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Some Inequalities for the dual p-quermassintegrals*

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Abstract: Based on the definitions of dual quermassintegrals, dual affine quermassintegrals and dual harmonic quermassintegrals, we generalize them to the dual p-quermassintegrals, such that the cases $p = 1, n$ and -1 just are the dual quermassintegrals, dual affine quermassintegrals and dual harmonic quermassintegrals, respectively. Further, we orderly establish the dual L_q Brunn-Minkowski type inequality, dual log-Brunn-Minkowski type inequality and Blaschke-Santaló type inequality for the dual p-quermassintegrals.

Keywords: Dual p-quermassintegral, dual L_q Brunn-Minkowski inequality, dual log-Brunn-Minkowski inequality, Blaschke-Santaló inequality.

1. Introduction and results

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}_o^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) containing the origin in their interiors in $\mathbb{R}^n$. Let $\mathcal{S}_o^n$ denote the

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set of star bodies (about the origin) in $\mathbb{R}^n$. Let B denote the n-dimensional Euclidean unit ball centered at the origin, and the surface of B is written S^{n-1}. We use $V(K)$ to denote the n-dimensional volume of a body K, and write $V(B) = \kappa_n$. Let $G(n,i)$ $(i = 0, 1, \cdots, n)$ denote the Grassmann manifold of i-dimensional subspaces of $\mathbb{R}^n$, and let μ_i denote the usual Haar measure on $G(n,i)$, normalized so that $\mu_i(G(n,i)) = 1$.

For each convex body K, $1 \leq i \leq n-1$ and any $\zeta \in G(n,i)$, the i-dimensional volume $V_i(K|\zeta)$ is called the projection function of K, where $K|\zeta$ is the orthogonal projection of K onto ζ. Based on the notion of projection function, the quermassintegrals (see [8, 11, 12]), affine quermassintegrals (see [11]), harmonic quermassintegrals (see [8, 12]) and the general forms (i.e., p-quermassintegrals) of the three aforementioned quermassintegrals (see [16]) were introduced, respectively. Duality, for each $K \in \mathcal{S}_o^n$, $1 \leq i \leq n-1$ and any $\zeta \in G(n,i)$, the i-dimensional volume $V_i(K \cap \zeta)$ is called the section function of K. According to the notion of section function, the dual quermassintegrals, dual affine quermassintegrals and dual harmonic quermassintegrals are respectively defined as follows:

Definition 1.A. *For $K \in \mathcal{S}_o^n$ and $i = 0, 1, \cdots, n$, the dual quermassintegrals, $\widetilde{W}_i(K)$, of K are defined by letting $\widetilde{W}_0(K) = V(K)$, $\widetilde{W}_n(K) = \kappa_n$ and for $0 < i < n$,*

$$\widetilde{W}_i(K) = \frac{\kappa_n}{\kappa_{n-i}} \int_{G(n,n-i)} V_{n-i}(K \cap \xi) d\mu_{n-i}(\xi),$$

where V_{n-i} denotes $(n-i)$-dimensional volume.

Definition 1.B. *For $K \in \mathcal{S}_o^n$ and $i = 0, 1, \cdots, n$, the dual affine quermassintegrals, $\widetilde{A}_i(K)$, of K are defined by letting $\widetilde{A}_0(K) = V(K)$, $\widetilde{A}_n(K) = \kappa_n$ and for $0 < i < n$,*

$$\widetilde{A}_i(K) = \frac{\kappa_n}{\kappa_{n-i}} \left(\int_{G(n,n-i)} V_{n-i}(K \cap \xi)^n d\mu_{n-i}(\xi) \right)^{\frac{1}{n}}.$$

Definition 1.C. *For $K \in \mathcal{S}_o^n$ and $i = 0, 1, \cdots, n$, the dual harmonic quermassintegrals, $\widetilde{H}_i(K)$, of K are defined by letting $\widetilde{H}_0(K) = V(K)$, $\widetilde{H}_n(K) = \kappa_n$ and for $0 < i < n$,*

$$\widetilde{H}_i(K) = \frac{\kappa_n}{\kappa_{n-i}} \left(\int_{G(n,n-i)} V_{n-i}(K \cap \xi)^{-1} d\mu_{n-i}(\xi) \right)^{-1}.$$

Note that the definitions of dual quermassintegrals and dual affine quermassintegrals see Lutwak's papers [9] and [12], the dual harmonic quermassintegrals was given by Yuan, Yuan and Leng (see [18]). In addition, Gardner (see [5]) extended the definitions of dual quermassintegrals and dual affine quermassintegrals from star bodies to a bounded Borel set. For the studies of quermassintegrals and dual quermassintegrals, a lot of research results were aggregated in books [4, 8, 14] and papers [1, 2, 3, 5, 7, 9, 10, 11, 12, 17, 18].

