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Künneth formulas for path homology

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Abstract: We study the path homology groups with coefficients in a general ring R, and show that such groups are always finitely generated. We further prove two versions of Eilenberg-Zilber theorem for the Cartesian product and the join of two regular path complexes over a commutative ring R. Hence Künneth formulas are derived for the two cases over a PID. Note that this generalizes the related results proved for regular path complexes over a field K in [7], whose proofs can not be carried over here parallelly.

Keywords: Directed graphs, path complexes, path (co)homology, cross product, Künneth formula.

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1. Introduction

Homological theory for graphs appears in a natural way, in fact, when considering digraphs (i.e., directed graphs), we can define their (co)homology groups in a similar way as that in topology (see [5]). These (co)homology groups can be used directly to study the relationship among digraphs and their related theories.

Historically, in order to study the topological structure of digraphs and further to classify them, there are many attempts to form a homological theory for digraphs. Among these approaches there are three of them being well-known: regarding a digraph as a special one-dimensional simplicial complex, considering all the cliques of a digraph as simplices of the corresponding dimensions ([1, 11]), or taking Hochschild cohomology of the path algebra of a digraph ([9]). But as it is commented in [5] that all these approaches have their emphasises and limitations. In view of this, the authors of [5] introduced the notions of path complexes and path homology over a field (while their cohomology version can be found in [4]). This new (co)homology theory for digraphs, including its sequel notions and results, not only shares many properties with the above approaches but also avoid many limitations. It is shown that one can use path homology to give a refined classification of digraphs via some homological invariants such as the dimensions of the homology groups, Euler characteristic and so on. On the other hand, as we can see that from [6], path cohomology theory is a powerful tool when one deals with the algebraic aspect of simplicial cohomology, in fact it allows a delicate proof to the isomorphism obtained in [2] without using Cohomology Comparison Theorem.

Furthermore, as one of the most fundamental functorial properties, the classic Eilenberg-Zilber theorem and Künneth formula holding for the Cartesian product space of two topological spaces have also analogues in the theory of path homology. In fact, the similar results hold for both the Cartesian product and the join of two regular path complexes over a field from digraphs ([7]).

Meanwhile, as we usually do in the theory of simplicial (co)homology, it seems that there is no need to confine the coefficients of path (co)homology in a field, since different coefficient rings usually induce different (co)homology groups. For instance, when one ignores the orientation of a simplicial complex one should consider directly its simplicial homology groups with coefficients in $\mathbb{Z}_2$ instead of $\mathbb{Z}$.

So in this paper, we start from the path (co)homology with coefficients in a general ring R. Our main goal is to generalize the Eilenberg-Zilber theorem to regular path complexes over any commutative ring, hence it enables us to obtain Künneth formula for regular path complexes over any PID. To be specified, the paper is organized as follows. We set off after reviewing some definitions and notations in Section 2, the connection between path homology groups with coefficients in a ring R and in $\mathbb{Z}$ is studied, as a conclusion we show that such path homology groups are finitely generated (see Theorem 2.3 and Corollary 2.5). In Section 3, analogues of the Eilenberg-Zilber theorem are obtained in a unified way for the Cartesian product and the join of two regular path complexes (Theorems 3.5 and 3.8) over any commutative ring. These imply respectively two general Künneth formulae for path homology with coefficients in principle ideal domains. Note that not only this generalizes the previous result in [7] obtained for path complexes over a field K, but also our proofs here go rather different from those given in [7].

2. Preliminaries

Throughout this paper, K denotes a field and R denotes an associative unitary ring if not specified. We recall from [5] some notations and definitions in this section, though most of them are defined temporarily in the case where K is a field, as we shall see that they can be easily extended to the case when one replaces K by a unital ring R.

2.1. Path complexes

Definition 2.1. Let V be an arbitrary non-empty finite set whose elements will be called *vertices*. For any non-negative integer p, an *elementary p-path* on a set V is any ordered sequence $\{i_k\}_{k=0}^{p}$ (or simply written as $i_0 \cdots i_p$) of $p+1$ vertices (needs not be distinct) of V. Furthermore, an elementary path $i_0 \cdots i_p$ is said to be *non-regular* if $i_{k-1} = i_k$ for some $1 \leq k \leq p$, and *regular* otherwise.

Denote by $\Lambda_p = \Lambda_p(V; K)$ the K-linear space that consists of all formal linear combinations of all elementary p-paths with the coefficients from K. The elements of Λ_p are called *p-paths* on V, and an elementary p-path $i_0 \cdots i_p$ as an element in Λ_p is written as $e_{i_0 \cdots i_p}$. Obviously the basis in Λ_p is the family of all elementary p-paths, and each element v in Λ_p has the following form:

$$v = \sum_{i_0, \cdots, i_p \in V} v^{i_0 \cdots i_p} e_{i_0 \cdots i_p}$$

where $v^{i_0\cdots i_p} \in K$. For any $p \geq -1$, consider the subspace of Λ_p spanned by the regular elementary paths: $\mathcal{R}_p = \mathcal{R}_p(V; K)$:=span$\{e_{i_0\cdots i_p}$: $i_0 \cdots i_p$ is regular$\}$, whose elements are called *regular p-paths*.

