 2$. Define $g := 1 + z\pi^{-1} - z$. Note that since $v(z) \geq 2$, the term $z\pi^{-1}$, and thus $z\pi^{-1} - z$, lies in \mathfrak{m} . Hence, $g \in U_F^1$. We compute

$$(1 - \pi) \cdot (1 - z) = 1 - \pi - z + \pi z = 1 - \pi(1 + z\pi^{-1} - z) = 1 - \pi g,$$

i.e. $1 - z = \frac{1 - \pi g}{1 - \pi}$. Now we compute

$$\{\pi, u\} = \{\pi, 1 - z\} = \left\{ \pi, \frac{1 - \pi g}{1 - \pi} \right\} = \{\pi, 1 - \pi g\}$$

by the Steinberg relation $\{\pi, 1 - \pi\} = 0$. Using the Steinberg relation a second time, we obtain

$$= \{\pi g, 1 - \pi g\} - \{g, 1 - \pi g\} = -\{g, 1 - \pi g\},$$

but $g \in U_F^1$ and $1 - \pi g \in U_F^1$, so take $a := g^{-1}$ and $b := 1 - \pi g$. \square

We need the following crucial observation of Hasse on the higher unit filtration and p -power maps. We let

$$e_1 := \frac{e}{p-1},$$

where $e := e_{F/\mathbb{Q}_p}$ is the absolute ramification index.

Lemma 4.9 (Hasse). *The filtration of the higher unit groups is compatible with the p -power filtration in the following sense:*

1. Suppose $i \geq 1$. Then for all sufficiently large k we have

$$(U_F^1)^{p^k} \subseteq U_F^i.$$

2. Suppose $i \geq 1$. Then whenever $k \geq 1$ and $i > pe_1 + (k-1)e$ one has the inclusion

$$U_F^i \subseteq (U_F^1)^{p^k}.$$

Proof. The crucial ingredient is a case-by-case study of the map $x \mapsto x^p$ which one can find explained in [FV02, Chapter I, (5.7) Proposition]. The cases are as follows: If $1 \leq t \leq e_1$, then $(U_F^t)^p \subseteq U_F^{pt}$ and if $t > e_1$, then $(U_F^t)^p \subseteq U_F^{t+e}$. As $p \geq 2$ and $e \geq 1$, it is clear that after taking p -th powers often enough, we land in the second case, and then can reach arbitrarily high up in the filtration. This settles (1). We could be more precise about how big k has to get, but we have no use for this information. For (2) one uses that once $t > e_1$ the p -power map

$$U_F^t/U_F^{t+1} \longrightarrow U_F^{t+e}/U_F^{t+e+1}, \quad x \mapsto x^p$$

is surjective, see [FV02, Chapter I, (5.7) Proposition, part (3)] for the proof. Thus, $U_F^{t+e} = (U_F^t)^p \cdot \underline{U_F^{t+e+1}}$. As the same holds for any greater t as well, we may apply the same computation to the underlined term, giving

$$U_F^{t+e} = (U_F^t)^p \cdot \underline{(U_F^{t+1})^p U_F^{t+e+2}}$$

and since $(U_F^{t+1})^p \subseteq (U_F^t)^p$, we get $U_F^{t+e} = (U_F^t)^p U_F^{t+e+l}$ for any $l \geq 1$. Letting $l \rightarrow +\infty$ and using $\bigcap_{c \geq t} U_F^c = \{1\}$, we obtain the crucial equality

$$U_F^{t+e} = (U_F^t)^p \quad (\text{once } t > e_1).$$

Hence, if $i - ke > e_1$, we get (by inductively running through the above identity from right to left for varying t)

$$(U_F^1)^{p^k} \supseteq (U_F^{i-ke})^{p^k} = (U_F^{i-ke+e})^{p^{k-1}} = \dots = U_F^i.$$

However, $i - ke > e_1$ is equivalent to $i > pe_1 + (k - 1)e$. This settles (2). \square

We write

$$h: K_2(F) \longrightarrow \mu(F).$$

for the Hilbert symbol. We had recalled its definition in §2.1.

Theorem 4.10 (Moore). *The map h is surjective and $\ker(h)$ is a divisible group.*

The proof is not terribly difficult. We refer to [FV02, Chapter IX, (4.3) Theorem] for a textbook proof. In fact, it is also known that the kernel is uniquely divisible, but this is much harder to show and we shall not need it.

Next, using our toolbox from Algebraic K -theory as in §A there is a localization sequence of spectra

$$(4.9) \quad K_{\mathfrak{m}}(\mathcal{O}_F) \longrightarrow K(\mathcal{O}_F) \longrightarrow K(F).$$

It induces a long exact sequence in K -theory groups and the boundary map

$$K_2(F) \xrightarrow{\partial} K_{\mathfrak{m},1}(\mathcal{O}_F) \xrightarrow{(\diamond)} K_1(\kappa) \cong \kappa^\times$$

for $\kappa := \mathcal{O}_F/\mathfrak{m}$ the residue field, is known as the *tame symbol*. An explicit formula for this map is known, see Equation 1.1. The isomorphism (\diamond) stems from devissage and is available because \mathcal{O}_F is regular. It has no counterpart for the rings R_m in general.

Lemma 4.11. *For $n := \#\mu(F)$ write $n = p^k n_0$ with $(n_0, p) = 1$. Then the diagram*

$$(4.10) \quad \begin{array}{ccc} K_2(F) & \xrightarrow{h} & \mu(F) \\ & \searrow \partial & \downarrow p^k \\ & & \kappa^\times \end{array}$$

is commutative. The downward arrow refers to taking the p^k -th power.

Proof. For any $m \mid n$ write $(x, y)_m := \frac{\text{Art}_F(x) \sqrt[m]{y}}{\sqrt[m]{y}}$ for the m -Hilbert symbol. This is the variant of the Hilbert symbol taking a root of possibly lesser order than would be maximally permitted to obtain a Kummer extension. Then $n = p^k(q - 1)$ with $q := \#\kappa$ and then $h(x, y)^{p^k} = (x, y)_n^{p^k} = (x, y)_{q-1}$ (by [FV02, Chapter IV, (5.1) Prop., (7)]) and $(x, y)_{q-1} = \partial\{x, y\}$ (by [FV02, Chapter IV, (5.3) Theorem, for n taken to be $q - 1$]). \square

Corollary 4.12. *The map $K_2(\mathcal{O}_F) \rightarrow K_2(F)$ is injective, so we may regard $K_2(\mathcal{O}_F)$ as a subgroup of $K_2(F)$. The restriction $h|_{K_2(\mathcal{O}_F)}$ is a surjective map*

$$h|_{K_2(\mathcal{O}_F)}: K_2(\mathcal{O}_F) \longrightarrow \mu(F)[p^\infty]$$

and the kernel is a divisible group.

Proof. Using the localization sequence of Equation 4.9, we obtain

$$\begin{array}{ccccccc} \dots & \xrightarrow{j} & K_2(\mathcal{O}_F) & \xrightarrow{i} & K_2(F) & \xrightarrow{\partial} & K_1(\kappa) \longrightarrow 0 \\ & & & & \downarrow h & \swarrow \text{dashed} & \\ & & & & \mu(F), & & \end{array}$$

where the dashed arrow is the Teichmüller lift, i.e. the fact that the multiplicative group of the residue field, \mathbb{F}_q^\times , can be identified isomorphically with the prime-to- p roots of unity $\langle \zeta_{q-1} \rangle$ in Equation 4.2. This is a section to the downward arrow on the right in Diagram 4.10. We see that the image of the dashed arrow amounts precisely to the prime-to- p torsion summand of $\mu(F)$. Since h is surjective, the image of $h|_{K_2(\mathcal{O}_F)}$ must surject onto the p -primary summand. Moreover, we have $j = 0$, so the map i is injective by [Wei13, Chapter V, Corollary 6.6.2]. Now suppose $\alpha \in K_2(\mathcal{O}_F)$ lies in the kernel of h . To check that α is a divisible element, say $\alpha = M\beta$ for a given $M \geq 1$, write $\alpha = M(q-1)\beta'$ for some β' in $K_2(F)$. This is possible since by Theorem 4.10 kernel elements are divisible in $K_2(F)$. As the codomain of ∂ has order $q-1$, we must have $\partial((q-1)\beta') = 1$, so $(q-1)\beta' \in K_2(\mathcal{O}_F)$. Thus, α can be written as an M -th multiple of an element in $K_2(\mathcal{O}_F)$. \square

