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On special generic maps of rational homology spheres into Euclidean spaces*

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Abstract: Special generic maps are smooth maps between smooth manifolds with only definite fold points as their singularities. The problem of whether a closed n-manifold admits a special generic map into Euclidean p-space for $1 \leq p \leq n$ was studied by several authors including Burlet, de Rham, Porto, Furuya, Èliašberg, Saeki, and Sakuma. In this paper, we study rational homology n-spheres that admit special generic maps into $\mathbb{R}^p$ for $p < n$. We use the technique of Stein factorization to derive a necessary homological condition for the existence of such maps for odd n. We examine our condition for concrete rational homology spheres including lens spaces and total spaces of linear S^3-bundles over S^4 to obtain new results on the nonexistence of special generic maps.

Keywords: Special generic map, definite fold point, Stein factorization, homology sphere, linking form, lens space, sphere bundle.

1. Introduction

Let $f \colon M^n \to \mathbb{R}^p$, $1 \leq p \leq n$, be a smooth map of a closed n-dimensional smooth manifold M into Euclidean p-space. A point $x \in M$ is called a *definite fold point* of f if there exist local coordinates $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_p)$ centered at x and $f(x)$, respectively, such that f takes the form

$$y_i \circ f = x_i, \qquad 1 \leq i \leq p-1,$$
$$y_p \circ f = x_p^2 + \cdots + x_n^2.$$

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The map f is called a *special generic map* if every singular point of f is a definite fold point. In this paper, we study special generic maps of rational homology spheres, i.e., closed manifolds with the rational homology groups of a sphere.

The notion of a special generic map seems to have first appeared in the literature in a paper of Calabi [4] under the name of quasisurjective mapping. As an important special case, we point out that special generic maps $M^n \to \mathbb{R}$ are the same as Morse functions of M^n with only maxima and minima as their critical points, whereby connected M^n are homeomorphic to the standard n-sphere S^n by the Reeb sphere theorem [15]. Special generic maps $M^n \to \mathbb{R}^2$ were studied for $n = 3$ by Burlet and de Rham [3], and for $n > 3$ by Porto and Furuya [14], and by Saeki [17]. Moreover, Sakuma [19] and Saeki [17] studied special generic maps $M^n \to \mathbb{R}^3$ under various assumptions on the source manifold M^n. Hara [9] studied the existence of special generic maps $M^n \to \mathbb{R}^p$ for $p \leq n/2$ by using L^2-Betti numbers of M^n. Èliašberg [7] showed that for orientable M^n, there is a special generic map $M^n \to \mathbb{R}^n$ if and only if M^n is stably parallelizable. Moreover, for not necessarily orientable M^n, Ando [1] proved that the existence of special generic maps $M^n \to \mathbb{R}^n$ is equivalent to the condition $\mathrm{span}^0(M^n) \geq n-1$. Here, $\mathrm{span}^0(M^n)$ denotes the stable span of the n-manifold M^n, i.e. the number s such that $s+1$ is the number of nowhere linearly dependent sections of the stable tangent bundle of M^n. Ando's condition is known to be equivalent to the vanishing $w_2(M^n) = 0$ of the second Stiefel-Whitney class of M^n for $n = 2$ by Èliašberg [7], and for $n = 3$ by Sakuma [20]. Furthermore, for $n = 4, 5, 6, 7$, Sadykov, Saeki, and Sakuma [16, Theorem 3.6] expressed Ando's condition as the vanishing $w_2(M^n) = 0$ plus a secondary obstruction that involves characteristic classes of vector bundles with a pin structure. In dimensions $n \geq 8$, a characterization of Ando's condition in terms of characteristic classes is to our knowledge a difficult open problem.

For a connected source manifold M^n, Saeki posed the problem to determine the dimension set $S(M^n)$ of all integers $1 \leq p \leq n$ for which there exists a special generic map $M^n \to \mathbb{R}^p$ (see Problem 5.3 in [18]). The dimension set $S(M^n)$ is a diffeomorphism invariant of manifolds that is in general far from being understood. An additional obstacle is the lack of a general diffeomorphism classification for manifolds, already for important classes of closed smooth manifolds like homotopy 4-spheres. Nevertheless, Kikuchi and Saeki showed in [12] that if (oriented) M^n are not smoothly (oriented) nullcobordant, then $S(M^n) = \emptyset$. Moreover, Saeki [18] discovered that $S(M^n) = \{1, \ldots, n\}$ if and only if M^n is diffeomorphic to the standard n-sphere S^n. Nishioka [13] determined the dimension set $S(M^5)$ for any simply connected closed 5-manifold. In [24], the author determined the dimension set $S(\Sigma^7)$ for 14 of Milnor's 27 exotic 7-spheres Σ^7.

In this paper, we study the dimension sets of rational homology spheres of odd dimension by using the technique of Stein factorization of special generic maps (see Section 2.2). Previously, Saeki [17] obtained the following characterization of homotopy spheres in terms of Stein factorization.

Theorem 1.1 (Proposition 4.1 in [17])**.** *Let $f\colon M^n \to \mathbb{R}^p$ $(1 \leq p < n)$ be a special generic map. Then M^n is a homotopy sphere if and only if the Stein factorization W_f is contractible.*

In Section 3, we show the following homological version of Theorem 1.1 for any commutative coefficient ring $R \neq 0$ with identity.

Theorem 1.2. *Let $f\colon M^n \to \mathbb{R}^p$ $(1 \leq p < n)$ be a special generic map. Suppose that M^n is R-orientable. If M^n is an R-homology n-sphere (see Definition 2.2), then the Stein factorization W_f is an R-homology p-ball. The converse implication holds under the additional assumption that R is a principal ideal domain (for example, $R = \mathbb{Z}$ or $R = k$ a field).*

We observe that Theorem 1.1 is a consequence of Theorem 1.2 for $R = \mathbb{Z}$. In fact, since M^n is simply connected if and only if the Stein factorization W_f is simply connected (see Proposition 3.9 in [17]), the homological version of the Whitehead theorem (see Corollary 4.33 in [10, p. 367]) can be applied to the constant map of W_f to a point and to a degree one map $M^n \to S^n$.