For any $p \geq 0$, define the boundary operator $\partial : \Lambda_p \to \Lambda_{p-1}$ as a linear operator that acts on elementary paths by

$$\partial e_{i_0\cdots i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0\cdots \widehat{i_q}\cdots i_p} \tag{2.1}$$

where the hat $\widehat{i_q}$ means omission of the index i_q. Note that such boundary operators make $\Lambda_* = \{\Lambda_p\}$ a chain complex (see [5, Lemma 2.4]). Similarly we can define the regular complex $\mathcal{R}_* = \{\mathcal{R}_p\}$ consisting of regular elements and with natural boundary operators, i.e., those boundary operators are defined by the induced maps of ∂ acting on the quotient space Λ_p over non-regular paths, and it is easy to check that $\mathcal{R}_* = \{\mathcal{R}_p\}$ is a chain complex under such boundary operators (see [5] for details). Let V, V' be two finite sets, by definition, any map $f : V \to V'$ gives rise to two natural morphisms $\Lambda_*(V) \to \Lambda_*(V')$ and $\mathcal{R}_*(V) \to \mathcal{R}_*(V')$.

The central concept in our study is the following.

Definition 2.2. A *path complex* over a finite set V is a non-empty collection $P(V)$ (abbreviated as P if no ambiguity) of elementary paths on V with the following property: for any $n \geq 0$, if $i_0 \cdots i_n \in P$ then also the truncated paths $i_0 \cdots i_{n-1}$ and $i_1 \cdots i_n$ belong to P. The elementary n-paths from P is denoted by P_n. If all the paths in P are regular, then P is called a *regular* path complex. P is called *finite* if $P_{\geq m}$ are all empty for some $m > 0$.

2.2. Path (co)homology

When a path complex P is fixed, all the n-paths of the form $\sum_{j=1}^{s} r_j e_{\mathbf{i}^{(j)}}$ with s a finite integer, each $r_j \in K$ and $\mathbf{i}^{(j)} = i_0^{(j)} i_1^{(j)} \cdots i_n^{(j)}$ such that $e_{\mathbf{i}^{(j)}} \in P_n$ are called *allowed*, otherwise are called *non-allowed*. The set of all allowed n-paths is denoted as $\mathcal{A}_n(P) = \mathcal{A}_n(P; K)$. Furthermore, for any $n \geq 0$ we define $\Omega_n(P)$ as follows:

$$\Omega_n(P) = \Omega_n(P; K) := \{p_n | p_n \in \mathcal{A}_n(P) \ \text{ and } \ \partial(p_n) \in \mathcal{A}_{n-1}(P).\}.$$

Apparently each $\Omega_n(P)$ is a K-module, namely a vector space over K. It is easy to verify that $\partial(\Omega_m(P)) \subseteq \Omega_{m-1}(P)$ and $\partial^2 = 0$, thus we obtain a chain complex of K-modules:

$$\Omega_*(P) = \Omega_*(P; K) := \cdots \to \Omega_n(P) \to \Omega_{n-1}(P) \to \cdots \to \Omega_0(P) \to 0.$$

Therefore, for any $n \geq 0$ we define the n-th *path homology group* of P as $H_n(\Omega_*(P))$, or denoted shortly by $H_n(P)$. If the path complex P is regular, which is the case we shall study in this paper, all the above definitions and notations have modified versions when one replaces the boundary operator by the modified boundary operator which is used to define $\mathcal{R}_*$.

The above definitions and notations also have dual versions. For any integer $p \geq -1$, denote by $\Lambda^p = \Lambda^p(V;K)$ the linear space of all K-valued functions on $(p+1)$-multiplicative product V^{p+1} of set V. Otherwise we set $\Lambda^{\leq -2} = \{0\}$. In particular, Λ^0 is the linear space of all K-valued functions on V, and Λ^{-1} is the space of all K-valued functions on $\Lambda^0 := \{0\}$, that is, one can identify Λ^{-1} with K. The elements of Λ^p are called *p-forms* on V, one can identify Λ^p with the dual space of Λ_p via the canonical identity $\Lambda^p \cong \mathrm{Hom}_K(\Lambda_p, K)$. The boundary operator (2.1) should be replaced now by exterior differential $d : \Lambda^p \to \Lambda^{p+1}$ given by

$$(d\omega)_{i_0\cdots i_{p+1}} = \sum_{q=0}^{p+1}(-1)^q\omega_{i_0\cdots\widehat{i_q}\cdots i_{p+1}} \tag{2.2}$$

for any $\omega \in \Lambda^p$. Similarly we define the space of *regular* p-forms $\mathcal{R}^p = \mathcal{R}^p(V) := \mathrm{Hom}_K(\mathcal{R}_p, K)$ (hereafter this means, any element in $\mathcal{R}^p$ always takes $\Lambda_p \setminus \mathcal{R}_p$, i.e., non-regular p-paths to 0). Given a path complex P, we define the space of *allowed* p-forms $\mathcal{A}^p(P) = \mathcal{A}^p(P;K) := \mathrm{Hom}_K(\mathcal{A}_p(P), K)$, also denote

