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Kerr stability for small angular momentum

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Abstract: This is our main paper in which we prove the full, unconditional, nonlinear stability of the Kerr family $Kerr(a, m)$ for small angular momentum, i.e. $|a|/m \ll 1$, in the context of asymptotically flat solutions of the Einstein vacuum equations (EVE). We rely on our GCM papers [40] and [41], the recently released [50], as well as on the general formalism contained in part I of [28], see also the older version in [27]. The recently released [28] also contains, in parts II, III, all hyperbolic type estimates needed in our work. Our work extends the strategy developed in [39], in which only axial polarized perturbations of Schwarzschild were treated, by developing new geometric and analytic ideas on how to deal with with general perturbations of Kerr. We note that the restriction to small angular momentum is needed only in [28], mainly for the Morawetz type estimates derived in that paper.

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1. INTRODUCTION

This is our main paper in a series of papers, see [40], [41], [28] and [50], in which we prove the full nonlinear stability of the Kerr family $Kerr(a, m)$ for small angular momentum, i.e. $|a|/m \ll 1$, in the context of asymptotically flat solutions of the Einstein vacuum equations (EVE),

$$\mathbf{R}_{\alpha\beta} = 0. \tag{1.1}$$

We recall that the Kerr family, discovered by R. Kerr [31] in 1963, consists of explicit, stationary, asymptotically flat, solutions of EVE. It is considered by physicists and astrophysicists to be the main mathematical model of a black hole.

1.1. Kerr stability conjecture

The discovery of black holes, first as explicit solutions of EVE and later as possible explanations of astrophysical phenomena, has not only revolutionized our understanding of the universe, it also gave mathematicians a monumental task: to test the physical reality of these solutions. This may seem nonsensical since physics tests the reality of its objects by experiments and observations and, as such, needs mathematics to formulate the theory and make quantitative predictions, not to test it. The problem, in this case, is that black holes are by definition non-observable and thus no direct experiments are possible. Astrophysicists ascertain the presence of such objects through indirect observations[1] and numerical experiments, but both are limited in scope to the range of possible observations or the specific initial conditions in which numerical simulations are conducted. One can rigorously check that the Kerr solutions have vanishing Ricci curvature, that is, their mathematical reality is undeniable. But to be real in a physical sense, they have to satisfy certain properties that can be neatly formulated in unambiguous mathematical language. Chief among them[2] is the problem of stability, that is, to show that if the precise initial data corresponding to Kerr are perturbed a bit, the basic features of the corresponding solutions do not change much[3]. This leads naturally to the following conjecture.

[1]The most recent Nobel prize in Physics was awarded to R. Penrose for his theoretical foundations and to R. Genzel and A. Ghez for providing observational evidence for the presence of super massive black holes in the center of our galaxy.

[2]Other such properties concern the rigidity of the Kerr family or the dynamical formations of black holes from regular configurations.

[3]If the Kerr family would be unstable under perturbations, black holes would be nothing more than mathematical artifacts.

Conjecture (Stability of Kerr conjecture)**.** *Vacuum, asymptotically flat, initial data sets, sufficiently close to* $Kerr(a, m)$, $|a|/m < 1$, *initial data, have maximal developments with complete future null infinity and with domain of outer communication*[4] *which approaches (globally) a nearby Kerr solution.*

In this section, we provide a brief introduction to the current state of the art concerning the Kerr stability conjecture. For a more in depth introduction to the problem, we refer the reader to [18] or the introduction of [39].

1.1.1. The Kerr stability problem in the physics literature The nonlinear stability of the Kerr family has become, ever since its discovery by R. Kerr [31] in 1963, a central topic in general relativity. The first stability results obtained by physicists in the context of the linearized EVE near a fixed member of the Kerr family were mode stability results. The metric perturbation point of view was initiated by Regge-Wheeler [47] who discovered the master Regge-Wheeler equation for odd-parity perturbations. An alternative approach via the Newman-Penrose (NP) formalism was first undertaken by Bardeen-Press [4]. This latter type of analysis was later extended to the Kerr family by Teukolsky [56] who made the important discovery that the extreme curvature components, relative to a principal null frame, satisfy decoupled, separable, wave equations. These extreme curvature components also turn out to be *gauge invariant* in the sense that small perturbations of the frame lead to quadratic errors in their expression. The full extent of what could be done by mode analysis, in both approaches, can be found in Chandrasekhar's book [11]. Chandrasekhar also introduced (see [10]) a transformation theory relating the two approaches. More precisely, he found a transformation which connects the Teukolsky equations to the Regge-Wheeler one. The full mode stability, i.e. lack of exponentially growing modes, for the Teukolsky equation in Kerr is due to Whiting [57] (see also [52] for a stronger quantitive version).

1.1.2. The scalar linear wave equation in Kerr Mode stability is far from establishing even boundedness of solutions to the linearized equations and falls thus far short of what is needed to understand nonlinear stability. To achieve that and, in addition, to derive realistic decay estimates, one needs an entirely different approach based on a far reaching extension of the classical vectorfield method[5] used in the proof of the nonlinear stability of Minkowski [17].

[4]This presupposes the existence of an event horizon. Note that the existence of such an event horizon can only be established upon the completion of the proof of the conjecture.

[5]Method based on the symmetries of Minkowski space to derive uniform, robust, decay for nonlinear wave equations, see [32], [34], [35], [16].

The new method, which has emerged in the last 18 years in connection to the study of boundedness and decay for the scalar wave equation in the Kerr space $\mathcal{K}(a,m)$, compensates for the lack of enough Killing and conformal Killing vectorfields on a Schwarzschild or Kerr background by introducing new vectorfields whose deformation tensors have coercive properties in various, not necessarily causal, regions of spacetime. The starting and most demanding part of the new method is the derivation of a global, simultaneous, *Energy-Morawetz* estimate which degenerates in the trapping region. This task is somewhat easier in Schwarzschild, or for axially symmetric solutions in Kerr, where the trapping region is restricted to a smooth hypersurface. The first such estimates, in Schwarzschild, were proved by Blue and Soffer in [5], [6] followed by a long sequence of further improvements in [7], [22], [44] etc.

In the absence of axial symmetry the derivation of an Energy-Morawetz estimate in $\mathcal{K}(a,m)$ for $|a/m| \ll 1$ requires a more refined analysis involving both the vectorfield method and Fourier or mode decompositions, see Tataru-Tohaneanu [55] for the first full quantitative decay result (see also Dafermos-Rodnianski [24] for boundedness of solutions). The derivation of such an estimate in the full sub-extremal case $|a| < m$ is even more subtle and was achieved by Dafermos-Rodnianski-Shlapentokh-Rothman [25]. A purely physical space proof the Energy-Morawetz estimate for small $|a/m|$, which extends the classical vectorfield method to include second order operators (in this case the Carter operator [9]) was pioneered by Andersson-Blue in [2]. Their approach has the usual advantages of the classical vectorfield method, i.e it is robust with respect to perturbations, which is the very reason we pursue it in [28].

Once the energy-Morawetz estimate is derived, one can combine it with local estimates near the horizon, based on its red shift properties, as introduced in [22], and r^p weighted estimates, first introduced[6] in [23], to derive realistic uniform decay properties of the solutions.

1.1.3. Stability of Schwarzschild The first application of the new vectorfield method to the linearized Einstein equation near Schwarzschild space, due to Dafermos, Holzegel and Rodnianski, appeared in [19]. The paper makes use of a physical space version of Chandrasekhar's transformation to provide realistic boundedness and decay of solutions of the Teukolsky equations using the new vectorfield method. This method, of estimating the extreme curvature

[6]A precursor of the method, based on introducing fractional power weights modifications of the Morawetz conformal Killing vectorfield of Minkowski space, which leads to positive bulk integrals, appeared earlier in [42].

components by passing from Teukolsky to a Regge-Wheeler type equation, to which the vectorfield method can be applied, is important in all future developments in the subject.

The first nonlinear stability result of the Schwarzschild space appears in [39]. In its simplest version, the result can be stated as follows.

Theorem 1.1 (Klainerman-Szeftel [39])**.** *The future globally hyperbolic development of an* axially symmetric, polarized*, asymptotically flat initial data set, sufficiently close (in a specified topology) to a Schwarzschild initial data set of mass* $m_0 > 0$*, has a complete future null infinity* $\mathcal{I}^+$ *and converges in its causal past* $\mathcal{J}^-(\mathcal{I}^+)$ *to another nearby Schwarzschild solution of mass* m_∞ *close to* m_0*.*

The restriction to axial polarized perturbations is the simplest assumption which ensures that the final state is itself Schwarzschild and thus avoids the additional complications of the Kerr stability problem which we discuss below. We note that in a just released preprint, the authors in [26] dispense of any symmetry assumptions by properly preparing a co-dimension 3 subset of the initial data such that the final state is still Schwarzschild.

1.1.4. The case of Kerr with small angular momentum The first breakthrough result on the linear stability of Kerr, for $|a|/m \ll 1$, is due to to Ma [43], see also [20]. Both results extend the method of [19], mentioned above, by providing estimates to the extreme linearized curvature components via a similar Chandrasekhar transformation which takes the Teukolsky equations to a generalized Regge-Wheeler (gRW) equation. The passage to a tensorial version of gRW equation, in the fully nonlinear setting, plays an essential role in our work, see [28] and the discussion below. The result of [20] was partially extended to the full subextremal range by Y. Shlapentokh-Rothman and R. Teixeira da Costa [53].

The first linear stability results for the full linearized Einstein vacuum equations near $Kerr(a, m)$, for $|a|/m \ll 1$, appear in [1] and [29]. The first paper, based on the NP formalism, builds on the results of [43] and [20] while the second paper is based on a version of the metric formalism. Though the ultimate relevance of these papers to nonlinear stability remains open they are both remarkable results in so far as they deal with difficulties that looked insurmountable even ten years ago.

Though it does not quite fit in the framework of our discussion, we would like to end this quick survey of results by mentioning the striking achievement of Hintz and Vasy [30] on the nonlinear stability of the stationary part of Kerr-de Sitter with small angular momentum, see [30]. The result does not concern EVE but rather the Einstein vacuum equation with a strictly positive cosmological constant

$$\mathbf{R}_{\alpha\beta} + \Lambda \mathbf{g}_{\alpha\beta} = 0, \qquad \Lambda > 0. \tag{1.2}$$

It is important to note that, despite the fact that, formally, (1.1) is the limit[7] of (1.2) as $\Lambda \to 0$, the global behavior of the corresponding solutions is radically different[8].

The main simplification in the case of stationary solutions of (1.2) is that the expected decay rates of perturbations near Kerr-de Sitter is exponential, while in the case $\Lambda = 0$ the decay is lower degree polynomial[9], with various components of tensorial quantities decaying at different rates, and the slowest decaying rate[10] being no better than t^{-1}. Despite this major simplification, the work of Hintz and Vasy is the first general nonlinear stability result in GR where one has to prove asymptotic stability towards a family of solutions, i.e. full quantitative convergence to a final state close, but different from the initial one[11]. It is also fair to say that the work of Hintz-Vasy deals with some of the geometric features of the black hole stability problem without having to worry about the considerable analytic difficulties of the physically relevant Kerr stability problem. On the other hand, as it is apparent in our work here, the geometric and analytic difficulties of the Kerr stability problem are highly entangled and cannot be neatly separated as in the $\Lambda > 0$ case. Thus the geometric framework of our work is very different from that of [30].

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[7]To pass to the limit requires one to understand all global in time solutions of (1.2) with $\Lambda = 1$, not only those which are small perturbations of Kerr-de Sitter, treated by [30].

[8]Major differences between formally close equations occur in many other contexts. For example, the incompressible Euler equations are formally the limit of the Navier-Stokes equations as the viscosity tends to zero. Yet, at fixed viscosity, the global properties of the Navier-Stokes equations are radically different from that of the Euler equations.

[9]While there is exponential decay in the stationary part treated in [30], note that lower degree polynomial decay is expected in connection to the stability of the complementary causal region (called cosmological or expanding) of the full Kerr-de Sitter space, see e.g. [49].

[10]Responsible for carrying gravitational waves at large distances so that they are detectable.

[11]The nonlinear stability of Schwarzschild result in [39] is, one the other hand, the first such result in the more demanding case of asymptotically flat solutions of EVE.

1.2. Kerr stability for small angular momentum

1.2.1. Simplest version of our main theorem The simplest version of our main theorem can be stated as follows.

Theorem 1.2 (Main Theorem, first version)**.** *The future globally hyperbolic development of a general, asymptotically flat, initial data set, sufficiently close (in a suitable topology) to a* $Kerr(a_0, m_0)$ *initial data set, for sufficiently small* a_0/m_0*, has a complete future null infinity* $\mathcal{I}^+$ *and converges in its causal past* $\mathcal{J}^-(\mathcal{I}^+)$ *to another nearby Kerr spacetime* $Kerr(a_\infty, m_\infty)$ *with parameters* (a_∞, m_∞) *close to the initial ones* (a_0, m_0)*.*

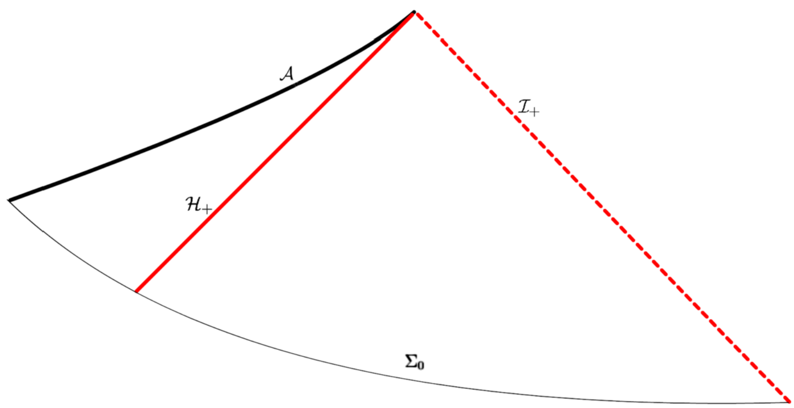

Figure 1: The Penrose diagram of the final space-time in the Main Theorem with complete future null infinity $\mathcal{I}^+$ and future event horizon $\mathcal{H}^+$.

Our proof rests on the following major ingredients.

1. A formalism to derive tensorial versions of the Teukolsky and Regge-Wheeler type equations in the full nonlinear setting.
2. An analytic mechanism to derive estimates for solutions of these.
3. A dynamical mechanism to identify the final values of (a_∞, m_∞).
4. A dynamical mechanism for finding the right gauge conditions in which convergence to the final state takes place.
5. A precisely formulated continuity argument, based on a grand bootstrap scheme, which assigns to all geometric quantities involved in the process specific decay rates, which can be dynamically recovered from the initial conditions by a long series of estimates, and thus ensure convergence to a final Kerr state.

6. The continuity argument is based on the crucial concept of finite, GCM admissible spacetimes $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M} \cup {}^{(top)}\mathcal{M}$, see Figure 2, whose defining characteristic is its spacelike, GCM boundary Σ_*. Note that the boundaries ${}^{(ext)}\mathcal{M} \cap {}^{(top)}\mathcal{M}$ and ${}^{(int)}\mathcal{M} \cap {}^{(top)}\mathcal{M}$ are timelike[12] and that ${}^{(top)}\mathcal{M}$ is needed to have the entire space $\mathcal{M}$ causal. The regions ${}^{(ext)}\mathcal{M}$ and ${}^{(int)}\mathcal{M}$ are separated by the timelike hypersurface $\mathcal{T}$ and the spacelike boundary $\mathcal{A}$ is beyond the future horizon $\mathcal{H}_+$ of the limiting space. Finally the region $\mathcal{L}_0$, is the initial data layer in which $\mathcal{M}$ is prescribed as a solution of the Einstein vacuum equations.

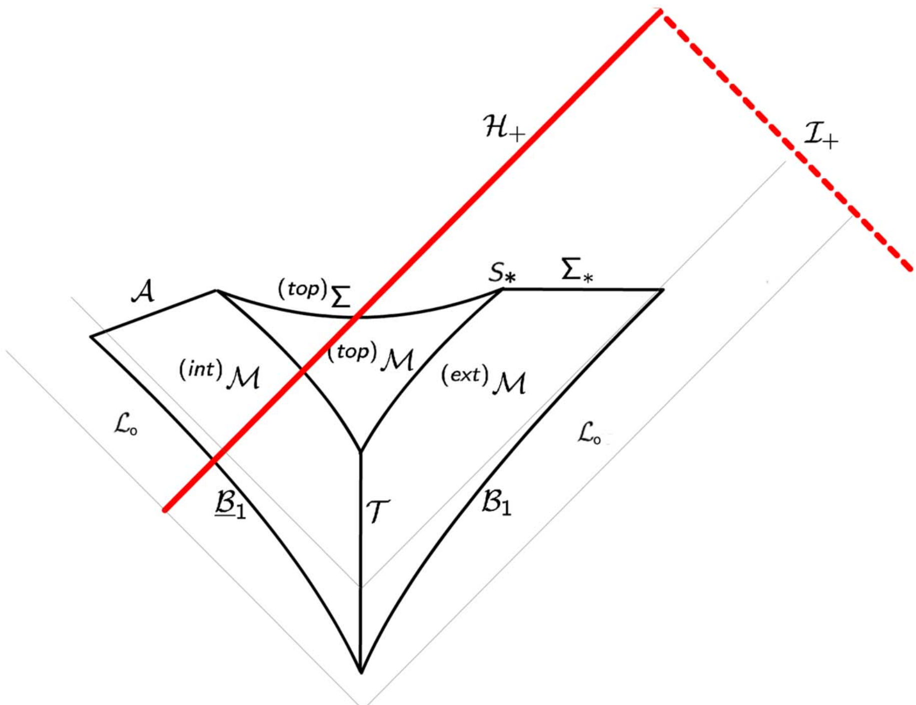

Figure 2: The GCM admissible space-time $\mathcal{M}$.

Remark 1.3. *As in [39] we construct spacetimes starting from the initial layer $\mathcal{L}_0$, see Figure 2. The initial layers we consider are those which arise from the evolution of asymptotically flat initial data sets[13], supported on a spacelike hypersurface Σ_0. Thus the future development of an initial layer $\mathcal{L}_0$*

[12] Asymptotically null as we pass to the limit.

[13] As constructed in the works [36], [37], [8] and [51].

should be interpreted as a future development of the corresponding initial data set, see Definition 3.20.

Remark 1.4. *As mentioned above the region* ${}^{(top)}\mathcal{M}$ *is only needed as causal completion to* ${}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$ *and can be easily determined by a standard local existence result once the geometry of* ${}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$ *is controlled. For that reason we will mostly ignore it in this introduction. We also note, as in [39], that* ${}^{(ext)}\mathcal{M}$ *is by far the harder region to control, even though* ${}^{(int)}\mathcal{M}$ *contains the degenerate region of trapped null geodesics.*

Here is a short summary of how we deal with these issues.

- In [40] and [41] we have provided a framework for dealing with the issue (4), by constructing generalized notions of generally covariant modulated (GCM) spheres[14] in the asymptotic region of a general perturbation of Kerr. The paper [41] also contains a definition of the angular momentum for GCM spheres. These results are needed here in connection to the construction of the essential boundary Σ_*, see also[15] [50].
- In part I of [28], see also the older version in [27], we deal with issue (1) by developing a geometric formalism of non-integrable horizontal structures, well adapted to perturbations of Kerr, and use it to derive the generalized Regge-Wheeler (gRW) equation in the context of general perturbations of Kerr. In the linear case, complex scalar versions of such equations were first derived in [43], see also [20], based on an extension of the physical space Chandrasekhar type transformation introduced in [10] and first exploited in [19], in the context of the linearized Einstein vacuum equations near Schwarzschild space.
- In Part II of [28] we deal with issue (2) by deriving estimates for gRW using an extension of the classical vectorfield method, based on commutation with second order operators. In the context of the standard scalar wave equation in Kerr, such an approach was developed by Andersson and Blue in their important paper [2]. We note that the results on decay in [43] and [20], on the other hand, depend heavily on mode decompositions for the linearized gRW equations in Kerr, an approach whose generalization to the full nonlinear setting seems to present substantial

[14]Generalizing those used in the nonlinear stability of Schwarzschild in the polarized case, see [39].

[15]The result in [50], where Σ_* is actually constructed from these GCM pieces, generalizes the construction of GCMH from [39] to the non-polarized case needed here.

difficulties. Such decompositions were also essential in the recent remarkable result [53] which derives decay estimates for solutions of the gRW equation in Kerr(a,m) for the full subextremal case $|a| < m$.

- The nonlinear terms present in the full version of the gRW equation derived in [28], see also [27], as well as those generated by commutation with vectorfields and second order Carter operator, are treated in a similar spirit as the treatment of the nonlinear terms in [39], by showing that they verify a favorable null type structure.
- In the present paper we state a precise version of our main Theorem 1.2, define the main objects and provide a roadmap for the entire proof. We also deal, in detail, with the issues (3) and (5) as follows:
  - We introduce the concept of PG structures (Chapter 2), which allows us to extend, in perturbations of Kerr, the main features embodied by the principal null frames in Kerr.
  - We define (Chapter 3) the notion of finite, GCM admissible, spacetimes $\mathcal{M}$, whose defining feature, as mentioned above, is given by their future, spacelike boundary Σ_*, see Figure 2. This hypersurface is foliated by GCM spheres, as defined in [40], [41], and is used to initialize the basic PG structure and sphere foliations[16] of $\mathcal{M}$.
  - We provide a full set of bootstrap assumptions (Chapter 3) on these admissible spacetimes. These are of two types: assumptions on decay, involving derivatives up to order k_{small} for all components of Ricci and curvature coefficients, relative to the adapted frame, and assumptions on boundedness, involving derivatives up to $k_{large} = 2k_{small} + 1$.
  - Relying on the estimates for the extreme components of the curvature, derived in Part II of [28], and the GCM conditions on Σ_* we derive here complete decay estimates for all other Ricci and curvature components, thus improving[17] the bootstrap assumptions on decay.
  - To improve the bootstrap assumptions on boundedness, we cannot rely on the PG frame, which loses derivatives, but need instead to

[16] We note that the null frames of the PG structure are not adapted to the sphere foliation, in the same way that the principal null frame in Kerr is not adapted to the $S(t,r)$ spheres in the Boyer-Lindquist coordinates. They do however verify specific compatibility assumptions described in this paper in connection to what we call principal geodesic structures, see Section 2.4.

[17] By showing that they depend only on the smallness of the initial perturbation.

use a different frame, which we call principal temporal (PT). In Chapter 9 of this paper, we show how to control the PT frame at the highest level of derivatives, conditional on boundedness estimates for the curvature. The estimates for the latter, hyperbolic in nature[18], are treated in [28].

– Finally, we show how $\mathcal{M}$ can be extended to a strictly larger GCM admissible spacetime $\widetilde{\mathcal{M}}$ and thus complete the continuation argument mentioned in item (5) above.

1.2.2. Short comparison with Theorem 1.1 Our proof follows the main outline of [39] in which we have settled the conjecture in the restricted class of polarized perturbations, see Theorem 1.1.

Besides fixing the angular momentum to be zero, the polarization assumption made in [39] led to important conceptual and technical simplifications. The most important challenges to extend the result in [39] to unconditional perturbations of Kerr are as follows.

a. The lack of integrability[19] of the PG structures of $\mathcal{M}$, which inherits the lack of integrability of the principal null frames in Kerr.
b. The structure and derivation of the gRW equations are considerable more complex.
c. The vectorfield approach used in [39] is no longer appropriate to Morawetz estimates in perturbations of Kerr.
d. The construction of GCM surfaces in the general setting is both conceptually and technically more involved than in the polarized case. For a comprehensive discussion of these we refer the reader to the introduction of [40] and [41].
e. The derivation of decay estimates in the general setting setting is both conceptually and technically more involved than in the polarized case. This is ultimately due to the lack of integrability of the PG structures which are incompatible with nonlocal estimates, such as integration of Hodge type elliptic systems on S-foliations, see Remark 1.5 below. To avoid this difficulty in our work we need to construct a secondary integrable structure and a mechanism to go back and forth from the integrable to the non-integrable one.

[18]The estimates for the PT frame, assuming the curvature as given, are based on the GCM assumptions on Σ_* and transport equations. The curvature estimates, derived in Part II of [28], are based instead on Energy-Morawetz and r^p-weighted estimates as well as by treating the null Bianchi equations as a Maxwell type system.

[19]We note that the PT frames, used in Chapter 9, are also non-integrable.

f. Unlike in [39], where both the decay and boundedness estimates are based on the same integrable frame, we use here two different[20] types of non-integrable frames: PG frames for decay and PT frames for boundedness.

We refer the reader to the introduction of [28] for a thorough discussion of the items b) and c) and [40], [41] for the item d).

Remark 1.5. *In connection to point e) above it is important to remark that various types of S-foliations and their adapted null frames play a a fundamental role in many of the major mathematical results in GR, starting with [17] but also [15], [36], [38], [19], [21], [39] and others. S-foliations also play an important role in applications to fluids such as pioneered by Christodoulou in [14]. Our work here is the first where S-foliations are replaced by the more complex geometric structures as mentioned in point e).*

In what follows we describe the main conceptual innovations to deal with a) and e) in this paper. We start by describing the geometric properties of our admissible spacetime $\mathcal{M}$ in Figure 2.

1.3. Main geometric structures

As mentioned above, both the results of [17] on the nonlinear stability of the Minkowski space and the result of [39] on the nonlinear stability of Schwarzschild under polarized perturbations rely on a geometric formalism based on *S-foliations*, i.e. foliations by topological 2-spheres, and *adapted null fames* $(e_3, e_4, \mathcal{H})$, with e_3, e_4 forming a null pair and $\mathcal{H}$, the horizontal space of vectors orthogonal to both, tangent to the S-foliation. In both works, this geometric structure was constructed such that it most resembles the situation in the unperturbed case. Thus, for example, in the proof of stability of the Minkowski case [17], all components of the curvature tensor, decomposed relative to the frame, converge to zero – albeit at different rates. The same holds true in [39], after the ρ components of the curvature is properly normalized by subtracting its Schwarzschild value.

By contrast, the principal null vectors (e_3, e_4) in Kerr, relative to which the curvature tensor takes a simple form, do not lead to integrable horizontal structures, i.e. the *horizontal* space of vectors $\mathcal{H}$ perpendicular to (e_3, e_4) is not integrable in the sense of Frobenius. Thus a geometric formalism based on S-foliations and adapted frames, as developed in [17] and used in many

[20] In fact, we use yet another frame, namely the integrable frame associated to Σ_*.

other important works in mathematical GR (see Remark 1.5), is no longer appropriate in perturbations of Kerr. The Newman-Penrose (NP), see [45], circumvents this difficulty by working with principal null pairs (e_3, e_4) and a specified basis[21] (e_1, e_2) for $\mathcal{H}$. It thus reduces all calculations to equations involving the Christoffel symbols of the frame. This un-geometric feature of the formalism makes it difficult to use it in the nonlinear setting of the Kerr stability problem. Indeed complex calculations, such those needed to derive the nonlinear analogue of gRW, mentioned above, depend on higher derivatives of all connection coefficients of the NP frame rather than only those which are geometrically significant. This seriously affects and complicates the structure of non-linear corrections and makes it difficult to avoid artificial gauge type singularities.

1.3.1. General horizontal formalism In our work we rely instead on a tensorial approach, based on horizontal structures which closely mimics the calculations done in integrable settings while maintaining the important diagonalizable properties of the principal directions. This allows us to maintain, with minimal changes, the geometric formalism of [17] widely used today in mathematical GR. The formalism, developed in detail in [28], is succinctly reviewed in Section 2.1.1. It is used[22] in Chapter 5 of [28] to derive a tensorial, nonlinear version of the gRW equation of [43]. The idea is very simple: we define Ricci coefficients exactly as in [17], relative to an arbitrary basis of vectors (e_1, e_2) of $\mathcal{H}$,

$$
\begin{aligned}
&\underline{\chi}_{ab} = \mathbf{g}(\mathbf{D}_a e_3, e_b), && \chi_{ab} = \mathbf{g}(\mathbf{D}_a e_4, e_b), && \zeta_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_a e_4, e_3),\\
&\underline{\eta}_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_3, e_a), && \eta_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_4, e_a), &&\\
&\underline{\xi}_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_3, e_a), && \xi_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_4, e_a), &&\\
&\underline{\omega} = \frac{1}{4}\mathbf{g}(\mathbf{D}_3 e_3, e_4), && \omega = \frac{1}{4}\mathbf{g}(\mathbf{D}_4 e_4, e_3), &&
\end{aligned}
$$

and remark that, due to the lack of integrability of $\mathcal{H}$, the null fundamental forms χ and $\underline{\chi}$ are no longer symmetric. They can be both decomposed as follows

$$
\chi_{ab} = \frac{1}{2}\mathrm{tr}\,\chi\delta_{ab} + \frac{1}{2}\in_{ab}\ {}^{(a)}\mathrm{tr}\chi + \widehat{\chi}_{ab}, \qquad \underline{\chi}_{ab} = \frac{1}{2}\mathrm{tr}\,\underline{\chi}\delta_{ab} + \frac{1}{2}\in_{ab}\ {}^{(a)}\mathrm{tr}\underline{\chi} + \widehat{\underline{\chi}}_{ab},
$$

[21] Or rather the complexified vectors $m = e_1 + ie_2$ and $\overline{m} = e_1 - ie_2$.
[22] See also [27].

where the new scalars ${}^{(a)}\mathrm{tr}\chi$, ${}^{(a)}\mathrm{tr}\underline{\chi}$ measure the lack of integrability of the horizontal structure. The null curvature components are also defined as in [17],

$$\alpha_{ab} = \mathbf{R}_{a4b4}, \qquad \beta_a = \frac{1}{2}\mathbf{R}_{a434}, \qquad \underline{\beta}_a = \frac{1}{2}\mathbf{R}_{a334}, \qquad \underline{\alpha}_{ab} = \mathbf{R}_{a3b3},$$
$$\rho = \frac{1}{4}\mathbf{R}_{3434}, \qquad {}^*\rho = \frac{1}{4}\,{}^*\mathbf{R}_{3434}.$$

The null structure and null Bianchi equations can then be derived as in the integrable case. The only new features are the presence of the scalars ${}^{(a)}\mathrm{tr}\chi$, ${}^{(a)}\mathrm{tr}\underline{\chi}$ in the equations. Finally we note that the equations acquire additional simplicity if we pass to complex notations[23],

$$\begin{aligned}
&A := \alpha + i\,{}^*\alpha, \qquad B := \beta + i\,{}^*\beta, \qquad P := \rho + i\,{}^*\rho, \qquad \underline{B} := \underline{\beta} + i\,{}^*\underline{\beta},\\
&\underline{A} := \underline{\alpha} + i\,{}^*\underline{\alpha},\\
&X := \chi + i\,{}^*\chi, \qquad \underline{X} := \underline{\chi} + i\,{}^*\underline{\chi}, \qquad H := \eta + i\,{}^*\eta, \qquad \underline{H} := \underline{\eta} + i\,{}^*\underline{\eta},\\
&Z := \zeta + i\,{}^*\zeta, \qquad \Xi := \xi + i\,{}^*\xi, \qquad \underline{\Xi} := \underline{\xi} + i\,{}^*\underline{\xi}.
\end{aligned}$$

Note that, in particular, $\mathrm{tr}X = \mathrm{tr}\,\chi - i\,{}^{(a)}\mathrm{tr}\chi$, $\mathrm{tr}\underline{X} = \mathrm{tr}\,\underline{\chi} - i\,{}^{(a)}\mathrm{tr}\underline{\chi}$.

1.3.2. Principal geodesic structures The geometric formalism based on these non-integrable frames, though perfectly adapted to calculations, is insufficient to derive estimates, which often involves the integration of Hodge type elliptic systems on S-foliations. It is for this reason that we develop here a more complex formalism which combines S-foliations with non-integrable frames. This approach requires in fact two pairs of frames, the non-integrable one which most resemble the principal frame of Kerr, and a secondary one which is adapted to the S-foliation. To estimate various quantities we need to constantly pass from one frame to the other. This is done according to the general change of frames formula

$$\begin{aligned}
\lambda^{-1}e_4' &= e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3,\\
e_a' &= \left(\delta_a^b + \frac{1}{2}\underline{f}_a f^b\right) e_b + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right) e_3, \qquad (1.3)\\
\lambda e_3' &= \left(1 + \frac{1}{2}f\cdot\underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right) e_3 + \left(\underline{f}^b + \frac{1}{4}|\underline{f}|^2 f^b\right) e_b + \frac{1}{4}|\underline{f}|^2 e_4,
\end{aligned}$$

[23] The dual here is taken with respect to the fully antisymmetric horizontal 1 tensor $\in_{ab}$.

where $f, \underline{f}$ are arbitrary 1 forms and λ is an arbitrary real scalar, see Lemma 2.10.

The transformation formulas (1.3) provide the most general way of passing between two different null frames. They play an essential role all through our work, most prominently in the construction of GCM surfaces in [40], [41].

At the heart of this dual geometric formalism lies the following crucial definition, see Definition 2.16.

Definition 1.6 (PG structure). *An outgoing principal geodesic (PG) structure consists of a null pair (e_3, e_4) and the induced horizontal structure $\mathcal{H}$, together with a scalar function r such that*

1. *e_4 is a null outgoing geodesic vectorfield, i.e. $\mathbf{D}_4 e_4 = 0$,*
2. *r is an affine parameter, i.e. $e_4(r) = 1$,*
3. *the gradient of r, given by $N = \mathbf{g}^{\alpha\beta}\partial_\beta r \partial_\alpha$, is perpendicular to $\mathcal{H}$.*

A similar concept of incoming PG structure is defined by interchanging the roles of e_3, e_4

1.3.3. Initialization of PG structures

Such structures are initialized in our work on the boundary Σ_*, see Figure 2. This leads to the following definitions, see details in Section 2.5.

Definition 1.7. *A framed hypersurface consists of a set $(\Sigma, r, (\mathcal{H}, e_3, e_4))$ where*

1. *Σ is smooth a hypersurface in $\mathcal{M}$,*
2. *(e_3, e_4) is a null pair on Σ such that e_4 is transversal to Σ, and $\mathcal{H}$, the horizontal space perpendicular on e_3, e_4, is tangent to Σ,*
3. *the function $r : \Sigma \to \mathbb{R}$ is a regular function on Σ such that $\mathcal{H}(r) = 0$.*

To define an appropriate initial data set we need also to prescribe an additional horizontal 1-form f as follows.

Definition 1.8 (PG-data set). *The boundary data of a PG structure (PG-data set) consists of:*

1. *a framed hypersurface $(\Sigma, r, (\mathcal{H}, e_3, e_4))$ as in Defintion 1.7,*
2. *a fixed 1-form f on the spheres S of the r-foliation of Σ verifying the condition*

$$b_\Sigma |f|^2 < 4 \quad on \quad \Sigma,$$

where b_Σ is such that $\nu = e_3 + b_\Sigma e_4$ is tangent to Σ.

The following is precisely Proposition 2.57 in the main text.

Proposition 1.9. *Given a PG data set $(\Sigma, r, (\mathcal{H}, e_3, e_4), f)$ as in Definition 1.8, there exists a unique PG structure $(r', (\mathcal{H}', e_3', e_4'))$ defined in a neighborhood of Σ such that the following hold true*

1. *The function r' is prescribed on Σ by $r' = r$.*
2. *Along Σ, the restriction of the spacetime PG null frame $(\mathcal{H}', e_3', e_4')$ and the given null frame $(\mathcal{H}, e_3, e_4)$ on Σ are related by the transformation formulas (1.3) with transition coefficients $(f, \underline{f}, \lambda)$, where (e_1, e_2) is a fixed, arbitrary, orthonormal basis of $\mathcal{H}$, where f is part of the PG-data set, and where $\underline{f}$ and λ are given by*

$$\lambda = 1, \qquad \underline{f} = -\frac{(\nu(r) - b_\Sigma)}{1 - \frac{1}{4} b_\Sigma |f|^2} f.$$

1.4. GCM initial data sets

The hypersurface Σ_* in Figure 2 is not only a framed hypersurface. It also verifies crucial general covariant modulated (GCM) conditions. Given the importance of these conditions we describe below the main ingredients needed in their definitions. We concentrate first on the boundary S_* of Σ_*, see Figure 2, on which various quantities are initialized and transported along Σ_*.

1.4.1. Last sphere S_* of Σ_* To define the geometry of S_* we need the effective uniformization results derived in [41], which we review in Section 5.1.1. Based on these results, we endow S_* with coordinates (θ, φ) such that the following conditions are verified.

i. The induced metric g on S_* takes the form

$$g = e^{2\phi} r^2 \Big((d\theta)^2 + \sin^2\theta (d\varphi)^2 \Big). \tag{1.4}$$

ii. The functions

$$J^{(0)} := \cos\theta, \quad J^{(-)} := \sin\theta \sin\varphi, \quad J^{(+)} := \sin\theta\cos\varphi, \tag{1.5}$$

verify the balanced conditions

$$\int_{S_*} J^{(p)} = 0, \quad p = 0, +, -. \tag{1.6}$$

Recall that Σ_* is assumed to be a framed hypersurface in the sense of Definition 1.7 and thus endowed with a frame $(e_3, e_4, \mathcal{H})$ and function r on it such that $\mathcal{H}(r) = 0$.

Definition 1.10. *We define the parameters (m, a) of S_* by the formulas*

$$\frac{2m}{r} = 1 + \frac{1}{16\pi}\int_{S_*} tr\chi tr\underline{\chi}, \tag{1.7}$$

and

$$a \quad := \quad \frac{r^3}{8\pi m}\int_{S_*} J^{(0)}\, curl\,\beta. \tag{1.8}$$

(1.7) *is the usual Hawking mass of S_* while* (1.8) *was introduced in [41].*

1.4.2. GCM conditions for Σ_* The coordinates (θ, φ) on S_* and the $\ell = 1$ basis $J^{(p)}$ are extended to Σ_* by setting

$$\nu(\theta) = \nu(\varphi) = 0, \qquad \nu(J^{(p)}) = 0, \quad p = 0, +, -, \tag{1.9}$$

where $\nu = e_3 + b_* e_4$ is tangent to Σ_* and normal to the r-foliation on Σ_*. We also extend the parameters (a, m) to be constant along Σ_*.

We are now ready to define the crucial concept of a GCM hypersurface.

Definition 1.11 (GCM hypersurface)**.** *Consider a framed hypersurface Σ_* with end sphere S_*, coordinates (θ, φ), and functions $J^{(0)}$, $J^{(+)}$ and $J^{(-)}$ defined as in* (1.4)*–*(1.9)*. Σ_* is called a GCM hypersurface if in addition the following conditions*[24] *are verified.*

1. *On any sphere S of the r-foliation of Σ_*, the following holds*

$$\begin{aligned}
tr\chi &= \frac{2}{r},\\
tr\underline{\chi} &= -\frac{2(1-\frac{2m}{r})}{r} + \underline{C}_0 + \sum_{p=0,+,-}\underline{C}_p J^{(p)},\\
\mu &= \frac{2m}{r^3} + M_0 + \sum_{p=0,+,-} M_p J^{(p)},\\
\int_S J^{(p)}\, div\,\eta = 0, &\qquad \int_S J^{(p)}\, div\,\underline{\xi} = 0, \qquad p = 0, +, -,
\end{aligned} \tag{1.10}$$

[24]The scalar $\mu := -\mathrm{div}\,\zeta - \rho + \frac{1}{4}\widehat{\chi}\cdot\widehat{\underline{\chi}}$ is the familiar mass aspect function, as in [17] and [39].

$$\overline{b_*} = -1 - \frac{2m}{r},$$

where $\underline{C}_0$, $\underline{C}_p$, M_0, M_p *are scalar functions on* Σ_* *constant along the leaves of the foliation, and* $\overline{b_*}$ *denotes the average of* b_* *on the spheres foliating* Σ_*.

2. *In addition, we have on the last sphere* S_* *of* Σ_*

$$(1.11) \qquad tr\underline{\chi} = -\frac{2(1-\frac{2m}{r})}{r}, \qquad \int_{S_*} J^{(p)}\, div\,\beta = 0, \quad p = 0, +, -,$$

as well as

$$(1.12) \qquad \int_{S_*} J^{(+)}\, curl\,\beta = 0, \qquad \int_{S_*} J^{(-)}\, curl\,\beta = 0.$$

Remark 1.12. *Given the five degrees of freedom of the transition parameters* $(f, \underline{f}, \lambda)$ *in the general change of frame formula* (1.3) *we expect to be able to impose five GCM conditions on a sphere* $S \subset \Sigma_*$. *Since the frame of* Σ_* *is tangent to its* S*-foliation we implicitly have* ${}^{(a)}tr\chi = {}^{(a)}tr\underline{\chi} = 0$. *It would be natural to impose Schwarzschildian values for* $tr\,\chi, tr\,\underline{\chi}$ *and* μ, *to account for the remaining three degrees of freedom. This would lead however to a differential system in* $(f, \underline{f}, \lambda)$ *which is not solvable, due to the presence of a kernel and a co-kernel at the level of* $\ell = 1$ *modes. We are thus obliged to relax these conditions by imposing, in the case of* $tr\underline{\chi}$ *and* μ, *Schwarzschildian values only for the* $\ell \geq 2$ *modes, see* (1.10). *The remaining degrees of freedom allow us to prescribe also the* $\ell = 1$ *modes of* $div\underline{\xi}$ *and* $div\eta$, *as in* (1.10). *These conditions on the* $\ell = 1$ *modes correspond in fact at the level of* $(f, \underline{f}, \lambda)$ *to ODEs for the* $\ell = 1$ *modes of* $div\, f$ *and* $div\,\underline{f}$ *along*[25] Σ_*. *As a consequence, we can freely prescribe these* $\ell = 1$ *modes on* S_*, *which allows us to obtain* (1.11) *on* S_*. *Using the additional freedom of rigid rotations for frames on* S_* *we can also ensure that* (1.12) *holds. The remaining condition on* b_* *is related to the freedom to choose the hypersurface* Σ_*.

1.5. GCM admissible spacetimes

We are now ready to define our GCM admissible spacetime, concept of fundamental importance in our proof. As can be seen in Figure 2, $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M} \cup {}^{(top)}\mathcal{M}$. Each of the domains ${}^{(ext)}\mathcal{M}$, ${}^{(int)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$ are endowed

[25] Our first GCM result, in [40], is based in fact on prescribing the $\ell = 1$ modes of $\mathrm{div}\, f$ and $\mathrm{div}\,\underline{f}$.

with a PG structure, all ultimately induced by Σ_*. The crucial structure is that of ${}^{(ext)}\mathcal{M}$. Once it is fixed, those of ${}^{(int)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$ can be easily derived.

1.5.1. The GCM-PG data set on Σ_* To initialize the PG structure of ${}^{(ext)}\mathcal{M}$, according to Proposition 1.9, we assume not only that Σ_*, in Figure 2, is a GCM hypersurface, as in Definition 1.11, but also that it is endowed with a 1-form f which makes it into a GCM-PG data set $\big(\Sigma_*, r, (e_3, e_4, \mathcal{H}), f\big)$. In addition Σ_* is specified by a function u such that $u = c_* - r$, for some constant c_* to be specified.

Here are therefore the main features of the boundary Σ_*:

- $\big(\Sigma_*, r, (e_3, e_4, \mathcal{H}), f\big)$ is a GCM-PG data set, in the sense of Definitions 1.8 and 1.11, with r decreasing from its value r_* on S_*.
- the parameters a, m are defined by (1.7) and (1.8),
- the transition parameter f is given by $f = \frac{a}{r}d\varphi$ on S_* and transported to Σ_* by $\nabla_\nu(rf) = 0$.
- Along Σ_* we have $u = c_* - r$ with $c_* = 1 + r(S_1)$ where $S_1 = \Sigma_* \cap \mathcal{B}_1$, see Figure 2.
- The function r verifies a dominance condition on S_*, see (3.8),

$$r_* \sim u_*^{1+\delta_{dec}}, \tag{1.13}$$

where u_* and r_* denote respectively the value of u and r on S_*.

1.5.2. The PG structures and S foliations of ${}^{(ext)}\mathcal{M}$, ${}^{(int)}\mathcal{M}$

- The outgoing PG structure on ${}^{(ext)}\mathcal{M}$ is fixed from the GCM-PG data set of Σ_*, with the help of Proposition 1.9. ${}^{(ext)}\mathcal{M}$ is also endowed with the $S(u, r)$ foliation where u is extended from Σ_* by setting $e_4(u) = 0$. The hypersurfaces of constant u are timelike[26]. Note also that $u = u_*$ is the hypersurface separating ${}^{(ext)}\mathcal{M}$ from ${}^{(top)}\mathcal{M}$ while $u = u_1$ is the boundary $\mathcal{B}_1$.
- ${}^{(ext)}\mathcal{M}$ terminates at the inner boundary $\mathcal{T} = {}^{(int)}\mathcal{M} \cap {}^{(ext)}\mathcal{M}$. ${}^{(int)}\mathcal{M}$ is endowed with an ingoing PG structure initialized at $\mathcal{T}$, defined starting by renormalizing e_3 on $\mathcal{T}$ and extending it geodesically in ${}^{(int)}\mathcal{M}$. We can also extend r from $\mathcal{T}$ in ${}^{(int)}\mathcal{M}$ by setting $e_3(r) = -1$. We define $\underline{u}$ in ${}^{(int)}\mathcal{M}$ such that it coincides with u on $\mathcal{T}$ and $e_3(\underline{u}) = 0$. The corresponding hypersurfaces are timelike.

[26]They become null at infinity.

- Note that ${}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}$ in Figure 2 is not a causal region. This is ultimately due to the fact that the functions $u, \underline{u}$ are not null but timelike. Thus, see Remark 1.4, the region ${}^{(top)}\mathcal{M}$ is needed as a completion of ${}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}$ to a causal region.
- The black hole parameters (a, m) are extended everywhere in $\mathcal{M}$ to be constant. We also define an ingoing PG structure on ${}^{(top)}\mathcal{M}$ suitably initialized from the outgoing PG structure of ${}^{(ext)}\mathcal{M}$ on $\{u = u_*\}$.

Remark 1.13. *it is important to note that ${}^{(ext)}\mathcal{M}$ comes equipped not only with the PG frame $(e_3, e_4, \mathcal{H})$ but also with the secondary, integrable, frame $(e_3', e_4', \mathcal{H}')$ adapted to the spheres $S(u, r)$, i.e. $\mathcal{H}'$ is tangent to the S spheres. We also have precise formulas*[27] *to pass form one frame to the other whenever needed.*

1.5.3. GCM admissible spacetimes We are now ready to define our central concept which, in addition to the geometric specifications made above for Σ_*, ${}^{(ext)}\mathcal{M}$, ${}^{(int)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$, contains information about decay and boundedness of the linearized[28] Ricci and curvature coefficients. As in [39], we divide these into the sets we denote by Γ_g, Γ_b. For example, Γ_g includes in particular $\widetilde{\mathrm{tr}X}, \widetilde{\mathrm{tr}\underline{X}}, \widehat{X}, Z$, as well as the curvature components[29] rA, rB, rP. The set Γ_b contains in particular the Ricci coefficient $\widehat{\underline{X}}, H, \check{\underline{\omega}}$ and the slow decaying curvature components $\underline{A}$ and $r\underline{B}$. We refer the reader to Definition 2.67 for the precise definition of Γ_b and Γ_g.

Definition 1.14. *A finite space $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M} \cup {}^{(top)}\mathcal{M}$ as in Figure 2 is called a GCM admissible spacetime with parameters (a, m) if the following hold true.*

1. *The boundary Σ_* is endowed with the PG-GCM data set described in Section 1.5.1.*
2. *The domains ${}^{(ext)}\mathcal{M}$, ${}^{(int)}\mathcal{M}$, ${}^{(top)}\mathcal{M}$ are endowed with the PG data sets and S foliations described in Section 1.5.2.*
3. *The linearized*[30] *Ricci and curvature coefficients verify bootstrap assumptions $(\mathbf{BA})_\epsilon$, in ${}^{(ext)}\mathcal{M}$, ${}^{(int)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$, measured in terms*

[27] The passage from the PG frame $(e_3, e_4, \mathcal{H})$ to the integrable one $(e_3', e_4', \mathcal{H}')$ is obtained by the transformation formulas (2.6) with parameters $(f, \underline{f}, \lambda)$ given by (2.12).

[28] Linearization consists for scalar quantities in subtracting Kerr values, but is slightly more subtle for 1-forms. See Definition 2.66.

[29] In fact A, B behave even better, see (3.37) (3.38).

[30] Obtained for scalars by subtracting their Kerr values, expressed in term of the scalar functions (r, θ). The case of 1-forms is slightly more subtle.

of a small parameter ϵ with $\epsilon \gg \epsilon_0$, with ϵ_0 the size of the original perturbation. The bootstrap assumptions are expressed in terms of:

- *uniform decay norms denoted here by $\mathcal{N}^{(Dec)}_{k_{small}}$, for a maximum of k_{small} derivatives,*
- *r^p-weighted supremum norms denoted by $\mathcal{N}^{(Sup)}_{k_{large}}$ for a maximum of k_{large} derivatives,*
- *the number k_{small} is sufficiently large and $k_{large} = 2k_{small} + 1$.*

Thus $(\mathbf{BA})_\epsilon$ can be expressed in the form

$$\mathcal{N}^{(Sup)}_{k_{large}} + \mathcal{N}^{(Dec)}_{k_{small}} \leq \epsilon. \tag{1.14}$$

Remark 1.15. *The bootstrap assumptions for decay $\mathcal{N}^{(Dec)}_{k_{small}} \leq \epsilon$ imply in particular the following decay rates[31] in ${}^{(ext)}\mathcal{M}$.*

$$|\Gamma_g| \leq \epsilon r^{-2} u^{-\frac{1}{2}-\delta_{dec}}, \qquad |\nabla_3 \Gamma_g| \leq \epsilon r^{-2} u^{-1-\delta_{dec}}, \qquad |\Gamma_b| \leq \epsilon r^{-1} u^{-1-\delta_{dec}}.$$

In addition each derivatives ∇, ∇_4 improve the decay in r while each additional ∇_3 derivative keeps the decay unchanged. We express this schematically in the form

$$|\mathfrak{d}^{\leq k}\Gamma_g| \leq \epsilon r^{-2} u^{-\frac{1}{2}-\delta_{dec}}, \qquad |\mathfrak{d}^{\leq k-1}\nabla_3 \Gamma_g| \leq \epsilon r^{-2} u^{-1-\delta_{dec}},$$
$$|\mathfrak{d}^{\leq k}\Gamma_b| \leq \epsilon r^{-1} u^{-1-\delta_{dec}},$$

where $\mathfrak{d} = (\nabla_3, r\nabla_4, r\nabla)$ and $\mathfrak{d}^{\leq k}$ refers to derivatives up to order $k \leq k_{small}$.

1.6. Principal temporal frames

As mentioned earlier, the PG structures are adequate for deriving decay estimates but deficient in terms of loss of derivatives and thus inadequate for deriving boundedness estimates for the top derivatives of the Ricci coefficients. Indeed the ∇_4 equations for $\mathrm{tr}\underline{X}$, $\widehat{\underline{X}}$ and $\underline{\Xi}$ in Proposition 2.19 contain angular derivatives[32] of other Ricci coefficients. Similarly, the same situation occurs for ingoing PG structures where the ∇_3 equations for $\mathrm{tr}X$, $\widehat{X}$, and Ξ are manifestly losing derivatives. Thus, in order to derive boundedness

[31] Here δ_{dec} is a small positive constant.

[32] This loss can be overcome for integrable foliations such as geodesic foliations and double null foliations relying on elliptic Hodge systems on 2-spheres of the foliation, but not for non-integrable structures such as PG structures.

estimates for the top derivatives of the Ricci coefficients, we are forced to introduce new frames which we call principal temporal (PT). These frames are used only in Chapter 9 where they play an essential role.

1.6.1. Outgoing PT structures

Definition 1.16. *An outgoing PT structure $\{(e_3, e_4, \mathcal{H}), r, \theta, \mathfrak{J}\}$ on $\mathcal{M}$ consists of a null pair (e_3, e_4), the induced horizontal structure $\mathcal{H}$, functions (r, θ), and a horizontal 1-form $\mathfrak{J}$ such that the following hold true:*

1. *e_4 is geodesic.*
2. *We have*

$$e_4(r) = 1, \qquad e_4(\theta) = 0, \qquad \nabla_4(q\mathfrak{J}) = 0, \qquad q = r + ai\cos\theta.$$

3. *We have*

$$\underline{H} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J}.$$

An extended outgoing PT structure possesses, in addition, a scalar function u verifying $e_4(u) = 0$.

Definition 1.17. *An outgoing PT initial data set consists of a hypersurface Σ transversal to e_4 together with a null pair (e_3, e_4), the induced horizontal structure $\mathcal{H}$, scalar functions (r, θ), and a horizontal 1-form $\mathfrak{J}$, all defined on Σ.*

The following is precisely Lemma 2.76 in the main text.