As an application of our Theorem 1.2, we show in Proposition 4.2 that if a rational homology sphere M^n of odd dimension $n = 2k + 1 \geq 5$ admits a special generic map into $\mathbb{R}^p$ for some $1 \leq p < n$, then the cardinality of the finite abelian group $H_k(M;\mathbb{Z})$ is the square of an integer. However, this is in general not a sufficient condition for the existence of special generic maps on M (see Remark 4.3). Proposition 4.2 can be considered as a torsion analog of the fact that a closed manifold which admits a special generic map into Euclidean p-space for some $1 \leq p < n$ has even Euler characteristic (see Corollary 3.8 in [17]).

As shown in Proposition 4.5, the square of a positive integer can always be realized as the cardinality of $H_k(M;\mathbb{Z})$ for some highly connected rational homology sphere M^n of suitable odd dimension $n = 2k + 1 \geq 5$ that admits a special generic map into $\mathbb{R}^p$ for some $1 \leq p < n$. On the other hand, there are plenty of rational homology n-spheres M^n for which the cardinality of $H_k(M;\mathbb{Z})$ is *not* the square of an integer, so that M^n admits no special generic maps into $\mathbb{R}^p$ for any $1 \leq p < n$ (or, equivalently, $S(M^n) \subset \{n\}$). For instance, we show that this is the case for many lens spaces whose dimension is congruent to 3 (mod 4) (see Example 4.7), and many total spaces of linear S^3-bundles over S^4 (see Example 4.8).

Notation

The cardinality of a set X is denoted by $|X|$. The symbol $\cong$ either means diffeomorphism of smooth manifolds or isomorphism of modules over a commutative ring. Reduced singular homology and cohomology will be denoted by $\widetilde{H}_*$ and $\widetilde{H}^*$. The singular locus of a smooth map f between smooth manifolds is denoted by $S(f)$. Let $D^p = \{x = (x_1, \ldots, x_p) \in \mathbb{R}^p; x_1^2 + \cdots + x_p^2 \leq 1\}$ denote the closed unit ball in Euclidean p-space with boundary $S^{p-1} = \partial D^p$ the standard $(p-1)$-sphere.

2. Preliminaries

In this preparatory section, we collect several basic facts on homology spheres and homology balls (see Section 2.1), and review the method of Stein factorization of special generic maps (see Section 2.2).

2.1. Homology spheres and homology balls

Let R be a commutative ring with identity. Recall that an orientable manifold is R-orientable for any R, whereas a nonorientable manifold is R-orientable if and only if $2 = 0$ in R (see e.g. [10, p. 235]). For later reference, we record here the following

Lemma 2.1. *If $2 = 0$ in R, then for every pair (X, A) of topological spaces we have an isomorphism*

$$H_*(X, A; R) \cong H_*(X, A; \mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}/2\mathbb{Z}} R$$

of vector spaces over $\mathbb{Z}/2\mathbb{Z}$, where R is a vector space over $\mathbb{Z}/2\mathbb{Z}$ by means of the unique ring homomorphism $\mathbb{Z}/2\mathbb{Z} \to R$.

Proof. The ring homomorphism $\mathbb{Z}/2\mathbb{Z} \to R$ is well-defined because $2 = 0$ in R by assumption. The claimed isomorphism follows from [10, p. 267] by taking C to be the singular chain complex of the space pair (X, A) with coefficient ring $\mathbb{Z}/2\mathbb{Z}$. $\square$

In the following definition, we recall the notions of R-homology sphere and R-homology ball for R-orientable manifolds. In the case that $R = \mathbb{Q}$ (which will be considered in Section 4) we replace the term "R-homology" by "rational homology".

Definition 2.2. A closed R-orientable topological n-manifold P^n is called an *R-homology n-sphere* if $\widetilde{H}_*(P;R) \cong \widetilde{H}_*(S^n;R)$ (where note that $\widetilde{H}_n(S^n;R) \cong R$ and $\widetilde{H}_i(S^n;R) = 0$ for $i \neq n$). A compact R-orientable topological p-manifold Q^p with boundary is called an *R-homology p-ball* if $\widetilde{H}_*(Q;R) \cong \widetilde{H}_*(D^p;R)$ $(= 0)$.

Remark 2.3. If $2R \neq 0$, then the assumption that P is R-orientable is redundant in Definition 2.2. In fact, if P^n is a closed topological manifold that is not R-orientable, then it follows from Theorem 3.26(b) in [10] that $H_n(P;R) \cong \{r \in R;\ 2r = 0\}$. But then, $R \cong \widetilde{H}_n(P;R) \subset H_n(P;R)$ implies that all elements of R have order ≤ 2.

Proposition 2.4. *Suppose that R is a principal ideal domain (for example, $R = \mathbb{Z}$ or $R = k$ a field). If Q^p is an R-homology p-ball of dimension $p \geq 1$, then ∂Q is an R-homology $(p-1)$-sphere.*

Proof. As Q^p is a compact R-orientable p-manifold, its boundary ∂Q is a closed R-orientable $(p-1)$-manifold. In order to show that $\widetilde{H}_*(\partial Q;R) \cong \widetilde{H}_*(S^{p-1};R)$, we remove the interior of a p-disk D^p embedded in the interior of Q to obtain a compact R-orientable p-manifold Q' with boundary the closed R-orientable $(p-1)$-manifold $\partial Q' = \partial Q \sqcup S^{p-1}$. Using excision and the homotopy axiom for homology, we note that $H_*(Q', S^{p-1};R) \cong \widetilde{H}_*(Q;R) = 0$ because Q is an R-homology p-ball. Since R is a principal ideal domain, we can apply the universal coefficient theorem as stated on the bottom of p. 196 in [10] to the R-module $G = R$ and the chain complex $C\colon \cdots \to C_1(Q', S^{p-1}) \otimes_{\mathbb{Z}} R \to C_0(Q', S^{p-1}) \otimes_{\mathbb{Z}} R \to 0$ of the pair (Q', S^{p-1}) with R-coefficients to conclude that $H^*(Q', S^{p-1};R) = 0$. Then, $H_*(Q', \partial Q;R) = 0$ by Lefschetz duality (see Theorem 3.43 in [10, p. 254]). Finally, from the reduced homology long exact sequences of the pairs $(Q', \partial Q)$ and (Q', S^{p-1}) we then see that $\widetilde{H}_*(\partial Q;R) \cong \widetilde{H}_*(Q';R) \cong \widetilde{H}_*(S^{p-1};R)$. □

Proposition 2.5. *For $R \neq 0$ we have:*

1. *Let P^n be an R-homology n-sphere. Then, $\widetilde{H}_i(P;\mathbb{Z})$ is a finite abelian group for $i < n$. If P^n is orientable, then P^n is a rational homology n-sphere.*
2. *Let Q^p be an R-homology p-ball. Then, $\widetilde{H}_i(Q;\mathbb{Z})$ is a finite abelian group for all $i \in \mathbb{Z}$. If Q^p is orientable, then Q^p is a rational homology p-ball.*