Corollary 4.13. *Suppose p is odd. Then the group $K_2(\mathcal{O}_0)$ is divisible.*

Proof. Apply Corollary 4.12 to the field $F := F_0$. Suppose p is odd. As F_0/\mathbb{Q}_p is unramified by construction, it does not contain any non-trivial p -power roots of unity (This is classical. Observe that otherwise $\mathbb{Q}_p(\zeta_p - 1)/\mathbb{Q}_p$ is a subextension, but its minimal polynomial $((T+1)^p - 1)/T$ is Eisenstein and thus this is a non-trivial degree $p-1 > 1$ totally ramified field extension), so $\mu(F_0)[p^\infty] = 0$. The case $p = 2$ is different and the claim would be false. \square

We are ready to prove a key property for our study of the K -theory of the rings R_m . For $n := \#\mu(F)$ write $n = p^k(q-1)$ with $k \geq 0$. Below, we shall refer to the value of k on several occasions.

Lemma 4.14. *Suppose p is odd. Then for all sufficiently large $m \geq 1$ it holds that for any $u, v \in 1 + \mathfrak{m}_m$ the Hilbert symbol $h(u, v)$ vanishes. If $k \geq 1$, then any*

$$m > pe_1 + (k - 1)e$$

is sufficiently large. For $k = 0$, any $m \geq 1$ works.

Proof. Suppose $k = 0$. Then F has no non-trivial p -power roots of unity. Hence, by Corollary 4.12 the group $K_2(\mathcal{O}_F)$ is divisible. By the factorization

$$(4.11) \quad K_2(R_m) \longrightarrow K_2(\mathcal{O}_F) \xrightarrow{h} \mu(F)$$

coming from $R_m \subseteq \mathcal{O}_F$, the Hilbert symbol must vanish (the image of a divisible element is divisible, but since $\mu(F)$ is a finite group, it follows that the map $h|_{K_2(\mathcal{O}_F)}$ is zero). This case was easy. For the rest of the proof we suppose $k \geq 1$. By Lemma 4.9 we have

$$(4.12) \quad U_F^m \subseteq (U_F^1)^{p^k}.$$

Now write

$$u = 1 - px + \pi^m y \quad (x \in \mathcal{O}_0 \text{ and } y \in \mathcal{O}_F)$$

using that $\mathfrak{m}_m = p\mathcal{O}_0 + \mathfrak{m}^m\mathcal{O}_F$ by Equation 4.4. Then

$$\begin{aligned} u &= (1 - px) \cdot \left(\frac{1 - px + \pi^m y}{1 - px} \right) = (1 - px) \cdot \left(1 + \frac{\pi^m y}{1 - px} \right) \\ &= (1 - px) \cdot \left(1 + \pi^m y \sum_{l \geq 0} (px)^l \right) = (1 - px) \cdot \beta. \end{aligned}$$

Note that $\beta \in 1 + \mathfrak{m}^m\mathcal{O}_F$, i.e. $\beta \in U_F^m$. We can perform an analogous decomposition for v , say $v = (1 - py) \cdot \beta'$ with $\beta' \in U_F^m$. Then

$$\{u, v\} = \{1 - px, 1 - py\} + \{1 - px, \beta'\} + \{\beta, 1 - py\} + \{\beta, \beta'\}.$$

We now work in $K_2(F)$. As $\beta \in U_F^m$, we may write $\beta = \beta_0^{p^k}$ by Equation 4.12, and as U_F^1 is a \mathbb{Z}_p -module, even $\beta = \beta_{00}^{p^k(q-1)} = \beta_{00}^n$. The same works for β' . Thus, the term $\{1 - px, \beta'\} + \{\beta, 1 - py\} + \{\beta, \beta'\}$ is an n -th multiple of something. Therefore, under h , which is a map to a group of order n , these elements must map to zero. It remains to check that

$$h(1 - px, 1 - py) = 0,$$

but this is clear since $1 - px, 1 - py \in \mathcal{O}_0$, so our element comes from the left under the composition $K_2(\mathcal{O}_0) \rightarrow K_2(R_m) \rightarrow K_2(F)$, but $K_2(\mathcal{O}_0)$ is divisible by Corollary 4.13, and mapping a divisible element to a group of order n again must map the element to zero. This is the same idea as we had used in Equation 4.11, but with a reversed direction of inclusion, $\mathcal{O}_0 \subseteq R_m$. This finishes the proof. \square

Proposition 4.15. *Suppose p is odd. For $m \geq 1$ sufficiently big, the Hilbert symbol vanishes on the image of $K_2(R_m) \rightarrow K_2(F)$. If $k \geq 1$, then any*

$$m > pe_1 + (k - 1)e$$

is sufficiently large. For $k = 0$, any $m \geq 1$ works.

Proof. We had shown that R_m is a local ring in Lemma 4.1. By Equation 4.5 we have

$$(4.13) \quad R_m^\times \cong \langle \zeta_{q-1} \rangle \times (1 + \tilde{\mathfrak{m}}_m).$$

Now by Proposition 4.6 the Hilbert symbol vanishes on $K_2(R_m)$ once it vanishes on all pure symbols $\{a, b\}$ with $a, b \in R_m^\times$. Using the decomposition in Equation 4.13 it suffices to check this for

$$(A) \{ \zeta_{q-1}, \zeta_{q-1} \}, \quad (B) \{ \zeta_{q-1}, u \}, \quad (C) \{ u, v \}$$

with $u, v \in 1 + \tilde{\mathfrak{m}}_m$. The symbols of type (A) and (B) are zero by adapting [FV02, Chapter IX, (4.1) Lemma] to R_m . We provide the details: (B) All pure symbols $\{ \zeta_{q-1}, u \}$ with $u \in 1 + \tilde{\mathfrak{m}}_m$ are zero. To see this, note that ζ_{q-1} is $(q - 1)$ -torsion, but since $1 + \tilde{\mathfrak{m}}_m$ is a \mathbb{Z}_p -module (Lemma 4.3) and $\frac{1}{q-1} \in \mathbb{Z}_p$, we can take a root,

$$\{ \zeta_{q-1}, u \} = \{ \zeta_{q-1}, (u^{\frac{1}{q-1}})^{q-1} \} = \{ \zeta_{q-1}^{q-1}, u^{\frac{1}{q-1}} \} = 0.$$

Moreover, the symbols (A) are zero. To this end, copy the proof that $K_2(\mathbb{F}_q) = 0$, which holds up to an error in $1 + p\mathcal{O}_0$ in \mathcal{O}_0 , but $1 + p\mathcal{O}_0$ is also a \mathbb{Z}_p -module under exponentiation, so the same argument as for (B) works. Thus, it remains to consider the symbols of type (C). This case is covered by Lemma 4.14. \square

Lemma 4.16. *Suppose $\alpha \in K_2(F)$ is a divisible element in the group. Then for any $c \geq 0$ the element lies in the image of $K_2(R_c) \rightarrow K_2(F)$ induced from the inclusion $R_c \subset F$.*

