Pure and Applied Mathematics Quarterly
Volume 19, Number 2, 451–485, 2023

Metrics on twisted pluricanonical bundles and finite generation of twisted canonical rings

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Abstract: In this paper, we first introduce the notion of admissible Bergman metrics. Then we establish a connection between singularities of admissible Bergman metrics and finite generation of twisted pluricanonical rings with m-multiplier ideal sheaves on smooth projective pairs. It involves an analytic approach to Boucksom's result about asymptotic multiplier ideal of a graded system of ideals. In the end, we give a few applications of our main theorem.

Keywords: Multiplier ideal sheaf, canonical ring, Siu-type metric, admissible Bergman metric, pluricanonical bundle.

1. Introduction

Let L be a line bundle over a compact complex manifold X equipped with a singular hermitian metric $h_L = e^{-\varphi_L}$ for some $\varphi_L \in L^1_{\text{loc}}$. L is called *pseudo-effective* if $i\Theta_{h_L}(L) \geq 0$ in the weak sense of currents, i.e. φ_L is plurisubharmonic. Nadel ([24]) proposed the notion of multiplier ideal sheaves associated with a plurisubharmonic function φ_L:

$$\mathcal{I}(\varphi_L)_x := \left\{ f \in \mathcal{O}_{X,x}; |f|^2 e^{-\varphi_L} \in L^1 \text{ near } x \right\} \tag{1}$$

for any $x \in X$. More generally, the so-called m-multiplier ideal sheaves are defined as

$$\mathcal{I}_m(\varphi_L)_x := \{ f \in \mathcal{O}_{X,x}; |f|^{2/m} e^{-\varphi_L} \in L^1 \text{ near } x \} \tag{2}$$

for any $m \in \mathbb{N}$. Sometimes, we adopt notations $\mathcal{I}(h_L)$ or $\mathcal{I}_m(h_L)$ to represent (1) or (2) for convenience. Both (1) and (2) are coherent analytic sheaves

Received July 6, 2022.
2010 Mathematics Subject Classification: 14C30, 14F18, 32J25, 32L10, 32U05.
*The second author is partially supported by the National Natural Science Foundation of China (grant no. 11688101 and no. 12288201).

([24], [32], [7]). The latter result is obtained by combining the former result, a regularization process for plurisubharmonic functions ([13]), and the strong openness property of multiplier ideal sheaves ([17]).

Let $h_1 = e^{-\varphi_L}$ and $h_2 = e^{-\psi_L}$ be two singular metrics with semi-positive curvature current on L. The metric h_1 is said to be more singular than h_2, denoted by $h_1 \succeq h_2$ (or $\varphi_L \succeq \psi_L$), if $\varphi_L \leq \psi_L + O(1)$ on X. Two metrics are said to be equivalent, denoted by $h_1 \sim h_2$ (or $\varphi_L \sim \psi_L$), if $\varphi_L = \psi_L + O(1)$ on X. There is a unique element up to equivalence among all the singular metrics on L with semi-positive curvature current, say $h_{\min} = e^{-\varphi_{\min}}$, having minimal singularities on X ([10] or Definition 1.4 in [13]).

Let (X, Δ) be a smooth projective *klt* pair, which means that X is a smooth projective variety, Δ is a $\mathbb{Q}$-effective divisor whose corresponding multiplier ideal sheaf is trivial on X. Its graded canonical section ring is defined as follows:

$$R(X, K_X + \Delta) = \bigoplus_{k \in \mathbb{N}} H^0(K_X, \lfloor kK_X + k\Delta \rfloor), \tag{3}$$

where $\lfloor \cdot \rfloor$ means round down. A main question in birational geometry is to ask whether (3) is finitely generated or not. In [4], the authors has demonstrated the well-known result:

Theorem 1.1. *If $K_X + \Delta$ is big, then $R(X, K_X + \Delta)$ is finitely generated.*

Here a $\mathbb{Q}$-divisor D is said to be *big* if $h^0(X, \mathcal{O}_X(kD)) \geq Ck^{\dim X}$ holds for some $C > 0$ and sufficiently large divisible k.

In [4], the authors also obtained the following result as a corollary of Theorem 1.1 by combining Theorem 5.2 in [16]:

Corollary 1.1. *For any smooth projective klt pair (X, Δ), $R(X, K_X + \Delta)$ is finitely generated.*

A projective *pseudo-klt* (Definition 2.7 in [21]) pair (X, L) means that (L, h_L) is pseudoeffective and $\mathcal{I}(h_L) = \mathcal{O}_X$. This notion can be generalized when L is a $\mathbb{Q}$-line bundle. Precisely speaking, (X, L) is called a *pseudo-klt* pair if the line bundle $k_0 L$ for some k_0 divisible enough admits a singular hermitian metric $h_{k_0} = e^{-k_0\varphi_L}$ with semi-positive curvature current and $\mathcal{I}(\varphi_L) = \mathcal{O}_X$. For example, when L is a line bundle and $m \in \mathbb{N}$, $(X, \frac{1}{m}L)$ is a pseudo-klt pair if there exists a singular metric h_L with semi-positive curvature current and $\mathcal{I}(h_L^{1/m}) = \mathcal{O}_X$.

Any klt pair must be pseudo-klt. For a pseudo-klt pair (X, L), the question then becomes to ask whether its graded canonical section ring

$$R(X, K_X + L) = \bigoplus_{k \in \mathbb{N}} H^0(K_X, \lfloor kK_X + kL \rfloor)$$

is finitely generated. The following result has been established thus far:

Theorem 1.2 (Remark 1.2 (b) in [8]). *If (X, L) is a smooth projective pseudo-klt pair and $K_X + L$ is big, then $R(X, K_X + L)$ is finitely generated.*

For completeness, we will recall the proof of Theorem 1.2 in Section 3.

Siu has provided an analytic approach to Theorem 1.1 when $\Delta = 0$ ([33], [34], [35]) by using Skoda's L^2 division theorem ([36]). His idea is to show that his construction of singular metric with semi-positive curvature current (which will be called Siu-type metrics later) has stable vanishing order everywhere. For this motivation, a descending inductive argument on the discrepancy subspaces, where stable vanishing order of Siu-type metrics is not yet known to be achieved, has been established.

The goal of this paper is to introduce the notion of admissible Bergman metrics and connect the singularities of admissible Bergman metrics on the $\mathbb{Q}$-bundle $K_X + L$ to finite generation property of a subgraded algebra (8) of $R(X, K_X + L)$ with m-multiplier ideal sheaves. For this purpose, we review the definition of Siu-type metrics, which play important roles in the proofs of invariance of plurigenera ([31], [32]) and finite generation of canonical rings.

By multiplying a sufficiently divisible integer $m \in \mathbb{N}$, we may restrict ourselves to working on the twisted pluricanonical bundles $mK_X + L$, where L is a genuine line bundle. Let

$$S := \bigoplus_{k \in \mathbb{N}} S_k \subset R(X, mK_X + L)$$

be a subgraded $\mathbb{C}$-algebra of $R(X, mK_X + L)$, which implies that each S_k is a linear subspace of $H^0(X, kmK_X + kL)$ and $S_k S_l \subset S_{k+l}$. Let $\mathfrak{b}_k$ be the base ideal associated to each S_k, which is defined as the image of the map

$$S_k \otimes_{\mathbb{C}} \mathcal{O}_X(-kmK_X - kL) \to \mathcal{O}_X.$$

One can define the metric on $mK_X + L$ induced by a basis of S_k, denoted by $|S_k|^{-2/k}$. Indeed, if $\dim S_k = q_k$ and $\{s_1^{(k)}, \ldots, s_{q_k}^{(k)}\}$ denote such basis, then

$$|S_k|^2 = |s_1^{(k)}|^2 + \cdots + |s_{q_k}^{(k)}|^2.$$

Note this definition is independent of the choice of basis (Proposition 3.1 in [20]) up to equivalence.

We can always arrange

$$|S_j|^{2/j} \le |S_{k!}|^{2/k!} \tag{4}$$

for any $j = 1, \dots, k$ and $j = (k-1)!$ upon scaling. This can be done due to the basic relationship $S_k S_l \subset S_{k+l}$.

Siu has constructed

$$h_{S,\varepsilon_\bullet} = e^{-\varphi_{S,\varepsilon_\bullet}}, \varphi_{S,\varepsilon_\bullet} := \log\left(\sum_{k=1}^{\infty} \varepsilon_k |S_k|^{2/k}\right) \tag{5}$$

as a singular metric on $mK_X + L$, where $\{\varepsilon_\bullet\}$ is chosen to be a sequence of positive numbers satisfying

$$\sum_{k=1}^{\infty} \varepsilon_k \sup\{g(S_k, S_k)^{2/k}; x \in X\} < \infty, \tag{6}$$

g is an arbitrary and auxiliary smooth metric on $mK_X + L$. Such $\{\varepsilon_\bullet\}$ satisfying (6) will be called *admissible* ([20], [30]) and it is clearly independent of the choice of g.

Theorem 1.3. *Let $S = R(X, mK_X + L)$ be the full graded linear system.*

1. (Theorem 6.5 in [5]) *If $mK_X + L$ is big, then S is finitely generated if and only if $\varphi_{S,\varepsilon_\bullet}$ has minimal singularities.*
2. ([33], [34]) *Let $m = 1$ and $L = 0$. If $\varphi_{S,\varepsilon_\bullet}$ is equivalent to a finite sum, then S is finitely generated.*

Now we recall the so-called m-Bergman metrics on $mK_X + L$. Let L be a line bundle over a projective manifold X equipped with a positively curved singular metric h_L. The m-Bergman kernel B_m of $mK_X + L$ is defined as

$$B_m(x) := \sup\left\{u(x) \otimes \overline{u}(x); u \in H^0(mK_X + L) \text{ and } \int_X |u|^{2/m} h_L^{1/m} \le 1\right\} \tag{7}$$

and its inverse B_m^{-1} can be thought of as a metric, known as the m-Bergman metrics on $mK_X + L$. According to a standard normal family argument (Proposition 28.3 in [19]), it has semi-positive curvature current. Indeed, lower semi-continuity of h_L will be sufficient to obtain this property.

If we set

$$S_k := H^0(X, k(mK_X + L) \otimes \mathcal{I}_{km}(h_L^{1/m})) \tag{8}$$

and use the Hölder inequality

$$\int_X |s^{(k)} \cdot s^{(l)}|^{2/(k+l)} h_L^{1/m} \leq \left(\int_X |s^{(k)}|^{2/k} h_L^{1/m}\right)^{\frac{k}{k+l}} \left(\int_X |s^{(l)}|^{2/l} h_L^{1/m}\right)^{\frac{l}{k+l}}$$

for any $s^{(k)} \in S_k$ and $s^{(l)} \in S_l$, we see that S actually becomes a subgraded algebra of $R(X, mK_X + L)$. The rank of the Kodaira map induced by S_k for sufficiently large and divisible k, is nothing but the generalized Kodaira dimension $\kappa(X, K_X + \frac{1}{m}L, h_L^{1/m})$ which was introduced in [42]. This viewpoint will lead to a more direct proof of Theorem 1.1 in [42]. We will give its details in a forthcoming paper.

From now on, we will keep notation (8) for S unless otherwise specified.