Proof. Since P^n and Q^p are compact topological manifolds, their integral homology groups are finitely generated in every degree by Corollary A.8 and Corollary A.9 in [10, p. 527]. Therefore, by applying the universal coefficient theorem for homology as stated in Theorem 3A.3 in [10, p. 264] to the augmented chain complex $C\colon \cdots \to C_1(P) \to C_0(P) \to \mathbb{Z} \to 0$ of P, we conclude from $H_i(C; R) = \widetilde{H}_i(P; R) = 0$ for $i < n$ that $\operatorname{rank} \widetilde{H}_i(P; \mathbb{Z}) = \operatorname{rank} H_i(C) = 0$ for $i < n$ because $R \neq 0$. Similarly, we conclude from $\widetilde{H}_i(Q; R) = 0$ for $i \in \mathbb{Z}$ that $\operatorname{rank} \widetilde{H}_i(Q; \mathbb{Z}) = 0$ for $i \in \mathbb{Z}$. Thus, $\widetilde{H}_i(P; \mathbb{Z})$, $i < n$, and $\widetilde{H}_i(Q; \mathbb{Z})$, $i \in \mathbb{Z}$, are finite abelian groups. Finally, if P^n and Q^p are in addition orientable, then, using $H_*(X; \mathbb{Q}) \cong H_*(X; \mathbb{Z}) \otimes \mathbb{Q}$ for any topological space X (see Corollary 3A.6(a) in [10, p. 266]), we conclude that P^n is a rational homology n-sphere, and Q^p is a rational homology p-ball. $\square$

2.2. Stein factorization of special generic maps

First, let us recall the notion of Stein factorization of an arbitrary continuous map.

Definition 2.6. Let $f\colon X \to Y$ be a continuous map between topological spaces. We define an equivalence relation $\sim_f$ on X as follows. Two points $x_1, x_2 \in X$ are called equivalent, $x_1 \sim_f x_2$, if there is a point $y \in Y$ such that x_1 and x_2 are contained in the same connected component of the fiber $f^{-1}(y)$. The equivalence relation $\sim_f$ on X gives rise to a unique factorization of f of the form

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q_f"'] & & Y \\ & W_f, \arrow[ur, "\overline{f}"'] & \end{tikzcd} \tag{2.1}$$

where $W_f := X/\sim_f$ is the quotient space equipped with the quotient topology, $q_f\colon X \to W_f$ is the continuous quotient map, and the map $\overline{f}\colon W_f \to Y$ is continuous. The diagram (2.1), or sometimes the space W_f, is called the *Stein factorization* of f.

Let $f\colon M^n \to \mathbb{R}^p$, $1 \leq p < n$, be a special generic map of a connected closed smooth n-manifold M into Euclidean p-space. In the following, we recall from [17] some important properties of the Stein factorization of f.

As explained in [17, p. 267], the Stein factorization W_f of f can be equipped with the structure of a compact parallelizable smooth p-manifold with boundary in such a way that the quotient map $q_f\colon M \to W_f$ is a smooth map which satisfies $q_f^{-1}(\partial W_f) = S(f)$, and restricts to a diffeomorphism $S(f) \cong \partial W_f$. Moreover, it is shown in the proof of Proposition 2.1 in [17] that $M \setminus S(f)$ is the total space of a smooth (not necessarily linear) S^{n-p}-bundle $\pi\colon M \setminus S(f) \to W_f \setminus \partial W_f$ over the interior of W_f. Furthermore, it is shown there that M is homeomorphic to $\partial \widetilde{E}$, where $\widetilde{E}$ is the total space of the topological D^{n-p+1}-bundle $\rho\colon \widetilde{E} \to W$ associated[1] with the S^{n-p}-bundle $\pi|\colon \pi^{-1}(W) \to W$ that is the restriction of π over the closure $W = \overline{W_f \setminus C}$ of $W_f \setminus C$ in W_f for a sufficiently small collar neighborhood $C \cong \partial W_f \times [0,1]$ of ∂W_f in W_f (compare Proposition 3.1 in [17]).

Let R be a commutative ring with identity. Since $\widetilde{E}$ is homotopy equivalent to W, and $W \cong W_f$ by construction, we have

$$H_*(\widetilde{E}; R) \cong H_*(W; R) \cong H_*(W_f; R) \tag{2.2}$$

and

$$H^*(\widetilde{E}; R) \cong H^*(W; R) \cong H^*(W_f; R). \tag{2.3}$$

From now on, let us assume that the compact topological $(n+1)$-manifold $\widetilde{E}$ is R-orientable. The smooth compact p-manifold W_f is R-orientable as well because it is parallelizable and hence orientable. Thus, Poincaré-Lefschetz duality (see Theorem 3.43 in [10, p. 254]) implies

$$H_*(\widetilde{E}, \partial \widetilde{E}; R) \cong H^{n+1-*}(\widetilde{E}; R), \tag{2.4}$$

$$H_*(W_f, \partial W_f; R) \cong H^{p-*}(W_f; R), \tag{2.5}$$

and

$$H_*(\partial W_f; R) \cong H^{p-1-*}(\partial W_f; R). \tag{2.6}$$

Analogously to Proposition 3.10 in [17], we have a long exact sequence of

[1]The transition functions $D^{n-p+1} \to D^{n-p+1}$ of ρ are obtained by extending the transition functions $S^{n-p} \to S^{n-p}$ of $\pi|$ radially.

the form

(2.7)

$$\begin{aligned}
\dots \longrightarrow H_{q+1}(M;R) &\longrightarrow H_{q+1}(W_f;R) \longrightarrow H^{n-q}(W_f;R) \\
\longrightarrow H_q(M;R) &\longrightarrow H_q(W_f;R) \longrightarrow H^{n-q+1}(W_f;R) \longrightarrow \dots \\
&\dots \\
\dots \longrightarrow H_1(M;R) &\longrightarrow H_1(W_f;R) \longrightarrow H^n(W_f;R) \longrightarrow 0.
\end{aligned}$$

(In order to derive (2.7), we start with the homology long exact sequence of the pair $(\widetilde{E}, \partial\widetilde{E})$. Then, we make the replacements $H_q(\partial\widetilde{E};R) \cong H_q(M;R)$ by using that M is homeomorphic to $\partial\widetilde{E}$, $H_q(\widetilde{E};R) \cong H_q(W_f;R)$ by (2.2), and $H_q(\widetilde{E}, \partial\widetilde{E};R) \cong H^{n+1-q}(\widetilde{E};R) \cong H^{n+1-q}(W_f;R)$ by using (2.4) and (2.3). The right end of the sequence (2.7) has the claimed form because M and W_f are both connected so that the map $H_0(\partial\widetilde{E};R) \to H_0(\widetilde{E};R)$ is an isomorphism.)

