Moment polytopes on Sasaki manifolds and volume minimization

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Abstract: We show that transverse coupled Kähler-Einstein metrics on toric Sasaki manifolds arise as a critical point of a volume functional. As a preparation for the proof, we re-visit the transverse moment polytopes and contact moment polytopes under the change of Reeb vector fields. Then we apply it to a coupled version of the volume minimization by Martelli-Sparks-Yau. This is done assuming the Calabi-Yau condition of the Kähler cone, and the non-coupled case leads to a known existence result of a transverse Kähler-Einstein metric and a Sasaki-Einstein metric, but the coupled case requires an assumption related to Minkowski sum to obtain transverse coupled Kähler-Einstein metrics.

1. Introduction

Sasaki-Einstein metrics drew much attention from theoretical physics and mathematics during last two decades. The first breakthrough was an irregular example found in physics literature by Gauntlett, Martelli, Sparks and Waldrum [23]. Then in the toric case the existence was proven in our paper [20] using volume minimization of Martelli, Sparks and Yau [30], [31]. More recently it has been shown that the existence is equivalent to a notion called K-stability by Collins and Székelyhidi [7], [8]. Sasaki manifolds are characterized by two Kähler structures, one on the Riemannian cone and the other on the local orbit spaces of the one parameter group of transformations, which we call the *Reeb flow*, generated by the Reeb vector field. In fact, the existence of a Sasaki-Einstein metric is equivalent to the existence of a Ricci-flat Kähler metric on the Kähler cone, and also equivalent to the existence of a transverse Kähler-Einstein metric of positive scalar curvature on local orbit spaces of the Reeb flow. Therefore there are two possible extensions of these studies, one on the Kähler cone and the other on the Kähler local orbit spaces of the Reeb flow. In [10], de Borbon and Legendre used the volume minimization argument to prove the existence on toric Kähler cone manifolds

of Ricci-flat Kähler cone metrics with cone angle along the boundary invariant divisors without assuming the Calabi-Yau condition of the Kähler cone. This Calabi-Yau condition will be explained in the paragraph after Proposition 1.1 below. The purpose of this paper is to study the possibility to prove the existence on toric Sasaki manifolds of transverse coupled Kähler-Einstein metrics in the sense of Hultgren and Witt Nystrom [27] assuming the Calabi-Yau condition of the Kähler cone by using the volume minimization argument of Martelli-Sparks-Yau. Our study shows that the non-coupled transverse Kähler-Einstein metric recovers the toric Sasaki-Einstein metrics as in [20] but the coupled case requires an additional Minkowski sum assumption to obtain transverse coupled Kähler-Einstein metrics.

The transverse coupled Kähler-Einstein metrics are defined as follows. A Sasaki manifold S is determined by contact distribution D, pseudo-convex CR-structure J on D and Reeb vector field ξ. The pseudo-convex CR-structure determines Kähler structures on the local orbit spaces of the Reeb flow. Differential forms on S obtained by pulling back from those local orbit spaces are called *basic forms*. Naturally ∂ and $\overline{\partial}$ operators can be considered to operate on basic forms, which we denote by ∂_B and $\overline{\partial}_B$, and we obtain Dolbeault theory, Hodge theory and Chern-Weil theory for basic forms. Suppose that the basic first Chern class $c_1^B(S)$ is positive, i.e. represented by a real closed positive $(1,1)$-basic form, and that we are given a decomposition

$$(1.1) \qquad\qquad 2\pi c_1^B(S) = \gamma_1 + \cdots + \gamma_k$$

of $2\pi c_1^B(S)$ into a sum of basic Kähler classes γ_α. Basic Kähler metrics $\omega_\alpha \in \gamma_\alpha$ are called *transverse coupled Kähler-Einstein metrics* if

$$(1.2) \qquad\qquad \rho^T(\omega_1) = \cdots = \rho^T(\omega_k) = \sum_{\beta=1}^{k} \omega_\beta$$

where

$$\rho^T(\omega_\alpha) = -i\partial_B\overline{\partial}_B \log \omega_\alpha^m$$

is the transverse Ricci form of ω_α. Naturally, by the Chern-Weil theory,

$$2\pi c_1^B(S) = [\rho^T(\omega_\alpha)]_B$$

where $[\cdot]_B$ denotes a basic cohomology class.

As a preparation, we study the relation of transverse moment map image and the contact moment map image, and how the decompositions of the basic first Chern class induce the Minkowski sum decompositions of the image of

the contact moment map. The linkage of transverse moment map and the contact moment map is played by the conditions $c_1^B(S) > 0$ and $c_1(D) = 0$ where D is the contact distribution with complex structure J. As will be shown in Lemma 4.1 these two conditions imply that

$$c_1^B(S) = \tau[d\eta_\xi]_B$$

for some positive constant τ where η_ξ is the contact form with respect to the Reeb vector field ξ. Since the transverse moment map is with respect to the basic Kähler class $c_1^B(S)$ and the contact moment map is with respect to the contact form η_ξ we can compare the two moment maps. The result we obtain about the comparison is stated as follows.

Proposition 1.1. *Let (S, D, J, ξ) be a Sasaki manifold such that $c_1^B(S) > 0$ and $c_1(D) = 0$. Suppose that a real compact torus T acts effectively on S preserving (D, J, ξ) and that the Lie algebra $\mathfrak{t}$ of T contains ξ (but we do not need to assume S is toric in this proposition). Suppose also that we are given a decomposition* (1.1).

(1) There is a unique point $\mathbf{o}$, which we call the origin, in the image $\mathcal{P}_\xi \subset \{p \in \mathfrak{t}^* \,|\, \langle p, \xi \rangle = 1\}$ of the contact moment map and a Minkowski sum decomposition

$$(1.3) \qquad\qquad \mathcal{P}_\xi = \mathcal{P}_{\xi,1} + \cdots + \mathcal{P}_{\xi,k}$$

into the sum of convex polytopes $\mathcal{P}_{\xi,\alpha} \subset \mathcal{P}_\xi$, where we regard the hyperplane $\{p \in \mathfrak{t}^* \,|\, \langle p, \xi \rangle = 1\}$ as a vector space by choosing the origin $\mathbf{o}$ to be zero, such that if there are transverse coupled Kähler-Einstein metrics then the sum of the barycenters of $\mathcal{P}_{\xi,\alpha}$ lies at the origin $\mathbf{o}$.

(2) The Minkowski sum decomposition in (1) is unique up to translations of $\mathcal{P}_{\xi,\alpha}$ to $\mathcal{P}_{\xi,\alpha} + c_\alpha$ with $c_\alpha \in \mathfrak{t}^*$ such that $\sum_{\alpha=1}^k c_\alpha = \mathbf{o}$.

(3) The Minkoswski sum decomposition of $\mathcal{P}_\xi$ in (1) determines a Minkowski sum decomposition of the contact moment cone $\mathcal{C}_\xi$

$$(1.4) \qquad\qquad \mathcal{C}_\xi = \mathcal{C}_{\xi,1} + \cdots + \mathcal{C}_{\xi,k}$$

into the sum of cones $\mathcal{C}_{\xi,\alpha} \subset \mathfrak{t}^*$ in such a way that the intersection of $\mathcal{C}_{\xi,\alpha}$ with $\mathcal{P}_\xi$ is $\mathcal{P}_{\xi,\alpha}$.

The origin $\mathbf{o}$ in fact corresponds to the zero of the transverse moment polytope as the proof shows.

A Sasaki manifold S of dimension $2m + 1$ is said to be *toric* if its Kähler cone $C(S)$ is toric. Thus a real compact torus T of dimension $m + 1$ acts

effectively on S preserving the contact distribution D, the pseudo-convex CR-structure J on D and the Reeb vector field ξ, and the Lie algebra $\mathfrak{t}$ of T contains ξ where the elements of $\mathfrak{t}$ are identified with vector fields on S. Let

$$\mathcal{C} = \{p \in \mathfrak{t}^* \backslash \{o\} \,|\, \langle p, \ell_a \rangle \geq 0, \ a = 1, \cdots, d\}$$

be the moment cone of $C(S)$, which is a convex rational polyhedral cone, where $\ell_a \in \mathfrak{t}$ such that $2\pi\ell_1, \cdots, 2\pi\ell_d$ are primitive elements of the kernel Λ of $\exp : \mathfrak{t} \to T$. For a compact toric Sasaki manifold S we have the following equivalent conditions, c.f. [6], Theorem 1.2:

(a) $c_1^B(S) > 0$ and $c_1(D) = 0$.
(b) There is a rational vector $\gamma \in \mathfrak{t}^*$ such that

$$\langle \gamma, \xi \rangle = -m - 1 \quad \text{and} \quad \ell_a(\gamma) = -1 \text{ for } a = 1, \cdots, d.$$

(c) The power of the canonical line bundle $K_{C(S)}^{\otimes \ell}$ of the cone $C(S)$ is a trivial line bundle for some integer ℓ.