Proof. Fix any integer $c \geq 2$. Pick $M := (q - 1)M_0$ such that M_0 is a sufficiently large p -power such that

$$\left(U_F^1\right)^{M_0} \subseteq U_F^c$$

holds. This is possible by Lemma 4.9. As Equation 4.2 asserts that $\mathcal{O}_F^\times \cong \langle \zeta_{q-1} \rangle \times U_F^1$, we deduce that $(\mathcal{O}_F^\times)^M \subseteq U_F^c$. As α is divisible, we find some $\beta \in K_2(F)$ such that

$$\alpha = M^3 \cdot \beta$$

holds in $K_2(F)$. We may write $\beta = \{\pi, u_0\} + \sum_{i=1}^r \{u_i, v_i\}$ for some $u_i, v_i \in \mathcal{O}_F^\times$ by Lemma 4.7. Now,

$$M^2 \cdot \{u_i, v_i\} = \{u_i^M, v_i^M\} \in \{U_F^c, U_F^c\}$$

and

$$M^3 \cdot \{\pi, u_0\} = M^2 \cdot \{\pi, u_0^M\} = M^2 \cdot \{a, b\} = \{a^M, b^M\} \in \{U_F^c, U_F^c\}$$

with $a, b \in U_F^1$ supplied by Lemma 4.8 (using that $u_0^M \in U_F^2$). Since $U_F^c = 1 + \mathfrak{m}^c \mathcal{O}_F \subseteq R_c$, we obtain that $\alpha = M^3 \beta \in \text{im } K_2(R_c)$. This proves our claim for $c \geq 2$. The smaller the c , the bigger the ring R_c , so we get the claim for all $c \geq 0$. \square

Since $(R_m, \tilde{\mathfrak{m}}_m)$ is a local ring with field of fractions F by Lemma 4.1, we get a localization fibration sequence

$$(4.14) \quad K_{\tilde{\mathfrak{m}}_m}(R_m) \longrightarrow K(R_m) \longrightarrow K(F)$$

of spectra. It induces a long exact sequence of homotopy groups

$$\cdots \longrightarrow K_2(R_m) \longrightarrow K_2(F) \xrightarrow{\partial} K_{\tilde{\mathfrak{m}}_m, 1}(R_m) \xrightarrow{\beta} K_1(R_m) \xrightarrow{\gamma} K_1(F) \longrightarrow \cdots,$$

where ∂ denotes the connecting homomorphism $\partial: K(F) \rightarrow \Sigma K_{\tilde{\mathfrak{m}}_m}(R_m)$. The greek letters serve the purpose to refer to these arrows below.

Theorem 4.17. *Suppose p is odd.*

1. *Then the sequence of subgroups*

$$\text{im } K_2(R_0) \supseteq \text{im } K_2(R_1) \supseteq \text{im } K_2(R_2) \supseteq \cdots$$

of $K_2(F)$ becomes stationary after finitely many steps. In particular, there is a unique minimal $m_0 \geq 0$ such that

$$(4.15) \quad \text{im } K_2(R_{m_0}) = \bigcap_{c \geq 0} \text{im } K_2(R_c).$$

2. The group in Equation 4.15 agrees with the kernel of the Hilbert symbol on $K_2(F)$.
3. If F/\mathbb{Q}_p is unramified, $m_0 = 0$.
4. For any $m \geq m_0$ there is a canonical isomorphism

$$\phi: K_{\tilde{m}_m,1}(R_m) \cong \mu(F)$$

such that

$$(4.16) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & K_2(R_m) & \longrightarrow & K_2(F) & \xrightarrow{\partial} & K_{\tilde{m}_m,1}(R_m) & \longrightarrow & 0 \\ & & & \searrow & \downarrow h & & \swarrow \phi & & \\ & & & 0 & \mu(F) & & & & \end{array}$$

commutes.

Proof. (Claim 1) By Proposition 4.15 for m sufficiently big we obtain that $h(\text{im } K_2(R_m)) = 0$. By Theorem 4.10 it follows that for any such m the elements in $\text{im}(K_2(R_m))$ are divisible elements in $K_2(F)$. By Lemma 4.16 any divisible element lies in $\bigcap_{c \geq 0} \text{im } K_2(R_c)$. Thus, after finitely many steps the sequence becomes stationary and then equals $\bigcap_{c \geq 0} \text{im } K_2(R_c)$. Pick m_0 to be the minimal such.

(Claim 2) Since any element in $\text{im } K_2(R_c)$ for c sufficiently large must be divisible by Proposition 4.15, but the Hilbert symbol maps to a finite group, the group in Equation 4.15 is contained in the kernel of h . Again by Theorem 4.10 the kernel of the Hilbert symbol consists of divisible elements, so Lemma 4.16 yields the reverse inclusion.

(Claim 3) If F/\mathbb{Q}_p is unramified, then $F = F_0$, so $R_0 = \mathcal{O}_F = \mathcal{O}_0$ and by Lemma 4.13 $K_2(R_0) = K_2(\mathcal{O}_0)$ is a divisible group, so already on the largest possible group in our filtration the Hilbert symbol must be trivial, i.e. $m_0 = 0$ already does the trick according to Claim 2.

(Claim 4) Since R_m is a local ring and F a field, we may use [Sri08, Example 1.6] to see that the determinant induces the downward isomorphisms

in

$$\begin{array}{ccc} K_1(R) & \xrightarrow{\gamma} & K_1(F) \\ \cong \downarrow & & \downarrow \cong \\ R^\times & \hookrightarrow & F^\times, \end{array}$$

showing that γ is injective, and thus $\beta = 0$. Hence, we obtain Figure 4.16. The top row now yields

$$K_2(F)/\text{im } K_2(R_m) \xrightarrow[\sim]{\partial} K_{\tilde{m}_m,1}(R_m).$$

However, by Claim 1 and Claim 2, the group $\text{im } K_2(R_m)$ is precisely $\ker(h)$, and the Hilbert symbol, the vertical arrow h , is surjective by Theorem 4.10, so the map ϕ exists by the universal property of cokernels and must be an isomorphism as well. \square

Definition 4.18 (Local optimal order). *If p is odd, define $\mathcal{R} := R_{m_0}$ with m_0 as in Theorem 4.17. For $p = 2$, take $\mathcal{R} := \mathcal{O}_F$. We call \mathcal{R} the optimal order in F .*

5. Global theory

We now transport the local theory to the global setting using adèles.

5.1. A family of global orders

Let F be a number field. If v is a place, we write F_v for the completion at v . The adèles are the restricted product

$$\mathbf{A}_F := \prod'_v F_v$$

over all places. If v is a finite place, let \mathcal{O}_v denote the ring of integers of F_v , otherwise $\mathcal{O}_v := F_v$.

There is an exact sequence

$$(5.1) \quad 0 \longrightarrow \mathcal{O}_F \xrightarrow{\text{diag}} F \oplus \prod_v \mathcal{O}_v \xrightarrow{\text{diff}} \mathbf{A}_F \longrightarrow 0$$

with $\text{diag}(x) := (x, x, \dots)$ the diagonal map.

Suppose $\underline{m} = (m_v)_v$ is an effective Weil divisor on $\text{Spec } \mathcal{O}_F$. We can define

$$(5.2) \quad \mathcal{R}_{\underline{m}} := F \cap \left(\prod_v R_{v,m_v} \right),$$

where the meaning of “ R_{v,m_v} ” is as follows: for a finite place we mean $R_{m_v} \subseteq \mathcal{O}_v$ in the sense of Equation 4.3, and for an infinite place we mean F_v . For the divisor $\underline{m} = 0$ we get $\mathcal{R}_{\underline{m}} = \mathcal{O}_F$ by Equation 5.1. If $\underline{m}' \geq \underline{m}$ are effective Weil divisors on $\text{Spec } \mathcal{O}_F$, we obtain a commutative diagram

$$(5.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{R}_{\underline{m}'} & \longrightarrow & F \oplus \prod_v R_{v,m'_v} & \longrightarrow & \mathbf{A}_F \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{R}_{\underline{m}} & \longrightarrow & F \oplus \prod_v R_{v,m_v} & \longrightarrow & \mathbf{A}_F \longrightarrow 0. \end{array}$$

The exactness of the rows is clear, except perhaps the surjectivity on the right. Suppose $(\alpha_v)_v \in \mathbf{A}_F$ is an adèle. Then for any integer $N \geq 1$ the vector $(\frac{1}{N}\alpha_v)_v$ is also an adèle, because the adèle ring \mathbf{A}_F is a \mathbb{Q} -algebra. Since the right arrow in Equation 5.1 is surjective, we find $\beta \in F$ and $(\beta_v)_v \in \prod \mathcal{O}_v$ such that $(\frac{1}{N}\alpha_v)_v = (\beta - \beta_v)_v$. In particular, $(N\beta - N\beta_v)_v$ maps to $(\alpha_v)_v$, but $N\beta \in F$ and once N is sufficiently divisible by prime factors lying below those places with $m_v \geq 0$, we get $N\beta_v \in R_{v,m_v}$ for all v .