In a manner similar to the construction of Siu-type metrics, we may also choose $\{\delta_\bullet\}$ to be a sequence of positive numbers satisfying that

$$\inf_{k \in \mathbb{N}} \delta_k^{-1} B_{km}^{-1/k} \geq C > 0 \tag{9}$$

locally on X, where B_{km} stands for the km-Bergman kernel of $kmK_X + kL$ as (7). Such $\{\delta_\bullet\}$ exists, for example, by taking all $\delta_k = 1$ (see Lemma 2.2 below), which is referred to as the canonical metric in [38] when $m = 1$ and $L = \mathcal{O}_X$. Such $\{\delta_\bullet\}$ satisfying the restrained condition (9) will be also called *admissible.*

Taking admissible $\{\delta_\bullet\}$, we can define an admissible Bergman metric on $mK_X + L$ as follows

$$B_{S,\delta_\bullet}^{-1} = e^{-\psi_{S,\delta_\bullet}} := (\inf_{k \in \mathbb{N}} \delta_k^{-1} B_{km}^{-1/k})^*, \tag{10}$$

where "$*$" stands for the lower semi-continuity regularization process. If we write $B_{km}^{-1/k} = e^{-\psi_k}$ with local plurisubharmonic weight ψ_k, then (10) is equivalent to saying that

$$\psi_{S,\delta_\bullet}(x) = \limsup_{y \to x} (\sup_{k \in \mathbb{N}} \log \delta_k + \psi_k(y)).$$

To be concrete, in Example 5.1 below, these asymptotic metrics are computed in some special cases.

We can now state the main result in the present paper as follows.

Theorem 1.4 (Main theorem)**.** *In the above settings, we have the followings:*

1. *If* $\lim \delta_k = 0$ *and* $B^{-1}_{S,\delta_\bullet}$ *has minimal singularities, then* S *is finitely generated.*
2. *If the admissible Bergman metric* $B^{-1}_{S,\delta_\bullet}$ *is equivalent to a finite infimum, then* S *is finitely generated.*
3. *Let* $\left(X, \frac{L}{m}\right)$ *be a smooth projective pseudo-klt pair for some* $m \in \mathbb{N}$*. Suppose that* $mK_X + L$ *is big, then* $B^{-1}_{S,\delta_\bullet}$ *has minimal singularities for all admissible* $\{\delta_\bullet\}$*.*

Remark 1.1. *The first two statements are analogues of Theorem 1.3* 1) *and* 2)*. The difference is that* S*, which may rely on* h_L*, does not have to be the full graded linear system. The last statement* 3) *of Theorem 1.4 is an analogue of the converse statement of Theorem 1.3* 1) *for admissible Bergman metrics.*

Remark 1.2. *Unlike 1.3* 1) *which assumes minimality of* $\varphi_{S,\varepsilon_\bullet}$ *without any additional restrained condition (except (6)) on* $\{\varepsilon_\bullet\}$*, the condition* $\lim \delta_k = 0$ *in Theorem 1.4* 1) *cannot be omitted (see Example 5.1 below).*

Remark 1.3. *From the proof of Theorem 1.4* 3)*, we actually show that* $B^{-1/k}_{km}$ *has minimal singularities for sufficiently large* $k \in \mathbb{N}$*. In this circumstance,* S *will be the full graded linear system.*

Remark 1.4. *In* 3)*, if we merely assume the weight of* h_L *to be quasi-plurisubharmonic (i.e.,* $\sqrt{-1}\Theta_{h_L}(L) \geq -C\omega$ *for a Kähler form* ω *and a large constant* $C > 0$*) and keep the assumptions that* $\mathcal{I}(h_L^{1/m}) = \mathcal{O}_X$ *and* $mK_X + L$ *is big, then one still obtains that* $B^{-1}_{S,\delta_\bullet}$ *always has semi-positive curvature current and minimal singularities for all* $\delta_k = 1$*. The reason is that, when* h_L *has analytic singularities, this result was already proved in Proposition 19.8 in [10] (take* $\varphi = \varphi_{min}$ *in Proposition 19.8); for the general case, one should only add the openness property ([1] or [17]) of multiplier ideal sheaves into its original proof. This observation will be useful in subsection 5.1 for a smooth metric* h_L*, and in subsection 5.3 and 5.4 for a singular metric* h_L *with semi-positive curvature current.*

A key ingredient for the proofs of Theorem 1.4 is to compare the singularities of the asymptotic metrics $h_{S,\varepsilon_\bullet}$ and $B^{-1}_{S,\delta_\bullet}$. It is worth noting that their singularities differ not only from the way they are constructed, but also from different choices of $\{\varepsilon_\bullet\}$ and $\{\delta_\bullet\}$. Nonetheless, using the strong openness property of multiplier ideal sheaves ([17]), we can deduce:

Theorem 1.5. *For any* $c > 0$ *and any admissible* $\{\varepsilon_\bullet\}$ *and* $\{\delta_\bullet\}$*,*

$$\mathcal{I}(c\varphi_{S,\varepsilon_\bullet}) = \mathcal{I}(c\psi_{S,\delta_\bullet}). \tag{11}$$

$\varphi_{S,\varepsilon_\bullet}$ and $\psi_{S,\delta_\bullet}$ satisfying (11) are said to be v-equivalent. We remark that this fact has been obtained by using the valuative characterization of multiplier ideals ([30]'s appendix or [6]). Our proof of Theorem 1.5 is analytic which essentially uses the strong openness property of multiplier ideals (see Lemma 2.2 below).

The rest of the proof for the first two parts of Theorem 1.4 is motivated by the idea in [5]. The main tool is a characterization of finite generation of S via singularities of the family of metrics $\{\phi_m := \log |S_{m!}|^{2/m!}\}$ (see Lemma 3.5 below). The final part 3) in Theorem 1.4 makes use of Theorem 1.2 and Theorem 1.5.

Let us turn to some applications of Theorem 1.4:

The first application (Corollary 5.1) is about comparing singularities of admissible Bergman metrics as in [20].

The second application (Corollary 5.2) is about giving another criterion of semi-ampleness for pseudo-klt pairs in terms of admissible Bergman metrics.

The third application is to show that, if $f : X \to Y$ is either an algebraic fiber space (Corollary 5.3) or a Kähler fiber space (Corollary 5.5), the relative twisted pluricanonical bundle is f-big and $L \to X$ is equipped with a positively curved singular metric h_L whose singularities are "mild" enough, then the general fiberwise metrics with minimal singularities can be glued as a global metric h on the relative twisted pluricanonical bundle with semi-positive curvature current on X. Furthermore, Corollary 5.4 (see also Remark 5.7) asserts that h has minimal singularities as well when h_L has analytic singularities. Note in Corollary 5.5, although the assumption on f has been relaxed, we require the singularities of the metric h_L on the twisted line bundle L to be even "milder". This type of results, meaning gluing fiberwise metric with minimal singularities as a global metric on the twisted relative canonical bundle with semi-positive curvature current, has been systematically studied in [38] (with respect to fiberwise supercanonical metrics) and in [8], [15] (with respect to fiberwise Kähler-Einstein metrics).

This paper will be organized as follows. In Section 2, we give a proof of Theorem 1.5. In Section 3, we list some lemmas which will be used in the proof of the main result. In Section 4, we give the proof of Theorem 1.4. In Section 5, we discuss some applications of Theorem 1.4.

2. Proof of theorem 1.5

In this section, we will give an analytic approach to Theorem 1.5 and compare the singularities of two types of metrics with semi-positive curvature currents, $h_{S,\varepsilon_\bullet}$ and $B^{-1}_{S,\delta_\bullet}$.

2.1. Multiplier ideal sheaves of Siu-type plurisubharmonic functions

Plurisubharmonic functions are useful in both several complex variables and complex geometry. Let φ be a plurisubharmonic germ at the origin $o \in \mathbb{C}^n$, then its Lelong number at o is defined by

$$\nu(\varphi, o) := \liminf_{z \to o} \frac{\varphi(z)}{\log |z|}$$

and its multiplier ideal at o is defined by

$$\mathcal{I}(\varphi)_o := \left\{ f \in \mathcal{O}_{\mathbb{C}^n,o}; |f|^2 e^{-\varphi} \in L^1 \text{ near } o \right\}.$$

Let ψ be another plurisubharmonic germ at o. Recall that φ is more singular than ψ (denoted by $\varphi \succeq \psi$) if $\varphi \le \psi + O(1)$. It is easy to see $\mathcal{I}(\varphi)_o \subset \mathcal{I}(\psi)_o$ and $\nu(\varphi, 0) \ge \nu(\psi, 0)$ if $\varphi \succeq \psi$. Their singularities are typically compared in the following two ways ([20]). One is saying that, φ and ψ are equivalent (denoted by $\varphi \sim \psi$) if $\varphi = \psi + O(1)$. The other is saying that, φ and ψ are v-equivalent (denoted by $\varphi \sim_v \psi$) if $\mathcal{I}(c\varphi)_o = \mathcal{I}(c\psi)_o$ for any $c > 0$.

An equivalent formulation of v-equivalence is that for all proper modifications $\pi : \tilde{U} \to U$ above o the Lelong numbers of their pull-backs at $\pi^{-1}(o)$ are equal, according to the strong openness property of multiplier ideals ([17]) and the main result of [6]. We deduce from this fact that

$$\varphi \sim \psi \Rightarrow \varphi \sim_v \psi \Rightarrow \nu(\varphi, o) = \nu(\psi, o) \tag{12}$$

holds. The converse of each "$\Rightarrow$" in (12) fails in general because one can take

$$\varphi = \psi - \sqrt{-\log |z|}$$

as a counterexample for the first implication and

$$\varphi = \max\{\log |z|, \log |w|^2\}, \psi = \max\{\log |z|, \log |w|\}$$

for the second.

We now recall plurisubharmonic functions associated with a graded system of ideals $\{\mathfrak{a}_\bullet\} = \{\mathfrak{a}_k\}_{k \in \mathbb{N}}$ on a complex manifold X, where each $\mathfrak{a}_k \subset \mathcal{O}_X$ is an ideal sheaf on X and $\mathfrak{a}_k \mathfrak{a}_l \subset \mathfrak{a}_{k+l}$. Assume that at any point $x \in X$, there exists a neighborhood U of x, where $\mathfrak{a}_k$ is generated by $g_1^{(k)}, \ldots, g_{q_k}^{(k)} \in \mathcal{O}(\bar{U})$. We will use the notation

$$|\mathfrak{a}_k|^{\frac{2}{k}} := |g_1^{(k)}|^{\frac{2}{k}} + \cdots + |g_{q_k}^{(k)}|^{\frac{2}{k}}.$$

for simplicity. Given an admissible sequence of positive numbers $\{\varepsilon_k\}_{k\in\mathbb{N}}$, which means that the inequality

$$\sum_{k=1}^{\infty} \varepsilon_k |\mathfrak{a}_k|^{\frac{2}{k}} = \sum_{k=1}^{\infty} \varepsilon_k \sum_{j=1}^{q_k} |g_j^{(k)}|^{\frac{2}{k}} < \infty$$

holds, there will be a well-defined plurisubharmonic function

$$\varphi = \varphi_{\mathfrak{a}_\bullet} := \log \left(\sum_{k=1}^{\infty} \varepsilon_k |\mathfrak{a}_k|^{\frac{2}{k}} \right)$$

on U.