Because of (c) we call these equivalent conditions *Calabi-Yau condition of the Kähler cone*. The condition (b) appeared in [30] as (2.57), (2.60). The existence of $-\gamma$ is also known in algebraic geometry of toric varieties, see [9], Theorem 4.2.8. The paper [10] also gives an account from the broader view points of what they call *angle cone*.

Theorem 1.2. *Let S be a toric Sasaki manifold satisfying Calabi-Yau condition of the Kähler cone. Then, in Proposition 1.1, we can take*

$$\mathbf{o} = -\frac{1}{m+1}\gamma.$$

Using Proposition 1.1 and Theorem 1.2 we apply the volume minimization argument of Martelli-Sparks-Yau in the following setting. Let S be a toric Sasaki manifold satisfying Calabi-Yau condition of the Kähler cone. We regard

$$
\begin{aligned}
(1.5) \qquad \Xi_{\mathbf{o}} \ &:= \ \{\xi' \in \mathcal{C}^* \subset \mathfrak{t} \,|\, \langle \xi', \mathbf{o} \rangle = 1\} \\
&= \ \{\xi' \in \mathcal{C}^* \subset \mathfrak{t} \,|\, \langle \xi', \gamma \rangle = -m - 1\}
\end{aligned}
$$

as the space of Reeb vector fields satisfying the Calabi-Yau conditions of the Kähler cone.

Let $\gamma_1, \cdots, \gamma_k$ be basic Kähler classes with respect to the Reeb vector field ξ. But we do not assume $c_1^B(S) = (\gamma_1 + \cdots + \gamma_k)/2\pi$ for the moment. Let $\mathcal{P}_{\xi,1}, \cdots, \mathcal{P}_{\xi,k}$ be compact convex polytopes corresponding to $\gamma_1, \cdots, \gamma_k$,

which are assumed to be subsets in the contact moment polytope P_ξ of S, and $\mathcal{C}_{\xi,1}, \cdots, \mathcal{C}_{\xi,k}$ be convex polyhedral cones in the contact moment convex cone $\mathcal{C}_\xi$ of the Kähler cone $C(S)$ of S such that $\mathcal{P}_{\xi,\alpha} = \mathcal{C}_{\xi,\alpha} \cap P_\xi$.

Choose $\xi' \in \Xi_\mathbf{o}$, and set for $\alpha = 1, \cdots, k$

$$\mathcal{P}_{\xi'} = \{p \in \mathcal{C}_\xi \mid \langle \xi', p \rangle = 1\},$$

$$\mathcal{P}_{\xi',\alpha} = \mathcal{C}_{\xi,\alpha} \cap P_{\xi'},$$

$$\Delta_{\xi',\alpha} = \{p \in \mathcal{C}_{\xi,\alpha} \mid \langle \xi', p \rangle \leq 1\}.$$

We now consider the functional $W : \Xi_\mathbf{o} \to \mathbf{R}$ defined by

$$(1.6) \qquad W(\xi') \quad := \quad \sum_{\alpha=1}^{k} \log \frac{\mathrm{Vol}(\mathcal{P}_{\xi',\alpha})}{|\xi'|}$$

$$(1.7) \qquad = \quad \sum_{\alpha=1}^{k} \log((m+1)\mathrm{Vol}(\Delta_{\xi'})).$$

Theorem 1.3. *Let S be a toric Sasaki manifold with Calabi-Yau condition of the Kähler cone, i.e. $c_1^B(S) > 0$ and $c_1(D) = 0$.*

(1) W is a strictly convex function on $\Xi_\mathbf{o}$.

(2) If we have a critical point $\xi' \in \Xi_\mathbf{o}$ such that

$$\mathcal{P}_{\xi'} = \mathcal{P}_{\xi',1} + \cdots + \mathcal{P}_{\xi',k}$$

then there exist transverse couple Kähler-Einstein metrics with respect to ξ'.

(3) In the case of $k = 1$, if we take $\gamma_1 = c_1^B(S)$ and $\mathcal{P}_{\xi,1} = \mathcal{P}_\xi$ then we have $\mathcal{P}_{\xi',1} = \mathcal{P}_{\xi'}$ for any $\xi' \in \Xi_\mathbf{o}$, and thus the assumption in (2) is always satisfied. Further, the functional W is a strictly convex proper function and always have a critical point.

The part (3) is due to Martelli-Sparks-Yau [30, 31], and the part (2) is an attempt to extend their argument to the coupled case. However, even if we assume $c_1^B(S) = (\gamma_1 + \cdots + \gamma_k)/2\pi$ and $\mathcal{P}_\xi = \mathcal{P}_{\xi,1} + \cdots + \mathcal{P}_{\xi,k}$ it is not clear whether $\mathcal{P}'_\xi = \mathcal{P}_{\xi',1} + \cdots + \mathcal{P}_{\xi',k}$ for other $\xi' \in \Xi_\mathbf{o}$, and can not conclude the existence of transverse coupled Kähler-Einstein metrics. In the last section, this will be explained using the CR f-twist of Apostolov-Calderbank [2].

The volume minimization arguments were used for the studies of non-linear problems extending Kähler-Einstein metrics in which Killing vector

fields or Killing potentials are involved: Kähler-Ricci solitons [32], Sasaki-Einstein metrics [30, 31] and conformally Kähler, Einstein-Maxwell metrics [18], see also the survey [19]. In all these cases, a volume functional is defined on the space of Killing vector fields, and the derivative is an obstruction to the existence of those metrics which extends the obstruction to the existence of Kähler-Einstein metrics [13].

After this introduction, the plan of this paper is as follows. In Section 2 we review basic facts about Sasaki manifolds. In Section 3 we review known facts on transverse Kähler geometry and the transverse moment map. In Section 4, we map a Minkowski sum decomposition of the image of the transverse moment map to the image of the contact moment map to obtain a Minkowski sum decomposition of of the contact moment map image. The properties of the latter is stated as Proposition 1.1. Then we prove Theorem 1.2. In Section 5 we use the volume minimization argument and prove Theorem 1.3.

2. Deformations of Sasakian structures

Let S be a $(2m + 1)$-dimensional smooth manifold. A *contact structure* on S is a $2m$-dimensional distribution $D \subset TS$ such that the *Levi form* $L_D :$ $D \times D \to TS/D$ defined by

$$L_D(X, Y) = -\eta_D([X, Y])$$

is non-degenerate where $\eta_D : TS \to TS/D$ is the projection. The pair (S, D) is called a *contact manifold*, and D is also called the *contact distribution*. We assume TS/D is an oriented real line bundle. If τ is a positive section of TS/D, then $\eta_\tau = \tau^{-1}\eta_D$ is a contact form, i.e. $d\eta_\tau|_D$ is non-degenerate. Then there is a unique vector field ξ, called the *Reeb vector field*, such that

$$i(\xi)\eta_\tau = 1, \qquad i(\xi)d\eta_\tau = 0$$

where $i(\xi)$ denotes the inner product by ξ. In this case $\eta_D(\xi) = \tau$. The flow on S generated by ξ, i.e. the one parameter group of transformations generated by ξ, is called the *Reeb flow*. Since $i(\xi)d\eta_\tau = 0$ and $L_\xi d\eta_\tau = 0$ where L_ξ denotes the Lie derivative by ξ, then $d\eta_\tau$ descends to a symplectic form on local orbit spaces of the Reeb flow. Let $\Omega_B^k(S)$ denote the set of all k-forms on S which are obtained by pulling back from the local orbit spaces of the Reeb flow. We call such forms *basic k-forms* with respect to ξ. Obviously a k-form α on S belongs to $\Omega_B^k(S)$ if and only if $i(\xi)\alpha = 0$ and $L_\xi \alpha = 0$. The 2-form $d\eta_\tau$ is a typical example of a basic 2-form.

A vector field X on S is said to be a *contact vector field* if $L_X C^\infty(D) \subset C^\infty(D)$.