In this paper, based on the definitions of dual quermassintegrals, dual affine quermassintegrals and dual harmonic quermassintegrals, we generalize them to the dual p-quermassintegrals, such that the special cases $p = 1, n$ and -1 just are the dual quermassintegrals, dual affine quermassintegrals and dual harmonic quermassintegrals, respectively. Whereafter, we orderly establish the dual L_q Brunn-Minkowski type inequality, dual log-Brunn-Minkowski type inequality and Blaschke-Santaló type inequality of the dual p-quermassintegrals. Here, we give definition of dual p-quermassintegrals as follows:

Definition 1.1. *Let $K \in \mathcal{S}_o^n$, $i = 0, 1, \cdots, n$ and p be any real. For $p \neq 0$, the dual p-quermassintegrals, $\widetilde{Q}_{i,p}(K)$, of K are defined by letting $\widetilde{Q}_{0,p}(K) = V(K)$, $\widetilde{Q}_{n,p}(K) = \kappa_n$ and for $0 < i < n$,*

$$\widetilde{Q}_{i,p}(K) = \frac{\kappa_n}{\kappa_{n-i}} \left(\int_{G(n,n-i)} V_{n-i}(K \cap \xi)^p d\mu_{n-i}(\xi) \right)^{\frac{1}{p}}. \tag{1.1}$$

For $p = 0$, we define $\widetilde{Q}_{0,0}(K) = V(K)$, $\widetilde{Q}_{n,0}(K) = \kappa_n$ and for $0 < i < n$,

$$\widetilde{Q}_{i,0}(K) = \lim_{p \to 0} \widetilde{Q}_{i,p}(K) = \frac{\kappa_n}{\kappa_{n-i}} \left(\exp \int_{G(n,n-i)} \ln V_{n-i}(K \cap \xi) d\mu_{n-i}(\xi) \right). \tag{1.2}$$

Note that for $p > 0$ and K is a bounded Borel set, definition (1.1) was given by Gardner (see [5]).

Clearly, the cases $p = 1, n, -1$ of Definition 1.1 successively are Definition 1.A, Definition 1.B and Definition 1.C, i.e.,

$$\widetilde{Q}_{i,1}(K) = \widetilde{W}_i(K), \quad \widetilde{Q}_{i,n}(K) = \widetilde{A}_i(K), \quad \widetilde{Q}_{i,-1}(K) = \widetilde{H}_i(K). \tag{1.3}$$

From the Jensen mean integral inequality, we easily know that: If $K \in \mathcal{S}_o^n$, $i = 0, 1, \cdots, n$, reals $p, q \neq 0$ and $p < q$, then

$$\widetilde{Q}_{i,p}(K) \leq \widetilde{Q}_{i,q}(K), \tag{1.4}$$

with equality for $0 \leq i < n$ if and only if $V_{n-i}(K \cap \xi)$ is a constant for all $\xi \in G(n, n-i)$.

For the dual quermassintegrals, dual affine quermassintegrals and dual harmonic quermassintegrals, the related dual Brunn-Minkowski inequalities for the radial Minkowski additions of star bodies were established as follows:
Theorem 1.A. *If $K, L \in \mathcal{S}_o^n$ and $0 \leq i < n$, then*

$$\widetilde{W}_i(K \widetilde{+} L)^{\frac{1}{n-i}} \leq \widetilde{W}_i(K)^{\frac{1}{n-i}} + \widetilde{W}_i(L)^{\frac{1}{n-i}}; \tag{1.5}$$

$$\widetilde{A}_i(K \widetilde{+} L)^{\frac{1}{n-i}} \leq \widetilde{A}_i(K)^{\frac{1}{n-i}} + \widetilde{A}_i(L)^{\frac{1}{n-i}}; \tag{1.6}$$

$$\widetilde{H}_i(K \widetilde{+} L)^{\frac{1}{n-i}} \leq \widetilde{H}_i(K)^{\frac{1}{n-i}} + \widetilde{H}_i(L)^{\frac{1}{n-i}}. \tag{1.7}$$

In inequality (1.5), equality holds for $0 \leq i < n-1$ if and only if K and L are dilated; for $i = n-1$, (1.5) is an equality. In inequalities (1.6) and (1.7), equality holds if and only if K and L are dilated. Here $K \widetilde{+} L$ denotes the radial Minkowski addition of K and L.

Note that inequality (1.5) can be found in [5], inequality (1.6) was established by Yuan and Leng (see [17], also see Gardner [5]), inequality (1.7) was due to Yuan, Yuan and Leng (see [18]).

Nextly, together with the L_q radial Minkowski combinations of star bodies, we establish the dual L_q Brunn-Minkowski type inequalities of dual p-quermassintegrals. If $p \neq 0$, we have result as follows:
Theorem 1.1. *Let $K, L \in \mathcal{S}_o^n$, $0 \leq i < n$, $\lambda, \mu \geq 0$ (not both zero) and reals $p, q \neq 0$. If $0 < q < n-i$ and $\frac{p(n-i)}{q} \geq 1$, then*

$$\widetilde{Q}_{i,p}(\lambda \cdot K \widetilde{+}_q \mu \cdot L)^{\frac{q}{n-i}} \leq \lambda \widetilde{Q}_{i,p}(K)^{\frac{q}{n-i}} + \mu \widetilde{Q}_{i,p}(L)^{\frac{q}{n-i}}; \tag{1.8}$$

if $q < 0$ and $\frac{p(n-i)}{q} \leq 1$, or $q > n-i$ and $\frac{p(n-i)}{q} \leq 1$, then

$$\widetilde{Q}_{i,p}(\lambda \cdot K \widetilde{+}_q \mu \cdot L)^{\frac{q}{n-i}} \geq \lambda \widetilde{Q}_{i,p}(K)^{\frac{q}{n-i}} + \mu \widetilde{Q}_{i,p}(L)^{\frac{q}{n-i}}. \tag{1.9}$$

In each case, equality holds if and only if K and L are dilated. Here $\lambda \cdot K \widetilde{+}_q \mu \cdot L$ denotes the L_q radial Minkowski combination of K and L.