Lemma 5.1. *The set $\mathcal{R}_{\underline{m}} \subseteq \mathcal{O}_F$ is a one-dimensional Noetherian domain of finite index $[\mathcal{O}_F : \mathcal{R}_{\underline{m}}] < \infty$ and with field of fractions F . It is a finitely generated free \mathbb{Z} -module of rank $[F : \mathbb{Q}]$.*

Proof. Harmless. Consider the quotient of the bottom row by the top row in Diagram 5.3 for the choice $\underline{m} = 0$ and \underline{m}' equal to the \underline{m} in the claim of this lemma. The resulting downward injections are an isomorphism on F and \mathbf{A}_F , so $[\mathcal{O}_F : \mathcal{R}_{\underline{m}}] = \prod_v [\mathcal{O}_v : R_{v,m_v}] < \infty$, showing that \mathcal{O}_F is a finite $\mathcal{R}_{\underline{m}}$ -algebra. It follows that $\mathcal{R}_{\underline{m}}$ is a Noetherian one-dimensional ring. For any $x \in F$, we can write $x = \frac{a}{b}$ with $a, b \in \mathcal{O}_F$ and then $x = \frac{ga}{gb}$ with $ga, gb \in \mathcal{R}_{\underline{m}}$, so $\text{Frac } \mathcal{R}_{\underline{m}} = F$. □

Now fix \underline{m} . Let $\underline{\gamma}$ be a further effective Weil divisor on $\text{Spec } \mathcal{O}_F$. Define

$$I_{\underline{\gamma}} := F \cap \left(\prod_v \tilde{\mathfrak{m}}_v^{\gamma_v} \right),$$

where at the finite places $\tilde{\mathfrak{m}}_v$ refers to the unique maximal ideal $\tilde{\mathfrak{m}}_{m_v}$ of the local ring R_{v,m_v} , and we let $\tilde{\mathfrak{m}}_v := F_v$ at the infinite places. Since $\underline{\gamma}$ is a Weil

divisor, we have $\tilde{\mathfrak{m}}_v^\gamma = (1)$ for all but finitely many places. It is clear that $I_\gamma \subseteq \mathcal{R}_{\underline{m}}$ is an ideal. We obtain a diagram, similar to Diagram 5.3,

$$(5.4) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & I_\gamma & \longrightarrow & F \oplus \prod_v \tilde{\mathfrak{m}}_v^\gamma & \longrightarrow & \mathbf{A}_F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & \mathcal{R}_{\underline{m}} & \longrightarrow & F \oplus \prod_v R_{v,m_v} & \longrightarrow & \mathbf{A}_F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & \mathcal{R}_{\underline{m}}/I_\gamma & \xrightarrow{\sim} & \prod_v R_{v,m_v}/\tilde{\mathfrak{m}}_v^\gamma, & & & & \end{array}$$

where the downward arrows between the first rows are just the inclusions. The exactness of the middle row holds for the same reason as before, since for any $c \geq 1$ we can also pick N sufficiently divisible such that $N\beta_v \in \tilde{\mathfrak{m}}_v^c$. The downward arrows induce isomorphisms on the summand F as well as on \mathbf{A}_F . It follows that the bottom horizontal arrow is indeed an isomorphism. Since $\tilde{\mathfrak{m}}_v^\gamma = (1)$ for all but finitely many places, the product in the bottom row is over a finite set, and moreover we deduce that $\mathcal{R}_{\underline{m}}/I_\gamma$ is a finite ring.

Lemma 5.2. *We keep \underline{m} fixed. The inclusion of $\mathcal{R}_{\underline{m}}$ into the product coming from Equation 5.2 induces an isomorphism of topological rings*

$$(5.5) \quad \varprojlim_{\underline{\gamma}} \mathcal{R}_{\underline{m}}/I_{\underline{\gamma}} \cong \prod_v R_{v,m_v},$$

where the finite rings $\mathcal{R}_{\underline{m}}/I_{\underline{\gamma}}$ are equipped with the discrete topology, $\underline{\gamma}$ runs through all effective Weil divisors on $\text{Spec } \mathcal{O}_F$ (partially ordered by \leq), the limit is given the inverse limit topology, R_{v,m_v} is taken with its natural subspace topology from Lemma 4.4, and \prod is given the product topology. Moreover, on the left side the inverse limit agrees with the profinite completion of $\mathcal{R}_{\underline{m}}$, seen as an abelian group.

Proof. The bottom isomorphism in Diagram 5.4 yields isomorphisms

$$(5.6) \quad \mathcal{R}_{\underline{m}}/I_{\underline{\gamma}} \cong \prod_v R_{v,m_v}/\tilde{\mathfrak{m}}_v^\gamma$$

and both sides carry the discrete topology (also on the right, using that $\tilde{\mathfrak{m}}_v^\gamma$ is clopen in R_{v,m_v}). Thus, running over the inverse system of effective Weil divisors, we have levelwise isomorphisms, and thus obtain an isomorphism in the limit. As \prod_v is a limit itself, limits commute with each other and

$\varprojlim_c R_{v,m_v}/\tilde{\mathfrak{m}}_v^c = R_{v,m_v}$, the first claim follows. Note that the topology on R_{v,m_v} comes from the valuation on \mathcal{O}_v , which can be sandwiched with the filtration by powers of $\tilde{\mathfrak{m}}_v^c$, so the inverse limit topology agrees with the valuation topology, and R_{v,m_v} is adically complete (it is clopen in \mathcal{O}_v by Lemma 4.4 and compact subspaces of complete metric spaces are themselves complete). Finally, to see that this inverse limit agrees with the profinite completion, we just need to use that each $\mathcal{R}_{\underline{m}}/I_{\underline{\gamma}}$ is a finite quotient, and conversely for any finite quotient $\mathcal{R}_{\underline{m}}/N$ (as an abelian group!), we can pick M to be the exponent and then the Weil divisor of M on \mathbb{Z} pulls back to a divisor on $\text{Spec } \mathcal{O}_F$ such that $I_{\underline{\gamma}} \subseteq (N)\mathcal{O}_F$. Thus, the $I_{\underline{\gamma}}$ are a cofinal family. \square

The ring $\mathcal{R}_{\underline{m}}$ is one-dimensional Noetherian, so every non-zero prime ideal is maximal. In particular, any two distinct non-zero prime ideals are automatically coprime. Thus, the Chinese Remainder Theorem yields isomorphisms $\mathcal{R}_{\underline{m}}/I_{\underline{\gamma}} \cong \prod_{\mathcal{I}} \mathcal{R}_{\underline{m}}/\mathcal{I}$, where \mathcal{I} runs through the minimal primes of the support of $\mathcal{R}_{\underline{m}}/I_{\underline{\gamma}}$, viewed as an $\mathcal{R}_{\underline{m}}$ -module. In the inverse limit, this yields a ring isomorphism

$$(5.7) \quad \varprojlim_{\underline{\gamma}} \mathcal{R}_{\underline{m}}/I_{\underline{\gamma}} \cong \prod_{\mathcal{P}} F_{\mathcal{P}}, \quad \text{where} \quad F_{\mathcal{P}} := \varprojlim_i \mathcal{R}_{\underline{m}}/\mathcal{I}_{\mathcal{P},i},$$

where \mathcal{P} runs through the maximal ideals of $\text{Spec } \mathcal{R}_{\underline{m}}$ and $\mathcal{I}_{\mathcal{P},i}$ is a family of ideals such that the finite modules $\mathcal{R}_{\underline{m}}/\mathcal{I}_{\mathcal{P},i}$ have support equal to the point \mathcal{P} in $\text{Spec } \mathcal{R}_{\underline{m}}$. Analogous to Lemma 5.2, the isomorphism in Equation 5.7 is seen to be a homeomorphism.