Denote multiplier ideal sheaves of $\mathfrak{a}_k$ with coefficient $c > 0$ by $\mathcal{J}(c \cdot \mathfrak{a}_k)$ (section 9.2 in [22]), which can be equivalently defined as

$$\mathcal{J}(c \cdot \mathfrak{a}_k) = \mathcal{I}(\log(|g_1^{(k)}| + \cdots + |g_{q_k}^{(k)}|)^{2c})$$

on U. The asymptotic multiplier ideal sheaf (Definition 11.1.15 in [22]) of $\{\mathfrak{a}_\bullet\}$ with coefficient $c > 0$, written as $\mathcal{J}(c \cdot \mathfrak{a}_\bullet)$, is defined to be the unique maximal member among the family of ideals

$$\left\{ \mathcal{J}\left(\frac{c}{p} \cdot \mathfrak{a}_p\right); p \in \mathbb{N} \right\}.$$

Thus $\mathcal{J}(c \cdot \mathfrak{a}_\bullet) = \mathcal{J}\left(\frac{c}{p} \cdot \mathfrak{a}_p\right)$ on a relative compact subset $V \Subset X$ for sufficiently large and divisible p.

Multiplier ideal sheaves of $\varphi = \varphi_{\mathfrak{a}_\bullet}$ and asymptotic multiplier ideal sheaves of $\{\mathfrak{a}_\bullet\}$ coincide. This result is helpful to construct toric plurisubharmonic functions with clusters of jumping numbers in [30].

Theorem 2.1 (Appendix in [30]). *For any $c > 0$, $\mathcal{J}(c \cdot \mathfrak{a}_\bullet) = \mathcal{I}(c \cdot \varphi_{\mathfrak{a}_\bullet})$.*

As Dano Kim told us, the valuative characterization of multiplier ideals which is used in Boucksom's original proof of Theorem 2.1 actually relies on the *strong openness* property of multiplier ideals.

Theorem 2.2 (Subsection 3.3 in [17]). *Let Δ^n be the unit polydisc in $\mathbb{C}^n$. Assume that $\varphi_j \in \mathrm{PSH}(\Delta^n)$ being a sequence of plurisubharmonic functions increasingly converging a.e. towards $\varphi \in \mathrm{PSH}(\Delta^n)$, then*

$$\bigcup_j \mathcal{I}(\varphi_j) = \mathcal{I}(\varphi).$$

The following fact will be a direct consequence of Theorem 2.2:

Corollary 2.1. *Let Δ^n be the unit polydisc in $\mathbb{C}^n$. Assume that $\varphi_j, \psi_j \in \mathrm{PSH}(\Delta^n)$ being two sequences of plurisubharmonic functions increasingly converging a.e. towards $\varphi \in \mathrm{PSH}(\Delta^n)$ and respectively $\psi \in \mathrm{PSH}(\Delta^n)$. Suppose moreover that $\mathcal{I}(\varphi_j) = \mathcal{I}(\psi_j)$ for each $j \in \mathbb{N}$, then*

$$\mathcal{I}(\varphi) = \mathcal{I}(\psi).$$

Now we are in position to give the

Proof of Theorem 2.1. Let $\varphi = \varphi_{\mathfrak{a}_\bullet}$, then it suffices to show $\mathcal{I}(c\varphi)_x = \mathcal{J}(c \cdot \mathfrak{a}_\bullet)_x$ for any $x \in \bigcap V(\mathfrak{a}_k)$, where $V(\mathfrak{a}_k)$ is the corresponding analytic subset. According to Definition 2.1 and the Noetherian property, one can find a large integer $N_0 > 0$ such that

$$\mathcal{J}(c \cdot \mathfrak{a}_\bullet)_x = \mathcal{J}\left(\frac{c}{N_0} \cdot \mathfrak{a}_{N_0}\right)_x \subset \mathcal{I}(c \cdot \varphi)_x. \tag{13}$$

To see the inverse direction, thanks to Theorem 2.2, there exists $N_1 > 0$ such that

$$\mathcal{I}\left(c \cdot \log\left(\sum_{k=1}^{N} \varepsilon_k |\mathfrak{a}_k|^{\frac{2}{k}}\right)\right)_x = \mathcal{I}(c \cdot \varphi)_x \tag{14}$$

as soon as $N \geq N_1$. If we fix an $N_2 \geq \max\{N_0, N_1\}$ and observe that

$$\mathfrak{a}_k^{1\cdot 2\cdot \ldots \cdot (k-1)\cdot (k+1)\cdot \ldots \cdot N_2} \subset \mathfrak{a}_{N_2!}$$

for $1 \leq k \leq N_2$, then it implies that the inequalities

$$\varepsilon_k |\mathfrak{a}_k|^{\frac{2}{k}} \leq C_k |\mathfrak{a}_{N_2!}|^{\frac{2}{N_2!}}, 1 \leq k \leq N_2 \tag{15}$$

hold on U for some $C_k > 0$ independent of x. A summation of inequalities (15) yields that

$$\log\left(\sum_{k=1}^{N_2}\varepsilon_k|\mathfrak{a}_k|^{\frac{2}{k}}\right) \le \frac{2}{N_2!}\log|\mathfrak{a}_{N_2!}| + \log C \tag{16}$$

hold on U, where $C = \sum_{k=1}^{N_2} C_k$ is independent of $x \in U$. As a consequence of (13), (14) and (16), we get

$$\mathcal{J}(c\cdot\mathfrak{a}_\bullet)_x = \mathcal{J}\left(\frac{c}{N_2!}\cdot\mathfrak{a}_{N_2!}\right)_x \supset \mathcal{I}\left(c\cdot\log\left(\sum_{k=1}^{N_2}\varepsilon_k|\mathfrak{a}_k|^{\frac{2}{k}}\right)\right)_x = \mathcal{I}(c\cdot\varphi)_x, \tag{17}$$

which finishes our proof. □

As pointed out in Corollary 2.3 in [30], $\varphi = \log\left(\sum_{k=1}^{\infty}\varepsilon_k\sum_{j=1}^{q_k}|g_j^{(k)}|^{\frac{2}{k}}\right)$ and $\varphi' = \log\left(\sum_{k=1}^{\infty}\varepsilon'_k\sum_{j=1}^{q_k}|g_j^{(k)}|^{\frac{2}{k}}\right)$ are v-equivalent for any two different choices of admissible $\{\varepsilon_\bullet\}$ and $\{\varepsilon'_\bullet\}$.

We now present a comparable formulation in terms of a graded system of linear series. Let us begin with a line bundle L over a compact complex n-fold X with non-negative Kodaira-Iitaka dimension

$$\limsup_{k\to\infty}\frac{\log h^0(X,kL)}{\log k} =: \kappa(X,L) \ge 0$$

and let

$$S := \bigoplus_{k\in\mathbb{N}} S_k \subset R(X,L) := \bigoplus_{k\in\mathbb{N}} H^0(X,kL)$$

be a subgraded section ring of L, where all S_k are linear subspaces of $H^0(X,kL)$ and $S_kS_l \subset S_{k+l}$.

Definition 2.1 (Definition 1.1.8 in [22]). *The base ideal of $|S_k|$, written as $\mathfrak{b}_k$ or $\mathfrak{b}(X,|S_k|)$, is defined to be the image of the map $S_k\otimes_{\mathbb{C}} L^{-1}\to\mathcal{O}_X$.*

Remark 2.1. *It is easy to check that $\{\mathfrak{b}_\bullet\}$ forms a graded system of ideals on X and $\varphi_{S,\varepsilon_\bullet}$ defined in (5) satisfies $\varphi_{S,\varepsilon_\bullet} = \log\left(\sum_{k=1}^{\infty}\varepsilon_k|\mathfrak{b}_k|^{2/k}\right)$.*

Now Theorem 2.1 immediately implies:

Corollary 2.2. *For any $c>0$, $\mathcal{J}(c\cdot\mathfrak{b}_\bullet) = \mathcal{I}(c\cdot\varphi_{S,\varepsilon_\bullet})$. In particular, all $\varphi_{S,\varepsilon_\bullet}$ are v-equivalent with any admissible choice of $\{\varepsilon_\bullet\}$.*

2.2. Comparing admissible Bergman metrics and Siu metrics

Let X be a compact complex manifold and L a line bundle over X equipped with a singular metric h_L with semi-positive curvature current. We will fix some $m \in \mathbb{N}$ and consider the line bundles $mK_X + L$ as before and keep the notation (7) and (8) in this subsection. For any $k \in \mathbb{N}$, B_{km} will stand for the km-Bergman kernels of the line bundle $kmK_X + kL$ as in (7), where kL is equipped with the metric h_L^k.

Lemma 2.1. *For $k \geq 1$, $|S_k|^{-2/k}$ is equivalent to $B_{km}^{-1/k}$ as metrics on $mK_X + L$.*

Proof. It suffices to show the $k = 1$ case. For any $\xi = \xi_x \in -(mK_X + L)_x$, one may write the dual m-Bergman metrics $|\xi|_{B_m}$ as

$$\sup\left\{|\xi \cdot u(x)|; u \in H^0(X, mK_X + L), \|u\|_m^{\frac{2}{m}} := \int_X |u \wedge \overline{u}|^{\frac{1}{m}} h_L^{\frac{1}{m}} \leq 1\right\}. \tag{18}$$

by (7). Let $s_1, \ldots, s_N$ be a basis of S_1, and

$$\max_{\sum_{j=1}^N |b_j|^2=1} \|\sum_{j=1}^N b_j s_j\|_m =: C_{\max} > 0,$$

$$\min_{\sum_{j=1}^N |b_j|^2=1} \|\sum_{j=1}^N b_j s_j\|_m =: C_{\min} > 0.$$

Our mission is to show $B_m \sim \sum_{j=1}^N |s_j|^2 \sim |S_1|^2$. It is easy to see

$$E_1 := \left\{u = \frac{\sum_{j=1}^N a_j s_j}{C_{\max}}; \sum_{j=1}^N |a_j|^2 \leq 1\right\} \subset \{u; \|u\|_m \leq 1\} \subset$$

$$\left\{u = \frac{\sum_{j=1}^N a_j s_j}{C_{\min}}; \sum_{j=1}^N |a_j|^2 \leq 1\right\} =: E_2. \tag{19}$$

On one side,

$$|\xi \cdot u(x)|^2 = \frac{|\sum_{j=1}^N a_j(\xi \cdot s_j(x))|^2}{C_{\max}^2} \leq \frac{\sum_{j=1}^N |\xi \cdot s_j(x)|^2}{C_{\max}^2} \tag{20}$$

holds for any $u = \frac{\sum_{j=1}^N a_j s_j}{C_{\max}}$ and $\sum_{j=1}^N |a_j|^2 \le 1$ by the Cauchy-Schwarz inequality. One can choose $\{a_j^0; j = 1, \ldots, N\}$ depending on ξ such that $\sum_{j=1}^N |a_j^0|^2 = 1$ and (20) becomes an equality, i.e.

$$|\xi \cdot u^0(x)|^2 = \frac{\sum_{j=1}^N |\xi \cdot s_j(x)|^2}{C_{\max}^2}. \tag{21}$$

The former inclusion of (19) implies the fact that

$$|\xi|^2_{B_m} \ge \sup\{|\xi \cdot u(x)|^2; u \in E_1\} \ge |\xi \cdot u^0(x)|^2. \tag{22}$$

Combining (21) and (22), we obtain that

$$|\xi|^2_{B_m} \ge \frac{\sum_{j=1}^N |\xi \cdot s_j(x)|^2}{C_{\max}^2}. \tag{23}$$

On the other side, the latter inclusion of (19) implies that

$$|\xi|^2_{B_m} \le \sup\{|\xi \cdot u(x)|^2; u \in E_2\} \le \frac{\sum_{j=1}^N |\xi \cdot s_j(x)|^2}{C_{\min}^2}, \tag{24}$$

where the last "$\le$" also comes from the Cauchy-Schwarz inequality. Our claim follows immediately from (23) and (24). ☐

Remark 2.2. *When $k = m = 1$ in Lemma 2.1, Bergman metric precisely equals the metric induced by any orthonormal basis of $H^0(X, (K_X + L) \otimes \mathcal{I}(\varphi_L))$.*

Let us consider a family of Bergman metrics $\{B_{km}^{-1/k}; k \in \mathbb{N}\}$ on $mK_X + L$. The following lemma will allow us to construct an asymptotic metric with "minimal" singularities among this family.