Lemma 1.18. *Any outgoing PT initial data set, as in Definition 1.17, can be locally extended to an outgoing PT structure.*

1.6.2. Ingoing PT structures

Definition 1.19. *An ingoing PT structure $\{(e_3, e_4, \mathcal{H}), r, \theta, \mathfrak{J}\}$ on $\mathcal{M}$ consists of a null pair (e_3, e_4), the induced horizontal structure $\mathcal{H}$, functions (r, θ), and a horizontal 1-form $\mathfrak{J}$ such that the following hold true:*

1. *e_3 is geodesic.*
2. *We have*

$$e_3(r) = -1, \qquad e_3(\theta) = 0, \qquad \nabla_3(\overline{q}\mathfrak{J}) = 0, \qquad q = r + ai\cos\theta.$$

3. We have

$$H = \frac{aq}{|q|^2}\mathfrak{J}.$$

An extended ingoing PT structure possesses, in addition, a function $\underline{u}$ verifying $e_3(\underline{u}) = 0$.

Definition 1.20. *An ingoing PT initial data set consists of a hypersurface Σ transversal to e_3 together with a null pair (e_3, e_4), the induced horizontal structure $\mathcal{H}$, scalar functions (r, θ), and a horizontal 1-form $\mathfrak{J}$, all defined on Σ.*

Lemma 1.21. *Any ingoing PT initial data set, as in Definition 1.20, can be locally extended to an ingoing PT structure.*

1.7. Outline of the proof of the main theorem

The detailed version of the main Theorem is found in Section 3.4.3. We sketch below the main steps in our proof. We refer the reader to sections 3.7.1 and 3.7.2 for more details. We also give an outline of the main conclusions of the Theorem.

1.7.1. Control of the initial data The main results on the initial data is stated in Theorem M0 and proved in Section 8.3, based on the initial data and bootstrap assumptions in the initial layer $\mathcal{L}_0$. The result provides estimates for the main linearized quantities restricted to the past boundary $\mathcal{B}_1 \cup \underline{\mathcal{B}}_1$ of our GCM admissible spacetime, see Figure 2. It is important to note that $\mathcal{B}_1, \underline{\mathcal{B}}_1$ are not causal, but rather timelike, with $\mathcal{B}_1$ asymptotically null. They are thus not to be regarded as fixed hypersurfaces where the initial data is prescribed. In fact they change throughout the continuation argument at the heart of the proof, while remaining constrained to the boundary layer $\mathcal{L}_0$. As in [39] the proof of Theorem M0 is quite subtle due to the fact that the spheres of the foliation induced by ${}^{(ext)}\mathcal{M}$ differ substantially from spheres of the initial data layer ${}^{(ext)}\mathcal{L}_0$ along the outgoing direction. This anomalous behavior reflects the difference between the center of mass frames of the final and initial Kerr states and is as such an important feature of our result.

1.7.2. Theorems M1–M5 Given a GCM admissible spacetime, Theorems M1–M5, stated in Section 3.7.1, improve the decay estimates for $k \leq k_{small}$ of the bootstrap assumptions $(\mathbf{BA})_\epsilon$ (see Definition 1.14), i.e. derive estimates in which ϵ is replaced[33] by ϵ_0.

Theorem M1. *Improved decay estimates for $\mathfrak{q}$ and A.* This is our main result concerning the improved decay estimates for A. This is achieved as follows:

- In Part I of [28] we derive a tensorial nonlinear version of the gRW equation. This is a tensorial wave equation for a 2-tensor $\mathfrak{q}$, derived from A by a Chandrasekhar transformation of the form $\mathfrak{q} = \nabla_3^2 A + C_1 \nabla_3 A + C_2 A$, for specific scalar functions C_1, C_2. Unlike in the Schwarzschild case, the wave equation for $\mathfrak{q}$ still contains linear terms in A. Thus, in reality, we have to deal with a coupled wave-transport system for the variables $(\mathfrak{q}, A)$. The linear theory for such systems, in a fixed Kerr background, was derived in [43], see also [20]. The first physical space version of the Chandrasekhar transformation has appeared in [19], in linear perturbations of Schwarzschild. An adapted nonlinear version of the transformation plays an important role in [39].
- It is important to note that, as in [39], the construction of $\mathfrak{q}$ and the estimates for $(\mathfrak{q}, A)$ mentioned below, need to be done in a global frame for $\mathcal{M}$ in which the component H has better decay in ${}^{(ext)}\mathcal{M}$ than the same component in the PG frame of ${}^{(ext)}\mathcal{M}$. Simple transformation formulas allow us to transfer results obtained in the global frame to results in the original PG frames and vice-versa.
- In [28], Part II, we derive boundedness and decay estimates for the coupled system mentioned above. The most demanding part is the derivation of a Morawetz type estimate for $\mathfrak{q}$, a step which requires a nonlinear adaptation of the Anderson-Blue [2] extension of the vectorfield method, mentioned earlier. The papers [43], [20] derive the corresponding estimate, in a fixed Kerr, by appealing to a mode decomposition, method which seems difficult to extend to realistic perturbations of Kerr.

Theorem M2. *Improved estimates[34] for $\underline{A}$ on Σ_*.* This is our main result concerning the improved decay estimates for $\underline{A}$. This is achieved as follows:

[33] Thus establishing that the bounds depend only on the initial conditions and universal constants.

[34] In [39] we relied instead on a nonlinear version of the well known Teukolsky-Starobinsky identity which relates ∇_3^2 derivatives of $\mathfrak{q}$ to four angular derivatives of $\underline{\alpha}$, see Proposition 2.3.15 in [39], from which we can, in principle, recover $\underline{\alpha}$. The non-integrable situation treated here requires in fact that we use both the gRW equations for $\underline{\mathfrak{q}}$ and an appropriate version of the Teukolsky-Starobinsky identities.

- At a linear level, $\underline{A}$ can be treated in a similar manner as A, i.e. we can pass from the Teukolsky equation for $\underline{A}$ to a gRW equations for a 2 tensor $\underline{\mathfrak{q}}$ derived from $\underline{A}$ by a similar second order transformation formula as for A, with e_3 replaced by e_4. The difficulty is that the nonlinear terms in the gRW equation are not so easy to control in view of their low decay in powers of r.
- We rely on a different global frame of $\mathcal{M}$ for which the component $\underline{H}$ has better decay in ${}^{(ext)}\mathcal{M}$ than the same component in the PG frame of ${}^{(ext)}\mathcal{M}$. Even with that property, the structure of the error terms turns out to be more subtle than that of the error terms of the gRW equation for $\mathfrak{q}$, see the relevant discussion in the introduction to [28]. In particular, after taking full advantage of the very special structure of the error term, we can only derive r^p weighted estimates for the couple $(\underline{A}, \underline{\mathfrak{q}})$ in the range $\delta < p < 1 - \delta$, rather than $\delta < p < 2 - \delta$ consistent with linear theory.
- To compensate for the r^p weighted loss mentioned above we take higher $\not{\mathcal{L}}_{\mathbf{T}}$ derivatives of $(\underline{\mathfrak{q}}, \underline{A})$ and show that their fluxes have better decay. Using this observation, we obtain suitable decay for the flux of $(\not{\mathcal{L}}_{\mathbf{T}}^2\underline{\mathfrak{q}}, \not{\mathcal{L}}_{\mathbf{T}}^2\underline{A})$ along Σ_*. Relying on this result, and some version of Teukolsky-Starobinsky providing an identity between $\mathfrak{q}$ and $\underline{A}$, we recover the desired decay estimate for $\underline{A}$ on Σ_* stated in Theorem M2.

We refer the reader to the introduction of [28] for more details.

Theorem M3. *Improved estimates for* (Γ_g, Γ_b) *on* Σ_*. Theorem M3, proved in Chapter 5 of this work, makes use of the improved estimates for α, $\underline{\alpha}$, and $\mathfrak{q}$ of Theorems M1 and M2, to derive improved estimates for all other Ricci and curvature components restricted to Σ_*. Together with Theorem M4, this is the most subtle part of the entire proof in that it depends crucially on the properties of Σ_*, mentioned above, and the difficult estimates of $\alpha, \underline{\alpha}$, and uses in fact almost all other elements of our overall scheme. Here are some of the key ideas in the proof.

- To derive decay estimates for all other quantities along Σ_* it is natural to make use of the transport equations along $\nu = e_3 + b_* e_4$ induced on Σ_* by the null structure and null Bianchi equations.
- Integrating these transport equations starting from $\mathcal{B}_1 \cap \Sigma_*$, where we have smallness information in terms of ϵ_0, is prohibitive since such an integration loses all decay with respect to the u factor. To integrate in the opposite direction, starting from S_*, we need initial conditions on S_*. This is, in a nut-shell, the very reason our GCM conditions were introduced.

– Using the propagation equations along Σ_*, the GCM conditions, in particular those on the final sphere S_*, the Hodge type equations on the S spheres and the information already derived for $\alpha, \underline{\alpha}, \mathfrak{q}$, one can derive improved estimates for all linearized Ricci and curvature coefficients (Γ_g, Γ_b) on Σ_*.

Theorem M4. *Improved estimates for* Γ_g, Γ_b *in* ${}^{(ext)}\mathcal{M}$. Theorem M4, proved in Chapter 6 of this work, extends the estimates proved of Theorem M3 on Σ_* to the entire region ${}^{(ext)}\mathcal{M}$. There are two type of difficulties. The first, type already encountered in [39], is to derive sufficient decay for Γ_g quantities in the regions near the black hole where r is just bounded. The second type of difficulties, are due to the lack of integrability of the PG structure of ${}^{(ext)}\mathcal{M}$. Here are some of the key ideas in the proof. For a more comprehensive discussion of this step we refer to Section 6.4.3.

– Ideally one would use the null structure and Bianchi equations in the e_4 direction to transport information from Σ_* to ${}^{(ext)}\mathcal{M}$. Unfortunately, as it turns out, many of these equations are strongly overshooting in r. As in [39] we devise new renormalized quantities which verify useful transport equations which can be integrated from Σ_* in the e_4 direction.
– In [39] we were able to combine these transport equations with elliptic Hodge systems on the leaves of the S-foliation to derive estimates for the remaining quantities. This becomes a problem in our case due to the lack of integrability of the PG structure. What we do instead is to go back and forth between the PG frame and the integrable frame associated to the $S(u, r)$ spheres, and perform our elliptic estimates on these S-spheres.
– The process generates additional derivatives in the direction of the vectorfield $\mathbf{T}$, analogous to the time translation of Kerr, which turns out to be almost Killing. Fortunately the equations obtained by commutations with $\mathbf{T}$ are no longer overshooting and thus can be integrated directly from Σ_*.
– We combine all these ingredients, making use of the fact that in ${}^{(ext)}\mathcal{M}$ the defining function r is also sufficiently large, to derive estimates for all elements of (Γ_g, Γ_b) in ${}^{(ext)}\mathcal{M}$.

Theorem M5. *Improved estimates for* Γ_g, Γ_b *in* ${}^{(int)}\mathcal{M} \cup {}^{(top)}\mathcal{M}$. This step, proved in Chapter 7, is significantly easier than Theorem M4 due to the fact that ${}^{(int)}\mathcal{M}$ is bounded in r and ${}^{(top)}\mathcal{M}$ is a local existence region. We first control the foliation of ${}^{(int)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$ from the one of ${}^{(ext)}\mathcal{M}$ respectively on $\mathcal{T}$ and $\{u = u_*\}$, and then propagate this control, using transport equations along e_3, respectively to ${}^{(int)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$ thanks to the equations of the corresponding ingoing PG structures.

1.7.3. Extension of GCM admissible spacetimes We end the proof by invoking a continuity argument as in [39], see Section 3.7.2. The argument requires a definition of a set $\mathcal{U}(u_*)$ of GCM admissible spacetimes verifying the bootstrap assumptions $\mathbf{BA}_\epsilon$ such that ϵ and the values (r_*, u_*) of (r, u) on S_* verify

$$\epsilon = \epsilon_0^{\frac{2}{3}}, \qquad r_* = \delta_* \epsilon_0^{-1} u_*^{1+\delta_{dec}}, \tag{1.15}$$

where $\delta_* > 0$ is a small constant satisfying $\delta_* \gg \epsilon$.

Theorem M6. *The set $\mathcal{U}(u_*)$ is not empty.* More precisely, we show that there exists $\delta_0 > 0$ small enough such that, for sufficiently small constants $\epsilon_0 > 0$ and $\epsilon > 0$ satisfying the constraint in (1.15),

$$[1, 1 + \delta_0] \subset \mathcal{U}(u_*).$$

Once the estimates assumed by $(\mathbf{BA})_\epsilon$ have been improved we extend $\mathcal{M}$ and its foliation to a larger GCM admissible spacetime $\widetilde{\mathcal{M}}$. This is achieved as follows.

Theorem M7. *Extension argument.* We show that any GCM admissible spacetime in $\mathcal{U}(u_*)$ for some $0 < u_* < +\infty$ has a GCM admissible extension in in $\mathcal{U}(u'_*)$ for some $u'_* > u_*$, initialized by Theorem M0, which verifies the improved decay bootstrap assumptions.

The main steps in the extension are, as in [39]:

- First extend $\mathcal{M}$ and its foliation to a strictly larger space $\mathcal{M}'$.
- To make sure that the extended spacetime is GCM admissible, one has to construct a new GCM hypersurface $\widetilde{\Sigma}_*$ in $\mathcal{M}' \setminus \mathcal{M}$ and use it to define a new extended GCM admissible spacetime $\widetilde{\mathcal{M}}$. It is at this stage that we have to prove the existence of GCM spheres in $\mathcal{M}' \setminus \mathcal{M}$. More precisely, using the bounds on the Ricci and curvature coefficients on $\mathcal{M}'$, defined by the extended foliation, we have to construct GCM spheres in $\mathcal{M}' \setminus \mathcal{M}$.
- The GCM spheres mentioned above are used as building blocks for the new spacelike hypersurface $\widetilde{\Sigma}_*$. The construction of $\widetilde{\Sigma}_*$, similar to that in [39], is explicitly done in our context in [50]. Once this is done we can also a construct a new GCM-PG data set on $\widetilde{\Sigma}_*$ and use it construct thus the desired GCM admissible extension $\widetilde{\mathcal{M}}$.
- One needs to check that relative to the new structure we improve the original bootstrap assumption for decay, i.e. $\mathcal{N}^{(Dec)}_{k_{small}} \lesssim \epsilon_0$.

Theorem M8. *Estimates for the top order derivatives.* The new admissible spacetime $\widetilde{\mathcal{M}}$ is strictly larger than $\mathcal{M}$ and verifies $\mathcal{N}^{(Dec)}_{k_{small}} \lesssim \epsilon_0$. It still remains to improve the second half of the bootstrap assumptions concerning $\mathcal{N}^{(Sup)}_{k_{large}}$ and show that $\mathcal{N}^{(Sup)}_{k_{large}} \lesssim \epsilon_0$. Both types of estimates are proved by an induction argument starting with the improved estimates for $k \leq k_{small}$, i.e. $\mathcal{N}^{(Dec)}_{k_{small}} \lesssim \epsilon_0$, derived in Theorems M0–M7.

- The proof for the first part of the induction argument is provided in Chapter 9 where, using the PT frame, we derive boundedness estimates for the top derivatives of the Ricci coefficients in terms of bounds for the top derivatives of the curvature coefficients.
- The proof for the second part of the induction argument where we derive boundedness estimates for the top derivatives of the Curvature coefficients is provided in Part III of [28] by deriving energy-Morawetz type estimates for $\check{P}$ and making use of the Bianchi identities for all other curvature coefficients. The r^p weighted estimates in ${}^{(ext)}\mathcal{M}$ are derived taking advantage of the Maxwell like character of the Bianchi equations.

1.7.4. Conclusions The precise version of our main theorem, see Section 3.4.3, states a few important conclusions. Here are some of them:

- The future null infinity $\mathcal{I}^+$ of the limiting space $\mathcal{M}_\infty$ is complete. The other future boundary of $\mathcal{M}_\infty$ is given by the spacelike hypersurface $\mathcal{A}$, which can be shown to belong to the complement of $\mathcal{J}^-(\mathcal{I}^+)$. In particular this establishes the existence of the event horizon $\mathcal{H}^+$.
- The spheres $S(u,r)$ converge to round spheres, i.e. $\lim_{r\to\infty} r^2 K(u,r) = 1$, where K is the Gauss curvature of S.
- The quantities a_∞, m_∞ can be determined by taking limits of well defined quasi-local quantities which we define below.

1.7.4.1. Limits of quasi-local quantities on $\mathcal{I}^+$ The quasi-local quantities appearing below are defined relative to the integrable frame of ${}^{(ext)}\mathcal{M}_\infty$, i.e. the frame $(e_3', e_4', \mathcal{H}')$ with $\mathcal{H}'$ tangent to the spheres $S(u,r)$. The first quantity is the well known Hawking mass. The second quantity was first defined in [41].

Definition 1.22. *We define the following quasi-local quantities on a given sphere $S = S(u,r) \subset {}^{(ext)}\mathcal{M}_\infty$ and its integrable frame $(e_3', e_4', \mathcal{H}')$.*

1. *We define the Hawking mass of S to be*

$$m_H(u,r) \quad = \quad \frac{|S(u,r)|^{1/2}}{4\pi^{1/2}}\left(1 + \frac{1}{16\pi}\int_{S(u,r)} tr\,\chi'\, tr\,\underline{\chi}'\right).$$

where $tr\chi', tr\underline{\chi}'$ *are calculated with respect to the integrable frame of* ${}^{(ext)}\mathcal{M}_\infty$.

2. *We define the quasi-local angular momentum of* S *to be the triplet*

$$\mathrm{j}_{\ell=1,p}(u,r) := \frac{r^5}{|S(u,r)|}\int_{S(u,r)}(curl'\beta')J^{(p)}, \qquad p=-,0,+.$$

where β', $curl'\beta'$ *are defined with respect to the integrable frame of* ${}^{(ext)}\mathcal{M}$ *and the triplet* $J^{(p)}$ *is defined by*[35]

$$J^0 = \cos\theta, \qquad J^+ = \sin\theta\cos\varphi, \qquad J^- = \sin\theta\sin\varphi$$

Proposition 1.23. *The following statements hold true*

1. *The Hawking mass has a limit as* $r\to\infty$, *called the Bondi mass*

$$M_B(u) = \lim_{r\to\infty} m_H(u,r).$$

2. *The Bondi mass has a limit as* $u\to\infty$

$$\lim_{u\to\infty} M_B(u) = m_\infty.$$

3. *The quasi-local angular momentum* $\mathrm{j}_{\ell=1,p}(u,r)$ *of* S *has a limit as* $r\to\infty$

$$\mathcal{J}_{\ell=1,p}(u) \quad = \quad \lim_{r\to\infty}\mathrm{j}_{\ell=1,p}(u,r).$$

4. *The triplet* $\mathcal{J}_{\ell=1,p}(u)$ *has a limit as* $u\to\infty$ *and*

$$\lim_{u\to\infty}\mathcal{J}_{\ell=1,0}(u) = 2a_\infty m_\infty, \qquad \lim_{u\to\infty}\mathcal{J}_{\ell=1,\pm}(u) = 0,$$

which defines a_∞.

We also note that other definitions of angular momentum have been proposed in the literature, see [54] for a comprehensive review, and [48] and [12] for other interesting proposals.

[35]Here θ and φ are such that $e_4(\theta) = e_4(\varphi) = 0$ initialized at $\mathcal{I}^+$ to be standard spherical coordinates.

1.8. Comments on the full subextremal case

Though our result is restricted to small angular momentum, there are reasons to hope that a full stability result, for the full subextremal case, is conceivable in the near future. To start with, the only important limitation in our work to small values of $|a|/m$ comes from the proof of the Morawetz type estimates for the gRW wave equations in part II of [28]. On the other hand, progress on the Morawetz estimates for gRW in Kerr, in the full subextremal range, has been made recently by R. Shlapentokh-Rothman and R. Teixeira da Costa in [53]. Their work rests however on mode decompositions, which rely strongly on the specific structure of Kerr. Thus the only remaining obstacle, while important, seems to be more of a technical nature rather than conceptual.

1.9. Organization of the paper

In Chapter 2 we provide a full descriptions of the main geometric structures needed in our work. Chapter 3 contains the precise version of our main theorem, its main conclusions, as well as a full strategy of its proof, divided in the nine supporting intermediate results, Theorems M0–M8. In Chapters 4 to 8, we then give complete proofs of Theorems, M0 and M3–M7. Finally, we provide in Chapter 9 a proof of Theorem M8 by assuming the curvature estimates derived in Part III of [28].

1.10. Acknowledgements

As we have pointed out in our introduction to [39], our results would be inconceivable without the remarkable achievements obtained, first, during the so called golden age of black hole physics, and then the equally golden age of mathematical GR in the last 30–40 years. In addition to the references made in [39], which have influenced our work on the nonlinear stability of Schwarzschild, we have to single out the works of Andersson-Blue in [2] and Ma in [43] which play an important role in our approach. We thank E. Giorgi, our co-author of [28], for her many useful comments about this work. We also thank D. Shen, the author of [50], for reading part of the manuscript. And last but not least, we thank our wives Anca and Emilie for their continuing, priceless, patience, understanding and support. We also thank Anca and Emilie for their help with the main drawings in our papers.

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2. PRELIMINARIES

2.1. A general formalism

We review the general formalism introduced in [28], see also the earlier version [27].

2.1.1. Null pairs and horizontal structures Consider a fixed null pair e_3, e_4, i.e. $\mathbf{g}(e_3,e_3) = \mathbf{g}(e_4,e_4) = 0$, $\mathbf{g}(e_3,e_4) = -2$, and denote by $\mathbf{O}(\mathcal{M})$ the vectorspace of horizontal vectorfields X on $\mathcal{M}$, i.e. $\mathbf{g}(e_3,X) = \mathbf{g}(e_4,X) = 0$. Given a fixed orientation on $\mathcal{M}$, with corresponding volume form $\in$, we define the induced volume form on $\mathbf{O}(\mathcal{M})$ by,

$$\in(X,Y) := \frac{1}{2} \in (X,Y,e_3,e_4).$$

A null frame on $\mathcal{M}$ consists of a choice of horizontal vectorfields e_1, e_2, such that[36]

$$\mathbf{g}(e_a,e_b) = \delta_{ab} \qquad a,b = 1,2.$$

The commutator $[X,Y]$ of two horizontal vectorfields may fail however to be horizontal. We say that the pair (e_3,e_4) is integrable if $\mathbf{O}(\mathcal{M})$ forms an integrable distribution, i.e. $X,Y \in \mathbf{O}(\mathcal{M})$ implies that $[X,Y] \in \mathbf{O}(\mathcal{M})$. As known the principal null pair in Kerr fails to be integrable. Given an arbitrary vectorfield X we denote by ${}^{(h)}X$ its horizontal projection, ${}^{(h)}X = X + \frac{1}{2}\mathbf{g}(X,e_3)e_4 + \frac{1}{2}\mathbf{g}(X,e_4)e_3$. A k-covariant tensor-field U is said to be horizontal, $U \in \mathbf{O}_k(\mathcal{M})$, if for any $X_1, \dots X_k$ we have

$$U(X_1,\dots X_k) = U({}^{(h)}X_1, \dots {}^{(h)}X_k).$$

We define the left and right duals of horizontal 1-forms ω and covariant 2-tensors U,

$${}^*\omega_a = \in_{ab} \omega_b, \quad \omega^*{}_a = \omega_b \in_{ba}, \quad ({}^*U)_{ab} = \in_{ac} U_{cb}, \qquad (U^*)_{ab} = U_{ac} \in_{cb}.$$

Note that ${}^*({}^*\omega) = -\omega$, ${}^*\omega = -\omega^*$, ${}^*({}^*U) = -U$. Moreover if U is symmetric, then ${}^*U = -U^*$ and if $U = \widehat{U}$ is symmetric, traceless, then, ${}^*\widehat{U} = -\widehat{U}^*$ is also symmetric traceless.

[36] We use greek indices for $1,2,3,4$ and latin indices a,b for $1,2$.

Definition 2.1. *We denote by $\mathcal{S}_1 = \mathcal{S}_1(\mathcal{M})$ the set of horizontal 1-forms on $\mathcal{M}$, and by $\mathcal{S}_2 = \mathcal{S}_2(\mathcal{M})$ the set of symmetric traceless horizontal 2-forms on $\mathcal{M}$.*

Given $\xi, \eta \in \mathcal{S}_1$, $U, V \in \mathcal{S}_2$ we denote[37]

$$\begin{aligned} \xi\cdot\eta &:= \delta^{ab}\xi_a\eta_b, & \xi\wedge\eta &:= \in^{ab}\xi_a\eta_b = \xi\cdot {}^*\eta,\\ (\xi\widehat{\otimes}\eta)_{ab} &:= \xi_a\eta_b + \xi_b\eta_a - \delta_{ab}\xi\cdot\eta, & (\xi\cdot U)_a &:= \delta^{bc}\xi_b U_{ac},\\ (U\wedge V)_{ab} &:= \epsilon^{ab}U_{ac}V_{cb}. \end{aligned}$$

For any $X, Y \in \mathbf{O}(\mathcal{M})$ we define the induced metric $g(X,Y) = \mathbf{g}(X,Y)$ and the null second fundamental forms

$$\chi(X,Y) = \mathbf{g}(\mathbf{D}_X e_4, Y), \qquad \underline{\chi}(X,Y) = \mathbf{g}(\mathbf{D}_X e_3, Y). \tag{2.1}$$

Observe that χ and $\underline{\chi}$ are symmetric if and only if the horizontal structure is integrable. Indeed this follows easily from the formulas,

$$\begin{aligned} \chi(X,Y) - \chi(Y,X) &= \mathbf{g}(\mathbf{D}_X e_4, Y) - \mathbf{g}(\mathbf{D}_Y e_4, X) = -\mathbf{g}(e_4, [X,Y]),\\ \underline{\chi}(X,Y) - \underline{\chi}(Y,X) &= \mathbf{g}(\mathbf{D}_X e_3, Y) - \mathbf{g}(\mathbf{D}_Y e_3, X) = -\mathbf{g}(e_3, [X,Y]). \end{aligned}$$

Note that we can view χ and $\underline{\chi}$ as horizontal 2-covariant tensor-fields by extending their definition to arbitrary vectorfields X, Y by setting $\chi(X,Y) = \chi({}^{(h)}X, {}^{(h)}Y)$, $\underline{\chi}(X,Y) = \underline{\chi}({}^{(h)}X, {}^{(h)}Y)$. Given an horizontal 2-tensor U we define its trace $\mathrm{tr}U$ and anti-trace ${}^{(a)}\mathrm{tr}U$

$$\mathrm{tr}(U) := \delta^{ab}U_{ab}, \qquad {}^{(a)}\mathrm{tr}U = \in^{ab} U_{ab}.$$

Accordingly we decompose $\chi, \underline{\chi}$ as follows,

$$\begin{aligned} \chi_{ab} &= \widehat{\chi}_{ab} + \frac{1}{2}\delta_{ab}\mathrm{tr}\,\chi + \frac{1}{2}\in_{ab} {}^{(a)}\mathrm{tr}\chi,\\ \underline{\chi}_{ab} &= \widehat{\chi}_{ab} + \frac{1}{2}\delta_{ab}\mathrm{tr}\,\chi + \frac{1}{2}\in_{ab} {}^{(a)}\mathrm{tr}\underline{\chi}. \end{aligned}$$

We define the horizontal covariant operator ∇ as follows. Given $X, Y \in \mathbf{O}(\mathcal{M})$

$$\nabla_X Y := {}^{(h)}(\mathbf{D}_X Y) = \mathbf{D}_X Y - \frac{1}{2}\underline{\chi}(X,Y)e_4 - \frac{1}{2}\chi(X,Y)e_3. \tag{2.2}$$

[37]Note that the definition of $\xi\widehat{\otimes}\eta$ is the same as in [17].

Note that,

$$\nabla_X Y - \nabla_Y X = [X,Y] - \frac{1}{2}\big({}^{(a)}\mathrm{tr}\underline{\chi}\, e_4 + {}^{(a)}\mathrm{tr}\chi\, e_3\big) \in (X,Y).$$

In particular,

$$[X,Y]^\perp = \frac{1}{2}\big({}^{(a)}\mathrm{tr}\underline{\chi}\, e_4 + {}^{(a)}\mathrm{tr}\chi\, e_3\big) \in (X,Y). \tag{2.3}$$

Also, for all $X, Y, Z \in \mathbf{O}(\mathcal{M})$,

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

Remark. In the integrable case, ∇ coincides with the Levi-Civita connection of the metric induced on the integral surfaces of $\mathbf{O}(\mathcal{M})$. Given X horizontal, $\mathbf{D}_4 X$ and $\mathbf{D}_3 X$ are in general not horizontal. We define $\nabla_4 X$ and $\nabla_3 X$ to be the horizontal projections of the former. More precisely,

$$\begin{aligned}
\nabla_4 X &:= {}^{(h)}(\mathbf{D}_4 X) = \mathbf{D}_4 X - \frac{1}{2}\mathbf{g}(X, \mathbf{D}_4 e_3) e_4 - \frac{1}{2}\mathbf{g}(X, \mathbf{D}_4 e_4) e_3,\\
\nabla_3 X &:= {}^{(h)}(\mathbf{D}_3 X) = \mathbf{D}_3 X - \frac{1}{2}\mathbf{g}(X, \mathbf{D}_3 e_3) e_3 - \frac{1}{2}\mathbf{g}(X, \mathbf{D}_3 e_4) e_3.
\end{aligned}$$

The definition can be easily extended to arbitrary $\mathbf{O}_k(\mathcal{M})$ tensor-fields U

$$\begin{aligned}
\nabla_4 U(X_1, \ldots, X_k) &= e_4(U(X_1, \ldots, X_k)) - \sum_i U(X_1, \ldots, \nabla_4 X_i, \ldots X_k),\\
\nabla_3 U(X_1, \ldots, X_k) &= e_3(U(X_1, \ldots, X_k)) - \sum_i U(X_1, \ldots, \nabla_3 X_i, \ldots X_k).
\end{aligned}$$

2.1.2. Ricci and curvature coefficients Given a null frame e_1, e_2, e_3, e_4 we define the general connection coefficients,

$$(\Lambda_\mu)_{\alpha\beta} : = \mathbf{g}\big(\mathbf{D}_{e_\mu} e_\beta, e_\alpha\big) \tag{2.4}$$

and the special ones

$$\begin{aligned}
\underline{\chi}_{ab} &= \mathbf{g}(\mathbf{D}_a e_3, e_b), & \chi_{ab} &= \mathbf{g}(\mathbf{D}_a e_4, e_b),\\
\underline{\xi}_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_3, e_a), & \xi_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_4, e_a),\\
\underline{\omega} &= \frac{1}{4}\mathbf{g}(\mathbf{D}_3 e_3, e_4), & \omega &= \frac{1}{4}\mathbf{g}(\mathbf{D}_4 e_4, e_3),
\end{aligned}$$

$$\underline{\eta}_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_3, e_a), \qquad \eta_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_4, e_a),$$
$$\zeta_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_{e_a} e_4, e_3).$$

Note that these account for all the connection coefficients except

$$(\Lambda_\mu)_{ab} := \mathbf{g}(\mathbf{D}_{e_\mu} e_b, e_a), \qquad \mu = 1,2,3,4, \quad a,b = 1,2.$$

We have the Ricci formulas

$$\begin{array}{rcl}
\mathbf{D}_a e_b &=& \nabla_a e_b + \frac{1}{2}\chi_{ab} e_3 + \frac{1}{2}\underline{\chi}_{ab} e_4,\\
\mathbf{D}_a e_4 &=& \chi_{ab} e_b - \zeta_a e_4,\\
\mathbf{D}_a e_3 &=& \underline{\chi}_{ab} e_b + \zeta_a e_3,\\
\mathbf{D}_3 e_a &=& \nabla_3 e_a + \eta_a e_3 + \underline{\xi}_a e_4,\\
\mathbf{D}_3 e_3 &=& -2\underline{\omega} e_3 + 2\underline{\xi}_b e_b,\\
\mathbf{D}_3 e_4 &=& 2\underline{\omega} e_4 + 2\eta_b e_b,\\
\mathbf{D}_4 e_a &=& \nabla_4 e_a + \underline{\eta}_a e_4 + \xi_a e_3,\\
\mathbf{D}_4 e_4 &=& -2\omega e_4 + 2\xi_b e_b,\\
\mathbf{D}_4 e_3 &=& 2\omega e_3 + 2\underline{\eta}_b e_b.
\end{array} \tag{2.5}$$

For a given horizontal 1-form ξ, we define the frame independent operators

$$\mathrm{div}\,\xi = \delta^{ab}\nabla_b \xi_a, \qquad \mathrm{curl}\,\xi = \in^{ab} \nabla_a \xi_b,$$
$$(\nabla\widehat{\otimes}\xi)_{ba} = \nabla_b\xi_a + \nabla_a\xi_b - \delta_{ab}(\mathrm{div}\,\xi).$$

We also define the usual curvature components, see [17],

$$\alpha_{ab} = \mathbf{R}_{a4b4}, \quad \beta_a = \frac{1}{2}\mathbf{R}_{a434}, \quad \rho = \frac{1}{4}\mathbf{R}_{3434}, \quad {}^*\rho = \frac{1}{4}\,{}^*\mathbf{R}_{3434},$$
$$\underline{\beta}_a = \frac{1}{2}\mathbf{R}_{a334}, \quad \underline{\alpha}_{ab} = \mathbf{R}_{a3b3},$$

where ${}^*\mathbf{R}$ denotes the Hodge dual of $\mathbf{R}$.

2.1.3. Commutation formulas

Lemma 2.2. *Let $U_A = U_{a_1\ldots a_k}$ be a general k-horizontal tensorfield.*

1. *We have*

$$[\nabla_3, \nabla_b]U_A = -\underline{\chi}_{bc}\nabla_c U_A + (\eta_b - \zeta_b)\nabla_3 U_A + \sum_{i=1}^k (\underline{\chi}_{a_i b}\eta_c - \underline{\chi}_{bc}\eta_{a_i})U_{a_1\ldots}{}^c{}_{\ldots a_k} + Err_{3bA}[U],$$

$$Err_{3bA}[U] = \sum_{i=1}^k \Big(\chi_{a_i c}\underline{\xi}_c - \chi_{bc}\,\underline{\xi}_{a_i} - \in_{a_i c}\ {}^*\underline{\beta}_b\Big)U_{a_1\ldots}{}^c{}_{\ldots a_k} + \underline{\xi}_b\nabla_4 U_A.$$

2. *We have*

$$[\nabla_4, \nabla_b]U_A = -\chi_{bc}\nabla_c U_A + (\underline{\eta}_b + \zeta_b)\nabla_4 U_A + \sum_{i=1}^k (\chi_{a_i b}\underline{\eta}_c - \chi_{bc}\underline{\eta}_{a_i})U_{a_1\ldots c\ldots a_k} + Err_{4bA}[U],$$

$$Err_{4bA}[U] = \sum_{i=1}^k \Big(\underline{\chi}_{a_i c}\xi_c - \underline{\chi}_{bc}\,\xi_{a_i} + \in_{a_i c}\ {}^*\beta_b\Big)U_{a_1\ldots}{}^c{}_{\ldots a_k} + \xi_b\nabla_3 U_A.$$

3. *We have*

$$[\nabla_4, \nabla_3]U_A = 2(\underline{\eta}_b - \eta_b)\nabla_b U_A + 2\sum_{i=1}^k (\eta_{a_i}\underline{\eta}_b - \underline{\eta}_{a_i}\eta_b - \in_{a_i b}\ {}^*\rho)U_{a_1\ldots}{}^b{}_{\ldots a_k} + 2\omega\nabla_3 U_A - 2\underline{\omega}\nabla_4 U_A + Err_{43A},$$

$$Err_{43A} = 2\sum_{i=1}^k (\underline{\xi}_{a_i}\xi_b - \xi_{a_i}\underline{\xi}_b)U_{a_1\ldots}{}^b{}_{\ldots a_k}.$$

Proof. See section 2.2.7 in [28]. □

2.1.4. Null structure and Bianchi equations The null structure equations are given in the following proposition of [28], see also [17] for the integrable case[38].

Proposition 2.3. *We have*

$$\begin{aligned}
\nabla_3 tr\underline{\chi} &= -|\widehat{\underline{\chi}}|^2 - \frac{1}{2}\big(tr\underline{\chi}^2 - {}^{(a)}tr\underline{\chi}^2\big) + 2\,div\underline{\xi} - 2\underline{\omega}\,tr\underline{\chi} \\
&\quad +2\underline{\xi}\cdot(\eta + \underline{\eta} - 2\zeta),\\
\nabla_3\, {}^{(a)}tr\underline{\chi} &= -tr\underline{\chi}\,{}^{(a)}tr\underline{\chi} + 2\,curl\underline{\xi} - 2\underline{\omega}\,{}^{(a)}tr\underline{\chi} + 2\underline{\xi}\wedge(-\eta + \underline{\eta} + 2\zeta),
\end{aligned}$$

[38] Note that the term $\underline{\xi}\wedge(-\eta+\underline{\eta}+2\zeta)$ in $\nabla_3\,{}^{(a)}\mathrm{tr}\underline{\chi}$ differs from that in [17], see (7.4.1c) on page 165.

$$
\begin{array}{rcl}
\nabla_3\widehat{\underline{\chi}} &=& -tr\underline{\chi}\,\widehat{\underline{\chi}}+\nabla\widehat{\otimes}\underline{\xi}-2\underline{\omega}\,\widehat{\underline{\chi}}+\underline{\xi}\widehat{\otimes}(\eta+\underline{\eta}-2\zeta)-\underline{\alpha},\\
\nabla_3 tr\chi &=& -\widehat{\underline{\chi}}\cdot\widehat{\chi}-\frac{1}{2}tr\underline{\chi}tr\chi+\frac{1}{2}{}^{(a)}tr\underline{\chi}\,{}^{(a)}tr\chi+2div\eta+2\underline{\omega}tr\chi\\
&& +2(\xi\cdot\underline{\xi}+|\eta|^2)+2\rho,\\
\nabla_3\,{}^{(a)}tr\chi &=& -\widehat{\underline{\chi}}\wedge\widehat{\chi}-\frac{1}{2}({}^{(a)}tr\underline{\chi}tr\chi+tr\underline{\chi}\,{}^{(a)}tr\chi)+2curl\eta+2\underline{\omega}\,{}^{(a)}tr\chi\\
&& +2\underline{\xi}\wedge\xi-2\,{}^*\rho,\\
\nabla_3\widehat{\chi} &=& -\frac{1}{2}(tr\chi\widehat{\underline{\chi}}+tr\underline{\chi}\widehat{\chi})-\frac{1}{2}(-\,{}^*\widehat{\underline{\chi}}\,{}^{(a)}tr\chi+\,{}^*\widehat{\chi}\,{}^{(a)}tr\underline{\chi})+\nabla\widehat{\otimes}\eta\\
&& +2\underline{\omega}\widehat{\chi}+\underline{\xi}\widehat{\otimes}\xi+\eta\widehat{\otimes}\eta,\\
\nabla_4 tr\underline{\chi} &=& -\widehat{\chi}\cdot\widehat{\underline{\chi}}-\frac{1}{2}tr\chi tr\underline{\chi}+\frac{1}{2}{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi}+2div\underline{\eta}+2\omega tr\underline{\chi}\\
&& +2(\xi\cdot\underline{\xi}+|\underline{\eta}|^2)+2\rho,\\
\nabla_4\,{}^{(a)}tr\underline{\chi} &=& -\widehat{\chi}\wedge\widehat{\underline{\chi}}-\frac{1}{2}({}^{(a)}tr\chi tr\underline{\chi}+tr\chi\,{}^{(a)}tr\underline{\chi})+2curl\underline{\eta}+2\omega\,{}^{(a)}tr\underline{\chi}\\
&& +2\xi\wedge\underline{\xi}+2\,{}^*\rho,\\
\nabla_4\widehat{\underline{\chi}} &=& -\frac{1}{2}(tr\underline{\chi}\widehat{\chi}+tr\chi\widehat{\underline{\chi}})-\frac{1}{2}(-\,{}^*\widehat{\chi}\,{}^{(a)}tr\underline{\chi}+\,{}^*\widehat{\underline{\chi}}\,{}^{(a)}tr\chi)+\nabla\widehat{\otimes}\underline{\eta}\\
&& +2\omega\widehat{\underline{\chi}}+\xi\widehat{\otimes}\underline{\xi}+\underline{\eta}\widehat{\otimes}\underline{\eta},\\
\nabla_4 tr\chi &=& -|\widehat{\chi}|^2-\frac{1}{2}(tr\chi^2-{}^{(a)}tr\chi^2)+2div\xi-2\omega tr\chi\\
&& +2\xi\cdot(\underline{\eta}+\eta+2\zeta),\\
\nabla_4\,{}^{(a)}tr\chi &=& -tr\chi\,{}^{(a)}tr\chi+2curl\xi-2\omega\,{}^{(a)}tr\chi+2\xi\wedge(-\underline{\eta}+\eta-2\zeta),\\
\nabla_4\widehat{\chi} &=& -tr\chi\,\widehat{\chi}+\nabla\widehat{\otimes}\xi-2\omega\widehat{\chi}+\xi\widehat{\otimes}(\underline{\eta}+\eta+2\zeta)-\alpha,\\
\nabla_3\zeta+2\nabla\underline{\omega} &=& -\widehat{\underline{\chi}}\cdot(\zeta+\eta)-\frac{1}{2}tr\underline{\chi}(\zeta+\eta)-\frac{1}{2}{}^{(a)}tr\underline{\chi}({}^*\zeta+{}^*\eta)\\
&& +2\underline{\omega}(\zeta-\eta)+\widehat{\chi}\cdot\underline{\xi}+\frac{1}{2}tr\chi\,\underline{\xi}+\frac{1}{2}{}^{(a)}tr\chi\,{}^*\underline{\xi}+2\omega\underline{\xi}-\underline{\beta},\\
\nabla_4\zeta-2\nabla\omega &=& \widehat{\chi}\cdot(-\zeta+\underline{\eta})+\frac{1}{2}tr\chi(-\zeta+\underline{\eta})+\frac{1}{2}{}^{(a)}tr\chi(-\,{}^*\zeta+{}^*\underline{\eta})\\
&& +2\omega(\zeta+\underline{\eta})-\widehat{\underline{\chi}}\cdot\xi-\frac{1}{2}tr\underline{\chi}\,\xi-\frac{1}{2}{}^{(a)}tr\underline{\chi}\,{}^*\xi-2\underline{\omega}\xi-\beta,\\
\nabla_3\underline{\eta}-\nabla_4\underline{\xi} &=& -\widehat{\underline{\chi}}\cdot(\underline{\eta}-\eta)-\frac{1}{2}tr\underline{\chi}(\underline{\eta}-\eta)+\frac{1}{2}{}^{(a)}tr\underline{\chi}({}^*\underline{\eta}-{}^*\eta)-4\omega\underline{\xi}+\underline{\beta},\\
\nabla_4\eta-\nabla_3\xi &=& -\widehat{\chi}\cdot(\eta-\underline{\eta})-\frac{1}{2}tr\chi(\eta-\underline{\eta})+\frac{1}{2}{}^{(a)}tr\chi({}^*\eta-{}^*\underline{\eta})-4\underline{\omega}\xi-\beta,
\end{array}
$$

$$\nabla_3\omega + \nabla_4\underline{\omega} \quad = \quad \rho + 4\omega\underline{\omega} + \xi\cdot\underline{\xi} + (\eta-\underline{\eta})\cdot\zeta - \eta\cdot\underline{\eta}.$$

We also have the Codazzi equations

$$\begin{aligned}
div\,\widehat{\chi} + \zeta\cdot\widehat{\chi} &= \frac{1}{2}\nabla tr\chi + \frac{1}{2}tr\chi\zeta - \frac{1}{2}\,{}^*\nabla\,{}^{(a)}tr\chi - \frac{1}{2}\,{}^{(a)}tr\chi\,{}^*\zeta - {}^{(a)}tr\chi\,{}^*\eta\\
&\quad - {}^{(a)}tr\chi\,{}^*\xi - \beta,\\
div\,\underline{\widehat{\chi}} - \zeta\cdot\underline{\widehat{\chi}} &= \frac{1}{2}\nabla tr\underline{\chi} - \frac{1}{2}tr\underline{\chi}\zeta - \frac{1}{2}\,{}^*\nabla\,{}^{(a)}tr\underline{\chi} + \frac{1}{2}\,{}^{(a)}tr\underline{\chi}\,{}^*\zeta - {}^{(a)}tr\underline{\chi}\,{}^*\underline{\eta}\\
&\quad - {}^{(a)}tr\underline{\chi}\,{}^*\underline{\xi} + \underline{\beta},
\end{aligned}$$

$$curl\,\zeta \quad = \quad -\frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}} + \frac{1}{4}\left(tr\chi\,{}^{(a)}tr\underline{\chi} - tr\underline{\chi}\,{}^{(a)}tr\chi\right) + \omega\,{}^{(a)}tr\underline{\chi} - \underline{\omega}\,{}^{(a)}tr\chi + {}^*\rho.$$

The null Bianchi equations are given below, see [28], and [17] for the integrable case.

Proposition 2.4. *We have*

$$\begin{aligned}
\nabla_3\alpha - \nabla\widehat{\otimes}\beta &= -\frac{1}{2}\left(tr\underline{\chi}\alpha + {}^{(a)}tr\underline{\chi}\,{}^*\alpha\right) + 4\underline{\omega}\alpha + (\zeta+4\eta)\widehat{\otimes}\beta\\
&\quad -3(\rho\widehat{\chi} + {}^*\rho\,{}^*\widehat{\chi}),\\
\nabla_4\beta - div\,\alpha &= -2(tr\chi\beta - {}^{(a)}tr\chi\,{}^*\beta) - 2\omega\beta + \alpha\cdot(2\zeta+\underline{\eta})\\
&\quad +3(\xi\rho + {}^*\xi\,{}^*\rho),\\
\nabla_3\beta - (\nabla\rho + {}^*\nabla\,{}^*\rho) &= -(tr\underline{\chi}\beta + {}^{(a)}tr\underline{\chi}\,{}^*\beta) + 2\underline{\omega}\,\beta + 2\underline{\beta}\cdot\widehat{\chi}\\
&\quad +3(\rho\eta + {}^*\rho\,{}^*\eta) + \alpha\cdot\underline{\xi},\\
\nabla_4\rho - div\,\beta &= -\frac{3}{2}(tr\chi\rho + {}^{(a)}tr\chi\,{}^*\rho) + (2\underline{\eta}+\zeta)\cdot\beta - 2\xi\cdot\underline{\beta}\\
&\quad -\frac{1}{2}\underline{\widehat{\chi}}\cdot\alpha,\\
\nabla_4\,{}^*\rho + curl\,\beta &= -\frac{3}{2}(tr\chi\,{}^*\rho - {}^{(a)}tr\chi\rho) - (2\underline{\eta}+\zeta)\cdot{}^*\beta - 2\xi\cdot{}^*\underline{\beta}\\
&\quad +\frac{1}{2}\underline{\widehat{\chi}}\cdot{}^*\alpha,\\
\nabla_3\rho + div\,\underline{\beta} &= -\frac{3}{2}(tr\underline{\chi}\rho - {}^{(a)}tr\underline{\chi}\,{}^*\rho) - (2\eta-\zeta)\cdot\underline{\beta} + 2\underline{\xi}\cdot\beta\\
&\quad -\frac{1}{2}\widehat{\chi}\cdot\underline{\alpha},\\
\nabla_3\,{}^*\rho + curl\,\underline{\beta} &= -\frac{3}{2}(tr\underline{\chi}\,{}^*\rho + {}^{(a)}tr\underline{\chi}\rho) - (2\eta-\zeta)\cdot{}^*\underline{\beta} - 2\underline{\xi}\cdot{}^*\beta\\
&\quad -\frac{1}{2}\widehat{\chi}\cdot{}^*\underline{\alpha},
\end{aligned}$$

$$
\begin{aligned}
\nabla_4\underline{\beta} + \nabla\rho - {}^*\nabla {}^*\rho &= -(tr\chi\underline{\beta} + {}^{(a)}tr\chi {}^*\underline{\beta}) + 2\omega\,\underline{\beta} + 2\beta\cdot\underline{\widehat{\chi}} \\
&\quad -3(\rho\underline{\eta} - {}^*\rho {}^*\underline{\eta}) - \underline{\alpha}\cdot\xi, \\
\nabla_3\underline{\beta} + div\,\underline{\alpha} &= -2(tr\underline{\chi}\,\underline{\beta} - {}^{(a)}tr\underline{\chi} {}^*\underline{\beta}) - 2\underline{\omega}\underline{\beta} - \underline{\alpha}\cdot(-2\zeta+\eta) \\
&\quad -3(\underline{\xi}\rho - {}^*\underline{\xi} {}^*\rho), \\
\nabla_4\underline{\alpha} + \nabla\widehat{\otimes}\underline{\beta} &= -\frac{1}{2}(tr\chi\underline{\alpha} + {}^{(a)}tr\chi {}^*\underline{\alpha}) + 4\omega\underline{\alpha} + (\zeta - 4\underline{\eta})\widehat{\otimes}\underline{\beta} \\
&\quad -3(\rho\underline{\widehat{\chi}} - {}^*\rho {}^*\underline{\widehat{\chi}}).
\end{aligned}
$$

2.1.5. Main equations in complex form In this section, we review the complex notations introduced in [28] that will allow us to simplify the main equations.

Definition 2.5.

$$
\begin{aligned}
&A := \alpha + i\,{}^*\alpha, \qquad B := \beta + i\,{}^*\beta, \qquad P := \rho + i\,{}^*\rho, \\
&\underline{B} := \underline{\beta} + i\,{}^*\underline{\beta}, \qquad \underline{A} := \underline{\alpha} + i\,{}^*\underline{\alpha},
\end{aligned}
$$

and

$$
\begin{aligned}
&X = \chi + i\,{}^*\chi, \quad \underline{X} = \underline{\chi} + i\,{}^*\underline{\chi}, \quad H = \eta + i\,{}^*\eta, \quad \underline{H} = \underline{\eta} + i\,{}^*\underline{\eta}, \\
&Z = \zeta + i\,{}^*\zeta, \quad \Xi = \xi + i\,{}^*\xi, \quad \underline{\Xi} = \underline{\xi} + i\,{}^*\underline{\xi}.
\end{aligned}
$$

In particular, note that

$$
trX = tr\chi - i\,{}^{(a)}tr\chi, \quad \widehat{X} = \widehat{\chi} + i\,{}^*\widehat{\chi}, \quad tr\underline{X} = tr\underline{\chi} - i\,{}^{(a)}tr\underline{\chi}, \quad \underline{\widehat{X}} = \underline{\widehat{\chi}} + i\,{}^*\underline{\widehat{\chi}}.
$$

Remark 2.6. *A, B, $\underline{A}$, $\underline{B}$, X, $\underline{X}$, H, $\underline{H}$, Ξ, $\underline{\Xi}$ and Z are anti-self dual tensors, i.e. they verify ${}^*U = -iU$.*

Definition 2.7. *We define derivatives of complex quantities as follows*

- *For two scalar functions a and b, we define*

$$
\mathcal{D}(a+ib) := (\nabla + i\,{}^*\nabla)(a+ib).
$$

- *For a 1-form f, we define*

$$
\mathcal{D}\cdot(f + i\,{}^*f) := (\nabla + i\,{}^*\nabla)\cdot(f + i\,{}^*f)
$$

  and

$$
\mathcal{D}\widehat{\otimes}(f + i\,{}^*f) := (\nabla + i\,{}^*\nabla)\widehat{\otimes}(f + i\,{}^*f).
$$

- *For a symmetric traceless 2-form u, we define*

$$\mathcal{D}\cdot(u+i\,{}^*u) \quad := \quad (\nabla+i\,{}^*\nabla)\cdot(u+i\,{}^*u).$$

These complex notations allow us to rewrite the null structure equations as follows, see section 2.4.3 in [28].