The product decompositions in Equations 5.5 and 5.7 are indexed over different sets: Maximal ideals of $\text{Spec } \mathcal{O}_F$ (namely the finite places v) versus the maximal ideals \mathcal{P} of $\text{Spec } \mathcal{R}_{\underline{m}}$.

Proposition 5.3. *The product decomposition of Equation 5.5 is a refinement of the decomposition in Equation 5.7: The factors R_{v,m_v} do not admit a further non-trivial decomposition as a product ring, while the $F_{\mathcal{P}}$ split as a finite direct product of rings, and the factors correspond (in a fashion which can be made canonical) to finite places v of \mathcal{O}_F lying over \mathcal{P} .*

To prove this, we need to discuss the map

$$j_{\underline{m}}: \text{Spec } \mathcal{O}_F \longrightarrow \text{Spec } \mathcal{R}_{\underline{m}}.$$

This is a finite morphism and by the same argument as in Lemma 4.5 it is the normalization map. For all maximal ideals \mathfrak{p} of \mathcal{O}_F coprime to the conductor ideal the map is a local isomorphism, i.e. $(\mathcal{R}_{\underline{m}})_{\mathfrak{p} \cap \mathcal{R}_{\underline{m}}} \xrightarrow{\sim} (\mathcal{O}_F)_{\mathfrak{p}}$.

For the finitely many maximal ideals not coprime to the conductor all sorts of phenomena can occur. It might be useful to think of the geometric analogue of a singular curve and a cusp resp. nodal singularity exhibiting analogous phenomena. We recall a crucial principle:

Proposition 5.4 ([Die67, Theorem 6.5]). *Suppose (R, \mathfrak{m}) is an excellent reduced Noetherian local ring. Then there is a canonical bijection between the maximal ideals of the normalization R' and the minimal primes of the completion $\widehat{R}_{\mathfrak{m}}$.*

We will not explain the precise construction of the bijection here. We also recall:

Lemma 5.5 ([HS06, Proposition 4.3.2]). *Suppose (R, \mathfrak{m}) is a complete local Noetherian ring and S a finite R -algebra. Then S is semilocal and there is an isomorphism of rings $S \cong \prod S_{\mathfrak{q}}$, where \mathfrak{q} runs through the maximal ideals of S .*

Proof of Prop. 5.3. As $j_{\underline{m}}$ is the normalization, so is the base change to any local ring $j_{\underline{m}} \otimes_{\mathcal{R}_{\underline{m}}} \mathcal{R}_{\underline{m}, \mathcal{P}}$ for a maximal ideal \mathcal{P} of $\mathcal{R}_{\underline{m}}$. Then Proposition 5.4 applies. The completion side, i.e. $(\widehat{\mathcal{R}_{\underline{m}, \mathcal{P}}})_{\mathcal{P}}$, then corresponds to a factor $F_{\mathcal{P}} = \varprojlim_i \mathcal{R}_{\underline{m}} / \mathcal{I}_{\mathcal{P}, i}$ in Equation 5.7. The proposition therefore identifies the minimal primes of $F_{\mathcal{P}}$ with the maximal ideals of the normalization. However, the latter exactly correspond to the points in the fiber of the normalization map $j_{\underline{m}}$ over the point $\mathcal{P} \in \text{Spec } \mathcal{R}_{\underline{m}}$. As $F_{\mathcal{P}}$ is a finite $(\widehat{\mathcal{R}_{\underline{m}, \mathcal{P}}})_{\mathcal{P}}$ -algebra, Lemma 5.5 yields a further decomposition. By lifting of idempotents, the factors correspond to those of $S/\mathfrak{m}S$ in the cited lemma, in particular each maximal ideal in Lemma 5.5 corresponds to a unique minimal ideal of Proposition 5.4. The factors R_{v, m_v} are domains, so they do not permit non-trivial decompositions as a product of rings. \square

Definition 5.6 (Global optimal order). *Suppose v is a finite place. Write $\mathcal{R}_v \subseteq \mathcal{O}_v$ for the optimal order according to Definition 4.18. For infinite places v , define $\mathcal{R}_v := \mathcal{O}_v = F_v$. We define*

$$\mathcal{R} := F \cap \left(\prod_v \mathcal{R}_v \right).$$

We call \mathcal{R} the optimal order in F .

Of course \mathcal{R} is an order of the type $\mathcal{R}_{\underline{m}}$ for \underline{m} suitably chosen.

5.2. K -theory computations

We shall imitate Gillet’s proof of Weil reciprocity, but with the twist of exploiting the special feature that our order, while typically not regular, is sufficiently small to make the wild part of the Hilbert symbol visible in the boundary maps.

As before, we refer to Appendix §A for the necessary background on Algebraic K -theory. Let $I = (N)$ be the principal ideal generated by some squarefree natural number $N \geq 1$. As a special instance of Equation A.2 we get the localization sequence

$$(5.8) \quad K_I(\mathcal{R}) \longrightarrow K(\mathcal{R}) \longrightarrow K(\mathcal{R}[\frac{1}{N}]).$$

We may factor $I = (p_1 p_2 \cdots p_r)$ into prime numbers, so that each pair $(p_i) + (p_j) = 1$ is coprime for $i \neq j$. Thus, relying on Equation A.3, we get

$$\bigoplus K_{(p_i)}(\mathcal{R}) \longrightarrow K(\mathcal{R}) \longrightarrow K(\mathcal{R}[\frac{1}{N}]).$$

These sequences sit in a family for $N \mid N'$. As K -theory commutes with filtering colimits, we obtain the fiber sequence

$$\bigoplus_p K_{(p)}(\mathcal{R}) \longrightarrow K(\mathcal{R}) \longrightarrow K(F),$$

where the sum runs over all prime numbers and we have used that $F = \text{colim}_N \mathcal{R}[\frac{1}{N}]$ (since $F = \mathbb{Q} \cdot \mathcal{R}$). For the next step, we invoke the equivalence of Equation A.4, showing that $K_{(p)}(\mathcal{R})$ agrees with $K_{(p)}(\widehat{\mathcal{R}}_p)$ of the completion. But $\widehat{\mathcal{R}}_p$ decomposes at the product over the completion of \mathcal{R} over all finite places v over p by Proposition 5.3, and combined with Lemma 5.2 this yields

$$K_{(p)}(\mathcal{R}) \cong \prod_{v|p} K_{\mathfrak{m}_v}(\mathcal{R}_v),$$

where v runs through the finite places of F lying over the prime number p , and \mathfrak{m}_v refers to the unique maximal ideal of \mathcal{R}_v . Plugging this into the previous fiber sequence, we arrive at the fiber sequence

$$(5.9) \quad \bigoplus_v K_{\mathfrak{m}_v}(\mathcal{R}_v) \longrightarrow K(\mathcal{R}) \longrightarrow K(F).$$

Remark 5.7. Instead of our somewhat peculiar choice of natural numbers N in Equation 5.8 we could also have taken any zero-dimensional ideal I in \mathcal{R} , the fiber sequence

$$K_I(\mathcal{R}) \longrightarrow K(\mathcal{R}) \longrightarrow K(\text{Spec } \mathcal{R} - \text{Spec } \mathcal{R}/I)$$

and then taken the filtering colimit over all such ideals partially ordered by inclusion. This yields the same colimit in Equation 5.9 (since the family of ideals N with $N \in \mathbb{Z}_{\geq 1}$ is a cofinal subsystem).