Lemma 2.2. *All $B_{km}^{1/k}$ are locally bounded above by a constant C independent of k.*

Proof. Choosing a coordinate $(z_1, \ldots, z_n)$ centered at $x \in X$ and a holomorphic frame e_L of L in some open set Ω, we can write any $u \in H^0(X, kmK_X + kL)$ in Ω as

$$u = u'(z) dz^{\otimes km} \otimes e_L^{\otimes k},$$

where u' is holomorphic, $dz = dz_1 \wedge \ldots \wedge dz_n$. We may assume that $h_L|_\Omega$ is written as

$$\langle e_L, e_L \rangle_{h_L} = e^{-\varphi_L}$$

and $\sup_\Omega \varphi_L := C_0 < \infty$ after shrinking Ω if necessary. Then by definition B_{km} can be trivialized as

$$B_{km}(z) = \sup_{\|u\|_{km} \leq 1} |u'(z)|^2.$$

If u is normalized by $\|u\|_{km} = 1$, then by applying the mean value inequality on some $\Omega' \Subset \Omega$ we obtain that

(25)
$$|u'(z)|^{\frac{2}{km}} \leq C_1 \operatorname{dist}(\Omega', \partial\Omega) e^{C_0/m} \int_\Omega |u'(z)|^{\frac{2}{km}} e^{-\frac{\varphi_L}{m}} d\lambda \leq C_2 \|u\|_{km}^{\frac{2}{km}} = C_2,$$

where $C_2 = C_1 \operatorname{dist}(\Omega', \partial\Omega) e^{C_0/m}$. This immediately yields our result if we replace C by C_2^m. □

Remark 2.3. *In this lemma, only lower semi-continuity of h_L has been used.*

We can construct

(26)
$$B_S^{-1} := (\inf_{k \in \mathbb{N}} B_{km}^{-1/k})^*$$

as a singular metric on $mK_X + L$ with semi-positive curvature current by using Lemma 2.2, where "$*$" stands for the lower semi-continuity regularization process. The metric in (26) is referred to as the canonical metric in [38] and [39] when $L = 0$ and $m = 1$. More generally, we will investigate metrics $B_{S,\delta_\bullet}^{-1} = e^{-\psi_{S,\delta}}$ as in (10) for an admissible choice of $\{\delta_\bullet\}$.

One may refer to Section 4.2 in [39] for constructing another type of the asymptotic metrics, which is called the super canonical metrics on canonical bundles of projective *klt* pairs. The author's method is replacing each S_k by $H^0(X, k(mK_X + L) + A)$, where A is an auxiliary ample line bundle. In this way, the super canonical metric no longer owns a sup in the envelope, but just a limsup. Through its construction, the author proved that it always has minimal singularities (Theorem 4.2 in [39]).

We are in position to show the

Proof of Theorem 1.5. Setting

$$\varphi_N := \log(\sum_{j=1}^{N} \varepsilon_j |S_j|^{2/j})$$

and

$$\psi_N := \sup_{1 \leq k \leq N} \log \delta_k + \psi_k,$$

we infer that $\varphi_N \sim \psi_N$ and $\mathcal{I}(\varphi_N) = \mathcal{I}(\psi_N)$ for any $N \in \mathbb{N}$ thanks to Lemma 2.1. Since φ_N (resp. ψ_N) increasingly converges towards $\varphi_{S,\varepsilon_\bullet}$ (resp. $\psi_{S,\delta_\bullet}$) as $N \to \infty$ almost everywhere, Corollary 2.1 finally asserts the desired equality. □

Remark 2.4. *Combining Theorem 1.5 and Corollary 2.2, one can conclude that $\mathcal{I}(c \cdot \psi_{S,\delta_\bullet}) = \mathcal{J}(c \cdot \mathfrak{b}_\bullet)$. In particular, all $\psi_{S,\delta_\bullet}$ are v-equivalent with any admissible choice of $\{\delta_\bullet\}$.*

It is however natural to ask when those two v-equivalent metrics $h_{S,\varepsilon_\bullet}$ and $B^{-1}_{S,\delta_\bullet}$ are genuinely equivalent. We will give a partial answer to this question:

Proposition 2.1. *Let $h_{S,\varepsilon_\bullet} = e^{-\varphi_{S,\varepsilon_\bullet}}$ and $B^{-1}_{S,\delta_\bullet} = e^{-\psi_{S,\delta_\bullet}}$ be two singular metrics on $mK_X + L$ for some admissible $\{\varepsilon_\bullet\}$ and $\{\delta_\bullet\}$. Then*

1. *There exists an appropriate choice of $\{\varepsilon'_\bullet\}$ such that $\varphi_{S,\varepsilon'_\bullet} \succeq \psi_{S,\delta_\bullet}$;*
2. *There exists an appropriate choice of $\{\delta'_\bullet\}$ such that $\psi_{S,\delta'_\bullet} \succeq \varphi_{S,\varepsilon_\bullet}$.*

Proof. We only give the proof of the first statement here since the other will be derived in a similar manner. Let

$$\varphi_N := \log\left(\sum_{k=1}^{N} \varepsilon'_k |S_k|^{\frac{2}{k}}\right)$$

and respectively

$$\psi_N := \sup_{1\le k\le N!} (\log \delta_k + \varphi_k)$$

for any $N \geq 1$. Thanks to Lemma 2.1 and the convention (4), we get

$$\varphi_N \le \log\left(\sum_{k=1}^{N} \varepsilon'_k\right)\left(\sum_{j=1}^{q_{N!}} |s_j^{(N!)}|^2\right)^{\frac{1}{N!}} \le$$

$$\log\left[\left(\sum_{k=1}^{N} \varepsilon'_k\right)\cdot C_N/\delta_{N!}\right] + (\log \delta_{N!} + \varphi_{N!}) \le C + \psi_N$$

provided that $\{\varepsilon'_\bullet\}$ satisfies

$$\sup_N \left(\sum_{k=1}^{N} \varepsilon'_k\right)\cdot C_N/\delta_{N!} := C < \infty.$$

Hence the first statement follows by letting $N \to \infty$ and taking the upper semi-continuity regularization process. □

3. Key lemmas

This section contains a list of key lemmas that will be used in the proof of the main theorem. The following lemma is about the finite generation of canonical ring of a projective pseudo-klt pair. For the sake of completeness, we give its proof originating from [8] here:

Lemma 3.1 (=Theorem 1.2)**.** *Let L be a $\mathbb{Q}$-line bundle over X. If (X, L) is a smooth projective* pseudo-klt *pair and $K_X + L$ is big, then $R(X, K_X + L)$ is finitely generated.*

Proof. Let us first assume that $K_X + L \sim_{\mathbb{Q}} A + E$ for an ample divisor A and an effective divisor E, where $\sim_{\mathbb{Q}}$ means $\mathbb{Q}$-linear equivalence. Thanks to Theorem 2.2 or the openness property ([1]),

$$\mathcal{I}(\varphi_L + k^{-1}\varphi_E) = \mathcal{I}(\varphi_L) = \mathcal{O}_X \tag{27}$$

for $k \gg 0$, where φ_E is a canonical weight attached to E.

Now we fix a Kähler form $\omega = i\Theta_{h_A}(A)$ on X with respect to a smooth metric h_A on A, as well as a smooth metric h_0 on L. Writing $e^{-\varphi_L} = h_L = h_0 e^{-\psi_L}$ for some quasi-plurisubharmonic function ψ_L, we understand that there exists a closed positive $(1, 1)$ current $T \in c_1(A + L)$ satisfying

$$T := ik^{-1}\Theta_{h_A}(A) + i\Theta_{h_L}(L) = ik^{-1}\Theta_{h_A}(A) + i\Theta_{h_0}(L) + i\partial\overline{\partial}\psi_L \geq k^{-1}\omega.$$

By Demailly's regularization process ([9] or Corollary 13.13 in [10]), there exists a quasi-plurisubharmonic function $\{\psi'_L\}$ with logarithmic poles such that

$$\psi_L \succeq \psi'_L \tag{28}$$

and simultaneously

$$T' := ik^{-1}\Theta_{h_A}(A) + i\Theta_{h_0}(L) + i\partial\overline{\partial}\psi'_L \geq \frac{1}{2k}\omega \tag{29}$$

holds.

Since finite generation of canonical ring serves as a birationally invariant property, we may assume

$$T' = [D] + \beta$$

after taking a log resolution $\mu : X' \to X$ of singularities of ψ'_L and E and replacing X by X', where β is a closed smooth positive $(1, 1)$ form outside the exceptional divisor $E' := Exc(\mu) = \sum a_j E'_j$ from (29) and $D + E' + E$ is an effective $\mathbb{Q}$-divisor with simple normal crossing support.

In view of (28) and (27), we also obtain that

$$\mathcal{I}(\varphi_D + \delta\varphi_{E'} + k^{-1}\varphi_E) = \mathcal{O}_X \tag{30}$$

for some small $\delta > 0$. It follows from Lemma 13.18 in [10] that one may pick a sequence of rational number $\delta_j \leq \delta$ and $\varepsilon_0 > 0$ and a smooth $(1,1)$ form u_j representing $c_1(E'_j)$ such that $\beta - \sum \delta_j a_j u_j = T' - [D] - \sum \delta_j a_j u_j \geq \varepsilon_0 \omega$ becomes a Kähler form.

In summary, we can write $L + k^{-1}A \sim_{\mathbb{Q}} B_k + H_k$ for some ample H_k and effective B_k with simple normal crossing support, where $B_k := D + \sum \delta_j a_j E'_j$ and $H_k := L + k^{-1}A - \sum \delta_j a_j E'_j - D$. Note that $(X, \Delta_k := B_k + k^{-1}E)$ is klt thanks to (30), the canonical ring of $K_X + \Delta_k + H_k$ is finitely generated by Theorem 1.1. From the following equivalence relation

$$(1 + k^{-1})(K_X + L) \sim_{\mathbb{Q}} K_X + \Delta_k + H_k$$

on X, we are done. □

Let $\Omega \subset \mathbb{C}^n$ be a domain and φ a plurisubharmonic function defined on Ω. We say that φ has *analytic singularities* if

$$\varphi = c \log(|f_1|^2 + \cdots + |f_N|^2) + O(1)$$

near any point $a \in \Omega$ for some holomorphic functions f_j and $c > 0$.

Lemma 3.2 (Theorem 4.3 in [20]). *Let φ, ψ be two plurisubharmonic functions on Ω. Assume that φ and ψ are v-equivalent and φ has analytic singularities, then φ is less singular than ψ.*

The proof of Lemma 3.2 is based primarily on Demailly's approximation of plurisubharmonic functions via Bergman kernels ([10]). Some details of the proof are omitted here.