Next, we note that

$$H^q(W_f;R) \overset{(2.5)}{\cong} H_{p-q}(W_f, \partial W_f;R) = 0, \qquad q \geq p, \tag{2.8}$$

where $H_0(W_f, \partial W_f;R) = 0$ holds because W_f is connected as the image of the connected space M under the surjective quotient map $q_f\colon M \to W_f$. Thus, using (2.8) and (2.7), we conclude that

$$H_q(M;R) \cong H_q(W_f;R), \qquad q \leq n-p. \tag{2.9}$$

3. Proof of Theorem 1.2

The proof of Theorem 1.2 is very similar to the proof of Proposition 4.1 in [17]. By a close examination we assure in the following that each step of the argument is valid for our purely homological formulation that uses R-coefficients instead of $\mathbb{Z}$-coefficients, and does not involve any assumptions about fundamental groups.

Let $f\colon M^n \to \mathbb{R}^p$ $(1 \leq p < n)$ be a special generic map. Suppose that M^n is R-orientable. Then, it follows that the compact topological $(n+1)$-manifold $\widetilde{E}$ introduced in Section 2.2 is R-orientable as well. (In fact, we only need to consider the case that $\widetilde{E}$ is nonorientable because orientable manifolds are R-orientable for any R. But then, M is nonorientable as well because orientability of M implies orientability of $\widetilde{E}$, as stated after the proof of Proposition 3.1 in [17]. Now, recall from Section 2.1 that a nonorientable manifold is R-orientable if and only if $2 = 0$ in R. Consequently, R-orientability of M implies R-orientability of $\widetilde{E}$.)

Let us first suppose that the Stein factorization W_f of f is an R-homology p-ball, with R being a principal ideal domain. Then, we have

$$H_*(\widetilde{E};R) \overset{(2.2)}{\cong} H_*(W_f;R) \cong H_*(D^p;R) \cong H_*(D^{n+1};R).$$

Since $\widetilde{E}$ is R-orientable as shown above, it follows that $\widetilde{E}$ is an R-homology $(n+1)$-ball. Hence, using that R is a principal ideal domain, we conclude from Proposition 2.4 that $\partial\widetilde{E}$ is an R-homology n-sphere. As M is homeomorphic to $\partial\widetilde{E}$, M is an R-homology n-sphere as well.

Conversely, we suppose that M is an R-homology n-sphere. In particular,

$$\widetilde{H}_q(M;R) \cong \widetilde{H}_q(S^n;R) = 0, \qquad q < n. \tag{3.1}$$

Combining (3.1) with the long exact sequence (2.7) (where note that $\widetilde{E}$ is R-orientable as shown above), we obtain

$$H_q(W_f;R) \cong H^{n-q+1}(W_f;R), \qquad 0 < q < n. \tag{3.2}$$

Next, we observe that

$$\widetilde{H}_q(W_f;R) = 0, \qquad q \leq n-p+1, \tag{3.3}$$

which follows for $q \leq n-p$ from (2.9) and (3.1), and for $q = n-p+1$ (> 0) from (3.2) and (2.8) (where we may assume that $q < n$ because W_f is a p-manifold).

In the following, we show by induction on q that $\widetilde{H}_q(W_f;R) = 0$ for all $q \leq p$. (Then, it follows immediately that the compact R-orientable p-manifold W_f is an R-homology p-ball, where note that W_f is orientable as a parallelizable manifold.) Taking (3.3) as the basis $q = n-p+2$ of the induction, we fix $n-p+2 \leq q \leq p$, and suppose that we have already shown

$$\widetilde{H}_i(W_f;R) = 0, \qquad i \leq q-1. \tag{3.4}$$

The induction step consists of showing that $H_q(W_f; R) = 0$. As $0 < q < n$, we have

(3.5)

$$H_q(W_f; R) \overset{(3.2)}{\cong} H^{n-q+1}(W_f; R) \overset{(2.5)}{\cong} H_{p-n+q-1}(W_f, \partial W_f; R) \cong \widetilde{H}_{p-n+q-2}(\partial W_f; R),$$

where the last isomorphism is a connecting homomorphism in the reduced homology long exact sequence of the pair $(W_f, \partial W_f)$, and is an isomorphism because $\widetilde{H}_{p-n+q-1}(W_f; R) = 0 = \widetilde{H}_{p-n+q-2}(W_f; R)$ by induction hypothesis (3.4). If $q = n - p + 2$, then we obtain as desired

$$H_q(W_f; R) \overset{(3.5)}{\cong} \widetilde{H}_0(\partial W_f; R) = 0, \tag{3.6}$$

where the last equality holds because we have $l = 1$ for the number $l > 0$ of connected components of $\partial W_f = S(f) \neq \emptyset$. (In fact, if M is orientable, then we have $l \leq 1 + \operatorname{rank} H_{p-1}(M; \mathbb{Z})$ by Proposition 3.15 in [17], and $\operatorname{rank} H_{p-1}(M; \mathbb{Z}) = 0$ holds by Proposition 2.51 as $1 < p < n$ and $R \neq 0$ by assumption. On the other hand, if M is nonorientable, then a slight modification of the proof of Proposition 3.15 in [17] based on the intersection product with $\mathbb{Z}/2\mathbb{Z}$ coefficients yields $l \leq 1 + \dim_{\mathbb{Z}/2\mathbb{Z}} H_{p-1}(M; \mathbb{Z}/2\mathbb{Z})$ (for the intersection product, see e.g. Theorem 11.9 in [2, p. 372], which applies also to $\mathbb{Z}/2\mathbb{Z}$ coefficients as pointed out in Example 11.14 in [2, p. 376]). Note that we have $2 = 0$ in R because M is nonorientable and R-orientable. Therefore, Lemma 2.1 implies that $H_{p-1}(M; \mathbb{Z}/2\mathbb{Z}) = 0$ because $0 = H_{p-1}(M; R) \cong H_{p-1}(M; \mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}/2\mathbb{Z}} R$ and $R \neq 0$.) If $q > n - p + 2$, then

$$H_q(W_f; R) \overset{(3.5)}{\cong} H_{p-n+q-2}(\partial W_f; R) \overset{(2.6)}{\cong} H^{n-q+1}(\partial W_f; R). \tag{3.7}$$