Lemma 2.1. *Sending a contact vector field X to $\eta_D(X) \in C^\infty(TS/D)$ gives an isomorphism between the Lie algebra of contact vector fields and $C^\infty(TS/D)$. If $\sigma = f\tau \in C^\infty(TS/D)$ for a smooth function $f \in C^\infty(S)$, then the corresponding contact vector field X with $\eta_D(X) = \sigma$ is expressed as*

$$(2.1) \qquad\qquad X = f\xi + K_f$$

for $K_f \in C^\infty(D)$ satisfying

$$(2.2) \qquad\qquad i(K_f)d\eta_\tau|_D = -df|_D$$

Proof. We only show (2.2). Other part of the proof is left to the reader, or see the proof of Lemma 1 in [2], p.1055. If X is given by (2.1) then $\eta_D(X) = f\tau = \sigma$. For $Y \in C^\infty(D)$ we have

$$(i(X)d\eta_\tau)(Y) = -Yf$$

since $L_X C^\infty(D) = C^\infty(D)$. This implies

$$(i(X)d\eta_\tau)|D = -df|_D.$$

On the other hand, using (2.1) we have

$$i(X)d\eta_\tau = i(K_f)d\eta_\tau.$$

Then (2.2) follows from the last two equations. $\qquad\square$

Lemma 2.2. *In Lemma 2.1, if $[X, \xi] = 0$ then f is a basic function with respect to ξ, i.e. $\xi f = 0$, and K_f descends to a Hamiltonian vector field of f on local orbit spaces of the Reeb flow.*

Proof. One can show that f is basic by

$$0 = (d\eta_\tau)(X, \xi) = X\eta_\tau(\xi) - \xi(\eta_\tau(X)) - \eta_\tau([X, \xi]).$$

One can also show $[K_f, \xi] = 0$ using (2.1) together with $[X, \xi] = 0$ and $\xi f = 0$. Thus K_f descends to the local orbit spaces of the Reeb flow, and (2.2) shows that K_f is the Hamiltonian vector field of f. $\qquad\square$

Let $J \in \mathrm{End}(D)$ be an almost complex structure of the contact distribution D, i.e. $J^2 = -\mathrm{id}$. We say that (D, J) is a CR-structure if

$$D^{1,0} := \{X - iJX \,|\, X \in D\}$$

is involutive, i.e. the set $C^\infty(D^{1,0})$ of smooth sections of $D^{1,0}$ is closed under the bracket. A *CR-structure* (D, J) is said to be strictly pseudo-convex if $d\eta_\tau(\,\cdot\,, J\cdot)|_D$ is a positive definite Hermitian form for a positive section τ. Note that this definition is independent of the choice of a positive section τ. Then the triple (S, D, J) is called a strictly pseudo-convex *CR*-manifold.

A contact vector field ξ is said to be a *CR-vector field* if $L_\xi J = 0$.

Definition 2.3. *If ξ is a CR-vector field on a strictly pseudo-convex CR-manifold and $\eta_D(\xi)$ gives a positive section then we call (D, J, ξ) a Sasaki structure and (S, D, J, ξ) a Sasaki manifold. The CR-vector field ξ is called the Reeb vector field of the Sasaki manifold. (This definition of Reeb vector field is compatible with the above definition if we take $\tau = \eta_D(\xi)$.)*

Since $L_\xi J = 0$ and $D^{1,0}$ is involutive then the local orbit spaces of the Reeb flow have a complex structure. Further, for the submersions $\pi_i : U_i \to V_i$ of a small open set $U_i \subset S$ onto a local orbit space V_i, the map

$$\pi_i \circ \pi_j^{-1}|_{\pi_j(U_i \cap U_j)} : \pi_j(U_i \cap U_j) \to \pi_i(U_i \cap U_j)$$

is biholomorphic. The collection of such (U_i, V_i, π_i)'s is called a *transverse holomorphic structure*. Further, by the property of strong pseudo-convexity, $\frac{1}{2}d\eta_\tau$ descends to V_i's to define Kähler forms ω_i's, and $\pi_i \circ \pi_j^{-1}|_{\pi_j(U_i \cap U_j)}$'s are Kähler isometries. We call the collection of such $(U_i, V_i, \pi_i, \omega_i)$'s a *transverse Kähler strucure*.

Remark 2.4. *The convention of $\frac{1}{2}d\eta_\tau$, but not $d\eta_\tau$, is the standard choice of the transverse Kähler form. This makes $\frac{1}{2}i\partial\overline{\partial}r^2$ the Kähler form on the cone $C(S)$, see the proof of Proposition 2.9.*

Lemma 2.5. *Let (S, D, J) be a strictly pseudo-convex CR-manifold. Suppose that ξ and ξ' be two commuting Reeb vector fields, i.e. $[\xi, \xi'] = 0$, giving rise to two Sasaki structures (D, J, ξ) and (D, J, ξ') on S. Then ξ' is expressed as*

$$(2.3) \qquad\qquad\qquad \xi' = f\xi + K_f$$

where $K_f \in C^\infty(D)$ descends to a Killing vector field on local orbit spaces of the Reeb flow of ξ and f is a positive basic function with respect to ξ which descends to a Killing potential of K_f.

Proof. A section $Y \in C^\infty(D)$ such that $L_\xi Y = 0$ descends to a vector field on local orbit spaces of the Reeb flow of ξ, which we denote by $Y^\vee$. For another $W \in C^\infty(D)$ such that $L_\xi W = 0$ we have

$$[Y^\vee, W^\vee] = ([Y, W] - \eta_\tau([Y, W])\xi)^\vee.$$

By Lemma 2.2 $K_f^\vee$ is a Hamiltonian vector field of the basic function f. To show that $K_f^\vee$ is a Killing vector field we need to show $L_{K_f^\vee} J^\vee = 0$. This is equivalent to

(2.4)
$$[K_f^\vee, J^\vee Y^\vee] = J^\vee [K_f^\vee, Y^\vee].$$

But this follows from the following three equalities:

$$([\xi', JY] - \eta_\tau([\xi', JY]\xi)^\vee = J^\vee[\xi', Y]^\vee,$$

$$
\begin{aligned}
([f\xi, JY] - \eta_\tau([f\xi, JY])\xi)^\vee &= f[\xi, JY] - ((JY)f)\xi - \eta_\tau(f[\xi, JY] \\
&\quad - ((JY)f)\xi)\xi \\
&= (fJ[\xi, Y] - ((JY)f)\xi + \eta_\tau(((JY)f)\xi)\xi)^\vee \\
&= (fJ[\xi, Y])^\vee
\end{aligned}
$$

and

$$
\begin{aligned}
J^\vee[f\xi, Y]^\vee &= (J([f\xi, Y] - \eta_\tau([f\xi, Y])\xi))^\vee \\
&= (J(f[\xi, Y] - (Yf)\xi - \eta_\tau(f[\xi, Y] - (Yf)\xi)\xi))^\vee \\
&= (fJ[\xi, Y])^\vee.
\end{aligned}
$$

Alternatively, one may simply argue that, since the flow generated by ξ' preserves J and this flow descends to the flow generated by $K_f^\vee$ preserving $J^\vee$, we obtain $L_{K_f^\vee} J^\vee = 0$.

Since both $\eta_\tau(\xi)$ and $\eta_\tau(\xi')$ give positive orientation then

$$f = \frac{\eta_\tau(\xi')}{\eta_\tau(\xi)} > 0.$$

This completes the proof of Lemma 2.5. $\qquad\square$

Associated with a Sasaki structure we have a Riemannian metric g defined by

$$g(\xi, \xi) = 1, \qquad g(\xi, D) = 0,$$

and

(2.5)
$$g_D = \frac{1}{2} d\eta_\tau(\cdot \, J\cdot).$$

The Riemannian manifold (S, g) associated with the Sasaki Structure (S, D, J, ξ) as above is often called a Sasaki manifold. The normalization of $g(\xi, \xi) =$

1 is the standard choice, and this choice determines various constants. For example, if (S, g) is an Einstein manifold, called a *Sasaki-Einstein manifold*, then the Ricci curvature Ricci satisfies

$$(2.6) \qquad\qquad\qquad \text{Ricci} = 2m\, g,$$

and in this case the transverse Kähler metric is Kähler-Einstein with the transverse Ricci curvature Ricci^T satisfying

$$(2.7) \qquad\qquad\qquad \text{Ricci}^T = (2m + 2)\, g^T$$

where g^T is the Kähler metric on the local orbit spaces of the Reeb flow naturally induced from g_D, called the transverse Kähler metric. Note that many authors use different conventions of Ricci curvature between Riemannian geometry and Kähler geometry because the trace is taken respect to an orthonormal basis of the real tangent bundle in Riemannian geometry while holomorphic tangent bundle in Kähler geometry, resulting in Kählerian Ricci curvature being a half of the Riemannian Ricci curvature. Ricci curvature in (2.7) is Riemannian Ricci curvature. To distinguish them we denote the Kählerian Ricci curvature by Ric while the Riemannian Ricci curvature by Ricci. Thus (2.7) is equivalent to

$$(2.8) \qquad\qquad\qquad \text{Ric}^T = (m + 1)\, g^T.$$

By the same reason the Kählerian scalar curvature is one fourth of Riemannian scalar curvature. In this paper we only deal with the Kählerian transverse scalar curvature, which we denote by R^T. Thus, for a Sasaki-Einstein manifold, we have

$$(2.9) \qquad\qquad\qquad R^T = m(m + 1).$$

Note also that the associated *transverse Kähler form* ω^T is given by

$$(2.10) \qquad\qquad\qquad \omega^T = \frac{1}{2} d\eta_\tau.$$

As the expression of (2.10) shows, the transverse Kähler form ω^T is identified with the differential 2-form $\frac{1}{2} d\eta_\tau$ on S as a basic 2-form.

Next we see that a Sasaki manifold is obtained as a link of a Kähler cone.