Proposition 2.8. *The following hold true:*

$$\begin{aligned}
\nabla_3 tr\underline{X}+\frac{1}{2}(tr\underline{X})^2+2\underline{\omega}\,tr\underline{X} &= \mathcal{D}\cdot\overline{\underline{\Xi}}+\underline{\Xi}\cdot\overline{\underline{H}}+\overline{\underline{\Xi}}\cdot(H-2Z)-\frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{\underline{X}}},\\
\nabla_3\widehat{\underline{X}}+\Re(tr\underline{X})\widehat{\underline{X}}+2\underline{\omega}\,\widehat{\underline{X}} &= \frac{1}{2}\mathcal{D}\widehat{\otimes}\underline{\Xi}+\frac{1}{2}\underline{\Xi}\widehat{\otimes}(H+\underline{H}-2Z)-\underline{A},
\end{aligned}$$

$$\begin{aligned}
\nabla_3 trX+\frac{1}{2}tr\underline{X}trX-2\underline{\omega}trX &= \mathcal{D}\cdot\overline{H}+H\cdot\overline{H}+2P+\underline{\Xi}\cdot\overline{\Xi}-\frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{X}},\\
\nabla_3\widehat{X}+\frac{1}{2}tr\underline{X}\,\widehat{X}-2\underline{\omega}\widehat{X} &= \frac{1}{2}\mathcal{D}\widehat{\otimes}H+\frac{1}{2}H\widehat{\otimes}H-\frac{1}{2}\overline{trX}\widehat{\underline{X}}+\frac{1}{2}\underline{\Xi}\widehat{\otimes}\Xi,
\end{aligned}$$

$$\begin{aligned}
\nabla_4 tr\underline{X}+\frac{1}{2}trX\,tr\underline{X}-2\omega tr\underline{X} &= \mathcal{D}\cdot\overline{\underline{H}}+\underline{H}\cdot\overline{\underline{H}}+2\overline{P}+\Xi\cdot\overline{\underline{\Xi}}-\frac{1}{2}\widehat{X}\cdot\overline{\widehat{\underline{X}}},\\
\nabla_4\widehat{\underline{X}}+\frac{1}{2}trX\,\widehat{\underline{X}}-2\omega\widehat{\underline{X}} &= \frac{1}{2}\mathcal{D}\widehat{\otimes}\underline{H}+\frac{1}{2}\underline{H}\widehat{\otimes}\underline{H}-\frac{1}{2}\overline{tr\underline{X}}\widehat{X}+\frac{1}{2}\Xi\widehat{\otimes}\underline{\Xi},
\end{aligned}$$

$$\begin{aligned}
\nabla_4 trX+\frac{1}{2}(trX)^2+2\omega trX &= \mathcal{D}\cdot\overline{\Xi}+\Xi\cdot\overline{H}+\overline{\Xi}\cdot(\underline{H}+2Z)-\frac{1}{2}\widehat{X}\cdot\overline{\widehat{X}},\\
\nabla_4\widehat{X}+\Re(trX)\widehat{X}+2\omega\widehat{X} &= \frac{1}{2}\mathcal{D}\widehat{\otimes}\Xi+\frac{1}{2}\Xi\widehat{\otimes}(\underline{H}+H+2Z)-A.
\end{aligned}$$

Also,

$$\begin{aligned}
\nabla_3 Z+\frac{1}{2}tr\underline{X}(Z+H)-2\underline{\omega}(Z-H) &= -2\mathcal{D}\underline{\omega}-\frac{1}{2}\widehat{\underline{X}}\cdot(\overline{Z}+\overline{H})\\
&\quad+\frac{1}{2}trX\underline{\Xi}+2\omega\underline{\Xi}-\underline{B}+\frac{1}{2}\overline{\underline{\Xi}}\cdot\widehat{X},\\
\nabla_4 Z+\frac{1}{2}trX(Z-\underline{H})-2\omega(Z+\underline{H}) &= 2\mathcal{D}\omega+\frac{1}{2}\widehat{X}\cdot(-\overline{Z}+\overline{\underline{H}})\\
&\quad-\frac{1}{2}tr\underline{X}\Xi-2\underline{\omega}\Xi-B-\frac{1}{2}\overline{\Xi}\cdot\widehat{\underline{X}},\\
\nabla_3\underline{H}-\nabla_4\underline{\Xi} &= -\frac{1}{2}\overline{tr\underline{X}}(\underline{H}-H)-\frac{1}{2}\widehat{\underline{X}}\cdot(\overline{\underline{H}}-\overline{H})\\
&\quad-4\omega\underline{\Xi}+\underline{B},\\
\nabla_4 H-\nabla_3\Xi &= -\frac{1}{2}\overline{trX}(H-\underline{H})-\frac{1}{2}\widehat{X}\cdot(\overline{H}-\overline{\underline{H}})\\
&\quad-4\underline{\omega}\Xi-B,
\end{aligned}$$

and

$$\nabla_3\omega + \nabla_4\underline{\omega} - 4\omega\underline{\omega} - \xi\cdot\underline{\xi} - (\eta - \underline{\eta})\cdot\zeta + \eta\cdot\underline{\eta} \;=\; \rho.$$

Also,

$$\begin{aligned}
\frac{1}{2}\overline{\mathcal{D}}\cdot\widehat{X} + \frac{1}{2}\widehat{X}\cdot\overline{Z} &= \frac{1}{2}\mathcal{D}\overline{trX} + \frac{1}{2}\overline{trX}Z - i\Im(trX)(H+\Xi) - B,\\
\frac{1}{2}\overline{\mathcal{D}}\cdot\underline{\widehat{X}} - \frac{1}{2}\underline{\widehat{X}}\cdot\overline{Z} &= \frac{1}{2}\mathcal{D}\overline{tr\underline{X}} - \frac{1}{2}\overline{tr\underline{X}}Z - i\Im(tr\underline{X})(\underline{H}+\underline{\Xi}) + \underline{B},
\end{aligned}$$

and,

$$curl\zeta \;=\; -\frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}} + \frac{1}{4}\left(tr\chi\,{}^{(a)}tr\underline{\chi} - tr\underline{\chi}\,{}^{(a)}tr\chi\right) + \omega\,{}^{(a)}tr\underline{\chi} - \underline{\omega}\,{}^{(a)}tr\chi + {}^*\rho.$$

The complex notations allow us to rewrite the Bianchi identities as follows, see section 2.4.3 in [28].

Proposition 2.9. *We have*[39]*,*

$$\begin{aligned}
\nabla_3 A - \frac{1}{2}\mathcal{D}\widehat{\otimes}B &= -\frac{1}{2}tr\underline{X}A + 4\underline{\omega}A + \frac{1}{2}(Z+4H)\widehat{\otimes}B - 3\overline{P}\widehat{X},\\
\nabla_4 B - \frac{1}{2}\overline{\mathcal{D}}\cdot A &= -2\overline{trX}B - 2\omega B + \frac{1}{2}A\cdot(\overline{2Z+\underline{H}}) + 3\overline{P}\,\Xi,\\
\nabla_3 B - \mathcal{D}\overline{P} &= -tr\underline{X}B + 2\underline{\omega}B + \overline{\underline{B}}\cdot\widehat{X} + 3\overline{P}H + \frac{1}{2}A\cdot\overline{\underline{\Xi}},\\
\nabla_4 P - \frac{1}{2}\mathcal{D}\cdot\overline{B} &= -\frac{3}{2}trXP + \frac{1}{2}(2\underline{H}+Z)\cdot\overline{B} - \overline{\Xi}\cdot\underline{B} - \frac{1}{4}\underline{\widehat{X}}\cdot\overline{A},\\
\nabla_3 P + \frac{1}{2}\overline{\mathcal{D}}\cdot\underline{B} &= -\frac{3}{2}\overline{tr\underline{X}}P - \frac{1}{2}(\overline{2H-Z})\cdot\underline{B} + \underline{\Xi}\cdot\overline{B} - \frac{1}{4}\overline{\widehat{X}}\cdot\underline{A},\\
\nabla_4\underline{B} + \mathcal{D}P &= -trX\underline{B} + 2\omega\underline{B} + \overline{B}\cdot\underline{\widehat{X}} - 3P\underline{H} - \frac{1}{2}\underline{A}\cdot\overline{\Xi},\\
\nabla_3\underline{B} + \frac{1}{2}\overline{\mathcal{D}}\cdot\underline{A} &= -2\overline{tr\underline{X}}\,\underline{B} - 2\underline{\omega}\,\underline{B} - \frac{1}{2}\underline{A}\cdot(\overline{-2Z+H}) - 3P\,\underline{\Xi},\\
\nabla_4\underline{A} + \frac{1}{2}\mathcal{D}\widehat{\otimes}\underline{B} &= -\frac{1}{2}trX\underline{A} + 4\omega\underline{A} + \frac{1}{2}(Z-4\underline{H})\widehat{\otimes}\underline{B} - 3P\underline{\widehat{X}}.
\end{aligned}$$

2.2. Null frame transformations

Consider two null frames (e_4, e_3, e_1, e_2) and (e'_4, e'_3, e'_1, e'_2) on $\mathcal{M}$ with $H = \{e_3, e_4\}^\perp$ and $H' = \{e'_3, e'_4\}^\perp$ the corresponding horizontal structures. We

[39]Here $\widehat{\otimes}$ is the usual standard notation from [17].

denote by Γ', Γ the connection coefficients relative to the two frames. We denote by $\nabla, \nabla\widehat{\otimes}, \mathrm{div}\,, \mathrm{curl}\,, \nabla_3, \nabla_4$ the standard operators corresponding to to H and by $\nabla', \nabla'\widehat{\otimes}, \mathrm{div}\,', \mathrm{curl}\,', \nabla_3', \nabla_4'$ those corresponding to H'. The goal is to establish transition formulas between the Ricci and curvature coefficients of the two frames.

2.2.1. Transformation between two null frames

Lemma 2.10. *The following transformation formulas hold true.*

1. *A general null transformation*[40] *between two null frames* (e_4, e_3, e_1, e_2) *and* (e_4', e_3', e_1', e_2') *on* $\mathcal{M}$ *can be written in the form,*

$$\begin{aligned} e_4' =&\lambda\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right),\\ e_a' =& \left(\delta_a^b + \frac{1}{2}\underline{f}_a f^b\right) e_b + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right) e_3,\\ e_3' =&\lambda^{-1}\left(\left(1 + \frac{1}{2}f\cdot\underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right) e_3\right.\\ &\left.+\left(\underline{f}^b + \frac{1}{4}|\underline{f}|^2 f^b\right) e_b + \frac{1}{4}|\underline{f}|^2 e_4\right), \end{aligned} \tag{2.6}$$

where λ, $f_a = f^a, \underline{f}_a = \underline{f}^a$ *are scalar functions, called the transition coefficients of the change of frame. Note, in particular,*

$$\begin{aligned} e_a' &= e_a + \frac{1}{2}\underline{f}_a\lambda^{-1}e_4' + \frac{1}{2}f_a e_3,\\ e_3' &= \lambda^{-1}\left(e_3 + \underline{f}^a e_a' - \frac{1}{4}|\underline{f}|^2\lambda^{-1}e_4'\right). \end{aligned}$$

2. *The inverse transformation is given by the formulas*

$$\begin{aligned} e_4 =&\lambda'\left(e_4' + f_b' e_b' + \frac{1}{4}|f'|^2 e_3'\right),\\ e_a =& \left(\delta_a^b + \frac{1}{2}\underline{f}_a' f'^b\right) e_b' + \frac{1}{2}\underline{f}_a' e_4' + \left(\frac{1}{2}f_a' + \frac{1}{8}|f'|^2\underline{f}_a'\right) e_3',\\ e_3 =&(\lambda')^{-1}\left(\left(1 + \frac{1}{2}f'\cdot\underline{f}' + \frac{1}{16}|f'|^2|\underline{f}'|^2\right) e_3'\right. \end{aligned} \tag{2.7}$$

[40] In full generality, one could also rotate e_1', e_2', but this would not change the horizontal structure and, as it turns out, is in fact not needed. The dot product and magnitude $|\cdot|$ are taken with respect to the standard euclidian norm of $\mathbb{R}^2$.

$$+\left(\underline{f}'^{b}+\frac{1}{4}|\underline{f}'|^2 f'^{b}\right)e'_b+\frac{1}{4}|\underline{f}'|^2 e'_4\Big),$$

where

$$\lambda' = \lambda^{-1}\left(1+\frac{1}{2}f\cdot\underline{f}+\frac{1}{16}|f|^2|\underline{f}|^2\right),$$

$$(2.8)\qquad f'_a = -\frac{\lambda}{1+\frac{1}{2}f\cdot\underline{f}+\frac{1}{16}|f|^2|\underline{f}|^2}\left(f_a+\frac{1}{4}|f|^2\underline{f}_a\right),$$

$$\underline{f}'_a = -\lambda^{-1}\left(\underline{f}_a+\frac{1}{4}|\underline{f}|^2 f_a\right).$$

Moreover

$$(2.9)\qquad \underline{f}'_a f'_b = \underline{f}_b f_a, \quad \lambda'|f'|^2 = \lambda|f|^2, \quad (\lambda')^{-1}|\underline{f}'|^2 = \lambda^{-1}|\underline{f}|^2.$$

Denoting $F = (f, \underline{f}, \lambda - 1)$, *we also write, for small* $|F|$,

$$\lambda' = \lambda^{-1}\left(1+\frac{1}{2}f\cdot\underline{f}\right)+O(|F|^3),$$

$$(2.10)\qquad f'_a = -\lambda f_a + O(|F|^3),$$

$$\underline{f}'_a = -\lambda^{-1}\underline{f}_a + O(|F|^3).$$

Proof. For the first part of the lemma, see section 3.1 in [40]. To check the second part we make use of (2.6) to deduce

$$\begin{aligned}
\mathbf{g}(e'_a, e_4) &= -f_a - \frac{1}{4}|f|^2\underline{f}_a,\\
\mathbf{g}(e'_3, e_4) &= -2\lambda^{-1}\left(1+\frac{1}{2}f\cdot\underline{f}+\frac{1}{16}|f|^2|\underline{f}|^2\right),\\
\mathbf{g}(e'_3, e_a) &= \lambda^{-1}\left(\underline{f}_a+\frac{1}{4}|\underline{f}|^2 f_a\right).
\end{aligned}$$

Similarly, using (2.7), we deduce

$$\begin{aligned}
\mathbf{g}(e_4, e'_a) &= \lambda' f'_a,\\
\mathbf{g}(e_4, e'_3) &= -2\lambda',\\
\mathbf{g}(e_a, e'_3) &= -\underline{f}'_a.
\end{aligned}$$

We infer

$$-f_a - \frac{1}{4}|f|^2\underline{f}_a \;=\; \mathbf{g}(e'_a, e_4) = \lambda' f'_a,$$

$$\begin{aligned} -2\lambda^{-1}\left(1+\frac{1}{2}f\cdot\underline{f}+\frac{1}{16}|f|^2|\underline{f}|^2\right) &= \mathbf{g}(e_3', e_4) = -2\lambda',\\ \lambda^{-1}\left(\underline{f}_a+\frac{1}{4}|\underline{f}|^2 f_a\right) &= \mathbf{g}(e_3', e_a) = -\underline{f}'_a, \end{aligned}$$

and hence

$$\begin{aligned} \lambda' f'_a &= -f_a - \frac{1}{4}|f|^2\underline{f}_a,\\ \lambda' &= \lambda^{-1}\left(1+\frac{1}{2}f\cdot\underline{f}+\frac{1}{16}|f|^2|\underline{f}|^2\right),\\ \underline{f}'_a &= -\lambda^{-1}\left(\underline{f}_a+\frac{1}{4}|\underline{f}|^2 f_a\right). \end{aligned}$$

In particular

$$\begin{aligned} f'_a &= -\lambda'^{-1}\left(f_a+\frac{1}{4}|f|^2\underline{f}_a\right)\\ &= -\frac{\lambda}{1+\frac{1}{2}f\cdot\underline{f}+\frac{1}{16}|f|^2|\underline{f}|^2}\left(f_a+\frac{1}{4}|f|^2\underline{f}_a\right). \end{aligned}$$

The identities (2.9) follow from the relations induced by $\mathbf{g}(e_a, e'_b)$, $\mathbf{g}(e_3.e'_3)$, $\mathbf{g}(e_4, e'_4)$. This concludes the proof of the lemma. □

Remark 2.11. *As a consequence of the above lemma both f and f' can be regarded as horizontal vectors on both $H = \{e_3, e_4\}^\perp$ and $H' = \{e'_3, e'_4\}^\perp$.*

2.2.2. Transformation formulas for Ricci and curvature coefficients

In the proposition below $\check{\Gamma}$ denotes any linearized Ricci coefficient, see for example Section 2.6.3 for the definition of linearized Ricci coefficients. Later on, the linearized Ricci coefficients are split into the sets Γ_b and Γ_g, which have different asymptotic behavior, so that $\check{\Gamma} = \Gamma_g \cup \Gamma_b$, see for example Section 2.6.4 for the definition of Γ_g and Γ_b.

Proposition 2.12. *Under a general transformation of type (2.6), the Ricci coefficients transform as follows:*

- *The transformation formula for ξ is given by*

$$\begin{aligned} \lambda^{-2}\xi' &= \xi + \frac{1}{2}\lambda^{-1}\nabla'_4 f + \frac{1}{4}(tr\chi f - {}^{(a)}tr\chi\, {}^*f) + \omega f + Err(\xi,\xi'),\\ Err(\xi,\xi') &= \frac{1}{2}f\cdot\widehat{\chi} + \frac{1}{4}|f|^2\eta + \frac{1}{2}(f\cdot\zeta)\,f - \frac{1}{4}|f|^2\underline{\eta}\\ &\quad + \lambda^{-2}\left(\frac{1}{2}(f\cdot\xi')\,\underline{f} + \frac{1}{2}(f\cdot\underline{f})\,\xi'\right) + \ l.o.t. \end{aligned}$$

- *The transformation formula for $\underline{\xi}$ is given by*

$$\lambda^2\underline{\xi}' = \underline{\xi} + \frac{1}{2}\lambda\nabla_3'\underline{f} + \underline{\omega}\,\underline{f} + \frac{1}{4}tr\underline{\chi}\,\underline{f} - \frac{1}{4}{}^{(a)}tr\underline{\chi}\,{}^*\underline{f} + Err(\underline{\xi},\underline{\xi}'),$$
$$Err(\underline{\xi},\underline{\xi}') = \frac{1}{2}\underline{f}\cdot\widehat{\underline{\chi}} - \frac{1}{2}(\underline{f}\cdot\zeta)\underline{f} + \frac{1}{4}|\underline{f}|^2\underline{\eta} - \frac{1}{4}|\underline{f}|^2\eta' + \ l.o.t.$$

- *The transformation formulas for χ are given by*

$$\begin{aligned}
\lambda^{-1}tr\chi' &= tr\chi + div'f + f\cdot\eta + f\cdot\zeta + Err(tr\chi, tr\chi')\\
Err(tr\chi, tr\chi') &= \underline{f}\cdot\xi + \frac{1}{4}\underline{f}\cdot\Big(f\,tr\chi - {}^*f\,{}^{(a)}tr\chi\Big) + \omega(f\cdot\underline{f})\\
&\quad - \underline{\omega}|f|^2 - \frac{1}{4}|f|^2 tr\underline{\chi} - \frac{1}{4}(f\cdot\underline{f})\lambda^{-1}tr\chi'\\
&\quad + \frac{1}{4}(\underline{f}\wedge f)\lambda^{-1}\,{}^{(a)}tr\chi' + \ l.o.t.,\\
\lambda^{-1}\,{}^{(a)}tr\chi' &= {}^{(a)}tr\chi + curl'f + f\wedge\eta + f\wedge\zeta\\
&\quad + Err({}^{(a)}tr\chi, {}^{(a)}tr\chi'),\\
Err({}^{(a)}tr\chi, {}^{(a)}tr\chi') &= \underline{f}\wedge\xi + \frac{1}{4}\Big(\underline{f}\wedge f\,tr\chi + (f\cdot\underline{f})\,{}^{(a)}tr\chi\Big)\\
&\quad + \omega f\wedge\underline{f} - \frac{1}{4}|f|^2\,{}^{(a)}tr\underline{\chi} - \frac{1}{4}(f\cdot\underline{f})\lambda^{-1}\,{}^{(a)}tr\chi'\\
&\quad + \frac{1}{4}\lambda^{-1}(f\wedge\underline{f})tr\chi' + \ l.o.t.,\\
\lambda^{-1}\widehat{\chi}' &= \widehat{\chi} + \nabla'\widehat{\otimes}f + f\widehat{\otimes}\eta + f\widehat{\otimes}\zeta + Err(\widehat{\chi},\widehat{\chi}'),\\
Err(\widehat{\chi},\widehat{\chi}') &= \underline{f}\widehat{\otimes}\xi + \frac{1}{4}\underline{f}\widehat{\otimes}\Big(f\,tr\chi - {}^*f\,{}^{(a)}tr\chi\Big) + \omega f\widehat{\otimes}\underline{f} - \underline{\omega}f\widehat{\otimes}f\\
&\quad - \frac{1}{4}|f|^2\,{}^{(a)}tr\underline{\chi} + \frac{1}{4}(f\widehat{\otimes}\underline{f})\lambda^{-1}tr\chi' + \frac{1}{4}({}^*f\widehat{\otimes}\underline{f})\lambda^{-1}\,{}^{(a)}tr\chi'\\
&\quad + \frac{1}{2}\underline{f}\widehat{\otimes}(f\cdot\lambda^{-1}\widehat{\chi}') + \ l.o.t.
\end{aligned}$$

- *The transformation formulas for $\underline{\chi}$ are given by*

$$\begin{aligned}
\lambda tr\underline{\chi}' &= tr\underline{\chi} + div'\underline{f} + \underline{f}\cdot\underline{\eta} - \underline{f}\cdot\zeta + Err(tr\underline{\chi}, tr\underline{\chi}'),\\
Err(tr\underline{\chi}, tr\underline{\chi}') &= \frac{1}{2}(f\cdot\underline{f})tr\underline{\chi} + f\cdot\underline{\xi} - |\underline{f}|^2\omega\\
&\quad + (f\cdot\underline{f})\underline{\omega} - \frac{1}{4}|\underline{f}|^2\lambda^{-1}tr\chi' + \ l.o.t.,
\end{aligned}$$

$$
\begin{aligned}
\lambda\, {}^{(a)}tr\underline{\chi}' &= {}^{(a)}tr\underline{\chi} + curl'\underline{f} + \underline{f}\wedge\underline{\eta} - \zeta\wedge\underline{f}\\
&\quad + Err({}^{(a)}tr\underline{\chi}, {}^{(a)}tr\underline{\chi}'),\\
Err({}^{(a)}tr\underline{\chi}, {}^{(a)}tr\underline{\chi}') &= \frac{1}{2}(f\cdot\underline{f})\, {}^{(a)}tr\underline{\chi} + f\wedge\underline{\xi} + (f\wedge\underline{f})\underline{\omega}\\
&\quad - \frac{1}{4}|\underline{f}|^2\lambda^{-1}\, {}^{(a)}tr\chi' + \ l.o.t.,\\
\lambda\widehat{\underline{\chi}}' &= \widehat{\underline{\chi}} + \nabla'\widehat{\otimes}\underline{f} + \underline{f}\widehat{\otimes}\underline{\eta} - \underline{f}\widehat{\otimes}\zeta + Err(\widehat{\underline{\chi}}, \widehat{\underline{\chi}}'),\\
Err(\widehat{\underline{\chi}}, \widehat{\underline{\chi}}') &= \frac{1}{2}(f\widehat{\otimes}\underline{f})tr\,\underline{\chi} + f\widehat{\otimes}\underline{\xi} - (\underline{f}\widehat{\otimes}\underline{f})\omega + (f\widehat{\otimes}\underline{f})\underline{\omega}\\
&\quad - \frac{1}{4}|\underline{f}|^2\lambda^{-1}\widehat{\chi}' + \ l.o.t.
\end{aligned}
$$

- *The transformation formula for* ζ *is given by*

$$
\begin{aligned}
\zeta' &= \zeta - \nabla'(\log\lambda) - \frac{1}{4}tr\underline{\chi}f + \frac{1}{4}\, {}^{(a)}tr\underline{\chi}\ {}^*f + \omega\underline{f} - \underline{\omega}f\\
&\quad + \frac{1}{4}\underline{f}tr\chi + \frac{1}{4}\ {}^*\underline{f}\, {}^{(a)}tr\chi + Err(\zeta,\zeta'),\\
Err(\zeta,\zeta') &= -\frac{1}{2}\widehat{\underline{\chi}}\cdot f + \frac{1}{2}(f\cdot\zeta)\underline{f} - \frac{1}{2}(f\cdot\underline{\eta})\underline{f} + \frac{1}{4}\underline{f}(f\cdot\eta) + \frac{1}{4}\underline{f}(f\cdot\zeta)\\
&\quad + \frac{1}{4}\ {}^*\underline{f}(f\wedge\eta) + \frac{1}{4}\ {}^*\underline{f}(f\wedge\zeta) + \frac{1}{4}\underline{f}div'f + \frac{1}{4}\ {}^*\underline{f}curl'f\\
&\quad + \frac{1}{2}\lambda^{-1}\underline{f}\cdot\widehat{\chi}' - \frac{1}{16}(f\cdot\underline{f})\underline{f}\lambda^{-1}tr\chi' + \frac{1}{16}(\underline{f}\wedge f)\underline{f}\lambda^{-1}\, {}^{(a)}tr\chi'\\
&\quad - \frac{1}{16}\ {}^*\underline{f}(f\cdot\underline{f})\lambda^{-1}\, {}^{(a)}tr\chi' + \frac{1}{16}\ {}^*\underline{f}\lambda^{-1}(f\wedge\underline{f})tr\chi' + \ l.o.t.
\end{aligned}
$$

- *The transformation formula for* η *is given by*

$$
\begin{aligned}
\eta' &= \eta + \frac{1}{2}\lambda\nabla_3'f + \frac{1}{4}\underline{f}tr\chi - \frac{1}{4}\ {}^*\underline{f}\, {}^{(a)}tr\chi - \underline{\omega}\,f + Err(\eta,\eta'),\\
Err(\eta,\eta') &= \frac{1}{2}(f\cdot\underline{f})\eta + \frac{1}{2}\underline{f}\cdot\widehat{\chi} + \frac{1}{2}f(\underline{f}\cdot\zeta) - (\underline{f}\cdot f)\eta'\\
&\quad + \frac{1}{2}\underline{f}(f\cdot\eta') + \ l.o.t.
\end{aligned}
$$

- *The transformation formula for* $\underline{\eta}$ *is given by*

$$
\begin{aligned}
\underline{\eta}' &= \underline{\eta} + \frac{1}{2}\lambda^{-1}\nabla_4'\underline{f} + \frac{1}{4}tr\underline{\chi}f - \frac{1}{4}\, {}^{(a)}tr\underline{\chi}\ {}^*f - \omega\underline{f} + Err(\underline{\eta},\underline{\eta}'),\\
Err(\underline{\eta},\underline{\eta}') &= \frac{1}{2}f\cdot\widehat{\underline{\chi}} + \frac{1}{2}(f\cdot\underline{\eta})\underline{f} - \frac{1}{4}(f\cdot\zeta)\underline{f} - \frac{1}{4}|\underline{f}|^2\lambda^{-2}\xi' + \ l.o.t.
\end{aligned}
$$

- *The transformation formula for* ω *is given by*

$$\begin{aligned}\lambda^{-1}\omega' &= \omega - \frac{1}{2}\lambda^{-1}e_4'(\log\lambda) + \frac{1}{2}f\cdot(\zeta-\underline{\eta}) + Err(\omega,\omega'),\\ Err(\omega,\omega') &= -\frac{1}{4}|f|^2\underline{\omega} - \frac{1}{8}tr\,\underline{\chi}|f|^2 + \frac{1}{2}\lambda^{-2}\underline{f}\cdot\xi' + \text{ l.o.t.}\end{aligned}$$

- *The transformation formula for* $\underline{\omega}$ *is given by*

$$\begin{aligned}\lambda\underline{\omega}' &= \underline{\omega} + \frac{1}{2}\lambda e_3'(\log\lambda) - \frac{1}{2}\underline{f}\cdot\zeta - \frac{1}{2}\underline{f}\cdot\eta + Err(\underline{\omega},\underline{\omega}'),\\ Err(\underline{\omega},\underline{\omega}') &= f\cdot\underline{f}\,\underline{\omega} - \frac{1}{4}|\underline{f}|^2\omega + \frac{1}{2}f\cdot\underline{\xi} + \frac{1}{8}(f\cdot\underline{f})tr\,\underline{\chi}\\ &\quad + \frac{1}{8}(\underline{f}\wedge f)\,{}^{(a)}tr\underline{\chi} - \frac{1}{8}|\underline{f}|^2 tr\,\chi - \frac{1}{4}\lambda\underline{f}\cdot\nabla_3' f\\ &\quad + \frac{1}{2}(\underline{f}\cdot f)(\underline{f}\cdot\eta') - \frac{1}{4}|\underline{f}|^2(f\cdot\eta') + \text{ l.o.t.}\end{aligned}$$

where, for the transformation formulas of the Ricci coefficients above, l.o.t. denote expressions of the type

$$\text{l.o.t.} \quad = O((f,\underline{f})^3)\Gamma + O((f,\underline{f})^2)\check{\Gamma}$$

containing no derivatives of f, $\underline{f}$, Γ *and* $\check{\Gamma}$.

Also, the curvature components transform as follows:

- *The transformation formula for* $\alpha, \underline{\alpha}$ *are given by*

$$\begin{aligned}\lambda^{-2}\alpha' &= \alpha + Err(\alpha,\alpha'),\\ Err(\alpha,\alpha') &= (f\widehat{\otimes}\beta - {}^*f\widehat{\otimes}\,{}^*\beta) + \left(f\widehat{\otimes}f - \frac{1}{2}\,{}^*f\widehat{\otimes}\,{}^*f\right)\rho\\ &\quad + \frac{3}{2}(f\widehat{\otimes}\,{}^*f)\ {}^*\rho + \text{ l.o.t.},\\ \lambda^2\underline{\alpha}' &= \underline{\alpha} + Err(\alpha,\alpha'),\\ Err(\underline{\alpha},\underline{\alpha}') &= -(\underline{f}\widehat{\otimes}\underline{\beta} - {}^*\underline{f}\widehat{\otimes}\,{}^*\underline{\beta}) + (\underline{f}\widehat{\otimes}\underline{f} - \frac{1}{2}\,{}^*\underline{f}\widehat{\otimes}\,{}^*\underline{f})\rho\\ &\quad + \frac{3}{2}(\underline{f}\widehat{\otimes}\,{}^*\underline{f})\ {}^*\rho + \text{ l.o.t.}\end{aligned}$$

- *The transformation formula for $\beta, \underline{\beta}$ are given by*

$$
\begin{aligned}
\lambda^{-1}\beta' &= \beta + \frac{3}{2}(f\rho + {}^*f\ {}^*\rho) + Err(\beta, \beta'),\\
Err(\beta, \beta') &= \frac{1}{2}\alpha \cdot \underline{f} + \ l.o.t.,\\
\lambda\underline{\beta}' &= \underline{\beta} - \frac{3}{2}(\underline{f}\rho + {}^*\underline{f}\ {}^*\rho) + Err(\underline{\beta}, \underline{\beta}'),\\
Err(\underline{\beta}, \underline{\beta}') &= -\frac{1}{2}\underline{\alpha} \cdot f + \ l.o.t.
\end{aligned}
$$

- *The transformation formula for ρ and ${}^*\rho$ are given by*

$$
\begin{aligned}
\rho' &= \rho + Err(\rho, \rho'),\\
Err(\rho, \rho') &= \underline{f} \cdot \beta - f \cdot \underline{\beta} + \frac{3}{2}\rho(f \cdot \underline{f}) - \frac{3}{2}\ {}^*\rho(f \wedge \underline{f}) + \ l.o.t.\\
{}^*\rho' &= {}^*\rho + Err({}^*\rho, {}^*\rho'),\\
Err({}^*\rho, {}^*\rho') &= -\underline{f} \cdot {}^*\beta - f \cdot {}^*\underline{\beta} + \frac{3}{2}\ {}^*\rho(f \cdot \underline{f}) + \frac{3}{2}\rho(f \wedge \underline{f}) + \ l.o.t.
\end{aligned}
$$

where, for the transformation formulas of the curvature components above, l.o.t. denote expressions of the type

$$
l.o.t. \ = O((f, \underline{f})^3)(\rho, {}^*\rho) + O((f, \underline{f})^2)(\alpha, \beta, \underline{\alpha}, \beta)
$$

containing no derivatives of f, $\underline{f}$, α, β, $(\rho, {}^*\rho)$, $\underline{\beta}$, and $\underline{\alpha}$.

Proof. See Proposition 3.3 in [40]. □

2.2.3. Transport equations for $(f, \underline{f}, \lambda)$

Corollary 2.13. *Under the assumption*

$$
\xi' = 0, \qquad \omega' = 0, \qquad \underline{\eta}' + \zeta' = 0,
$$

we have the following transport equations for $(f, \underline{f}, \lambda)$

$$
\begin{aligned}
\nabla_{\lambda^{-1}e_4'} f + \frac{1}{2}(tr\chi f - {}^{(a)}tr\chi\ {}^*f) + 2\omega f &= -2\xi - f \cdot \widehat{\chi} + E_1(f, \Gamma),\\
\lambda^{-1}e_4'(\log\lambda) &= 2\omega + f \cdot (\zeta - \underline{\eta}) + E_2(f, \Gamma),\\
\nabla_{\lambda^{-1}e_4'} \underline{f} + \frac{1}{2}(tr\chi \underline{f} + {}^{(a)}tr\chi\ {}^*\underline{f}) &= -2(\underline{\eta} + \zeta) + 2\nabla'(\log\lambda) + 2\underline{\omega} f
\end{aligned}
$$

$$+E_3(\nabla'^{\leq 1}f, \underline{f}, \Gamma, \lambda^{-1}\chi'),$$

where $E_1(f,\Gamma)$, $E_2(f,\Gamma)$ *and* $E_3(\nabla'^{\leq 1}f, \underline{f}, \Gamma, \lambda^{-1}\chi')$ *are given by*

$$\begin{aligned}
E_1(f,\Gamma) &= -(f\cdot\zeta)f - \frac{1}{2}|f|^2\eta + \frac{1}{2}|f|^2\underline{\eta} + O(f^3\Gamma),\\
E_2(f,\Gamma) &= -\frac{1}{2}|f|^2\underline{\omega} - \frac{1}{4}tr\,\underline{\chi}|f|^2 + O(f^3\Gamma + f^2\widehat{\underline{\chi}}),
\end{aligned}$$

and

$$\begin{aligned}
E_3(\nabla'^{\leq 1}f, \underline{f}, \Gamma, \lambda^{-1}\chi') &= \frac{1}{4}(f\cdot\eta)\underline{f} + \frac{1}{2}(f\cdot\zeta)\underline{f}\\
&+\frac{1}{4}\,{}^*\underline{f}(f\wedge\eta) + \frac{1}{4}\,{}^*\underline{f}(f\wedge\zeta) + \frac{1}{4}\underline{f}\,div' f\\
&+\frac{1}{4}\,{}^*\underline{f}\,curl' f + \frac{1}{2}\lambda^{-1}\underline{f}\cdot\widehat{\chi}'\\
&+O\Big((\lambda^{-1}tr\chi', \lambda^{-1}\,{}^{(a)}tr\chi')(f,\underline{f})^3 + (f,\underline{f})^3\Gamma\Big).
\end{aligned}$$

Proof. See Section A.1. ∎

Corollary 2.14. *Assume that we have*

$$\Xi' = 0, \qquad \omega' = 0, \qquad \underline{H}' + Z' = 0.$$

We introduce

$$F := f + i\,{}^*f, \qquad \underline{F} := \underline{f} + i\,{}^*\underline{f}.$$

Then, we have

$$\begin{aligned}
\nabla_{\lambda^{-1}e_4'}F + \frac{1}{2}\overline{trX}F + 2\omega F &= -2\Xi - \widehat{\chi}\cdot F + E_1(f,\Gamma),\\
\lambda^{-1}\nabla_4'(\log\lambda) &= 2\omega + f\cdot(\zeta - \underline{\eta}) + E_2(f,\Gamma),\\
\nabla_{\lambda^{-1}e_4'}\underline{F} + \frac{1}{2}trX\underline{F} &= -2(\underline{H} + Z) + 2\mathcal{D}'(\log\lambda) + 2\underline{\omega}F\\
&\quad +E_3(\nabla'^{\leq 1}f, \underline{f}, \Gamma, \lambda^{-1}\chi').
\end{aligned}$$

Moreover, introducing a complex valued scalar function q *satisfying* $e_4(q) = 1$, *we have*

$$\nabla_{\lambda^{-1}e_4'}(\overline{q}F) = -2\overline{q}\omega F - 2\overline{q}\Xi + E_4(f,\Gamma),$$

$$\begin{aligned}\nabla_{\lambda^{-1}e'_4}\Big[q\Big(\underline{F}-2q\mathcal{D}'(\log\lambda)\Big)+\overline{q}e_3(r)F\Big] &= -2q(\,\underline{H}+Z)\\ &\quad -2q^2\mathcal{D}'\Big(2\omega+f\cdot(\zeta-\underline{\eta})\Big)\\ &\quad +e_3(r)\left(-2\overline{q}\omega F-2\overline{q}\Xi\right)\\ &\quad +2\underline{\omega}(q-\overline{q})F\\ &\quad +E_5(\nabla'^{\leq 1}f,\underline{f},\nabla'^{\leq 1}\lambda,\mathbf{D}^{\leq 1}\Gamma),\end{aligned}$$

where

$$\begin{aligned}E_4(f,\Gamma) &:= -\frac{1}{2}\overline{q}\left(\overline{trX}-\frac{2}{\overline{q}}\right)F-\overline{q}\widehat{\chi}\cdot F+\overline{q}E_1(f,\Gamma)+f\cdot\nabla(\overline{q})F\\ &\quad +\frac{1}{4}|f|^2e_3(\overline{q})F\end{aligned}$$

and

$$\begin{aligned}&E_5(\nabla'^{\leq 1}f,\underline{f},\nabla'^{\leq 1}\lambda,\mathbf{D}^{\leq 1}\Gamma)\\ =\;& -\frac{q}{2}\left(trX-\frac{2}{q}\right)\underline{F}+qE_3(\nabla'^{\leq 1}f,\underline{f},\Gamma,\lambda^{-1}\chi')-2q^2\mathcal{D}'\left(E_2(f,\Gamma)\right)\\ & -2q[\nabla_{\lambda^{-1}e'_4},q\mathcal{D}']\log\lambda+\left(f\cdot\nabla(q)+\frac{1}{4}|f|^2e_3(q)\right)\Big(\underline{F}-2q\mathcal{D}'(\log\lambda)\Big)\\ & +e_3(r)E_4(f,\Gamma).\end{aligned}$$

Proof. See Section A.2. □

Remark 2.15. *In practice, we will integrate first the transport equations for F, then for λ. Finally, we will integrate the renormalized transport equation for $\underline{F}$ and we recover one less derivative than for F and $\log(\lambda)$. Note that the renormalization of the transport equation for $\underline{F}$ is needed in order to avoid a potential log-loss due to the terms $\mathcal{D}'(\log\lambda)$ and $\underline{\omega}F$ on the RHS.*

2.3. Principal geodesic structures

2.3.1. Principal outgoing geodesic structures

Definition 2.16 (PG structure)**.** *An outgoing PG structure consists of a null pair (e_3, e_4) and the induced horizontal structure $\mathcal{H} = \mathbf{O}(\mathcal{M})$, together with a scalar function r such that:*

1. *e_4 is a null outgoing geodesic vectorfield, i.e. $\mathbf{D}_4 e_4 = 0$,*
2. *r is an affine parameter, i.e. $e_4(r) = 1$,*
3. *the gradient of r, given by $N = \mathbf{g}^{\alpha\beta}\partial_\beta r\partial_\alpha$, is perpendicular to $\mathcal{H}$.*

Lemma 2.17. *Given a PG structure as above, we have*

$$\omega = 0, \qquad \xi = 0, \qquad \underline{\eta} + \zeta = 0.$$

Proof. Since e_4 is geodesic, we have $\xi = \omega = 0$. Also, in view of the Ricci formulas (2.5),

$$[e_a, e_4] \;=\; \chi_{ab} e_b - \nabla_4 e_a - (\zeta + \underline{\eta})_a e_4,$$

with $(e_a)_{a=1,2}$ an orthonormal basis of $\mathcal{H}$. Applying the commutator formula to r, and using $e_4(r) = 1$, $e_1(r) = e_2(r) = 0$, and $(\nabla_4 e_a)(r) = 0$, we infer that

$$\underline{\eta} + \zeta \;=\; 0$$

as stated. □

In view of the above, the following relations hold for PG structure

$$\xi = 0, \qquad \omega = 0, \qquad \underline{\eta} + \zeta = 0, \qquad e_4(r) = 1, \qquad \nabla(r) = 0.$$

The following lemma shows how to initialize a PG structure on a hypersurface of $\mathcal{M}$ transversal to e_4.

Lemma 2.18. *Consider a hypersurface Σ in $\mathcal{M}$. Let a scalar function r and a null pair (e_4, e_3) be both defined on Σ, with $\mathcal{H}$ the corresponding horizontal space. Assume that e_4 is transversal to Σ, and impose the transversality condition*

$$e_4(r) = 1 \quad \text{on} \;\; \Sigma.$$

Under this transversality condition[41]*, we assume that*[42] *$\mathcal{H}(r) = 0$. Then, we can extend r and the null frame (e_4, e_3) uniquely to a PG structure.*

Proof. Since e_4 is transversal to Σ, we may extend it geodesically, i.e. $\mathbf{D}_{e_4} e_4 = 0$, to a neighborhood of Σ. We then extend r in a neighborhood of Σ such that $e_4(r) = 1$. Since $\mathcal{H}(r) = 0$ on Σ we have $\mathcal{H}$ orthogonal to $N = \mathbf{g}^{\alpha\beta}\partial_\beta r \partial_\alpha$ on Σ. Since e_4 and N are well defined in a neighborhood of Σ, we extend $\mathcal{H}$ by choosing it to be orthogonal to both. We then choose e_3 the unique null vector perpendicular to $\mathcal{H}$ and such that $\mathbf{g}(e_3, e_4) = -2$. Thus r and the frame (e_3, e_4) define a PG structure in a neighborhood of Σ coinciding with the given one on Σ. □

[41]Note that we need to take the transversality condition into account since $\mathcal{H}$ is not necessarily included in the tangent space of Σ.

[42]That is $X(r) = 0$ for any $X \in \mathcal{H}$.

2.3.2. Null structure and Bianchi identities for an outgoing PG structure In an outgoing PG structure, Propositions 2.8 and 2.9 take the following form.

Proposition 2.19.

$$\begin{aligned}
\nabla_4 trX + \frac{1}{2}(trX)^2 &= -\frac{1}{2}\widehat{X}\cdot\overline{\widehat{X}},\\
\nabla_4\widehat{X} + \Re(trX)\widehat{X} &= -A,\\
\nabla_4 tr\underline{X} + \frac{1}{2}trX\, tr\underline{X} &= -\mathcal{D}\cdot\overline{Z} + Z\cdot\overline{Z} + 2\overline{P} - \frac{1}{2}\widehat{X}\cdot\overline{\widehat{\underline{X}}},\\
\nabla_4\widehat{\underline{X}} + \frac{1}{2}trX\,\widehat{\underline{X}} &= -\frac{1}{2}\mathcal{D}\widehat{\otimes}Z + \frac{1}{2}Z\widehat{\otimes}Z - \frac{1}{2}\overline{tr\underline{X}}\widehat{X},\\
\nabla_3 tr\underline{X} + \frac{1}{2}(tr\underline{X})^2 + 2\underline{\omega}\, tr\underline{X} &= \mathcal{D}\cdot\overline{\underline{\Xi}} - \underline{\Xi}\cdot\overline{Z} + \overline{\underline{\Xi}}\cdot(H-2Z) - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{\underline{X}}},\\
\nabla_3\widehat{\underline{X}} + \Re(tr\underline{X})\widehat{\underline{X}} + 2\underline{\omega}\,\widehat{\underline{X}} &= \frac{1}{2}\mathcal{D}\widehat{\otimes}\underline{\Xi} + \frac{1}{2}\underline{\Xi}\widehat{\otimes}(H-3Z) - \underline{A},\\
\nabla_3 trX + \frac{1}{2}tr\underline{X}trX - 2\underline{\omega}trX &= \mathcal{D}\cdot\overline{H} + H\cdot\overline{H} + 2P - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{X}},\\
\nabla_3\widehat{X} + \frac{1}{2}tr\underline{X}\,\widehat{X} - 2\underline{\omega}\widehat{X} &= \frac{1}{2}\mathcal{D}\widehat{\otimes}H + \frac{1}{2}H\widehat{\otimes}H - \frac{1}{2}\overline{trX}\widehat{\underline{X}}.
\end{aligned}$$

Also,

$$\begin{aligned}
\nabla_4 Z + trXZ &= -\widehat{X}\cdot\overline{Z} - B,\\
\nabla_4 H + \frac{1}{2}\overline{trX}(H+Z) &= -\frac{1}{2}\widehat{X}\cdot(\overline{H}+\overline{Z}) - B,\\
\nabla_3 Z + \frac{1}{2}tr\underline{X}(Z+H) - 2\underline{\omega}(Z-H) &= -2\mathcal{D}\underline{\omega} - \frac{1}{2}\widehat{\underline{X}}\cdot(\overline{Z}+\overline{H}) + \frac{1}{2}trX\underline{\Xi}\\
&\quad -\underline{B} + \frac{1}{2}\overline{\underline{\Xi}}\cdot\widehat{X},\\
\nabla_3 Z + \nabla_4\underline{\Xi} &= -\frac{1}{2}\overline{tr\underline{X}}(Z+H) - \frac{1}{2}\widehat{\underline{X}}\cdot(\overline{Z}+\overline{H})\\
&\quad -\underline{B},
\end{aligned}$$

and

$$\nabla_4\underline{\omega} - (2\eta+\zeta)\cdot\zeta = \rho.$$

Also,

$$\frac{1}{2}\overline{\mathcal{D}}\cdot\widehat{X} + \frac{1}{2}\widehat{X}\cdot\overline{Z} = \frac{1}{2}\mathcal{D}\overline{trX} + \frac{1}{2}\overline{trX}Z - i\Im(trX)H - B,$$

$$\frac{1}{2}\overline{\mathcal{D}}\cdot\underline{\widehat{X}}-\frac{1}{2}\underline{\widehat{X}}\cdot\overline{Z} = \frac{1}{2}\mathcal{D}\overline{tr\underline{X}}-\frac{1}{2}\overline{tr\underline{X}}Z-i\Im(tr\underline{X})(-Z+\underline{\Xi})+\underline{B}.$$

Also,

$$\begin{aligned}
\nabla_3 A-\frac{1}{2}\mathcal{D}\widehat{\otimes}B &= -\frac{1}{2}tr\underline{X}A+4\underline{\omega}A+\frac{1}{2}(Z+4H)\widehat{\otimes}B-3\overline{P}\widehat{X},\\
\nabla_4 B-\frac{1}{2}\overline{\mathcal{D}}\cdot A &= -2\overline{trX}B+\frac{1}{2}A\cdot\overline{Z},\\
\nabla_3 B-\mathcal{D}\overline{P} &= -tr\underline{X}B+2\underline{\omega}B+\overline{\underline{B}}\cdot\widehat{X}+3\overline{P}H+\frac{1}{2}A\cdot\overline{\underline{\Xi}},\\
\nabla_4 P-\frac{1}{2}\mathcal{D}\cdot\overline{B} &= -\frac{3}{2}trXP-\frac{1}{2}Z\cdot\overline{B}-\frac{1}{4}\underline{\widehat{X}}\cdot\overline{A},\\
\nabla_3 P+\frac{1}{2}\overline{\mathcal{D}}\cdot\underline{B} &= -\frac{3}{2}\overline{tr\underline{X}}P-\frac{1}{2}(\overline{2H-Z})\cdot\underline{B}+\underline{\Xi}\cdot\overline{B}-\frac{1}{4}\overline{\widehat{X}}\cdot\underline{A},\\
\nabla_4\underline{B}+\mathcal{D}P &= -trX\underline{B}+\overline{B}\cdot\underline{\widehat{X}}+3PZ,\\
\nabla_3\underline{B}+\frac{1}{2}\overline{\mathcal{D}}\cdot\underline{A} &= -2\overline{tr\underline{X}}\,\underline{B}-2\underline{\omega}\,\underline{B}-\frac{1}{2}\underline{A}\cdot(\overline{-2Z+H})-3P\,\underline{\Xi},\\
\nabla_4\underline{A}+\frac{1}{2}\mathcal{D}\widehat{\otimes}\underline{B} &= -\frac{1}{2}\overline{trX}\underline{A}+\frac{5}{2}Z\widehat{\otimes}\underline{B}-3P\underline{\widehat{X}}.
\end{aligned}$$

Proof. We are simply making use of the fact that, in an outgoing PG structure, we have $\omega=\xi=0$, $\underline{\eta}=-\zeta$, and hence also $\Xi=0$, $\underline{H}=-Z$. □

2.3.3. Coordinates associated to an outgoing PG structure

Definition 2.20. *Assume given $\{r,(e_3,e_4),\mathcal{H}\}$ an outgoing principal geodesic structure. In addition to r, we define scalar functions (u,θ,φ) such that*

$$e_4(u)=e_4(\theta)=e_4(\varphi)=0. \tag{2.11}$$

Proposition 2.21. *The following equations hold true for the coordinates (u,r,θ,φ) associated to an outgoing PG structure*

$$\begin{aligned}
e_4(e_3(r)) &= -2\underline{\omega},\\
\nabla_4\mathcal{D}u+\frac{1}{2}trX\mathcal{D}u &= -\frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}u,\\
e_4(e_3(u)) &= -\Re\Big((Z+H)\cdot\overline{\mathcal{D}}u\Big),\\
\nabla_4(\mathcal{D}\cos\theta)+\frac{1}{2}trX(\mathcal{D}\cos\theta) &= -\frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}(\cos\theta),\\
e_4(e_3(\cos\theta)) &= -\Re\Big((Z+H)\cdot\overline{\mathcal{D}}(\cos\theta)\Big).
\end{aligned}$$

Proof. For a scalar function f, we have the commutator formulas

$$\begin{aligned}[]
[\nabla_4, \mathcal{D}]f &= -\frac{1}{2}\mathrm{tr}X\mathcal{D}f - \frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}f,\\
[\nabla_4, \nabla_3]f &= -\Re\Big((Z+H)\cdot\overline{\mathcal{D}}f\Big) - 2\underline{\omega}\nabla_4 f.
\end{aligned}$$

Hence, for a scalar function f such that $\nabla_4 f = 1$ of $\nabla_4 f = 0$, we have

$$\begin{aligned}
\nabla_4\mathcal{D}f + \frac{1}{2}\mathrm{tr}X\mathcal{D}f &= -\frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}f,\\
e_4(e_3(f)) &= -\Re\Big((Z+H)\cdot\overline{\mathcal{D}}f\Big) - 2\underline{\omega}\nabla_4 f.
\end{aligned}$$

We then apply these transport equations respectively with the choice $f = r$, $f = u$ and $f = \cos\theta$. □

2.3.4. Integrable frame adapted to a PG structure Given a PG structure[43] $\{r, (e_3, e_4, e_1, e_2)\}$, see Definition 2.16, and a coordinate u transported by $e_4(u) = 0$, we construct in the following lemma a new frame (e_3', e_4', e_1', e_2') adapted to the topological spheres $S(u, r)$, i.e. with (e_1', e_2') tangent to $S(u, r)$.

Lemma 2.22. *Consider a PG structure $\{r, (e_3, e_4, e_1, e_2)\}$, and a coordinate u transported by $e_4(u) = 0$, and assume that*

$$e_3(u) > 0, \qquad (e_3(u))^2 + 4|\nabla u|^2 e_3(r) > 0.$$

Then, there exists a frame transformation of type (2.6), taking (e_1, e_2, e_3, e_4) into a frame (e_3', e_4', e_1', e_2'), and verifying the following conditions

1. *The horizontal vectors (e_1', e_2') are tangent to $S(u, r)$.*
2. *We have $\mathbf{g}(e_3', e_4) = -2$.*

Moreover, the coefficients $(\lambda, f, \underline{f})$ of the frame transformation are given by[44]

[43]Here, (e_1, e_2) denotes an arbitrary orthonormal basis of the horizontal space $\mathcal{H}$.