Let X be a projective manifold, L a line bundle over X endowed with a positively curved metric h_L and S as in (8), $\varphi_{S,\varepsilon_\bullet}$, $\psi_{S,\delta_\bullet}$ defined as in (5), (10).

Lemma 3.3. *If S is finitely generated, then for a suitable choice of $\{\varepsilon_\bullet\}$* (resp. $\{\delta_\bullet\}$), *$\varphi_{S,\varepsilon_\bullet}$* (resp. $\psi_{S,\delta_\bullet}$) *has analytic singularities.*

Proof. Assume that S is finitely generated by $\bigoplus_{k=1}^{N} S_k$, then

$$S_p = \sum_{j_1+2j_2+\cdots+Nj_N=p} \prod_{k=1}^{N} S_k^{j_k}$$

holds for each $p > N$.

Thanks to this fact, for any $s^{(p)} \in S_p$, it can be expressed as

$$s^{(p)} = \sum_{j_1+2j_2+\cdots+Nj_N=p} \left(\sum_{0\le t_k\le q_k, 1\le k\le N} a_{t_1,\ldots,t_k} \prod_{l=1}^{N} (s_{t_l}^{(l)})^{j_l} \right),$$

where all $a_{t_1,\ldots,t_k}$ are complex numbers, $\{s_{t_l}^{(l)}\}_{0\le t_l\le q_l}$ forms a basis of S_l. The geometric-arithmetic mean inequality asserts that

$$|s^{(p)}|^{2/p} \le C_{1,p} \sum_{j_1+2j_2+\cdots+Nj_N=p} \left(\sum_{0\le t_k\le q_k, 1\le k\le N} a_{t_1,\ldots,t_k} \prod_{l=1}^{N} |s_{t_l}^{(l)}|^{2j_l/p} \right) \le$$

$$C_{2,p} \sum_{0\le t_l\le q_l, 1\le l\le N} |s_{t_l}^{(l)}|^{2/l} \tag{31}$$

for some $C_{1,p}, C_{2,p} > 0$.

As a consequence of (31), we obtain

$$|S_p|^{2/p} \le C_p \sum_{0\le t_l\le q_l, 1\le l\le N} |s_{t_l}^{(l)}|^{2/l} \tag{32}$$

for some $C_p > 0$. If our choice of $\{\varepsilon_\bullet\}$ ensures that

$$\sum_{p>N} \varepsilon_p C_p = C < \infty, \tag{33}$$

then it will be clear that $\varphi_{S,\varepsilon_\bullet}$ is equivalent to $\log\left(\sum_{j=1}^{N} |S_j|^{2/j}\right)$ from (32) and (33), which finishes the first part of our proof. As for $\psi_{S,\delta_\bullet}$, we can show the statement in the same way. □

Remark 3.1. *There is a similar argument for a graded system of ideals* $\{\mathfrak{a}_\bullet\}$ *and one may refer to Lemma 5.9 in [23] for more details.*

We now assume that L is big. Combining the strong openness property ([17]) of multiplier ideal sheaves and Theorem 1.11 in [11], it is known that:

Lemma 3.4. *$\mathcal{J}(c \cdot \|L\|) = \mathcal{I}(c \cdot \varphi_{\min})$ for any $c > 0$.*

Remark 3.2. *If we take $S = R(X, L)$ to be the full graded linear system, then Proposition 9.2.22 in [22] tells us that*

$$\mathcal{J}(c \cdot \mathfrak{b}_\bullet) = \mathcal{J}(c \cdot \|L\|).$$

Therefore, $\mathcal{J}(c \cdot \mathfrak{b}_\bullet) = \mathcal{I}(c \cdot \varphi_{\min})$ when L is big.

The following lemma characterizes finite generation of S via singularities of the family of metrics $\{\phi_m := \log |S_{m!}|^{2/m!}\}$. By convention (4), we already know that $\phi_m \geq \phi_l$ for any $m \geq l$.

Lemma 3.5 (Theorem 6.6 in [5])**.** *$\phi_m = \phi_l + O(1)$ for any $m \geq l$ if and only if S is finitely generated.*

Remark 3.3. *The original statement in [5] considers S to be the full linear system, while it still works for general S through a similar proof.*

4. Proof of theorem 1.4

Proof.

1. By contradiction, we assume that S is not finitely generated. We claim that for each $l \in \mathbb{N}$, $\sup_{1\leq k\leq l} \psi_k$ cannot have minimal singularities. In fact, If $\sup_{1\leq k\leq l} \psi_k$ has minimal singularities for some l, Lemma 2.1 tells us that ϕ_l is less singular than $\sup_{1\leq k\leq l} \psi_k$ and therefore ϕ_l is minimal. By the basic fact that $\phi_m \geq \phi_l$ for any $m \geq l$, we understand that ϕ_m is likewise minimal and hence $\phi_m = \phi_l + O(1)$ holds. Applying Lemma 3.5 it contradicts the fact that S is not finitely generated.
   Let

$$E_{\delta_\bullet} := \{x \in X; \sup_{k\geq 1} \delta_k B_{km}^{1/k}(x) \neq B_{S,\delta_\bullet}(x)\} \tag{34}$$

   be a measure zero subset of X. By our assumption, it suggests that there exist $\{x_l\} \subset X \backslash E_{\delta_\bullet}$ and $C_l \to +\infty$ such that

$$\sup_{1\leq k\leq l} \psi_k(x_l) \leq \varphi_{\min}(x_l) - C_l. \tag{35}$$

   Therefore, one obtains an inequality

$$\sup_{k\geq 1} \delta_k B_{km}^{1/k}(x_l) \leq e^{\varphi_{\min}(x_l)} \max\{(\sup_{1\leq k\leq l} \delta_k) e^{-C_l}, \sup_{k\geq l} \delta_k B_{km}^{1/k}(x_l) e^{-\varphi_{\min}(x_l)}\} \tag{36}$$

from (35). According to Lemma 2.2,

$$B_{km}^{1/k} e^{-\varphi_{\min}} \le (\sup_{k\ge 1} B_{km}^{1/k})^* e^{-\varphi_{\min}} \le e^C \tag{37}$$

for some $C > 0$ independent of k. Note that

$$(\sup_{1\le k\le l} \delta_k) e^{-C_l} \to 0, \sup_{k\ge l} \delta_k \to 0$$

as $l \to \infty$, we infer

$$\log B_{S,\delta_\bullet}(x_l) \le \varphi_{\min}(x_l) + \max\{(\sup_{1\le k\le l} \delta_k) e^{-C_l}, C + \log(\sup_{k\ge l} \delta_k)\} \to -\infty$$

as $l \to \infty$ by (36) and (37), which contradicts the fact that $B_{S,\delta_\bullet}^{-1}$ has minimal singularities.

2. Thanks to Proposition 2.1 and Lemma 2.1, there exists a Siu-type metric $\varphi_{S,\varepsilon'}$ such that $\varphi_{S,\varepsilon'} \succeq \psi_{S,\delta} \sim \phi_{m_0}$ for some m_0. By the construction of Siu-type metric, It follows that $\phi_l \sim \phi_{m_0}$ for all $l \ge m_0$. Therefore, our claim follows from Lemma 3.5.
3. By Theorem 1.2, we infer that S is finitely generated. According to Lemma 3.3, $\psi_{S,\delta_\bullet^0}$ has analytic singularities for an appropriate choice of $\{\delta_\bullet^0\}$. Thanks to Lemma 3.4 and Theorem 1.5 (see also Remark 2.4), we get $\psi_{S,\delta_\bullet^0} \sim_v \varphi_{\min}$. Now applying Lemma 3.2 we can show $\psi_{S,\delta_\bullet^0} \sim \varphi_{\min}$. Furthermore, by the fact that $\psi_{S,\delta_\bullet^0}$ is actually equivalent to a finite supreme, we obtain $\psi_{S,\delta_\bullet^0} \succeq \psi_{S,\delta_\bullet^1}$ for any admissible choice of $\{\delta_\bullet^1\}$. Therefore, $\psi_{S,\delta_\bullet^1}$ has minimal singularities as well. □

5. Applications

5.1. Equivalence of admissible Bergman metrics

Let X be a projective manifold, L a line bundle over X. We endow $L - K_X$ with a smooth metric h_{L-K_X} and $S = R(X, L)$ as in (8), $\varphi_{S,\varepsilon_\bullet}$, $\psi_{S,\delta_\bullet}$ defined as (5), (10). In this subsection, we discuss how the singularities of $\psi_{S,\delta_\bullet}$ are affected by the choice of $\{\delta_\bullet\}$. In [20], the author has already shown that there exists an infinite family of Siu-type metrics having non-equivalent singularities for some L.

Theorem 5.1 (Theorem 3.5 in [20]). *Fix an admissible $\{\varepsilon_\bullet\}$ and assume $S = R(X, L)$ is not finitely generated. Then there always exists another Siu-type metric $h' = e^{-\varphi_{S,\varepsilon'_\bullet}}$ such that $\varphi_{S,\varepsilon'_\bullet}$ is strictly singular than $\varphi_{S,\varepsilon_\bullet}$* (denoted by $\varphi_{S,\varepsilon'_\bullet} \succ \varphi_{S,\varepsilon_\bullet}$).

The idea of proof of Theorem 5.1 mainly follows from the argument of the proof of Proposition 6.5 in [5]. As stated in [20], such an example of L due to Zariski is precisely given as follows

Example 5.1 (Section 2.3.A in [22])**.** *Let X be the blow up of $\mathbb{P}^2$ at 12 generic points belonging to an elliptic curve $C_0 \subset \mathbb{P}^2$, L given as in Section 2.3.A in [22]. Then L is big and nef, but $R(X, L)$ is not finitely generated. The line bundle L has the property that there exists a curve C such that $|kL - C|$ is free for any $k \geq 1$. Indeed, C is the strict transformation of C_0. If C is written as $\{z = 0\}$ in a local chart $(U, (z, w))$, then*

$$\varphi_{R(X,L),\varepsilon_\bullet} = \log \sum_{k=1}^{\infty} \varepsilon_k |z|^{2/k} + O(1)$$

and

$$\psi_{R(X,L),\delta_\bullet} = \sup_{k\in\mathbb{N}}{}^{*}(\log \delta_k + k^{-1} \log |z|^2) + O(1) \tag{38}$$

along $\{z = 0\}$. If we choose $\{\delta_k^0 = 1\}$, then $\psi_{R(X,L),\delta_\bullet^0}$ becomes a bounded weight with minimal singularities (see Remark 1.4). It is clear that $\varphi_{R(X,L),\varepsilon_\bullet}$ is not equivalent to any finite sum as $\log \sum_{k=1}^{k_0} \varepsilon_k |z|^{2/k}$ and will never have minimal singularities. If we choose $\{\delta_k^1 = e^{-k/2}\}$ for instance, then

$$\psi_{R(X,L),\delta_\bullet^1} \leq -2\sqrt{-\log |z|} + O(1)$$

along C from (38) and is likewise non-minimal. Note $\psi_{R(X,L),\delta_\bullet}$ is not equivalent to any finite supremum as $\sup_{1\leq k\leq k_0}(\log \delta_k + k^{-1} \log |z|^2)$ for any admissible $\{\delta_\bullet\}$.

We can also establish an analogue of Theorem 5.1 with respect to admissible Bergman metrics via a similar method as in [20], as well as the proof of Theorem 1.4 1). This family seems to be "bigger" because it contains a metric with minimal singularities as illustrated in Example 5.1.