Now consider the long exact sequence of homotopy groups attached to Equation 5.9. Around π_1 and π_2 it specializes to

$$\begin{aligned} K_2(\mathcal{R}) \longrightarrow K_2(F) \longrightarrow \bigoplus_v K_{\tilde{m}_v,1}(\mathcal{R}_v) \longrightarrow K_1(\mathcal{R}) \xrightarrow{\iota} \dots \\ \dots \xrightarrow{\iota} K_1(F) \longrightarrow \bigoplus_v K_{\tilde{m}_v,0}(\mathcal{R}_v) \longrightarrow K_0(\mathcal{R}) \longrightarrow K_0(F) \longrightarrow \dots \end{aligned}$$

Note that since we use non-connective K -theory, we do not per se know that the sequence terminates in three π_0 -groups, there might well be negative homotopy groups. This need not concern us however. The middle part at the arrow ι is easy to analyze. We recall the definition of SK_1 :

$$SK_1(R) := \ker(\det : K_1(R) \longrightarrow R^\times).$$

The arrow ι thus induces a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & SK_1(\mathcal{R}) & \longrightarrow & K_1(\mathcal{R}) & \xrightarrow{\det} & \mathcal{R}^\times \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & SK_1(F) & \longrightarrow & K_1(F) & \xrightarrow{\det} & F^\times \longrightarrow 1. \end{array}$$

The horizontal surjections on the right are split. As F is a field, we have $SK_1(F) = 1$. We deduce that the sequence is spliced from two exact sequences (5.10)

$$0 \longrightarrow \mathcal{R}^\times \longrightarrow K_1(F) \longrightarrow \bigoplus_v K_{\tilde{m}_v,0}(\mathcal{R}_v) \longrightarrow K_0(\mathcal{R}) \longrightarrow K_0(F) \longrightarrow \dots$$

and

$$(5.11) \quad \dots \longrightarrow K_2(\mathcal{R}) \longrightarrow K_2(F) \longrightarrow \bigoplus_v K_{\tilde{m}_v,1}(\mathcal{R}_v) \longrightarrow SK_1(\mathcal{R}) \longrightarrow 1.$$

Although interesting in its own right, we have no immediate use for Equation 5.10. Equation 5.11 yields

$$(5.12) \quad \cdots \longrightarrow K_2(\mathcal{R}) \longrightarrow K_2(F) \xrightarrow{\partial} \bigoplus_v K_{\mathfrak{m},1}(\mathcal{R}_v) \longrightarrow SK_1(\mathcal{R}) \longrightarrow 1.$$

From Theorem 4.17 we know that the boundary map ∂ on each summand v agrees canonically with the full Hilbert symbol at the place v .

Below, we shall use the term “up to 2-primary torsion”. It means that we work in the abelian category

$$(5.13) \quad Q := \frac{\text{AbelianGroups}}{2\text{-primary-torsion}}.$$

To set this up in detail note that the 2-primary torsion abelian groups form a Serre subcategory in all abelian groups, so the above quotient exists and is an abelian category itself. A complex of abelian groups becomes exact in this quotient category if and only if its homology is made of 2-primary torsion groups. For example, a map $A \rightarrow B$ is a monomorphism in the quotient category if and only if its kernel is 2-primary torsion.

As an alternative characterization, a complex of abelian groups becomes exact in Q if and only if becomes exact when tensoring it with the flat \mathbb{Z} -module $\mathbb{Z}[\frac{1}{2}]$. In fact, the category Q is equivalent to the category of $\mathbb{Z}[\frac{1}{2}]$ -modules.

Theorem 5.8. *Suppose F is any number field. Suppose \mathcal{R} denotes the optimal order. Then, in abelian groups up to 2-primary torsion, the diagram*

$$\begin{array}{ccccccc} K_2(\mathcal{R}) & \longrightarrow & K_2(F) & \xrightarrow{\partial} & \bigoplus_v \text{finite} & K_{\mathfrak{m},1}(\mathcal{R}_v) & \longrightarrow SK_1(\mathcal{R}) \longrightarrow 0 \\ & & \parallel & & & \downarrow \oplus_v \phi & \downarrow \tau \\ & & K_2(F) & \xrightarrow{h_v} & \bigoplus_v \text{noncomplex} & \mu(F_v) & \xrightarrow{\frac{m_v}{m}} \mu(F) \longrightarrow 0, \end{array}$$

commutes, has exact rows, and the downward arrows are isomorphisms. In particular, the Hilbert reciprocity law gets identified with a sequence coming directly from a K -theory localization sequence (up to 2-primary torsion). We deduce

$$SK_1(\mathcal{R}) \cong \mu(F),$$

up to 2-primary torsion.

Recall that it is known that $SK_1(\mathcal{O}_F) = 0$ by the work of Bass–Milnor–Serre [BMS67].

Proof. The exactness of the top row comes from the localization sequence. The bottom row is the Moore sequence (Theorem 2.1). We employ Theorem 4.17 for all finite places over odd primes, showing that each such ϕ is an isomorphism. Finite places over $p = 2$ have the full (local) ring of integers as their local optimal order. Thus, at these the boundary map is just the tame symbol and not the honest Hilbert symbol. But the tame symbol misses only the 2-primary torsion summand of $\mu(F_v)$ as we are over $p = 2$, which disappears after inverting two. Complex places play no role in the Hilbert reciprocity law. Real places contribute factors $\{\pm 1\}$, which also disappear by inverting two. Thus, even though in principle the sum of the bottom row should also include real places, it makes no difference to consider only the finite places here as well. The snake lemma implies the claim about SK_1 . \square

Corollary 5.9. *Let F be a number field with $\sqrt{-1} \notin F$. Then the statement of Hilbert reciprocity up to sign, i.e.*

$$\prod_{v \text{ noncomplex}} h_v(\alpha, \beta)^{\frac{m_v}{m}} = \pm 1 \quad \text{for all } \alpha, \beta \in F^\times,$$

can be phrased as the property of being a complex ($d^2 = 0$) for a localization sequence in K -theory.

Proof. Theorem 5.8 shows that the said product is zero in $\mu(F) [\frac{1}{2}]$, but by our assumption the 2-torsion part of $\mu(F)$ is only $\{\pm 1\}$. \square

One might also ask about the kernel of the leftmost arrow in Moore’s sequence. It is known as the *wild kernel* WK_2 :

$$0 \longrightarrow WK_2(F) \longrightarrow K_2(F) \longrightarrow \bigoplus_{v \text{ noncomplex}} \mu(F_v) \xrightarrow{\frac{m_v}{m}} \mu(F) \longrightarrow 0.$$

Following earlier work of Tate, Hutchinson has proven that $WK_2(F)/K_2(F)_{div} \in \{0, \mathbb{Z}/2\}$ along with a precise criterion which case occurs for what number fields F (both cases occur), [Hut01, Hut04]. Note that our Theorem 5.8 shows that $K_2(\mathcal{R})$ surjects onto the wild kernel after inverting 2. This suggests that one can probably also define the global optimal order analogously to Theorem 4.17 as the largest \mathbb{Z} -order in \mathcal{O}_F such that $K_2(\mathcal{R}) [\frac{1}{2}]$ is divisible. This would mirror our characterization of local optimal orders in a rather neat way.

We shall explain the relation to the picture of Kapranov–Smirnov [KS95] in a future text.

6. Is this a self-contained proof of Hilbert reciprocity?

Given that Gillet’s proof of Hilbert reciprocity for function fields uses nothing more than localization, the reader might (and should) ask to what extent the method in this text gives a self-contained proof in the number field case. First, we have freely used the definition of the Hilbert symbol, which itself uses local class field theory. Using only this, we then could establish the exactness of

$$K_2(F) \xrightarrow{\partial} \bigoplus_{v \text{ noncomplex}} \mu(F_v) \longrightarrow SK_1(\mathcal{R}) \longrightarrow 0$$

up to inverting 2. The issues at $p = 2$, a familiar nightmare of any number theorist (and homotopy theorist...) do not appear easy to remove. At places over $p = 2$ Theorem 4.17 fails, and the real places do not even possess a valuation ideal (how to pinch a singularity in the order when there is no maximal order?). Let us accept these shortcomings. Then the more crucial issue is that our Hilbert reciprocity law is about the vanishing of terms in the group $SK_1(\mathcal{R})$, which remains elusive without further work (see Remark 6.1).