Let $q = 1$, $\lambda = \mu = 1$ in Theorem 1.1, we have the following dual Brunn-Minkowski type inequality of dual p-quermassintegrals.
Corollary 1.1. *If $K, L \in \mathcal{S}_o^n$, $0 \leq i < n$ and real $p \geq \frac{1}{n-i}$, then*

$$\widetilde{Q}_{i,p}(K \widetilde{+} L)^{\frac{1}{n-i}} \leq \widetilde{Q}_{i,p}(K)^{\frac{1}{n-i}} + \widetilde{Q}_{i,p}(L)^{\frac{1}{n-i}}, \tag{1.10}$$

with equality if and only if K and L are dilated.

Taking $p = 1, n$ in Corollary 1.1, then inequality (1.10) yields inequality (1.5) and inequality (1.6), respectively.

In Theorem 1.1, if $p = 1, n, -1$, then by (1.3) we may orderly get the dual L_q Brunn-Minkowski type inequalities of dual quermassintegrals, dual affine quermassintegrals and dual harmonic quermassintegrals.

If $p = 0$, we have the following dual L_q Brunn-Minkowski type inequalities for the dual p-quermassintegrals.

Theorem 1.2. *Let $K, L \in \mathcal{S}_o^n$, $0 \le i < n$, real $q \ne 0$ and $\lambda \in [0,1]$. If $q > n - i$, then*

$$\widetilde{Q}_{i,0}(\lambda \cdot K \widetilde{+}_q (1-\lambda) \cdot L) \ge \widetilde{Q}_{i,0}(K)^{\lambda} \widetilde{Q}_{i,0}(L)^{1-\lambda}; \tag{1.11}$$

if $q < 0$, then

$$\widetilde{Q}_{i,0}(\lambda \cdot K \widetilde{+}_q (1-\lambda) \cdot L) \le \widetilde{Q}_{i,0}(K)^{\lambda} \widetilde{Q}_{i,0}(L)^{1-\lambda}. \tag{1.12}$$

In each case, equality holds for $\lambda \in (0,1)$ if and only if $K = L$; for $\lambda = 0$ or $\lambda = 1$, inequalities (1.11) and (1.12) both are equalities.

Further, based on Wang and Liu's dual log-Brunn-Minkowski inequality (see [15]), we give the dual p-quermassintegrals form of dual log-Brunn-Minkowski inequality.

Theorem 1.3. *For $K, L \in \mathcal{S}_o^n$, $\lambda \in [0,1]$, $0 \le i < n$ and real p, if $p \ge 0$, then*

$$\widetilde{Q}_{i,p}(\lambda \cdot K \widetilde{+}_0 (1-\lambda) \cdot L) \le \widetilde{Q}_{i,p}(K)^{\lambda} \widetilde{Q}_{i,p}(L)^{1-\lambda}, \tag{1.13}$$

with equality for $\lambda \in (0,1)$ if and only if K and L are dilated. For $\lambda = 0$ or $\lambda = 1$, inequality (1.13) becomes an equality. Here, $\lambda \cdot K \widetilde{+}_0 (1-\lambda) \cdot L$ denotes the log-radial combination of $K, L \in \mathcal{S}_o^n$.

Let $p = 1, n$ in Theorem 1.3, we respectively obtain the dual log-Brunn-Minkowski type inequalities for the dual quermassintegrals and dual affine quermassintegrals as follows:

Corollary 1.2. *If $K, L \in \mathcal{S}_o^n$, $\lambda \in [0,1]$ and $0 \le i < n$, then*

$$\widetilde{W}_i(\lambda \cdot K \widetilde{+}_0 (1-\lambda) \cdot L) \le \widetilde{W}_i(K)^{\lambda} \widetilde{W}_i(L)^{1-\lambda};$$

$$\widetilde{A}_i(\lambda \cdot K \widetilde{+}_0 (1-\lambda) \cdot L) \le \widetilde{A}_i(K)^{\lambda} \widetilde{A}_i(L)^{1-\lambda}.$$

In each inequality, equality holds for $\lambda \in (0,1)$ if and only if K and L are dilated; for $\lambda = 0$ or $\lambda = 1$, above inequalities become equalities.

Finally, according to the well-known Blaschke-Santaló inequality, we obtain related dual p-quermassintegrals version. Recall that Lutwak ([13]) improved the Blaschke-Santaló inequality as follows:

Theorem 1.B. *If $K \in \mathcal{S}_o^n$ whose centroid is at the origin, then*

$$V(K)V(K^*) \leq \kappa_n^2, \tag{1.14}$$

with equality if and only if K is an ellipsoid centered at the origin. Here K^* denotes the polar of K.

From inequality (1.14), we have the following Blaschke-Santaló type inequality for the dual p-quermassintegrals.

Theorem 1.4. *If K is an origin-symmetric star body, $0 \leq i < n$ and real $p \leq n-1$, then*

$$\widetilde{Q}_{i,p}(K)\widetilde{Q}_{i,p}(K^*) \leq \kappa_n^2, \tag{1.15}$$

with equality for $i = 0$ if and only if K is an ellipsoid centered at the origin, for $0 < i < n$ if and only if K is a ball centered at the origin.

Let $p = 1, -1$ in Theorem 1.4 and by equality (1.3), we may obtain the Blaschke-Santaló type inequalities for dual quermassintegrals and dual harmonic quermassintegrals.

Corollary 1.3. *If K is an origin-symmetric star body and $0 \leq i < n$, then*

$$\widetilde{W}_i(K)\widetilde{W}_i(K^*) \leq \kappa_n^2; \tag{1.16}$$

$$\widetilde{H}_i(K)\widetilde{H}_i(K^*) \leq \kappa_n^2. \tag{1.17}$$

In each inequality, equality holds for $i = 0$ if and only if K is an ellipsoid centered at the origin, for $0 < i < n$ if and only if K is a ball centered at the origin.