Corollary 5.1. *Fix an admissible $\{\delta_\bullet\}$ and assume L is not finitely generated. Then there always exists another metric $h' = e^{-\psi_{S,\delta'_\bullet}}$ such that $\psi_{S,\delta'_\bullet} \succ \psi_{S,\delta_\bullet}$, which means $\psi_{S,\delta'_\bullet}$ is strictly singular than $\psi_{S,\delta_\bullet}$.*

Proof. By our assumption and Theorem 1.4 2), $\psi_{S,\delta_\bullet}$ is not equivalent to any finite supreme $\sup_{1\leq k\leq k_0} \psi_k$ for any $k_0 \in \mathbb{N}$. First choose $\{\delta'_\bullet\}$ such that lim

$\delta'_k/\delta_k = 0$ and let $E_{\delta'_\bullet}$ be defined as in (34). By this assumption, there exist $\{x_l\} \subset X \backslash E_{\delta'_\bullet}$ and $C_l \to +\infty$ such that

$$\sup_{1 \le k \le l} \psi_k(x_l) \le \psi_{S,\delta_\bullet}(x_l) - C_l. \tag{39}$$

Then it follows from (39) that:

$$\sup_{k \ge 1} \delta'_k B_{km}^{1/k}(x_l) \le e^{\psi_{S,\delta_\bullet}(x_l)} \max\{(\sup_{1 \le k \le l} \delta'_k) e^{-C_l}, \sup_{k \ge l}(\delta'_k/\delta_k)\} \le$$

$$e^{\psi_{S,\delta_\bullet}(x_l)} \sup_{k \ge l}(\delta'_k/\delta_k) \to 0 \tag{40}$$

as $l \to \infty$, provided that C_l growing fast enough such that

$$(\sup_{1 \le k \le l} \delta'_k) e^{-C_l} \le \sup_{k \ge l}(\delta'_k/\delta_k)$$

holds. Now (40) meets what we want. □

5.2. A criterion for semi-ampleness

Let X be a projective manifold, L a line bundle over X endowed with a positively curved metric h_L, S as in (8), $\varphi_{S,\varepsilon_\bullet}$, $\psi_{S,\delta_\bullet}$ defined as in (5), (10) and $m \in \mathbb{N}$. The numerical dimension of L (Remark 2.3.17 in [22]) denoted by $n(L)$ is defined to be the largest integer $p \in \mathbb{N}$ such that $L^p \cdot V > 0$ for some p-dimensional subvariety V.

Definition 5.1.

1. *L is called* nef *if $L \cdot C \ge 0$ for any curve C in X.*
2. *L is called* good (or abundant) *if $n(L) = \kappa(L)$.*
3. *L is called* semi-ample *if $L^{\otimes k}$ is free for some $k \in \mathbb{N}$.*

Assume that $\left(X, \frac{L}{m}\right)$ is a pseudo-klt pair and $mK_X + L$ is $\mathbb{Q}$-effective, then $S = R(X, mK_X + L)$. The following result illustrates when its canonical bundle is semi-ample.

Theorem 5.2 (Theorem 1.4 in [21])**.** *$mK_X + L$ is semi-ample if and only if $\varphi_{S,\varepsilon_\bullet}$ is locally bounded for some admissible $\{\varepsilon_\bullet\}$.*

Remark 5.1. *In the above theorem, the last statement is also equivalent to the statement that $\varphi_{S,\varepsilon_\bullet}$ is locally bounded for* any *admissible $\{\varepsilon_\bullet\}$ by its proof.*

The proof of Theorem 5.2 relies on a result of Russo ([29]):

Proposition 5.1 (Theorem 2 in [29]). *$mK_X + L$ is nef and good if and only if $\mathcal{J}(c \cdot \|mK_X + L\|) = \mathcal{O}_X$ for c sufficiently large.*

Using Theorem 1.4 we can also give the following analytic characterization of semi-ampleness in terms of admissible Bergman metrics.

Corollary 5.2. *$mK_X + L$ is semi-ample if and only if $\psi_{S,\delta_\bullet}$ is locally bounded for some $\lim \delta_k = 0$.*

Proof. One direction is clear. It suffices to show that $mK_X + L$ is nef, good and finitely generated when $\psi_{S,\delta_\bullet}$ is locally bounded. According to Theorem 1.5, we get that $\mathcal{J}(c \cdot \|mK_X + L\|) = \mathcal{I}(c \cdot \psi_{S,\delta_\bullet}) = \mathcal{O}_X$ for any $c > 0$. Hence $mK_X + L$ is nef and good from Proposition 5.1. On the other side, since $\psi_{S,\delta_\bullet}$ has minimal singularities we infer that $mK_X + L$ is finitely generated thanks to Theorem 1.4 1), which accomplishes our proof. ☐

Remark 5.2. *The last statement can be also replaced by "$\psi_{S,\delta_\bullet}$ is locally bounded for* any *$\lim \delta_k = 0$".*

5.3. Variation of metrics with minimal singularities on f-big line bundle $mK_{X/Y} + L$ with klt singularities of h_L under projective deformation

In this subsection, we focus on the problem of gluing fiberwise closed positive $(1, 1)$ currents with minimal singularities as a global closed positive $(1, 1)$ current with minimal singularities. This type of results has been systematically studied in [38], [8], [15], etc.

The well-known Ohsawa-Takegoshi L^2 extension theorem [25] will be a useful tool for us. For the sake of brevity, we will only look at $L^{2/m}$ extension theorems for regular hypersurfaces, though they can be generalized to higher codimensional cases.

Lemma 5.1 (Theorem 0.3 in [3]). *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex open set and $\sigma : \Omega \to \mathbb{D}$ a holomorphic function. Assume that $d\sigma$ does not vanish on the hypersurface $V := \sigma^{-1}(0)$. Suppose that φ is a plurisubharmonic function on Ω such that $\varphi|_V \not\equiv -\infty$. Then for each holomorphic section $f \in H^0(V, K_V^{\otimes m})$ with $m \in \mathbb{N}$ and*

$$\int_V |f|^{2/m} e^{-\varphi} < \infty,$$

there is a section $F \in H^0(\Omega, K_\Omega^{\otimes m})$ such that $F|_V = f \wedge d\sigma^{\otimes m}$ and

$$\int_\Omega |F|^{2/m} e^{-\varphi} \leq C_0 \int_V |f|^{2/m} e^{-\varphi}$$

holds, where C_0 is the same constant in Ohsawa-Takegoshi extension theorem.

Let us recall some classical settings to define the relative Bergman metrics.

Let X, Y be connected complex manifolds and $f : X \to Y$ a proper surjective holomorphic map with connected fibers. Let $L \to X$ be a pseudoeffective line bundle equipped with a singular metric h_L with semi-positive curvature current. Let $K_{X/Y} := K_X - f^*K_Y$ be the relative canonical bundle and let Y_0 be the set of regular values of f,

$$Y_{m,\text{ext}} := \{y \in Y_0; h^0(X_y, mK_{X_y} + L_{X_y}) = \text{rank}\, f_*(mK_{X/Y} + L)\},$$

and respectively $X_0 := f^{-1}(Y_0)$, $X_{m,\text{ext}} = f^{-1}(Y_{m,\text{ext}})$. Let $Y_h := \{y \in Y_0; h|_{X_y} \not\equiv +\infty\}$ be a set whose complement has zero measure in Y. It is well known that $Y_{m,\text{ext}}$ is a Zariski open subset of Y. Assume that

$$H^0(X_y, (mK_{X_y} + L_{X_y}) \otimes \mathcal{I}_m(h_L^{1/m})) \neq 0$$

holds for some $y \in Y_{m,\text{ext}} \bigcap Y_h$. We can now assign the m-Bergman metric (7), denoted by $B_{m,y}^{-1}$, to each fiber X_y where $y \in Y_{m,ext}$. Gluing them together we may endow $(mK_{X/Y} + L)|_{X_{m,\text{ext}}}$ with a metric $B_{m,X/Y}^{-1}$, which is known as the relative m-Bergman metric ([2], [27], [41], etc).

Theorem 5.3 ([2], [27], and [18] for $m = 1$)**.** *If f is projective, then $B_{m,X/Y}^{-1}$ is lower semi-continuous, positively curved. Moreover, it can be naturally extended across $X \backslash X_{m,\text{ext}}$ as a new metric with semi-positive curvature current.*

Remark 5.3. *When $f : X \to Y$ is merely supposed to be a Kähler fiber space, which means, f is a proper surjective holomorphic map with connected fibers and the total space X is Kähler, this result was obtained in [40] for $m = 1$ by using Guan-Zhou method in [18]. For $m \geq 2$, this result was independently obtained in [7] when X, Y are compact, and in [41] (see Theorem 5.4 below) for the general case by overcoming some extension difficulties for $m \geq 2$ pointed out in Remark 4.2.4 in [27]. Both methods involve using the optimal L^2 and $L^{2/m}$ extension results (i.e., the constant $C_0 = 2\pi$ in Lemma 5.1). One might also refer to [14] for another new proof of the log-plurisubharmonicity of the relative m-Bergman kernels.*

From now on, we assume that f is projective and denote $B_{m,X/Y}^{-1}$ by its natural extension according to Theorem 5.3.

Lemma 5.2 (Corollary 4.3.2 in [27])**.** *Let $B_{km,X/Y}^{-1}$ be the relative km-Bergman metric defined on $(kmK_{X/Y} + kL)$. Then there is a constant C independent* *of k such that*

$$B^{1/k}_{km,X/Y} \leq C$$

locally on X.

This proof originally dates back to Theorem 4.2.7 in [27], which essentially uses Lemma 5.1 to give the uniform upper bound of the weight. Now the metric

$$B^{-1}_{m,X/Y,\infty} := (\inf_{k\in\mathbb{N}} B^{-1/k}_{km,X/Y})^* \tag{41}$$

on $mK_{X/Y} + L$ is well-defined and we can prove the following result.