$$\mathcal{N}^p = \Lambda^p \setminus \mathcal{A}^p(P) \quad \text{and} \quad \mathcal{J}^p = \mathcal{N}^p + d\mathcal{N}^{p-1},$$

and define

$$\Omega^p(P) = \mathcal{A}^p/(\mathcal{A}^p \cap \mathcal{J}^p).$$

Actually, it follows from [5, Lemma 3.19] that $\Omega^p(P)$ is the dual space of $\Omega_p(P)$ while d is dual to ∂, that is to say, one has

$$\Omega^p(P) \cong \mathrm{Hom}_K(\Omega_p(P), K) \quad \text{and} \quad d \cong \mathrm{Hom}_K(\partial, K). \tag{2.3}$$

It can be shown that $\{\Omega^p(P)\}$ amounts to a cochain complex with the differential operator given by (2.2), whereas the n-th *path cohomology group* of P for any $n \geq 0$ is referred to the n-th cohomology group $H^n(\Omega^*(P))$ of this cochain complex, which is denoted shortly by $H^n(P)$.

All of the above definitons can be easily carried over to the case when replacing K by any ring R (see, e.g., [3]). In this paper, path (co)homology is understood to be with coefficients in a ring R, and we shall omit "R"

in the notation if there is no ambiguity. As a preparation for proving the following interesting result, for any path $p = \sum_{k=1}^{s} r_k e_k \in \Omega_n(P;R)$, define the *support* of p to be the set consisting of e_k with nonzero coefficient r_k, and denote it by $Supp(p)$. Moreover, p is called *minimal* if $p \neq p_1 + p_2$ whenever $Supp(p_1) \subsetneq Supp(p)$ and $Supp(p_2) \subsetneq Supp(p)$. By reduction of the cardinal of support, one sees easily that any path in $\Omega_n(P;R)$ is a R-linear combination of minimal n-paths, though may not be unique (comparing with the definitions above [10, Lemma 2.1]).

Theorem 2.3. *For any path complex P over a finite set V and ring R, there is a chain isomorphism*

$$\Omega_*(P;R) \cong R \otimes_{\mathbb{Z}} \Omega_*(P;\mathbb{Z}).$$

Proof. It needs only to show that $\Omega_n(P;R) \cong R \otimes_{\mathbb{Z}} \Omega_n(P;\mathbb{Z})$ for any $n \geq 0$ since the operator ∂ acts on both of them in an obvious way. We claim that the canonical map $f : R \otimes_{\mathbb{Z}} \Omega_n(P;\mathbb{Z}) \to \Omega_n(P;R)$ induced by $r_i \otimes z_i e_i \mapsto (r_i z_i) e_i$ with $r_i \in R$, $z_i \in \mathbb{Z}$ and $e_i \in P_n$ gives the desired isomorphism. First note that f is obviously injective. To show that it is also surjective, suppose

$$p = \sum_{e_i \in I} r_i e_i \tag{2.4}$$

is a path in $\Omega_n(P;R)$ where I is a finite subset of P_n and all $r_i \neq 0$. Moreover, suppose that p is minimal, since otherwise one can write p as a linear combination of minimal n-paths.

We aim to show that all r_i's in (2.4) differ up to a sign. The proof is purely an exercise of linear algebra. To see this, first note that any e_i in (2.4) is not in $\Omega_n(P;R)$ since p is minimal, so each e_i has some boundary $(n-1)$-path which is not in P_{n-1}. For clarity, one can rewrite (2.4) as $p = \sum_{i=1}^{m} r_i e_i$, where m is the total number of elementary n-paths in I. Then starting with e_1, pick one of its boundary $(n-1)$-paths which is not in P_{n-1}, so it must also occur as the boundary $(n-1)$-paths of some other elementary n-paths in I since $\partial(p) \in \mathcal{A}_{n-1}(P;R)$, this cancelling relation leads to a equation

$$\varepsilon_{11} r_1 + \varepsilon_{12} r_2 + \cdots + \varepsilon_{1m} r_m = 0,$$

where each ε_{1i} is ± 1 or 0 up to whether and how the sign of this elementary $(n-1)$-path occurs in the boundary of corresponding e_i. Next we repeat this procedure for the rest finite boundary $(n-1)$-paths of e_1 (if such path exists) which are not in P_{n-1}, and obtain an equation for each one of them:

$$\varepsilon_{j1} r_1 + \varepsilon_{j2} r_2 + \cdots + \varepsilon_{jm} r_m = 0$$

with each $\varepsilon_{ji} \in \{\pm 1, 0\}$ and $j \geq 2$. After this, one can continue this procedure for $e_2, e_3, \cdots$ and so on until it is finished. Hence we get a system of homogenous linear equations whose coefficients matrix is (ε_{kl}) with $0 < l \leq m$ and $0 < k \leq t$ for some integer t. It is apparent that any p with the form of (2.4) is in $\Omega_n(P; R)$ only if this system of equations has at least a nonzero solution, or equivalently, the rank s of (ε_{kl}) is less than m. We shall show that it is exactly $m - 1$.