[44]Note that the formula for λ implies $\lambda > 0$. Indeed, using $e_3(u) > 0$ and $(e_3(u))^2 + 4|\nabla u|^2 e_3(r) > 0$, and the above formulas for λ, f and $\underline{f}$, we have

$$\begin{aligned}
\lambda &= 1 - \frac{4|\nabla u|^2 e_3(r)}{\Big(e_3(u) + \sqrt{(e_3(u))^2 + 4|\nabla u|^2 e_3(r)}\Big)\sqrt{(e_3(u))^2 + 4|\nabla u|^2 e_3(r)}}\\
&\quad + \frac{1}{16}|f|^2|\underline{f}|^2\\
&\geq \frac{(e_3(u))^2}{(e_3(u))^2 + 4|\nabla u|^2 e_3(r)} > 0.
\end{aligned}$$

$$
\begin{aligned}
f &= -\frac{4}{e_3(u)+\sqrt{(e_3(u))^2+4|\nabla u|^2 e_3(r)}}\nabla u,\\
\underline{f} &= \frac{2e_3(r)}{\sqrt{(e_3(u))^2+4|\nabla u|^2 e_3(r)}}\nabla u, \qquad (2.12)\\
\lambda &= 1+\frac{1}{2}f\cdot\underline{f}+\frac{1}{16}|f|^2|\underline{f}|^2.
\end{aligned}
$$

Remark 2.23. *In Kerr, we have, see Section 2.4,*

$$
|\nabla u|^2 = \frac{a^2(\sin\theta)^2}{|q|^2}, \qquad e_3(r) = -\frac{\Delta}{|q|^2}, \qquad e_3(u) = \frac{2(r^2+a^2)}{|q|^2},
$$

and hence

$$
e_3(u) > 0, \qquad (e_3(u))^2 + 4|\nabla u|^2 e_3(r) = \frac{4\Sigma^2}{|q|^4} > 0
$$

as well as

$$
f = -\frac{2|q|^2}{r^2+a^2+\Sigma}\nabla u, \qquad \underline{f} = -\frac{\Delta}{\Sigma}\nabla u.
$$

Proof. Recall the frame transformation (2.6)

$$
\begin{aligned}
e_4' &= \lambda\left(e_4+f^b e_b+\frac{1}{4}|f|^2 e_3\right),\\
e_a' &= \left(\delta_a^b+\frac{1}{2}\underline{f}_a f^b\right)e_b+\frac{1}{2}\underline{f}_a e_4+\left(\frac{1}{2}f_a+\frac{1}{8}|f|^2\underline{f}_a\right)e_3,\\
e_3' &= \lambda^{-1}\left(\left(1+\frac{1}{2}f\cdot\underline{f}+\frac{1}{16}|f|^2|\underline{f}|^2\right)e_3+\left(\underline{f}^b+\frac{1}{4}|\underline{f}|^2 f^b\right)e_b+\frac{1}{4}|\underline{f}|^2 e_4\right).
\end{aligned}
$$

In particular, the choice for λ is equivalent to the normalization $\mathbf{g}(e_3', e_4) = -2$. It thus suffices to look for $(f, \underline{f})$ such that (e_1', e_2') is tangent to $S(u,r)$, which is equivalent to $e_a'(r) = e_a'(u) = 0$. Now, recall that

$$
e_4(r) = 1, \qquad e_4(u) = 0, \qquad e_1(r) = e_2(r) = 0.
$$

Thus, we have $e_a'(r) = e_a'(u) = 0$ if and only if

$$
\begin{aligned}
0 &= \underline{f}_a + \left(f_a + \frac{1}{4}|f|^2\underline{f}_a\right)e_3(r),\\
0 &= \left(\delta_a^b+\frac{1}{2}\underline{f}_a f^b\right)e_b(u)+\left(\frac{1}{2}f_a+\frac{1}{8}|f|^2\underline{f}_a\right)e_3(u),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\underline{f} &= -\frac{e_3(r)}{1+\frac{1}{4}|f|^2e_3(r)}f,\\
0 &= \nabla u+\frac{1}{2}(f\cdot\nabla u)\underline{f}+\left(\frac{1}{2}f+\frac{1}{8}|f|^2\underline{f}\right)e_3(u).
\end{aligned}
$$

Plugging the first equation in the second, we infer

$$
-(f\cdot\nabla u)e_3(r)f+e_3(u)f+2\left(1+\frac{1}{4}|f|^2e_3(r)\right)\nabla u = 0.
$$

We look for f under the form

$$
f=h\nabla u
$$

where h is a scalar function to be determined. Plugging in the equation above, we obtain

$$
\left(-h^2|\nabla u|^2e_3(r)+he_3(u)+2\left(1+\frac{1}{4}h^2|\nabla u|^2e_3(r)\right)\right)\nabla u = 0.
$$

Thus, (e_1', e_2') is tangent to $S(u,r)$ if and only if h satisfies

$$
-\frac{1}{2}|\nabla u|^2e_3(r)h^2+e_3(u)h+2=0.
$$

The discriminant is given by

$$
Disc = (e_3(u))^2+4|\nabla u|^2e_3(r)>0,
$$

where the strict positivity comes from the assumptions, and the roots are given by

$$
h_\pm = \frac{-e_3(u)\pm\sqrt{(e_3(u))^2+4|\nabla u|^2e_3(r)}}{-|\nabla u|^2e_3(r)}.
$$

We choose the root h_+, i.e.

$$
\begin{aligned}
h &= \frac{-e_3(u)+\sqrt{(e_3(u))^2+4|\nabla u|^2e_3(r)}}{-|\nabla u|^2e_3(r)}\\
&= \frac{4|\nabla u|^2e_3(r)}{-|\nabla u|^2e_3(r)\big(e_3(u)+\sqrt{(e_3(u))^2+4|\nabla u|^2e_3(r)}\big)}
\end{aligned}
$$

$$= -\frac{4}{e_3(u) + \sqrt{(e_3(u))^2 + 4|\nabla u|^2 e_3(r)}},$$

where we recall that assumption $e_3(u) > 0$. We infer

$$f = h\nabla u = -\frac{4}{e_3(u) + \sqrt{(e_3(u))^2 + 4|\nabla u|^2 e_3(r)}}\nabla u$$

and

$$\begin{aligned}\underline{f} &= -\frac{e_3(r)}{1 + \frac{1}{4}|f|^2 e_3(r)} f = -\frac{h e_3(r)}{1 + \frac{1}{4}h^2|\nabla u|^2 e_3(r)}\nabla u = -\frac{2h e_3(r)}{4 + e_3(u)h}\nabla u\\ &= \frac{2e_3(r)}{\sqrt{(e_3(u))^2 + 4|\nabla u|^2 e_3(r)}}\nabla u\end{aligned}$$

as desired. □

Definition 2.24. *We refer to the frame introduced in Lemma 2.22 as the associated integrable frame to the PG structure.*

Corollary 2.25. *Let (e_3, e_4, e_1, e_2) be a PG frame and denote (e_3', e_4', e_1', e_2') the associated integrable frame. Consider the frame transformation of type (2.6), taking (e_3', e_4', e_1', e_2') into (e_3, e_4, e_1, e_2), and let $(\lambda', f', \underline{f}')$ be the corresponding coefficients. Then, we have $\lambda' = 1$.*

Proof. The frame transformation of type (2.6), taking (e_3', e_4', e_1', e_2') into (e_3, e_4, e_1, e_2), is the inverse transformation of the one of Lemma 2.22. Thus, the proof follows immediately from the properties of $(\lambda, f, \underline{f})$ in Lemma 2.22 and the identities (2.8) relating $(\lambda, f, \underline{f})$ and $(\lambda', f', \underline{f}')$. □

2.4. Canonical outgoing PG structure in Kerr

2.4.1. Boyer-Lindquist coordinates The Kerr metric in standard Boyer-Lindquist coordinates (t, r, θ, ϕ) is given by

$$\mathbf{g} = -\frac{|q|^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{|q|^2}\left(d\phi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{|q|^2}{\Delta}(dr)^2 + |q|^2(d\theta)^2,$$

where

$$\begin{cases} q &= r + ia\cos\theta,\\ \Delta &= r^2 - 2mr + a^2,\\ \Sigma^2 &= (r^2 + a^2)|q|^2 + 2mra^2(\sin\theta)^2 = (r^2 + a^2)^2 - a^2(\sin\theta)^2\Delta.\end{cases}$$

We also note that

$$\mathbf{T} = \partial_t, \qquad \mathbf{Z} = \partial_\phi, \tag{2.13}$$

are both Killing and $\mathbf{T}$ is only time-like in the complement of the ergoregion, i.e. in $|q|^2 > 2mr$. The domain of outer communication is given by,

$$\mathcal{R} = \{(\theta, r, t, \phi) \in (0, \pi) \times (r_+, \infty) \times \mathbb{R} \times \mathbb{S}^1\},$$

where $r_+ := m + \sqrt{m^2 - a^2}$, the larger root of Δ, corresponds to the event horizon.

Lemma 2.26. *The following principal null directions are canonical in Kerr.*

1. *The null pair* (e_3, e_4) *for which* e_4 *is geodesic is given by*

$$\begin{aligned} e_4 &= \frac{r^2 + a^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi, \\ e_3 &= \frac{r^2 + a^2}{|q|^2}\partial_t - \frac{\Delta}{|q|^2}\partial_r + \frac{a}{|q|^2}\partial_\phi. \end{aligned} \tag{2.14}$$

2. *The null pair* (e_3, e_4) *for which* e_3 *is geodesic is given by*

$$\begin{aligned} e_3 &= \frac{r^2 + a^2}{\Delta}\partial_t - \partial_r + \frac{a}{\Delta}\partial_\phi, \\ e_4 &= \frac{r^2 + a^2}{|q|^2}\partial_t + \frac{\Delta}{|q|^2}\partial_r + \frac{a}{|q|^2}\partial_\phi. \end{aligned} \tag{2.15}$$

3. *In both cases, a principal null frame can be obtained by adding the vectorfields*

$$e_1 = \frac{1}{|q|}\partial_\theta, \quad e_2 = \frac{a\sin\theta}{|q|}\partial_t + \frac{1}{|q|\sin\theta}\partial_\phi.$$

Proof. Straightforward verification. □

Definition 2.27. *We refer to the null frame obtained by adding* (e_1, e_2) *to* (e_3, e_4) *in* (2.14) *as the canonical outgoing principal null frame. We refer to the null frame obtained by adding* (e_1, e_2) *to* (e_3, e_4) *in* (2.15) *as the canonical ingoing principal null frame.*

Remark 2.28. *The canonical ingoing principal null frame, i.e. the null frame obtained by adding* (e_1, e_2) *to* (e_3, e_4) *in* (2.15)*, is regular towards the future for all* $r > 0$.

Remark 2.29. *In the remaining of Section 2.4, we will only consider the outgoing PG structure of Kerr.*

2.4.2. Canonical complex 1-form $\mathfrak{J}$

Definition 2.30. *We define the following complex 1-form $\mathfrak{J}$ in Kerr as follows*

$$\mathfrak{J} := j + i\,{}^*j, \tag{2.16}$$

where the real 1-form j is defined by

$$j_1 = 0, \qquad j_2 = \frac{\sin\theta}{|q|}. \tag{2.17}$$

Hence

$$\mathfrak{J}_1 = \frac{i\sin\theta}{|q|}, \qquad \mathfrak{J}_2 = \frac{\sin\theta}{|q|}.$$

Remark 2.31. *The relevance of the complex 1-form $\mathfrak{J}$ is due to its link to the complex 1-forms H, $\underline{H}$ and Z, see (2.24).*

Lemma 2.32. *The following identities hold true in Kerr:*

- *We have*

$$^*\mathfrak{J} = -i\mathfrak{J}, \qquad \mathfrak{J}\cdot\overline{\mathfrak{J}} = \frac{2(\sin\theta)^2}{|q|^2}. \tag{2.18}$$

- *We have*

$$\nabla_4\mathfrak{J} + \frac{1}{2}trX\mathfrak{J} = 0, \qquad \nabla_3\mathfrak{J} + \frac{1}{2}tr\underline{X}\mathfrak{J} = 0. \tag{2.19}$$

- *The complex 1-form $\mathfrak{J}$ verifies*

$$\overline{\mathcal{D}}\cdot\mathfrak{J} = \frac{4i(r^2+a^2)\cos\theta}{|q|^4}, \qquad \mathcal{D}\widehat{\otimes}\mathfrak{J} = 0. \tag{2.20}$$

- *We have*

$$\mathcal{D}(q) = -a\mathfrak{J}, \qquad \mathcal{D}(\overline{q}) = a\mathfrak{J}. \tag{2.21}$$

Proof. Straightforward computations. Note that (2.19) holds true in both the ingoing and outgoing frame. In the outgoing frame it becomes

$$\nabla_4\mathfrak{J} = -\frac{1}{q}\mathfrak{J}, \qquad \nabla_3\mathfrak{J} = \frac{\Delta q}{|q|^4}\mathfrak{J}.$$

To check the third identity it helps to first check the following identities for $j = \Re(\mathfrak{J})$

$$\begin{aligned}
&\operatorname{div} j = 0, \qquad \operatorname{curl} j = \frac{2(r^2+a^2)\cos\theta}{|q|^4}, \qquad \nabla\widehat{\otimes} j = 0,\\
&\nabla_a j_b = \frac{(r^2+a^2)\cos\theta}{|q|^4} \in_{ab}.
\end{aligned}$$

The last identity is a straightforward verification. □

Lemma 2.33. *In Kerr, the Killing vectorfield given by* $\mathbf{T} = \partial_t$*, in BL coordinates, can be expressed in terms of the outgoing PG frame by*

$$\mathbf{T} = \frac{1}{2}\left(e_3 + \frac{\Delta}{|q|^2}e_4 - 2a\Re(\mathfrak{J})^b e_b\right). \tag{2.22}$$

Proof. Using

$$\frac{\sin\theta}{|q|}e_2 = j^b e_b = \Re(\mathfrak{J})^b e_b$$

yields the formula. □

2.4.3. Canonical outgoing PG structure in Kerr Consider the canonical principal outgoing null frame in BL coordinates

$$\begin{aligned}
&e_4 = \frac{r^2+a^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi, \quad e_3 = \frac{r^2+a^2}{|q|^2}\partial_t - \frac{\Delta}{|q|^2}\partial_r + \frac{a}{|q|^2}\partial_\phi,\\
&e_1 = \frac{1}{|q|}\partial_\theta, \quad e_2 = \frac{a\sin\theta}{|q|}\partial_t + \frac{1}{|q|\sin\theta}\partial_\phi.
\end{aligned} \tag{2.23}$$

Lemma 2.34. *The function* r *together with the principal frame* (2.23) *defines an outgoing PG structure in Kerr. Moreover*

1. *The complex curvature null components w.r.t. the frame are given by*

$$A = B = \underline{B} = \underline{A} = 0, \quad P = -\frac{2m}{q^3}.$$

2. *The vanishing complex Ricci coefficients in Kerr are*

$$\widehat{X} = \underline{\widehat{X}} = \Xi = \underline{\Xi} = \omega = 0.$$

3. *The non-vanishing complex Ricci coefficients are*[45]

$$\begin{aligned} & trX = \frac{2}{q}, \qquad tr\underline{X} = -\frac{2\Delta q}{|q|^4}, \\ & \underline{H} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J}, \qquad H = \frac{aq}{|q|^2}\mathfrak{J}, \qquad Z = \frac{a\overline{q}}{|q|^2}\mathfrak{J}, \\ & \underline{\omega} = \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right), \end{aligned} \tag{2.24}$$

with the 1-form $\mathfrak{J}$ of Definition 2.30.

4. *We have*

$$\begin{aligned} & (\Lambda_1)_{12} = 0, \qquad (\Lambda_2)_{12} = -\frac{r^2+a^2}{|q|^3}\cot\theta, \\ & (\Lambda_3)_{12} = -\frac{a\Delta\cos\theta}{|q|^4}, \qquad (\Lambda_4)_{12} = -\frac{a\cos\theta}{|q|^2}. \end{aligned} \tag{2.25}$$

Proof. Straightforward computations. See also the material on Kerr in Chapter 3 of [28]. □

2.4.4. Outgoing PG coordinates in Kerr Note that relative to the BL coordinates we have

$$e_4(r) = 1, \qquad e_4(\theta) = 0, \qquad e_4(\phi) = \frac{a}{\Delta}.$$

To derive the coordinate system associated to the outgoing PG structure, we need to introduce in addition to (r, θ)

$$u := t - f(r), \quad f'(r) = \frac{r^2+a^2}{\Delta}, \qquad \varphi := \phi - h(r), \quad h'(r) = \frac{a}{\Delta},$$

which satisfies

$$e_4(u) = 0, \qquad e_4(\varphi) = 0.$$

[45]Note that all complex 1-forms are expressed with respect to $\mathfrak{J}$.

Lemma 2.35. *The coordinate system given by (u, r, θ, φ), with*

$$u = t - f(r), \quad f'(r) = \frac{r^2 + a^2}{\Delta}, \qquad \varphi = \phi - h(r), \quad h'(r) = \frac{a}{\Delta},$$

is the canonical coordinate system associated to the canonical outgoing PG structure of Kerr, called the outgoing Eddington-Finkelstein (EF) coordinate system of Kerr. Moreover,

1. *The action of the outgoing PG frame on the coordinates (u, r, θ, φ) is given by*

   (2.26)

$$\begin{array}{llll}
e_4(r) = 1, & e_4(u) = 0, & e_4(\theta) = 0, & e_4(\varphi) = 0, \\
e_3(r) = -\frac{\Delta}{|q|^2}, & e_3(u) = \frac{2(r^2+a^2)}{|q|^2}, & e_3(\theta) = 0, & e_3(\varphi) = \frac{2a}{|q|^2}, \\
e_1(r) = 0, & e_1(u) = 0, & e_1(\theta) = \frac{1}{|q|}, & e_1(\varphi) = 0, \\
e_2(r) = 0, & e_2(u) = \frac{a\sin\theta}{|q|}, & e_2(\theta) = 0, & e_2(\varphi) = \frac{1}{|q|\sin\theta}.
\end{array}$$

2. *In the outgoing EF coordinates, the metric takes the form*

$$\begin{aligned}
\mathbf{g} &= -\left(1 - \frac{2mr}{|q|^2}\right)(du)^2 - 2drdu + 2a(\sin\theta)^2 dr d\varphi \\
&\quad - \frac{4mra(\sin\theta)^2}{|q|^2} du d\varphi + |q|^2 (d\theta)^2 + \frac{\Sigma^2(\sin\theta)^2}{|q|^2}(d\varphi)^2.
\end{aligned}$$

3. *In the outgoing EF coordinates, the determinant on the metric is given by*

$$\det(\mathbf{g}) = -|q|^4 (\sin\theta)^2. \tag{2.27}$$

4. *In the outgoing EF coordinates, the inverse metric coefficients $\mathbf{g}^{\alpha\beta}$ are given by*

$$\mathbf{g}^{rr} = \frac{\Delta}{|q|^2}, \qquad \mathbf{g}^{ru} = -\frac{r^2+a^2}{|q|^2}, \qquad \mathbf{g}^{r\theta} = 0, \qquad \mathbf{g}^{r\varphi} = -\frac{a}{|q|^2},$$

$$\mathbf{g}^{uu} = \frac{a^2(\sin\theta)^2}{|q|^2}, \qquad \mathbf{g}^{u\theta} = 0, \qquad \mathbf{g}^{u\varphi} = \frac{a}{|q|^2},$$

$$\mathbf{g}^{\theta\theta} = \frac{1}{|q|^2}, \qquad \mathbf{g}^{\theta\varphi} = 0, \qquad \mathbf{g}^{\varphi\varphi} = \frac{1}{|q|^2(\sin\theta)^2}.$$

Proof. Straightforward verification. □

2.4.5. Canonical basis of $\ell = 1$ modes in Kerr

Definition 2.36. *We define the canonical basis of $\ell = 1$ modes of Kerr to be the triplet of scalar functions*[46] $J^{(p)}$, $p \in \{0, +, -\}$,

$$J^{(0)} := \cos\theta, \qquad J^{(-)} := \sin\theta\sin\varphi, \qquad J^{(+)} := \sin\theta\cos\varphi.$$

Remark 2.37. *Note that we have*

$$e_4(J^{(p)}) = 0.$$

Lemma 2.38. *The following identities hold true.*

$$\begin{aligned} \Delta J^{(0)} &= -\frac{2(r^2+a^2)}{|q|^4} J^{(0)}, \\ \Delta J^{(\pm)} &= -\frac{2r^2}{|q|^4} J^{(\pm)}. \end{aligned} \tag{2.28}$$

Also

$$\mathcal{D}\widehat{\otimes}\mathcal{D} J^{(p)} = 0. \tag{2.29}$$

Remark 2.39. *To avoid any confusion concerning the notation Δ in Sections 2.4.5 and 2.4.6, notice that:*

- *In the statement of Lemma 2.38 and in its proof, as well as in the proof of Lemma 2.45, Δ denotes the horizontal Laplacian.*
- *In the statement of Lemma 2.44 and in its proof, as well as in the statement of Lemma 2.45, Δ denotes the scalar function given by $r^2 - 2mr + a^2$.*

Proof. Given a scalar function h we have, in view of $(\Lambda_1)_{12} = 0$ and the formula for $(\Lambda_2)_{12}$ in (2.25),

$$\Delta h = e_1(e_1 h) + e_2(e_2 h) + \frac{r^2+a^2}{|q|^3}\cot\theta e_1(h)$$

[46] Note that they are regular everywhere including the axis of symmetry.

and (2.28) follows by choosing $h = J^{(p)}$ and recalling the relations, see (2.26),

$$e_1(\theta) = \frac{1}{|q|}, \quad e_1(\varphi) = 0, \quad e_2(\theta) = 0, \quad e_2(\varphi) = \frac{1}{|q|\sin\theta}.$$

Next, we focus on (2.29). First, note that for a scalar function h, we have

$$\mathcal{D}\widehat{\otimes}\mathcal{D}h = 2\nabla\widehat{\otimes}\nabla h + 2i\,{}^*(\nabla\widehat{\otimes}\nabla h)$$

so that it suffices to prove $\nabla\widehat{\otimes}\nabla J^{(p)} = 0$. Also, given a scalar function h we have, in view of $(\Lambda_1)_{12} = 0$ and the formula for $(\Lambda_2)_{12}$ in (2.25),

$$\begin{aligned}
(\nabla\widehat{\otimes}\nabla h)_{11} &= e_1(e_1 h) - e_2(e_2 h) - \frac{r^2+a^2}{|q|^3}\cot\theta e_1(h),\\
(\nabla\widehat{\otimes}\nabla h)_{12} &= e_1(e_2 h) + e_2(e_1 h) - \frac{r^2+a^2}{|q|^3}\cot\theta e_2(h),
\end{aligned}$$

and we obtain $\nabla\widehat{\otimes}\nabla J^{(p)} = 0$ by choosing $h = J^{(p)}$ so that (2.29) follows. □

Lemma 2.40. *The following identities hold true*

$$e_3(J^{(0)}) = 0, \qquad e_3(J^{(+)}) + \frac{2a}{|q|^2}J^{(-)} = 0, \qquad e_3(J^{(-)}) - \frac{2a}{|q|^2}J^{(+)} = 0.$$

Proof. Straightforward calculation. □

Lemma 2.41. *We have*

$$\mathcal{D}J^{(0)} = i\mathfrak{J}. \tag{2.30}$$

Proof. We check that

$$\nabla J^{(0)} = -\Im(\mathfrak{J}), \qquad {}^*\nabla J^{(0)} = \Re(\mathfrak{J}).$$

Indeed, recalling the definition of $\mathfrak{J}$,

$$\begin{aligned}
e_1(J^{(0)}) &= -\frac{\sin\theta}{|q|} = -\Im(\mathfrak{J}_1), \qquad e_2(J^{(0)}) = 0 = -\Im(\mathfrak{J}_2),\\
{}^*\nabla J^{(0)} &= -\Im({}^*\mathfrak{J}) = -\Im(-i\Re\mathfrak{J}) = \Re(\mathfrak{J}).
\end{aligned}$$

Therefore

$$\mathcal{D}J^{(0)} = \nabla J^{(0)} + i\,{}^*\nabla J^{(0)} = -\Im(\mathfrak{J}) + i\Re(\mathfrak{J}) = i\mathfrak{J}$$

as stated. □

It remains to also calculate $\mathcal{D}J^{(\pm)}$. To do that, we introduce in the next section the complex 1-forms $\mathfrak{J}_\pm$, as counterparts to $\mathfrak{J}$.

2.4.6. Canonical complex 1-forms $\mathfrak{J}_\pm$

Definition 2.42. *We define the complex 1-forms $\mathfrak{J}_\pm$ as follows*

$$\mathfrak{J}_\pm = j_\pm + i\,{}^*j_\pm, \tag{2.31}$$

where the real 1-forms $j_\pm$ are given by

$$\begin{aligned}(j_+)_1 &= \frac{1}{|q|}\cos\theta\cos\varphi, & (j_+)_2 &= -\frac{1}{|q|}\sin\varphi,\\ (j_-)_1 &= \frac{1}{|q|}\cos\theta\sin\varphi, & (j_-)_2 &= \frac{1}{|q|}\cos\varphi.\end{aligned} \tag{2.32}$$

We can now obtain the following analog of Lemma 2.41.

Lemma 2.43. *The complex 1-forms $\mathfrak{J}_\pm$ are anti-selfadjoint, i.e. ${}^*\mathfrak{J}_\pm = -i\mathfrak{J}_\pm$, and verify*

$$\mathcal{D}J^{(+)} = \mathfrak{J}_+, \qquad \mathcal{D}J^{(-)} = \mathfrak{J}_-. \tag{2.33}$$

Proof. To prove $\mathcal{D}J^{(\pm)} = \mathfrak{J}_\pm$ we check the following

$$\nabla J^{(\pm)} = \Re(\mathfrak{J}_\pm), \qquad {}^*\nabla J^{(\pm)} = \Im(\mathfrak{J}_\pm).$$

Indeed, in view of Definition 2.42 and Definition 2.36,

$$\begin{aligned}e_1(J^{(+)}) &= e_1(\sin\theta\cos\varphi) = \frac{1}{|q|}\cos\theta\cos\varphi = (j_+)_1 = \big(\Re(\mathfrak{J}_+)\big)_1,\\ e_2(J^{(+)}) &= e_2(\sin\theta\cos\varphi) = -\frac{1}{|q|}\sin\varphi = (j_+)_2 = \big(\Re(\mathfrak{J}_+)\big)_2,\\ e_1(J^{(-)}) &= e_1(\sin\theta\sin\varphi) = \frac{1}{|q|}\cos\theta\sin\varphi = (j_-)_1 = \big(\Re(\mathfrak{J}_-)\big)_1,\\ e_2(J^{(-)}) &= e_2(\sin\theta\sin\varphi) = \frac{1}{|q|}\cos\varphi = (j_-)_2 = \big(\Re(\mathfrak{J}_-)\big)_2.\end{aligned}$$

Also

$$\begin{aligned}{}^*\nabla J^{(+)} &= \Re({}^*\mathfrak{J}_+) = \Re(-i\mathfrak{J}_+) = \Im(\mathfrak{J}_+),\\ {}^*\nabla J^{(-)} &= \Re({}^*\mathfrak{J}_-) = \Re(-i\mathfrak{J}_-) = \Im(\mathfrak{J}_-).\end{aligned}$$

Thus

$$
\begin{aligned}
\mathcal{D}J^{(+)} &= \nabla J^{(+)} + i\,{}^*\nabla J^{(+)} = \Re(\mathfrak{J}_+) + i\Im(\mathfrak{J}_+) = \mathfrak{J}_+,\\
\mathcal{D}J^{(-)} &= \nabla J^{(-)} + i\,{}^*\nabla J^{(-)} = \Re(\mathfrak{J}_-) + i\Im(\mathfrak{J}_-) = \mathfrak{J}_-,
\end{aligned}
$$

as stated. □

Lemma 2.44. *Let Ψ be a complex 1-form in Kerr of the form*

$$\Psi = \frac{1}{|q|}(\psi + i\,{}^*\psi)$$

where the real 1-form ψ is such that $e_4(\psi_1) = e_4(\psi_2) = 0$. Then

$$\nabla_4\Psi + \frac{1}{q}\Psi = 0.$$

Also, we have

$$(\nabla_3\Psi)_a = \frac{\Delta q}{|q|^4}\Psi_a + \frac{1}{|q|}e_3(\psi_a + i\,{}^*\psi_a), \quad a = 1, 2.$$

Proof. Since $(\Lambda_4)_{12} = -\frac{a\cos\theta}{|q|^2}$ and $(\Lambda_3)_{12} = -\frac{a\Delta\cos\theta}{|q|^4}$ in view of (2.25), we easily check, using also $e_4(\psi_1) = e_4(\psi_2) = 0$,

$$\nabla_4\psi = -\frac{a\cos\theta}{|q|^2}\,{}^*\psi, \qquad \nabla_3\psi_a = -\frac{a\Delta\cos\theta}{|q|^4}\,{}^*\psi_a + e_3(\psi_a),$$

and thus

$$
\begin{aligned}
\nabla_4(\psi + i\,{}^*\psi) &= i\frac{a\cos\theta}{|q|^2}(\psi + i\,{}^*\psi),\\
\nabla_3(\psi + i\,{}^*\psi)_a &= i\frac{a\Delta\cos\theta}{|q|^4}(\psi + i\,{}^*\psi)_a + e_3(\psi_a + i\,{}^*\psi_a).
\end{aligned}
$$

The conclusion then easily follows using $e_4(r) = 1$, $e_4(\theta) = e_3(\theta) = 0$ and $e_3(r) = -\frac{\Delta}{|q|^2}$ to compute $e_4(|q|^2)$ and $e_3(|q|^2)$. □

Lemma 2.45. *The complex 1-forms $\mathfrak{J}_\pm$ verify*

$$\nabla_4\mathfrak{J}_\pm = -\frac{1}{q}\mathfrak{J}_\pm, \qquad \nabla_3\mathfrak{J}_\pm = \frac{\Delta q}{|q|^4}\mathfrak{J}_\pm \mp \frac{2a}{|q|^2}\mathfrak{J}_\mp, \tag{2.34}$$

$$\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm = 0, \qquad \overline{\mathcal{D}}\cdot\mathfrak{J}_\pm = -\frac{4r^2}{|q|^4}J^{(\pm)} \mp \frac{4ia^2\cos\theta}{|q|^4}J^{(\mp)}, \tag{2.35}$$

$$\begin{aligned}\mathfrak{J}_+\cdot\overline{\mathfrak{J}_+} &= \frac{2(\cos\theta)^2(\cos\varphi)^2+2(\sin\varphi)^2}{|q|^2},\\ \mathfrak{J}_-\cdot\overline{\mathfrak{J}_-} &= \frac{2(\cos\theta)^2(\sin\varphi)^2+2(\cos\varphi)^2}{|q|^2},\end{aligned} \tag{2.36}$$

and

$$\Re(\mathfrak{J}_+)\cdot\Re(\mathfrak{J}) = -\frac{1}{|q|^2}J^{(-)}, \qquad \Re(\mathfrak{J}_-)\cdot\Re(\mathfrak{J}) = \frac{1}{|q|^2}J^{(+)}. \tag{2.37}$$

Proof. (2.34) is an immediate consequence of Lemma 2.44 using that $e_4(\theta) = e_4(\varphi) = e_3(\theta) = 0$ and $e_3(\varphi) = \frac{2a}{|q|^2}$.

To check (2.35), recall from Lemma 2.43 that $\mathfrak{J}_\pm = \mathcal{D}J^{(\pm)}$ which yields

$$\begin{aligned}\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm &= \mathcal{D}\widehat{\otimes}\mathcal{D}J^{(\pm)} = 0,\\ \overline{\mathcal{D}}\cdot\mathfrak{J}_\pm &= \overline{\mathcal{D}}\cdot\mathcal{D}J^{(\pm)} = 2\Delta J^{(\pm)} + 2i(\nabla_1\nabla_2-\nabla_2\nabla_1)J^{(\pm)}\\ &= -\frac{4r^2}{|q|^4}J^{(\pm)} + 2i(\nabla_1\nabla_2-\nabla_2\nabla_1)J^{(\pm)},\end{aligned}$$

where we used (2.28) and (2.29). This yields the first identity in (2.35), while the second one follows from the above together with the following identity for a scalar function h only depending on (θ,φ)

$$(\nabla_1\nabla_2-\nabla_2\nabla_1)h = e_1(e_2h)-e_2(e_1h)+\frac{r^2+a^2}{|q|^3}\cot\theta e_2(h) = \frac{2a^2\cos\theta}{|q|^4}\partial_\varphi(h)$$

which is then applied to $h = J^{(\pm)}$ noticing that $\partial_\varphi(J^{(\pm)}) = \mp J^{(\mp)}$.

The remaining estimates (2.36) and (2.37) are straightforward verifications from the definitions. □

2.4.7. Additional coordinates system Recall from (2.27) that the determinant of the Kerr metric in the (u,r,θ,φ) coordinates is given by $\det(\mathbf{g}) = -|q|^4(\sin\theta)^2$ which illustrates that the (r,u,θ,φ) coordinates system is singular at the axis, i.e. at $\theta=0$ and $\theta=\pi$. In this section, we introduce a coordinates system which is regular at the axis

$$(u,r,x^1,x^2), \qquad x^1 := J^{(+)}, \qquad x^2 := J^{(-)}. \tag{2.38}$$

We start by showing that the coordinates (x^1,x^2) defined above form a coordinates system of the spheres $S(u,r)$ which is regular at the axis.

Lemma 2.46. *The metric g induced by $\mathbf{g}$ on $S(u,r)$ has the following form in Kerr in the (x^1,x^2) coordinates system*

$$\begin{aligned} g &= |q|^2\Bigg[\left(\frac{1-(x^2)^2}{1-|x|^2}+\frac{a^2(x^2)^2}{|q|^2}\left(1+\frac{2mr}{|q|^2}\right)\right)(dx^1)^2 \\ &\quad +\left(\frac{2x^1x^2}{1-|x|^2}-\frac{2a^2x^1x^2}{|q|^2}\left(1+\frac{2mr}{|q|^2}\right)\right)dx^1dx^2 \\ &\quad +\left(\frac{1-(x^1)^2}{1-|x|^2}+\frac{a^2(x^1)^2}{|q|^2}\left(1+\frac{2mr}{|q|^2}\right)\right)(dx^2)^2\Bigg]. \end{aligned}$$

Proof. Straightforward verification. □

Remark 2.47. *The coordinates (x^1,x^2) verify the following properties*

1. *We have $|(x^1,x^2)|\le 1$, with $|(x^1,x^2)|=1$ corresponding to the equator, i.e. $\theta=\frac{\pi}{2}$, and $|(x^1,x^2)|=0$ corresponding to the poles.*
2. *(x^1,x^2) are regular coordinates away from the equator, i.e. for $|(x^1,x^2)|<1$, or equivalently for $\theta\neq\frac{\pi}{2}$*
3. *Since the (θ,φ) coordinates system is regular for $\theta\neq 0,\pi$, and the (x^1,x^2) coordinates system is regular for $\theta\neq\frac{\pi}{2}$, the spheres $S(u,r)$ are covered by these two coordinates systems (or rather 3 since one needs one (x^1,x^2) coordinates system per hemisphere).*

Remark 2.48. *Since we have $e_4(\theta)=e_4(\varphi)=0$, (x^1,x^2) satisfy*

$$e_4(x^1)=e_4(x^2)=0.$$

Using the above coordinates on $S(u,r)$, we consider the coordinates (u,r,x^1,x^2) on Kerr in the following lemma.

Lemma 2.49. *Let coordinates on $S(u,r)$ be given by $(x^1,x^2)=(J^{(+)},J^{(-)})$. Then,*

1. *The Kerr metric is given in the (u,r,x^1,x^2) coordinates system by*

$$\begin{aligned} \mathbf{g} &= -\left(1-\frac{2mr}{|q|^2}\right)(du)^2-2drdu+2adr\big(-x^2dx^1+x^1dx^2\big) \\ &\quad -\frac{4mra}{|q|^2}du\big(-x^2dx^1+x^1dx^2\big)+g, \end{aligned}$$

where g denotes the induced metric by $\mathbf{g}$ on $S(u,r)$, which is given by Lemma 2.46 in the (x^1,x^2) coordinates system.

2. *In the (u, r, x^1, x^2) coordinates, the determinant of the metric is given by*

$$\det(\mathbf{g}) = \frac{|q|^4}{(\cos\theta)^2}. \tag{2.39}$$

3. *In the (u, r, x^1, x^2) coordinates, the inverse metric coefficients $\mathbf{g}^{\alpha\beta}$ are given by*

$$\begin{aligned}
&\mathbf{g}^{rr} = \frac{\Delta}{|q|^2}, \quad \mathbf{g}^{ru} = -\frac{r^2+a^2}{|q|^2}, \quad \mathbf{g}^{rx^1} = \frac{ax^2}{|q|^2}, \quad \mathbf{g}^{rx^2} = -\frac{ax^1}{|q|^2},\\
&\mathbf{g}^{uu} = \frac{a^2(\sin\theta)^2}{|q|^2}, \quad \mathbf{g}^{ux^1} = -\frac{ax^2}{|q|^2}, \quad \mathbf{g}^{ux^2} = \frac{ax^1}{|q|^2},\\
&\mathbf{g}^{x^1x^1} = \frac{(\cos\theta\cos\varphi)^2 + (\sin\varphi)^2}{|q|^2},\\
&\mathbf{g}^{x^1x^2} = \frac{((\cos\theta)^2 - 1)\sin\varphi\cos\varphi}{|q|^2},\\
&\mathbf{g}^{x^2x^2} = \frac{(\cos\theta\sin\varphi)^2 + (\cos\varphi)^2}{|q|^2}.
\end{aligned}$$

Proof. Straightforward verification. □

Remark 2.50. *In view of the above lemma, away from the horizon, the outer domain of communication of Kerr can be covered by the following three coordinates systems*

- *(u, r, θ, φ) away from the poles $\theta = 0$ and $\theta = \pi$, e.g. in $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$,*
- *two copies of $(u, r, J^{(+)}, J^{(-)})$ away from the equator $\theta = \frac{\pi}{2}$, e.g. one copy in $0 \le \theta < \frac{\pi}{3}$ and another copy in $\frac{2\pi}{3} < \theta \le \pi$.*

2.4.8. Asymptotic for the outgoing PG structure in Kerr We denote by j^{Sw} the following real 1-form

$$j^{Sw}_1 = 0, \qquad j^{Sw}_2 = \frac{\sin\theta}{r}.$$

Corollary 2.51. *In Kerr, relative to its canonical outgoing PG frame (see Section 2.4.3), the following identities hold asymptotically for large r:*

1. *We have,*

$$j = (1 + O(r^{-2}))j^{Sw}.$$

2. *We have*

$$\nabla u = aj = a\big(1+O(r^{-2})\big)j^{Sw},$$

and

$$e_3(r) = -\Upsilon + O(r^{-2}), \qquad e_3(u) = 2 + O(r^{-2}),$$

where we have used the notation

$$\Upsilon \ := \ 1 - \frac{2m}{r}.$$

3. *The real components of the non-vanishing Ricci scalars are given by*

$$\begin{aligned} tr\chi &= \frac{2}{r} + O(r^{-3}),\\ tr\underline{\chi} &= -\frac{2\Upsilon}{r} + O(r^{-3}),\\ {}^{(a)}tr\chi &= \frac{2a\cos\theta}{r^2} + O(r^{-4}),\\ {}^{(a)}tr\underline{\chi} &= \frac{2a\Upsilon\cos\theta}{r^2} + O(r^{-4}). \end{aligned}$$

Also

$$\underline{\omega} \ = \ \frac{m}{r^2} + O(r^{-3}).$$

4. *The Ricci coefficient 1-forms are given by*

$$\begin{aligned} \underline{\eta} = -\zeta &= -\frac{a}{r^2}\Big(j^{Sw} + \frac{a\cos\theta}{r}\ {}^*j^{Sw}\Big)\big(1+O(r^{-2})\big),\\ \eta &= \frac{a}{r^2}\Big(j^{Sw} - \frac{a\cos\theta}{r}\ {}^*j^{Sw}\Big)\big(1+O(r^{-2})\big), \end{aligned}$$

and, relative to the frame e_1, e_2,

$$\begin{aligned} \sin\theta(\Lambda_2)_{12} &= -\frac{\cos\theta}{r} + O(r^{-3}), \quad (\Lambda_3)_{21} = \frac{a\Upsilon\cos\theta}{r^2} + O(r^{-4}),\\ (\Lambda_4)_{21} &= \frac{a\cos\theta}{r^2} + O(r^{-4}). \end{aligned}$$

5. *The real non-vanishing curvature components are given by*

$$\rho \ = \ -\frac{2m}{r^3} + O(r^{-5}),$$

$$ {}^*\rho = \frac{6am\cos\theta}{r^4} + O(r^{-5}). $$

Proof. Straightforward verification. □

2.4.9. Asymptotic of the associated integrable frame in Kerr We consider now the associated integrable frame (e_3', e_4', e_1', e_2') as defined by Lemma 2.22.

Lemma 2.52. *Consider the change of frame transformation in Kerr from the PG frame (e_3, e_4, e_1, e_2) to the associated integrable frame (e_3', e_4', e_1', e_2') as defined by Lemma 2.22. The following statements hold true for large r.*

1. *We have $\lambda = 1 + O(r^{-2})$, $f_1 = \underline{f}_1 = 0$, and*

$$ f_2 = -\frac{a\sin\theta}{r} + O\big(\sin\theta r^{-3}\big), \qquad \underline{f}_2 = -\frac{a\sin\theta\Upsilon}{r} + O\big(\sin\theta r^{-3}\big). $$

2. *The connection coefficients $(\Lambda_\mu')_{12} = \mathbf{g}(\mathbf{D}_{e_\mu'} e_2', e_1')$, $\mu = 1, 2, 3, 4$, verify*

$$ (\Lambda_1')_{12} = O(\sin\theta r^{-4}), \qquad \sin\theta(\Lambda_2')_{21} = \frac{\cos\theta}{r} + O(r^{-3}), $$

and

$$ (\Lambda_3')_{21} = O\big(r^{-4}\big), \qquad (\Lambda_4')_{21} = O\big(r^{-4}\big). $$

3. *The curvature components are given by*

$$ \begin{aligned} \alpha' &= O(\sin^2\theta r^{-5}), \\ \beta_1' &= O(\sin\theta r^{-5}), \qquad \beta_2' = \frac{3am\sin\theta}{r^4} + O(\sin\theta r^{-5}), \\ \rho' &= -\frac{2m}{r^3} + O(r^{-5}), \qquad {}^*\rho' = \frac{6am\cos\theta}{r^4} + O(\sin^2\theta r^{-5}), \\ \underline{\beta}' &= O\big(\sin\theta r^{-4}\big), \\ \underline{\alpha}' &= O\big(\sin^2\theta r^{-5}\big). \end{aligned} $$

4. *We have,*

$$ \begin{aligned} div'\beta' &= O(r^{-6}), \\ curl'\beta' &= \frac{6am\cos\theta}{r^5} + O(r^{-6}). \end{aligned} $$

5. *We have,*

$$
\begin{aligned}
div' f &= O(\sin^2\theta r^{-5}), \qquad & \nabla'\widehat{\otimes} f &= O(\sin\theta r^{-5}),\\
div' \underline{f} &= O(\sin^2\theta r^{-5}), \qquad & \nabla'\widehat{\otimes} \underline{f} &= O(\sin\theta r^{-5}).
\end{aligned}
$$

Also,

$$
\begin{aligned}
\nabla'_3(f_1, \underline{f}_1) + \frac{1}{2} tr\underline{\chi}(f_1, \underline{f}_1) &= O\big(\sin\theta r^{-5}\big),\\
\nabla'_3(f_2, \underline{f}_2) + \frac{1}{2} tr\underline{\chi}(f_2, \underline{f}_2) &= O\big(\sin\theta r^{-3}\big),\\
\nabla'_4(f_1, \underline{f}_1) + \frac{1}{2} tr\chi(f_1, \underline{f}_1) &= O\big(\sin\theta r^{-5}\big),\\
\nabla'_4(f_2, \underline{f}_2) + \frac{1}{2} tr\chi(f_2, \underline{f}_2) &= O\big(\sin\theta r^{-3}\big).
\end{aligned}
$$

6. *The connection coefficients behave as follows*

$$
\begin{aligned}
tr\chi' &= \frac{2}{r} + O(r^{-3}), \qquad & \widehat{\chi}' &= O(r^{-3}),\\
tr\underline{\chi}' &= -\frac{2\Upsilon}{r} + O(r^{-3}), \qquad & \widehat{\underline{\chi}}' &= O(r^{-3}),\\
\zeta' &= O(\sin\theta r^{-3}), \qquad & \eta' &= O(\sin\theta r^{-3}).
\end{aligned}
$$

Also,

$$
\begin{aligned}
\xi' &= O\big(\sin\theta r^{-3}\big), \qquad & \underline{\xi}' &= O\big(\sin\theta r^{-3}\big),\\
\omega' &= O\big(r^{-3}\big), \qquad & \underline{\omega}' &= \frac{m}{r^2} + O\big(r^{-3}\big).
\end{aligned}
$$

Proof. Straightforward verification in view of Lemma 2.22, Corollary 2.51 and the transformation formulas of Proposition 2.12. □

Remark 2.53. *Recall from Lemma 2.34 that $\alpha = \underline{\alpha} = \beta = \underline{\beta} = 0$ in the PG structure of Kerr. Notice from Lemma 2.52 that this does not hold for the associated integrable frame, denoted by prime.*

We end this section with the following proposition.

Proposition 2.54. *Consider the linearized quantities*

$$
\widetilde{tr\chi}' := tr\chi' - \frac{2}{r'}, \qquad \widetilde{tr\underline{\chi}}' := tr\underline{\chi}' + \frac{2\,(1-\frac{2m}{r'})}{r'},
$$

where r' denotes the area radius of the spheres $S(u,r)$ in Kerr. We have, for large r,

$$\int_{S(u,r)} \widetilde{tr\chi}' J^{(p)} = O\left(\frac{1}{r^2}\right), \quad p = 0, +, -,$$
$$\int_{S(u,r)} \widetilde{tr\underline{\chi}}' J^{(p)} = O\left(\frac{1}{r^2}\right), \quad p = 0, +, -,$$

where $J^{(p)}$, $p = 0, +, -$, denote the standard $\ell = 1$ spherical harmonics, i.e.

$$J^{(-)} = \sin\theta \sin\varphi, \qquad J^{(0)} = \cos\theta, \qquad J^{(+)} = \sin\theta\cos\varphi.$$

Proof. See Section A.3. □

2.5. Initialization of PG structures on a hypersurface

2.5.1. Framed hypersurfaces

Definition 2.55. *A framed hypersurface consists of a set $(\Sigma, r, (\mathcal{H}, e_3, e_4))$ where:*

1. *Σ is a smooth hypersurface in $\mathcal{M}$,*
2. *(e_3, e_4) is a null pair on Σ such that e_4 is transversal to Σ, and $\mathcal{H}$, the horizontal space perpendicular on e_3, e_4, is tangent to Σ,*
3. *the function $r : \Sigma \to \mathbb{R}$ is a regular function on Σ such that $\mathcal{H}(r) = 0$.*

For a given framed hypersurface $(\Sigma, r, (\mathcal{H}, e_3, e_4))$, we denote by ν the vectorfield tangent to Σ, normal to the r-foliation, and normalized by the condition[47] $\mathbf{g}(\nu, e_4) = -2$. Thus,

$$\nu = e_3 + b_\Sigma e_4 \tag{2.40}$$

with b_Σ a given scalar function on Σ.

2.5.2. Initialization of PG structures In what follows, we consider natural initial data structures on hypersurfaces Σ of $\mathcal{M}$ which generate PG structures.

Definition 2.56 (PG-data set)**.** *The boundary data of a PG structure (PG-data set) consists of:*

1. *a framed hypersurface $(\Sigma, r, (\mathcal{H}, e_3, e_4))$ as in Definition 2.55,*

[47] This is always possible since e_4 is transversal to Σ.

2. *a fixed 1-form f on the spheres S of the r-foliation of Σ verifying the condition*

$$b_\Sigma|f|^2 < 4 \quad on \quad \Sigma, \tag{2.41}$$

where b_Σ appears in (2.40).

Proposition 2.57. *Given a PG data set $\big(\Sigma, r, (\mathcal{H}, e_3, e_4), f\big)$ as in Definition 2.56, there exists a unique PG structure $\big(r', (\mathcal{H}', e_3', e_4')\big)$ defined in a neighborhood of Σ such that the following hold true:*

1. *The function r' is prescribed on Σ by $r' = r$.*
2. *Along Σ, the restriction of the spacetime null frame $(\mathcal{H}', e_3', e_4')$ and the given null frame $(\mathcal{H}, e_3, e_4)$ on Σ are related by the transformation formulas, where (e_1, e_2) is a fixed orthonormal basis of $\mathcal{H}$,*

$$\begin{aligned} e_4' &= e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3,\\ e_a' &= \left(\delta_a^b + \frac{1}{2}\underline{f}_a f^b\right) e_b + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right) e_3,\\ e_3' &= \left(1 + \frac{1}{2}f\cdot\underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right) e_3 + \left(\underline{f}^b + \frac{1}{4}|\underline{f}|^2 f^b\right) e_b\\ &\quad + \frac{1}{4}|\underline{f}|^2 e_4, \end{aligned} \tag{2.42}$$

where $\underline{f}$ is chosen such that[48]

$$\underline{f} = -\frac{(\nu(r) - b_\Sigma)}{1 - \frac{1}{4}b_\Sigma|f|^2} f. \tag{2.43}$$

Proof. First, note that in view of (2.40), we have on Σ

$$e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3 \;=\; \left(1 - \frac{1}{4}b_\Sigma|f|^2\right) e_4 + f^b e_b + \frac{1}{4}|f|^2\nu.$$

Since ν and e_2 are tangent to Σ, and e_4 is transversal to Σ, under the condition (2.41), the vectorfield $e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3$ is transversal to Σ. Hence, in view of (2.42), e_4' is transversal to Σ.

e_4' being transversal to Σ, we introduce the following transversality condition on Σ for r'

$$e_4'(r') = 1 \quad \text{on} \quad \Sigma. \tag{2.44}$$

[48] Note that $\underline{f}$ is well defined thanks to the condition (2.41).

The transversality condition (2.44) allows us to specify all derivatives of r' on Σ. In particular, we would like to compute $e_1'(r)$ and $e_2'(r)$ on Σ. In view of (2.42), we have on Σ

$$e_a'(r') = \left(\delta_a^b + \frac{1}{2}\underline{f}_a f^b\right) e_b(r') + \frac{1}{2}\underline{f}_a e_4(r') + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right) e_3(r').$$

Since (e_1, e_2) are tangent to Σ, since $r' = r$ on Σ, and since $e_1(r) = e_2(r) = 0$ on Σ, we have on Σ

$$e_a(r') = e_a(r) = 0$$

and hence

$$\begin{aligned} e_a'(r') &= \frac{1}{2}\underline{f}_a e_4(r') + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right) e_3(r') \\ &= \frac{1}{2}\left(e_4(r') + \frac{1}{4}|f|^2 e_3(r')\right)\underline{f}_a + \frac{1}{2}e_3(r') f_a. \end{aligned}$$

Together with the definition (2.43) of $\underline{f}$, we obtain on Σ

$$e_a'(r') = \frac{1}{2}\left(e_3(r') - \frac{\nu(r) - b_\Sigma}{1 - \frac{1}{4}b_\Sigma|f|^2}\left(e_4(r') + \frac{1}{4}|f|^2 e_3(r')\right)\right) f_a. \tag{2.45}$$

In view of (2.45), we need to compute $e_4(r')$ and $e_3(r')$. In view of (2.42) and the transversality condition (2.44), we have on Σ

$$\begin{aligned} 1 &= e_4'(r') = e_4(r') + f^b e_b(r') + \frac{1}{4}|f|^2 e_3(r') \\ &= e_4(r') + \frac{1}{4}|f|^2 e_3(r'). \end{aligned}$$

Also, since ν is tangent to Σ and $r' = r$ on Σ, we have

$$\nu(r) = \nu(r') = e_3(r') + b_\Sigma e_4(r').$$

We infer from both identities, using also (2.41),

$$e_4(r') = \frac{1 - \frac{1}{4}\nu(r)|f|^2}{1 - \frac{1}{4}b_\Sigma|f|^2}, \qquad e_3(r') = \frac{\nu(r) - b_\Sigma}{1 - \frac{1}{4}b_\Sigma|f|^2}, \quad \text{on} \quad \Sigma.$$

Together with (2.45), we infer on Σ

$$
\begin{aligned}
e'_a(r') &= \frac{1}{2}\left(e_3(r') - \frac{\nu(r)-b_\Sigma}{1-\frac{1}{4}b_\Sigma|f|^2}\left(e_4(r')+\frac{1}{4}|f|^2 e_3(r')\right)\right)f_a\\
&= \frac{1}{2}\left(e_3(r') - \frac{\nu(r)-b_\Sigma}{1-\frac{1}{4}b_\Sigma|f|^2}\right)f_a\\
&= 0.
\end{aligned}
$$

Hence, we have finally obtained

$$
e'_1(r')=0, \qquad e'_2(r')=0, \quad \text{on} \quad \Sigma. \tag{2.46}
$$

Finally, on the hypersurface Σ, we are given a scalar function r' and a null frame (e'_1, e'_2, e'_3, e'_4), such that e'_4 is transversal to Σ and r' satisfies the transversality condition (2.44) on Σ. Also, under this transversality condition, (2.46) holds. We are thus in position to apply Lemma 2.18, according to which there exists a unique PG structure $\big(r', (e'_1, e'_2, e'_3, e'_4)\big)$ defined in a neighborhood of Σ. This concludes the proof of the proposition. □

2.5.3. GCM hypersurfaces In this section, we introduce the concept of GCM hypersurfaces and GCM-PG data sets which are at the core of the construction of our bootstrap regions in Chapter 3, see Section 3.2.3. The construction starts with framed hypersurfaces $\big(\Sigma_*, r, (\mathcal{H}, e_3, e_4)\big)$ which terminate in a boundary S_* on which the given function r is constant, i.e. S_* is a leaf of the r-foliation of Σ_*.