Note that inequality (1.16) can be found in [6], inequality (1.17) was established by Yuan, Yuan and Leng (see [18]).

2. Background materials

If K is a compact star shaped subset (about the origin) in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \backslash \{0\} \to [0, \infty)$, is defined by (see [4, 14])

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \backslash \{0\}.$$

If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Two star bodies K and L are said to be dilated (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

Based on the radial function, we have the following polar coordinate formula of volume:

$$V(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n du. \tag{2.1}$$

For the radial function, we see that if $K \in \mathcal{S}_o^n$, $1 \le i \le n$ and ζ is the i-dimensional subspace of $\mathbb{R}^n$, then for any $u \in S^{n-1} \cap \zeta$ (see [4]),

$$\rho(K \cap \zeta, u) = \rho(K,u). \tag{2.2}$$

For $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \ge 0$ (not both zero) and real $q \neq 0$, the L_q radial Minkowski combination, $\lambda \cdot K \widetilde{+}_q \mu \cdot L \in \mathcal{S}_o^n$, of K and L is defined by (see [14])

$$\rho(\lambda \cdot K \widetilde{+}_q \mu \cdot L, \cdot) = [\lambda \rho(K,\cdot)^q + \mu \rho(L,\cdot)^q]^{\frac{1}{q}}. \tag{2.3}$$

If $\lambda = \mu = 1$, then $K \widetilde{+}_q L$ is called the L_q radial Minkowski addition of K and L. In particular, $K \widetilde{+}_1 L = K \widetilde{+} L$ is the radial Minkowski addition of K and L.

In (2.3), let $\mu = 1 - \lambda$ ($\lambda \in [0,1]$) and $q \to 0$, then

$$\lim_{q\to 0} \rho(\lambda \cdot K \widetilde{+}_q (1-\lambda) \cdot L, \cdot) = \lim_{q \to 0} [\lambda \rho(K,\cdot)^q + (1-\lambda)\rho(L,\cdot)^q]^{\frac{1}{q}} = \rho(K,\cdot)^{\lambda} \rho(L,\cdot)^{1-\lambda}.$$

Thereout, Wang and Liu ([15]) introduced the notion of log-radial combination as follows: For $K, L \in \mathcal{S}_o^n$ and $\lambda \in [0,1]$, the log-radial combination, $\lambda \cdot K \widetilde{+}_0 (1-\lambda) \cdot L$, of K and L is defined by

$$\rho(\lambda \cdot K \widetilde{+}_0 (1-\lambda) \cdot L, \cdot) = \rho(K,\cdot)^{\lambda} \rho(L,\cdot)^{1-\lambda}. \tag{2.4}$$

If E is a nonempty set in $\mathbb{R}^n$, the polar duality of E, E^*, is defined by (see [4])

$$E^* = \{x : x \cdot y \le 1, y \in E\}, \quad x \in \mathbb{R}^n.$$

From above definition, we easily know that E^* is convex, and for $K \in \mathcal{K}_o^n$ and ζ is a subspace of $\mathbb{R}^n$ (see [4]),

$$K^* \cap \zeta = (K|\zeta)^*. \tag{2.5}$$

3. Dual L_q Brunn-Minkowski type inequalities

In this section, we start to prove the dual L_q Brunn-Minkowski type inequalities of dual p-quermassintegrals for the L_q radial Minkowski combinations. The following dual L_q-Brunn-Minkowski inequality is essential.

Lemma 3.1 ([3]). *If $K, L \in \mathcal{S}_o^n$, real $q \neq 0$ and $\lambda, \mu \geq 0$ (not both zero), then for $0 < q < n$,*

$$V(\lambda \cdot K \widetilde{+}_q \mu \cdot L)^{\frac{q}{n}} \leq \lambda V(K)^{\frac{q}{n}} + \mu V(L)^{\frac{q}{n}}; \tag{3.1}$$

for $q < 0$ or $q > n$,

$$V(\lambda \cdot K \widetilde{+}_q \mu \cdot L)^{\frac{q}{n}} \geq \lambda V(K)^{\frac{q}{n}} + \mu V(L)^{\frac{q}{n}}. \tag{3.2}$$

In each case, equality holds if and only if K and L are dilated.

Lemma 3.2. *If $K, L \in \mathcal{S}_o^n$, real $q \neq 0$, $\lambda, \mu \geq 0$ (not both zero) and $\xi \in G(n, n-i)$, then*

$$(\lambda \cdot K \widetilde{+}_q \mu \cdot L) \cap \xi = \lambda \cdot (K \cap \xi) \widetilde{+}_q \mu \cdot (L \cap \xi). \tag{3.3}$$

Proof. According to (2.2) and (2.3), since $\xi \in G(n, n-i)$, thus we have for any $u \in S^{n-1} \cap \xi$,

$$\rho((\lambda \cdot K \widetilde{+}_q \mu \cdot L) \cap \xi, u)^q = \rho(\lambda \cdot K \widetilde{+}_q \mu \cdot L, u)^q = \lambda \rho(K, u)^q + \mu \rho(L, u)^q$$

$$= \lambda \rho(K \cap \xi, u)^q + \mu \rho(L \cap \xi, u)^q = \rho(\lambda \cdot (K \cap \xi) \widetilde{+}_q \mu \cdot (L \cap \xi), u)^q.$$

This gives (3.3). □

Proof of Theorem 1.1. For $0 < q < n-i$, from dual L_q-Brunn-Minkowski inequality (3.1) for $(n-i)$-dimensional case, we get for any $\xi \in G(n, n-i)$,

$$V_{n-i}(\lambda \cdot (K \cap \xi) \widetilde{+}_q \mu \cdot (L \cap \xi))^{\frac{q}{n-i}} \leq \lambda V_{n-i}(K \cap \xi)^{\frac{q}{n-i}} + \mu V_{n-i}(L \cap \xi)^{\frac{q}{n-i}}, \tag{3.4}$$

and equality holds if and only if $K \cap \xi$ and $L \cap \xi$ are dilated for any $\xi \in G(n, n-i)$, i.e., K and L are dilated.