Note that since any two paths e_i and e_j in I have at most one common elementary $(n-1)$-path appeared in their boundary (perhaps with different signs), the matrix (ε_{kl}) admits a property that $i)$ all elements are in $\{\pm 1, 0\}$; $ii)$ the elements at the four intersections of any two rows and two columns of it could not be all ± 1, i.e., one of them must be 0. In order to obtain the row standard simplest form of (ε_{kl}), one needs only to exchange two rows or add ± 1 times (chosen properly so as to cancel 1 or -1 in some columns) of some row to another row successively, also note that the above property is still preserved when applying these row transformations. Now if $s < m - 1$, then there is a finite basis $\{r_i\}$ consisting of $m - s$ elements such that any other r_j can be written as a linear combination of them, thus p can be written as a sum of $m - s$ terms each with its coefficient in $\{r_i\}$. It applies that each such term is in $\Omega_n(P; R)$ since $\{r_i\}$ can be arbitrary chosen while the linear equations are still satisfied (let one of them be 1 and others be 0), thus p is not minimal, a contradiction. Hence $s = m - 1$, but the above property shows exactly that all r_i's differ up to a sign, since any one of the first m elements in the last column of the row standard simplest form of (ε_{kl}) could not be 0 for all $r_i \neq 0$, they can only be ± 1. Hence all r_i's are $\pm r_t$ for some fixed r_t. □

Let $R = \mathbb{Z}$, one obtains immediately the following result.

Corollary 2.4. *For each $n \in \mathbb{N}$, $\Omega_n(P; \mathbb{Z})$ is finitely free generated by a $\mathbb{Z}$-basis with the form of $\{\sum_i \pm e_i\}$.*

Proof. Since $\Omega_n(P; \mathbb{Z})$ is a submodule of finitely free $\mathbb{Z}$-module $\mathcal{A}_n(P; \mathbb{Z})$ and $\mathbb{Z}$ is a PID, $\Omega_n(P; \mathbb{Z})$ is finitely free genereated. The result follows then from the proof of Theorem 2.3. □

This corollary generalizes [10, Corollary 2.4], since the word regular has a stricter meaning there. Also we deduce a result that will be used in Section 3.

Corollary 2.5. *For each $n \in \mathbb{N}$, $\Omega_n(P; R)$ is finitely generated as an R-module by the generators $\{\sum_i \pm e_i\}$.*

It can be shown that the module $\Omega_n(P;R)$ is actually a finitely free R-module with the basis $\{\sum_i \pm e_i\}$, by using a similar discussion as that in the proof of Theorem 2.3. We omit the detail since we do not need this more delicate result.

3. Künneth formulas for path homology

Now we turn to the functorial properties of path homology. Similar to simplicial homological theory, the authors of [7] defined the Cartesian product for two path complexes (see [7, Definition 4.5]), and furthermore gave the analogues of the Eilenberg-Zilber theorem and Künneth formula for regular path complexes with coefficients in a field K.

In this section, we will generalize these results to a more general setting, i.e., for regular path complexes with coefficients in a commutative ring R. With the same assumption, we show that the pattern of the proof can also be used to obtain the Künneth formula for the join of two regular path complexes (see Definition 3.7 below) over any PID.

3.1. The case of Cartesian product

For our purpose, let us first recall the definition of the *cross product* of two path complexes from [7].

Let X, Y be two finite sets. For two elementary m and n-paths $e_{i_0\cdots i_m} \in \mathcal{R}_m(X)$ and $e_{j_0\cdots j_n} \in \mathcal{R}_n(Y)$. Naturally, we have $(m+1)(n+1)$ pairs of vertices of $(i_0,j_0),(i_0,j_1),\cdots,(i_1,j_0),\cdots,(i_m,j_n)$ in $X\times Y$. We now view the pairs $(k,\ l)$ with $0\le k\le m$ and $0\le l\le n$ as the vertices of an $m\times n$ rectangle grid in $\mathbb{R}^2$, which are assigned the order "$<$" such that $(k,\ l)<(p,\ q)$ for $k\le p,\ l<q$ or $k<p,\ l\le q$. Then for each $m+n$ step-like edgepath σ with vertices $(0,0),\cdots,(k,l),\cdots,(m,n)$, we can associate it with an elementary $(m+n)$-path $e_\sigma = e_{(i_0,j_0)\cdots(i_k,j_l)\cdots(i_m,j_n)}$. Furthermore, we define the *cross product* of two elementary paths $e_{i_0\cdots i_m}$ and $e_{j_0\cdots j_n}$ as

$$e_{i_0\cdots i_m}\times e_{j_0\cdots j_n}=\sum_\sigma(-1)^{|\sigma|}e_\sigma,$$

where σ runs through all the possible $m+n$ step-like paths in the grid from (0,0) to (m,n), and $|\sigma|$ is the number of squares in the grid lying below the path σ, also note that '$\times$' here means the cross product. This formula can be extended bilinearly to give the cross product $u\times v$ of any two paths

$u \in \mathcal{R}_m(X)$ and $v \in \mathcal{R}_n(Y)$, and the differential operators are given by the following formula (see [5, Proposition 7.2]):

$$\partial(u \times v) = \partial(u) \times v + (-1)^m u \times \partial(v). \tag{3.1}$$

More generally, we have the following definition.