The definition of GCM hypersurfaces below includes in particular conditions on the Ricci coefficients η and $\underline{\xi}$, see (2.54). Now, a priori, the only well defined Ricci coefficients on Σ_* are $\mathrm{tr}\,\chi$, $\mathrm{tr}\,\underline{\chi}$, $\widehat{\chi}$, $\widehat{\underline{\chi}}$ and ζ. To make sense of all Ricci coefficients on Σ_*, including in particular η and $\underline{\xi}$, we need to choose transversality conditions on Σ_*. We choose them to be compatible with an outgoing geodesic foliation initialized on Σ_*, i.e., we choose the following transversality conditions on Σ_*

$$
\xi=0, \qquad \omega=0, \qquad \underline{\eta}=-\zeta, \qquad \text{on} \quad \Sigma_*. \tag{2.47}
$$

The GCM hypersurfaces below also involve a basis of $\ell=1$ modes $J^{(p)}$, $p=0,+,-$. This basis is defined as follows

1. There exist coordinates (θ, φ) on S_* such that the induced metric g on S_* takes the form (see [41] for results on effective uniformization and

canonical $\ell = 1$ modes; a short review of the main results on this topic can also be found in Section 5.1.1)

$$g = e^{2\phi} r^2 \Big((d\theta)^2 + \sin^2\theta (d\varphi)^2 \Big). \tag{2.48}$$

2. The functions

$$J^{(0)} := \cos\theta, \quad J^{(-)} := \sin\theta \sin\varphi, \quad J^{(+)} := \sin\theta\cos\varphi, \tag{2.49}$$

verify the balanced conditions

$$\int_{S_*} J^{(p)} = 0, \qquad p = 0, +, -. \tag{2.50}$$

3. We choose (e_1, e_2) on S_* as follows

$$e_1 := \frac{1}{re^{\phi}} \partial_\theta, \qquad e_2 := \frac{1}{r\sin\theta e^{\phi}} \partial_\varphi. \tag{2.51}$$

4. The above (θ, φ) coordinates on S_* are extended to Σ_* by setting

$$\nu(\theta) = \nu(\varphi) = 0, \tag{2.52}$$

where ν is the normal to the r-foliation on Σ_* defined in (2.40).
5. We also extend the $J^{(p)}$ functions to Σ_* by setting

$$\nu(J^{(p)}) = 0, \quad p = 0, +, -. \tag{2.53}$$

Definition 2.58 (GCM hypersurface)**.** *Consider a framed hypersurface with end sphere S_*, transversality conditions (2.47), coordinates (θ, φ) and functions $J^{(0)}$, $J^{(+)}$ and $J^{(-)}$ defined as in (2.48)–(2.53). We call such a framed hypersurface a GCM hypersurface if in addition the following GCM conditions are verified.*

1. *We have on any sphere of the foliation induced by r,*

$$\begin{aligned}
tr\chi &= \frac{2}{r}, \\
tr\underline{\chi} &= -\frac{2\Upsilon}{r} + \underline{C}_0 + \sum_{p=0,+,-} \underline{C}_p J^{(p)}, \\
\mu &= \frac{2m}{r^3} + M_0 + \sum_{p=0,+,-} M_p J^{(p)},
\end{aligned} \tag{2.54}$$

$$\int_S J^{(p)} div\,\eta = 0, \qquad \int_S J^{(p)} div\,\underline{\xi} = 0, \qquad p = 0, +, -,$$
$$\overline{b_{\Sigma_*}} = -1 - \frac{2m}{r},$$

where $\underline{C}_0$, $\underline{C}_p$, M_0, M_p *are scalar functions on* Σ_* *constant along the leaves of the foliation, and* $\overline{b_{\Sigma_*}}$ *denotes the average of* b_{Σ_*} *on the spheres foliating* Σ_*.

2. *In addition, we have on the end sphere* S_*

$$\widecheck{tr\underline{\chi}} = 0, \qquad \int_{S_*} J^{(p)}\, div\,\beta = 0, \qquad p = 0, +, -, \tag{2.55}$$

as well as

$$\int_{S_*} J^{(+)}\, curl\,\beta = 0, \qquad \int_{S_*} J^{(-)}\, curl\,\beta = 0. \tag{2.56}$$

Definition 2.59. *We define the parameters* (m, a) *of a GCM hypersurface to be constant on* Σ_* *with* m *being the Hawking mass of* S_*, *i.e.*

$$\frac{2m}{r} = 1 + \frac{1}{16\pi}\int_{S_*} tr\chi tr\underline{\chi}, \tag{2.57}$$

and with a *given by*

$$a \quad := \quad \frac{r^3}{8\pi m}\int_{S_*} J^{(0)}\, curl\,\beta. \tag{2.58}$$

Definition 2.60. *A GCM-PG data set is a PG data set* $(\Sigma_*, r, (\mathcal{H}, e_3, e_4), f)$, *as in Definition 2.56 such that the framed hypersurface* $(\Sigma_*, r, (\mathcal{H}, e_3, e_4))$ *is a GCM hypersurface, see Definition 2.58.*

2.6. Linearization of outgoing PG structures

In this section, we assume given an outgoing principal geodesic structure with associated PG coordinates (u, r, θ, φ). To compare various quantities with the corresponding ones in Kerr we also assume given two constants m and $|a| < m$. With these fixed values we define, as in Kerr, the following functions of (r, θ)

$$\begin{aligned} q &:= r + ia\cos\theta, \\ \Delta &:= r^2 + a^2 - 2mr, \\ \Sigma^2 &:= (r^2 + a^2)|q|^2 + 2mra^2(\sin\theta)^2. \end{aligned} \tag{2.59}$$

2.6.1. Adapted basis of $\ell = 1$ modes Recall that the coordinates (θ, φ) verify $e_4(\theta) = e_4(\varphi) = 0$. We define[49] the basis of $\ell = 1$ modes $J^{(p)}$, $p \in \{0, +, -\}$, adapted to the PG structure according to

$$J^{(0)} := \cos\theta, \qquad J^{(+)} := \sin\theta\cos\varphi, \qquad J^{(-)} := \sin\theta\sin\varphi. \tag{2.60}$$

Clearly,

$$e_4(J^{(p)}) = 0. \tag{2.61}$$

The $\ell = 1$ modes of a scalar function on $S(u, r)$ are defined as follows.

Definition 2.61. *Given a scalar function f on a sphere $S = S(u, r)$, we define the $\ell = 1$ modes of f to be the triplet of numbers*

$$(f)_{\ell=1} \quad := \quad \left(\frac{1}{|S|}\int_S f J^{(0)}, \frac{1}{|S|}\int_S f J^{(+)}, \frac{1}{|S|}\int_S f J^{(-)}\right).$$

2.6.2. The auxiliary complex 1-forms $\mathfrak{J}$, $\mathfrak{J}_\pm$ To linearize 1-forms, we rely on a complex horizontal 1-form $\mathfrak{J}$ verifying the following properties

$$\nabla_4\mathfrak{J} = -\frac{1}{q}\mathfrak{J}, \qquad {}^*\mathfrak{J} = -i\mathfrak{J}, \qquad \mathfrak{J}\cdot\overline{\mathfrak{J}} = \frac{2(\sin\theta)^2}{|q|^2}. \tag{2.62}$$

Remark 2.62. *In Kerr, we have, see Definition 2.30,*

$$\mathfrak{J}_1 = \frac{i\sin\theta}{|q|}, \qquad \mathfrak{J}_2 = \frac{\sin\theta}{|q|}.$$

In addition to $\mathfrak{J}$, we also introduce the complex 1-forms $\mathfrak{J}_\pm$ verifying

$$\nabla_4\mathfrak{J}_\pm = -\frac{1}{q}\mathfrak{J}_\pm, \qquad {}^*\mathfrak{J}_\pm = -i\mathfrak{J}_\pm, \tag{2.63}$$

and

$$\begin{aligned}\mathfrak{J}_+\cdot\overline{\mathfrak{J}_+} &= \frac{2(\cos\theta)^2(\cos\varphi)^2 + 2(\sin\varphi)^2}{|q|^2},\\ \mathfrak{J}_-\cdot\overline{\mathfrak{J}_-} &= \frac{2(\cos\theta)^2(\sin\varphi)^2 + 2(\cos\varphi)^2}{|q|^2}.\end{aligned} \tag{2.64}$$

[49]See definition 2.36 in Kerr.

Remark 2.63. *In Kerr, see Definition 2.42, we have* $\mathfrak{J}_\pm = j_\pm + i\,{}^*j_\pm$ *where*

$$(j_+)_1 = \frac{1}{|q|}\cos\theta\cos\varphi, \qquad (j_+)_2 = -\frac{1}{|q|}\sin\varphi,$$
$$(j_-)_1 = \frac{1}{|q|}\cos\theta\sin\varphi, \qquad (j_-)_2 = \frac{1}{|q|}\cos\varphi.$$

The following lemma shows that, upon a suitable choice for $\mathfrak{J}, \mathfrak{J}_\pm$ on a hypersurface Σ transversal to e_4, it suffices in fact to transport $\mathfrak{J}, \mathfrak{J}_\pm$ using the equations

$$\nabla_4\mathfrak{J} = -\frac{1}{q}\mathfrak{J}, \qquad \nabla_4\mathfrak{J}_\pm = -\frac{1}{q}\mathfrak{J}_\pm. \tag{2.65}$$

Lemma 2.64. *Assume* $\mathfrak{J}, \mathfrak{J}_\pm$ *are initialized on a hypersurface* Σ *transversal to* e_4 *such that we have, along* Σ,

$${}^*\mathfrak{J} = -i\mathfrak{J}, \qquad \mathfrak{J}\cdot\overline{\mathfrak{J}} = \frac{2(\sin\theta)^2}{|q|^2}, \tag{2.66}$$

$$\begin{aligned} {}^*\mathfrak{J}_\pm &= -i\mathfrak{J}_\pm,\\ \mathfrak{J}_+\cdot\overline{\mathfrak{J}_+} &= \frac{2(\cos\theta)^2(\cos\varphi)^2 + 2(\sin\varphi)^2}{|q|^2},\\ \mathfrak{J}_-\cdot\overline{\mathfrak{J}_-} &= \frac{2(\cos\theta)^2(\sin\varphi)^2 + 2(\cos\varphi)^2}{|q|^2}, \end{aligned} \tag{2.67}$$

and

$$\Re(\mathfrak{J}_+)\cdot\Re(\mathfrak{J}) = -\frac{1}{|q|^2}J^{(-)}, \qquad \Re(\mathfrak{J}_-)\cdot\Re(\mathfrak{J}) = \frac{1}{|q|^2}J^{(+)}, \tag{2.68}$$

and extend $\mathfrak{J}, \mathfrak{J}_\pm$ *to a neighborhood of* Σ *according to* (2.65)*. Then, the identities* (2.66), (2.67) *and* (2.68) *are propagated to a neighborhood of* Σ*.*

Proof. In view of the transport equation along e_4 satisfied by $\mathfrak{J}$, we have

$$\nabla_4({}^*\mathfrak{J} + i\mathfrak{J}) + \frac{1}{q}({}^*\mathfrak{J} + i\mathfrak{J}) \;=\; 0$$

and, using also $e_4(q) = e_4(\overline{q}) = 1$ and $e_4(\theta) = 0$,

$$\nabla_4\left(\mathfrak{J}\cdot\overline{\mathfrak{J}} - \frac{2(\sin\theta)^2}{|q|^2}\right) + \left(\frac{1}{q} + \frac{1}{\overline{q}}\right)\left(\mathfrak{J}\cdot\overline{\mathfrak{J}} - \frac{2(\sin\theta)^2}{|q|^2}\right) \;=\; 0$$

from which (2.66) is propagated to a neighborhood of Σ as stated. The identities (2.67) can be propagated to a neighborhood of Σ in the same manner.

Next, we focus on (2.68). We write, setting $\Re(\mathfrak{J}) = j, \Re(\mathfrak{J}_\pm) = j_\pm$,

$$\begin{aligned}
\nabla_4 j_+ &= \nabla_4 \Re(\mathfrak{J}_+) = \Re\left(\nabla_4 \mathfrak{J}_+\right) = -\Re\left(\frac{1}{q}\mathfrak{J}\right) = -\frac{1}{|q|^2}\Re(\overline{q}\mathfrak{J}_+) \\
&= -\frac{1}{|q|^2}(rj_+ + a\cos\theta\, {}^*j_+), \\
\nabla_4 j &= \nabla_4 \Re(\mathfrak{J}) = -\frac{1}{|q|^2}\Re(\overline{q}\mathfrak{J}) = -\frac{1}{|q|^2}(rj + a\cos\theta\, {}^*j).
\end{aligned}$$

We deduce

$$\begin{aligned}
\nabla_4(j_+\cdot j) &= -\frac{1}{|q|^2}(rj_+ + a\cos\theta\, {}^*j_+)\cdot j - \frac{1}{|q|^2}(rj + a\cos\theta\, {}^*j)\cdot j_+ \\
&= -\frac{2r}{|q|^2} j_+\cdot j.
\end{aligned}$$

Hence, since $e_4(J^{(-)}) = 0$,

$$\nabla_4\Big(j_+\cdot j + \frac{1}{|q|^2}J^{(-)}\Big) = -\frac{2r}{|q|^2}j_+\cdot j - \frac{2r}{|q|^4}J^{(-)} = -\frac{2r}{|q|^2}\Big(j_+\cdot j + \frac{1}{|q|^2}J^{(-)}\Big).$$

We deduce

$$\nabla_4\Big(\Re(\mathfrak{J}_+)\cdot\Re(\mathfrak{J}) + \frac{1}{|q|^2}J^{(-)}\Big) + \frac{2r}{|q|^4}\Big(\Re(\mathfrak{J}_+)\cdot\Re(\mathfrak{J}) + \frac{1}{|q|^2}J^{(-)}\Big) = 0.$$

Thus, the first identity of (2.68) is propagated to a neighborhood of Σ as stated. The second identity of (2.68) can be propagated to a neighborhood of Σ in the same manner. □

Remark 2.65. *Note that the relations of Lemma 2.64 are verified in Kerr, see Lemma 2.45.*

2.6.3. Definition of linearized quantities for an outgoing PG structure Recall that, given an outgoing PG structure with associated coordinates (u, r, θ, φ) the following hold true.

1. The following identities hold

$$\begin{aligned}
&\xi = 0, \quad \omega = 0, \quad e_4(r) = 1, \quad e_4(u) = e_4(\theta) = e_4(\varphi) = 0, \\
&e_1(r) = e_2(r) = 0.
\end{aligned}$$

In addition, we have

$$\underline{H} = -Z.$$

2. The quantities

$$\widehat{X}, \quad \underline{\widehat{X}}, \quad \underline{\Xi}, \quad A, \quad B, \quad \underline{B}, \quad \underline{A},$$

vanish in Kerr and therefore are small in perturbations.

We renormalize below all other quantities, not vanishing in Kerr[50], by subtracting their Kerr(a, m) values for suitably chosen constants[51] (a, m).

Definition 2.66. *Let $\mathfrak{J}$ be a complex horizontal 1-form which verifies* (2.62). *We define the following renormalizations, for given constants* (a, m)*,*

1. *Linearization of Ricci and curvature coefficients.*

$$\widecheck{trX} := trX - \frac{2}{q}, \qquad \widecheck{tr\underline{X}} := tr\underline{X} + \frac{2q\Delta}{|q|^4},$$
$$\check{Z} := Z - \frac{a\overline{q}}{|q|^2}\mathfrak{J}, \qquad \check{H} := H - \frac{aq}{|q|^2}\mathfrak{J},$$
$$\check{\underline{\omega}} := \underline{\omega} - \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right), \qquad \check{P} := P + \frac{2m}{q^3}.$$

2. *Linearization of derivatives of* r, q, u*.*

$$\widecheck{\mathcal{D}q} := \mathcal{D}q + a\mathfrak{J}, \qquad \widecheck{\mathcal{D}\overline{q}} := \mathcal{D}\overline{q} - a\mathfrak{J},$$
$$\widecheck{e_3(r)} := e_3(r) + \frac{\Delta}{|q|^2},$$
$$\widecheck{\mathcal{D}u} := \mathcal{D}u - a\mathfrak{J}, \qquad \widecheck{e_3(u)} := e_3(u) - \frac{2(r^2+a^2)}{|q|^2}.$$

3. *Linearization for* $\mathfrak{J}$ *and* $\mathfrak{J}_\pm$*.*

$$\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}} := \overline{\mathcal{D}}\cdot\mathfrak{J} - \frac{4i(r^2+a^2)\cos\theta}{|q|^4}, \qquad \widecheck{\nabla_3\mathfrak{J}} := \nabla_3\mathfrak{J} - \frac{\Delta q}{|q|^4}\mathfrak{J},$$
$$\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm} := \overline{\mathcal{D}}\cdot\mathfrak{J}_\pm + \frac{4}{r^2}J^{(\pm)} \pm \frac{4ia^2\cos\theta}{|q|^4}J^{(\mp)},$$

[50] Since $\underline{H} = -Z$, $\underline{H}$ does not need to be included in Definition 2.66.

[51] The precise values of these constants will be defined in Section 3.2, see also Definition 2.59.

$$\widetilde{\nabla_3\mathfrak{J}_\pm} := \nabla_3\mathfrak{J}_\pm - \frac{\Delta q}{|q|^4}\,\mathfrak{J}_\pm \pm \frac{2a}{|q|^2}\mathfrak{J}_\mp.$$

4. *Linearization for* $J^{(p)}$.

$$\widetilde{\mathcal{D}J^{(0)}} := \mathcal{D}J^{(0)} - i\mathfrak{J}, \qquad \widetilde{\mathcal{D}(J^{(\pm)})} := \mathcal{D}(J^{(\pm)}) - \mathfrak{J}_\pm,$$
$$\widetilde{e_3(J^{(+)})} := e_3(J^{(+)}) + \frac{2a}{|q|^2}J^{(-)}, \qquad \widetilde{e_3(J^{(-)})} := e_3(J^{(-)}) - \frac{2a}{|q|^2}J^{(+)}.$$

2.6.4. Definition of the notations Γ_b and Γ_g for error terms

Definition 2.67. *The set of all linearized quantities is of the form* $\Gamma_g \cup \Gamma_b$ *with* Γ_g, Γ_b *defined as follows.*

1. *The set* Γ_g *with*

$$\Gamma_g = \Big\{\widetilde{trX}, \quad \widehat{X}, \quad \check{Z}, \quad \widetilde{tr\underline{X}}, \quad r\check{P}, \quad rB, \quad rA\Big\}. \tag{2.69}$$

2. *The set* $\Gamma_b = \Gamma_{b,1} \cup \Gamma_{b,2} \cup \Gamma_{b,3} \cup \Gamma_{b,4}$ *with*

$$\begin{aligned}
\Gamma_{b,1} &= \Big\{\check{H}, \quad \widehat{\underline{X}}, \quad \check{\underline{\omega}}, \quad \underline{\Xi}, \quad r\underline{B}, \quad \underline{A}\Big\},\\
\Gamma_{b,2} &= \Big\{r^{-1}\widetilde{e_3(r)}, \quad \widecheck{\mathcal{D}q}, \quad \widecheck{\mathcal{D}\overline{q}}, \quad \widecheck{\mathcal{D}u}, \quad r^{-1}\widetilde{e_3(u)}\Big\},\\
\Gamma_{b,3} &= \Big\{\widetilde{\mathcal{D}(J^{(0)})}, \quad \widetilde{\mathcal{D}(J^{(\pm)})}, \quad e_3(J^{(0)}), \quad \widetilde{e_3(J^{(\pm)})}\Big\},\\
\Gamma_{b,4} &= \Big\{r\,\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}, \quad r\,\mathcal{D}\widehat{\otimes}\mathfrak{J}, \quad r\,\widetilde{\nabla_3\mathfrak{J}}, \quad r\,\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm}, \quad r\,\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm, \quad r\,\widetilde{\nabla_3\mathfrak{J}_\pm}\Big\}.
\end{aligned} \tag{2.70}$$

We also define, with the help of the weighted derivatives $\mathfrak{d} = \{\nabla_3, r\nabla_4, \not{\mathfrak{d}} = r\nabla\}$,

$$\Gamma_g^{(s)} = \mathfrak{d}^{\le s}\Gamma_g, \qquad \Gamma_b^{(s)} = \mathfrak{d}^{\le s}\Gamma_b.$$

Remark 2.68. *The justification for the above decompositions has to do with the expected decay properties of the linearized components of the outgoing PG structure. More precisely, we expect that, see Sections 3.3 and 3.5 for details,*

$$\begin{aligned}
|\Gamma_g^{(s)}| &\lesssim \epsilon \min\Big\{r^{-2}u^{-1/2-\delta_{dec}}, \, r^{-1}u^{-1-\delta_{dec}}\Big\},\\
|\nabla_3\Gamma_g^{(s-1)}| &\lesssim \epsilon r^{-2}u^{-1-\delta_{dec}},\\
|\Gamma_b^{(s)}| &\lesssim \epsilon r^{-1}u^{-1-\delta_{dec}},
\end{aligned} \tag{2.71}$$

for a small constant $\delta_{dec} > 0$.

2.6.5. Approximate Killing vectorfield T Given an outgoing PG structure on $\mathcal{M}$ with adapted coordinates (u, r, θ, φ), i.e. $e_4(r) = 1$ and $e_4(u) = e_4(\theta) = e_4(\varphi) = 0$, we define a vectorfield $\mathbf{T}$ as follows.

Definition 2.69. *The vectorfield* $\mathbf{T}$ *is defined by*

$$\mathbf{T} \quad := \quad \frac{1}{2}\left(e_3 + \frac{\Delta}{|q|^2}e_4 - 2a\Re(\mathfrak{J})^b e_b\right).$$

Note that we have

$$\begin{aligned}\mathbf{T}(u) = 1 + \frac{1}{2}\left(\widecheck{e_3(u)} - 2a\Re(\mathfrak{J})\cdot\widecheck{\nabla u}\right), \qquad \mathbf{T}(r) = \frac{1}{2}\widecheck{e_3(r)},\\ \mathbf{T}(\cos\theta) = \frac{1}{2}\left(e_3(\cos\theta) - 2a\Re(\mathfrak{J})\cdot\widecheck{\nabla\cos\theta}\right),\end{aligned} \tag{2.72}$$

and[52]

$$\mathbf{g}(\mathbf{T}, \mathbf{T}) \quad = \quad -1 + \frac{2mr}{|q|^2}. \tag{2.73}$$

The following proposition shows that $\mathbf{T}$ is an approximate Killing vectorfield.

Proposition 2.70. *We have* ${}^{(\mathbf{T})}\pi_{44} = 0$, ${}^{(\mathbf{T})}\pi_{4a} \in \Gamma_g$ *and all other components of* ${}^{(\mathbf{T})}\pi$ *are in* Γ_b. *Moreover*

$$g^{ab}\,{}^{(\mathbf{T})}\pi_{ab} \quad = \quad \Gamma_g.$$

In addition

$$\mathbf{g}(\mathbf{D}_a\mathbf{T}, e_4),\ \mathbf{g}(\mathbf{D}_4\mathbf{T}, e_a) \in \Gamma_g, \qquad \mathbf{g}(\mathbf{D}_a\mathbf{T}, e_3),\ \mathbf{g}(\mathbf{D}_3\mathbf{T}, e_a) \in \Gamma_b,$$

and

$$\mathbf{g}(\mathbf{D}_a\mathbf{T}, e_b) = -\frac{2amr\cos\theta}{|q|^4} \in_{ab} + \Gamma_b.$$

Proof. See Section A.4. □

[52] We use in particular the identity

$$|\Re(\mathfrak{J})|^2 = \frac{(\sin\theta)^2}{|q|^2}.$$

2.7. Ingoing PG structures

2.7.1. Definition of ingoing PG structures Ingoing PG structures are PG structures where the roles of e_3 and e_4 are reversed compared to outgoing ones. In particular, we have for ingoing PG structures

$$\mathbf{D}_3 e_3 = 0, \qquad e_3(r) = -1, \qquad \nabla(r) = 0, \tag{2.74}$$

and hence

$$\underline{\xi} = 0, \qquad \underline{\omega} = 0, \qquad \eta = \zeta. \tag{2.75}$$

Also, in addition to r, we define the ingoing PG coordinates $(\underline{u}, \theta, \varphi)$ such that

$$e_3(\underline{u}) = e_3(\theta) = e_3(\varphi) = 0. \tag{2.76}$$

Finally, we introduce horizontal complex 1-forms $\mathfrak{J}$, $\mathfrak{J}_\pm$ satisfying

$$\nabla_3 \mathfrak{J} = \frac{1}{\overline{q}}\mathfrak{J}, \qquad \nabla_3 \mathfrak{J}_\pm = \frac{1}{\overline{q}}\mathfrak{J}_\pm, \tag{2.77}$$

and we define the adapted basis of $\ell = 1$ modes $J^{(p)}$, $p = 0, +, -$, by

$$J^{(0)} = \cos\theta, \qquad J^{(+)} = \sin\theta\cos\varphi, \qquad J^{(-)} = \sin\theta\sin\varphi, \tag{2.78}$$

so that we have in particular $e_3(J^{(p)}) = 0$ for $p = 0, +, -$.

All the equations of Proposition 2.19 and Proposition 2.21 for outgoing PG structures have a counterpart for ingoing PG structures. The equations can be easily deduced from the ones of Proposition 2.19 and Proposition 2.21 by performing the following substitutions

$$\begin{aligned}
&u \to \underline{u}, \quad r \to r, \quad \theta \to \theta, \quad \varphi \to \varphi, \quad e_4 \to e_3, \quad e_3 \to e_4, \quad e_a \to e_a,\\
&\alpha \to \underline{\alpha}, \quad \beta \to -\underline{\beta}, \quad \rho \to \rho, \quad {}^*\rho \to -\,{}^*\rho, \quad \underline{\beta} \to -\beta, \quad \underline{\alpha} \to \alpha,\\
&\xi \to \underline{\xi}, \quad \omega \to \underline{\omega}, \quad \chi \to \underline{\chi}, \quad \eta \to \underline{\eta}, \quad \underline{\eta} \to \eta, \quad \zeta \to -\zeta, \quad \underline{\chi} \to \chi,\\
&\underline{\omega} \to \omega, \quad \underline{\xi} \to \xi.
\end{aligned}$$

2.7.2. Ingoing PG structures in Kerr In Kerr, relative to BL coordinates (t, r, θ, ϕ), the ingoing principal null pair is given by

$$e_4 = \frac{r^2 + a^2}{|q|^2}\partial_t + \frac{\Delta}{|q|^2}\partial_r + \frac{a}{|q|^2}\partial_\phi, \qquad e_3 = \frac{r^2 + a^2}{\Delta}\partial_t - \partial_r + \frac{a}{\Delta}\partial_\phi.$$

Also, the functions $(\underline{u}, \varphi)$ are given by

$$\underline{u} := t + f(r), \quad f'(r) = \frac{r^2 + a^2}{\Delta}, \qquad \varphi := \phi + h(r), \quad h'(r) = \frac{a}{\Delta}.$$

Remark 2.71. *Note that we have indeed*

$$e_3(r) = -1, \qquad e_3(\underline{u}) = e_3(\theta) = e_3(\varphi) = 0.$$

$\mathfrak{J}$, $\mathfrak{J}_\pm$ and $J^{(p)}$ are still defined according respectively to Definition 2.30, Definition 2.42 and Definition 2.36.

2.7.3. Linearization of ingoing PG structures To linearize 1-forms, we rely on a complex horizontal 1-form $\mathfrak{J}$ verifying the following properties

$$\nabla_3 \mathfrak{J} = \frac{1}{q}\mathfrak{J}, \qquad {}^*\mathfrak{J} = -i\mathfrak{J}, \qquad \mathfrak{J} \cdot \overline{\mathfrak{J}} = \frac{2(\sin\theta)^2}{|q|^2}. \tag{2.79}$$

Definition 2.72. *Let a complex horizontal 1-form $\mathfrak{J}$ satisfying (2.79). We define the following renormalizations, for given constants (a, m),*

1. *Linearization of Ricci and curvature coefficients.*

$$\widecheck{trX} := trX - \frac{2\overline{q}\Delta}{|q|^4}, \qquad \widecheck{tr\underline{X}} := tr\underline{X} + \frac{2}{q},$$
$$\check{Z} := Z - \frac{aq}{|q|^2}\mathfrak{J}, \qquad \check{\underline{H}} := \underline{H} + \frac{a\overline{q}}{|q|^2}\mathfrak{J},$$
$$\check{\omega} := \omega + \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right), \qquad \check{P} := P + \frac{2m}{q^3}.$$

2. *Linearization of derivatives of $r, q, \underline{u}$.*

$$\widecheck{\mathcal{D}q} := \mathcal{D}q + a\mathfrak{J}, \qquad \widecheck{\mathcal{D}\overline{q}} := \mathcal{D}\overline{q} - a\mathfrak{J},$$
$$\widecheck{e_4(r)} := e_4(r) - \frac{\Delta}{|q|^2},$$
$$\widecheck{\mathcal{D}\underline{u}} := \mathcal{D}\underline{u} - a\mathfrak{J}, \qquad \widecheck{e_4(\underline{u})} := e_4(\underline{u}) - \frac{2(r^2 + a^2)}{|q|^2}.$$

3. *Linearization for $\mathfrak{J}$ and $\mathfrak{J}_\pm$.*

$$\widecheck{\overline{\mathcal{D}} \cdot \mathfrak{J}} := \overline{\mathcal{D}} \cdot \mathfrak{J} - \frac{4i(r^2 + a^2)\cos\theta}{|q|^4}, \qquad \widecheck{\nabla_4 \mathfrak{J}} := \nabla_4 \mathfrak{J} + \frac{\Delta\overline{q}}{|q|^4}\mathfrak{J},$$

$$\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm} := \overline{\mathcal{D}}\cdot\mathfrak{J}_\pm + \frac{4}{r^2}J^{(\pm)} \pm \frac{4ia^2\cos\theta}{|q|^4}J^{(\mp)},$$
$$\widetilde{\nabla_4\mathfrak{J}_\pm} := \nabla_4\mathfrak{J}_\pm + \frac{\Delta\overline{q}}{|q|^4}\mathfrak{J}_\pm \pm \frac{2a}{|q|^2}\mathfrak{J}_\mp.$$

4. *Linearization for $J^{(p)}$.*

$$\widetilde{\mathcal{D}J^{(0)}} := \mathcal{D}J^{(0)} - i\mathfrak{J}, \qquad \widetilde{\mathcal{D}(J^{(\pm)})} := \mathcal{D}(J^{(\pm)}) - \mathfrak{J}_\pm,$$
$$\widetilde{e_4(J^{(+)})} := e_4(J^{(+)}) + \frac{2a}{|q|^2}J^{(-)}, \qquad \widetilde{e_4(J^{(-)})} := e_4(J^{(-)}) - \frac{2a}{|q|^2}J^{(+)}.$$

Definition 2.73. *The set of all linearized quantities is of the form $\Gamma_g \cup \Gamma_b$ with Γ_g, Γ_b defined as follows.*

1. *The set $\Gamma_g = \Gamma_{g,1} \cup \Gamma_{g,2}$ with*

$$\Gamma_{g,1} = \Big\{\Xi, \quad \check{\omega}, \quad \widetilde{trX}, \quad \widehat{X}, \quad \check{Z}, \quad \underline{\check{H}}, \quad \widetilde{tr\underline{X}}, \quad r\check{P}, \quad rB, \quad rA\Big\},$$
$$\Gamma_{g,2} = \Big\{r\widetilde{e_4(r)}, \quad r\widetilde{e_4(\underline{u})}, \quad re_4(J^{(0)}), \quad r\widetilde{e_4(J^{(\pm)})}, \quad r^2\widetilde{\nabla_4\mathfrak{J}}, \quad r^2\widetilde{\nabla_4\mathfrak{J}_\pm}\Big\}.$$

2. *The set $\Gamma_b = \Gamma_{b,1} \cup \Gamma_{b,2}$ with*

$$\Gamma_{b,1} = \Big\{\underline{\widehat{X}}, \quad r\underline{B}, \quad \underline{A}, \quad \widetilde{\mathcal{D}q}, \quad \widetilde{\mathcal{D}\overline{q}}, \quad \widetilde{\mathcal{D}\underline{u}}\Big\},$$
$$\Gamma_{b,2} = \Big\{\widetilde{\mathcal{D}(J^{(0)})}, \quad \widetilde{\mathcal{D}(J^{(\pm)})}, \quad r\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}, \quad r\,\mathcal{D}\widehat{\otimes}\mathfrak{J}, \quad r\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm}, \quad r\,\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm\Big\}.$$

2.8. Principal temporal structures

The PG structures we have studied so far are perfectly adequate for deriving decay estimates but are deficient in terms of loss of derivatives and thus inadequate for deriving boundedness estimates for the top derivatives of the Ricci coefficients. Indeed the ∇_4 equations for $\mathrm{tr}\underline{X}$, $\underline{\widehat{X}}$ and $\underline{\Xi}$ in Proposition 2.19 contain angular derivatives[53] of other Ricci coefficients. Similarly, the same situation occurs for ingoing PG structures where the ∇_3 equations for $\mathrm{tr}X$, $\widehat{X}$, and Ξ are manifestly losing derivatives. Thus, in order to derive boundedness estimates for the top derivatives of the Ricci coefficients, we are forced to introduce new frames which we call principal temporal (PT). We first introduce outgoing PT structures, and then ingoing PT structures.

[53]This loss can be overcome for integrable foliations such as geodesic foliations and double null foliations relying on elliptic Hodge systems on 2-spheres of the foliation, but not for non-integrable structures such as PG structures.

2.8.1. Outgoing PT structures

Definition 2.74. *An outgoing PT structure* $\{(e_3, e_4, \mathcal{H}), r, \theta, \mathfrak{J}\}$ *on* $\mathcal{M}$ *consists of a null pair* (e_3, e_4)*, the induced horizontal structure* $\mathcal{H}$*, functions* (r, θ)*, and a horizontal 1-form* $\mathfrak{J}$ *such that the following hold true:*

1. e_4 *is geodesic.*
2. *We have*

$$e_4(r) = 1, \qquad e_4(\theta) = 0, \qquad \nabla_4(q\mathfrak{J}) = 0, \qquad q = r + ai\cos\theta.$$

3. *We have*

$$\underline{H} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J}.$$

An extended outgoing PT structure possesses, in addition, a scalar function u *verifying* $e_4(u) = 0$.

Definition 2.75. *An outgoing PT initial data set consists of a hypersurface* Σ *transversal to* e_4 *together with a null pair* (e_3, e_4)*, the induced horizontal structure* $\mathcal{H}$*, scalar functions* (r, θ)*, and a horizontal 1-form* $\mathfrak{J}$*, all defined on* Σ.

Lemma 2.76. *Any outgoing PT initial data set, as in Definition 2.75, can be locally extended to an outgoing PT structure.*

Proof. Extend first e_4 in a neighborhood of Σ such that $\mathbf{D}_{e_4}e_4 = 0$. Also, extend e_a, $a = 1, 2$, such that $\mathbf{D}_{e_4}e_a = 0$. Finally, extend e_3 to be the unique null companion of e_4 orthogonal to (e_1, e_2). Thus,

$$\xi = 0, \qquad \omega = 0, \qquad \underline{\eta} = 0.$$

We also extend (r, θ) and $\mathfrak{J}$ so that,

$$e_4(r) = 1, \qquad e_4(\theta) = 0, \qquad \nabla_4(q\mathfrak{J}) = 0, \qquad q = r + ia\cos\theta.$$

Since $\underline{\eta} = 0$, (e_1, e_2, e_3, e_4) is not a PT frame. We look for a new frame (e_1', e_2', e_3', e_4') related to (e_1, e_2, e_3, e_4) by the formula (2.6) with transition parameters $(f = 0, \underline{f}, \lambda = 1)$, i.e.

$$\begin{aligned}
e_4' &= e_4,\\
e_a' &= e_a + \frac{1}{2}\underline{f}_a e_4,\\
e_3' &= e_3 + \underline{f}^b e_b + \frac{1}{4}|\underline{f}|^2 e_4,
\end{aligned}$$

where $\underline{f}$ will be chosen later. Since $e_4' = e_4$, and since e_4 is geodesic, note first that we have in the frame (e_1', e_2', e_3', e_4')

$$\xi' = 0, \qquad \omega' = 0.$$

Next, we define the functions (r', θ') and the horizontal 1-form $\mathfrak{J}'$, for the horizontal structure $\mathcal{H}'$ induced by (e_3', e_4'), by

$$r' = r, \qquad \theta' = \theta, \qquad q' = r' + i\cos(\theta') = q, \qquad \mathfrak{J}'_{e_a'} = \mathfrak{J}_{e_a}, \quad a = 1, 2.$$

Since $e_4' = e_4$, we have in particular

$$e_4'(r') = 1, \qquad e_4'(\theta') = 0.$$

Also, we compute $\nabla_4'(q'\mathfrak{J}')$. Since $\mathfrak{J}'_{e_a'} = \mathfrak{J}_{e_a}$, $e_4' = e_4$, $q' = q$, and $\nabla_4(q\mathfrak{J}) = 0$, we have

$$\begin{aligned}
\nabla_4'(q'\mathfrak{J}')_a &= e_4'(q'\mathfrak{J}_a') - \mathbf{g}(\mathbf{D}_{e_4'}e_a', e_b')\mathfrak{J}_b' = e_4(q\mathfrak{J}_a) - \mathbf{g}(\mathbf{D}_{e_4'}e_a', e_b')\mathfrak{J}_b \\
&= \nabla_4(q\mathfrak{J})_a - \Big(\mathbf{g}(\mathbf{D}_{e_4'}e_a', e_b') - \mathbf{g}(\mathbf{D}_{e_4}e_a, e_b)\Big)\mathfrak{J}_b \\
&= -\Big(\mathbf{g}(\mathbf{D}_{e_4'}e_a', e_b') - \mathbf{g}(\mathbf{D}_{e_4}e_a, e_b)\Big)\mathfrak{J}_b.
\end{aligned}$$

Since

$$\begin{aligned}
\mathbf{g}(\mathbf{D}_{e_4'}e_a', e_b') &= \mathbf{g}\left(\mathbf{D}_{e_4}\left(e_a + \frac{1}{2}\underline{f}_a e_4\right), e_b + \frac{1}{2}\underline{f}_b e_4\right) \\
&= \mathbf{g}(\mathbf{D}_{e_4}e_a, e_b) - \underline{f}_b\xi_a + \underline{f}_a\xi_b = \mathbf{g}(\mathbf{D}_{e_4}e_a, e_b)
\end{aligned}$$

where we used the fact that $\xi = 0$, we infer

$$\nabla_4'(q'\mathfrak{J}') = 0.$$

In view of the above, in order for $\{(e_3', e_4', \mathcal{H}'), r', \theta', \mathfrak{J}'\}$ to satisfy all the properties of an outgoing PT structure, it remains to obtain the desired identity for $\underline{\eta}'$ which is related to the choice of $\underline{f}$. To this end, we compute[54]

$$2\underline{\eta}_a' = \mathbf{g}(\mathbf{D}_{e_4'}e_3', e_a') = \mathbf{g}\left(\mathbf{D}_{e_4}\left(e_3 + \underline{f}^b e_b + \frac{1}{4}|\underline{f}|^2 e_4\right), e_a + \frac{1}{2}\underline{f}_a e_4\right)$$

[54]We use here a more precise transformation formula for $\underline{\eta}$ than the one derived in Proposition 2.12.

$$
\begin{aligned}
&= \mathbf{g}\left(\mathbf{D}_{e_4}e_3, e_a+\frac{1}{2}\underline{f}_a e_4\right)+e_4(\underline{f}_a)+\underline{f}^b\mathbf{g}\left(\mathbf{D}_{e_4}e_b, e_a+\frac{1}{2}\underline{f}_a e_4\right)\\
&\quad +\frac{1}{4}|\underline{f}|^2\mathbf{g}\left(\mathbf{D}_{e_4}e_4, e_a+\frac{1}{2}\underline{f}_a e_4\right)\\
&= 2\underline{\eta}_a-2\omega\underline{f}_a+\nabla_4\underline{f}_a-\underline{f}_b\underline{f}_a\xi_b+\frac{1}{2}|\underline{f}|^2\xi_a
\end{aligned}
$$

and since $\xi=0$, $\omega=0$ and $\underline{\eta}=0$, we infer

$$
2\underline{\eta}' \quad = \quad \nabla_4\underline{f}.
$$

We now fix $\underline{f}$, and hence the frame (e_1', e_2', e_3', e_4'), as the solution of the following transport equation

$$
\nabla_4\underline{f}=-2\Re\left(\frac{a\overline{q'}}{|q'|^2}\mathfrak{J}'\right), \qquad \underline{f}_{|\Sigma}=0.
$$

In view of the above transformation formula for $\underline{\eta}'$, we infer $\underline{\eta}'=-\Re(\frac{a\overline{q'}}{|q'|^2}\mathfrak{J}')$, and hence

$$
\underline{H}'=-\frac{a\overline{q'}}{|q'|^2}\mathfrak{J}'.
$$

Thus, $\{(e_3', e_4', \mathcal{H}'), r, \theta, \mathfrak{J}'\}$ satisfies all the properties of an outgoing PT structure, and, since $\underline{f}=0$ on Σ, coincides with the initial outgoing PT data set on Σ. This ends the proof of Lemma 2.76. □

2.8.2. Null structure equations in an outgoing PT frame

Proposition 2.77. *Consider an outgoing PT structure. Then, the equations in the e_4 direction for the Ricci coefficients of the outgoing PT frame take the form*

$$
\begin{aligned}
\nabla_4 trX+\frac{1}{2}(trX)^2 &= -\frac{1}{2}\widehat{X}\cdot\overline{\widehat{X}},\\
\nabla_4\widehat{X}+\Re(trX)\widehat{X} &= -A,\\
\nabla_4 tr\underline{X}+\frac{1}{2}trX\,tr\underline{X} &= -\mathcal{D}\cdot\left(\frac{aq}{|q|^2}\overline{\mathfrak{J}}\right)+\frac{a^2}{|q|^2}|\mathfrak{J}|^2+2\overline{P}-\frac{1}{2}\widehat{X}\cdot\overline{\widehat{\underline{X}}},\\
\nabla_4\widehat{\underline{X}}+\frac{1}{2}trX\,\widehat{\underline{X}} &= -\mathcal{D}\widehat{\otimes}\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)+\frac{a^2(\overline{q})^2}{|q|^4}\mathfrak{J}\widehat{\otimes}\mathfrak{J}-\frac{1}{2}\overline{tr\underline{X}}\widehat{X},\\
\nabla_4 Z+\frac{1}{2}trXZ &= -\frac{1}{2}trX\frac{a\overline{q}}{|q|^2}\mathfrak{J}-\frac{1}{2}\widehat{X}\cdot\left(\overline{Z}+\frac{aq}{|q|^2}\overline{\mathfrak{J}}\right)-B,
\end{aligned}
$$

$$
\begin{aligned}
\nabla_4\underline{\Xi} &= -\nabla_3\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right) - \frac{1}{2}\overline{tr\underline{X}}\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J} + H\right)\\
&\quad -\frac{1}{2}\widehat{\underline{X}}\cdot\left(\frac{aq}{|q|^2}\overline{\mathfrak{J}} + \overline{H}\right) - \underline{B},\\
\nabla_4 H &= -\frac{1}{2}\overline{trX}\left(H + \frac{a\overline{q}}{|q|^2}\mathfrak{J}\right) - \frac{1}{2}\widehat{X}\cdot\left(\overline{H} + \frac{aq}{|q|^2}\overline{\mathfrak{J}}\right) - B,\\
\nabla_4\underline{\omega} &= \left(\eta + \Re\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)\right)\cdot\zeta + \eta\cdot\Re\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right) + \rho.
\end{aligned}
$$

Remark 2.78. *The main feature of the PT gauge choice is that treating all equations in Proposition 2.77 as transport equations in e_4 does not lose derivatives. Indeed, the RHS of all the equations depend only on Ricci and curvature coefficients, as well as first order derivatives of q and $\mathfrak{J}$ (in the RHS of the equations for $tr\underline{X}$, $\widehat{\underline{X}}$ and $\underline{\Xi}$). The point is that first order derivatives of q and $\mathfrak{J}$ can be controlled at the same level of regularity*[55] *than the Ricci coefficients of the PT frame.*

Proof. These equations follow immediately from plugging the identities

$$
\Xi = 0, \qquad \omega = 0, \qquad \underline{H} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J},
$$

satisfied by an outgoing PT frame in the general formulas of Proposition 2.8. □

2.8.3. Linearized quantities for outgoing PT structures Given an extended outgoing PT structure $\{(e_3, e_4, \mathcal{H}), r, \theta, u, \mathfrak{J}\}$, the following holds true:

1. We have

$$
\xi = \omega = 0, \quad e_4(r) = 1, \quad e_4(u) = e_4(\theta) = 0, \qquad \nabla_4(r\mathfrak{J}) = 0.
$$

   In addition, we have

$$
\underline{H} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J}.
$$

2. The quantities

$$
\widehat{X}, \quad \widehat{\underline{X}}, \quad \underline{\Xi}, \quad A, \quad B, \quad \underline{B}, \quad \underline{A}, \quad \mathcal{D}r, \quad e_3(\cos\theta), \quad \mathcal{D}\widehat{\otimes}\mathfrak{J},
$$

[55]In the case of PG frames, these terms are replaced by first order derivatives of Z leading to a loss of derivative.

vanish in Kerr and therefore are small in perturbations.

We renormalize below all other quantities, not vanishing in Kerr[56], by subtracting their Kerr(a,m) values.

Definition 2.79. *We consider the following renormalizations, for given constants* (a,m)*,*

$$\widetilde{trX} := trX - \frac{2}{q}, \qquad \widetilde{tr\underline{X}} := tr\underline{X} + \frac{2q\Delta}{|q|^4},$$
$$\check{Z} := Z - \frac{a\overline{q}}{|q|^2}\mathfrak{J}, \qquad \check{H} := H - \frac{aq}{|q|^2}\mathfrak{J}, \tag{2.80}$$
$$\check{\underline{\omega}} := \underline{\omega} - \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right), \qquad \check{P} := P + \frac{2m}{q^3},$$

as well as

$$\widetilde{e_3(r)} := e_3(r) + \frac{\Delta}{|q|^2}, \qquad \widetilde{\mathcal{D}(\cos\theta)} := \mathcal{D}(\cos(\theta)) - i\mathfrak{J},$$
$$\widetilde{\mathcal{D}u} := \mathcal{D}u - a\mathfrak{J}, \qquad \widetilde{e_3(u)} := e_3(u) - \frac{2(r^2+a^2)}{|q|^2}, \tag{2.81}$$

and

$$\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}} := \overline{\mathcal{D}}\cdot\mathfrak{J} - \frac{4i(r^2+a^2)\cos\theta}{|q|^4}, \qquad \widetilde{\nabla_3\mathfrak{J}} := \nabla_3\mathfrak{J} - \frac{\Delta q}{|q|^4}\mathfrak{J}. \tag{2.82}$$

2.8.4. Transport equations for $(f, \underline{f}, \lambda)$ We will need to compare PG structures with PT structures. To this end, we will control coefficients $(f, \underline{f}, \lambda)$ corresponding to the change of frame between PG frames and PT frames. We derive in this section transport equations for $(f, \underline{f}, \lambda)$ in the case where the second frame is an outgoing PT frame[57].

The following is the analog of Corollary 2.13.

Corollary 2.80. *Under the assumption*

$$\xi' = 0, \qquad \omega' = 0,$$

we have the following transport equations for $(f, \underline{f}, \lambda)$

[56] Since $\underline{H} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J}$, $\underline{H}$ does not need to be included in Definition 2.79.

[57] This corresponds to the analog of Section 2.2.3 where transport equations for $(f, \underline{f}, \lambda)$ are derived in the case where the second frame is an outgoing PG frame.

$$\begin{aligned}
\nabla_{\lambda^{-1}e_4'}f+\frac{1}{2}(tr\chi f-{}^{(a)}tr\chi\,{}^*f)+2\omega f &= -2\xi-f\cdot\widehat{\chi}+E_1(f,\Gamma),\\
\lambda^{-1}e_4'(\log\lambda) &= 2\omega+f\cdot(\zeta-\underline{\eta})+E_2(f,\Gamma),\\
\nabla_{\lambda^{-1}e_4'}\underline{f} &= 2(\underline{\eta}'-\underline{\eta})-\frac{1}{2}(tr\underline{\chi}f-{}^{(a)}tr\underline{\chi}\,{}^*f)\\
&\quad +2\omega\underline{f}-f\cdot\widehat{\underline{\chi}}+E_6(f,\underline{f},\Gamma),
\end{aligned}$$

where $E_1(f,\Gamma)$, $E_2(f,\Gamma)$ *and* $E_6(f,\underline{f},\Gamma)$ *are given by*

$$\begin{aligned}
E_1(f,\Gamma) &= -(f\cdot\zeta)f-\frac{1}{2}|f|^2\eta+\frac{1}{2}|f|^2\underline{\eta}+O(f^3\Gamma),\\
E_2(f,\Gamma) &= -\frac{1}{2}|f|^2\underline{\omega}-\frac{1}{4}tr\underline{\chi}|f|^2+O(f^3\Gamma+f^2\widehat{\underline{\chi}}),
\end{aligned}$$

and

$$E_6(f,\underline{f},\Gamma) = -(f\cdot\underline{\eta})\underline{f}+\frac{1}{2}(f\cdot\zeta)\underline{f}+O((f,\underline{f})^3\Gamma).$$

Proof. The transport equations for f and λ have already been derived in Corollary 2.13. We thus focus on the one for $\underline{f}$. Assuming that $\xi'=0$, we have in view of the frame transformation (2.6)

$$\begin{aligned}
2\underline{\eta}_a' &= \mathbf{g}(\mathbf{D}_{e_4'}e_3',e_a')=\mathbf{g}(\mathbf{D}_{\lambda^{-1}e_4'}(\lambda e_3'),e_a')\\
&= \mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}\left(e_3+\underline{f}^be_b'-\frac{1}{4}|\underline{f}|^2\lambda^{-1}e_4'\right),e_a'\right)\\
&= \mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_3,e_a'\right)+\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}\left(\underline{f}^be_b'\right),e_a'\right)-\frac{1}{2}|\underline{f}|^2\lambda^{-2}\xi_a'\\
&= \mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_3,e_a'\right)+\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}\left(\underline{f}^be_b'\right),e_a'\right).
\end{aligned}$$

We compute the terms on the right-hand side

$$\begin{aligned}
\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_3,e_a'\right) &= \mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_3,\left(\delta_a^b+\frac{1}{2}\underline{f}_af^b\right)e_b+\frac{1}{2}\underline{f}_ae_4\right)\\
&= \left(\delta_a^b+\frac{1}{2}\underline{f}_af^b\right)\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_3,e_b\right)+\frac{1}{2}\underline{f}_a\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_3,e_4\right)\\
&= \left(\delta_a^b+\frac{1}{2}\underline{f}_af^b\right)\mathbf{g}\left(\mathbf{D}_{e_4+f^ce_c+\frac{1}{4}|f|^2e_3}e_3,e_b\right)\\
&\quad +\frac{1}{2}\underline{f}_a\mathbf{g}\left(\mathbf{D}_{e_4+f^be_b+\frac{1}{4}|f|^2e_3}e_3,e_4\right)\\
&= \left(\delta_a^b+\frac{1}{2}\underline{f}_af^b\right)\left(2\underline{\eta}_b+f^c\underline{\chi}_{cb}+\frac{1}{2}|f|^2\underline{\xi}_b\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\underline{f}_a\big(-4\omega - f\cdot\zeta\big) + O((f,\underline{f})^3\Gamma)\\
=\;& 2\underline{\eta}_a + (f\cdot\underline{\eta})\underline{f}_a + f^c\underline{\chi}_{ca} + \frac{1}{2}|f|^2\underline{\xi}_a\\
&+\frac{1}{2}\underline{f}_a\big(-4\omega - f\cdot\zeta\big) + O((f,\underline{f})^3\Gamma)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}\left(\underline{f}^b e_b'\right), e_a'\right) &= \lambda^{-1}e_4'(\underline{f}_a) + \underline{f}^b\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_b', e_a'\right)\\
&= \lambda^{-1}e_4'(\underline{f}_a) - \underline{f}^b\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_a', e_b'\right)\\
&= \nabla_{\lambda^{-1}e_4'}\underline{f}_a.
\end{aligned}
$$

We infer

$$
\begin{aligned}
2\underline{\eta}'_a &= \mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_3, e_a'\right) + \mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}\left(\underline{f}^b e_b'\right), e_a'\right)\\
&= 2\underline{\eta}_a + \nabla_{\lambda^{-1}e_4'}\underline{f}_a + \frac{1}{2}\mathrm{tr}\,\underline{\chi} f_a - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\;{}^*f_a - 2\omega\underline{f}_a\\
&\quad + f^c\widehat{\underline{\chi}}_{ca} + (f\cdot\underline{\eta})\underline{f}_a - \frac{1}{2}\underline{f}_a(f\cdot\zeta) + O((f,\underline{f})^3\Gamma)
\end{aligned}
$$

and hence

$$
\nabla_{\lambda^{-1}e_4'}\underline{f} \;=\; 2(\underline{\eta}' - \underline{\eta}) - \frac{1}{2}(\mathrm{tr}\,\underline{\chi} f - {}^{(a)}\mathrm{tr}\underline{\chi}\;{}^*f) + 2\omega\underline{f} - f\cdot\widehat{\underline{\chi}} + E_6(f,\underline{f},\Gamma)
$$

where

$$
E_6(f,\underline{f},\Gamma) \;=\; -(f\cdot\underline{\eta})\underline{f} + \frac{1}{2}(f\cdot\zeta)\underline{f} + O((f,\underline{f})^3\Gamma)
$$

as desired. This concludes the proof of Corollary 2.80. □

The following is the analog of Corollary 2.14.