Definition 2.6. *Let* (S, g) *be a Riemannian manifold. The Riemannian cone* (V, g_V) *of* (S, g) *is the pair of the product manifold* $V = \mathbf{R}_+ \times S$ *and the warped product metric*

$$g_V = dr^2 + r^2 g$$

on V *where* r *is the standard coordinate of* $\mathbf{R}_+$.

Let (V, g_V) be a Riemannian cone of (S, g) as above. If (V, g_V) is Kähler then (S, g) is a Sasaki manifold in the following way. We identify (S, g) as the submanifold $\{r = 1\}$ in (V, g_V). Then

$$\eta = d^c r|_{S=\{r=1\}}$$

is a contact form where our convention of d^c is

$$(d^c f)(Y) = df(-JY) = (i(\bar{\partial} - \partial)f)(Y),$$

$D = \ker \eta$ is a contact distribution, and $\xi := Jr \frac{\partial}{\partial r}|_{r=1}$ is the Reeb vector field.

Proposition 2.7. *Let* (S, D, J, ξ) *be a Sasaki manifold as defined in Definition 2.3. Then there is a Kähler cone* (V, g_V) *such that the Sasakian structure on* $\{r = 1\} \subset V$ *described as above is isomorphic to* (S, D, J, ξ).

Proof. Let (V, g_V) be the Riemannian cone of (S, g). We show that the almost complex structure J on the CR-structure D extends to an integrable complex structure J on V such that (g_V, J) is Kähler. Extend J on D to TV by

$$J(r \frac{\partial}{\partial r}) = \xi.$$

Consider $d^c r = dr(-J \cdot)$ with respect to J on V. Then one can check

$$d^c r|_{\{r=1\}} = \eta_\tau, \qquad \tau = \eta_D(\xi).$$

We extend η_τ to V by

$$\eta_\tau = d^c \log r.$$

One can then show that $dr + ir\eta_\tau$ and $C^\infty(D^{1,0*})$ generates sections of the type $(1, 0)$ cotangent bundle of V and that they form a differential ideal. Thus J on V is integrable.

Then since $g_V = dr^2 + r^2 g_S$ its fundamental 2-form ω_V is computed by

$$\omega_V = dr(J \cdot) \wedge dr + r^2 g_S(J \cdot, \cdot)$$

$$= dr \wedge d^c r + \frac{r^2}{2} d\eta_\tau$$

$$= \frac{1}{2} \partial \overline{\partial} r^2.$$

This shows that g_V is a Kähler metric. □

Thus we have an equivalent definition of Sasaki manifolds:

Definition 2.8. *An odd dimensional Riemannian manifold (S, g) is called a Sasaki manifold if its Riemannain cone (V, g_V) is Kähler.*

3. Deformations of the transverse Kähler structures

As described in the previous section, a Sasakian structure on a differentiable manifold S is given by the triple of (D, J, ξ) where (D, J) is a strongly pseudo-convex CR-structure and ξ is a Reeb vector field. When we consider deformations of Sasakian structures we may separate into two types of deformations, one fixing (D, J) and the other fixing ξ. The case fixing (D, J) was already considered in Lemma 2.5, in which ξ and ξ' are commutative. This leads us to the setting where a compact real torus T of dimension n acts effectively on a Sasaki manifold (S, D, J, ξ) in such a manner that T preserves (D, J) and the Lie algebra $\mathfrak{t}$ of T contains ξ. Then T-action extends naturally as isometries of the Kähler cone associated to (S, D, J, ξ), and the image of its moment map $\mu_\xi : V \to \mathfrak{t}^*$ is a convex polyhedral cone ([28]). If further $n = m+1$, which is the maximal dimension of the effective torus action, V is a toric manifold and the complex structure of V is invariant under T-invariant deformations of (D, J, ξ).

Motivated by this toric setting, when we consider deformations fixing ξ we restrict ourselves to the situation where the complex structure of the Kähler cone V is fixed. Then the transverse holomorphic structure on the local orbit spaces of the flow generated by ξ is also fixed. Thus we try to deform the transverse Kähler form

$$\omega^T = \frac{1}{2} d\eta_\xi$$

into another transverse Kähler form with different contact structure where we have written η_ξ instead of η_τ with $\tau = \eta_D(\xi)$ and keep this new notation hereafter. As in Kähler geometry one may deform the transverse Kähler form ω^T into

$$\omega_\varphi^T = \omega^T + i\partial_B \overline{\partial}_B \varphi$$

using a basic smooth function $\varphi \in C_B^\infty(S)$. Here ∂_B and $\overline{\partial}_B$ are basic ∂ and $\overline{\partial}$ operators which are naturally defined on complex valued basic forms, and $\overline{\partial}_B$ naturally defines basic Dolbeault cohomology $H_B^{p,q}(S)$. Then we have

$$
\begin{aligned}
\omega_\varphi^T &= \frac{1}{2} dd^c \log re^{2\varphi} \\
&= \frac{1}{2} d(\eta_\xi + 2d^c\varphi).
\end{aligned}
$$

Thus

(3.1) $$D' := \ker(\eta_\xi + 2d^c\varphi)$$

is a variation of D with fixed Reeb vector field ξ. This deformation of D into D' is regarded as a deformation of the cone manifold structure by changing the radial function r into $re^{2\varphi}$. As argued in [25, 7, 5, 3], if we fix D and deform ξ we can take $\{r' = 1\} = \{r = 1\} = S$, but when ξ is fixed and D is deformed the submanifold $\{r = 1\}$ has to change into $\{re^{2\varphi} = 1\}$.

Let ω be arbitrary basic Kähler form (not necessarily equal to $\omega^T = \frac{1}{2}d\eta_\xi$). Denote by $\rho^T(\omega)$ the transverse Ricci form associated with the transverse Ricci curvature $\mathrm{Ric}^T(\omega)$:

$$
\rho^T(\omega) = -i\partial_B \overline{\partial}_B \log \omega^m.
$$

Then the basic cohomology class represented by $\rho^T(\omega)/2\pi$ is independent of the choice of the Kähler form ω. We call this basic cohomology class the *basic first Chern class* and denote it by $c_1^B(S)$.

Next we recall known results about an obstruction to the existence of transverse Kähler-Einstein metrics and transverse coupled Kähler-Einstein metrics studied in [13], [14], [15], [20], [16], [21], [22], [17]. Suppose $c_1^B(S) > 0$ and choose this basic class as a basic Kähler class. Unless we assume $c_1(D) = 0$ as in Lemma 4.1 in Section 4, this class may not be a positive multiple of the standard transverse Kähler class $[\omega^T]_B = [\frac{1}{2}d\eta_\xi]_B$ where $[\cdot]_B$ denotes the basic cohomology class, but for later applications we have in mind the case when $c_1^B(S) = (m+1)[\frac{1}{2}d\eta_\xi]_B$.

Let T be a real compact torus acting effectively as CR-automorphisms of (D, J) such that ξ is contained in the Lie algebra $\mathfrak{t}$ of T. We identify a Lie algebra element $X \in \mathfrak{t}$ as a smooth vector field on S. Further, by Lemma 2.5, X descends to a holomorphic Killing potential on local orbit spaces of the Reeb flow of ξ, thus as a transverse holomorphic vector field. We take this

view point below. Let ω be a T-invariant basic Kähler form in $\frac{1}{m+1}c_1^B(S)$. Let F be a T-invariant basic smooth function on S such that

$$(3.2) \qquad\qquad \rho^T(\omega) = (m+1)\omega + i\partial_B\overline{\partial}_B F.$$

Then since $c_1^B(S) > 0$, for any $X \in \mathfrak{t}$ there exists a smooth basic function v such that

$$(3.3) \qquad\qquad i(X)\omega = -dv.$$

Note that for $\xi \in \mathfrak{t}$, $i(\xi)\omega = 0$ since ω is basic and thus $v = 0$. Then with the normalization of v, by

$$(3.4) \qquad\qquad \int_S v\, e^F \omega^m \wedge \eta_\xi = 0$$

the same arguments as in Proposition 4.1 in [14] (see also Theorem 2.4.3 in [15]) one can show that v satisfies

$$(3.5) \qquad\qquad \Delta_B v + v^i F_i + (m+1)v = 0$$

where Δ_B denotes the $\overline{\partial}_B$-Laplacian. Note that

$$
\begin{aligned}
v^i F_i &= g^{i\overline{j}} v_{\overline{j}} F_i \\
&= \frac{i}{2}(X - iJX)F \\
&= \frac{1}{2}(JX)F
\end{aligned}
$$

since F is T-invariant. Define $\mathrm{Fut} : \mathfrak{t}/\mathbf{R}\xi \to \mathbf{R}$ by

$$
\begin{aligned}
(3.6) \qquad\qquad \mathrm{Fut}(X) &= \int_S v^i F_i \,\omega^m \wedge \eta_\xi \\
&= \int_S \frac{1}{2}(JX)F\,\omega^m \wedge \eta_\xi.
\end{aligned}
$$

Then as in [13], [20], Fut is independent of choice of ω in $\frac{1}{m+1}c_1^B(S)$, and the non-vanishing of Fut obstructs the existence of a transverse Kähler-Einstein metric in $\frac{1}{m+1}c_1^B(S)$ by (3.2). This invariant can be expressed in terms of the transverse moment map $\mu^T : S \to (\mathfrak{t}/\mathbf{R}\xi)^*$

$$(3.7) \qquad\qquad \langle \mu^T(x), X \rangle = v(x)$$

where v is related with X by (3.3) with the normalization (3.4). The image of μ^T is a compact convex polytope, and this polytope is unchanged even if the Kähler form ω is changed in the same cohomology class. This can be checked by noting that the vertices of the polytopes are the critical values of v's and that if ω changes to $\omega_\varphi = \omega + i\partial_B\overline{\partial}_B\varphi$ then v changes to $v + v^\alpha\varphi_\alpha$, and the critical values do not change. Notice that (3.4) and (3.5) are also preserved under these changes. Then since

$$(3.8) \qquad \mathrm{Fut}(X) = -(m+1)\int_S v\,\omega^m \wedge \eta_\xi$$

by (3.5) it follows that Fut vanishes if and only if the barycenter of the image of the moment map μ^T lies at zero.