If $\frac{p(n-i)}{q} \geq 1$, by (1.1), (3.3), (3.4) and the Minkowski integral inequality, we have that

$$\widetilde{Q}_{i,p}(\lambda \cdot K \widetilde{+}_q \mu \cdot L)^{\frac{q}{n-i}}$$
$$= \left(\frac{\kappa_n}{\kappa_{n-i}}\right)^{\frac{q}{n-i}} \left[\int_{G(n,n-i)} V_{n-i}((\lambda \cdot K \widetilde{+}_q \mu \cdot L) \cap \xi)^p d\mu_{n-i}(\xi)\right]^{\frac{q}{p(n-i)}}$$

$$
\begin{aligned}
=&\left(\frac{\kappa_n}{\kappa_{n-i}}\right)^{\frac{q}{n-i}}\left[\int_{G(n,n-i)}\left(V_{n-i}(\lambda\cdot(K\cap\xi)\widetilde{+}_q\mu\cdot(L\cap\xi))^{\frac{q}{n-i}}\right)^{\frac{p(n-i)}{q}}d\mu_{n-i}(\xi)\right]^{\frac{q}{p(n-i)}}\\
\leq&\left(\frac{\kappa_n}{\kappa_{n-i}}\right)^{\frac{q}{n-i}}\left[\int_{G(n,n-i)}\left(\lambda V_{n-i}(K\cap\xi)^{\frac{q}{n-i}}+\mu V_{n-i}(L\cap\xi)^{\frac{q}{n-i}}\right)^{\frac{p(n-i)}{q}}d\mu_{n-i}(\xi)\right]^{\frac{q}{p(n-i)}}\\
\leq&\lambda\left(\frac{\kappa_n}{\kappa_{n-i}}\right)^{\frac{q}{n-i}}\left[\int_{G(n,n-i)}V_{n-i}(K\cap\xi)^p d\mu_{n-i}(\xi)\right]^{\frac{q}{p(n-i)}}\\
&+\mu\left(\frac{\kappa_n}{\kappa_{n-i}}\right)^{\frac{q}{n-i}}\left[\int_{G(n,n-i)}V_{n-i}(L\cap\xi)^p d\mu_{n-i}(\xi)\right]^{\frac{q}{p(n-i)}}\\
=&\lambda\widetilde{Q}_{i,p}(K)^{\frac{q}{n-i}}+\mu\widetilde{Q}_{i,p}(L)^{\frac{q}{n-i}}.
\end{aligned}
$$

Therefore, inequality (1.8) is proved.

According to the equality conditions of inequality (3.4) and the Minkowski integral inequality, we see that equality holds in inequality (1.8) if and only if K and L are dilated.

For $q<0$ or $q>n-i$, applying dual L_q-Brunn-Minkowski inequality (3.2) to $(n-i)$-dimensional case, we know that for any $\xi\in G(n,n-i)$,

$$
V_{n-i}(\lambda\cdot(K\cap\xi)\widetilde{+}_q\mu\cdot(L\cap\xi))^{\frac{q}{n-i}}\geq\lambda V_{n-i}(K\cap\xi)^{\frac{q}{n-i}}+\mu V_{n-i}(L\cap\xi)^{\frac{q}{n-i}}, \tag{3.5}
$$

and the equality condition is the same as (3.4), i.e., equality holds in (3.5) if and only if K and L are dilated.

From this, similar to the proof of inequality (1.8), if $\frac{p(n-i)}{q}\leq 1$, then by (1.1), (3.3), (3.5) and the Minkowski integral inequality, we can obtain inequality (1.9) and its equality condition. □

Proof of Theorem 1.2. For $q>n-i(>0)$ or $q<0$, let $\mu=1-\lambda$ ($\lambda\in[0,1]$) in inequality (3.5), then by (1.2), (3.3), (3.5) and notice that function $f(x)=\ln x$ is concave on $x\in(0,+\infty)$, we have that for $q>n-i$,

(3.6)