Corollary 2.81. *Assume that we have*

$$
\Xi' = 0, \qquad \omega' = 0, \qquad \underline{H}' = -\frac{a\overline{q'}}{|q'|^2}\mathfrak{J}'.
$$

We introduce

$$
F := f + i\,{}^*f, \qquad \underline{F} := \underline{f} + i\,{}^*\underline{f}.
$$

Then, we have

$$\begin{aligned}
\nabla_{\lambda^{-1}e'_4}F+\frac{1}{2}\overline{trX}F+2\omega F &= -2\Xi-\widehat{\chi}\cdot F+E_1(f,\Gamma),\\
\lambda^{-1}\nabla'_4(\log\lambda) &= 2\omega+f\cdot(\zeta-\underline{\eta})+E_2(f,\Gamma),\\
\nabla_{\lambda^{-1}e'_4}\underline{F} &= -2\left(\frac{a\overline{q'}}{|q'|^2}\mathfrak{J}'-\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)-2\left(\underline{H}+\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)\\
&\quad -\frac{1}{2}\overline{trX}F+2\omega\underline{F}-F\cdot\widehat{\underline{\chi}}+E_6(f,\underline{f},\Gamma).
\end{aligned}$$

Moreover, introducing a complex valued scalar function q satisfying $e_4(q)=1$, we have

$$\nabla_{\lambda^{-1}e'_4}(\overline{q}F) = -2\overline{q}\omega F-2\overline{q}\Xi+E_4(f,\Gamma),$$

where

$$\begin{aligned}
E_4(f,\Gamma) &= -\frac{1}{2}\overline{q}\left(\overline{trX}-\frac{2}{\overline{q}}\right)F-\overline{q}\widehat{\chi}\cdot F+\overline{q}E_1(f,\Gamma)+f\cdot\nabla(\overline{q})F\\
&\quad +\frac{1}{4}|f|^2e_3(\overline{q})F.
\end{aligned}$$

Proof. The transport equations for F and λ have already been derived in Corollary 2.14. We thus focus on the one for $\underline{F}$. Since $\xi'=0$, recall from Corollary 2.80 that we have

$$\nabla_{\lambda^{-1}e'_4}\underline{f} = 2(\underline{\eta}'-\underline{\eta})-\frac{1}{2}(\mathrm{tr}\,\underline{\chi}f-{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*f)+2\omega\underline{f}-f\cdot\widehat{\underline{\chi}}+E_6(f,\underline{f},\Gamma).$$

In view of the definition of F and $\underline{F}$, this yields

$$\nabla_{\lambda^{-1}e'_4}\underline{F} = 2(\underline{H}'-\underline{H})-\frac{1}{2}\overline{\mathrm{tr}X}F+2\omega\underline{F}-F\cdot\widehat{\underline{\chi}}+E_6(f,\underline{f},\Gamma).$$

Plugging $\underline{H}'=-\frac{a\overline{q'}}{|q'|^2}\mathfrak{J}'$, we obtain

$$\begin{aligned}
\nabla_{\lambda^{-1}e'_4}\underline{F} &= -2\left(\frac{a\overline{q'}}{|q'|^2}\mathfrak{J}'-\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)-2\left(\underline{H}+\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)-\frac{1}{2}\overline{\mathrm{tr}X}F+2\omega\underline{F}\\
&\quad -F\cdot\widehat{\underline{\chi}}+E_6(f,\underline{f},\Gamma)
\end{aligned}$$

as desired. □

Remark 2.82. *In practice, we will integrate first the transport equations for F, then the one for λ, and finally the one for $\underline{F}$. Note that the transport* *equation for $\underline{F}$ in Corollary 2.81 is at the same regularity level that the one for F and λ, while the one for $\underline{F}$ in Corollary 2.14 loses one derivative. This is another manifestation of the fact that, unlike the PT frame, the PG frame exhibits a loss of derivative.*

2.8.5. Ingoing PT structures

Definition 2.83. *An ingoing PT structure $\{(e_3, e_4, \mathcal{H}), r, \theta, \mathfrak{J}\}$ on $\mathcal{M}$ consists of a null pair (e_3, e_4), the induced horizontal structure $\mathcal{H}$, functions (r, θ), and a horizontal 1-form $\mathfrak{J}$ such that the following hold true:*

1. *e_3 is geodesic.*
2. *We have*

$$e_3(r) = -1, \qquad e_3(\theta) = 0, \qquad \nabla_3(\overline{q}\mathfrak{J}) = 0, \qquad q = r + ai\cos\theta.$$

3. *We have*

$$H = \frac{aq}{|q|^2}\mathfrak{J}.$$

An extended ingoing PT structure possesses, in addition, a function $\underline{u}$ verifying $e_3(\underline{u}) = 0$.

Definition 2.84. *An ingoing PT initial data set consists of a hypersurface Σ transversal to e_3 together with a null pair (e_3, e_4), the induced horizontal structure $\mathcal{H}$, scalar functions (r, θ), and a horizontal 1-form $\mathfrak{J}$, all defined on Σ.*

Lemma 2.85. *Any ingoing PT initial data set, as in Definition 2.84, can be locally extended to an ingoing PT structure.*

Proof. Straightforward adaptation of the proof of Lemma 2.76. □

2.8.6. Null structure equations in an ingoing PT frame

Proposition 2.86. *Consider an ingoing PT structure. Then, the equations in the e_3 direction for the Ricci coefficients of the ingoing PT frame take the form*

$$\begin{aligned}
\nabla_3 tr\underline{X} + \frac{1}{2}(tr\underline{X})^2 &= -\frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{\underline{X}}},\\
\nabla_3\widehat{\underline{X}} + \Re(tr\underline{X})\widehat{\underline{X}} &= -\underline{A},
\end{aligned}$$

$$\begin{aligned}
\nabla_3 trX + \frac{1}{2}tr\underline{X}trX &= \mathcal{D}\cdot\left(\frac{a\overline{q}}{|q|^2}\overline{\mathfrak{J}}\right) + \frac{a^2}{|q|^2}|\mathfrak{J}|^2 + 2P - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{X}},\\
\nabla_3\widehat{X} + \frac{1}{2}tr\underline{X}\,\widehat{X} &= \frac{1}{2}\mathcal{D}\widehat{\otimes}\left(\frac{aq}{|q|^2}\mathfrak{J}\right) + \frac{1}{2}\frac{a^2q^2}{|q|^4}\mathfrak{J}\widehat{\otimes}\mathfrak{J} - \frac{1}{2}\overline{trX}\widehat{\underline{X}},\\
\nabla_3 Z + \frac{1}{2}tr\underline{X}Z &= -\frac{1}{2}tr\underline{X}\frac{aq}{|q|^2}\mathfrak{J} - \frac{1}{2}\widehat{\underline{X}}\cdot\left(\overline{Z} + \frac{a\overline{q}}{|q|^2}\overline{\mathfrak{J}}\right) - \underline{B},\\
\nabla_3\,\underline{H} &= -\frac{1}{2}\overline{tr\underline{X}}\left(\underline{H} - \frac{aq}{|q|^2}\mathfrak{J}\right) - \frac{1}{2}\widehat{\underline{X}}\cdot\left(\overline{\underline{H}} - \frac{a\overline{q}}{|q|^2}\overline{\mathfrak{J}}\right) + \underline{B},\\
\nabla_3\Xi &= \nabla_4\left(\frac{aq}{|q|^2}\mathfrak{J}\right) + \frac{1}{2}\overline{trX}\left(\frac{aq}{|q|^2}\mathfrak{J} - \underline{H}\right)\\
&\quad + \frac{1}{2}\widehat{X}\cdot\left(\frac{a\overline{q}}{|q|^2}\overline{\mathfrak{J}} - \overline{\underline{H}}\right) - B,\\
\nabla_3\omega &= \left(\Re\left(\frac{aq}{|q|^2}\mathfrak{J}\right) - \underline{\eta}\right)\cdot\zeta - \Re\left(\frac{aq}{|q|^2}\mathfrak{J}\right)\cdot\underline{\eta} + \rho.
\end{aligned}$$

Proof. These equations follow immediately from plugging the identities

$$\underline{\Xi} = 0, \qquad \underline{\omega} = 0, \qquad H = \frac{aq}{|q|^2}\mathfrak{J},$$

satisfied by an ingoing PT frame in the general formulas of Proposition 2.8. □

2.8.7. Linearized quantities in an ingoing PT frame Given an extended ingoing PT structure $\{(e_3, e_4, \mathcal{H}), r, \theta, \underline{u}, \mathfrak{J}\}$, the following hold true:

1. We have

$$\underline{\xi} = \underline{\omega} = 0, \quad e_3(r) = -1, \quad e_3(\underline{u}) = e_3(\theta) = 0, \qquad \nabla_3(\overline{q}\mathfrak{J}) = 0.$$

   In addition, we have

$$H = \frac{aq}{|q|^2}\mathfrak{J}.$$

2. The quantities

$$\widehat{X}, \quad \widehat{\underline{X}}, \quad \Xi, \quad A, \quad B, \quad \underline{B}, \quad \underline{A}, \quad \mathcal{D}r, \quad e_4(\cos\theta), \quad \mathcal{D}\widehat{\otimes}\mathfrak{J},$$

   vanish in Kerr and therefore are small in perturbations.

We renormalize below all other quantities, not vanishing in Kerr[58], by subtracting their Kerr(a, m) values.

Definition 2.87. *We consider the following renormalizations, for given constants* (a, m),

$$\widetilde{trX} := trX - \frac{2\overline{q}\Delta}{|q|^4}, \qquad \widetilde{tr\underline{X}} := tr\underline{X} + \frac{2}{\overline{q}},$$
$$\check{Z} := Z - \frac{aq}{|q|^2}\mathfrak{J}, \qquad \check{\underline{H}} := \underline{H} + \frac{a\overline{q}}{|q|^2}\mathfrak{J}, \tag{2.83}$$
$$\check{\omega} := \omega + \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right), \qquad \check{P} := P + \frac{2m}{q^3},$$

as well as

$$\widetilde{e_4(r)} := e_4(r) - \frac{\Delta}{|q|^2}, \qquad \mathcal{D}(\widetilde{\cos\theta}) := \mathcal{D}(\cos(\theta)) - i\mathfrak{J},$$
$$\widetilde{\mathcal{D}\underline{u}} := \mathcal{D}\underline{u} - a\mathfrak{J}, \qquad \widetilde{e_4(\underline{u})} := e_4(\underline{u}) - \frac{2(r^2 + a^2)}{|q|^2}, \tag{2.84}$$

and

$$\widetilde{\overline{\mathcal{D}} \cdot \mathfrak{J}} := \overline{\mathcal{D}} \cdot \mathfrak{J} - \frac{4i(r^2 + a^2)\cos\theta}{|q|^4}, \qquad \widetilde{\nabla_4\mathfrak{J}} := \nabla_4\mathfrak{J} + \frac{\Delta\overline{q}}{|q|^4}\mathfrak{J}. \tag{2.85}$$

[58] Since $H = \frac{aq}{|q|^2}\mathfrak{J}$, H does not need to be included in Definition 2.87.

3. GCM ADMISSIBLE SPACETIMES

In this chapter we introduce the crucial notion of general covariant modulated (GCM) admissible spacetimes, define our main norms and state the main results.

3.1. Initial data layer

We consider a spacetime region $(\mathcal{L}_0, \mathbf{g})$, sketched below in figure 3, where

- The Lorentzian metric $\mathbf{g}$ is close to the metric[59] of $Kerr(a_0, m_0)$, $|a_0| < m_0$, in a suitable topology[60].
- $\mathcal{L}_0 = {}^{(ext)}\mathcal{L}_0 \cup {}^{(int)}\mathcal{L}_0$.
- The intersection ${}^{(ext)}\mathcal{L}_0 \cap {}^{(int)}\mathcal{L}_0$ is non trivial.

$(\mathcal{L}_0, g)$ is called the initial data layer if it satisfies the properties in Sections 3.1.1–3.1.3 below.

3.1.1. Boundaries The future and past boundaries of $\mathcal{L}_0$ are given by

$$\begin{aligned} \partial^+\mathcal{L}_0 &= \mathcal{A}_0 \cup \mathcal{B}_{(3,0)} \cup \underline{\mathcal{B}}_{(3,0)}, \\ \partial^-\mathcal{L}_0 &= \mathcal{B}_{(0,0)} \cup \underline{\mathcal{B}}_{(0,0)}, \end{aligned}$$

where

1. The past non-spacelike outgoing boundary of the far region ${}^{(ext)}\mathcal{L}_0$ is denoted by $\mathcal{B}_{(0,0)}$.
2. The past non-spacelike incoming boundary of the near region ${}^{(int)}\mathcal{L}_0$ is denoted by $\underline{\mathcal{B}}_{(0,0)}$.
3. ${}^{(ext)}\mathcal{L}_0$ is unbounded in the future outgoing directions.
4. The future non-spacelike outgoing boundary of the far region ${}^{(ext)}\mathcal{L}_0$ is denoted by $\mathcal{B}_{(3,0)}$.
5. The future non-spacelike outgoing boundary of the near region ${}^{(int)}\mathcal{L}_0$ is denoted by $\underline{\mathcal{B}}_{(3,0)}$.
6. The future spacelike boundary of the near region ${}^{(int)}\mathcal{L}_0$ is denoted by $\mathcal{A}_0$.

[59] We will write $\mathcal{L}_0(a_0, m_0)$ whenever we need to emphasize the dependence on (a_0, m_0).

[60] This topology will be specified in our initial data layer assumptions, see (3.52) as well as Section 3.3.6.

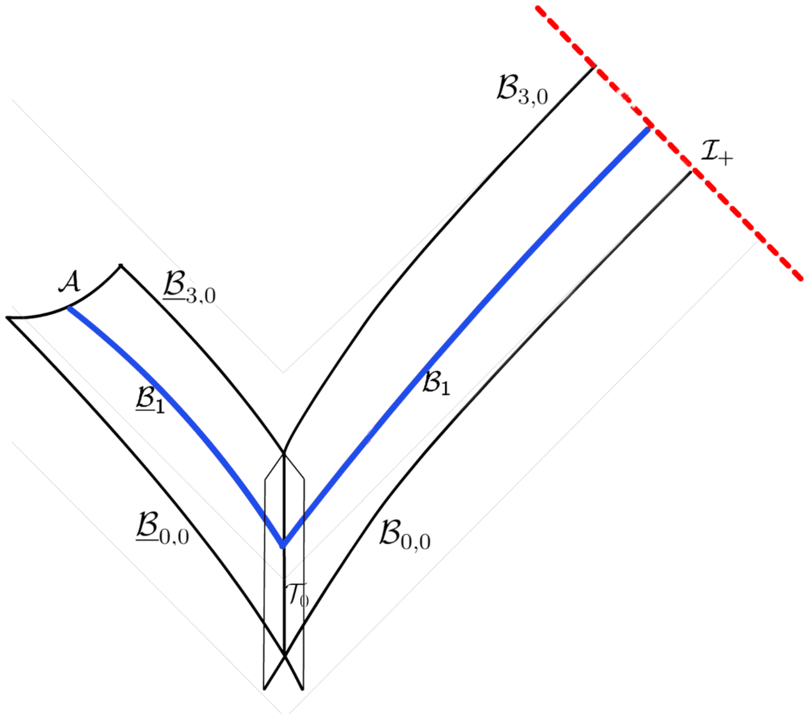

Figure 3: The initial data layer $\mathcal{L}_0$.

Remark 3.1. *Below, we make use of the following constants:*

- *the constants $m_0 > 0$ and $|a_0| < m_0$ are the given mass and the angular momentum of the perturbed Kerr solution,*
- *$\epsilon_0 > 0$ is a small constant measuring the size of the perturbation of the initial data,*
- *$\delta_{\mathcal{H}} > 0$ and $\delta_* > 0$ are sufficiently small constants,*
- *r_0 is a sufficiently large constant.*

See Section 3.4.1 for the specification of these constants.

3.1.2. Foliations of $\mathcal{L}_0$ and adapted null frames The spacetime $\mathcal{L}_0 = {}^{(ext)}\mathcal{L}_0 \cup {}^{(int)}\mathcal{L}_0$ is foliated as follows.

3.1.2.1. Foliations of ${}^{(ext)}\mathcal{L}_0$ The far region ${}^{(ext)}\mathcal{L}_0$ is covered by an outgoing PG structure, i.e. it is foliated by two functions $(u_{\mathcal{L}_0}, {}^{(ext)}r_{\mathcal{L}_0})$ such that

- We have

$$L_0(u_{\mathcal{L}_0}) = 0, \qquad L_0({}^{(ext)}r_{\mathcal{L}_0}) = 1,$$

where the vectorfield L_0 is null and satisfies $\mathbf{D}_{L_0} L_0 = 0$.
- We denote by $({}^{(ext)}(e_0)_3, {}^{(ext)}(e_0)_4, {}^{(ext)}(e_0)_1, {}^{(ext)}(e_0)_2)$ the null frame satisfying on ${}^{(ext)}\mathcal{L}_0$

$${}^{(ext)}(e_0)_4 = L_0, \qquad {}^{(ext)}(e_0)_1({}^{(ext)}r_{\mathcal{L}_0}) = {}^{(ext)}(e_0)_2({}^{(ext)}r_{\mathcal{L}_0}) = 0.$$

- The outgoing future non-spacelike boundary $\mathcal{B}_{(3,0)}$ and the past outgoing non-spacelike boundary $\mathcal{B}_{(0,0)}$ are given by

$$\mathcal{B}_{(3,0)} = \{u_{\mathcal{L}_0} = 3\}, \qquad \mathcal{B}_{(0,0)} = \{u_{\mathcal{L}_0} = 0\}.$$

- The foliation by $u_{\mathcal{L}_0}$ of ${}^{(ext)}\mathcal{L}_0$ terminates at the timelike boundary

$$\left\{ {}^{(ext)}r_{\mathcal{L}_0} = r_0 - 1 \right\},$$

where r_0 has been introduced in Remark 3.1.

3.1.2.2. Foliations of ${}^{(int)}\mathcal{L}_0$ The near region ${}^{(int)}\mathcal{L}_0$ is foliated by two functions $(\underline{u}_{\mathcal{L}_0}, {}^{(int)}r_{\mathcal{L}_0})$ such that

- We have

$$\underline{L}_0(\underline{u}_{\mathcal{L}_0}) = 0, \qquad \underline{L}_0({}^{(int)}r_{\mathcal{L}_0}) = -1,$$

where the vectorfield $\underline{L}_0$ is null and satisfies $\mathbf{D}_{\underline{L}_0} \underline{L}_0 = 0$.
- We denote by $({}^{(int)}(e_0)_3, {}^{(int)}(e_0)_4, {}^{(int)}(e_0)_1, {}^{(int)}(e_0)_2)$ the null frame satisfying on ${}^{(int)}\mathcal{L}_0$

$${}^{(int)}(e_0)_3 = \underline{L}_0, \qquad {}^{(int)}(e_0)_1({}^{(int)}r_{\mathcal{L}_0}) = {}^{(int)}(e_0)_2({}^{(int)}r_{\mathcal{L}_0}) = 0.$$

- The $(\underline{u}_{\mathcal{L}_0}, {}^{(int)}r)$ foliation is initialized on ${}^{(ext)}r_{\mathcal{L}_0} = r_0$ as it will be made precise below.
- The foliation by $\underline{u}_{\mathcal{L}_0}$, of ${}^{(int)}\mathcal{L}_0$ terminates at the space like boundary

$$\mathcal{A}_0 = \left\{ {}^{(int)}r_{\mathcal{L}_0} = \left(m_0 + \sqrt{m_0^2 - a_0^2}\right)(1 - 2\delta_{\mathcal{H}}) \right\}$$

where m_0, a_0 and $\delta_{\mathcal{H}}$ have been introduced in Remark 3.1.

- The future non-spacelike ingoing boundary $\underline{\mathcal{B}}_{(3,0)}$ and the past incoming non-spacelike boundary $\underline{\mathcal{B}}_{(0,0)}$ are given by

$$\underline{\mathcal{B}}_{(3,0)} = \{\underline{u}_{\mathcal{L}_0} = 3\}, \qquad \underline{\mathcal{B}}_{(0,0)} = \{\underline{u}_{\mathcal{L}_0} = 0\}.$$

- The foliation by $\underline{u}_{\mathcal{L}_0}$ of ${}^{(int)}\mathcal{L}_0$ terminates at the time like boundary

$$\left\{ {}^{(int)}r_{\mathcal{L}_0} = r_0 + 1 \right\}$$

  where r_0 has been introduced in Remark 3.1.

3.1.3. Definition of additional scalars and 1-forms in $\mathcal{L}_0$ We introduce the following scalars and 1-forms adapted to the initial data layer $\mathcal{L}_0$ defined above.

1. In ${}^{(ext)}\mathcal{L}_0$, we consider coordinates $({}^{(ext)}\theta_{\mathcal{L}_0}, {}^{(ext)}\varphi_{\mathcal{L}_0})$ satisfying

$$ {}^{(ext)}(e_0)_4({}^{(ext)}\theta_{\mathcal{L}_0}) = {}^{(ext)}(e_0)_4({}^{(ext)}\varphi_{\mathcal{L}_0}) = 0. \tag{3.1}$$

   We also consider scalar functions ${}^{(ext)}J^{(p)}$, $p = 0, +, -$ defined by

$$\begin{aligned} {}^{(ext)}J^{(0)} &= \cos\left({}^{(ext)}\theta_{\mathcal{L}_0}\right), \\ {}^{(ext)}J^{(+)} &= \sin\left({}^{(ext)}\theta_{\mathcal{L}_0}\right)\cos\left({}^{(ext)}\varphi_{\mathcal{L}_0}\right), \\ {}^{(ext)}J^{(-)} &= \sin\left({}^{(ext)}\theta_{\mathcal{L}_0}\right)\cos\left({}^{(ext)}\varphi_{\mathcal{L}_0}\right), \end{aligned} \tag{3.2}$$

   and a complex 1-form ${}^{(ext)}\mathfrak{J}$ satisfying

$$\nabla_{{}^{(ext)}(e_0)_4}{}^{(ext)}\mathfrak{J} = 0. \tag{3.3}$$

2. In ${}^{(int)}\mathcal{L}_0$, we consider coordinates $({}^{(int)}\theta_{\mathcal{L}_0}, {}^{(int)}\varphi_{\mathcal{L}_0})$ satisfying

$$ {}^{(int)}(e_0)_3({}^{(int)}\theta_{\mathcal{L}_0}) = {}^{(int)}(e_0)_3({}^{(int)}\varphi_{\mathcal{L}_0}) = 0. \tag{3.4}$$

   We also consider scalar functions ${}^{(int)}J^{(p)}$, $p = 0, +, -$ defined by

$$\begin{aligned} {}^{(int)}J^{(0)} &= \cos\left({}^{(ext)}\theta_{\mathcal{L}_0}\right), \\ {}^{(int)}J^{(+)} &= \sin\left({}^{(int)}\theta_{\mathcal{L}_0}\right)\cos\left({}^{(int)}\varphi_{\mathcal{L}_0}\right), \\ {}^{(int)}J^{(-)} &= \sin\left({}^{(int)}\theta_{\mathcal{L}_0}\right)\cos\left({}^{(int)}\varphi_{\mathcal{L}_0}\right), \end{aligned} \tag{3.5}$$

and a complex 1-form ${}^{(int)}\mathfrak{J}$ satisfying

$$\nabla_{{}^{(int)}(e_0)_3} {}^{(int)}\mathfrak{J} = 0. \tag{3.6}$$

3.1.4. Initializations of the foliation on ${}^{(int)}\mathcal{L}_0$ The $(\underline{u}_{\mathcal{L}_0}, {}^{(int)}r_{\mathcal{L}_0})$ foliation is initialized on ${}^{(ext)}r_{\mathcal{L}_0} = r_0$ by setting,

$$\underline{u}_{\mathcal{L}_0} = u_{\mathcal{L}_0}, \qquad {}^{(int)}r_{\mathcal{L}_0} = {}^{(ext)}r_{\mathcal{L}_0},$$

and,

$$\begin{aligned} &{}^{(int)}(e_0)_4 = \lambda_0 \, {}^{(ext)}(e_0)_4, \qquad {}^{(int)}(e_0)_3 = \lambda_0^{-1} \, {}^{(ext)}(e_0)_3, \\ &{}^{(int)}(e_0)_b = {}^{(ext)}(e_0)_b, \;\; b = 1, 2, \end{aligned}$$

where

$${}^{(ext)}\lambda_0 = \frac{r_0^2 - 2m_0 r_0 + a_0^2}{r_0^2 + a_0^2(\cos({}^{(ext)}\theta_{\mathcal{L}_0}))^2}.$$

We also initialize the coordinates $({}^{(int)}\theta_{\mathcal{L}_0}, {}^{(int)}\varphi_{\mathcal{L}_0})$ on ${}^{(ext)}r_{\mathcal{L}_0} = r_0$ as follows

$${}^{(int)}\theta_{\mathcal{L}_0} = {}^{(ext)}\theta_{\mathcal{L}_0}, \qquad {}^{(int)}\varphi_{\mathcal{L}_0} = {}^{(ext)}\varphi_{\mathcal{L}_0}.$$

3.2. GCM admissible spacetimes

We consider a spacetime $(\mathcal{M}, \mathbf{g})$, sketched below in figure 4, where

- The Lorentzian spacetime metric $\mathbf{g}$ is close to Kerr in a suitable topology[61].
- $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M} \cup {}^{(top)}\mathcal{M}$.
- $\mathcal{T} = {}^{(ext)}\mathcal{M} \cap {}^{(int)}\mathcal{M}$ is a time-like hyper-surface.
- ${}^{(ext)}\mathcal{M} \cap {}^{(top)}\mathcal{M}$ and ${}^{(int)}\mathcal{M} \cap {}^{(top)}\mathcal{M}$ are essentially time-like hyper-surfaces.

$(\mathcal{M}, \mathbf{g})$ is called a general covariant modulated admissible (or shortly GCM-admissible) spacetime if it is defined as in Sections 3.2.1–3.2.6 below.

[61]This topology will be specified in our bootstrap assumptions, see Section 3.5 as well as Section 3.3.

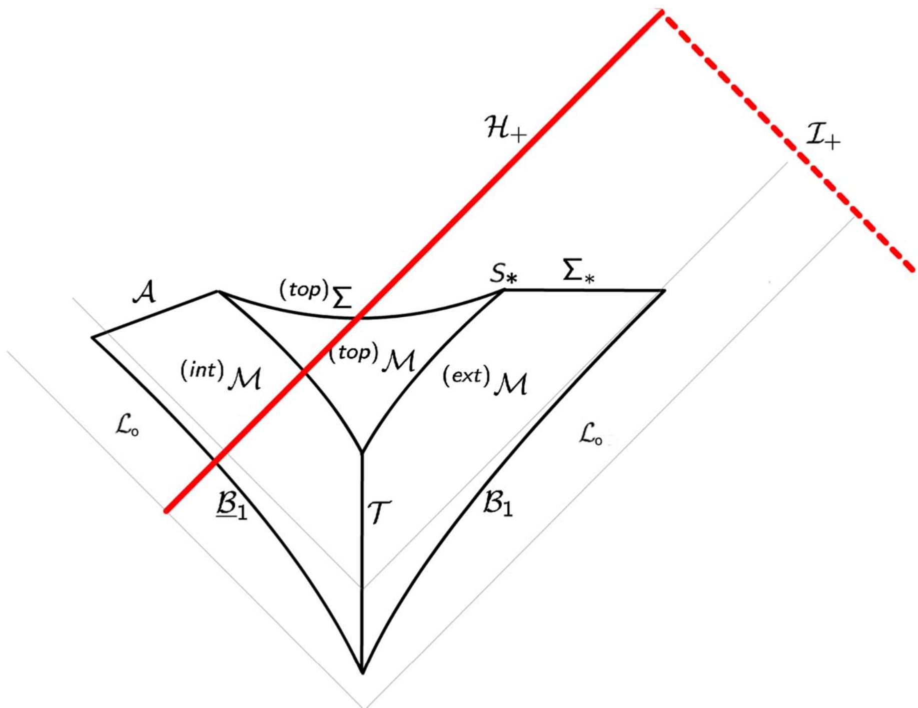

Figure 4: The GCM admissible space-time $\mathcal{M}$.

3.2.1. Boundaries The future and past boundaries of $\mathcal{M}$ are given by

$$\begin{aligned} \partial^+\mathcal{M} &= \mathcal{A} \cup {}^{(top)}\Sigma \cup \Sigma_*, \\ \partial^-\mathcal{M} &= \mathcal{B}_1 \cup \underline{\mathcal{B}}_1, \end{aligned}$$

where, see figure 4,

1. The past boundary $\mathcal{B}_1 \cup \underline{\mathcal{B}}_1$ is included in the initial data layer[62] $\mathcal{L}_0$.
2. The future spacelike boundary of the far region ${}^{(ext)}\mathcal{M}$ is denoted by Σ_*.
3. The future spacelike boundary of the top region ${}^{(top)}\mathcal{M}$ is denoted by ${}^{(top)}\Sigma$.
4. The future spacelike boundary of the near region ${}^{(int)}\mathcal{M}$ is denoted by $\mathcal{A}$.

[62] Recall that $\mathcal{L}_0$, defined in Section 3.1, is a spacetime region in which the metric on $\mathcal{M}$ is specified to be a small perturbation of the Kerr data.

5. The time-like hyper-surface $\mathcal{T}$, separating ${}^{(ext)}\mathcal{M}$ from ${}^{(int)}\mathcal{M}$, starts at $\underline{\mathcal{B}}_1 \cap \mathcal{B}_1$ and terminates at ${}^{(ext)}\mathcal{M} \cap {}^{(int)}\mathcal{M} \cap {}^{(top)}\mathcal{M}$.

3.2.2. Principal geodesic structures on $\mathcal{M}$ The spacetime region $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M} \cup {}^{(top)}\mathcal{M}$ admits the following PG structures.

3.2.2.1. Principal geodesic structure on ${}^{(ext)}\mathcal{M}$ The far region ${}^{(ext)}\mathcal{M}$ is endowed with an outgoing PG structure defined by a scalar functions ${}^{(ext)}r$ and null frame

$$\left({}^{(ext)}e_3, {}^{(ext)}e_4, {}^{(ext)}e_1, {}^{(ext)}e_2\right)$$

with ${}^{(ext)}e_4$ null geodesic outgoing and such that we have on ${}^{(ext)}\mathcal{M}$, see Section 2.3.1,

$${}^{(ext)}e_4\left({}^{(ext)}r\right) = 1, \qquad {}^{(ext)}e_1\left({}^{(ext)}r\right) = {}^{(ext)}e_2\left({}^{(ext)}r\right) = 0.$$

Moreover

1. We introduce in addition a scalar function u satisfying on ${}^{(ext)}\mathcal{M}$

$${}^{(ext)}e_4(u) = 0.$$

2. The $(u, {}^{(ext)}r)$ foliation is initialized on Σ_* as it will be made precise below.
3. The outgoing non-spacelike past boundary $\mathcal{B}_1$ corresponds precisely to $u = 1$.
4. The foliation by u of ${}^{(ext)}\mathcal{M}$ terminates at the timelike boundary

$$\mathcal{T} = \left\{ {}^{(ext)}r = r_0 \right\},$$

where r_0 has been introduced in Remark 3.1.

3.2.2.2. Principal geodesic structure on ${}^{(int)}\mathcal{M}$ The near region ${}^{(int)}\mathcal{M}$ is endowed with an ingoing PG structure defined by a scalar function ${}^{(int)}r$ and null frame

$$\left({}^{(int)}e_3, {}^{(int)}e_4, {}^{(int)}e_1, {}^{(int)}e_2\right)$$

with ${}^{(int)}e_3$ null geodesic ingoing and such that we have on ${}^{(int)}\mathcal{M}$, see Section 2.7,

$${}^{(int)}e_3\left({}^{(int)}r\right) = -1, \qquad {}^{(int)}e_1\left({}^{(int)}r\right) = {}^{(int)}e_2\left({}^{(int)}r\right) = 0.$$

Moreover

1. We introduce in addition a scalar function $\underline{u}$ satisfying on ${}^{(int)}\mathcal{M}$
$$ {}^{(int)}e_3(\underline{u}) = 0. $$
2. The $(\underline{u}, {}^{(int)}r)$ foliation is initialized on $\mathcal{T}$ as it will be made precise below.
3. The ingoing non-spacelike past boundary $\underline{\mathcal{B}}_1$ corresponds precisely to $\underline{u} = 1$.
4. The foliation by $\underline{u}$ of ${}^{(int)}\mathcal{M}$ terminates at the space like boundary
$$ \mathcal{A} = \left\{ {}^{(int)}r = \left(m_0 + \sqrt{m_0^2 - a_0^2} \right) (1 - \delta_{\mathcal{H}}) \right\} $$
where m_0 and $\delta_{\mathcal{H}}$ have been defined above.
5. We have $\underline{u} = u_*$ on $\mathcal{A} \cap {}^{(top)}\Sigma$.

3.2.2.3. Principal geodesic structure on ${}^{(top)}\mathcal{M}$ The region ${}^{(top)}\mathcal{M}$ is endowed with an ingoing PG structure defined by a scalar function ${}^{(top)}r$ and a null frame
$$ ({}^{(top)}e_3, {}^{(top)}e_4, {}^{(top)}e_1, {}^{(top)}e_2) $$
where ${}^{(top)}e_3$ is null ingoing geodesic and such that we have on ${}^{(top)}\mathcal{M}$
$$ {}^{(top)}e_3({}^{(top)}r) = -1, \qquad {}^{(top)}e_1({}^{(top)}r) = {}^{(top)}e_2({}^{(top)}r) = 0. $$

Moreover

1. We introduce in addition a scalar function $\underline{u}$ satisfying on ${}^{(top)}\mathcal{M}$
$$ {}^{(top)}e_3(\underline{u}) = 0. $$
2. ${}^{(int)}\mathcal{M} \cap {}^{(top)}\mathcal{M} = \{\underline{u} = u_*\}$.
3. The function $\underline{u}$ is continuous across ${}^{(int)}\mathcal{M} \cap {}^{(top)}\mathcal{M}$.
4. The function ${}^{(top)}r$ extends the function ${}^{(int)}r$ across ${}^{(int)}\mathcal{M} \cap {}^{(top)}\mathcal{M}$ continuously.
5. The null vectorfield ${}^{(top)}e_3$ is a continuous extension of ${}^{(int)}e_3$ across ${}^{(int)}\mathcal{M} \cap {}^{(top)}\mathcal{M}$.
6. The $(\underline{u}, {}^{(top)}r)$ foliation is initialized on $\{u = u_*\}$ as it will be made precise below.

7. The foliation by $\underline{u}$ of ${}^{(top)}\mathcal{M}$ terminates at the boundary

$$ {}^{(top)}\Sigma \quad = \quad \Big\{\underline{u} + \sigma_{top}\left({}^{(top)}r\right) = u_*\Big\}, $$

where the function σ_{top} may be chosen such that[63]

   (a) ${}^{(top)}\Sigma$ is spacelike,
   (b) ${}^{(top)}\Sigma$ starts at $\mathcal{A}\cap\{\underline{u}=u_*\}$ and terminates at S_*,
   (c) denoting by $r_{+,top}(\underline{u})$ and $r_{-,top}(\underline{u})$ respectively the maximum and the minimum of ${}^{(top)}r$ along a level hypersurface of $\underline{u}$ in ${}^{(top)}\mathcal{M}(r \geq r_0)$, there holds

$$ 0 \leq r_{+,top}(\underline{u}) - r_{-,top}(\underline{u}) \leq 4m_0 \tag{3.7} $$

   uniformly in $\underline{u}$.

3.2.3. The GCM-PG data set on Σ_* To initialize the PG structure of ${}^{(ext)}\mathcal{M}$ on its future spacelike boundary Σ_*, we assume given a GCM-PG data set $\big(\Sigma_*, r, (e_1, e_2, e_3, e_4), f\big)$ as defined in Definition 2.60, i.e.

1. r is a scalar function on Σ_* whose level sets are 2-spheres foliating Σ_*, (e_1, e_2, e_3, e_4) is a null frame defined on Σ_*, and f is a 1-form tangent to the spheres of the r-foliation of Σ_*.
2. e_4 is transversal[64] to Σ_*, (e_1, e_2) are tangent to Σ_*, and $e_1(r) = e_2(r) = 0$, so that (e_1, e_2) are tangent to the leaves of the r-foliation.
3. The sphere S_* is the final sphere on Σ_*.
4. The frame (e_1, e_2, e_3, e_4) satisfies the transversality conditions (2.47).
5. We are given on Σ_* coordinates (θ, φ) and a basis of $\ell = 1$ modes $J^{(p)}$, $p = 0, +, -$, that are defined as follows

[63] The particular choice of the function $\sigma_{top}(r)$ satisfying the desired constraints is irrelevant for the proof. One could for example make the following suitable choice

$$ \begin{aligned} \sigma_{top}(r) &= \frac{m^2}{r} + c_1 \quad \text{for} \quad r \leq r_0, \\ \sigma_{top}(r) &= -2(r - r_0) - 4m\log\left(\frac{r}{r_0}\right) - \frac{m^2}{r} + c_2 \quad \text{for} \quad r \geq r_0 + m, \end{aligned} $$

pick the constants c_1 and c_2 such that ${}^{(top)}\Sigma$ starts at $\mathcal{A}\cap\{\underline{u}=u_*\}$ and terminates at S_*, and smoothly extend $\sigma_{top}(r)$ to (r_0, r_0+m) so that ${}^{(top)}\Sigma$ is everywhere spacelike. See Section D.3, and in particular Proposition D.5 and Lemma D.9, for explicit computations in Kerr.

[64] This is in fact automatic since Σ_* is spacelike.

(a) (θ, φ) and $J^{(p)}$, $p = 0, +, -$, are initialized on S_* as in Section 2.5.3,

(b) (θ, φ) and $J^{(p)}$, $p = 0, +, -$, are propagated to Σ_* by $\nu(\theta) = \nu(\varphi) = 0$, and $\nu(J^{(p)}) = 0$, $p = 0, +, -$, where $\nu = e_3 + b_* e_4$ denotes the unique vectorfield on Σ_* orthogonal to the r-foliation such that $\mathbf{g}(\nu, e_4) = -2$.

6. The GCM conditions (2.54)–(2.56) of Definition 2.58 are verified.
7. The constants (a, m) are specified according to Definition 2.59.

In addition, we assume the following.

(a) Let r_* such that $S_* = S(r_*)$. Then, r is monotonically increasing from r_*.

(b) The function r verifies the dominance condition[65] on S_*

$$r_* \gg u_*^{1+\delta_{dec}}, \tag{3.8}$$

where u_* denotes the value of the function u on S_*, with u is specified below, see (3.13).

(c) The 1-form f on Σ_* is chosen by

$$f_1 = 0, \quad f_2 = \frac{a \sin\theta}{r}, \text{ on } S_*, \quad \nabla_\nu(rf) = 0 \text{ on } \Sigma_*, \tag{3.9}$$

where (e_1, e_2) are specified on S_* by (2.51).

3.2.4. Definition of (m, a) in $\mathcal{M}$ Given a GCM admissible spacetime $\mathcal{M}$, we define the values (a, m) associated to $\mathcal{M}$ to be the constants (a, m) associated to S_* according to Definition 2.59. Thus each GCM admissible spacetime $\mathcal{M}$ is naturally equipped with constants (a, m). Note that these constants depend on $\mathcal{M}$, i.e. two different GCM admissible spacetimes have in general different constants (a, m) associated to them.

3.2.5. Initialization of the PG structures of $\mathcal{M}$

3.2.5.1. Initialization of the PG structure of ${}^{(ext)}\mathcal{M}$ The PG structure of ${}^{(ext)}\mathcal{M}$ is initialized on Σ_* by the above GCM-PG data set according to Proposition 2.57, i.e.

[65] A precise condition will be given later in (3.50).

1. Along Σ_*, the restriction of the PG frame $({}^{(ext)}e_3, {}^{(ext)}e_4, {}^{(ext)}e_1, {}^{(ext)}e_2)$ of ${}^{(ext)}\mathcal{M}$ is prescribed by the transformation formulas

$$
\begin{aligned}
{}^{(ext)}e_4 &= e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3,\\
{}^{(ext)}e_a &= \left(\delta_a^b + \frac{1}{2}\underline{f}_a f^b\right) e_b + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right) e_3,\\
{}^{(ext)}e_3 &= \left(1 + \frac{1}{2}f\cdot\underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right) e_3 + \left(\underline{f}^b + \frac{1}{4}|\underline{f}|^2 f^b\right) e_b\\
&\quad + \frac{1}{4}|\underline{f}|^2 e_4,
\end{aligned}
\tag{3.10}
$$

where (e_1, e_2, e_3, e_4) is the null frame of the GCM-PG data set of Σ_*, and the 1-forms f and $\underline{f}$ are given respectively by (3.9) and (2.43), i.e.

$$
f_1 = 0, \quad f_2 = \frac{a\sin\theta}{r}, \text{ on } S_*, \quad \nabla_\nu(rf) = 0 \text{ on } \Sigma_*,
\tag{3.11}
$$

and[66]

$$
\underline{f} = -\frac{(\nu(r) - b_*)}{1 - \frac{1}{4}b_*|f|^2} f \quad \text{on} \quad \Sigma_*,
\tag{3.12}
$$

where the scalar functions (r, θ, b_*) on Σ_*, the constant a, and the vectorfield ν tangent to Σ_* are part of the above GCM-PG data set.

2. The functions ${}^{(ext)}r$ is prescribed on Σ_* by ${}^{(ext)}r = r$ where r belongs to the GCM-PG data set of Σ_*.
3. The function u is prescribed on Σ_* by

$$
u = c_* - {}^{(ext)}r,
\tag{3.13}
$$

where c_* is a constant that will be fixed in Remark 8.39. We then set u_* to be the value of u on S_*.

3.2.5.2. Initialization of the PG structure of ${}^{(int)}\mathcal{M}$

1. The $(\underline{u}, {}^{(int)}r)$ foliation is initialized on $\mathcal{T}$ such that,

$$
\underline{u} = u, \qquad {}^{(int)}r = {}^{(ext)}r.
\tag{3.14}
$$

In particular $S(\underline{u}, {}^{(int)}r)$ coincides on $\mathcal{T}$ with $S(u, {}^{(ext)}r)$.

[66] The fact that $\underline{f}$ is well defined will follow from our bootstrap assumptions.

2. The null frame $({}^{(int)}e_3, {}^{(int)}e_4, {}^{(int)}e_1, {}^{(int)}e_2)$ is defined on $\mathcal{T}$ by the following renormalization,

$$ {}^{(int)}e_4 = \lambda^{(ext)}e_4, \quad {}^{(int)}e_3 = \lambda^{-1(ext)}e_3, \quad {}^{(int)}e_a = {}^{(ext)}e_a, \; a = 1, 2, \; \text{ on } \mathcal{T} $$

where

$$ \lambda = {}^{(ext)}\lambda = \frac{{}^{(ext)}\Delta}{|{}^{(ext)}q|^2}. $$

3.2.5.3. Initialization of the PG structure of ${}^{(top)}\mathcal{M}$

1. The $(\underline{u}, {}^{(top)}r)$-foliation of ${}^{(top)}\mathcal{M}$ is initialized on $\{u = u_*\}$ such that,

$$ \underline{u} = u + 2\int_{r_0}^{{}^{(ext)}r} \frac{\tilde{r}^2 + a^2}{\tilde{r}^2 - 2m\tilde{r} + a^2} d\tilde{r}, \qquad {}^{(top)}r = {}^{(ext)}r. \tag{3.15} $$

In particular, the 2-spheres $S(\underline{u}, {}^{(top)}r)$ coincide on $\{u = u_*\}$ with $S(u, {}^{(ext)}r)$.

Remark 3.2. *The initialization of* $\underline{u}$ *in* (3.15) *agrees with the relation between* $\underline{u}$ *and* u *in Kerr, see Section 2.4.4 and Section 2.7.2. Note also that* $\underline{u} = u$ *at* $r = r_0$.

2. Moreover, the null frame $({}^{(top)}e_3, {}^{(top)}e_4, {}^{(top)}e_1, {}^{(top)}e_2)$ is prescribed on $\{u = u_*\}$ by the transformation formulas

$$ \begin{aligned} {}^{(top)}e_4 &= \lambda\left({}^{(ext)}e_4 + f^b\,{}^{(ext)}e_b + \frac{1}{4}|f|^2\,{}^{(ext)}e_3\right), \\ {}^{(top)}e_a &= {}^{(ext)}e_a + \frac{1}{2}f_a\,{}^{(ext)}e_3, \quad a = 1, 2, \\ {}^{(top)}e_3 &= \lambda^{-1(ext)}e_3, \end{aligned} \tag{3.16} $$

where

$$ \lambda = {}^{(ext)}\lambda = \frac{{}^{(ext)}\Delta}{|{}^{(ext)}q|^2}, \quad f = {}^{(ext)}h\,\widetilde{{}^{(ext)}e_3(r)}\,{}^{(ext)}\nabla(u), \tag{3.17} $$

with the scalar function ${}^{(ext)}h$ given by[67]

$$ {}^{(ext)}h = -\frac{2}{{}^{(ext)}\lambda\,{}^{(ext)}e_3(u)}. \tag{3.18} $$

[67]The fact that ${}^{(ext)}h$ is well defined will follow from our bootstrap assumptions.

Remark 3.3. *The choice* (3.18) *above ensures* ${}^{(top)}\nabla({}^{(top)}r) = 0$ *on* $\{u = u_*\}$*, see Lemma 7.13, which is a necessary condition for the ingoing foliation of* ${}^{(top)}\mathcal{M}$ *to be a PG structure.*

3.2.6. Definition of coordinates (θ, φ) in $\mathcal{M}$ We introduce the following (θ, φ) coordinates adapted to the PG structures defined above.

1. In ${}^{(ext)}\mathcal{M}$, we initialize $({}^{(ext)}\theta, {}^{(ext)}\varphi)$ on Σ_* by

$$ {}^{(ext)}\theta = \theta, \qquad {}^{(ext)}\varphi = \varphi, \tag{3.19}$$

where (θ, φ) is associated to the GCM-PG data set as above, and we propagate it to ${}^{(ext)}\mathcal{M}$ by

$$ {}^{(ext)}e_4({}^{(ext)}\theta) = {}^{(ext)}e_4({}^{(ext)}\varphi) = 0. \tag{3.20}$$

2. In ${}^{(int)}\mathcal{M}$, we initialize $({}^{(int)}\theta, {}^{(int)}\varphi)$ on $\mathcal{T}$ by

$$ {}^{(int)}\theta = {}^{(ext)}\theta, \qquad {}^{(int)}\varphi = {}^{(ext)}\varphi, \tag{3.21}$$

and we propagate it to ${}^{(int)}\mathcal{M}$ by

$$ {}^{(int)}e_3({}^{(int)}\theta) = {}^{(int)}e_3({}^{(int)}\varphi) = 0. \tag{3.22}$$

3. In ${}^{(top)}\mathcal{M}$, we initialize ${}^{(top)}\theta$ on $\{u = u_*\}$ by

$$ {}^{(top)}\theta = {}^{(ext)}\theta. \tag{3.23}$$

and we propagate it to ${}^{(top)}\mathcal{M}$ by

$$ {}^{(top)}e_3({}^{(top)}\theta) = 0. \tag{3.24}$$

Remark 3.4. *Note that we do not need to define a coordinate* ${}^{(top)}\varphi$*. Indeed, the coordinate* φ*, in the various regions of* $\mathcal{M}$*, plays an auxiliary role in the proof, and is in fact only needed to control coordinates system on* $\mathcal{M}$ *in the limit* $u_* \to +\infty$ *where the region* ${}^{(top)}\mathcal{M}$ *actually disappears.*

3.3. Main norms

We define our main norms on a given GCM admissible spacetime $\mathcal{M}$. The norms involve in particular the constants (a, m) specified in Section 3.2.4.

3.3.1. Main norms on Σ_* All quantities appearing in this section are defined relative to the GCM frame of Σ_*, the scalar function r of Σ_*, and the constant m introduced in Section 3.2.3.

We introduce the function u as in (3.13), i.e. we have

$$u = c_* - r \tag{3.25}$$

where c_* is a fixed constant. Also, recall that we have the following properties for angular derivatives r on Σ_*

$$e_1(r) = e_2(r) = 0. \tag{3.26}$$

Finally, recall that the frame (e_1, e_2, e_3, e_4) of Σ_* satisfies the transversality conditions (2.47), to which we add transversality conditions for $e_4(r)$ and $e_4(u)$

$$\xi = 0, \quad \omega = 0, \quad \underline{\eta} = -\zeta, \quad e_4(r) = 1, \quad e_4(u) = 0, \quad \text{on} \quad \Sigma_*. \tag{3.27}$$

Remark 3.5. *The transversality conditions (3.27) allow us to make sense of all the Ricci coefficients for the frame of Σ_* and all first order derivatives of r and u.*

Definition 3.6. *For the GCM foliation of the boundary Σ_* of $\mathcal{M}$, the linearized quantities are defined as follows*

$$\begin{array}{rclrcl}
\widetilde{tr\chi} & := & tr\chi - \dfrac{2}{r}, & \widetilde{tr\underline{\chi}} & := & tr\underline{\chi} + \dfrac{2\Upsilon}{r},\\
\check{\underline{\omega}} & := & \underline{\omega} - \dfrac{m}{r^2}, & \check{\rho} & := & \rho + \dfrac{2m}{r^3},\\
\widetilde{e_3(r)} & := & e_3(r) + \Upsilon, & \widetilde{e_3(u)} & := & e_3(u) - 2,\\
\check{b}_* & := & b_* + 1 + \dfrac{2m}{r}, & & &
\end{array} \tag{3.28}$$

where $\Upsilon = 1 - \frac{2m}{r}$.

With these normalizations we define the sets Γ_g^*, Γ_b^* as follows.

Definition 3.7. *The set of all linearized quantities on Σ_* is of the form $\Gamma_g^* \cup \Gamma_b^*$ defined as follows.*

1. *The set Γ_g^* contains*

$$\Gamma_g^* := \Big\{\widetilde{tr\chi},\, \widehat{\chi},\, \zeta,\, \widetilde{tr\underline{\chi}},\, r\alpha,\, r\beta,\, r(\check{\rho},\ \ {}^*\rho)\Big\}. \tag{3.29}$$

2. *The set* Γ_b^* *contains*

$$\Gamma_b^* \quad := \quad \Big\{\eta,\, \underline{\widehat{\chi}},\, \underline{\check{\omega}},\, \underline{\xi},\, r\underline{\beta},\, \underline{\alpha},\, r^{-1}\widecheck{e_3(r)}\, r^{-1}\widecheck{e_3(u)},\, r^{-1}\check{b}_*\Big\}. \tag{3.30}$$

To define higher order derivatives norms on Σ_*, we use the weighted derivative operators $\mathfrak{d}_*$ tangential to Σ_* defined as follows

$$\mathfrak{d}_* := \{\nabla_\nu,\, \not{\mathfrak{d}}\}. \tag{3.31}$$

3.3.1.1. Boundedness norms on Σ_* For any $k \geq 0$, we introduce the following norms

$${}^*\mathfrak{B}_k := \sup_{\Sigma_*}\Big\{r^2|\mathfrak{d}_*^{\leq k}\Gamma_g^*| + r|\mathfrak{d}_*^{\leq k}\Gamma_b^*|\Big\}. \tag{3.32}$$

3.3.1.2. Decay norms on Σ_* Let $\delta_{dec} > 0$ be a small constant to be specified later. For any $k \geq 1$, we introduce the following norms

$$\begin{aligned} {}^*\mathfrak{D}_k \quad := \quad \sup_{\Sigma_*}\Big\{ & r^2u^{\frac{1}{2}+\delta_{dec}}|\mathfrak{d}_*^{\leq k}\Gamma_g^*| + r^2u^{1+\delta_{dec}}|\mathfrak{d}_*^{\leq k-1}\nabla_\nu\Gamma_g^*| \\ & + ru^{1+\delta_{dec}}|\mathfrak{d}_*^{\leq k}\Gamma_b^*|\Big\}. \end{aligned} \tag{3.33}$$

3.3.2. Main norms in $^{(ext)}\mathcal{M}$ All quantities appearing in this section are defined relative to the outgoing PG structure of $^{(ext)}\mathcal{M}$. As there is no danger of confusion we will drop the prefixes $^{(ext)}$ in what follows.