To express this moment polytope of μ^T we let K_S denote the complex line bundle over S consisting of basic $(m,0)$-forms. Then $c_1^B(S) = c_1(K_S^{-1})$. The moment polytope of μ^T is associated with the basic Kähler class $\frac{1}{m+1}K_S^{-1}$. Thus we express the moment polytope of μ^T by $\frac{1}{m+1}\mathcal{P}_{-K_S}$.

Next we recall an obstruction to the existence of transverse coupled Kähler-Einstein metrics. Suppose $c_1^B(S) > 0$. A *decomposition* of $c_1^B(S)$ is a sum

$$c_1^B(S) = (\gamma_1 + \cdots + \gamma_k)/2\pi$$

of positive basic $(1,1)$ classes $\gamma_\alpha/2\pi$. If we choose basic Kähler forms ω_α representing γ_α, there exist smooth basic functions F_α such that

$$(3.9) \qquad \rho^T(\omega_\alpha) - \sqrt{-1}\partial_B\overline{\partial}_B F_\alpha = \sum_{\beta=1}^{k} \omega_\beta, \quad \alpha = 1, \cdots, k.$$

We say ω_α's are *transverse coupled Kähler-Einstein metrics* if F_α is constant so that transverse coupled Kähler-Einstein metrics satisfy

$$\rho^T(\omega_1) = \cdots = \rho^T(\omega_k) = \sum_{\beta=1}^{k} \omega_\beta.$$

We will also call transverse coupled Kähler-Einstein metrics *coupled Sasaki-Einstein metrics* when we further assume $c_1(D) = 0$ since, for $k = 1$, $\frac{1}{m+1}\omega_1$ is a Sasaki-Einstein metric with respect to a modified contact form as in (3.1).

We choose ω_α in γ_α and normalize F_α so that

$$(3.10) \qquad e^{F_1}\omega_1^m = \cdots = e^{F_k}\omega_k^m$$

and put

$$(3.11) \qquad\qquad dV := e^{F_\alpha} \omega_\alpha^m \wedge \eta_\xi.$$

For $X \in \mathfrak{t}$, let v_α be the basic function satisfying

$$(3.12) \qquad\qquad i(X)\omega_\alpha = -dv_\alpha$$

with normalization condition

$$(3.13) \qquad\qquad \int_S (v_1 + \cdots + v_k)\, dV = 0.$$

Then as proved in [21], Theorem 3.3, v_α's satisfy

(a) $\nabla_\alpha^i v_\alpha = \nabla_\beta^i v_\beta$ for $i = 1, 2, \ldots, n$ and $\alpha, \beta = 1, 2, \ldots, k$.

(b) $\Delta_\alpha v_\alpha + v_\alpha^i F_{\alpha i} + \sum_{\beta=1}^k v_\beta = 0$ for $\alpha = 1, 2, \ldots, k$, where $\Delta_\alpha = -\overline{\partial}_\alpha^* \overline{\partial}$ is the
Laplacian with respect to the Kähler form ω_α.

It is also shown in [21], Theorem 5.2, that (3.13) is equivalent to the Minkowski sum relation

$$\sum_{\alpha=1}^k \mathcal{P}_\alpha = \mathcal{P}_{-K_S}$$

where $\mathcal{P}_\alpha$ is the moment polytope for ω_α. This of course means

$$(3.14) \qquad\qquad \sum_{\alpha=1}^k \frac{1}{m+1}\mathcal{P}_\alpha = \frac{1}{m+1}\mathcal{P}_{-K_S},$$

the right hand side being the moment polytope for μ^T. Define $\mathrm{Fut}^{cpld} : \mathfrak{t} \to \mathbf{R}$ by

$$(3.15) \qquad\qquad \mathrm{Fut}^{cpld}(X) = \sum_{\alpha=1}^k \frac{\int_S v_\alpha\, \omega_\alpha^m \wedge \eta_\xi}{\int_S \omega_\alpha^m \wedge \eta_\xi}.$$

Then Fut^{cpld} is independent of the choice of ω_α in γ_α, the nonvanishing of Fut^{cpld} obstructs the existence of transverse coupled Kähler-Einstein metrics and Fut^{cpld} vanishes if and only if the sum of the barycenters of $\mathcal{P}_\alpha$ lies at zero ([21], Theorem 1.4). In the toric case where $\dim T = m + 1$, the vanishing of Fut^{cpld} is a sufficient condition for the existence of transverse coupled Kähler-Einstein metrics, which is a straightforward extension of the celebrated results of Wang-Zhu [34], Donaldson [12], Hultgren [26].

As a summary of this section, the obstructions Fut, Fut^{cpld} and the transverse moment polytopes of μ^T depend only on the Reeb vector field ξ, its basic first Chern class $c_1^B(S)$ and its decomposition. But as mentioned above we have in mind the case $c_1^B(S) = (m+1)[\frac{1}{2}d\eta_\xi]_B$, in which case Fut, Fut^{cpld} are independent of the choice of D' of the form (3.1). As a conclusion of this section, when we study Fut, Fut^{cpld} and moment polytopes under the variation of Reeb vector fields we may choose an arbitrary strongly pseudo-convex CR-structure (D, J), fix it, and thus are in the position of Lemma 2.5.

4. The transverse moment map and the contact moment map

The equality (2.8) shows that a necessary condition for the existence of a Sasaki-Einstein metric is

$$(4.1) \qquad c_1^B(S) = (m+1)[\frac{1}{2}d\eta_\xi]_B$$

as a basic cohomology. Here $[d\eta_\xi]_B$ denotes the basic cohomology class which is a positive $(1,1)$-class as a basic class though it is zero as a de Rham class of S. Recall that the basic first Chern class is said to be positive, denoted $c_1^B(S) > 0$, if it is represented by a positive basic $(1,1)$-form, i.e. a basic Kähler form. This is an obvious necessary condition for the existence of Sasaki-Einstein metric by (4.1). The following lemma is well-known and important for us, see [4].

Lemma 4.1. *If $c_1^B(S) > 0$ and $c_1(D) = 0$ then by changing r into r^a for a positive constant a if necessary we can assume (4.1) is satisfied. (The transformation from r into r^a is called the D-homothetic transformation.)*

Proof. If $c_1(D) = 0$, then $c_1^B = \tau[d\eta_\xi]$ for some constant τ by [4], Corollary 7.5.26. Since $c_1^B(S) > 0$ we must have $\tau > 0$. Then we may take $a = (m+1)/\tau$. $\qquad\square$

The summary at the end of previous section is an observation concerning the transverse moment map μ^T, but we may compare it with the contact moment map $\mu^{con} : V \to \mathfrak{t}^*$ defined by

$$(4.2) \qquad \langle \mu^{con}(x), X \rangle = (r^2 \eta_\xi(X))(x).$$

The linkage between the transverse moment map μ^T, which is defined with respect to the transverse Kähler class $c_1^B(S)$, and the contact moment map

μ^{con}, which is defined with respect to the contact form η_ξ, is played by the conditions $c_1(D) = 0$ and $c_1^B(S) > 0$ in Lemma 4.1. The relation of v in (3.7) and $\eta_\xi(X)$ in (4.2) is that

$$v = \frac{m+1}{2}\eta_\xi(X) + c$$

where c is determined by the normalization (3.4).

By [28], the image of μ^{con} is a convex polyhedral cone, which we denote by $\mathcal{C}$. Identifying S with $\{r = 1\}$ we have the moment map of S by restricting μ^{con} to $\{r = 1\}$. The image of S is

$$\mathcal{P}_\xi := \text{Image}(\mu^{con}) \cap \{p \in \mathfrak{t}^* \mid \langle p, \xi \rangle = 1\}$$

since $\eta_\xi(\xi) = 1$. This set is called the *characteristic hyperplane* in $\mathcal{C}$.