$$
\begin{aligned}
&\widetilde{Q}_{i,0}(\lambda\cdot K\widetilde{+}_q(1-\lambda)\cdot L)\\
=&\frac{\kappa_n}{\kappa_{n-i}}\left(\exp\int_{G(n,n-i)}\ln V_{n-i}((\lambda\cdot K\widetilde{+}_q(1-\lambda)\cdot L)\cap\xi)d\mu_{n-i}(\xi)\right)\\
=&\frac{\kappa_n}{\kappa_{n-i}}\left[\exp\int_{G(n,n-i)}\frac{n-i}{q}\right.\\
&\left.\times\ln V_{n-i}(\lambda\cdot(K\cap\xi)\widetilde{+}_q(1-\lambda)\cdot(L\cap\xi))^{\frac{q}{n-i}}d\mu_{n-i}(\xi)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{\kappa_n}{\kappa_{n-i}}\Bigg[\exp\int_{G(n,n-i)}\frac{n-i}{q}\\
&\qquad\times\ln\left(\lambda V_{n-i}(K\cap\xi)^{\frac{q}{n-i}}+(1-\lambda)V_{n-i}(L\cap\xi)^{\frac{q}{n-i}}\right)d\mu_{n-i}(\xi)\Bigg]\\
&\geq \frac{\kappa_n}{\kappa_{n-i}}\Bigg[\exp\int_{G(n,n-i)}\frac{n-i}{q}\\
&\qquad\times\left(\lambda\ln V_{n-i}(K\cap\xi)^{\frac{q}{n-i}}+(1-\lambda)\ln V_{n-i}(L\cap\xi)^{\frac{q}{n-i}}\right)d\mu_{n-i}(\xi)\Bigg]\\
&=\frac{\kappa_n}{\kappa_{n-i}}\Bigg[\exp\int_{G(n,n-i)}\Big(\lambda\ln V_{n-i}(K\cap\xi)+(1-\lambda)\ln V_{n-i}(L\cap\xi)\Big)d\mu_{n-i}(\xi)\Bigg]\\
&=\left(\frac{\kappa_n}{\kappa_{n-i}}\exp\int_{G(n,n-i)}\ln V_{n-i}(K\cap\xi)d\mu_{n-i}(\xi)\right)^{\lambda}\\
&\quad\cdot\left(\frac{\kappa_n}{\kappa_{n-i}}\exp\int_{G(n,n-i)}\ln V_{n-i}(L\cap\xi)d\mu_{n-i}(\xi)\right)^{1-\lambda}\\
&=\widetilde{Q}_{i,0}(K)^{\lambda}Q_{i,0}(L)^{1-\lambda}.
\end{aligned}
$$

This is just inequality (1.11).

If $q<0$, similar to the proof of inequality (1.11), we easily prove inequality (3.6) is reverse. From this, inequality (1.12) is obtained.

According to the equality conditions of inequality (3.5) and the definition of concave function, we see that if $\lambda\in(0,1)$, then equality hold in (1.11) and (1.12) if and only if K and L are dilated and $V_{n-i}(K\cap\xi)=V_{n-i}(L\cap\xi)$ for any $\xi\in G(n,n-i)$, i.e., equality hold in (1.11) and (1.12) if and only if $K=L$. If $\lambda=0$ or $\lambda=1$, inequality (1.11) and inequality (1.12) clearly are equalities. □

4. Dual Log-Brunn-Minkowski type inequality

In order to prove Theorem 1.3, we need the following dual log-Brunn-Minkowski inequality which be established by Wang and Liu (see [15]).

Lemma 4.1. *If $K,L\in\mathcal{S}_o^n$ and $\lambda\in[0,1]$, then*

$$V(\lambda\cdot K\widetilde{+}_0(1-\lambda)\cdot L)\leq V(K)^{\lambda}V(L)^{1-\lambda}, \tag{4.1}$$

with equality for $\lambda\in(0,1)$ if and only if K and L are dilated. For $\lambda=0$ or $\lambda=1$, (4.1) becomes an equality.

Lemma 4.2. *If $K, L \in \mathcal{S}_o^n$, $\lambda \in [0,1]$ and $0 \leq i < n$, then for any $\xi \in G(n, n-i)$,*

$$(\lambda \cdot K \widetilde{+}_0 (1-\lambda) \cdot L) \cap \xi = \lambda \cdot (K \cap \xi) \widetilde{+}_0 (1-\lambda) \cdot (L \cap \xi). \tag{4.2}$$

Proof. By (2.2) and (2.4) we have that for any $u \in S^{n-1} \cap \xi$,

$$\rho((\lambda \cdot K \widetilde{+}_0 (1-\lambda) \cdot L) \cap \xi, u) = \rho(\lambda \cdot K \widetilde{+}_0 (1-\lambda) \cdot L, u)$$

$$= \rho(K,u)^\lambda \rho(L,u)^{1-\lambda} = \rho(K \cap \xi, u)^\lambda \rho(L \cap \xi, u)^{1-\lambda}$$

$$= \rho(\lambda \cdot (K \cap \xi) \widetilde{+}_0 (1-\lambda) \cdot (L \cap \xi), u).$$

This provides (4.2). □

Proof of Theorem 1.3. For $0 \leq i < n$, applying inequality (4.1) to $(n-i)$-dimensional case, then by (4.2) we know that for $\lambda \in [0,1]$ and $\xi \in G(n, n-i)$,

$$V_{n-i}((\lambda \cdot K \widetilde{+}_0 (1-\lambda) \cdot L) \cap \xi) = V_{n-i}(\lambda \cdot (K \cap \xi) \widetilde{+}_0 (1-\lambda) \cdot (L \cap \xi))$$

$$\leq V_{n-i}(K \cap \xi)^\lambda V_{n-i}(L \cap \xi)^{1-\lambda}. \tag{4.3}$$

According to the equality condition of inequality (4.1), we know that equality holds in inequality (4.3) for $\lambda \in (0,1)$ if and only if $K \cap \xi$ and $L \cap \xi$ are dilated for any $\xi \in G(n, n-i)$, i.e., K and L are dilated. For $\lambda = 0$ or $\lambda = 1$, (4.3) becomes an equality.