Definition 3.1 ([7],Definition 4.2)**.** For any two path complexes $P(X)$, $P(Y)$, their *Cartesian product* $P(X) \boxplus P(Y)$ (or $P(X \times Y)$ for short) are defined as the path complex over $X \times Y$ with each elementary k-path of $P(X \times Y)$ of the form e_σ, where the k step-like path σ comes from the above construction for any elementary s-path $e_\alpha = e_{i_0 i_1 \cdots i_s} \in P_s(X)$ and elementary $(k-s)$-path $e_\beta = e_{j_0 j_1 \cdots j_{k-s}} \in P_{k-s}(Y)$ with all $i_m \in X$ and $j_n \in Y$, while its differential operators are given by (3.1).

To simplify notations, we shall abbreviate $\mathcal{A}_n(P(X))$ (resp. $\Omega_n(P(X))$) as $\mathcal{A}_n(X)$ (resp. $\Omega_n(X)$) if there is no confusion. Also recall that $\mathcal{A}_n(X \times Y)$ is the set of all R-linear combinations of elements in $P_n(X \times Y)$ and $\Omega_n(X \times Y)$ is defined as the set $\{z_n | z_n \in \mathcal{A}_n(X \times Y) \text{ and } \partial_n(z_n) \in \mathcal{A}_{n-1}(X \times Y).\}$.

Lemma 3.2. $\Omega_s(X) \times \Omega_{k-s}(Y) \subseteq \Omega_k(X \times Y)$.

Proof. It can be proved easily from the boundary formula for cross products. □

The following lemma comes from [5, Proposition 7.12], whose proof can be carried over to the case where R is a commutative ring without any change.

Lemma 3.3. *Any path $w \in \Omega_*(X \times Y)$ admits a representation*

$$w = \sum_{e_x \in P(X),\ e_y \in P(Y)} c^{xy}(e_x \times e_y)$$

with finitely many nonzero coefficients $c^{xy} \in R$ which are uniquely determined by w. Furthermore, the cross products $\{e_x \times e_y\}$ across all $e_x \in P(X)$ and $e_y \in P(Y)$ are R-linearly independent.

Our proof in the sequel depends on the following key lemma (comparing its proof with that of [7, Theorem 5.1]).

Lemma 3.4. *Any path $w \in \Omega_n(X \times Y)$ can be written as a finite sum:*

$$w = \sum_k \sum_{i \le n} (p_i^k(X) \times q_{n-i}^k(Y))$$

where k runs a finite set, each $p_i^k(X) \in \Omega_i(X)$ and $q_{n-i}^k(Y) \in \Omega_{n-i}(Y)$.

Proof. Let w be an n-path in $\Omega_n(X\times Y)$, by Lemma 3.3 we can write it as a finite sum:

$$w=\sum_{i=s}^{n}\Big(\sum_{e_x\in P_i(X),\ e_y\in P_{n-i}(Y)} c^{xy}(e_x\times e_y)\Big), \tag{3.2}$$

and let $J_i(X)$ and $J_{n-i}(Y)$ be respectively subsets of $P_i(X)$ and $P_{n-i}(Y)$ consisting of all paths e_x and e_y such that the coefficients $c^{xy}\in R$ of $e_x\times e_y$ in (3.2) are non-zero, while $s\geq 0$ is the lowest index of e_x appeared in the expression. We do the proof by induction on the total number a of the elementary paths in $\bigcup_i J_{n-i}(Y)$.

If $a=1$, that is to say, $w=\big(\sum_{e_x}c^{xy}e_x\big)\times e_y\in\Omega_n(X\times Y)$ where $e_x\in J_s(X)$ and $\{e_y\}=J_{n-s}(Y)$ for some fixed integer $s\leq n$. Now

$$\partial(w)=\partial\big(\sum_{e_x}c^{xy}e_x\big)\times e_y+(-1)^s\big(\sum_{e_x}c^{xy}e_x\big)\times\partial(e_y)\in\mathcal{A}_{n-1}(X\times Y)$$

and Lemma 3.3 imply that $\partial\big(\sum_{e_x}c^{xy}e_x\big)\in\mathcal{A}_{s-1}(X)$ and $\partial(e_y)\in\mathcal{A}_{n-s-1}(Y)$. Set $k=1$, then $p_s^1=c^{xy}e_x$ and $q_{n-s}^1=e_y$ give the desired result.