Since the Hamiltonian functions for the basis of $\mathfrak{t}/\mathbf{R}\xi$ determine affine coordinates on the images of μ^T and μ^{con}, the map

$$(4.3) \qquad \Phi := \mu^T \circ (\mu^{con})^{-1}|_{\mathcal{P}_\xi} : \mathcal{P}_\xi \to \frac{1}{m+1}\mathcal{P}_{-K_S}$$

is an affine map in terms of those affine coordinates. Note that $\frac{1}{m+1}\mathcal{P}_{-K_S}$ is in $(\mathfrak{t}/\mathbf{R}\xi)^*$ which is a vector space and contains the origin 0 but that $\mathcal{P}_\xi$ is in a hyperplane in the cone $\mathcal{C} \subset \mathfrak{t}^*$.

Proof of Proposition 1.1. We take $\mathbf{o}$ and $\mathcal{P}_{\xi,\alpha}$ so that to be $\Phi(\mathbf{o}) = 0$ and $\Phi(\mathcal{P}_{\xi,\alpha}) = \frac{1}{m+1}\mathcal{P}_\alpha$ in (3.14). Then we obtain the Minkowski decomposition as claimed in (1). Further, the barycenter of the domain polytope is mapped to the barycenter of the image polytope by an affine map, and thus the last claim in (1) follows.

By (3.13), $\mathcal{P}_\alpha$'s are unique up to translations $\mathcal{P}_\alpha + c_\alpha$ satisfying $c_1 + \cdots + c_k = 0$, from which (2) follows.

There is a unique cone $\mathcal{C}_{\xi,\alpha}$ such that $\mathcal{C}_{\xi,\alpha} \cap \mathcal{P}_\xi = \mathcal{P}_{\xi,\alpha}$. Then (1.3) implies (1.4). This completes the proof of Proposition 1.1. □

Suppose now that the dimension of the torus T is $m+1$ so that the Kähler cone $V = C(S)$ is a toric manifold. Then the moment map image is a convex polyhedral cone which is *good* in the sense of Lerman [29] described as follows. Let $\mathcal{C}$ be the convex rational polyhedral cone described as

$$\mathcal{C} = \{p \in \mathfrak{t}^*\setminus\{o\} \mid \langle p, \ell_a \rangle \geq 0, \ a = 1, \cdots, d\}$$

where $\ell_a \in \mathfrak{t}$ such that $2\pi\ell_1, \cdots, 2\pi\ell_d$ are primitive elements of the kernel Λ of $\exp : \mathfrak{t} \to T$. We regard ℓ_a as a linear function on $\mathfrak{t}^*$ and write

$$\ell_a(p) = \langle p, \ell_a \rangle \quad \text{for} \ \ p \in \mathfrak{t}^*.$$

We say $\mathcal{C}$ is good if for any face $F = \cap_{j=1}^k \{\ell_{a_j} = 0\}$ of codimension k we have

(4.4) $$(\mathbf{R}\ell_{a_1} + \cdots + \mathbf{R}\ell_{a_k}) \cap \mathbf{Z}^{m+1} = \mathbf{Z}\ell_{a_1} + \cdots + \mathbf{Z}\ell_{a_k}.$$

It is shown in [29] that if S is smooth then $\mathcal{C} := \mu^{con}(C(S))$ is a good convex rational polyhedral cone and conversely if $\mathcal{C}$ is a good convex rational polyhedral cone then there is a smooth toric Sasaki manifold S such that $\mu^{con}(C(S)) = \mathcal{C}$.

For a toric Sasaki manifold there are descriptions using the action-angle coordinates [24], [1], [30], [31]. As explained in the Introduction, a salient fact when $c_1^B(S) > 0$ and $c_1(D) = 0$ is that there is a distinguished point $\gamma \in \mathcal{P}_\xi$ with the property that

(4.5) $$\langle \gamma, \xi \rangle = -(m+1) \qquad \text{and} \qquad \ell_a(\gamma) = -1 \quad \text{for} \ a = 1, \cdots, d,$$

see [30], [6]. Theorem 1.2 claims that the origin $\mathbf{o}$ coincides with $q := -\frac{1}{m+1}\gamma$.

Proof of Theorem 1.2. By the proof of Proposition 1.1, we have only to show $\Phi(q) = 0$ for the affine map Φ defined (4.3). This proof is motivated by the computations in [10]. First of all, by Donaldson's expression of the obstruction in [11]

(4.6) $$\text{Fut}(X) = \int_{\partial \mathcal{P}_\xi} y\sigma_\xi - \frac{\int_{\partial \mathcal{P}_\xi} \sigma_\xi}{\int_{\mathcal{P}_\xi} d\tilde{x}} \int_{\mathcal{P}_\xi} y\, d\tilde{x}$$

where y is an affine function corresponding to $X \in \mathfrak{t}$, $\tilde{x} = (\tilde{x}^1, \cdots, \tilde{x}^m)$ are affine coordinates on the hyperplane $\{p \in \mathfrak{t}^* \,|\, \langle p, \xi \rangle = 1\}$, and

$$d\ell_a \wedge \sigma_\xi = -d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^m \quad \text{on the facet } F_a \cap P_\xi.$$

Note that $\frac{\int_{\partial \mathcal{P}_\xi} \sigma_\xi}{\int_{\mathcal{P}_\xi} d\tilde{x}}$ is the average (Kähler geometers') scalar curvature, which is equal to $m(m+1)$ for $\omega \in 2\pi c_1^B(S)/(m+1)$. As shown in [10], Lemma 3.8, σ_ξ is expressed using the distinguished point q by

(4.7) $$\sigma_\xi = \frac{1}{m+1} \sum_{i=1}^m (-1)^{i+1}(\tilde{x}^i - q^i)d\tilde{x}^1 \wedge \cdots \wedge \widehat{d\tilde{x}^i} \wedge \cdots \wedge d\tilde{x}^m.$$

Using $d\sigma_\xi = m(m+1)d\tilde{x}$ and Stokes theorem, one can show

$$(4.8) \qquad \int_{\partial \mathcal{P}_\xi} \sigma_\xi = m(m+1) \int_{\mathcal{P}_\xi} d\tilde{x}$$

as expected to get average scalar curvature and

$$(4.9) \qquad \begin{aligned} \int_{\mathcal{P}_\xi} \tilde{x}^i d\tilde{x} &= \frac{1}{m(m+1)} \int_{\mathcal{P}_\xi} \tilde{x}^i d\sigma_\xi \\ &= \frac{1}{m(m+1)} \int_{\partial \mathcal{P}_\xi} \tilde{x}^i \sigma_\xi - \frac{1}{m} \int_{\mathcal{P}_\xi} (\tilde{x}^i - q^i) d\tilde{x}. \end{aligned}$$

It follows from (4.8) and (4.9) that

$$(4.10) \qquad \int_{\partial P_\xi} \tilde{x}^i \sigma_\xi - \frac{\int_{\partial \mathcal{P}_\xi} \sigma_\xi}{\int_{\mathcal{P}_\xi} d\tilde{x}} \int_{\mathcal{P}_\xi} \tilde{x}^i \, d\tilde{x} = -(m+1) \int_{\mathcal{P}_\xi} \tilde{x}^i d\tilde{x} + (m+1)q^i.$$

Comparing (4.6) and (4.10) we see that Fut vanishes if and only if the barycenter of $\mathcal{P}_\xi$ lies at the distinguished point q. By the affine map Φ, the barycenter of $\mathcal{P}_\xi$ is mapped to the barycenter of $\frac{1}{m+1}\mathcal{P}_{-K_S}$. Therefore, in view of (3.8), q is mapped to the origin 0 in $\frac{1}{m+1}\mathcal{P}_{-K_S}$. $\qquad\square$

5. Volume minimization

In this section, we prove Theorem 1.3. Let $\mathcal{C}$ be one of $\mathcal{C}_\xi$ or $\mathcal{C}_{\xi,\alpha}$'s, and $\mathcal{C}^*$ be

$$\mathcal{C}^* = \{y \in \mathfrak{t} \mid \langle y, p \rangle \geq 0 \ \text{ for all } p \in \mathcal{C}\}.$$

For a Reeb vector field ξ we put

$$\mathcal{P}_\xi = \{p \in \mathcal{C} \subset \mathfrak{t}^* \mid \langle p, \xi \rangle = 1\}.$$

Let $q \in P_\xi$ be fixed (we have q in the proof of Theorem 1.2 in mind which is also equal to the origin $\mathbf{o}$ in the statement of Theorem 1.2), and put

$$\Xi_q = \{\xi' \in \mathcal{C}^* \subset \mathfrak{t} \mid \langle \xi', q \rangle = 1\}.$$