If $p > 0$, by (1.1), inequality (4.3) and the Hölder integral inequality, we have that for $\lambda \in (0,1)$,

$$\left[\widetilde{Q}_{i,p}(K)^\lambda \widetilde{Q}_{i,p}(L)^{1-\lambda}\right]^p$$

$$= \left[\left(\frac{\kappa_n}{\kappa_{n-i}}\right)^p \int_{G(n,n-i)} V_{n-i}(K \cap \xi)^p d\mu_{n-i}(\xi)\right]^\lambda$$

$$\cdot \left[\left(\frac{\kappa_n}{\kappa_{n-i}}\right)^p \int_{G(n,n-i)} V_{n-i}(L \cap \xi)^p d\mu_{n-i}(\xi)\right]^{1-\lambda}$$

$$= \left(\frac{\kappa_n}{\kappa_{n-i}}\right)^p \left[\int_{G(n,n-i)} \left(V_{n-i}(K \cap \xi)^{p\lambda}\right)^{\frac{1}{\lambda}} d\mu_{n-i}(\xi)\right]^\lambda$$

$$
\begin{aligned}
&\cdot\left[\int_{G(n,n-i)}\left(V_{n-i}(L\cap\xi)^{p(1-\lambda)}\right)^{\frac{1}{1-\lambda}}d\mu_{n-i}(\xi)\right]^{1-\lambda}\\
&\geq\left(\frac{\kappa_n}{\kappa_{n-i}}\right)^p\int_{G(n,n-i)}\left[V_{n-i}(K\cap\xi)^{\lambda}V_{n-i}(L\cap\xi)^{1-\lambda}\right]^p d\mu_{n-i}(\xi)\\
&\geq\left(\frac{\kappa_n}{\kappa_{n-i}}\right)^p\int_{G(n,n-i)}V_{n-i}((\lambda\cdot K\widetilde{+}_0(1-\lambda)\cdot L)\cap\xi)^p d\mu_{n-i}(\xi)\\
&=\widetilde{Q}_{i,p}(\lambda\cdot K\widetilde{+}_0(1-\lambda)\cdot L)^p.
\end{aligned}
$$

This yields the case $p>0$ of inequality (1.13).

From the equality conditions of inequality (4.3) and the Hölder integral inequality, we see that equality holds in the case $p>0$ of inequality (1.13) for $\lambda\in(0,1)$ if and only if K and L are dilated.

If $p=0$, then (1.2) and (4.3) give that for $\lambda\in[0,1]$,

$$
\begin{aligned}
&\widetilde{Q}_{i,0}(\lambda\cdot K\widetilde{+}_0(1-\lambda)\cdot L)\\
&=\frac{\kappa_n}{\kappa_{n-i}}\left[\exp\int_{G(n,n-i)}\ln V_{n-i}((\lambda\cdot K\widetilde{+}_0(1-\lambda)\cdot L)\cap\xi)d\mu_{n-i}(\xi)\right]\\
&\leq\frac{\kappa_n}{\kappa_{n-i}}\left[\exp\int_{G(n,n-i)}\left(\ln V_{n-i}(K\cap\xi)^{\lambda}+\ln V_{n-i}(L\cap\xi)^{1-\lambda}\right)d\mu_{n-i}(\xi)\right]\\
&=\frac{\kappa_n}{\kappa_{n-i}}\left[\exp\left(\lambda\int_{G(n,n-i)}\ln V_{n-i}(K\cap\xi)d\mu_{n-i}(\xi)\right.\right.\\
&\qquad\left.\left.+(1-\lambda)\int_{G(n,n-i)}\ln V_{n-i}(L\cap\xi)d\mu_{n-i}(\xi)\right)\right]\\
&=\left[\frac{\kappa_n}{\kappa_{n-i}}\left(\exp\int_{G(n,n-i)}\ln V_{n-i}(K\cap\xi)d\mu_{n-i}(\xi)\right)\right]^{\lambda}\\
&\quad\cdot\left[\frac{\kappa_n}{\kappa_{n-i}}\left(\exp\int_{G(n,n-i)}\ln V_{n-i}(L\cap\xi)d\mu_{n-i}(\xi)\right)\right]^{1-\lambda}\\
&=\widetilde{Q}_{i,0}(K)^{\lambda}\widetilde{Q}_{i,0}(L)^{1-\lambda}.
\end{aligned}
$$

From this, we obtain the case $p=0$ of inequality (1.13).

According to the equality condition of inequality (4.3), we see that if $\lambda\in(0,1)$, then equality holds in the case $p=0$ of inequality (1.13) if and only if K and L are dilated.

For $\lambda=0$ or $\lambda=1$, (1.13) obviously becomes an equality. $\square$

5. Blaschke-Santaló type inequality

Theorem 1.4 shows the Blaschke-Santaló type inequality for dual p-quermassintegrals. Here, we complete its proof.

Lemma 5.1 (see [5], Theorem 7.4) *If $K, L \in \mathcal{S}_o^n$, $0 \le i < j < n$, real p satisfies $0 < p \le n - i - 1$, then*

$$\left(\frac{\widetilde{Q}_{i,p}(K)}{\kappa_n}\right)^{\frac{1}{n-i}} \ge \left(\frac{\widetilde{Q}_{j,p}(K)}{\kappa_n}\right)^{\frac{1}{n-j}}, \tag{5.1}$$

with equality if and only if K is a ball centered at the origin.

Lemma 5.2. *If $K \in \mathcal{K}_o^n$, $0 < i < n$, and $p < 0$ or $0 < p \le n - 1$, then*

$$\widetilde{Q}_{i,p}(K) \le \kappa_n^{\frac{i}{n}} V(K)^{\frac{n-i}{n}}, \tag{5.2}$$

with equality if and only if K is a ball centered at the origin.