Now suppose the total number of the elementary paths in $\bigcup_i J_{n-i}(Y)$ is b ($b>1$), and suppose that for any $a<b$ the result holds, we shall show that the result also holds for $a=b$. We rewrite w in two forms as follows:

$$\begin{aligned} w&=\sum_{i=s}^{n}\sum_{e_y\in J_{n-i}(Y)}\Big(\big(\sum_{e_x\in J_i^y(X)}c^{xy}e_x\big)\times e_y\Big)\\ &=\sum_{i=s}^{n}\sum_{e_x\in J_i(X)}\Big(e_x\times\sum_{e_y\in J_{n-i}^x(Y)}c^{xy}e_y\Big)\end{aligned} \tag{3.3}$$

where each set $J_i^y(X)\subseteq J_i(X)$ in the first equality runs over all the e_x's accompanied by each e_y in the cross product of (3.2) (so is decided by e_y), and each set $J_{n-i}^x(Y)\subseteq J_{n-i}(Y)$ in the second equality runs over all the e_y's accompanied by each e_x in a symmetric manner (so is decided by e_x). Fix some $e_{\bar{y}}\in J_{n-s}(Y)$, it follows from $J_s^{\bar{y}}(X)\subseteq J_s(X)$ and the first equality of (3.3) that

$$p_s^1=\sum_{e_x\in J_s^{\bar{y}}(X)}c^{x\bar{y}}e_x\in\mathcal{A}_s(X).$$

For the sake of simplicity, given any elementary n-path $e_v=e_{0\cdots n}\in P_n(V)$ and integer $0\leq l\leq n$, we denote $e_{v(\widehat{l})}=e_{0\cdots(l-1)\widehat{l}(l+1)\cdots n}$. On the other hand by using the first and the second equalities of (3.3) alternatively we can compute as follows:

$$
\begin{aligned}
\partial(w) = & \sum_{e_y \in J_{n-s}(Y)} \Big(\sum_{e_x \in J_s^y(X)} c^{xy} \partial(e_x) \Big) \times e_y \\
& + \sum_{e_x \in J_s(X)} \Big(e_x \times (-1)^s \sum_{e_y \in J_{n-s}^x(Y)} c^{xy} \partial(e_y) \Big) \\
& + \sum_{e_{x'} \in J_{s+1}(X)} \Big(\partial(e_{x'}) \times \sum_{e_{y'} \in J_{n-s-1}^{x'}(Y)} c^{x'y'} e_{y'} \Big) + R(Z)
\end{aligned}
$$

(3.4)

$$
\begin{aligned}
= & \sum_{e_y \in J_{n-s}(Y)} \Big(\sum_{e_x \in J_s^y(X)} c^{xy} \partial(e_x) \Big) \times e_y \\
& + \sum_{e_x \in J_s(X)} e_x \times \Big((-1)^s \sum_{e_y \in J_{n-s}^x(Y)} c^{xy} \partial(e_y) + (-1)^l \sum_{\substack{e_{x'(\widehat{l})} = e_x \\ e_{\tilde{y}'} \in J_{n-s-1}^{x'}(Y)}} c^{x'y'} e_{y'} \Big) \\
& + \sum_{\substack{e_{x'(\widehat{l'})} \notin J_s(X) \\ e_{x'} \in J_{s+1}(X)}} \Big((-1)^{l'} e_{x'(\widehat{l'})} \times \sum_{e_{y'} \in J_{n-s-1}^{x'}(Y)} c^{x'y'} e_{y'} \Big) + R(Z)
\end{aligned}
$$

where $R(Z)$ denotes the remained summand of the form $\sum_{e_x \in J_r(X), e_y \in J_t(Y)} c^{xy} e_x \times e_y$ with $s < r \leq n-1$ and $t = n-1-r$. Thus it follows from $\partial(w) \in \mathcal{A}_{n-1}(X \times Y)$, (3.4) and Lemma 3.3 that

$$
\partial(p_s^1) = \sum_{e_x \in J_s^{\bar{y}}(X)} c^{x\bar{y}} \partial(e_x) \in \mathcal{A}_{s-1}(X)
$$

and for each $e_x \in J_s(X)$ one has

$$
(-1)^s \sum_{e_y \in J_{n-s}^x(Y)} c^{xy} \partial(e_y) + (-1)^l \sum_{\substack{e_{x'(\widehat{l})} = e_x \\ e_{\tilde{y}'} \in J_{n-s-1}^{x'}(Y)}} c^{x'y'} e_{y'} \in \mathcal{A}_{n-s-1}(Y).
$$

Namely, one has

$$
p_s^1 \in \Omega_s(X) \quad \text{and} \quad \sum_{e_y \in J_{n-s}^x(Y)} c^{xy} \partial(e_y) \in \mathcal{A}_{n-s-1}(Y) \tag{3.5}
$$

for each $e_x \in J_s(X)$. Now for some fixed $e_{\bar{x}} \in J_s^{\bar{y}}(X)$ we always have $e_{\bar{y}} \in$

$J^{\bar{x}}_{n-s}(Y)$, so if we denote

$$q_{n-s} = \sum_{e_y \in J^{\bar{x}}_{n-s}(Y)} c^{\bar{x}y} e_y, \tag{3.6}$$

then $\partial(q_{n-s}) \in \mathcal{A}_{n-s-1}(Y)$ and hence $q_{n-s} \in \Omega_{n-s}(Y)$. Therefore by Corollary 2.5 there exists some generator

$$q^1_{n-s} = \sum_i \pm e_{y_i} \in \Omega_{n-s}(Y) \tag{3.7}$$

with $e_{y_i} = e_{\bar{y}}$ for some i. Moreover, one can make that all $e_{y_i} \in J^{\bar{x}}_{n-s}(Y) \subset J_{n-s}(Y)$, just note that q^1_{n-s} can be obtained by replacing (2.4) and I with (3.6) and $J^x_{n-s}(Y)$ respectively in the proof of Theorem 2.3.