We regard Ξ_q as the space of Reeb vector fields, and look for $\xi' \in \Xi_q$ which minimizes the volume functional defined as follows. For $\xi' \in \Xi_q$ we define

$$(5.1) \qquad \mathcal{P}_{\xi'} = \{p \in \mathcal{C} \mid \langle \xi', p \rangle = 1\},$$

$$\Delta_{\xi'} = \{p \in \mathcal{C} \mid \langle p, \xi \rangle \leq 1\}$$

and the volume functional $\mathrm{Vol} : \Xi_q \to \mathbf{R}$ by

$$\mathrm{Vol}(\xi') \;\; := \;\; \frac{1}{|\xi'|}\mathrm{Vol}(\mathcal{P}_{\xi'})$$

$$(5.2) \qquad\qquad = \;\; (m+1)\mathrm{Vol}(\Delta_{\xi'}).$$

Proposition 5.1. *Let $\xi_t = \xi' + t\nu$ be a path in Ξ_q. Then*

$$\left.\frac{d}{dt}\mathrm{Vol}(\xi_t)\right|_{t=0} = -\frac{m+1}{|\xi'|}\int_{\mathcal{P}_{\xi'}} \langle p, \nu\rangle d\sigma$$

where $d\sigma$ is the natural measure on $\mathcal{P}_{\xi'}$ induced from $\mathcal{P}_{\xi'} \subset \mathfrak{t}^* \cong \mathbf{R}^{m+1}$ where the last isomorphism is induced from the property (4.4) *of good convex polyhedral cone.*

Proof. Let ξ' be in Ξ_q and p be in $P_{\xi'}$ so that $\langle p, \xi' \rangle = 1$. For the path $\xi_t = \xi' + t\nu$ in Ξ_q we have $\langle q, \nu\rangle = 0$. Write $p' \in P_{\xi_t}$ as $p' = sp$. Then

$$\begin{aligned} s \;\; &= \;\; \frac{1}{1 + t\langle p, \nu\rangle} \\ &= \;\; 1 - t\langle p, \nu\rangle + t^2(\langle p, \nu\rangle)^2 + O(t^3). \end{aligned}$$

Thus

$$\left.\frac{d}{dt}\right|_{t=0} d(sp_1) \wedge \cdots \wedge d(sp_{m+1}) = -(m+2)\langle p, \nu\rangle dp_1 \wedge \cdots \wedge dp_{m+1}$$

From this

$$\left.\frac{d}{dt}\mathrm{Vol}(\Delta_{\xi_t})\right|_{t=0} = -(m+2)\int_{\Delta_{\xi'}} \langle p, \nu\rangle dp_1 \wedge \cdots \wedge dp_{m+1}.$$

Using the Stokes theorem we have

$$\begin{aligned} (m+2)\int_{\Delta_{\xi'}} p_i dp_1 \wedge \cdots \wedge dp_{m+1} \;\; &= \;\; \int_{\Delta_{\xi'}} (\sigma_j \frac{\partial}{\partial p_j}(p_j p_i)) dp_1 \wedge \cdots \wedge dp_{m+1} \\ &= \;\; \frac{1}{|\xi'|}\int_{P_{\xi'}} p_i d\sigma \end{aligned}$$

Note that (5.2) can also be proved similarly. Thus we obtain

$$\left.\frac{d}{dt}\mathrm{Vol}(\xi_t)\right|_{t=0} \;\; = \;\; (m+1)\left.\frac{d}{dt}\mathrm{Vol}(\Delta_{\xi_t})\right|_{t=0}$$

$$= -(m+1)(m+2)\int_{\Delta_{\xi'}} \langle p,\nu\rangle dp_1 \wedge \cdots \wedge dp_{m+1}$$

$$= -\frac{m+1}{|\xi'|}\int_{P_{\xi'}} \langle p,\nu\rangle d\sigma.$$

This completes the proof of Proposition 5.1.   □

Proposition 5.2. *In the same situation as in Proposition 5.1*

$$\frac{d^2}{dt^2}\mathrm{Vol}(\xi_t)\bigg|_{t=0} = \frac{(m+1)(m+2)}{|\xi'|}\int_{\mathcal{P}_{\xi'}} \langle p,\nu\rangle^2 dp_1 \wedge \cdots \wedge dp_{m+1},$$

Proof. By similar computations as in the proof of the previous proposition we obtain the desired equality from

$$\frac{d^2}{dt^2}\mathrm{Vol}(\xi_t)\bigg|_{t=0} = (m+1)(m+2)(m+3)\int_{\Delta_{\xi'}} \langle p,\nu\rangle^2 d\sigma$$

and

$$(m+3)\int_{\Delta_{\xi'}} p_i p_j dp_1 \wedge \cdots \wedge dp_{m+1} = \frac{1}{|\xi'|}\int_{P_{\xi'}} p_i p_j d\sigma.$$

This completes the proof of Proposition 5.2.   □

Proof of Theorem 1.3. Let us take $\mathcal{C}$ in Proposition 5.1 and 5.2 to be one of $\mathcal{C}_{\xi,1}, \cdots, \mathcal{C}_{\xi,k}$, and consider the functional $W : \Xi_q \to \mathbf{R}$ defined by

$$W(\xi') = \sum_{\alpha=1}^{k} \log \frac{\mathrm{Vol}(\mathcal{P}_{\xi',\alpha})}{|\xi'|}.$$

This means that we take $\mathcal{P}_{\xi'}$ in (5.1) to be

$$\mathcal{P}_{\xi',\alpha} := \{p \in \mathcal{C}_{\xi,\alpha} \mid \langle p,\xi'\rangle = 1\}$$

and apply the subsequent computations of Proposition 5.1 and 5.2. For a path $\xi_t = \xi' + t\nu$, we obtain by Proposition 5.1, Proposition 5.2 and Schwarz inequality,

$$\frac{d^2}{dt^2}W(\xi_t)\bigg|_{t=0} \geq (m+1)\sum_{\alpha=1}^{k} \frac{\int_{\mathcal{P}_{\xi',\alpha}} \langle p,\nu\rangle^2 dp_1 \wedge \cdots \wedge dp_{m+1}}{\mathrm{Vol}(\mathcal{P}_{\alpha,\xi'})}.$$

This shows that W is strictly convex. This completes the proof of (1).

Suppose that we have a critical point ξ' at which the Minkowski sum we have the Minkowski sum decomposition

$$\mathcal{P}_{\xi'} = \mathcal{P}_{\xi',1} + \cdots + \mathcal{P}_{\xi',k}$$

holds. Then Proposition 5.1 shows that

$$\sum_{\alpha=1}^{k} \frac{\int_{\mathcal{P}_{\xi',\alpha}} \langle p - q, \nu \rangle d\sigma}{\mathrm{Vol}(\mathcal{P}_{\xi',\alpha})} = 0$$

for any ν with $\langle q, \nu \rangle = 0$. This implies that the sum of the barycenters of $P_{\xi',\alpha}$'s lie at q, which is the origin $\mathbf{o}$. Since the Minkowski sum decomposition $\mathcal{P}_{\xi'} = \mathcal{P}_{\xi',1} + \cdots + \mathcal{P}_{\xi',k}$ corresponds to the decomposition of the positive basic first Chern class $c_1^B(S, \xi')$

$$2\pi c_1^B(S, \xi') = \gamma_1' + \cdots + \gamma_k'$$

with respect to ξ' and the corresponding Minkowski sum decomposition of the transverse moment map image $\sum_{\alpha=1}^{k} \frac{1}{m+1} \mathcal{P}_\alpha'$ as in (3.14), this further implies that the sum of the barycenters of the transverse moment polytopes P_α''s for ξ' lie at zero. It follows from [21], Theorem 1.4, that there exist transverse coupled Kähler-Einstein metrics ω_α' in γ_α'. This completes the proof of (2).

In the case of $k = 1$, if we take $\mathcal{P}_{\xi,1} = \mathcal{P}_\xi$ then $\mathcal{C}_{\xi,1} = \mathcal{C}_\xi$ (which is indeed a convex polyhedral cone of the contact toric manifold and independent of ξ). Hence

$$\mathcal{P}_{\xi',1} = \mathcal{C}_\xi \cap \mathcal{P}_{\xi'} = \mathcal{P}_{\xi'}.$$

As shown in (2), W is a strictly convex function. W is a proper function because as ξ' tends to the boundary of $\Xi_{\mathbf{o}}$, $\mathcal{P}_{\xi'}$ tends to plane passing through $\mathbf{o}$ parallel to a facet, and the volume tends to infinity. This completes the proof of (3). $\qquad\square$

With fixed Reeb vector field ξ, any Reeb vector field $\xi' \in \mathfrak{t}^*$ such that $\langle \mathbf{o}, \xi' \rangle = 1$ determines a toric Sasaki structure satisfying the Calabi-Yau condition of the Kähler cone. As was remarked in the introduction, even if we assume $c_1^B(S) = (\gamma_1 + \cdots + \gamma_k)/2\pi$ and $\mathcal{P}_\xi = \mathcal{P}_{\xi,1} + \cdots + \mathcal{P}_{\xi,k}$ we do not in general obtain

(5.3) $$\mathcal{P}_\xi' = \mathcal{P}_{\xi',1} + \cdots + \mathcal{P}_{\xi',k}$$

for other $\xi' \in \Xi_{\mathbf{o}}$, nor a decomposition of the basic first Chern class $c_1^B(S, \xi')$ with respect to ξ' in the form

$$(5.4) \qquad 2\pi c_1^B(S, \xi') = \gamma_1' + \cdots + \gamma_k'.$$

The failure of getting a Minkowski sum decomposition (5.3) can be seen from the non-linearity of the CR f-twist of Apostolov-Calderbank [2], see also [33]. For this we use Lemma 2.5.