This is really similar to a completely different approach to Hilbert reciprocity originating from the papers [AB19, BHv21]. These papers produce, if we only use local class field theory as input, an exact sequence

$$(6.1) \quad K_2(F) \xrightarrow{\partial} \bigoplus_v \mu(F_v) \longrightarrow K_2(\mathbf{LCA}_F)_{/div} \longrightarrow 0,$$

(literally, without having to invert 2), where v runs over all non-complex places and \mathbf{LCA}_F is the category of locally compact topological F -vector spaces, $G_{/div}$ refers to G/G_{div} , the quotient by the subgroup of divisible elements. The exactness of Equation 6.1 gets proven loc. cit. in an *entirely different fashion* than the methods of the present text. There are no singular orders involved. Even though localization sequences play a role in the proof, one never localizes with respect to zero-dimensional subschemes. Loc. cit. does *not* realize the Hilbert symbol as a boundary map. Sequence 6.1 also turns out to be equivalent to Moore’s sequence [BHv21, Corollary 9.6], but again with the issue that the identification $K_2(\mathbf{LCA}_F)_{/div} \cong \mu(F)$ remains elusive without using global class field theory.

Remark 6.1. I see several possible approaches to compute $SK_1(\mathcal{R})$. In this paper, in Theorem 5.8 we have just used Moore’s sequence. This was easy, but would be circular to be an independent approach to Hilbert reciprocity. Alternatively, I would believe one could use the work of Bass–Milnor–Serre on the congruence subgroup problem [BMS67]. All elements of $SK_1(\mathcal{R})$ can be

expressed through Mennicke symbols associated to invertible ideals. As \mathcal{R} is generally not Dedekind, finitely many ideals will fail to be invertible, but the others suffice to generate the group. However, the next step then would be to find relations between these Mennicke symbols, which would probably require to connect these to power reciprocity symbols as in Bass–Milnor–Serre and then again rely on properties also equivalent to already having Hilbert reciprocity available. Another idea might be Iwasawa theory, because inspecting the corresponding group in Gillet’s proof in the function field case is linked to the Jacobian of the curve after base extension to the algebraic closure. Iwasawa has taught us that the mixed characteristic counterpart of this should be \mathbb{Z}_p -extensions. But again, relying on Iwasawa theory very quickly necessitates relying on tools from global class field theory. This remains to be investigated.

7. Applications & complements

7.1. Metaplectic extensions

Using the above results, one can provide a quick construction of the local and global metaplectic coverings, including the proof of the reciprocity property of the global covering vaguely resembling Gillet’s proof of Weil reciprocity (§3). We briefly recall the background.

The classical theta function is given by

$$\theta(z) := \sum_{n \in \mathbb{Z}} q^{n^2} \quad \text{with} \quad q := e^{2\pi iz}$$

for z in the complex upper half-plane. Any even power θ^{2m} is a modular form of weight m for $\Gamma_0(4)$ and a certain nebentype. The odd powers therefore were classically understood as something like a half-integral weight modular form, having a more complicated transformation behaviour involving the quadratic residue symbol. Weil [Wei64] then proposed to view these odd powers, and in particular θ itself, as automorphic forms on a central extension

$$(7.1) \quad 1 \longrightarrow \mu_2 \longrightarrow \widehat{G} \longrightarrow \mathrm{GL}_2 \longrightarrow 1$$

as opposed to living on the original group GL_2 . The appearance of quadratic residue symbols in the transformation formulas then translates to the defining cocycle $H_{\mathrm{grp}}^2(\mathrm{GL}_2, \mu_2)$ be given in terms of these. However, once thinking about such central extensions, one can construct more complicated ones. Following Kubota [Kub69], if F is a number field and \mathbf{A}_F its adèle ring, one can

construct a central extension

$$(7.2) \quad 1 \longrightarrow \mu(F) \longrightarrow \widehat{G} \longrightarrow \mathrm{GL}_2(\mathbf{A}_F) \longrightarrow 1,$$

which for $F = \mathbb{Q}$ specializes to Equation 7.1. In fact its restriction to $\mathrm{SL}_2(\mathbf{A}_F) \subset \mathrm{GL}_2(\mathbf{A}_F)$ is the universal topological central extension of SL_2 (we will not explain this in detail, but using the topology of the adèles, $\mathrm{SL}_2(\mathbf{A}_F)$ is naturally a locally compact group and one can consider the category of central extensions of locally compact topological groups. Algebraically, the universal central extension is strictly bigger, [Moo68]). The extension in Equation 7.2 has the following property: If $F \hookrightarrow \mathbf{A}_F$ is the diagonal embedding, the pullback of the central extension trivializes (i.e. it splits). This property turns out to be equivalent to Hilbert reciprocity. The pullback of the extension along the inclusion of a local field $F_v \hookrightarrow \mathbf{A}_F$ gives rise to local counterparts of the metaplectic extension. The theory of metaplectic extensions has branched into several separate lines of development [KPS84, BD01]. We just follow one thread here.

Suppose R is any ring. Then the group of elementary matrices $\mathrm{E}(R) \subseteq \mathrm{SL}(R)$ has a universal central extension, the *Steinberg group* $\mathrm{St}(R)$. Its center agrees with the group $K_2(R)$.

$$(7.3) \quad 1 \longrightarrow K_2(R) \longrightarrow \mathrm{St}(R) \longrightarrow \mathrm{E}(R) \longrightarrow 1.$$

Thus, for any group homomorphism $\gamma : K_2(R) \rightarrow A$ to an abelian group A , one can take the pushout of this central extension along γ and get a new central extension of $\mathrm{E}(R)$. If we describe the isomorphism class of a central extension of $\mathrm{E}(R)$ by $K_2(R)$ through its group cocycle in $H_{\mathrm{grp}}^2(\mathrm{E}(R), K_2(R))$, this operation just amounts to the functoriality in coefficients

$$\gamma_* : H_{\mathrm{grp}}^2(\mathrm{E}(R), K_2(R)) \longrightarrow H_{\mathrm{grp}}^2(\mathrm{E}(R), A).$$

For F/\mathbb{Q}_p a finite extension with p odd, take $R := F$ and $(\mathcal{R}, \tilde{\mathfrak{m}})$ the local optimal order. We may use Theorem 4.17 and the boundary map $\partial : K_2(F) \rightarrow K_{\tilde{\mathfrak{m}},1}(\mathcal{R}_m)$ to get a central extension

$$(7.4) \quad 1 \longrightarrow \mu(F) \longrightarrow \widehat{G} \longrightarrow \mathrm{SL}(F) \longrightarrow 1$$

because for fields the inclusion $\mathrm{E}(F) \subseteq \mathrm{SL}(F)$ is the identity. One can now pull this back along $\mathrm{SL}_n(F) \hookrightarrow \mathrm{SL}(F)$ to obtain the classical Kubota metaplectic

extension for $n = 2$. The story for GL_n is more complicated in general, but one can get Kubota’s extension for GL_2 by pulling back along

$$GL_2(F) \hookrightarrow SL_3(F) \quad M \mapsto \begin{pmatrix} M & \\ & \det(M)^{-1} \end{pmatrix}.$$

We can also do all this globally. Suppose F is a number field and \mathbf{A}_F its adèle ring. It is easy to set up a map

(7.5)

$$\begin{array}{ccc} K_2(\mathbf{A}_F) & \longrightarrow & \left\{ (\alpha_v)_v \in \prod_v K_2(F_v) \mid \begin{array}{l} \text{with } \alpha_v \in \text{im } K_2(\mathcal{O}_v) \text{ for all } v \\ \text{but finitely many } v \end{array} \right\} \\ & & \downarrow \partial \\ & & \bigoplus_v K_{\mathfrak{m},1}(\mathcal{R}_v) \\ & & \downarrow \\ & & SK_1(\mathcal{R}), \end{array}$$

where the first downward arrow uses the boundary map with respect to the local optimal order and the adelic finiteness condition ensures that the image lands in a direct sum. The map to SK_1 then comes from our global formalism.