Apparently, one has $p^1_s \times q^1_{n-s} \in \Omega_s(X) \times \Omega_{n-s}(Y) \subseteq \Omega_n(X \times Y)$ from (3.5) and (3.7). Denote $w' = w - p^1_s \times q^1_{n-s}$, then $w' \in \Omega_n(X \times Y)$ and rewrite it as

$$w' = \sum_{i=s}^{n} \Big(\sum_{e_{x'} \in P_i(X),\ e_{y'} \in P_{n-i}(Y)} c^{x'y'} (e_{x'} \times e_{y'}) \Big).$$

Similarly this expression of w' decides respectively two subsets $J'_i(X)$ and $J'_{n-i}(Y)$ of $P_i(X)$ and $P_{n-i}(Y)$ consisting of all paths $e_{x'}$ and $e_{y'}$ such that the coefficients $c^{x'y'} \in R$ of $e_{x'} \times e_{y'}$ in it are non-zero. Comparing $w = p^1_s \times q^1_{n-s} + w'$ with (3.2), it then follows by Lemma 3.3 that $J'_{n-i}(Y) \subsetneqq J_{n-i}(Y)$ since $e_{\bar{y}} \notin J'_{n-i}(Y)$, thus by inductive hypothesis one has $w' = \sum_{k \geq 2} \sum_{i \leq n} (p^k_i(X) \times q^k_{n-i}(Y))$ with each $p^k_i(X) \in \Omega_i(X)$ and $q^k_{n-i}(Y) \in \Omega_{n-i}(Y)$, and one gets the desired result for w. □

We are able to prove the first main result now:

Theorem 3.5. *Let $P(X)$ and $P(Y)$ be two regular path complexes and R a commutative ring. Then for their Cartesian product $P(X \times Y) = P(X) \boxplus P(Y)$ the following isomorphism of chain complexes holds:*

$$\Omega_*(X) \otimes_R \Omega_*(Y) \cong \Omega_*(X \times Y) \tag{3.8}$$

whose mapping is given by $u \otimes v \mapsto u \times v$.

Proof. Let us inspect the map (3.8)

$$F : \Omega_*(X) \otimes_R \Omega_*(Y) \to \Omega_*(X \times Y)$$

defined by the formula $F(u \otimes v) = u \times v$ for any $u \in \Omega_m(X)$ and $v \in \Omega_n(Y)$. Lemma 3.3 shows that F is injective, and Lemma 3.4 implies that F is surjective, so it only needs to show that F is a chain map, but this follows easily from the differential operators of tensor product of two chain complexes defined as follows in n-degree:

$$\partial(u \otimes v) = \partial(u) \otimes v + (-1)^m u \otimes \partial(v)$$

with u in m-degree and v in $(n-m)$-degree, and the differential operators of Cartesian products of path complexes defined by (3.1). $\square$

Künneth formula is used to compute the (co)homology of a product space in terms of the (co)homology of the factors. For path complexes over a field, a type of Künneth formula also holds (see, [7, Theorem 4.7]). In fact, for any principle ideal domain R, we have the following more general result.

Corollary 3.6. *Let $P(X)$ and $P(Y)$ be two regular path complexes and R a PID. Then, for each n, there holds a Künneth formula by the following natural splitting short exact sequence*

$$\begin{aligned} 0 \to \oplus_i \big(H_i(X) \otimes_R H_{n-i}(Y)\big) &\to H_n(X \times Y) \\ &\to \oplus_i \, Tor_1^R\big(H_i(X), H_{n-i-1}(Y)\big) \to 0. \end{aligned}$$

Proof. By Theorem 3.5 one has

$$\mathrm{H}_n(X \times Y) = \mathrm{H}_n\big(\Omega_*(X \times Y)\big) \cong \mathrm{H}_n\big(\Omega_*(X) \otimes_R \Omega_*(Y)\big).$$

Note that each $\mathcal{A}_n(X)$ $(n \geq 0)$ is a finitely generated free R-module, thus $\Omega_n(X)$ is also a free R-module since R is a PID. Therefore, the result follows from [8, Theorem 3B.5]. $\square$

One sees that both Theorem 2.3 and Theorem 3.5 show a resemblance between the theory of path homology and that of singular homology in topology.

3.2. The case of join

Comparing with the approach of considering the Cartesian product, there is another way to derive the Künneth formula via an operation called the *join* (see Definition 3.7 below) of two regular path complexes. For path complexes over a field K, this is exactly what [7, Theorem 3.3] says. In fact, the general result also holds when one replaces the field K by any commutative ring R. Before we set off to prove this, let us do some preparation.