Since $\partial W_f = S(f)$ is a closed subset of the closed n-manifold M^n, we have

$$H^{n-q+1}(S(f); R) \cong H_{q-1}(M, M \setminus S(f); R)$$

by Poincaré-Lefschetz duality as stated in Corollary 8.4 in [2, p. 352], where note that the Čech cohomology can be replaced by singular cohomology because $S(f)$ is a manifold. (For nonorientable M, we apply Poincaré-Lefschetz duality with coefficient ring $\mathbb{Z}/2\mathbb{Z}$ to obtain $H^{n-q+1}(S(f); \mathbb{Z}/2\mathbb{Z}) \cong H_{q-1}(M, M \setminus S(f); \mathbb{Z}/2\mathbb{Z})$, and then carry this isomorphism over to coefficient ring R by means of Lemma 2.1. Here, we have $2 = 0$ in R because M is nonorientable and R-orientable.) Next, we observe that $H_{q-1}(M, M \setminus S(f); R) \cong$

$H_{q-2}(M \setminus S(f); R)$, which is a connecting homomorphism in the reduced homology long exact sequence of the pair $(M, M \setminus S(f))$, and is an isomorphism because $\widetilde{H}_{q-1}(M; R) = 0 = \widetilde{H}_{q-2}(M; R)$ by (3.1), where $q - 1 < n$ because $q \leq p < n$. Altogether, using $\partial W_f \cong S(f)$, we conclude that for $q > n-p+2$,

(3.8)
$$H_q(W_f; R) \overset{(3.7)}{\cong} H^{n-q+1}(\partial W_f; R) \cong H_{q-1}(M, M\setminus S(f); R) \cong H_{q-2}(M\setminus S(f); R).$$

Now, assuming for the moment that the manifold M is orientable, we recall from Section 2.2 that there is a smooth S^{n-p}-bundle $\pi\colon M \setminus S(f) \to W_f \setminus \partial W_f$ whose total space $M \setminus S(f)$ and base space $W_f \setminus \partial W_f$ are orientable manifolds. Hence, π is an orientable sphere bundle in the sense of [10, p. 442] (i.e., for every loop in the base space the induced homeomorphism $\pi^{-1}(x) \to \pi^{-1}(x)$ over the basepoint x induces the identity map on $H_{n-p}(\pi^{-1}(x); \mathbb{Z})$). Let $D(\pi)\colon E' \to W_f \setminus \partial W_f$ be the orientable topological D^{n-p+1}-bundle associated with π. Then, the Thom isomorphism yields $H^*(W_f; \mathbb{Z}) \cong H^{*+(n-p+1)}(E', M\setminus S(f); \mathbb{Z})$ (see Corollary 4D.9 in [10, p. 441]). Consequently, $H_*(W_f; \mathbb{Z}) \cong H_{*+(n-p+1)}(E', M \setminus S(f); \mathbb{Z})$ by Corollary 3.3 in [10, p. 196]. Then, the universal coefficient theorem for homology (see Corollary 3A.4 in [10, p. 264]) yields an isomorphism

$$H_*(W_f; R) \cong H_{*+(n-p+1)}(E', M \setminus S(f); R) \tag{3.9}$$

(of $\mathbb{Z}$-modules, not of R-modules). On the other hand, if M is nonorientable, then the disk bundle $D(\pi)$ still has a Thom class with $\mathbb{Z}/2\mathbb{Z}$ coefficients by Theorem 4D.10 in [10], and the Thom isomorphism yields $H^*(W_f; \mathbb{Z}/2\mathbb{Z}) \cong H^{*+(n-p+1)}(E', M \setminus S(f); \mathbb{Z}/2\mathbb{Z})$. Then, Lemma 2.1 allows us to carry this isomorphism over to coefficient ring R, which yields (3.9) also in the case that M is nonorientable. (Here, note that we have $2 = 0$ in R because M is nonorientable and R-orientable.)

Let us consider the following part of the homology long exact sequence of the pair $(E', M \setminus S(f))$:

$$H_{q-1}(E', M \setminus S(f); R) \to H_{q-2}(M \setminus S(f); R) \to H_{q-2}(E'; R). \tag{3.10}$$

Since E' is homotopy equivalent to W_f, we have $H_{q-2}(E'; R) \cong H_{q-2}(W_f; R) = 0$ by induction hypothesis (3.4), where note that $q - 2 > 0$ because $q > n - p + 2 > 2$. Furthermore, we have

$$H_{q-1}(E', M \setminus S(f); R) \overset{(3.9)}{\cong} H_{p-n+q-2}(W_f; R) \overset{(3.4)}{=} 0,$$

where we can apply the induction hypothesis because $0 < p - n + q - 2 < q$. Finally, in view of (3.10), we obtain

$$H_q(W_f; R) \overset{(3.8)}{\cong} H_{q-2}(M \setminus S(f); R) = 0.$$

This completes the proof of Theorem 1.2.

4. An application in odd dimensions

In order to derive Proposition 4.2 below from Theorem 1.2 we need the following

Lemma 4.1. *Let $\cdots \to A_{i-1} \to A_i \to A_{i+1} \to \dots$ be a long exact sequence of finite abelian groups such that $|A_i| = 1$ for all but a finite number of $i \in \mathbb{Z}$. If $|A_{-i}| = |A_i|$ for all $i \in \mathbb{Z}$, then $|A_0| = k^2$ for some integer k.*

Proof. Every map $\alpha_i \colon A_i \to A_{i+1}$ of the given long exact sequence gives rise to a short exact sequence

$$0 \to \ker(\alpha_i) \to A_i \to \operatorname{im}(\alpha_i) \to 0$$

of finite abelian groups. Since A_i and A_{i+1} are finite, we obtain

$$|A_i| = |\ker(\alpha_i)| \cdot |\operatorname{im}(\alpha_i)|, \qquad i \in \mathbb{Z}. \tag{4.1}$$

Using that $\ker(\alpha_i) = \operatorname{im}(\alpha_{i-1})$ by exactness of the given sequence, we have

$$\prod_{i \text{ odd}} |A_i| \overset{(4.1)}{=} \prod_{i \text{ odd}} |\operatorname{im}(\alpha_{i-1})| \cdot |\ker(\alpha_{i+1})| = \prod_{i \text{ even}} |\operatorname{im}(\alpha_i)| \cdot |\ker(\alpha_i)| \overset{(4.1)}{=} \prod_{i \text{ even}} |A_i|,$$

where note that the products are finite because $|A_i| = 1$ for almost all $i \in \mathbb{Z}$.