Proof. Let $i = 0$ in Lemma 5.1, replace j by i and notice that $Q_{0,p}(K) = V(K)$, then inequality (5.1) gives the case $0 < p \le n - 1$ of inequality (5.2), i.e., for $0 < p \le n - 1$,

$$\widetilde{Q}_{i,p}(K) \le \kappa_n^{\frac{i}{n}} V(K)^{\frac{n-i}{n}}, \tag{5.3}$$

with equality if and only if K is a ball centered at the origin.

If $p < 0$, choose real q such that $0 < q \le n - 1$, then by inequalities (1.4) and (5.3), we get that for $p < 0$,

$$\widetilde{Q}_{i,p}(K) \le \widetilde{Q}_{i,q}(K) \le \kappa_n^{\frac{i}{n}} V(K)^{\frac{n-i}{n}},$$

with equality if and only if K is a ball centered at the origin. □

Lemma 5.3. *If K is origin-symmetric star body, $0 < i < n$ and $\xi \in G(n, n-i)$, then*

$$V_{n-i}(K|\xi)V_{n-i}((K|\xi)^*) \le \kappa_{n-i}^2, \tag{5.4}$$

with equality if and only if K is an ellipsoid centered at the origin.

Proof. Since $0 < i < n$ and K is an origin-symmetric star body, thus $K \cap \xi$ is also an origin-symmetric star body for any $\xi \in G(n, n-i)$. Thereout, applying Blaschke-Santaló inequality (1.14) to $(n-i)$-dimensional case, we have for $\xi \in G(n, n-i)$,

$$V_{n-i}(K|\xi)V_{n-i}((K|\xi)^*) \le \kappa_{n-i}^2.$$

This just inequality (5.4). And equality holds in inequality (5.4) if and only if $K|\xi$ is an ellipsoid centered at the origin for any $\xi \in G(n, n-i)$, i.e., K is an ellipsoid centered at the origin. $\square$

Proof of Theorem 1.4. For $i = 0$, according to Definition 1.1, inequality (1.15) is just inequality (1.14).

For $0 < i < n$, if $p < 0$ or $0 < p \leq n-1$, then by inequality (5.2) and Blaschke-Santaló inequality (1.14) we get that

$$\widetilde{Q}_{i,p}(K)\widetilde{Q}_{i,p}(K^*) \leq \left(\kappa_n^{\frac{i}{n}} V(K)^{\frac{n-i}{n}}\right)\left(\kappa_n^{\frac{i}{n}} V(K^*)^{\frac{n-i}{n}}\right)$$

$$= \kappa_n^{\frac{2i}{n}} [V(K)V(K^*)]^{\frac{n-i}{n}} \leq \kappa_n^{\frac{2i}{n}} (\kappa_n^2)^{\frac{n-i}{n}} = \kappa_n^2.$$

This yields the case $p < 0$ or $0 < p \leq n-1$ of inequality (1.15). And the equality conditions of inequalities (5.2) and (1.14) show that equality holds in the case $p < 0$ or $0 < p \leq n-1$ of inequality (1.15) if and only if K is a ball centered at the origin.

If $p = 0$, since $K \cap \xi \subseteq K|\xi$, thus by (1.2), (2.5) and (5.4) we have that

$$\begin{aligned}
&\widetilde{Q}_{i,0}(K)\widetilde{Q}_{i,0}(K^*) \\
&= \frac{\kappa_n}{\kappa_{n-i}}\left(\exp\int_{G(n,n-i)} \ln V_{n-i}(K\cap\xi)d\mu_{n-i}(\xi)\right) \\
&\quad \cdot \frac{\kappa_n}{\kappa_{n-i}}\left(\exp\int_{G(n,n-i)} \ln V_{n-i}(K^*\cap\xi)d\mu_{n-i}(\xi)\right) \\
&= \left(\frac{\kappa_n}{\kappa_{n-i}}\right)^2 \exp\left[\int_{G(n,n-i)} \left(\ln V_{n-i}(K\cap\xi) + \ln V_{n-i}(K^*\cap\xi)\right)d\mu_{n-i}(\xi)\right] \\
&= \left(\frac{\kappa_n}{\kappa_{n-i}}\right)^2 \exp\left[\int_{G(n,n-i)} \left(\ln V_{n-i}(K\cap\xi) + \ln V_{n-i}((K|\xi)^*)\right)d\mu_{n-i}(\xi)\right] \\
&\leq \left(\frac{\kappa_n}{\kappa_{n-i}}\right)^2 \exp\left[\int_{G(n,n-i)} \left(\ln V_{n-i}(K|\xi) + \ln V_{n-i}((K|\xi)^*)\right)d\mu_{n-i}(\xi)\right] \\
&= \left(\frac{\kappa_n}{\kappa_{n-i}}\right)^2 \exp\left[\int_{G(n,n-i)} \ln\left(V_{n-i}(K|\xi)V_{n-i}((K|\xi)^*)\right)d\mu_{n-i}(\xi)\right] \\
&\leq \left(\frac{\kappa_n}{\kappa_{n-i}}\right)^2 \exp\left[\int_{G(n,n-i)} \ln\left(\kappa_{n-i}^2\right)d\mu_{n-i}(\xi)\right] \\
&= \left(\frac{\kappa_n}{\kappa_{n-i}}\right)^2 \kappa_{n-i}^2 = \kappa_n^2.
\end{aligned}$$

This gives the case $p = 0$ of inequality (1.15). And equality holds in the case $p = 0$ of inequality (1.15) if and only if K is a ball centered at the origin. $\square$

Competing interests

The authors declare that they have no competing interests.

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