Definition 3.7 ([7],Definition 3.1)**.** Given two disjoint finite sets X, Y and their path complexes $P(X)$, $P(Y)$, define a path complex $P(X) * P(Y)$ (or $P(X * Y)$ for short) consisting of all paths of the form $uv := e_{i_0 i_1 \cdots i_m j_0 j_1 \cdots j_n} \in P_{m+n+1}(X * Y)$ where $u = e_{i_0 i_1 \cdots i_m} \in P_m(X)$ and $v = e_{j_0 j_1 \cdots j_n} \in P_n(Y)$. The path complex $P(X * Y)$ is called the *join* of $P(X)$ and $P(Y)$.

Given any two paths $u \in P_{i-1}(X)$ and $v \in P_{n-i}(Y)$, it is easy to check that, the differential operator acting on the join $uv \in P_n(X * Y)$ is given by the following formula:

$$\partial(uv) = \partial(u)v + (-1)^i u\partial(v).$$

For more properties and examples of the join of two regular path complexes the reader may refer to [7]. To prove the asserted Künneth formula, we proceed by a parallel way as that of proving Theorem 3.5. The key idea is to simply replace the symbol "$\times$" of Cartesian product by the symbol "$*$" of join in the previous proofs, and treat carefully the corresponding degrees.

In details, suppose we are given two regular path complexes $P(X)$ and $P(Y)$, let $P(X * Y)$ be their join. We see that by definition $\Omega_{s-1}(X) * \Omega_{k-s}(Y) \subseteq \Omega_k(X * Y)$ (comparing with Lemma 3.2) and each path $w \in \Omega_n(X * Y)$ admits a representation

$$w = \sum_{i=1}^{n} \sum_{e_x \in P_{i-1}(X),\ e_y \in P_{n-i}(Y)} c^{xy}(e_x * e_y)$$

with finitely nonzero coefficients $c^{xy} \in R$, which are uniquely determined by w since obviously $e_{xy} = e_x * e_y$ across all $e_x \in P(X)$ and $e_y \in P(Y)$ are R-linearly independent (comparing with Lemma 3.3). Now for the chain complex $\Omega_*(X)$, we consider a new chain complex $\Omega'_*(X)$ which is defined by the formula $\Omega'_i(X) = \Omega_{i-1}(X)$ and $\partial'_i(-) = \partial_{i-1}(-)$. With this trick one is able to prove the following result.

Theorem 3.8. *Let $P(X)$ and $P(Y)$ be two regular path complexes and R a commutative ring. Then for their join $P(X*Y) = P(X)*P(Y)$ the following isomorphism of chain complexes holds:*

$$\Omega'_*(X) \otimes_R \Omega_*(Y) \cong \Omega_*(X * Y) \tag{3.9}$$

whose mapping is given by $u \otimes v \mapsto u * v$.

If furthermore R is a PID. Then there holds a Künneth formula by the following natural splitting short exact sequence

$$0 \to \oplus_i (H_{i-1}(X) \otimes_R H_{n-i}(Y)) \to H_n(X * Y)$$

$$\rightarrow \oplus_i Tor_1^R(H_{i-1}(X), H_{n-i-1}(Y)) \rightarrow 0. \tag{3.10}$$

for each n.

Proof. First note that as in Corollary 3.6, (3.10) follows directly from (3.9). To obtain the isomorphism (3.9), let us consider the map

$$F : \Omega'_*(X) \otimes \Omega_*(Y) \to \Omega_*(X * Y)$$

given by $u \otimes v \mapsto u * v$ for any $u \in \Omega'_i(X)$ and $v \in \Omega_{n-i}(Y)$. One sees that the basis of $\Omega'_i(X) \otimes \Omega_{n-i}(Y)$ consists of all elements of the form $e_x \otimes e_y$ where e_x and e_y are some elementary paths in $\Omega'_i(X) = \Omega_{i-1}(X)$ and $\Omega_{n-i}(Y)$ respectively. Apparently F is injective since all $e_{xy} = e_x * e_y$ are R-linearly independent.

Now we are done if we can show that the map F is surjective, but this follows directly from the proof of Lemma 3.4. To see this one needs only to replace the symbol "$\times$" by "$*$", similarly $\Omega_i(X)$, $P_i(X)$ and $J_i(X)$ etc. by $\Omega'_i(X)$, $P'_i(X)$ and $J'_i(X)$ etc., respectively, where $P'_i(X) := P_{i-1}(X)$ and $J'_i(X) := J_{i-1}(X)$. Then the proof still validates and this gives that F is surjective. □

Remark 3.9. If $R = K$ is a field, one immediately obtains [7, Theorems 3.3 and 4.7] by Theorem 3.5, Corollary 3.6 and Theorem 3.8. But note that the proofs in [7] used the dimension theory of vector spaces over a field (see [7, Theorem 5.1]), which no more works for R-modules when R is a commutative ring, so our proofs of Theorems 3.5, 3.8 and the proofs of [7, Theorems 3.3 and 4.7] have different points.

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