If $|A_{-i}| = |A_i|$ for all $i \in \mathbb{Z}$, then we can write

$$|A_0| = \frac{\prod_{i \text{ odd}} |A_i|}{\prod_{0 \neq i \text{ even}} |A_i|} = \frac{(\prod_{j \geq 0} |A_{2j+1}|)^2}{(\prod_{j \geq 1} |A_{2j}|)^2}.$$

Thus, the positive integers $a = |A_0|$, $x = \prod_{j \geq 0} |A_{2j+1}|$ and $y = \prod_{j \geq 1} |A_{2j}|$ satisfy $ay^2 = x^2$. By comparing the exponents of prime numbers in the prime factorizations of a, x, and y, we conclude that $|A_0| = k^2$ for some integer k. □

The main result of this section is the following

Proposition 4.2. *Let $f\colon M^n \to \mathbb{R}^p$ $(1 \le p < n)$ be a special generic map on a smooth rational homology n-sphere M^n of odd dimension $n = 2k+1 \ge 5$. Then the cardinality of the finite abelian group $H_k(M;\mathbb{Z})$ is the square of an integer.*

Proof. We conclude from Theorem 1.2 that the Stein factorization W_f is a rational homology p-ball. By Proposition 2.5, $\widetilde{H}_i(M;\mathbb{Z})$, $i < n$, and $\widetilde{H}_i(W_f;\mathbb{Z})$, $i \in \mathbb{Z}$, are finite abelian groups. Hence, taking C to be the augmented chain complexes of M and W_f in Corollary 3.3 in [10, p. 196], we obtain

$$\widetilde{H}^i(M;\mathbb{Z}) \cong \widetilde{H}_{i-1}(M;\mathbb{Z}), \qquad i < n, \tag{4.2}$$

and

$$\widetilde{H}^i(W_f;\mathbb{Z}) \cong \widetilde{H}_{i-1}(W_f;\mathbb{Z}), \qquad i \in \mathbb{Z}, \tag{4.3}$$

respectively. Poincaré duality for M^n yields

$$H_i(M;\mathbb{Z}) \cong H^{n-i}(M;\mathbb{Z}) \overset{(4.2)}{\cong} H_{n-i-1}(M;\mathbb{Z}), \qquad 1 \le i \le n-2. \tag{4.4}$$

In view of $H^{n-1}(W_f;\mathbb{Z}) = 0$ (see (2.8) applied for $q = n-1 \ge p$) and (4.3), the long exact sequence (2.7) takes for $n \ge 5$ and $R = \mathbb{Z}$ the form
(4.5)

$$
\begin{aligned}
&0 = H_{n-2}(W_f;\mathbb{Z}) \longrightarrow H_2(W_f;\mathbb{Z}) \\
&\longrightarrow H_{n-3}(M;\mathbb{Z}) \longrightarrow H_{n-3}(W_f;\mathbb{Z}) \longrightarrow H_3(W_f;\mathbb{Z}) \longrightarrow \dots \\
&\dots \\
&\dots \longrightarrow H_{q+1}(M;\mathbb{Z}) \longrightarrow H_{q+1}(W_f;\mathbb{Z}) \longrightarrow H_{n-q-1}(W_f;\mathbb{Z}) \\
&\longrightarrow H_q(M;\mathbb{Z}) \longrightarrow H_q(W_f;\mathbb{Z}) \longrightarrow H_{n-q}(W_f;\mathbb{Z}) \longrightarrow \dots \\
&\dots \\
&\dots \longrightarrow H_2(M;\mathbb{Z}) \longrightarrow H_2(W_f;\mathbb{Z}) \longrightarrow H_{n-2}(W_f;\mathbb{Z}) = 0.
\end{aligned}
$$

By assumption, $n = 2k+1$ is odd. Writing $\dots \to A_{i-1} \to A_i \to A_{i+1} \to \dots$ with $A_0 = H_k(M;\mathbb{Z})$ for the above exact sequence (4.5), our claim will follow from Lemma 4.1 once we show that $|A_{-i}| = |A_i|$ for all integers $i > 0$. If $i \equiv 1 \pmod 3$, say $i = 3s+1$, then we see that $A_i = H_{k-s}(W_f;\mathbb{Z}) = A_{-i}$

for $k-s \geq 2$, and $A_i = 0 = A_{-i}$ for $k-s < 2$. If $i \equiv 2 \pmod 3$, say $i = 3s+2$, then we see that $A_i = H_{k+1+s}(W_f;\mathbb{Z}) = A_{-i}$ for $k+1+s \leq n-2$, and $A_i = 0 = A_{-i}$ for $k+1+s > n-2$. If $i \equiv 0 \pmod 3$, say $i = 3s$, then we see by means of (4.4) that $A_i = H_{k-s}(M;\mathbb{Z}) \cong H_{k+s}(M;\mathbb{Z}) = A_{-i}$ for $k-s \geq 2$, and $A_i = 0 = A_{-i}$ for $k-s < 2$. All in all, we have shown that $|A_{-i}| = |A_i|$ for all integers $i > 0$, which completes the proof of Proposition 4.2. □

Remark 4.3. Our homological condition in Proposition 4.2 is in general not sufficient for a smooth rational homology sphere M of odd dimension $n \geq 5$ to admit a special generic map into $\mathbb{R}^p$ for some $1 \leq p < n$. In fact, the real projective space $\mathbb{R}P^5$ satisfies $H_2(\mathbb{R}P^5;\mathbb{Z}) = 0$, but there does not exist a special generic map $f\colon \mathbb{R}P^5 \to \mathbb{R}^p$ for any $1 \leq p < 5$. (Otherwise, the universal cover $\pi\colon S^5 \to \mathbb{R}P^5$ would induce a 2-sheeted covering $W_{f\circ\pi} \to W_f$ of Stein factorizations of the special generic maps $f \circ \pi$ and f by Proposition 2.6 in [9]. Thus, the space $W_{f\circ\pi}$ would have even Euler characteristic $\chi(W_{f\circ\pi}) = 2\chi(W_f)$ while being contractible by Theorem 1.1.)