3.3.2.1. Definition of complex horizontal 1-form $\mathfrak{J}$ *on* $^{(ext)}\mathcal{M}$ Recall that the PG frame of $^{(ext)}\mathcal{M}$ satisfies

$$\Xi = \omega = 0, \qquad \underline{H} = -Z,$$

as well as,

$$e_4(r) = 1, \quad e_4(u) = e_4(\theta) = e_4(\varphi) = 0, \quad e_1(r) = e_2(r) = 0.$$

To define linearized quantities in $^{(ext)}\mathcal{M}$ as in Definition 2.66, we need to use the complex horizontal 1-form $\mathfrak{J}$ introduced in Section 2.6.2. Recall that $\mathfrak{J}$ is defined in that section up to an initialization on a hypersurface transversal to e_4. We provide such initialization on the hypersurface Σ_* below[68].

[68]Note that $q = r + ia\cos\theta$ is defined with respect to the constant a of $^{(ext)}\mathcal{M}$.

Definition 3.8 (Definition of $\mathfrak{J}$ in ${}^{(ext)}\mathcal{M}$)**.** *We define the complex horizontal 1-form $\mathfrak{J}$ on ${}^{(ext)}\mathcal{M}$ by*[69]

$$\begin{aligned} &\mathfrak{J}_1 = \frac{i\sin\theta}{|q|}, \quad \mathfrak{J}_2 = \frac{\sin\theta}{|q|}, \quad \textit{on} \quad S_*,\\ &\nabla_\nu(|q|\mathfrak{J}) = 0 \quad \textit{on} \quad \Sigma_*,\\ &\nabla_4\mathfrak{J} = -\frac{1}{q}\mathfrak{J} \quad \textit{on} \quad {}^{(ext)}\mathcal{M}. \end{aligned} \tag{3.34}$$

The complex horizontal 1-form $\mathfrak{J}$ satisfies the following lemma.

Lemma 3.9. *The complex horizontal 1-form $\mathfrak{J}$ of Definition 3.8 satisfies on ${}^{(ext)}\mathcal{M}$ the identities (2.62), i.e.*

$$\nabla_4\mathfrak{J} = -\frac{1}{q}\mathfrak{J}, \qquad {}^*\mathfrak{J} = -i\mathfrak{J}, \qquad \mathfrak{J}\cdot\overline{\mathfrak{J}} = \frac{2(\sin\theta)^2}{|q|^2}. \tag{3.35}$$

Remark 3.10. *Since $\mathfrak{J}$ satisfies (2.62), it can be used to define the linearized quantities in ${}^{(ext)}\mathcal{M}$ as in Definition 2.66.*

Proof. The first identity holds true by Definition 3.8. The two other identities hold true on S_* by Definition 3.8., and are then immediately transported to Σ_* using the fact that $\nu(|q|\mathfrak{J}) = 0$ and $\nu(\theta) = 0$ on Σ_*. Finally, they are transported to ${}^{(ext)}\mathcal{M}$ using $\nabla_4\mathfrak{J} = -q^{-1}\mathfrak{J}$ and Lemma 2.64. □

Recall[70] the set of quantities Γ_g, Γ_b, see Definition 2.67. Finally, we use the weighted derivative operator

$$\mathfrak{d} := \{\nabla_3, r\nabla_4, \not{\mathfrak{d}}\}. \tag{3.36}$$

3.3.2.2. Boundedness norms in ${}^{(ext)}\mathcal{M}$ Let $\delta_B > 0$ be a small constant to be specified later. For any $k \geq 1$, we introduce the following norms

$$\begin{aligned} {}^{(ext)}\mathfrak{B}_k := \sup_{{}^{(ext)}\mathcal{M}} &\Big\{ r^2|\mathfrak{d}^{\leq k}\Gamma_g| + r|\mathfrak{d}^{\leq k}\Gamma_b| + r^{\frac{7}{2}+\frac{\delta_B}{2}}\left(|\mathfrak{d}^{\leq k}A| + |\mathfrak{d}^{\leq k}B|\right)\\ &+ r^{\frac{9}{2}+\delta_{dec}}|\mathfrak{d}^{\leq k-1}\nabla_3 A| + r^4|\mathfrak{d}^{\leq k-1}\nabla_3 B|\Big\} + \left(\int_{\Sigma_*}|\mathfrak{d}^{\leq k}\check{H}|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{3.37}$$

[69] Note that $\mathfrak{J}$ is related on Σ_* to the 1-form f introduced in (3.11) as follows

$$\mathfrak{J} = \frac{r}{|q|}(f + i\,{}^*f) \quad \text{on} \quad \Sigma_*.$$

[70] We stress the fact that the linearized quantities appearing in the definitions of Γ_g and Γ_b are done with respect to the constants (a, m) of ${}^{(ext)}\mathcal{M}$.

3.3.2.3. Decay norms in ${}^{(ext)}\mathcal{M}$ Let $\delta_{dec} > 0$ be a small constant to be specified later. We define for $k \geq 1$,

(3.38)
$$
\begin{aligned}
{}^{(ext)}\mathfrak{D}_k := & \sup_{{}^{(ext)}\mathcal{M}} \Big(ru^{1+\delta_{dec}} + r^2 u^{\frac{1}{2}+\delta_{dec}}\Big)|\mathfrak{d}^{\leq k}\Gamma_g| + \sup_{{}^{(ext)}\mathcal{M}} ru^{1+\delta_{dec}}|\mathfrak{d}^{\leq k}\Gamma_b| \\
& + \sup_{{}^{(ext)}\mathcal{M}} r^4 u^{\frac{1}{2}+\delta_{dec}}\Big(|\mathfrak{d}^{\leq k-1}\nabla_3 A| + |\mathfrak{d}^{\leq k-1}\nabla_3 B|\Big) \\
& + \sup_{{}^{(ext)}\mathcal{M}} r^2 u^{1+\delta_{dec}}|\mathfrak{d}^{\leq k-1}\nabla_3\Gamma_g| + \left(\int_{\Sigma_*} u^{2+2\delta_{dec}}|\mathfrak{d}^{\leq k}\check{H}|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Remark 3.11. *The integral bootstrap assumption on Σ_* for $\check{H}$ will only be needed in the proof of Proposition 3.26 and recovered in Proposition 5.42. In fact, other components satisfy an analog integral estimate on Σ_*: this is the case of $\underline{\widehat{X}}$, $\underline{\Xi}$ and $r\underline{B}$, see Proposition 5.42. But $\check{H}$ is the only component for which we need to make this type of bootstrap assumption.*

3.3.3. Main norms in ${}^{(int)}\mathcal{M}$ All quantities appearing in this section are defined relative to the ingoing PG structure of ${}^{(int)}\mathcal{M}$. As there is no danger of confusion we will drop the prefixes ${}^{(int)}$ in what follows.

3.3.3.1. Definition of complex horizontal 1-form $\mathfrak{J}$ on ${}^{(int)}\mathcal{M}$ To define linearized quantities in ${}^{(int)}\mathcal{M}$ as in Definition 2.72, we need to use the complex horizontal 1-form $\mathfrak{J}$ introduced in Section 2.7. Recall that $\mathfrak{J}$ is defined in that section up to an initialization on a hypersurface transversal to e_3. We provide such initialization on the hypersurface $\mathcal{T}$ below.

Definition 3.12 (Definition of $\mathfrak{J}$ in ${}^{(int)}\mathcal{M}$)**.** *We define the complex horizontal 1-form $\mathfrak{J}$ on ${}^{(int)}\mathcal{M}$ by*

(3.39)
$$
\begin{aligned}
\mathfrak{J} &= {}^{(ext)}\mathfrak{J} \quad on \quad \mathcal{T}, \\
\nabla_3\mathfrak{J} &= \frac{1}{\overline{q}}\mathfrak{J} \quad on \quad {}^{(int)}\mathcal{M}.
\end{aligned}
$$

Recall also, see Definition 2.73, the set of quantities Γ_g, Γ_b for ingoing PG structures.

3.3.3.2. Boundedness norms in ${}^{(int)}\mathcal{M}$ For any $k \geq 1$, we introduce the following norms

(3.40)
$$
{}^{(int)}\mathfrak{B}_k := \sup_{{}^{(int)}\mathcal{M}} \Big\{|\mathfrak{d}^{\leq k}\Gamma_g| + |\mathfrak{d}^{\leq k}\Gamma_b|\Big\}.
$$

3.3.3.3. Decay norms in ${}^{(int)}\mathcal{M}$ We define for $k \geq 1$,

$$
{}^{(int)}\mathfrak{D}_k := \sup_{{}^{(int)}\mathcal{M}} \underline{u}^{1+\delta_{dec}}\Big(|\mathfrak{d}^{\leq k}\Gamma_g| + |\mathfrak{d}^{\leq k}\Gamma_b|\Big). \tag{3.41}
$$

3.3.4. Main norms in ${}^{(top)}\mathcal{M}$ All quantities appearing in this section are defined relative to the ingoing PG structure of ${}^{(top)}\mathcal{M}$. As there is no danger of confusion we will drop the prefixes ${}^{(top)}$ in what follows.

3.3.4.1. Definition of complex horizontal 1-form $\mathfrak{J}$ *on* ${}^{(top)}\mathcal{M}$ To define linearized quantities in ${}^{(top)}\mathcal{M}$ as in Definition 2.72, we need to use the complex horizontal 1-form $\mathfrak{J}$ introduced in Section 2.7. Recall that $\mathfrak{J}$ is defined in that section up to an initialization on a hypersurface transversal to e_3. We provide such initialization on the hypersurface $\{u = u_*\}$ below.

Definition 3.13 (Definition of $\mathfrak{J}$ in ${}^{(top)}\mathcal{M}$)**.** *We define the complex horizontal 1-form* $\mathfrak{J}$ *on* ${}^{(top)}\mathcal{M}$ *by*

$$
\begin{aligned}
\mathfrak{J} &= {}^{(ext)}\mathfrak{J} \quad on \quad \{u = u_*\},\\
\nabla_3\mathfrak{J} &= \frac{1}{q}\mathfrak{J} \quad on \quad {}^{(top)}\mathcal{M}.
\end{aligned} \tag{3.42}
$$

Remark 3.14. *We do not introduce the scalar function* $J^{(\pm)}$ *and the complex 1-forms* $\mathfrak{J}_\pm$ *in* ${}^{(top)}\mathcal{M}$*. Thus, the quantities* Γ_g, Γ_b *in* ${}^{(top)}\mathcal{M}$ *correspond to the ones in Definition 2.73 where all linearized quantities based on* $J^{(\pm)}$ *and* $\mathfrak{J}_\pm$ *have been removed.*

3.3.4.2. Boundedness norms in ${}^{(top)}\mathcal{M}$ For any $k \geq 1$, we introduce the following norms

$$
\begin{aligned}
{}^{(top)}\mathfrak{B}_k := \sup_{{}^{(top)}\mathcal{M}} \Big\{ & r^2|\mathfrak{d}^{\leq k}\Gamma_g| + r|\mathfrak{d}^{\leq k}\Gamma_b| + r^{\frac{7}{2}+\frac{\delta_B}{2}}\big(|\mathfrak{d}^{\leq k}A| + |\mathfrak{d}^{\leq k}B|\big)\\
& + r^{\frac{9}{2}+\delta_{dec}}|\mathfrak{d}^{\leq k-1}\nabla_3 A| + r^4|\mathfrak{d}^{\leq k-1}\nabla_3 B|\Big\}.
\end{aligned} \tag{3.43}
$$

3.3.4.3. Decay norms in ${}^{(top)}\mathcal{M}$ To describe decay norms in ${}^{(top)}\mathcal{M}(r \geq r_0)$, we introduce the scalar function ${}^{(top)}u$ as follows

$$
{}^{(top)}u \quad := \quad \underline{u} - 2\int_{r_0}^{r} \frac{\tilde{r}^2 + a^2}{\tilde{r}^2 - 2m\tilde{r} + a^2} d\tilde{r}. \tag{3.44}
$$

Remark 3.15. *Note in view of* (3.15) *that* ${}^{(top)}u = u$ *on* $u = u_*$.

We define for $k \geq 1$,

$$
\begin{aligned}
{}^{(top)}\mathfrak{D}_k &:= {}^{(top)}\mathfrak{D}_k^{\leq r_0} + {}^{(top)}\mathfrak{D}_k^{\geq r_0},\\
{}^{(top)}\mathfrak{D}_k^{\leq r_0} &:= \sup_{{}^{(top)}\mathcal{M}(r\leq r_0)} \underline{u}^{1+\delta_{dec}}\Big(|\mathfrak{d}^{\leq k}\Gamma_g| + |\mathfrak{d}^{\leq k}\Gamma_b|\Big),\\
{}^{(top)}\mathfrak{D}_k^{\geq r_0} &:= \sup_{{}^{(top)}\mathcal{M}(r\geq r_0)} \Big(r({}^{(top)}u)^{1+\delta_{dec}} + r^2({}^{(top)}u)^{\frac{1}{2}+\delta_{dec}}\Big)|\mathfrak{d}^{\leq k}\Gamma_g|\\
&\quad + \sup_{{}^{(top)}\mathcal{M}(r\geq r_0)} r({}^{(top)}u)^{1+\delta_{dec}}|\mathfrak{d}^{\leq k}\Gamma_b|\\
&\quad + \sup_{{}^{(top)}\mathcal{M}(r\geq r_0)} r^2({}^{(top)}u)^{1+\delta_{dec}}|\mathfrak{d}^{\leq k-1}\nabla_3\Gamma_g|\\
&\quad + \sup_{{}^{(top)}\mathcal{M}(r\geq r_0)} r^4({}^{(top)}u)^{\frac{1}{2}+\delta_{dec}}\Big(|\mathfrak{d}^{\leq k-1}\nabla_3 A| + |\mathfrak{d}^{\leq k-1}\nabla_3 B|\Big).
\end{aligned}
\tag{3.45}
$$

3.3.5. Combined norms We define the following norms $\mathcal{M}$ by combining our above norms on Σ_*, ${}^{(ext)}\mathcal{M}$, ${}^{(int)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$

$$
\begin{aligned}
\mathfrak{N}_k^{(Sup)} &:= {}^*\mathfrak{B}_k + {}^{(ext)}\mathfrak{B}_k + {}^{(int)}\mathfrak{B}_k + {}^{(top)}\mathfrak{B}_k,\\
\mathfrak{N}_k^{(Dec)} &:= {}^*\mathfrak{D}_k + {}^{(ext)}\mathfrak{D}_k + {}^{(int)}\mathfrak{D}_k + {}^{(top)}\mathfrak{D}_k.
\end{aligned}
$$

3.3.6. Initial layer norm Recall the notations of Section 3.1 concerning the initial data layer $\mathcal{L}_0 = \mathcal{L}_0(a_0, m_0)$. Recall that the constants $m_0 > 0$ and $|a_0| < m_0$ are the mass and angular momentum of the initial Kerr spacetime relative to which our initial perturbation is measured. We define the initial layer norm to be[71]

$$
\mathfrak{I}_k := {}^{(ext)}\mathfrak{I}_k + {}^{(int)}\mathfrak{I}_k + \mathfrak{I}'_k,
$$

where

$$
{}^{(ext)}\mathfrak{I}_0 := \sup_{{}^{(ext)}\mathcal{L}_0}\Big\{r^2|\Gamma_g| + r|\Gamma_b| + r^{\frac{7}{2}+\frac{\delta_B}{2}}(|A| + |B|)\Big\},
$$

$$
{}^{(int)}\mathfrak{I}_0 := \sup_{{}^{(int)}\mathcal{L}_0}\Big\{|\Gamma_g| + |\Gamma_b|\Big\},
$$

$$
\mathfrak{I}'_0 := \sup_{{}^{(int)}\mathcal{L}_0\cap{}^{(ext)}\mathcal{L}_0}\Big(|f| + |\underline{f}| + |\log(\lambda_0^{-1}\lambda)|\Big), \quad \lambda_0 = {}^{(ext)}\lambda_0 = 1 - \frac{2m_0}{{}^{(ext)}r_{\mathcal{L}_0}},
$$

and

[71] Recall that the initial data layer foliations satisfy $\underline{\eta}+\zeta = 0$, as well as $\xi = \omega = 0$ on ${}^{(ext)}\mathcal{L}_0$, and $\eta = \zeta$ as well as $\underline{\xi} = \underline{\omega} = 0$ on ${}^{(int)}\mathcal{L}_0$.

- for ${}^{(ext)}\mathfrak{I}_0$, (Γ_g, Γ_b) is given by Definition 2.67 for the outgoing PG structure of ${}^{(ext)}\mathcal{L}_0$,
- for ${}^{(int)}\mathfrak{I}_0$, (Γ_g, Γ_b) is given by Definition 2.73 for the ingoing PG structure of ${}^{(int)}\mathcal{L}_0$,
- for $\mathfrak{I}'_0$, $(f, \underline{f}, \lambda)$ denote the transition coefficients of Lemma 2.10 from the frame of the outgoing PG structure of ${}^{(ext)}\mathcal{L}_0$ to the frame of the ingoing PG structure of ${}^{(int)}\mathcal{L}_0$ in the region ${}^{(int)}\mathcal{L}_0 \cap {}^{(ext)}\mathcal{L}_0$,

with $\mathfrak{I}_k$ the corresponding higher derivative norms obtained by replacing each component by $\mathfrak{d}^{\leq k}$ of it.

Remark 3.16. *Note that in the definition of ${}^{(ext)}\mathfrak{I}_k$ we allow a higher power of r in front α, β and their derivatives than what it is consistent with the results of [17] and [36]. The additional r^{δ_B} power, for δ_B small, is consistent instead with the result of [37]. See also Remark 3.19.*

3.4. Main theorem

3.4.1. Smallness constants Before stating our main theorem, we first introduce the following constants that will be involved in its statement.

- The constants $m_0 > 0$ and $|a_0| < m_0$ are the mass and the angular momentum of the Kerr solution relative to which our initial perturbation is measured.
- The integer k_{large} which corresponds to the maximum number of derivatives of the solution.
- The size of the initial data layer norm is measured by $\epsilon_0 > 0$.
- The size of the bootstrap assumption norms is measured by $\epsilon > 0$.
- $r_0 > 0$ is tied to the definition of $\mathcal{T}$, i.e. $\mathcal{T} = \{r = r_0\}$.
- $\delta_{\mathcal{H}} > 0$ measures the width of the region $|r - m_0 - \sqrt{m_0^2 - a_0^2}| \leq 2m_0\delta_{\mathcal{H}}$ where the redshift estimate holds.
- δ_{dec} is tied to decay estimates in u, $\underline{u}$ for $\check{\Gamma}$ and $\check{R}$.
- δ_B is involved in the r-power of the supremum estimates for high derivatives of α and β.
- δ_* is involved in the behavior of r on S_*, see (3.50) below.

In what follows, m_0 and a_0 are fixed constants with $0 \leq |a_0| < m_0$, $\delta_{\mathcal{H}}$, δ_B, and δ_{dec} are fixed, sufficiently small, universal constants, and r_0 and k_{large} are fixed, sufficiently large, universal constants, chosen such that

$$
\begin{aligned}
&0 < \delta_{\mathcal{H}},\ \delta_{dec},\ \delta_B,\ \delta_* \ll \min\{m_0 - |a_0|, 1\}, \qquad \delta_B > 2\delta_{dec},\\
&r_0 \gg \max\{m_0, 1\}, \qquad k_{large} \gg \frac{1}{\delta_{dec}}.
\end{aligned} \tag{3.46}
$$

Then, ϵ and ϵ_0 are chosen such that

$$0 < \epsilon_0, \epsilon \ll \min\left\{\delta_{\mathcal{H}}, \delta_{dec}, \delta_B, \delta_*, \frac{1}{r_0}, \frac{1}{k_{large}}, m_0 - |a_0|, 1\right\}, \tag{3.47}$$

$$\epsilon_0, \epsilon \ll |a_0| \quad \text{in the case } a_0 \neq 0, \tag{3.48}$$

and

$$\epsilon = \epsilon_0^{\frac{2}{3}}. \tag{3.49}$$

Remark 3.17. *Note that we may always assume (3.48), even if* $0 < |a_0| \lesssim \epsilon_0$. *Indeed, in that case, an initial data layer assumption of the type*[72]

$$\mathfrak{I}_k \lesssim \epsilon_0$$

remains true by setting $a_0 = 0$.

Using the definition of ϵ_0, we can now make precise the condition (3.8) of r on S_*

$$r_* = \delta_* \epsilon_0^{-1} u_*^{1+\delta_{dec}}, \tag{3.50}$$

where we recall that r_* and u_* denote respectively the value of r and u on S_*.

Also, we introduce the integer k_{small} which corresponds to the number of derivatives for which the solution satisfies decay estimates. It is related to k_{large} by

$$k_{small} = \left\lfloor \frac{1}{2} k_{large} \right\rfloor + 1. \tag{3.51}$$

From now on, in the rest of the paper, $\lesssim$ means bounded by a constant depending only on geometric universal constants (such as Sobolev embeddings, elliptic estimates,...) as well as the constants

$$m_0,\ a_0,\ \delta_{\mathcal{H}},\ \delta_{dec},\ \delta_B,\ \delta_*,\ r_0,\ k_{large}$$

but not on ϵ and ϵ_0.

3.4.2. Admissible future null complete spacetimes We introduce in this section the notion of admissible future null complete spacetimes which

[72]See Section 3.3.6 for the definition of the initial data layer norm $\mathfrak{I}_k$.

corresponds formally to the limit as $u_* \to +\infty$ of the GCM admissible spacetimes of Section 3.2.

Definition 3.18. *We say that an initial data layer $\mathcal{L}_0 = \mathcal{L}_0(a_0, m_0)$, defined as in Section 3.1, is admissible if it lies in the future of an asymptotically flat initial data set, supported on a spacelike hypersurface Σ_0, of ADM mass m_0 and angular momentum a_0. For a representation see the lower part of the Figure 5.*

In addition, we say that $\mathcal{L}_0$ is (ϵ_0, k)-admissible if it verifies the bound[73]

$$\mathfrak{I}_k \leq \epsilon_0^2,$$

with $\mathfrak{I}_k$ defined as in Section 3.3.6.

Remark 3.19. *The results in [36], [37], [8] and [51] identify large classes of initial data sets on Σ_0 which generate (ϵ_0, k)-admissible initial data layers.*

We define a development of an admissible initial data layer $\mathcal{L}_0$ as follows.

Definition 3.20. *We say that a spacetime is a future development of an admissible initial data layer $\mathcal{L}_0$ if it is in fact a future development of the initial data set supported on Σ_0.*

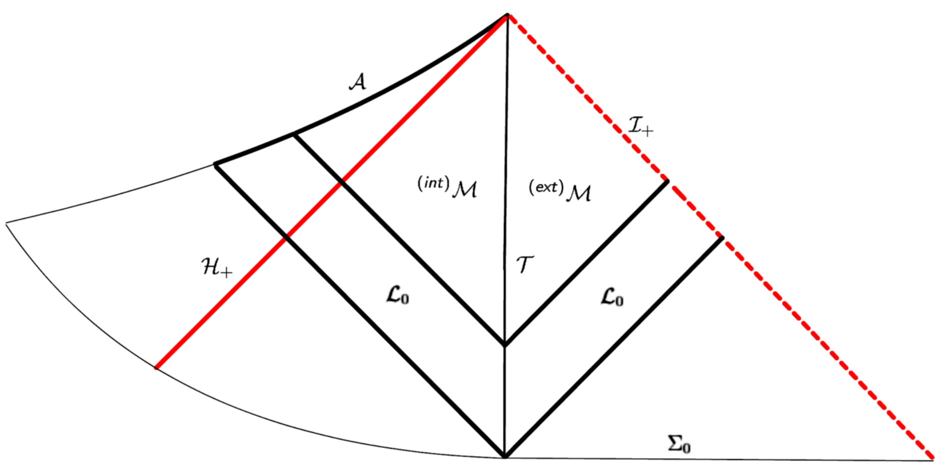

Figure 5: Penrose diagram of an admissible future complete spacetime.

[73]One expects in principal the weaker bound $\mathfrak{I}_k \leq \epsilon_0$. See Remark 3.24 for an explanation of the need of the stronger bound $\mathfrak{I}_k \leq \epsilon_0^2$.

Definition 3.21. *We say that an asymptotically flat Einstein vacuum spacetime $\mathcal{M}$, as in Figure 5, is admissible future null complete, if it verifies the following properties.*

- *It is a future development of an admissible initial data layer set $\mathcal{L}_0 = \mathcal{L}_0(a_0, m_0)$, in the sense of Definition 3.20.*
- *The future null infinity $\mathcal{I}^+$ of $\mathcal{M}$ is complete. The other future boundary of $\mathcal{M}$ is given by the spacelike hypersurface $\mathcal{A}$ and belongs to the complement of $\mathcal{J}^-(\mathcal{I}^+)$.*
- $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$.
- $\mathcal{T} = {}^{(ext)}\mathcal{M} \cap {}^{(int)}\mathcal{M}$ *is time-like.*
- ${}^{(ext)}\mathcal{M}$ *comes equipped with an outgoing PG structure, as in Definition 2.16, given by the function r and null frame*[74] $(e_3, e_4, \mathcal{H})$*, and a function u verifying $e_4(u) = 0$.*
- *Each $S = S(u, r)$ sphere in ${}^{(ext)}\mathcal{M}$ comes equipped with an adapted, integrable, frame $(e_3', e_4', \mathcal{H}')$. The passage from the PG frame to the integrable one is obtained by the transformation formulas (2.6) with parameters $(f, \underline{f}, \lambda)$ given by (2.12).*
- *The timelike surface $\mathcal{T}$ is given by $\{r = r_0\}$, and ${}^{(int)}\mathcal{M}$ comes equipped with an ingoing PG structure and function $\underline{u}$ initialized on $\mathcal{T}$ as in Section 3.2.5.*

Definition 3.22. *Given an admissible future null complete, spacetime $\mathcal{M}$ as in Definition 3.21, and constants (a_∞, m_∞), $|a_\infty| < m_\infty$, we define the norms*[75] $\mathfrak{N}^{(Sup)}_{k_{large}}(a_\infty, m_\infty)$, $\mathfrak{N}^{(Dec)}_{k_{small}}(a_\infty, m_\infty)$ *as in Section 3.3.5, with the constants (a, m), which appear in the definition of the linearized quantities replaced by (a_∞, m_∞).*

3.4.3. Statement of the main theorem We are now ready to give the following precise version of our main theorem.

Main Theorem (Main theorem, version 2)**.** *Let $\mathcal{L}_0 = \mathcal{L}_0(a_0, m_0)$ be an $(\epsilon_0, k_{large}+10)$-admissible initial data layer as in Definition 3.18, with $|a_0|/m_0$ sufficiently small, k_{large} sufficiently large, and $\epsilon_0 > 0$ sufficiently small. In particular, we assume*[76]

$$\mathfrak{I}_{k_{large}+10} \leq \epsilon_0^2. \tag{3.52}$$

[74]Recall that $\mathcal{H}(r) = 0$, $e_4(r) = 1$.

[75]Note that $\mathcal{M}$ does not contain a region ${}^{(top)}\mathcal{M}$, so that there are no norms corresponding to ${}^{(top)}\mathcal{M}$ in the definition of $\mathfrak{N}^{(Sup)}_{k_{large}}(a_\infty, m_\infty)$ and $\mathfrak{N}^{(Dec)}_{k_{small}}(a_\infty, m_\infty)$.

[76]One expects in principal the weaker bound $\mathfrak{I}_{k_{large}+10} \leq \epsilon_0$. See Remark 3.24 for an explanation of the need of the stronger bound $\mathfrak{I}_{k_{large}+10} \leq \epsilon_0^2$.

Then $\mathcal{L}_0 = \mathcal{L}_0(a_0, m_0)$ possesses an admissible future complete development $\mathcal{M}_\infty$ as in Definition 3.21. Moreover:

1. *There exist constants (a_∞, m_∞), $|a_\infty| \ll m_\infty$, such that the following estimates hold true, relative to the norms $\mathfrak{N}^{(Sup)}_{k_{large}} = \mathfrak{N}^{(Sup)}_{k_{large}}(a_\infty, m_\infty)$, $\mathfrak{N}^{(Dec)}_{k_{small}} = \mathfrak{N}^{(Dec)}_{k_{small}}(a_\infty, m_\infty)$ defined above,*

$$\mathfrak{N}^{(Sup)}_{k_{large}} + \mathfrak{N}^{(Dec)}_{k_{small}} + |a_\infty - a_0| + |m_\infty - m_0| \leq C\epsilon_0 \tag{3.53}$$

   where C is a universal constant sufficiently large and $k_{small} = \lfloor \frac{1}{2} k_{large} \rfloor + 1$.
2. *The space $\mathcal{M}_\infty$ is a limit of finite GCM admissible spacetimes[77].*

The estimates (3.53) imply, in particular:

– *On ${}^{(ext)}\mathcal{M}_\infty$, we have*

$$\begin{aligned}
|\alpha|, |\beta| &\lesssim \min\left\{\frac{\epsilon_0}{r^3(u+2r)^{\frac{1}{2}+\delta_{dec}}}, \frac{\epsilon_0}{r^2(u+2r)^{1+\delta_{dec}}}\right\},\\
|\check{\rho}|, |\widecheck{{}^*\rho}| &\lesssim \min\left\{\frac{\epsilon_0}{r^3 u^{\frac{1}{2}+\delta_{dec}}}, \frac{\epsilon_0}{r^2 u^{1+\delta_{dec}}}\right\},\\
|\underline{\beta}| &\lesssim \frac{\epsilon_0}{r^2 u^{1+\delta_{dec}}},\\
|\underline{\alpha}| &\lesssim \frac{\epsilon_0}{r u^{1+\delta_{dec}}},
\end{aligned}$$

and

$$\begin{aligned}
|\widecheck{tr\chi}|, |\widecheck{{}^{(a)}tr\chi}| &\lesssim \frac{\epsilon_0}{r^2 u^{1+\delta_{dec}}},\\
|\widehat{\chi}|, |\check{\zeta}|, |\widecheck{tr\underline{\chi}}|, |\widecheck{{}^{(a)}tr\underline{\chi}}| &\lesssim \min\left\{\frac{\epsilon_0}{r^2 u^{\frac{1}{2}+\delta_{dec}}} \frac{\epsilon_0}{r u^{1+\delta_{dec}}}\right\},\\
|\check{\eta}|, |\widehat{\underline{\chi}}|, |\check{\underline{\omega}}|, |\underline{\xi}| &\lesssim \frac{\epsilon_0}{r u^{1+\delta_{dec}}}.
\end{aligned}$$

– *On ${}^{(int)}\mathcal{M}_\infty$ we have, for any linearized quantity $\check{\psi}$,*

$$|\check{\psi}| \lesssim \frac{\epsilon_0}{\underline{u}^{1+\delta_{dec}}}.$$

Note that analog statements of the above estimates also hold for $\mathfrak{d}^k$ derivatives with $k \leq k_{small}$.

[77] More precisely they are the limits of the GCM family $\mathcal{U}(u_*)$, as in Definition 3.37. Note also that ${}^{(top)}\mathcal{M}$ disappears in the limit.

Moreover the following other statements hold true.

1. *Let $m_H(u,r)$ denote the Hawking mass adapted to the spheres $S = S(u,r)$ of ${}^{(ext)}\mathcal{M}_\infty$, i.e.*

$$m_H(u,r) = \sqrt{\frac{|S(u,r)|}{4\pi}}\left(1+\frac{1}{16\pi}\int_{S(u,r)} tr\chi' tr\underline{\chi}'\right),$$

where $tr\chi', tr\underline{\chi}'$ are calculated with respect to the integrable[78] frame (e'_1, e'_2, e'_3, e'_4) of ${}^{(ext)}\mathcal{M}$. Then:

– *The Bondi mass exists and is given by*

$$M_B(u) := \lim_{r\to\infty} m_H(u,r).$$

– *$M_B(u)$ has a limit as $u\to\infty$ and*

$$\lim_{u\to\infty} M_B(u) = m_\infty, \tag{3.54}$$

i.e. m_∞ coincides with the final Bondi mass.

– *The Bondi mass law formula (3.84) holds true. In particular $m_\infty < m_0$.*

2. *We define the quasi-local angular momentum for a sphere $S(u,r)$ to be the triplet*

$$\mathrm{j}_{\ell=1,p}(u,r) := \frac{r^5}{|S(u,r)|}\int_{S(u,r)} (curl'\beta')J^{(p)}, \qquad p = 0,+,-.$$

with $curl'\beta'$ defined relative the the integrable frame of ${}^{(ext)}\mathcal{M}$. Then

– *The triplet $\mathrm{j}_{\ell=1,p}(u,r)$ has a limit as $r\to\infty$ at fixed u given by*

$$\mathcal{J}_{\ell=1,p}(u) = \lim_{r\to\infty} \mathrm{j}_{\ell=1,p}(u,r). \tag{3.55}$$

– *The triplet $\mathcal{J}_{\ell=1,p}(u)$ has a limit as $u\to\infty$ and*

$$\lim_{u\to\infty}\mathcal{J}_{\ell=1,0}(u) = 2a_\infty m_\infty, \qquad \lim_{u\to\infty}\mathcal{J}_{\ell=1,\pm}(u) = 0.$$

3. *${}^{(ext)}\mathcal{M}$ is covered by three regular coordinates patches:*

[78] See Definition 3.21. The integrable frame is such that (e'_1, e'_2) is tangent to $S(u,r)$. Recall, on the other hand, that the outgoing PG frame of ${}^{(ext)}\mathcal{M}$ is non-integrable.

– *in the* (u, r, θ, φ) *coordinates system, we have, for* $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$*,*

$$\mathbf{g} = \mathbf{g}_{a_\infty, m_\infty} + \Big(du, dr, rd\theta, r\sin\theta d\varphi\Big)^2 O\left(\frac{\epsilon_0}{u^{1+\delta_{dec}}}\right),$$

– *in the* (u, r, x^1, x^2) *coordinates system, with* $x^1 = J^{(+)}$ *and* $x^2 = J^{(-)}$*, we have, for* $0 \leq \theta < \frac{\pi}{3}$ *and for* $\frac{2\pi}{3} < \theta \leq \pi$*,*

$$\mathbf{g} = \mathbf{g}_{a_\infty, m_\infty} + \Big(du, dr, rdx^1, rdx^2\Big)^2 O\left(\frac{\epsilon_0}{u^{1+\delta_{dec}}}\right),$$

where in each case, $\mathbf{g}_{a_\infty, m_\infty}$ *denotes the Kerr metric expressed in the corresponding coordinates system of Kerr, see Lemma 2.35 and Lemma 2.49.*

4. ${}^{(int)}\mathcal{M}$ *is covered by three regular coordinates patches:*

– *in the* $(\underline{u}, r, \theta, \varphi)$ *coordinates system, we have, for* $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$*,*

$$\mathbf{g} = \mathbf{g}_{a_\infty, m_\infty} + \Big(d\underline{u}, dr, rd\theta, r\sin\theta d\varphi\Big)^2 O\left(\frac{\epsilon_0}{\underline{u}^{1+\delta_{dec}}}\right),$$

– *in the* $(\underline{u}, r, x^1, x^2)$ *coordinates system, with* $x^1 = J^{(+)}$ *and* $x^2 = J^{(-)}$*, we have, for* $0 \leq \theta < \frac{\pi}{3}$ *and for* $\frac{2\pi}{3} < \theta \leq \pi$*,*

$$\mathbf{g} = \mathbf{g}_{a_\infty, m_\infty} + \Big(d\underline{u}, dr, rdx^1, rdx^2\Big)^2 O\left(\frac{\epsilon_0}{\underline{u}^{1+\delta_{dec}}}\right),$$

where in each case, $\mathbf{g}_{a_\infty, m_\infty}$ *denotes the Kerr metric expressed in the corresponding coordinates system of Kerr, i.e. the analog for ingoing PG structures of Lemma 2.35 and Lemma 2.49.*

Remark 3.23. *By far the most demanding part in the proof of the theorem is to establish the crucial estimates* (3.53)*. The items 1–5 are important corollaries of those estimates, to be treated as conclusions in Section 3.8. In Section 3.7, we discuss the main intermediary results, Theorems M0–M8, in the proof of the estimates* (3.53)*.*

Remark 3.24. *The assumption* (3.52) *for the initial data layer norm* $\mathfrak{I}_{k_{large}+10}$ *can in fact be replaced by its following weaker analog*

$$\mathfrak{I}_{k_{large}+10} \leq \epsilon_0, \qquad {}^{(ext)}\mathfrak{I}_3 \leq \epsilon_0^2, \tag{3.56}$$

see also Remark 8.21. The estimate ${}^{(ext)}\mathfrak{I}_3 \leq \epsilon_0^2$ *in* (3.56) *is then used at two instances in the proof of the main theorem:*

- *the proof of Theorem M0, see Section 8.3, where it is used to infer the $O(\epsilon_0)$ control of the curvature components of the PG structure of ${}^{(ext)}\mathcal{M}$ and ${}^{(int)}\mathcal{M}$ respectively on the past boundaries $\mathcal{B}_1$ and $\underline{\mathcal{B}}_1$ of $\mathcal{M}$,*
- *the proof of Theorem M6, see Section 8.4, where it is used to exhibit in the initial data layer a spacetime $\mathcal{M}$ satisfying our bootstrap assumptions, hence initiating the continuity argument in the proof of the main theorem.*

The precise places where we need the estimate ${}^{(ext)}\mathfrak{I}_3 \leq \epsilon_0^2$ in (3.56) are outlined in Remark 8.33 for the proof of Theorem M0, and Remark 8.38 for the proof of Theorem M6.

3.5. Main bootstrap assumptions

Given a GCM admissible spacetime $\mathcal{M}$, as defined in Section 3.2, we assume that the combined norms[79] $\mathfrak{N}_k^{(Sup)}$ and $\mathfrak{N}_k^{(Dec)}$, defined in Section 3.3, verifies the following bounds

BA-B *(Bootstrap Assumptions on r-weighted sup norms)*

$$\mathfrak{N}^{(Sup)}_{k_{large}} + |m - m_0| + |a - a_0| \leq \epsilon, \tag{3.57}$$

BA-D *(Bootstrap Assumptions on decay)*

$$\mathfrak{N}^{(Dec)}_{k_{small}} \leq \epsilon. \tag{3.58}$$

We shall often refer to them in the text as $\mathbf{BA}_\epsilon$.

3.6. Global null frames

3.6.1. The quantities $\mathfrak{q}$ and $\underline{\mathfrak{q}}$ We will need to control complex horizontal symmetric traceless 2-tensors $\mathfrak{q}$ and $\underline{\mathfrak{q}}$ of the form

$$\begin{aligned}
\mathfrak{q} &= q\overline{q}^3\Big((\nabla_3 - 2\underline{\omega})(\nabla_3 - 4\underline{\omega})A + C_1(\nabla_3 - 4\underline{\omega})A + C_2 A\Big),\\
\underline{\mathfrak{q}} &= \overline{q}q^3\Big((\nabla_4 - 2\omega)(\nabla_4 - 4\omega)\underline{A} + \underline{C}_1(\nabla_4 - 4\omega)\underline{A} + \underline{C}_2\underline{A}\Big),
\end{aligned}$$

for specific complex scalar functions[80] $C_1, C_2, \underline{C}_1$ and $\underline{C}_2$. The quantities $\mathfrak{q}$ and $\underline{\mathfrak{q}}$ are introduced in chapter 5 of [28], where Regge Wheeler type equations are

[79] Recall that the norms are defined with respect to linearized quantities which involve the constants (a, m) specified in Section 3.2.4.

[80] See Definitions 5.2.2 and 5.3.4 in [28] for the choice of C_1, C_2, $\underline{C}_1$ and $\underline{C}_2$.

derived for these quantities. Based on these Regge Wheeler type equations, we derive estimates for $\mathfrak{q}$ and A in Theorem M1 and for $\underline{A}$ in Theorem M2, see Section 3.7.1.

As in [39], the quantity $\mathfrak{q}$ has to be defined relative to a global frame, smooth in the entire region $\mathcal{M}$, in which[81] $\check{H}$ behaves like Γ_g. Also, $\underline{\mathfrak{q}}$ has to be defined relative to another global frame, smooth in the entire region $\mathcal{M}$, in which $\underline{\check{H}}$ behaves like[82] $r^{-1}\Gamma_g$. To achieve this in a way which does not affect the other quantities we proceed in two steps:

1. We construct in Section 3.6.2 a second frame on ${}^{(ext)}\mathcal{M}$ for which $\check{H}$ belongs to Γ_g, and in Section 3.6.3 a third frame of ${}^{(ext)}\mathcal{M}$ for which $\underline{\check{H}}$ belongs to $r^{-1}\Gamma_g$.
2. We use these second and third frame of ${}^{(ext)}\mathcal{M}$ to construct in Section 3.6.4 a first global frame in $\mathcal{M}$ for which $\check{H}$ belongs to Γ_g, and in Section 3.6.5 a second global frame in $\mathcal{M}$ for which $\underline{\check{H}}$ belongs to $r^{-1}\Gamma_g$ in ${}^{(ext)}\mathcal{M}(u \leq u_* - 1)$.

These global frames are used respectively to analyze the decay properties of $\mathfrak{q}$ in Theorem M1 and $\underline{\mathfrak{q}}$ in Theorem M2, see also discussion in the introduction.

3.6.2. Construction of a second null frame in ${}^{(ext)}\mathcal{M}$ We denote in this section by (e_1, e_2, e_3, e_4) the outgoing PG frame of the region ${}^{(ext)}\mathcal{M}$, and by (e'_1, e'_2, e'_3, e'_4) the second frame of ${}^{(ext)}\mathcal{M}$ constructed below. The primed frame of ${}^{(ext)}\mathcal{M}$ is obtained by performing a transformation of the form

$$
\begin{aligned}
e'_4 &= e_4 + f^a e_a + \frac{1}{4}|f|^2 e_3,\\
e'_a &= e_a + \frac{1}{2} f_a e_3, \quad a = 1, 2,\\
e'_3 &= e_3,
\end{aligned}
\tag{3.59}
$$

such that

$$
\check{H}' \in \Gamma'_g,
\tag{3.60}
$$

where Γ'_g, Γ'_b are defined below.

[81]This was η and ξ in [39].

[82]In fact, this will be achieved only in ${}^{(ext)}\mathcal{M}(u \leq u_* - 1)$ which turns out to be sufficient.

Definition 3.25. *Let (e'_1, e'_2, e'_3, e'_4) be the second frame of ${}^{(ext)}\mathcal{M}$ constructed in Proposition 3.26 below. With respect to that frame, we introduce the notations Γ'_g, Γ'_b as follows:*

- *the linearized quantities for the frame (e'_1, e'_2, e'_3, e'_4) are defined in the same way as Definition 2.66 for the outgoing PG frame of ${}^{(ext)}\mathcal{M}$, with respect to the coordinates (r, θ) and the complex 1-form $\mathfrak{J}$ of the PG structure*[83]*,*
- *in addition, we introduce the following linearized quantities which are trivial for an outgoing PG structure*[84]

$$\widecheck{\underline{H}}' := \underline{H}' + \frac{a\overline{q}}{|q|^2}\mathfrak{J}, \qquad \widecheck{e'_4(r)} := e'_4(r) - 1, \qquad \widecheck{\nabla'_4\mathfrak{J}} := \nabla'_4\mathfrak{J} + \frac{1}{q}\mathfrak{J},$$

- *the notation Γ'_b is the one of Definition 2.67, except that $\widecheck{H}'$ does not belong to Γ'_b,*
- *the notation Γ'_g is given by*

$$\Gamma'_g \quad = \quad \Gamma'_{g,1} \cup \Gamma'_{g,2},$$

where $\Gamma'_{g,1}$ is the one of Definition 2.67, and where $\Gamma'_{g,2}$ is given by[85]

$$\Gamma'_{g,2} := \Big\{\omega',\ \ \Xi',\ \ \widecheck{\underline{H}}',\ \ \widecheck{H}',\ \ \widecheck{e'_4(r)},\ \ e'_4(u),\ \ e'_4(J^{(0)}),\ \ r^{-1}\nabla'(r),\ \ \widecheck{\nabla'_4\mathfrak{J}}\Big\}.$$

Using these notations we can define the decay norms $\mathfrak{D}'_k$ exactly as the norms $\mathfrak{D}_k$ in Section 3.3.2. In the proposition below, however, we derive estimates for $\mathfrak{D}'_k$ norms for values of $k \leq k_{small} + 129$, with the particular choice 129 being sufficiently large to absorb possible losses of derivatives later on in the process of proving our main theorem. This requires an interpolation between the decay norms, for $k \leq k_{small}$ and the boundedness norms for $k \leq k_{large}$. The interpolation leads to a slight loss of decay which affects the small constant δ_{dec} in the definition of the decay norms. Thus, in the statement of the proposition below, the norms $\mathfrak{D}'_k$ are defined exactly as $\mathfrak{D}_k$ with

[83] Thus, for example, $\widecheck{\mathrm{tr} X}' = \mathrm{tr} X' - \frac{2}{q}$, $\widecheck{H}' = H' - \frac{aq}{|q|^2}\mathfrak{J}$, $\widecheck{e'_3(r)} = e'_3(r) - \frac{\Delta}{|q|^2}$, $\widecheck{\mathcal{D}' J^{(0)}} = \mathcal{D}' J^{(0)} - i\mathfrak{J}$, $\widecheck{\nabla'_3\mathfrak{J}} = \nabla'_3\mathfrak{J} - \frac{\Delta q}{|q|^4}\mathfrak{J}$,...

[84] Except $\widecheck{\underline{H}}$ which satisfies instead $\underline{H} = -Z$.

[85] Note that all quantities in $\Gamma'_{g,2}$ vanish identically in the case of an outgoing PG structure except $\widecheck{H}'$ and $\widecheck{\underline{H}}'$.

δ_{dec} replaced by $\delta'_{dec} = \delta_{dec} - 2\delta_0$. The precise definition of δ_0 is as follows[86]

$$\delta_0 := \frac{130}{k_{large} - k_{small}}, \qquad 0 < \delta_0 \leq \frac{\delta_{dec}}{3}. \tag{3.61}$$

Proposition 3.26. *Let (e_4, e_3, e_1, e_2) be the outgoing PG frame of ${}^{(ext)}\mathcal{M}$. There exists another frame (e'_4, e'_3, e'_1, e'_2) of ${}^{(ext)}\mathcal{M}$ given by (3.59) such that:*

1. *The 1-form f appearing in (3.59) vanishes in ${}^{(ext)}\mathcal{M}(r \leq u^{\frac{1}{2}})$. In particular, the frame (e'_4, e'_3, e'_1, e'_2) coincides with the outgoing PG frame (e_4, e_3, e_1, e_2) of ${}^{(ext)}\mathcal{M}$ in ${}^{(ext)}\mathcal{M}(r \leq u^{\frac{1}{2}})$.*
2. *The 1-form f appearing in (3.59) vanishes also on $\{u = u_*\}$. In particular, the frame (e'_4, e'_3, e'_1, e'_2) coincides with the outgoing PG frame (e_4, e_3, e_1, e_2) of ${}^{(ext)}\mathcal{M}$ on $\{u = u_*\}$.*
3. *The norms ${}^{(ext)}\mathfrak{D}'_k$ defined as ${}^{(ext)}\mathfrak{D}_k$, with Γ_g, Γ_b replaced by Γ'_g, Γ'_b, $\mathfrak{d}$ replaced by $\mathfrak{d}'$, δ_{dec} replaced by $\delta'_{dec} = \delta_{dec} - 2\delta_0$, and Γ'_g, Γ'_b given by Definition 3.25, verify the estimates*

$$\max_{0 \leq k \leq k_{small}+129} {}^{(ext)}\mathfrak{D}'_k \lesssim \epsilon. \tag{3.62}$$

4. *The horizontal 1-form f verifies, for $k \leq k_{small} + 130$ on ${}^{(ext)}\mathcal{M}$,*

$$\begin{aligned} |(\mathfrak{d}')^k f| &\lesssim \frac{\epsilon}{r u^{\frac{1}{2}+\delta'_{dec}} + u^{1+\delta'_{dec}}}, \\ |(\mathfrak{d}')^{k-1} \nabla'_3 f| &\lesssim \frac{\epsilon}{r u^{1+\delta'_{dec}}}. \end{aligned} \tag{3.63}$$

5. *In addition to the control induced by $\xi' \in \Gamma'_g$, we have, for $k \leq k_{small} + 129$ on ${}^{(ext)}\mathcal{M}$,*

$$|(\mathfrak{d}')^k \xi'| \lesssim \frac{\epsilon}{r^{3+\delta'_{dec}}}, \qquad |(\mathfrak{d}')^{k-1} \nabla'_3 \xi'| \lesssim \frac{\epsilon}{r^3 u^{\frac{1}{2}+\delta'_{dec}}}. \tag{3.64}$$

Remark 3.27. *The crucial point of Proposition 3.26 is that in the new frame (e'_4, e'_3, e'_1, e'_2) of ${}^{(ext)}\mathcal{M}$, $\check{H}'$ belongs to Γ'_g, see Definition 3.25, and thus displays a better decay in r^{-1} than $\check{H}$ corresponding to the outgoing PG frame*

[86] Note that we have in view of (3.46) and (3.51)

$$\delta_{dec}(k_{large} - k_{small}) \geq \frac{1}{2}\delta_{dec} k_{large} - \delta_{dec} \gg 1,$$

and we may thus assume $\delta_{dec}(k_{large} - k_{small}) \geq 390$ so that we have indeed $\delta_0 \leq \frac{\delta_{dec}}{3}$.

(e_4, e_3, e_1, e_2) *of* ${}^{(ext)}\mathcal{M}$. *Also, since* Γ'_b *and* Γ'_g *satisfy the same estimates in* ${}^{(ext)}\mathcal{M}(r \lesssim u^{\frac{1}{2}})$, $\check{H}$ *displays the correct behavior in such regions. This is why we may choose the frame* (e'_4, e'_3, e'_1, e'_2) *to coincide with the outgoing PG frame* (e_4, e_3, e_1, e_2) *of* ${}^{(ext)}\mathcal{M}$ *in* ${}^{(ext)}\mathcal{M}(r \le u^{\frac{1}{2}})$.

3.6.3. Construction of a third null frame in $\boldsymbol{{}^{(ext)}\mathcal{M}}$ We denote in this section by (e_1, e_2, e_3, e_4) the outgoing PG frame of the region ${}^{(ext)}\mathcal{M}$, and by (e'_1, e'_2, e'_3, e'_4) the third frame of ${}^{(ext)}\mathcal{M}$ constructed below. The primed frame of ${}^{(ext)}\mathcal{M}$ is obtained by performing a transformation of the form

$$
\begin{aligned}
e'_4 &= e_4,\\
e'_a &= e_a + \frac{1}{2}\underline{f}_a e_4, \quad a = 1, 2, \\
e'_3 &= e_3 + \underline{f}^a e_a + \frac{1}{4}|\underline{f}|^2 e_4,
\end{aligned}
\tag{3.65}
$$

such that

$$
\underline{\breve{H}}' = 0, \tag{3.66}
$$

where Γ'_g, Γ'_b are defined below.

Definition 3.28. *Let* (e'_1, e'_2, e'_3, e'_4) *be the third frame of* ${}^{(ext)}\mathcal{M}$ *constructed in Proposition 3.29 below. With respect to that frame, the notations* Γ'_g, Γ'_b *are the analog of the corresponding ones in Definition 3.25, except that the notation* Γ'_b *is the one of Definition 2.67, i.e.* $\check{H}' \in \Gamma'_b$.

Proposition 3.29. *Let* (e_4, e_3, e_1, e_2) *be the outgoing PG frame of* ${}^{(ext)}\mathcal{M}$. *There exists another frame* (e'_4, e'_3, e'_1, e'_2) *of* ${}^{(ext)}\mathcal{M}$ *given by* (3.65) *such that:*

1. *The 1-form* $\underline{f}$ *appearing in* (3.65) *vanishes on* Σ_*. *In particular, the frame* (e'_4, e'_3, e'_1, e'_2) *coincides with the outgoing PG frame* (e_4, e_3, e_1, e_2) *of* ${}^{(ext)}\mathcal{M}$ *on* Σ_*.
2. *The norms* ${}^{(ext)}\mathfrak{D}'_k$ *defined as* ${}^{(ext)}\mathfrak{D}_k$, *with* Γ_g, Γ_b *replaced by* Γ'_g, Γ'_b, $\mathfrak{d}$ *replaced by* $\mathfrak{d}'$, δ_{dec} *replaced by* $\delta'_{dec} = \delta_{dec} - \frac{5}{2}\delta_0$, *and* Γ'_g, Γ'_b *given by Definition 3.28, verify the estimates*

$$
\max_{0 \le k \le k_{small}+129} {}^{(ext)}\mathfrak{D}'_k \lesssim \epsilon. \tag{3.67}
$$

3. *The horizontal 1-form* $\underline{f}$ *verifies, for* $k \le k_{small} + 130$ *on* ${}^{(ext)}\mathcal{M}$,

$$|(\mathfrak{d}')^k \underline{f}| \lesssim \frac{\epsilon}{ru^{\frac{1}{2}+\delta'_{dec}} + u^{1+\delta'_{dec}}},$$
$$|(\mathfrak{d}')^{k-1}\nabla'_3 \underline{f}| \lesssim \frac{\epsilon}{ru^{1+\delta'_{dec}}}. \tag{3.68}$$

4. The following identities holds on ${}^{(ext)}\mathcal{M}$

$$\xi' = 0, \qquad \omega' = 0, \qquad \underline{\widecheck{H}}' = 0. \tag{3.69}$$

Remark 3.30. *The crucial point of Proposition 3.29 is that in the new frame* (e'_4, e'_3, e'_1, e'_2) *of* ${}^{(ext)}\mathcal{M}$*, we have the identities* $\underline{\widecheck{H}}' = 0$ *and* $\xi' = 0$.