Now push out Equation 7.3 for $R := \mathbf{A}_F$ by the map in Equation 7.5 to obtain a central extension

$$(7.6) \quad 1 \longrightarrow SK_1(\mathcal{R}) \longrightarrow \widehat{G} \longrightarrow SL(\mathbf{A}_F) \longrightarrow 1.$$

By Theorem 5.8 up to a zig-zag of isogenies of 2-power orders, this extension agrees with the global metaplectic extension (as an elaboration: The isomorphism $SK_1(\mathcal{R}) [\frac{1}{2}] \cong \mu(F) [\frac{1}{2}]$ implies the existence of isogenies of abelian groups with 2-power order. Pushouts along these maps on the centers of the extension provide isogenies between the middle terms \widehat{G} of the same order). Moreover, this construction trivially has all the properties as in Kubota’s construction. Pulling back the extension in Equation 7.6 along $F_v \hookrightarrow \mathbf{A}_F$ (for v any finite place over an odd prime) retrieves the local metaplectic extension of Equation 7.4. The pullback along the diagonal embedding $\iota: F \hookrightarrow \mathbf{A}_F$ trivializes.

To see the latter, we only need to understand the isomorphism class of the extension, i.e. we can work with the group 2-cocycles. The pullback amounts to pulling back the cocycle, i.e if γ denotes the extension in Equation 7.6, we

consider

$$\iota^* : H_{\text{grp}}^2(\text{SL}(\mathbf{A}_F), SK_1(\mathcal{R})) \longrightarrow H_{\text{grp}}^2(\text{SL}(F), SK_1(\mathcal{R})),$$

but the cocycle came from pushing out along a map on the level of K_2 , which for the pullback $\iota^*\gamma$ can be factored as

$$K_2(F) \longrightarrow K_2(\mathbf{A}_F) \xrightarrow{\text{Eq. 7.5}} SK_1(\mathcal{R}),$$

but this is the composition of two successive arrows in the localization sequence (Equation 5.12). Thus, $\iota^*\gamma$ is the zero cohomology class. This class corresponds to a trivial central extension. And, in view of Theorem 5.8 this property is again (up to the 2-power isogeny) equivalent to Hilbert reciprocity. Note that, without worrying about 2, the central extension by $SK_1(\mathcal{R})$ exists unconditionally and trivializes after pulling back to $\text{SL}(F)$. This is some variant of Hilbert reciprocity and of the usual metaplectic covering.

Remark 7.1 ([BHv21]). The group on the upper right in Diagram 7.5 can be shown to be isomorphic to $K_2(\text{LCA}_F)$. The horizontal arrow then merely amounts to be induced by the functor sending the projective generator \mathbf{A}_F to itself, equipped with its standard locally compact topology. Moreover, the downward arrow, here denoted by ∂ , can be characterized as the cokernel under quotienting out the subgroup of divisible elements in the source group.

Appendix A. K -theory in the singular situation

In this section we summarize some background on Algebraic K -theory, focussing on what we need in this paper. As we will specifically deal with non-regular rings, we need to use a sufficiently broad framework to handle such singular situations. We will use the framework of Thomason–Trobeaugh [TT90].

For a scheme⁶ we write $K(X)$ for its non-connective K -theory spectrum. This can be defined as the non-connective K -theory of the category of perfect complexes on X , or equivalently as the one of vector bundles on X . We mostly deal with rings, so following common practice we write $K(R) := K(\text{Spec } R)$ for the K -theory of a ring. Equivalently, this could be defined as the K -theory of the category of finitely generated projective R -modules.

If I is an ideal in R , we write $K_I(R)$ for what would otherwise be denoted by

$$K_{\text{Spec } R/I}(R) \quad \text{or} \quad K(\text{Spec } R \text{ on } \text{Spec } R/I).$$

⁶We tacitly assume that all our schemes are Noetherian and separated over \mathbb{Z} .

This can be defined as the K -theory of perfect complexes of R -modules which become acyclic after pulling them back to the open $\text{Spec } R - \text{Spec } R/I$.

We shall only need a few general facts: First, there is the general localization theorem, a fiber sequence of spectra

$$(A.1) \quad K_I(R) \longrightarrow K(R) \longrightarrow K(\text{Spec } R - \text{Spec } R/I).$$

This holds for any commutative unital ring R and any ideal I . This is [TT90, Theorem 7.4]. If $I = (f)$ is a principal ideal, the last term agrees with $K(\text{Spec } R[f^{-1}])$. In this case $K_I(R)$ can also be defined as the K -theory of perfect complexes P_\bullet whose (exact) basechange $P_\bullet \otimes_R R[f^{-1}]$ is acyclic:

$$(A.2) \quad K_I(R) \longrightarrow K(R) \longrightarrow K\left(R \left[\frac{1}{f} \right] \right).$$

We return to the general case. If I, J are coprime ideals, i.e. $I + J = R$, then

$$(A.3) \quad K_{IJ}(R) \cong K_I(R) \oplus K_J(R).$$

This is [TT90, Corollary 8.1.4]: As the ideals are coprime, we have $\text{Spec } R/I \cap \text{Spec } R/J = \emptyset$, when regarded as closed subschemes in $\text{Spec } R$, so the upper left corner in the cartesian square loc. cit. is the zero spectrum.

We also need a certain compatibility with completions. Suppose R is a Noetherian ring and I any ideal. Then there is an equivalence

$$(A.4) \quad K_I(R) \xrightarrow{\sim} K_{I\widehat{R}_I}(\widehat{R}_I),$$

where \widehat{R}_I denotes the I -adic completion of R , and $I\widehat{R}_I$ is the extended ideal. This is [TT90, Proposition 3.19, in the format of Exercise 3.19.2].

Finally, if R is a Noetherian regular ring, then the inclusion of the category of finitely generated projective R -modules into the category of all finitely generated R -modules induces an equivalence

$$K(R) \xrightarrow{\sim} K(\text{Mod}_{fg}(R)).$$

In this situation we have devissage: Suppose R is a regular Noetherian ring such that R/I is also regular. Then

$$K_I(R) \xrightarrow{\sim} K(R/I).$$

Finally, let us note that K -theory commutes with finite products of rings and with filtering colimits.

Appendix B. The residue theorem using Gillet’s method

Besides wanting to prove Hilbert reciprocity using Gillet’s method from §3, one can also prove the residue theorem on curves using his technique. This is well-known among specialists, but perhaps not too well recorded in the literature.

Theorem B.1 (Residue theorem). *Let k be a field (of any characteristic). Let X/k be a geometrically integral smooth proper curve with function field $F := k(X)$. Then the composition*

$$\Omega_{F/k}^1 \xrightarrow{\partial} \bigoplus_v H_v^1(X, \Omega_{X/k}^1) \xrightarrow{\mathrm{Tr}_{\{v\}/k}} k$$

is zero, where ∂ denotes the residue of a rational 1-form. Here H_v^1 denotes coherent cohomology with support⁷ in the closed point belonging to the place v .

Proof. As in Gillet’s proof, we use the localization sequence, but this time for Hochschild homology over k . To this end note that Hochschild homology is also a localizing invariant in the sense of Blumberg–Gepner–Tabuada [BGT13]. We get the fiber sequence

$$(B.1) \quad HH_Z(X) \longrightarrow HH(X) \longrightarrow HH(X - Z).$$

As before, we may split Z into its disjoint connected components and get $HH_Z(X) = \bigoplus_P HH_P(X)$, where P runs through the finitely many points of Z . Finally, by the Hochschild–Kostant–Rosenberg (HKR) isomorphism we have $HH_n(X) \cong \Omega_{X/k}^n$. As $\dim X = 1$, we only get a non-trivial statement for $n = 1$. For unravelling the Hochschild homology with support in a closed point and showing that the boundary map agrees with the residue, see the HKR theorem with support from [BW19, Prop. 2.0.1], noting that Hochschild homology with support is denoted by H^Z loc. cit. \square

One can also use cyclic homology over k . This also proves the residue theorem.

⁷The right derived functors of the functor of global sections with support in the given closed point, $R\Gamma_v$.

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