If $\xi' \in \mathfrak{t}$ is another Reeb vector field then by Lemma 2.5 there is a positive Killing potential f of ξ' with respect to ξ satisfying (2.3). This implies

$$\eta_{\xi'} = \eta_D(\xi')^{-1}\eta_D = \frac{1}{f}\eta_\xi.$$

If $x^1, \cdots, x^n$ and $x'^1, \cdots, x'^n$ are affine coordinates in terms of a basis of $\mathfrak{t}$ on $\mathcal{P}_\xi$ and $\mathcal{P}_{\xi'}$ respectively such that $\mathbf{o}$ is $(0, \cdots, 0)$ in both of the coordinates then

$$x'^i = \frac{x^i}{f}.$$

Let $\widetilde{\mathcal{P}}_\xi$ and $\widetilde{\mathcal{P}}_{\xi,\alpha}$ be the f-twist of $\mathcal{P}_\xi$ and $\mathcal{P}_{\xi,\alpha}$. If $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$ for some $\mathbf{x}_\alpha \in \mathcal{P}_{\xi,\alpha}$ the f-twist of $\mathbf{x}$ is

$$\begin{aligned}
\widetilde{\mathbf{x}} &= \frac{\mathbf{x}}{f(\mathbf{x})} \\
&= \frac{\mathbf{x}_1 + \cdots + \mathbf{x}_k}{f(\mathbf{x})} \\
&\neq \sum_{i=1}^k \frac{\mathbf{x}_i}{f(\mathbf{x}_i)} \\
&= \widetilde{\mathbf{x}}_1 + \cdots + \widetilde{\mathbf{x}}_k
\end{aligned}$$

The last inequality explains the failure of getting a Minkowski sum

$$\widetilde{\mathcal{P}}_\xi' = \widetilde{\mathcal{P}}_{\xi',1} + \cdots + \widetilde{\mathcal{P}}_{\xi',k}$$

by the f-twist.

References

[1] M. ABREU: Kähler geometry of toric manifolds in symplectic coordinates, Fields Institute Comm. vol. 35, 1–24, AMS, 2003. MR1969265

[2] V. Apostolov and D.M.J. Calderbank: The CR geometry of weighted extremal Kähler and Sasaki metrics. Math. Ann. **379** (2021), 1047–1088. MR4238260

[3] V. Apostolov, D.M.J. Calderbank and E. Legendre: Weighted K-stability of polarized varieties and extremality of Sasaki manifolds. Adv. Math. 391 (2021), Paper No. 107969, 63 pp. MR4300921

[4] C.P. Boyer and K. Galicki: Sasakian geometry. Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008. MR2382957

[5] C.P. Boyer and C. van Coevering: Relative K-stability and extremal Sasaki metrics, Math. Res. Lett. **28** (2018) 1–19. MR3818612

[6] K. Cho, A. Futaki and H. Ono: Uniqueness and examples of toric Sasaki-Einstein metrics, Comm. Math. Phys. **277** (2008) 439–458. MR2358291

[7] T. Collins, G. Székelyhidi, K-semistability for irregular Sasaki manifolds, J. Diffe. Geom. **109** (2018) 81–109. MR3798716

[8] T. Collins, G. Székelyhidi, Sasaki-Einstein metrics and K-stability, Geom. Topol. **23** (2019) 1339–1413. MR3956894

[9] D. Cox, B. Little and H. Schenck, Toric varieties, Graduate Studies in Mathematics 124, AMS, (2011). MR2810322

[10] M. de Borbon and E. Legendre: Toric Sasaki-Einstein metrics with conical singularities. Preprint arXiv:2005.03502 (2020). MR4418387

[11] S.K. Donaldson: Scalar curvature and stability of toric varieties, J. Differential Geometry **62** (2002) 289–349. MR1988506

[12] S.K. Donaldson: Kähler geometry on toric manifolds, and some other manifolds with large symmetry, Handbook of geometric analysis. No. 1, 29–75, Adv. Lect. Math. (ALM), 7, Int. Press, Somerville, MA, 2008. MR2483362

[13] A. Futaki: An obstruction to the existence of Einstein Kähler metrics, Invent. Math. **73** (1983) 437–443. MR0718940

[14] A. Futaki: The Ricci curvature of symplectic quotients of Fano manifolds. Tohoku Math. J., **39** (1987), no. 3, 329–339. MR0902573

[15] A. Futaki: Kähler-Einstein metrics and integral invariants, Lecture Notes in Math., vol.1314, Springer-Verlag, Berlin-Heidelberg-New York, (1988). MR0947341

[16] A. FUTAKI: Toric Sasaki-Einstein Geometry, Fourth International Congress of Chinese Mathematicians (eds. L. JI et al.), AMS/IP Studies in Advanced Mathematics, Vol.**48**(2010), 107–125. MR2744219

[17] A. FUTAKI: Irregular Eguchi-Hanson type metrics and their soliton analogues. Pure Appl. Math. Q. **17**(2021), 27–53. MR4257580

[18] A. FUTAKI and H. ONO: Volume minimization and conformally Kähler, Einstein-Maxwell geometry, J. Math. Soc. Japan. Vol. **70**, No. 4 (2018) pp. 1493–1521. MR3868215

[19] A. FUTAKI and H. ONO: Volume minimization and obstructions to solving some problems in Kähler geometry, ICCM Notices, **6**(2018), 51–60. MR3961490

[20] A. FUTAKI, H. ONO, G. WANG, Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds. J. Differential Geom. **83** (2009), no. 3, 585–635. MR2581358

[21] A. FUTAKI and Y. Y. ZHANG: Coupled Sasaki-Ricci solitons. Science China Math., **64**(2021), 1447–1462. MR4280363

[22] A. FUTAKI and Y. Y. ZHANG: Residue formula for an obstruction to Coupled Kähler-Einstein metrics. J. Math. Soc. Japan. **73**(2021), 389–401. MR4255082

[23] J. P. GAUNTLETT, D. MARTELLI, J. SPARKS and D. WALDRAM: Sasaki-Einstein metrics on $S^2 \times S^3$, Adv. Theor. Math. Phys. **8** (2004), 711–734. MR2141499

[24] V. GUILLEMIN: Kaehler structures on toric varieties. Journal of differential geometry, **40**(1994), 285–309. MR1293656

[25] W. HE and S. SUN: Frankel conjecture and Sasaki geometry, Adv. Math. **291**(2016) 912–960. MR3459033

[26] J. HULTGREN: Coupled Kähler-Ricci solitons on toric manifolds. Anal. PDE., **12**(2019), no. 8, 2067–2094. MR4023975

[27] J. HULTGREN and D. WITT NYSTRÖM: Coupled Kähler-Einstein metrics. Int. Math. Res. Not. IMRN **2019**, no. 21, 6765–6796. MR4027565

[28] E. LERMAN, A convexity theorem for torus actions on a contact manifold, Ill. J. Math. **46** (2002) 171–184. MR1936083

[29] E. LERMAN: Contact toric manifolds, J. Symplectic Geom. **1** (2003), no. 4, 785–828. MR2039164

[30] D. MARTELLI, J. SPARKS and S.-T. YAU: The geometric dual of a-maximisation for toric Sasaki-Einstein manifolds, Comm. Math. Phys. **268**, 39–65 (2006). MR2249795

[31] D. MARTELLI, J. SPARKS and S.-T. YAU: Sasaki-Einstein manifolds and volume minimisation, Comm. Math. Phys. **280** (2008), no. 3, 611–673. MR2399609

[32] G. TIAN and X.-H. ZHU: A new holomorphic invariant and uniqueness of Kähler-Ricci solitons, Comment. Math. Helv. **77**(2002), 297–325. MR1915043

[33] I. VIZA DE SOUZA: Conformally Kähler, Einstein-Maxwell metrics on Hirzebruch surfaces. Ann. Global Anal. Geom. **59**.2 (2021), 263–284. MR4213767

[34] X.-J. WANG and X. ZHU: Kähler-Ricci solitons on toric manifolds with positive first Chern class, Adv. Math. **188** (2004), no. 1, 87–103. MR2084775

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