Remark 4.4. Suppose that $n = 4l+1$ for some integer $l \geq 1$. After Seifert [21], the linking form $b\colon TH_{2l}(N;\mathbb{Z}) \times TH_{2l}(N;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ on the torsion subgroup of the homology group $H_{2l}(N;\mathbb{Z})$ of a closed oriented topological n-manifold N is a nondegenerate skew-symmetric bilinear form. Wall has shown (see Theorem 3 in [22]) that

$$TH_{2l}(N;\mathbb{Z}) \cong \begin{cases} H \oplus H, & \text{if } b(x,x) = 0 \text{ for all } x \in TH_{2l}(N;\mathbb{Z}), \\ H \oplus H \oplus \mathbb{Z}/2\mathbb{Z}, & \text{else}, \end{cases} \tag{4.6}$$

for a suitable finite abelian group H. Thus, if M is a smooth rational homology n-sphere that admits a special generic map into $\mathbb{R}^p$ for some $1 \leq p < n$, then Proposition 4.2 implies that the first alternative holds for $H_{2l}(M;\mathbb{Z})$ in (4.6).

Next, we show that the homological condition in Proposition 4.2 is generally optimal.

Proposition 4.5. *Let $m > 0$ be an integer. There exist an integer $k > 1$ and a $(k-1)$-connected smooth rational homology $(2k+1)$-sphere M with $|H_k(M;\mathbb{Z})| = m^2$ which admits a special generic map into $\mathbb{R}^p$ for some $1 \leq p < 2k+1$.*

Proof. Let W^p be a compact parallelizable smooth p-manifold with boundary. If $n > p > 0$, then by Proposition 2.1 in [17], there exists a special generic map $f\colon M^n \to \mathbb{R}^p$, where the closed smooth n-manifold M is diffeomorphic to the boundary of the product $W \times D^{n-p+1}$ (after smoothing the corners), and the Stein factorization of f is diffeomorphic to W.

Now, we suppose in addition that $n = 2k + 1$ for some integer $k > 1$, and that W is a simply connected rational homology p-ball whose only non-vanishing integral homology group in positive degree is $H_k(W; \mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$. Then, it follows from Theorem 1.2 that M is a smooth rational homology n-sphere, and M is simply connected by Proposition 3.9 in [17]. Moreover, using the assumptions on the homology of W in the long exact sequence (4.5) from the proof of Proposition 4.2, we see that $H_q(M; \mathbb{Z}) = 0$ for $2 \leq q \leq k-1$, and that there is a short exact sequence

$$0 \to H_k(W; \mathbb{Z}) \to H_k(M; \mathbb{Z}) \to H_k(W; \mathbb{Z}) \to 0.$$

Thus, M is $(k-1)$-connected by the Hurewicz theorem, and we have $|H_k(M; \mathbb{Z})| = |H_k(W; \mathbb{Z})|^2 = m^2$. This shows that M will have all the desired properties.

Thus, it remains to construct a manifold W having all of the above properties. For this purpose, we consider a finite connected simplicial complex K embedded in some $\mathbb{R}^a$, and whose only non-vanishing integral homology group in positive degree is $H_1(K; \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$. (Such a simplicial complex K can be obtained by an embedded 3-dimensional lens space $L(m, l) = L_m(1, l)$ (compare Example 4.7 below) with a small open 3-disk removed.) Then, by taking r-fold suspension, we obtain a finite connected simplicial complex L embedded in $\mathbb{R}^{a+r}$ whose only non-vanishing integral homology group in positive degree is $H_{r+1}(L; \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$. It is well-known that L is the deformation retract of a regular neighborhood V in $\mathbb{R}^{a+r}$ that is a compact smoothly embedded $(a + r)$-manifold (which is in particular parallelizable). Then, by choosing $r > 0$ so large that $n = 2k + 1 > p$ with $k = r + 1$ and $p = a + r$, the manifold $W = V$ will have all of the desired properties. (In particular, note that W is simply connected by the Freudenthal suspension theorem.)

This completes the proof of Proposition 4.5. □

Remark 4.6. Concerning the choice of an embedded simplicial complex $K \subset \mathbb{R}^a$ in the proof of Proposition 4.5, we note that $a = 5$ is sufficient because any orientable closed 3-manifold can be embedded in $\mathbb{R}^5$ according to a result of Hirsch [11]. Moreover, by results of Zeeman [25] and Epstein [6], the 3-dimensional punctured lens space $L(m, l) \setminus \mathrm{pt}$ can be embedded into $\mathbb{R}^4$ if and only if m is odd. Consequently, in Proposition 4.5 we can realize all values $k \geq 4$, and also $k = 3$ when m is odd. We do not know if there exists a special generic map $M^7 \to \mathbb{R}^p$, $1 \leq p < 7$, where M is a smooth rational homology 7-sphere such that $|H_3(M; \mathbb{Z})|$ is even.

We conclude with applications of Proposition 4.2 to determine the dimension sets of some rational homology spheres.

Example 4.7 (Lens spaces)**.** For an integer $m > 1$ and integers $l_1, \ldots, l_{k+1}$ ($k \geq 0$) relatively prime to m, the lens space $L_m(l_1, \ldots, l_{k+1})$ (see e.g. Example 2.43 in [10, p. 144]) is a closed smooth $(2k+1)$-manifold whose integral homology groups are given by

$$H_i(L_m(l_1, \ldots, l_{k+1}); \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0, 2k+1, \\ \mathbb{Z}/m\mathbb{Z} & \text{for } i \text{ odd, } 0 < i < 2k+1, \\ 0 & \text{otherwise.} \end{cases}$$

If $k \geq 3$ is odd, and $m = |H_k(L_m(l_1, \ldots, l_{k+1}); \mathbb{Z})|$ is not the square of an integer, then Proposition 4.2 implies that $L_m(l_1, \ldots, l_{k+1})$ does not admit a special generic map into $\mathbb{R}^p$ for any $1 \leq p < 2k+1$. Furthermore, Èliašberg [7] has shown that there is a special generic map $L_m(l_1, \ldots, l_{k+1}) \to \mathbb{R}^{2k+1}$ if and only if $L_m(l_1, \ldots, l_{k+1})$ is stably parallelizable. Hence, we have $S(L_m(l_1, \ldots, l_{k+1})) = \{2k+1\}$ if $L_m(l_1, \ldots, l_{k+1})$ is stably parallelizable, and $S(L_m(l_1, \ldots, l_{k+1})) = \emptyset$ else. If m is an odd prime and $1 \leq l_i \leq m-1$ for all i, then it follows from [8] that $S(L_m(l_1, \ldots, l_{k+1})) = \{2k+1\}$ if and only if $k < m$ and $l_1^{2j} + \cdots + l_{k+1}^{2j}$ is divisible by m for $j = 1, \ldots, \lfloor k/2 \rfloor$, where $\lfloor x \rfloor$ denotes the biggest integer $\leq x$ for a real number x.