Corollary 5.3. *Let X, Y be connected complex manifolds and $f : X \to Y$ a proper surjective projective morphism with connected fibers. Let $L \to X$ be a pseudoeffective line bundle equipped with a singular metric h_L with semi-positive curvature current. Assume that*

$$Y^0 := \{y \in Y_0; \mathcal{I}(h_L|_{X_y}) = \mathcal{O}_{X_y}\} \tag{42}$$

is a Zariski open subset of Y for some $m \in \mathbb{N}$. Assume also that $mK_{X_y}+L_{X_y}$ is big for each $y \in Y^0$, then there exists a singular metric h on $mK_{X/Y} + L$ such that:

1. *h has semi-positive curvature current.*
2. *$h|X_y$ has minimal singularities for any $y \in Y^0$.*

Proof. Let

$$h := B^{-1}_{m,X/Y,\infty}$$

be a singular metric as in (41) on $mK_{X/Y} + L$ with semi-positive curvature current, according to Theorem 5.3. In this setting, we get $Y^0 \subset Y_{km,\mathrm{ext}}$ by invariance of plurigenera (Theorem 16.2 in [10]). Therefore,

$$h|X_y = (\inf_{k\in\mathbb{N}} B^{-1/k}_{km,X/Y})^*|_{X_y} \leq (\inf_{k\in\mathbb{N}} B^{-1/k}_{km,X/Y}|_{X_y})^* = (\inf_{k\in\mathbb{N}} B^{-1/k}_{km,y})^* \tag{43}$$

holds for all $y \in Y^0$. The right hand side of (43) has minimal singularities thanks to the fact that $\mathcal{I}(h_L|X_y) = \mathcal{O}_{X_y} \Rightarrow \mathcal{I}(h_L^{1/m}|X_y) = \mathcal{O}_{X_y}$ and Theorem 1.4 3). Therefore, $h|X_y$ has minimal singularities for any $y \in Y^0$. □

Remark 5.4. *We have the following remarks:*

1. *If $\mathcal{I}(h_L) = \mathcal{O}_X$ and h_L has analytic singularities, (42) will be satisfied (Section 9.5.D in [22]).*

2. *Invariance of plurigenera can also be deduced from the conclusion of Corollary 5.3 and the Ohsawa-Takegoshi L^2 extension theorem ([25]).*
3. *If condition (42) is replaced by the following assumption: $\frac{1}{m}L$ is still a genuine line bundle and $Y^0 := \{y \in Y_0; \mathcal{I}(h_L^{1/m}|_{X_y}) = \mathcal{O}_{X_y}\}$ is a Zariski open subset of Y for some $m \in \mathbb{N}$, then the conclusion of Corollary 5.3 still holds via the same proof as above.*
4. *An alternative way to show that $h|X_y$ has minimal singularities has been discussed in Remark 1.4.*

It is tempting to ask whether $h = B^{-1}_{m,X/Y,\infty}$ has minimal singularities on $X^0 = f^{-1}(Y^0)$ as in [15]. To be more precise, we suppose that $Y = Y^0 = \mathbb{B}^m$ is a unit ball in $\mathbb{C}^m$, $V \subset\subset Y$ a relative compact open subset. Fix any smooth metric h_∞ on $mK_{X/Y} + L$ and set

$$h_0 := \inf\{h : \sqrt{-1}\Theta_h(mK_{X/Y} + L) \geq 0, h \geq h_\infty\}, h_0 = e^{-\varphi_0} \tag{44}$$

to be the lower envelope of all positively curved metric h on $mK_{X/Y}+L$. Thus h_0 has minimal singularities on $f^{-1}(V)$, which does not rely on the choice of h_∞. Our answer to this question in some special cases are as follows.

Corollary 5.4. *Let the settings be as in Corollary 5.3 or Remark 5.4* 3)*. Assume moreover that h_L has analytic singularities, then there exists a constant $C = C_V$, such that*

$$C^{-1}h_0 \leq h \leq C \cdot h_0 \tag{45}$$

holds on $f^{-1}(V)$.

Proof. The left hand side is just an easy consequence of the first statement of Corollary 5.3 and (44). Therefore, it remains to show the right hand side inequality. First note that $h_0|X_y$ is a well-defined singular metric on $mK_{X_y}+L$ with semi-positive curvature current. By our assumption on h_L, Lemma 3.2 in [12] and the fact that φ_0 is locally bounded above, there exists a constant $C = C_V$ independent of y such that

$$\int_{X_y} e^{\frac{\varphi_0 - \varphi_L}{m}} \leq C \tag{46}$$

holds for any $y \in V$. Thanks to (46) and Proposition 19.8 in [10], one may find a sequence of sections $\sigma_{k,y} \in H^0(X_y, kmK_{X_y} + kL|_{X_y})$ such that

$$\left(\frac{\varphi_0}{m} - \log C\right)|_{X_y} = \left(\limsup_{k\to\infty} \frac{1}{mk} \log |\sigma_{k,y}|^2\right)^*, \tag{47}$$

where

$$\int_{X_y} |\sigma_{k,y}|^{2/km} h_L^{1/m} \le 1. \tag{48}$$

From the definition of km-Bergman metrics, (47) and (48), we obtain that

$$\frac{\varphi_0}{m} - \log C \le \frac{1}{m} \log B_{m,X/Y,\infty} \tag{49}$$

holds on $f^{-1}(V)$. Eventually, (45) comes from taking exponential on both sides of (49) and replacing C^m by a new constant C. □

5.4. Variation of metrics with minimal singularities on f-big line bundle $mK_{X/Y} + L$ with mild singularities of h_L under Kähler fibration

Our goal in this subsection is to extend Corollary 5.3 in the Kähler fiber space setting. Precisely speaking, let $f : X \to Y$ be a proper surjective holomorphic map from a Kähler manifold X to a connected complex manifold Y with connected fibers. Let $L \to X$ be a pseudoeffective line bundle equipped with a singular metric h_L with semi-positive curvature current. In addition, we require the singularities of metric h_L of the twisted line bundle L to be "mild" enough.

Before stating the main result of this subsection, let us first list some important results which we will use later. Let $X_{m,ext}$ and the relative Bergman metric $B^{-1}_{m,X/Y}$ be defined as in subsection 5.3. As mentioned in Remark 5.3, Zhou and Zhu has obtained that

Theorem 5.4 (Theorem 1.5 in [41]). *$B^{-1}_{m,X/Y}$ is lower semi-continuous, positively curved. Moreover, it can be naturally extended across $X\backslash X_{m,\text{ext}}$ as a new metric with semi-positive curvature current.*

The proof of Theorem 5.4 is essentially based on the fact (part of Theorem 1.4 in [41]) that, once $F \in H^0(X_y, mK_{X_y} + L|_{X_y})$ has an extension F_1 on a neighborhood of X_y, it has another extension F_2 with an additional optimal $L^{2/m}$ estimate. This improvement of [41] is crucial for the proof of Theorem 5.4 because we only know this fact under the additional assumption that F_1 is locally $L^{2/m}$ integrable near X_y before ([3], [27], [14], etc.).

We still denote $B^{-1}_{m,X/Y}$ by its natural extension on X and use Lemma 5.1 to give the uniform upper bound of the family of the Bergman kernels $\{B^{1/k}_{km,X/Y}\}_{k\in\mathbb{N}}$. The metric as in (41) on $mK_{X/Y} + L$ is also well-defined.

One main difficulty in this context is that, invariance of (twisted) plurigenera might be not that clear since f is not a projective morphism anymore. However, to conquer this, the following result in [28] tells us that f is not that far from a projective morphism if there exists a fiberwise big line bundle in the total space.

Lemma 5.3 (A particular case of Theorem 1.8 in [28])**.** *Let $f : X \to Y = \Delta$ be a smooth family (i.e. a surjective proper holomorphic submersion over the unit disc) and assume that there exists a line bundle $\mathcal{L} \to X$ such that $\mathcal{L}|_{X_y}$ is big for any $y \in \Delta$. Then for some $N \in \mathbb{N}$, there exists a bimeromorphic map $\Phi : X \dashrightarrow X'$ over Δ, to a subvariety of $\mathbb{P}^N \times \Delta$ with every fiber X'_t being a projective variety of $\mathbb{P}^N$.*

In other words, $f : X \to \Delta$ in Lemma 5.3 is a Moishezon family (see [28]), which means bimeromorphic to a projective family (i.e. a surjective projective morphism over the unit disk). In [28], the authors have applied this result to answering a question proposed by Demailly on invariance of plurigenera.

Another significant result considering invariance of plurigenera in the context of Moishezon family and fiberwise canonical singularities at worst, was obtained in [37]. Note that the canonical singularity assumption aims at getting the upper semi-continuity of the function $y \mapsto h^0(X_y, mK_{X_y})$, where

$$h^0(X_y, mK_{X_y}) := \dim H^0(X_y, mK_{X_y}).$$

The lower semi-continuity part mainly follows from Siu's approach ([32]), with a slight modification of the use of Ohsawa-Takegoshi L^2 extension theorem (see Theorem 5.5 below).

The following result about birational invariance of plurigenera is well-known.

Proposition 5.2. *Let X and X' be two non-singular projective varieties and let $\mu : X' \to X$ be a proper modification, $L \to X$ be a line bundle. Then for any $m \in \mathbb{N}$, equality*

$$H^0(X, mK_X + L) = H^0(X', mK_{X'} + \mu^* L)$$

holds.

Proof. Since μ has connected fibers, we know that $\mu_* \mathcal{O}_{X'} = \mathcal{O}_X$. It is also easy to see that $\mu_* \mathcal{O}_{X'}(mK_{X'}) = \mathcal{O}_X(mK_X)$. Therefore, according to the projection formula we obtain that

$$H^0(X', mK_{X'} + \mu^* L) = H^0(X, \mu_*(mK_{X'} + \mu^* L)) = H^0(X, mK_X + L),$$

which completes the proof. □

We proceed to generalize Theorem 1.2 in [37] with a twisted line bundle L. Up till now, this generalization has to suppose h_L to be "mild" enough. The reason is that, for an arbitrary holomorphic map $f : X' \to X$ between two complex manifolds X', X and a plurisubharmonic function φ on X, one always has the inclusion $\mathcal{I}(f^*\varphi) \subset f^*\mathcal{I}(\varphi)$ by the restriction formula of multiplier ideal sheaves (Proposition 14.3 in [10]).

Lemma 5.4. *Let $f : X \to \Delta$ be a smooth Moishezon family. Let $L \to X$ be a pseudoeffective line bundle equipped with a singular metric h_L with semi-positive curvature current. Assume the Lelong number of $h_L|_{X_y}$ equals zero for each $y \in \Delta$, then the function $y \mapsto h^0(X_y, mK_{X_y} + L_{X_y})$ is constant.*

Proof. By Grauert's upper semi-continuity theorem, it suffices to show that $y \mapsto h^0(X_y, mK_{X_y}+L_y)$ is lower semi-continuous at $y = 0$. Denote the central fiber $f^{-1}(0)$ by X_0. As the proof of Theorem 1.2 in [37], since f is a Moishezon family, there exists a proper modification $\mu : X' \to X$ such that the following holds:

1. $f \circ \mu : X' \to \Delta$ is a projective family;
2. X'_0 has simple normal crossing supports;
3. X'_0 has a prime decomposition $X'_0 = Z_0 + \sum_{j=1}^{N} k_j Z_j$, where Z_0 is the strict transform of X_0, $Z_j (1 \le j \le N)$ is a family of smooth divisors and $k_j \ge 1$ are positive integers.

Note μ also induces a proper modification on each smooth fiber near $y = 0$, then by Proposition 5.2 we reduce our claim to the following inequality

$$h^0(X'_y, mK_{X'_y} + (\mu^*L)_{X_y}) \le h^0(Z_0, mK_{Z_0} + (\mu^*L)_{Z_0}) \tag{50}$$

for $y \in \Delta\backslash\{0\}$, after shrinking the unit disk if necessary.

Actually, (50) can be obtained by the following argument:

(51)

Any section of $\sigma \in H^0(Z_0, (mK_{Z_0} + \mu^*L) \otimes \mathcal{I}((\mu^*h_L)|_{Z_0}))$ extends to X.

The proof of the statement (51) is by almost the same method as Păun's one-tower proof which is explained in Theorem 16.2 in [10] (see also [26]). The only difference between Păun's proof and our proof here is that, in order to construct an asymptotic singular metric H (the same notation as p.176 in [10]) with semi-positive curvature current on $mK_{X'} + \mu^*L$ and extend σ via L^2 extension theory, Păun's proof strongly depends on Ohsawa-Takegoshi L^2 extension theorem (Lemma 16.3 in [10]) from smooth central fibers. While in our case the central fiber X'_0 is of simple normal crossing support, what we need is just extending pluricanonical sections on Z_0 and using the following L^2 extension Theorem 5.5 to substitute Lemma 16.3 in [10].