Example 4.8 (Linear S^3-bundles over S^4)**.** As explained in [5], fiber bundles over S^4 with fiber S^3 and structure group $SO(4)$ are classified by elements of $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$. Moreover, the nontrivial integral homology groups of the total space $M_{m,n}$ corresponding to $(m, n) \in \pi_3(SO(4))$ are $H_0(M_{m,n}; \mathbb{Z}) \cong H_7(M_{m,n}; \mathbb{Z}) \cong \mathbb{Z}$ and $H_3(M_{m,n}; \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. We note that $M_{m,n}$ is a rational homology 7-sphere for $n \neq 0$. Hence, by Proposition 4.2, $M_{m,n}$ does not admit a special generic map into $\mathbb{R}^p$ for any $1 \leq p < 7$ whenever $|n|$ is not the square of an integer. Moreover, Èliašberg [7] has shown that there is a special generic map $M_{m,n} \to \mathbb{R}^7$ if and only if $M_{m,n}$ is stably parallelizable. According to Wilkens [23], this is equivalent to the vanishing of an obstruction $\widehat{\beta} \in H^4(M_{m,n}; \pi_3(SO)) \cong \mathbb{Z}/n\mathbb{Z}$. This obstruction has been determined to be $\widehat{\beta} = \frac{p_1}{2}(M_{m,n}) \equiv 2m \pmod{n}$ in [5, p. 365]. All in all, if $|n|$ is not the square of an integer, then

$$S(M_{m,n}) = \begin{cases} \{n\}, & n | 2m, \\ \emptyset, & \text{else.} \end{cases}$$

References

[1] Y. Ando, *Existence theorems of fold-maps*, Japan. J. Math. **30** (2004), 29–73. MR2070370

[2] G.E. Bredon, *Topology and Geometry*, Springer-Verlag New York, Graduate Texts in Mathematics 139 (1993). DOI: 10.1007/978-1-4757-6848-0. MR1224675

[3] O. Burlet, G. de Rham, *Sur certaines applications génériques d'une variété close à trois dimensions dans le plan*, Enseign. Math. **20** (1974), 275–292. MR0365594

[4] E. Calabi, *Quasi-surjective mappings and a generalization of Morse theory*, Proc. U.S.-Japan Seminar in Differential Geometry, Kyoto (1965), 13–16. MR0216513

[5] D. Crowley, C.M. Escher, *A classification of S^3-bundles over S^4*, Differential Geometry and its Applications **18** (2003), 363–380. DOI: 10.1016/S0926-2245(03)00012-3. MR1975035

[6] D.B.A. Epstein, *Embedding punctured manifolds*, Proc. Amer. Math. Soc. **16** (1965), 175–176. MR0208606

[7] J.M. Èliašberg, *On singularities of folding type*, Math. USSR-Izv. **4** (1970), 1119–1134. MR0278321

[8] J. Ewing, S. Moolgavkar, L. Smith, R.E. Stong, *Stable parallelizability of lens spaces*, Journal of Pure and Applied Algebra **10** (1977), 177–191. MR0478152

[9] Y. Hara, *Special generic maps and L^2-Betti numbers*, Osaka J. Math. **34** (1997), 151–167. MR1439003

[10] A.E. Hatcher, *Algebraic topology*, Cambridge Univ. Press, 2002. MR1867354

[11] M.W. Hirsch, *The Imbedding of Bounding Manifolds in Euclidean Space*, Annals of Mathematics **74** (1961), 494–497. MR0133136

[12] S. Kikuchi, O. Saeki, *Remarks on the topology of folds*, Proceedings of the American Mathematical Society **123** (1995), 905–908. MR1283556

[13] M. Nishioka, *Special generic maps of 5-dimensional manifolds*, Rev. Roumaine Math. Pures Appl. **60** (2015), 507–517. MR3436208

[14] P. Porto, Y.K.S. Furuya, *On special generic maps from a closed manifold into the plane*, Topology Appl. **35** (1990), 41–52. MR1049860

[15] G. Reeb, *Sur certaines propriétés topologiques des variétés feuilletées*, Actualités Scientifiques et Industrielles 1183 (Hermann, Paris, 1952), 91–154. MR0055692

[16] R. Sadykov, O. Saeki, K. Sakuma, *Obstructions to the existence of fold maps*, J. London Math. Soc. **81** (2010), 338–354. MR2602999

[17] O. Saeki, *Topology of special generic maps of manifolds into Euclidean spaces*, Topology Appl. **49** (1993), 265–293. MR1208678

[18] O. Saeki, *Topology of special generic maps into* $\mathbb{R}^3$, in: Workshop on Real and Complex Singularities (São Carlos, 1992), Mat. Contemp. 5 (1993), 161–186. MR1305180

[19] K. Sakuma, *On special generic maps of simply connected* $2n$*-manifolds into* $\mathbb{R}^3$, Topology Appl. **50** (1993), 249–261. MR1227553

[20] K. Sakuma, *Existence problem for fold maps*, Real and complex singularities, Proceedings of the Australian-Japanese Workshop (World Scientific, Singapore, 2007), 342–387. MR2336695

[21] H. Seifert, *Verschlingungsinvarianten*, Sitzungsber. Preuß. Akad. Wiss., Phys.-Math. Kl. **1933**, No. 26-29 (1933), 811–828.

[22] C.T.C. Wall, *Quadratic forms on finite groups, and related topics*, Topology **2** (1963), 281–298. MR0156890

[23] D.L. Wilkens, *Closed* $(s-1)$*-connected* $(2s+1)$*-manifolds,* $s = 3, 7$, Bull. Lond. Math. Soc. **4** (1972), 27–31. MR0307258

[24] D.J. Wrazidlo, *Standard special generic maps of homotopy spheres into Euclidean spaces*, Topology Appl. **234** (2018), 348–358. MR3739476

[25] E.C. Zeeman, *Twisting spun knots*, Trans. Amer. Math. Soc. **115** (1965), 471–495. MR0195085

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