Theorem 5.5 (Lemma 3.6 in [37]). *For any holomorphic line bundle $F \to X'$ with a singular metric h_F of semi-positive curvature, and any $s \in H^0(Z_0, K_{Z_0} + F_{Z_0})$ with $\int_{Z_0} |s|^2 h_F < \infty$, there exists a uniform constant $C > 0$ and $\tilde{s} \in H^0(X', K_{X'} + F)$, such that $\tilde{s} = s \wedge dt$ on Z_0 with*

$$\int_X |\tilde{s}|^2 h_F \leq C \int_{Z_0} |s|^2 h_F.$$

Finally, applying Theorem 5.5 with

$$(F, h_F) = ((m-1)K_{X'} + \mu^* L, H^{\frac{m-1}{m}} (\mu^* h_L)^{\frac{1}{m}}),$$

one infers statement (51). By our assumption on h_L, we deduce that the Lelong numbers of $\mu^* h_L$ restricted on Z_0 are zero because $\varphi_L|_{X_0}$ is v-equivalent to constant functions. Therefore, $\mathcal{I}((\mu^* h_L)|_{Z_0})$ is trivial and $H^0(Z_0, mK_{Z_0} + \mu^* L) = H^0(Z_0, (mK_{Z_0} + \mu^* L) \otimes \mathcal{I}((\mu^* h_L)|_{Z_0}))$ and (50) holds true. □

Remark 5.5. *When L is semi-positive, which means, there exists a smooth metric h_L with semi-positive $(1,1)$ curvature form, it is clear that the assumption in Lemma 5.4 is satisfied. When L is trivial, one may refer to Lemma 5.2 in [15] for another proof of this result.*

Now we are in position to show the main result of this subsection.

Corollary 5.5. *Let X be a Kähler manifold, Y a connected complex manifold and let $f : X \to Y$ be a proper surjective holomorphic map with connected fibers. Let $L \to X$ be a pseudoeffective line bundle equipped with a singular metric h_L with semi-positive curvature current. Assume that $mK_{X_y} + L_{X_y}$ is big and the Lelong number of $h_L|_{X_y}$ equals zero for each $y \in Y^0$, where Y^0 is contained in the regular values of f and $Y \backslash Y^0$ is an analytic subset of Y. Then there exists a singular metric h on $mK_{X/Y} + L$ such that:*

1. *h has semi-positive curvature current.*
2. *$h|X_y$ has minimal singularities for any $y \in Y^0$.*

Proof. Let

$$h := B^{-1}_{m,X/Y,\infty}$$

be a singular metric as in (41) on $mK_{X/Y} + L$ with semi-positive curvature current, according to Theorem 5.4. In this setting, f is a smooth Moishezon family thanks to Lemma 5.3. Then we also obtain that $y \mapsto h^0(X_y, kmX_y + kL_{X_y})$ is a constant function on Y^0 for any $k \in \mathbb{N}$, thanks to Lemma 5.4. Therefore, by Grauert's base change theorem, we obtain that $Y^0 \subset Y_{km,\mathrm{ext}}$. As a result,

(52)
$$B^{-1}_{m,X/Y,\infty}|_{X_y} = (\inf_{k\in\mathbb{N}} B^{-1/k}_{km,X/Y})^*|_{X_y} \leq (\inf_{k\in\mathbb{N}} B^{-1/k}_{km,X/Y}|_{X_y})^* = (\inf_{k\in\mathbb{N}} B^{-1/k}_{km,y})^*$$

holds for all $y \in Y^0$. The right hand side of (52) has minimal singularities thanks to Theorem 1.4 3) (or Remark 1.4). The fact that $h|X_y$ has minimal singularities comes from (52). □

Remark 5.6. *Under the assumption of Corollary 5.5, a paralleled result of Theorem 1.7 in [8] asserts that there exists a singular metric h_{KE} on $mK_{X/Y}+L$ with semi-positive curvature current such that it is the unique singular Kähler-Einstein metric with minimal singularities when restricting to each fiber X_y for $y \in Y^0$. The reason is that, our assumption already implies that h_L has zero Lelong numbers at each point of X.*

Remark 5.7. *When L is trivial, we can argue as in the proof of Corollary 5.4 to obtain that h has minimal singularities on $f^{-1}(Y^0)$ as well, which can be seen as a parallel result of Theorem C in [15].*

The key difference between the proof of Corollary 5.5 and Corollary 5.3 is that, when f is not necessarily a projective holomorphic map, invariance of twisted plurigenera with mild singularities still works and $B^{-1}_{km,X/Y}$ on $X_{km,ext}$ still has semi-positive curvature current for any integer k.

References

[1] B. Berndtsson, The openness conjecture for plurisubharmonic functions. ArXiv:1305.5781.

[2] B. Berndtsson and M. Păun, Bergman kernels and pseudoeffectivity of relative canonical bundles. *Duke Math. J.* **145** (2008), 341–378. MR2449950

[3] B. Berndtsson and M. Păun, A Bergman kernel proof of the Kawamata adjunction theorem. ArXiv:0804.3884v2.

[4] C. Bikar, P. Cascini, C. Hacon and J. McKernan, Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.* **23** (2010), 405–468. MR2601040

[5] S. Boucksom, P. Eyssidieux, V. Guedj and A. Zeriahi, Monge-Ampère equations in big cohomology classes. *Acta Math.* **205** (2010), 199–262. MR2746347

[6] S. Boucksom, C. Favre and M. Jonsson, Valuation and plurisubharmonic singularities. *Publ. Res. Inst. Math. Sci.* **44** (2008), 449–494. MR2426355

[7] J. Y. Cao, Ohsawa-Takegoshi extension theorem for compact Kähler manifolds and applications. *Springer INdAM Ser.* **21** (2017), 19–38. MR3645303

[8] J. Y. Cao, H. Guenancia and M. Păun, Variation of singular Kähler-Einstein metrics: positive Kodaira dimension. *J. Reine Angew. Math.* **779** (2021), 1–36. MR4319060

[9] J. P. Demailly, Regularization of closed positive currents and intersection theory. *J. Algebraic Geom.* **1** (1992), 361–409. MR1158622

[10] J. P. Demailly, *Analytic methods in algebraic geometry*. International Press. (2012). MR2978333

[11] J. P. Demailly, L. Ein and R. Lazarsfeld, A subadditivity property of multiplier ideals. *Michigan Math. J.* **48** (2000), 137–156. MR1786484

[12] J. P. Demailly and J. Kóllar, Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds. *Ann. Sci. École Norm. Sup.* **34** (2001), 525–556. MR1852009

[13] J. P. Demailly, T. Peternell and M. Schneider, Pseudo-effective line bundles on compact Kähler manifolds. *Internat. J. Math.* **12** (2001), 689–741. MR1875649

[14] F. S. Deng, Z. W. Wang, L. Y. Zhang and X. Y. Zhou, New characterizations of plurisubharmonic functions and positivity of direct image sheaves. ArXiv:1809.10371v1.

[15] E. D. Nezza, V. Guedj and H. Guenancia, Families of singular Kähler-Einstein metrics. ArXiv:2003.08178v2.

[16] O. Fujino and S. Mori, A canonical bundle formula. *J. Differential Geom.* **56** (2000), 167–188. MR1863025

[17] Q. A. Guan and X. Y. Zhou, A proof of Demailly's strong openness conjecture. *Ann. Math.* **182** (2015), 605–616. MR3418526

[18] Q. A. Guan and X. Y. Zhou, A solution of an L^2 extension problem with an optimal estimate and applications. *Ann. Math.* **181** (2015), 1139–1208. MR3296822

[19] C. Hacon, M. Popa and C. Schnell, Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Păun. *Contemp. Math.* **712** (2018), 143–195. MR3832403

[20] D. Kim, Equivalence of plurisubharmonic singularities and Siu-type metrics. *Monatsh. Math.* **178** (2015), 85–95. MR3384891

[21] D. Kim, Analytic characterization of nef and good line bundles. *Bull. Belg. Math. Soc.* **27** (2020), 535–535. MR4177392

[22] R. Lazarsfeld, *Positivity in Algebraic Geometry I and II.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, 48–49, Springer-Verlag, Berlin, 2004. MR2095471 and MR2095472

[23] Z. Q. Li, A note on multiplier ideal sheaves on complex spaces with singularities. ArXiv:2003.11717v1.

[24] A. M. Nadel, Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature. *Ann. Math.* **132** (1990), 549–596. MR1078269

[25] T. Ohsawa and K. Takegoshi, On the extension of L^2 holomorphic functions. *Math. Z.* **195** (1987), 197–204. MR0892051

[26] M. Păun, Siu's invariance of plurigenera: a one-tower proof. *J. Differential Geom.* **76** (2007), 485–493. MR2331528

[27] M. Păun and S. Takayama, Positivity of twisted relative pluricanonical bundles and their direct images. *J. Algebraic Geom.* **27** (2018), 211–272. MR3764276

[28] S. Rao and I. H. Tsai, Invariance of plurigenera and Chow-type lemma. ArXiv:2011.03306.

[29] F. Russo, A characterization of nef and good divisors by asymptotic multiplier ideals. *Bull. Belg. Math. Soc.* **16** (2009), 943–951. MR2574371

[30] H. Seo and D. Kim, Jumping numbers of analytic multiplier ideals (with an appendix by Sébastien Boucksom). *Ann. Polon. Math.* **124** (2020), 257–280. MR4085107

[31] Y. T. Siu, Invariance of plurigenera. *Invent. Math.* **134** (1998), 661–673. MR1660941

[32] Y. T. Siu, Invariance of plurigenera and torsion-freeness of direct image sheaves of pluricanonical bundles. *Adv. Complex Anal. Appl.* (2004), 45–83. MR2058399

[33] Y. T. Siu, Finite generation of canonical ring by analytic method. *Sci. China Ser. A* **51** (2008), 481–502. MR2395400

[34] Y. T. Siu, A general non-vanishing theorem and an analytic proof of the finite generation of the canonical ring. ArXiv:math.AG/0610740.

[35] Y. T. Siu, Additional explanatory notes on the analytic proof of the finite generation of the canonical ring. ArXiv:0704.1940.

[36] H. Skoda, Prolongement des courants positifs fermés de masse finie. *Invent. Math.* **66** (1982), 361–376. MR0662596

[37] S. Takayama, On the invariance and lower semi-continuity of plurigenera of algebraic varieties. *J. Algebraic Geom.* **16** (2007), 1–18. MR2257317

[38] H. Tsuji, Canonical singular Hermitian metrics on relative canonical bundles. *Amer. J. Math.* **133** (2011), 1469–1501. MR2863368

[39] H. Tsuji, Canonical singular Hermitian metrics on relative canonical bundles. ArXiv:1008.1466v2.

[40] X. Y. Zhou and L. F. Zhu, An optimal L^2 extension theorem on weakly pseudoconvex Kähler manifolds. *J. Differential Geom.* **110** (2018), 135–186. MR3851746

[41] X. Y. Zhou and L. F. Zhu, Siu's lemma, optimal L^2 extension and applications to twisted pluricanonical sheaves. *Math. Ann.* **377** (2020), 675–722. MR4099619

[42] X. Y. Zhou and L. F. Zhu, Subadditivity of generalized Kodaira dimensions and extension theorems. *Internat. J. Math.* **31** (2020), 36 pp. MR4184430

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