3.6.4. Construction of a first global null frame We start with the definition of the region where the frames of ${}^{(int)}\mathcal{M}$ and a conformal renormalization of the second frame of ${}^{(ext)}\mathcal{M}$ (i.e. the one of Proposition 3.26) will be matched.

Definition 3.31. *We define the matching region as the spacetime region*

$$\begin{aligned} Match \;:=\; & \Big({}^{(ext)}\mathcal{M} \cap (\{{}^{(ext)}r \leq r_0 + 1\} \cup \{u \geq u_* - 1\})\Big) \\ & \cup \Big({}^{(int)}\mathcal{M} \cap \{\underline{u} \geq u_* - 1\}\Big). \end{aligned} \tag{3.70}$$

Also, we introduce the notations ${}^{(glo)}\Gamma_g$, ${}^{(glo)}\Gamma_b$.

Definition 3.32. *Let* $({}^{(glo)}e_4, {}^{(glo)}e_3, {}^{(glo)}e_1, {}^{(glo)}e_2)$ *the global frame of* $\mathcal{M}$ *constructed in Proposition 3.33 below, together with the corresponding pair of scalars* $({}^{(glo)}r, {}^{(glo)}J^{(0)})$ *and the complex 1-form* ${}^{(glo)}\mathfrak{J}$*. With respect to that frame, we introduce the notations* ${}^{(glo)}\Gamma_g$, ${}^{(glo)}\Gamma_b$ *as follows, where we drop* ${}^{(glo)}$ *after the first item to ease the notations:*

- *the linearized quantities for the frame* $({}^{(glo)}e_4, {}^{(glo)}e_3, {}^{(glo)}e_1, {}^{(glo)}e_2)$ *are defined in the same way as Definition 2.72 for ingoing PG structures, with respect to the scalars* $({}^{(glo)}r, {}^{(glo)}J^{(0)})$ *and the complex 1-form* ${}^{(glo)}\mathfrak{J}$,
- *in addition, we introduce the following linearized quantities which are trivial for an ingoing PG structure*[87]

$$\widecheck{H} := H - \frac{aq}{|q|^2}\mathfrak{J}, \qquad \widecheck{e_3(r)} := e_3(r) + 1, \qquad \widecheck{\nabla_3\mathfrak{J}} := \nabla_3\mathfrak{J} - \frac{1}{\overline{q}}\mathfrak{J},$$

- *the notation* Γ_b *is given by*

[87] In fact H satisfies $H = -Z$ for an ingoing PG structure.

$$\Gamma_b = \Gamma_{b,1} \cup \Gamma_{b,2},$$

where $\Gamma_{b,1}$ *is the one of Definition 2.73, and* $\Gamma_{b,2}$ *is given by*[88]

$$\Gamma_{b,2} := \Big\{ \check{H},\ \underline{\omega},\ \underline{\Xi},\ r^{-1}\widecheck{e_3(r)},\ e_3(J^{(0)}),\ r\widecheck{\nabla_3\mathfrak{J}} \Big\},$$

- *the notation* Γ_g *is given by*

$$\Gamma_g = \Gamma_{g,1} \cup \{r^{-1}\nabla(r)\},$$

where $\Gamma_{g,1}$ *is the one of Definition 2.73.*

Here is our main proposition concerning our first global frame.

Proposition 3.33. *Let* $\delta_0 > 0$ *be the small constant which satisfies* (3.61). *There exist*

- *a global null frame* $({}^{(glo)}e_4, {}^{(glo)}e_3, {}^{(glo)}e_1, {}^{(glo)}e_2)$*,*
- *a pair of scalars* $({}^{(glo)}r, {}^{(glo)}J^{(0)})$*, and a complex 1-form* ${}^{(glo)}\mathfrak{J}$*,*

all defined on $\mathcal{M}$ *such that:*

(a) In ${}^{(ext)}\mathcal{M} \setminus Match$*, we have*

$$\begin{aligned} &({}^{(glo)}e_4, {}^{(glo)}e_3, {}^{(glo)}e_1, {}^{(glo)}e_2) \\ = &\left({}^{(ext)}\lambda\, {}^{(ext)}e_4', {}^{(ext)}\lambda^{-1}\, {}^{(ext)}e_3', {}^{(ext)}e_1', {}^{(ext)}e_2'\right), \end{aligned}$$

where $({}^{(ext)}e_4', {}^{(ext)}e_3', {}^{(ext)}e_1', {}^{(ext)}e_2')$ *denotes the second frame of* ${}^{(ext)}\mathcal{M}$*, i.e. the fame of Proposition 3.26, and where* ${}^{(ext)}\lambda := \frac{{}^{(ext)}\Delta}{|{}^{(ext)}q|^2}$*. We also have* ${}^{(glo)}r = {}^{(ext)}r$*,* ${}^{(glo)}J^{(0)} = \cos({}^{(ext)}\theta)$*, and* ${}^{(glo)}\mathfrak{J} = {}^{(ext)}\mathfrak{J}$*.*

(b) In ${}^{(int)}\mathcal{M} \setminus Match$*, we have*

$$({}^{(glo)}e_4, {}^{(glo)}e_3, {}^{(glo)}e_1, {}^{(glo)}e_2) = ({}^{(int)}e_4, {}^{(int)}e_3, {}^{(int)}e_1, {}^{(int)}e_2),$$

as well as ${}^{(glo)}r = {}^{(int)}r$*,* ${}^{(glo)}J^{(0)} = \cos({}^{(int)}\theta)$*, and* ${}^{(glo)}\mathfrak{J} = {}^{(int)}\mathfrak{J}$*.*

(c) In ${}^{(top)}\mathcal{M}$*, we have*

$$\max_{0\leq k\leq k_{small}+125} \sup_{{}^{(top)}\mathcal{M}(r\leq r_0)} \underline{u}^{1+\delta_{dec}-2\delta_0} \left|\mathfrak{d}^k\left({}^{(glo)}\Gamma_g, {}^{(glo)}\Gamma_b\right)\right| \lesssim \epsilon,$$

and

[88] Note that all quantities in $\Gamma_{b,2}$ vanish identically in the case of an ingoing PG structure.

$$\max_{0\leq k\leq k_{small}+125}\sup_{{}^{(top)}\mathcal{M}(r\geq r_0)}\Bigg\{ \left(r^2({}^{(top)}u)^{\frac{1}{2}+\delta_{dec}-2\delta_0}+r({}^{(top)}u)^{1+\delta_{dec}-2\delta_0}\right)\left|\mathfrak{d}^k\,{}^{(glo)}\Gamma_g\right| +r({}^{(top)}u)^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^k\,{}^{(glo)}\Gamma_b\right| +r^2({}^{(top)}u)^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^{k-1}\nabla_{{}^{(glo)}e_3}{}^{(glo)}\Gamma_g\right| +r^{\frac{7}{2}+\frac{\delta_B}{2}}\left|\mathfrak{d}^k({}^{(glo)}A,{}^{(glo)}B)\right| +r^4({}^{(top)}u)^{\frac{1}{2}+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^{k-1}\nabla_{{}^{(glo)}e_3}{}^{(glo)}B\right| +\left(r^{\frac{9}{2}+\frac{\delta_B}{2}}+r^4({}^{(top)}u)^{\frac{1}{2}+\delta_{dec}-2\delta_0}\right)\left|\mathfrak{d}^{k-1}\nabla_{{}^{(glo)}e_3}{}^{(glo)}A\right|\Bigg\} \lesssim \epsilon,$$

where ${}^{(glo)}\Gamma_g$ and ${}^{(glo)}\Gamma_b$ are given by Definition 3.32, and where ${}^{(top)}u$ is given by (3.44).

(d) *In the matching region, we have*

$$\max_{0\leq k\leq k_{small}+125}\sup_{Match\cap{}^{(int)}\mathcal{M}}\underline{u}^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^k({}^{(glo)}\Gamma_g,{}^{(glo)}\Gamma_b)\right| \lesssim \epsilon,$$

and

$$\max_{0\leq k\leq k_{small}+125}\sup_{Match\cap{}^{(ext)}\mathcal{M}}\Bigg\{ \left(r^2u^{\frac{1}{2}+\delta_{dec}-2\delta_0}+ru^{1+\delta_{dec}-2\delta_0}\right)\left|\mathfrak{d}^k\,{}^{(glo)}\Gamma_g\right| +ru^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^k\,{}^{(glo)}\Gamma_b\right|+r^2u^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^{k-1}\nabla_{{}^{(glo)}e_3}{}^{(glo)}\Gamma_g\right| +r^{\frac{7}{2}+\frac{\delta_B}{2}}\left|\mathfrak{d}^k({}^{(glo)}A,{}^{(glo)}B)\right|+r^4u^{\frac{1}{2}+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^{k-1}\nabla_{{}^{(glo)}e_3}{}^{(glo)}B\right| +\left(r^{\frac{9}{2}+\frac{\delta_B}{2}}+r^4u^{\frac{1}{2}+\delta_{dec}-2\delta_0}\right)\left|\mathfrak{d}^{k-1}\nabla_{{}^{(glo)}e_3}{}^{(glo)}A\right|\Bigg\} \lesssim \epsilon,$$

where ${}^{(glo)}\Gamma_g$ and ${}^{(glo)}\Gamma_b$ are given by Definition 3.32.

(e) *In the matching region, we have*

$$\max_{0\leq k\leq k_{small}+125}\sup_{Match\cap{}^{(int)}\mathcal{M}}\underline{u}^{1+\delta_{dec}-2\delta_0} \left|\mathfrak{d}^k\left({}^{(glo)}r-{}^{(int)}r,\ {}^{(glo)}J^{(0)}-\cos({}^{(int)}\theta),\ {}^{(glo)}\mathfrak{J}-{}^{(int)}\mathfrak{J}\right)\right| \lesssim \epsilon,$$

and

$$\max_{0\leq k\leq k_{small}+125}\sup_{Match\cap\,{}^{(ext)}\mathcal{M}} u^{1+\delta_{dec}-2\delta_0}\Big(\Big|\mathfrak{d}^k\big({}^{(glo)}r-{}^{(ext)}r\big)\Big| +r\Big|\mathfrak{d}^k\big({}^{(glo)}J^{(0)}-\cos({}^{(ext)}\theta)\big)\Big|+r^2\Big|\mathfrak{d}^k\big({}^{(glo)}\mathfrak{J}-{}^{(ext)}\mathfrak{J}\big)\Big|\Big) \lesssim \epsilon.$$

Also, we have in ${}^{(top)}\mathcal{M}$

$$\max_{0\leq k\leq k_{small}+125}\sup_{{}^{(top)}\mathcal{M}(r\leq r_0)} \underline{u}^{1+\delta_{dec}-2\delta_0} \Big|\mathfrak{d}^k\big({}^{(glo)}r-{}^{(top)}r,\ {}^{(glo)}J^{(0)}-\cos({}^{(top)}\theta),{}^{(glo)}\mathfrak{J}-{}^{(top)}\mathfrak{J}\big)\Big| \lesssim \epsilon,$$

and

$$\max_{0\leq k\leq k_{small}+125}\sup_{{}^{(top)}\mathcal{M}(r\geq r_0)} \big({}^{(top)}u\big)^{1+\delta_{dec}-2\delta_0}\Big(\Big|\mathfrak{d}^k\big({}^{(glo)}r-{}^{(top)}r\big)\Big| +r\Big|\mathfrak{d}^k\big({}^{(glo)}J^{(0)}-\cos({}^{(top)}\theta)\big)\Big|+r^2\Big|\mathfrak{d}^k\big({}^{(glo)}\mathfrak{J}-{}^{(top)}\mathfrak{J}\big)\Big|\Big) \lesssim \epsilon.$$

(f) Let $(f,\underline{f},\lambda)$ *denote the change of frame coefficients from the frame of* ${}^{(int)}\mathcal{M}$ *to the global frame, and let* $(f',\underline{f}',\lambda')$ *denote the change of frame coefficients from the second frame of* ${}^{(ext)}\mathcal{M}$*, i.e. the fame of Proposition 3.26, to the global frame. Then, in the matching region, we have*

$$\max_{0\leq k\leq k_{small}+125}\sup_{Match\cap\,{}^{(int)}\mathcal{M}} \underline{u}^{1+\delta_{dec}-2\delta_0}\Big|\mathfrak{d}^k(f,\underline{f},\lambda-1)\Big| \lesssim \epsilon,$$

and

$$\max_{0\leq k\leq k_{small}+125}\sup_{Match\cap\,{}^{(ext)}\mathcal{M}} ru^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^k\left(f',\underline{f}',\log\left(\frac{|{}^{(ext)}q|^2}{{}^{(ext)}\Delta}\lambda'\right)\right)\right|\lesssim\epsilon.$$

Also, let $(f'',\underline{f}'',\lambda'')$ *denote the change of frame coefficients from the frame of* ${}^{(top)}\mathcal{M}$ *to the global frame. Then, in* ${}^{(top)}\mathcal{M}$*, we have*

$$\begin{aligned}&\max_{0\leq k\leq k_{small}+125}\sup_{{}^{(top)}\mathcal{M}(r\leq r_0)} \underline{u}^{1+\delta_{dec}-2\delta_0}\Big|\mathfrak{d}^k(f'',\underline{f}'',\lambda''-1)\Big|\\ &+\max_{0\leq k\leq k_{small}+125}\sup_{{}^{(top)}\mathcal{M}(r\geq r_0)}\\ &\Big(r\big({}^{(top)}u\big)^{\frac{1}{2}+\delta_{dec}-2\delta_0}+\big({}^{(top)}u\big)^{1+\delta_{dec}-2\delta_0}\Big)\Big|\mathfrak{d}^k(f'',\underline{f}'',\lambda''-1)\Big|\\ \lesssim\ &\epsilon.\end{aligned}$$

(g) *In addition to the control induced by* ${}^{(glo)}\check{H} \in {}^{(glo)}\Gamma_b$ *and* ${}^{(glo)}\xi \in {}^{(glo)}\Gamma_g$, *we have, for* $k \leq k_{small} + 125$, *on* ${}^{(ext)}\mathcal{M}(u \leq u_* - 1)$

$$
\begin{aligned}
&{}^{(glo)}\check{H} \in {}^{(glo)}\Gamma_g,\\
&|\mathfrak{d}^k {}^{(glo)}\xi| \lesssim \frac{\epsilon}{r^{3+\delta_{dec}-2\delta_0}}, \qquad |\mathfrak{d}^{k-1}\nabla_{{}^{(glo)}e_3} {}^{(glo)}\xi| \lesssim \frac{\epsilon}{r^3 u^{\frac{1}{2}+\delta_{dec}-2\delta_0}},
\end{aligned}
$$

and on ${}^{(top)}\mathcal{M}(r \geq r_0)$

$$
{}^{(glo)}\check{H} \in {}^{(glo)}\Gamma_g, \qquad {}^{(glo)}\Xi \in r^{-1} {}^{(glo)}\Gamma_g.
$$

Remark 3.34. *Recall that the global frame on* $\mathcal{M}$ *of Proposition 3.33 will be needed to derive decay estimates for the quantity* $\mathfrak{q}$ *in Theorem M1 (stated in Section 3.7.1), see also the discussion in Section 1.7.*

3.6.5. Construction of a second global null frame We consider the same matching region (3.70) as well as Definition 3.32 for the quantities associated to the second global frame of $\mathcal{M}$.

Here is our main proposition concerning our second global frame.

Proposition 3.35. *Let* $\delta_0 > 0$ *be the small constant which satisfies* (3.61). *There exist*

- *a global null frame* $({}^{(glo')}e_4, {}^{(glo')}e_3, {}^{(glo')}e_1, {}^{(glo')}e_2)$,
- *a pair of scalars* $({}^{(glo')}r, {}^{(glo')}J^{(0)})$, *and a complex 1-form* ${}^{(glo')}\mathfrak{J}$,

all defined on $\mathcal{M}$ *such that:*

(a) *In* ${}^{(ext)}\mathcal{M} \setminus$ *Match, we have*

$$
\begin{aligned}
&({}^{(glo')}e_4, {}^{(glo')}e_3, {}^{(glo')}e_1, {}^{(glo')}e_2)\\
&= \left({}^{(ext)}\lambda\, {}^{(ext)}e'_4, {}^{(ext)}\lambda^{-1} {}^{(ext)}e'_3, {}^{(ext)}e'_1, {}^{(ext)}e'_2\right),
\end{aligned}
$$

where $({}^{(ext)}e'_4, {}^{(ext)}e'_3, {}^{(ext)}e'_1, {}^{(ext)}e'_2)$ *denotes the third frame of* ${}^{(ext)}\mathcal{M}$, *i.e. the fame of Proposition 3.29, and where* ${}^{(ext)}\lambda := \frac{{}^{(ext)}\Delta}{|{}^{(ext)}q|^2}$. *We also have* ${}^{(glo')}r = {}^{(ext)}r$, ${}^{(glo')}J^{(0)} = \cos({}^{(ext)}\theta)$, *and* ${}^{(glo')}\mathfrak{J} = {}^{(ext)}\mathfrak{J}$.

(b) *The second global frame of* $\mathcal{M}$ *satisfies properties (b)–(e) of Proposition 3.33.*

(c) *Let* $(f, \underline{f}, \lambda)$ *denote the change of frame coefficients from the frame of* ${}^{(int)}\mathcal{M}$ *to the second global frame, and let* $(f', \underline{f}', \lambda')$ *denote the change of frame coefficients from the third frame of* ${}^{(ext)}\mathcal{M}$, *i.e. the frame of*

Proposition 3.29, to the second global frame. Then, in the matching region, we have

$$\max_{0\le k\le k_{small}+125}\sup_{Match\cap{}^{(int)}\mathcal{M}}\underline{u}^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^k(f,\underline{f},\lambda-1)\right| \lesssim \epsilon,$$

and

$$\max_{0\le k\le k_{small}+125}\sup_{Match\cap{}^{(ext)}\mathcal{M}} ru^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^k\left(f',\underline{f}',\log\left(\frac{|{}^{(ext)}q|^2}{{}^{(ext)}\Delta}\lambda'\right)\right)\right| \lesssim \epsilon.$$

Also, let $(f'',\underline{f}'',\lambda'')$ denote the change of frame coefficients from the frame of ${}^{(top)}\mathcal{M}$ to the global frame. Then, in ${}^{(top)}\mathcal{M}$, we have

$$\begin{aligned}&\max_{0\le k\le k_{small}+125}\sup_{{}^{(top)}\mathcal{M}(r\le r_0)}\underline{u}^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^k(f'',\underline{f}'',\lambda''-1)\right|\\&+\max_{0\le k\le k_{small}+125}\sup_{{}^{(top)}\mathcal{M}(r\ge r_0)}\\&\left(r({}^{(top)}u)^{\frac{1}{2}+\delta_{dec}-2\delta_0}+({}^{(top)}u)^{1+\delta_{dec}-2\delta_0}\right)\left|\mathfrak{d}^k(f'',\underline{f}'',\lambda''-1)\right| \lesssim \epsilon.\end{aligned}$$

(d) *In addition to the control induced by ${}^{(glo)}\xi$, ${}^{(glo)}\widecheck{\underline{H}}\in{}^{(glo)}\Gamma_g$, we have, for $k\le k_{small}+125$ on ${}^{(ext)}\mathcal{M}\cap(\{{}^{(ext)}r\ge r_0+1\}\cap\{u\le u_*-1\})$,*

$${}^{(glo')}\xi=0,\qquad {}^{(glo')}\widecheck{\underline{H}}=0.$$

Remark 3.36. *Recall that the global frame on $\mathcal{M}$ of Proposition 3.35 will be needed to derive decay estimates for the quantity $\underline{\mathfrak{q}}$ in Theorem M2 (stated in Section 3.7.1), see also the discussion in Section 1.7.*

3.7. Proof of the main theorem

3.7.1. Main intermediate results We are ready to state our main intermediary results.

Theorem M0. *Assume that the initial data layer $\mathcal{L}_0$, as defined in Section 3.1, satisfies*

$$\mathfrak{I}_{k_{large}+10}\le\epsilon_0.$$

Then under the bootstrap assumptions **BA-B** *and* **BA-D**, *the following holds true on the initial data hypersurface* $\mathcal{B}_1 \cup \underline{\mathcal{B}}_1$,

$$\max_{0\leq k\leq k_{large-2}} \left\{ \sup_{\mathcal{B}_1} \left[r^{\frac{7}{2}+\delta_B} \left(|\mathfrak{d}^k \, {}^{(ext)}A| + |\mathfrak{d}^k \, {}^{(ext)}B| \right) + r^{\frac{9}{2}+\delta_B} |\mathfrak{d}^{k-1} \nabla_3 ({}^{(ext)}A)| \right] \right.$$
$$\left. + \sup_{\mathcal{B}_1} \left[r^3 \left| \mathfrak{d}^k \left({}^{(ext)}P + \frac{2m}{q^3} \right) \right| + r^2 |\mathfrak{d}^k \, {}^{(ext)}\underline{B}| + r |\mathfrak{d}^k \, {}^{(ext)}\underline{A}| \right] \right\} \lesssim \epsilon_0,$$

$$\max_{0\leq k\leq k_{large-2}} \sup_{\underline{\mathcal{B}}_1} \left[|\mathfrak{d}^k \, {}^{(int)}A| + |\mathfrak{d}^k \, {}^{(int)}B| + \left| \mathfrak{d}^k \left({}^{(int)}P + \frac{2m}{q^3} \right) \right| \right.$$
$$\left. + |\mathfrak{d}^k \, {}^{(int)}\underline{B}| + |\mathfrak{d}^k \, {}^{(int)}\underline{A}| \right] \lesssim \epsilon_0,$$

and

$$\sup_{\mathcal{B}_1 \cup \underline{\mathcal{B}}_1} \Big(|m - m_0| + |a - a_0| \Big) \lesssim \epsilon_0.$$

Theorem M1. *Assume given a GCM admissible spacetime* $\mathcal{M}$ *as defined in Section 3.2 verifying the bootstrap assumptions* **BA-B** *and* **BA-D** *for some sufficiently small* $\epsilon > 0$. *Then, if* $\epsilon_0 > 0$ *is sufficiently small, there exists* $\delta_{extra} > \delta_{dec}$ *such that we have the following estimates in* $\mathcal{M}$[89], *with respect to the global frame of Proposition 3.33,*

1. *The quantity* $\mathfrak{q}$ *verifies the estimates*

$$\max_{0\leq k\leq k_{small}+100} \sup_{\Sigma_*} \Big\{ \left(r u^{\frac{1}{2}+\delta_{extra}} + u^{1+\delta_{extra}} \right) |\mathfrak{d}^k \mathfrak{q}| + r u^{1+\delta_{extra}} |\mathfrak{d}^{k-1} \nabla_3 \mathfrak{q}| \Big\} \lesssim \epsilon_0.$$

Moreover, $\mathfrak{q}$ *also satisfies the following estimate*

$$\max_{0\leq k\leq k_{small}+100} u^{2+2\delta_{extra}} \int_{\Sigma_*(\geq u)} |\mathfrak{d}^{k-1} \nabla_3 \mathfrak{q}|^2 \lesssim \epsilon_0^2.$$

[89] Even though we only state below estimates on ${}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}$, which is not causal, the actual estimates are in fact proved on the full spacetime $\mathcal{M}$ which is itself causal.

2. *The quantity A verifies the estimate, for all $k \leq k_{small}+100$,*

$$\begin{aligned}\sup_{^{(ext)}\mathcal{M}}\Big(r^2u^{1+\delta_{extra}}+r^3(2r+u)^{\frac{1}{2}+\delta_{extra}}\Big)|\mathfrak{d}^kA|\\ +\sup_{^{(ext)}\mathcal{M}}\Big(r^3u^{1+\delta_{extra}}+r^4u^{\frac{1}{2}+\delta_{extra}}+r^{\frac{9}{2}+\delta_{dec}}\Big)|\mathfrak{d}^{k-1}\nabla_3A|\\ +\sup_{^{(ext)}\mathcal{M}}\Big(r^4u^{1+\delta_{extra}}+r^{\frac{9}{2}+\delta_{dec}}\Big)|\mathfrak{d}^{k-2}\nabla_3^2A|\\ +\sup_{^{(int)}\mathcal{M}}r^2\underline{u}^{1+\delta_{extra}}|\mathfrak{d}^kA| \quad\lesssim\quad \epsilon_0.\end{aligned}$$

Theorem M2. *Under the same assumptions as above we have the following decay estimates for $\underline{\alpha}$, with respect to the global frame of Proposition 3.35,*

$$\max_{0\leq k\leq k_{small}+80}\int_{\Sigma_*}u^{2+2\delta_{dec}}|\mathfrak{d}^k\underline{\alpha}|^2\lesssim\epsilon_0^2.$$

Theorem M3. *Under the same assumptions as above we have the following decay estimates on Σ_**

$${}^*\mathfrak{D}_{k_{small}+60}\lesssim\epsilon_0.$$

Theorem M4. *Under the same assumptions as above we have the following decay estimates on ${}^{(ext)}\mathcal{M}$*

$${}^{(ext)}\mathfrak{D}_{k_{small}+40}\lesssim\epsilon_0.$$

Theorem M5. *Under the same assumptions as above we also have the following decay estimates in ${}^{(int)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$*

$${}^{(int)}\mathfrak{D}_{k_{small}+20}+{}^{(top)}\mathfrak{D}_{k_{small}+20}\lesssim\epsilon_0.$$

Note that, as an immediate consequence of Theorem M1 to Theorem M5, we have obtained, under the same assumptions as above, the following improvement of our bootstrap assumptions on decay

$$\mathfrak{N}^{(Dec)}_{k_{small}+20}\lesssim\epsilon_0. \tag{3.71}$$

3.7.2. End of the proof of the main theorem We end the proof by invoking a continuity argument as in [39]. The argument is based on Definition 3.37 below of the set $\mathcal{U}(u_*)$ of GCM admissible spacetimes verifying bootstrap assumptions $\mathbf{BA}_\epsilon$ with $\epsilon=\epsilon_0^{\frac{2}{3}}$.

Definition 3.37 (Definition of $\mathcal{U}(u_*)$)**.** *Let $\epsilon_0 > 0$ and $\epsilon = \epsilon_0^{\frac{2}{3}}$ be given small constants, and let $\mathcal{U}(u_*)$ be the set of all GCM admissible spacetimes $\mathcal{M}$ defined in Section 3.2 such that*

- *u_* is the value of u on the last sphere S_* of Σ_*,*
- *u_* satisfies (see (3.50))*

$$r_* = \delta_* \epsilon_0^{-1} u_*^{1+\delta_{dec}},$$

- *relative to the combined norms defined in Section 3.3.5, we have*[90]

$$\mathfrak{N}^{(Sup)}_{k_{large}} \leq \epsilon, \qquad \mathfrak{N}^{(Dec)}_{k_{small}} \leq \epsilon.$$

Definition 3.38. *We define $\mathcal{U}$ to be the set of all values of $u_* \geq 0$ for which the spacetime $\mathcal{U}(u_*)$ exists.*

The following theorem shows that $\mathcal{U}$ is not empty.

Theorem M6. *There exists $\delta_0 > 0$ small enough such that for a sufficiently small constant $\epsilon_0 > 0$ we have $[1, 1+\delta_0] \subset \mathcal{U}$.*

In view of Theorem M6, we may define U_* as the supremum over all value of u_* that belongs to $\mathcal{U}$

$$U_* := \sup_{u_* \in \mathcal{U}} u_*.$$

Assume by contradiction that

$$U_* < +\infty.$$

Then, by the continuity of the flow, $U_* \in \mathcal{U}$. According to (3.71), the bootstrap assumptions on decay (3.58) on any spacetime of $\aleph(U_*)$ are improved by

$$\mathfrak{N}^{(Dec)}_{k_{small}+20} \lesssim \epsilon_0.$$

To reach a contradiction, we still need an extension procedure for spacetimes in $\aleph(u_*)$ to larger values of u, as well as to improve our bootstrap assumptions on boundedness (3.57). This is done in two steps.

[90] I.e. the bootstrap assumptions (3.57) (3.58) hold true.

Theorem M7. *Any GCM admissible spacetime in* $\aleph(u_*)$ *for some* $0 < u_* < +\infty$ *such that*

$$\mathfrak{N}^{(Dec)}_{k_{small}+20} \lesssim \epsilon_0$$

has a GCM admissible extension verifying (3.50)*, with* $u'_* > u_*$*, initialized by Theorem M0, which verifies*

$$\mathfrak{N}^{(Dec)}_{k_{small}} \lesssim \epsilon_0.$$

Theorem M8. *The GCM admissible spacetime exhibited in Theorem M7 satisfies in addition*

$$\mathfrak{N}^{(Sup)}_{k_{large}} \lesssim \epsilon_0$$

and therefore belongs to $\aleph(u'_*)$*. In particular* u'_* *belongs to* $\mathcal{U}$*.*

In view of Theorem M8, we have reached a contradiction, and hence

$$U_* = +\infty$$

so that the spacetime may be continued forever. This concludes the proof of the main theorem.

3.7.3. List of all null frames used in this work

For the convenience of the reader, we list in this section the various null frames used in this work:

- Null frame adapted to Σ_*: this null frame is part of the GCM-PG data set on Σ_* introduced in section 3.2.3. It appears both in the statement of Theorem M3 in section 3.7.1 and in its proof in Chapter 5.
- Outgoing PG frame of ${}^{(ext)}\mathcal{M}$: this null frame is attached to the outgoing PG structure of ${}^{(ext)}\mathcal{M}$ introduced in section 3.2.2. It appears both in the statement of Theorem M4 in section 3.7.1 and in its proof in Chapter 6.
- Ingoing PG frame of ${}^{(int)}\mathcal{M}$: this null frame is attached to the ingoing PG structure of ${}^{(int)}\mathcal{M}$ introduced in section 3.2.2. It appears both in the statement of Theorem M5 in section 3.7.1 and in its proof in Chapter 7.
- Ingoing PG frame of ${}^{(top)}\mathcal{M}$: this null frame is attached to the ingoing PG structure of ${}^{(top)}\mathcal{M}$ introduced in section 3.2.2. It appears both in the statement of Theorem M5 in section 3.7.1 and in its proof in Chapter 7.

- First global frame: this global null frame on $\mathcal{M}$ is constructed in section 3.6.4, see Proposition 3.33. It appears both in the statement of Theorem M1 in section 3.7.1 and in its proof in Part II of [28].
- Second global frame: this global null frame on $\mathcal{M}$ is constructed in section 3.6.5, see Proposition 3.35. It appears both in the statement of Theorem M2 in section 3.7.1 and in its proof in Part II of [28].
- Other null frames on ${}^{(ext)}\mathcal{M}$: the second null frame on ${}^{(ext)}\mathcal{M}$, constructed in section 3.6.2, see Proposition 3.26, is an auxiliary frame used in the construction of the first global null frame of section 3.6.4. Also, the third null frame on ${}^{(ext)}\mathcal{M}$, constructed in section 3.6.3, see Proposition 3.29, is an auxiliary frame used in the construction of the second global null frame of section 3.6.5.

Remark 3.39. *In addition to the above null frames, note that we will introduce other null frames in the proof of Theorem M8 in Chapter 9:*

- *Three null frames attached respectively to one outgoing and two ingoing PT structures in $\mathcal{M}$, introduced in section 9.1.3 and used throughout the proof of Theorem M8 in Chapter 9.*
- *An additional global null frame on $\mathcal{M}$, introduced in section 9.6.1, and used for the curvature estimates of Theorem M8 proved in Part III of [28].*

3.8. Conclusions

We denote by $\mathcal{M}$ our global spacetime obtained in the limit $u_* \to +\infty$. Note that in the limit $u_* \to +\infty$, the region ${}^{(top)}\mathcal{M}$ disappears[91] so that

$$\mathcal{M} = {}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}, \tag{3.72}$$

where ${}^{(int)}\mathcal{M}$ is covered by an ingoing PG structure, and ${}^{(ext)}\mathcal{M}$ by an outgoing PG structure. In particular, note that

$$\begin{aligned} {}^{(int)}\mathcal{M} &= \{1 \le \underline{u} < +\infty\} \cup \{r_+ - \delta_{\mathcal{H}} \le {}^{(int)}r \le r_0\}, \\ {}^{(ext)}\mathcal{M} &= \{1 \le u < +\infty\} \cup \{r_0 \le {}^{(ext)}r < +\infty\}, \end{aligned} \tag{3.73}$$

where we recall that

$$r_+ = m_0 + \sqrt{m_0^2 - a_0^2}. \tag{3.74}$$

[91] Indeed, recall that $u \ge u_*$ and $\underline{u} \ge u_*$ on ${}^{(top)}\mathcal{M}$ by the construction of our GCM admissible spacetime.

Also, as a consequence of Theorem M0, the parameters (m_∞, a_∞) obtained in the limit $u_* \to +\infty$ satisfy

$$|m_\infty - m_0| + |a_\infty - a_0| \lesssim \epsilon_0. \tag{3.75}$$

This implies in particular $m_\infty > 0$ and $|a_\infty| \ll m_\infty$.

3.8.1. The Penrose diagram of $\mathcal{M}$ Complete future null infinity. We first deduce from our estimate that our spacetime $\mathcal{M}$ has a complete future null infinity $\mathcal{I}^+$. Let us denote by (e_4, e_3, e_1, e_2) and (u, r) the null frame and scalar functions associated to our outgoing PG structure of ${}^{(ext)}\mathcal{M}$. Recall also that the outgoing PG structure of ${}^{(ext)}\mathcal{M}$ comes together with a scalar function θ and a complex 1-form $\mathfrak{J}$. The scalar function u of $\mathcal{M}$ satisfies, using $e_4(u) = 0$,

$$\mathbf{g}(\mathbf{D}u, \mathbf{D}u) \quad = \quad |\nabla u|^2.$$

Recalling that (see Definition 2.66) $\nabla u = a\Re(\mathfrak{J}) + \widecheck{\nabla u} = a\Re(\mathfrak{J}) + \Gamma_b$, as well as the identity $\Re(\mathfrak{J})\cdot\Re(\mathfrak{J}) = \frac{(\sin\theta)^2}{|q|^2}$, we infer on ${}^{(ext)}\mathcal{M}$, using also our control for Γ_b induced by the estimates $\mathfrak{N}^{(Dec)}_{k_{small}} \lesssim \epsilon_0$ of our main theorem,

$$\mathbf{g}(\mathbf{D}u, \mathbf{D}u) \quad = \quad \frac{a^2(\sin\theta)^2}{|q|^2} + O\left(\frac{\epsilon_0}{r^2 u^{2+2\delta_{dec}}}\right).$$

In particular, the leaves of the u-foliation of ${}^{(ext)}\mathcal{M}$ are asymptotically null as $r \to +\infty$, so that the portion of null infinity of $\mathcal{M}$ corresponds to the limit $r \to +\infty$ along the leaves of the u-foliation of the outgoing PG structure of ${}^{(ext)}\mathcal{M}$. As these leaves exist for all $u \geq 1$ with suitable estimates, it suffices to prove that u is an affine parameter of $\mathcal{I}^+$. To this end, recall from our main theorem that the estimates $\mathfrak{N}^{(Dec)}_{k_{small}} \lesssim \epsilon_0$ hold which implies in particular

$$\sup_{{}^{(ext)}\mathcal{M}} u^{1+\delta_{dec}} \left(r|\underline{\xi}| + r\left|\underline{\omega} - \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right)\right| + \left|\widecheck{e_3(u)}\right| \right) \quad \lesssim \quad \epsilon_0. \tag{3.76}$$

We infer that

$$\lim_{r\to+\infty} \underline{\xi}, \underline{\omega} = 0 \text{ for all } 1 \leq u < \infty.$$

In view of the identity

$$\mathbf{D}_3 e_3 \quad = \quad -2\underline{\omega} e_3 + 2\underline{\xi}_b e_b,$$

we infer that e_3 is a null geodesic generator of $\mathcal{I}^+$. Since we have

$$e_3(u) = \frac{2(r^2+a^2)}{|q|^2} + \widetilde{e_3(u)} = \frac{2(r^2+a^2)}{|q|^2} + O\left(\frac{\epsilon_0}{u^{1+\delta_{dec}}}\right) = 2 + O\left(\frac{1}{r^2} + \epsilon_0\right)$$

in view of (3.76), u is an affine parameter of $\mathcal{I}^+$ so that $\mathcal{I}^+$ is indeed complete.

Existence of a future event horizon. Let us denote by (e_4, e_3, e_1, e_2) and $(\underline{u}, r)$ the null frame and scalar functions associated to our ingoing PG structure of ${}^{(int)}\mathcal{M}$. Note that the estimates $\mathfrak{N}^{(Dec)}_{k_{small}} \lesssim \epsilon_0$ imply

$$\sup_{{}^{(int)}\mathcal{M}} \underline{u}^{1+\delta_{dec}} \left(|e_3(r)+1| + \left|e_4(r) - \frac{\Delta_\infty}{|q|^2}\right|\right) \lesssim \epsilon_0,$$

where $\Delta_\infty = \Delta(a_\infty, m_\infty)$.

In particular, considering the spacetime region $r \le r_+(1-\delta_{\mathcal{H}}/2)$ of ${}^{(int)}\mathcal{M}$, and in view of the estimate $|m_\infty - m_0| \lesssim \epsilon_0$, we infer, for all $r \le r_+(1-\delta_{\mathcal{H}}/2)$, that

$$\begin{aligned}
\Delta_\infty &= \Delta(a_\infty, m_\infty) = r^2 - 2m_\infty r + a_\infty^2 \\
&= (r - r_+(a_\infty, m_\infty))(r - r_-(a_\infty, m_\infty)) \\
&= (r - r_+)(r - r_-) + O(m_\infty - m_0) + O(a_\infty - a_0) \\
&\le -\frac{\delta_{\mathcal{H}}}{2} r_+ \left(r_+ - r_- - \frac{\delta_{\mathcal{H}} r_+}{2}\right) + O(\epsilon_0) \\
&\lesssim -\delta_{\mathcal{H}} < 0
\end{aligned}$$

and hence

$$e_3(r) \le -\frac{1}{2} < 0, \qquad e_4(r) \lesssim -\delta_{\mathcal{H}} < 0 \quad \text{on} \quad {}^{(int)}\mathcal{M}\,(r \le r_+(1-\delta_{\mathcal{H}}/2)).$$

Consider now $\gamma(s)$ any future directed null geodesic emanating from a point of the region ${}^{(int)}\mathcal{M}(r \le r_+(1-\delta_{\mathcal{H}}/2))$. Since $\dot\gamma$ is a null vector, there exists at any point of $\gamma(s)$ in ${}^{(int)}\mathcal{M}$ a scalar λ and a 1-form f such that

$$\dot\gamma = \lambda\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right),$$

where $\lambda > 0$ (since $\dot\gamma$ is future directed). Since $\nabla(r) = 0$, we infer

$$\frac{dr}{ds} = \mathbf{D}_{\dot\gamma} r = \lambda\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right) r = \lambda\left(e_4(r) + \frac{1}{4}|f|^2 e_3(r)\right).$$

Since $e_3(r) < 0$ and $e_4(r) < 0$ in ${}^{(int)}\mathcal{M}(r \le r_+(1-\delta_{\mathcal{H}}/2))$ in view of the above, and since $\lambda > 0$ and $|f|^2 \ge 0$, we deduce that r decreases along $\gamma(s)$ so that, in particular, $\gamma(s)$ cannot reach $\mathcal{I}^+$. Thus the past of $\mathcal{I}^+$ does not contain this region and hence $\mathcal{M}$ contains the event horizon $\mathcal{H}_+$ of a black hole in its interior. Moreover, since any point on the timelike hypersurface $\mathcal{T} = {}^{(int)}\mathcal{M} \cap {}^{(ext)}\mathcal{M}$ is on an outgoing null geodesic in ${}^{(ext)}\mathcal{M}$ of geodesic generator e_4 with $e_4(r) = 1$ and defined for all $r \ge r_0$, $\mathcal{T}$ is in the past of $\mathcal{I}^+$. Hence, since $\mathcal{T}$ is one of the boundaries of ${}^{(int)}\mathcal{M}$, $\mathcal{H}_+$ is actually located in the interior of the region ${}^{(int)}\mathcal{M}$.

Asymptotic stationarity of $\mathcal{M}$. Recall that we have introduced a vectorfield $\mathbf{T}$ in ${}^{(ext)}\mathcal{M}$ as well as one in ${}^{(int)}\mathcal{M}$ by

$$\begin{aligned}\mathbf{T} &= \frac{1}{2}\left(e_3 + \frac{\Delta}{|q|^2}e_4 - 2a\Re(\mathfrak{J})^b e_b\right) \quad \text{in} \quad {}^{(ext)}\mathcal{M},\\ \mathbf{T} &= \frac{1}{2}\left(e_4 + \frac{\Delta}{|q|^2}e_3 - 2a\Re(\mathfrak{J})^b e_b\right) \quad \text{in} \quad {}^{(int)}\mathcal{M}.\end{aligned}$$

Also, see Proposition 2.70, all components of ${}^{(\mathbf{T})}\pi$ belong, at least, to Γ_b. Thus, making use of the estimate $\mathfrak{N}^{(Dec)}_{k_{small}} \lesssim \epsilon_0$ of our main theorem, we deduce,

$$|{}^{(\mathbf{T})}\pi| \lesssim \frac{\epsilon_0}{ru^{1+\delta_{dec}}} \quad \text{in} \quad {}^{(ext)}\mathcal{M} \quad \text{and} \quad |{}^{(\mathbf{T})}\pi| \lesssim \frac{\epsilon_0}{\underline{u}^{1+\delta_{dec}}} \quad \text{in} \quad {}^{(int)}\mathcal{M}.$$

In particular, $\mathbf{T}$ is an asymptotically Killing vectorfield and hence our spacetime $\mathcal{M}$ is asymptotically stationary.

The above conclusions regarding $\mathcal{I}^+$ and $\mathcal{H}_+$ allow us to draw the Penrose diagram of $\mathcal{M}$, see figure 6 below.

3.8.2. Limits at $\mathcal{I}^+$

3.8.2.1. Integrable frame We denote by (e_4, e_3, e_1, e_2) and (u, r) the null frame and defining functions associated to our outgoing PG structure of ${}^{(ext)}\mathcal{M}$. Recall that (e_1, e_2) is not tangent to the spheres $S(u, r)$ since $\nabla(u) \neq 0$. Recall that we have exhibited in Lemma 2.22 a frame transformation taking (e_3, e_4, e_1, e_2) into an integrable null frame (e'_3, e'_4, e'_1, e'_2), i.e. such that the horizontal vectors (e'_1, e'_2) are tangent to $S(u, r)$. The corresponding frame coefficients $(f, \underline{f}, \lambda)$ satisfy, see (2.12) (6.32),

$$\begin{aligned}f &= -\Big(1 + O(r^{-2}) + r\Gamma_b\Big)\nabla u,\\ \underline{f} &= -\left(1 - \frac{2m}{r} + O(r^{-2}) + r\Gamma_b\right)\nabla u,\end{aligned} \tag{3.77}$$

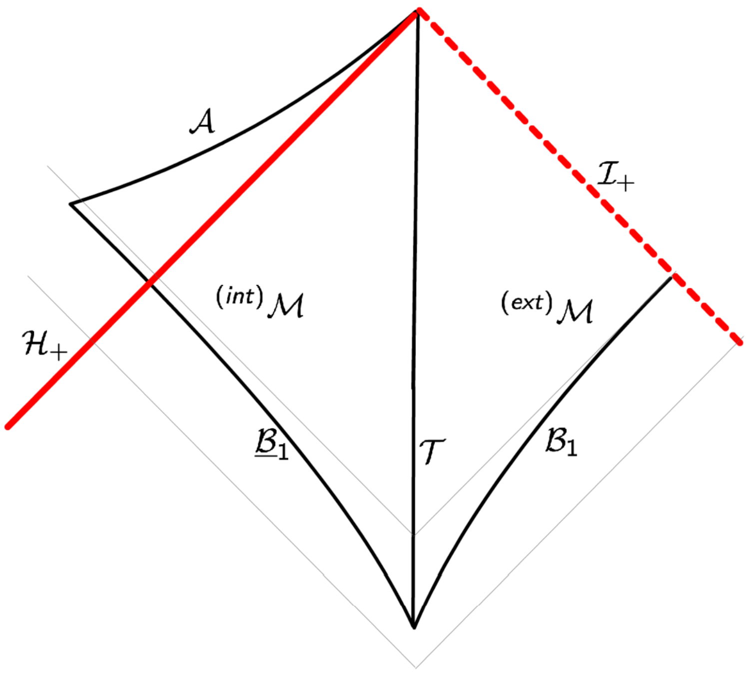

Figure 6: The Penrose diagram of the space-time $\mathcal{M}$ with past boundary $\underline{\mathcal{B}}_1 \cup \mathcal{B}_1$.

$$\lambda = 1 + O(r^{-2}) + r^{-1}\Gamma_b.$$

Then, relying on the frame transformations formulas of Proposition 2.12 and the above control of $(f, \underline{f}, \lambda)$, we obtain[92] for the frame (e_3', e_4', e_1', e_2')

$$\text{(3.78)}\qquad \begin{aligned} \operatorname{tr}\chi' &= \frac{2}{r} + O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_g, & \widehat{\chi}' &= O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_g,\\ \operatorname{tr}\underline{\chi}' &= -\frac{2\left(1-\frac{2m_\infty}{r}\right)}{r} + O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_g, & \underline{\widehat{\chi}}' &= O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_b,\\ \zeta' &= O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_g, \end{aligned}$$

$$\text{(3.79)}\qquad \begin{aligned} \xi' &= O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b, & \underline{\xi}' &= O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_b,\\ \omega' &= O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b, & \underline{\omega}' &= \frac{m_\infty}{r^2} + O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_b,\\ \underline{\eta}' &= O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_g, & \eta' &= O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_b, \end{aligned}$$

[92]See Lemma 2.52 where the corresponding passage from the PG frame to the integrable frame is done in details in the particular case of Kerr.

and

$$\begin{aligned} &\alpha' = \alpha + O(r^{-5}) + r^{-2}\Gamma_g, && \beta' = \beta + O(r^{-4}) + r^{-2}\Gamma_g, \\ &\rho' = -\frac{2m_\infty}{r^3} + O(r^{-5}) + r^{-1}\Gamma_g, && {}^*\rho' = O(r^{-4}) + r^{-1}\Gamma_g, \\ &\underline{\beta}' = O(r^{-4}) + r^{-1}\Gamma_b, && \underline{\alpha}' = O(r^{-5}) + \Gamma_b. \end{aligned} \tag{3.80}$$

3.8.2.2. The spheres at null infinity are round Recalling that the integrable frame (e_3', e_4', e_1', e_2') is such that the horizontal vectors (e_1', e_2') are tangent to $S(u,r)$, the Gauss curvature K of the spheres $S(u,r)$ is given by the Gauss equation

$$K = -\rho' - \frac{1}{4}\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}' + \frac{1}{2}\widehat{\chi}'\cdot\widehat{\underline{\chi}}'.$$

In view of (3.78) and (3.80), we infer

$$K - \frac{1}{r^2} = O(r^{-4}) + r^{-1}\Gamma_g.$$

Thus, in view of our estimates in ${}^{(ext)}\mathcal{M}$ for Γ_g, we deduce

$$\left|K - \frac{1}{r^2}\right| \lesssim \frac{1}{r^4} + \frac{\epsilon_0}{r^3 u^{\frac{1}{2}+\delta_{dec}}}$$

so that

$$\lim_{r\to+\infty} r^2 K = 1.$$

In particular the spheres at null infinity are round.

Also, using Gauss-Bonnet, we have

$$\begin{aligned} 4\pi &= \int_S K = \int_S \left(\frac{1}{r^2} + O\left(\frac{1}{r^4} + \frac{\epsilon_0}{r^3 u^{\frac{1}{2}+\delta_{dec}}}\right)\right) \\ &= \frac{|S|}{r^2} + O\left(\frac{|S|}{r^4} + \frac{|S|\epsilon_0}{r^3 u^{\frac{1}{2}+\delta_{dec}}}\right) \end{aligned}$$

and hence

$$\sqrt{\frac{|S|}{4\pi}} = r\left(1 + O(r^{-2}) + O(\epsilon_0 r^{-1} u^{-\frac{1}{2}-\delta_{dec}})\right) \tag{3.81}$$

which shows that r is a good approximation of the area radius of the spheres $S(u,r)$.

3.8.2.3. Limits at null infinity and Bondi mass Recall the definition of the Hawking mass, associated here to the spheres $S(u,r)$

$$m_H = \frac{\sqrt{\frac{|S|}{4\pi}}}{2}\left(1+\frac{1}{16\pi}\int_S \mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}'\right).$$

In view of (3.81) and (3.78), we infer

$$m_H = m_\infty\Big(1+O(r^{-1})+O(\epsilon_0 u^{-\frac{1}{2}-\delta_{dec}})\Big). \tag{3.82}$$

Also, we differentiate the identity for m_H w.r.t. the vectorfield e_4 of the PG structure of ${}^{(ext)}\mathcal{M}$ and find

$$e_4(m_H) = \frac{m_H}{2|S|}e_4\left(|S|\right)+\frac{1}{32\pi}\sqrt{\frac{|S|}{4\pi}}e_4\left(\int_S \mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}'\right).$$

Next, we make use of the following corollary of Lemma 6.31.

Lemma 3.40. *Given a scalar function h we have*

$$e_4\left(\int_{S(u,r)}h\right) = \int_{S(u,r)}\Big(e_4(h)+tr\,\chi' h + f\cdot\nabla' h\Big)+O(r^{-1})h.$$

Proof. According to Lemma 6.31

$$e_4\left(\int_{S(u,r)}h\right) = \int_{S(u,r)}\Big(e_4(h)+\delta^{ab}\mathbf{g}(\mathbf{D}_{e'_a}e_4,e'_b)h\Big).$$

In view of the precise formula for $\delta^{ab}\mathbf{g}(\mathbf{D}_{e'_a}e_4,e'_b)$ in that lemma, the estimates for $f,\underline{f}$ in (3.77)

$$\delta^{ab}\mathbf{g}(\mathbf{D}_{e'_a}e_4,e'_b) = (1+O(r^{-2}))\mathrm{tr}\,\chi+O(r^{-3})+O(r^{-2})\mathfrak{d}^{\leq 1}\Gamma_b.$$

Next we need to replace $\mathrm{tr}\,\chi$ with $\mathrm{tr}\,\chi'$. To this end, we make use[93] of the transformation formula for $\mathrm{tr}\,\chi'$ in Proposition 2.12. Together with (3.77) we deduce

$$\mathrm{tr}\,\chi' = \mathrm{tr}\,\chi+\mathrm{div}'f+O(r^{-3}).$$

[93]Note that the formula $\mathrm{tr}\,\chi=\big(1+O(r^{-2})\big)\mathrm{tr}\,\chi'+O(r^{-3})+O(r^{-2})\mathfrak{d}^{\leq 1}\Gamma_b$ is not good enough.

Consequently

$$\delta^{ab}\mathbf{g}(\mathbf{D}_{e'_a}e_4, e'_b) = \mathrm{tr}\,\chi' - \mathrm{div}'f + O(r^{-3}).$$

Hence

$$\begin{aligned} e_4\left(\int_{S(u,r)} h\right) &= \int_{S(u,r)} \Big(e_4(h) + \mathrm{tr}\,\chi' h - (\mathrm{div}'f)h\Big) + O(r^{-1})h \\ &= \int_{S(u,r)} \Big(e_4(h) + \mathrm{tr}\,\chi' h + f\nabla' h\Big) + O(r^{-1})h \end{aligned}$$

as stated. □

Using the above identity with the choice $h=1$ and $h = \mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}'$, we infer, using also (3.78), $f = O(r^{-1})$ and the gain in powers of r for $\nabla'(\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}')$,

$$\begin{aligned} e_4(|S|) &= \int_S \mathrm{tr}\,\chi' + O(r^{-1}) = \frac{2|S|}{r} + O(1), \\ e_4\left(\int_S \mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}'\right) &= \int_S \Big(e_4(\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}') + (\mathrm{tr}\,\chi')^2\mathrm{tr}\,\underline{\chi}'\Big) + O(r^{-3}). \end{aligned}$$

Plugging in the above identity for $e_4(m_H)$, we infer

$$e_4(m_H) = \frac{m_H}{r} + \frac{1}{32\pi}\sqrt{\frac{|S|}{4\pi}}\int_S \Big(e_4(\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}') + (\mathrm{tr}\,\chi')^2\mathrm{tr}\,\underline{\chi}'\Big) + O(r^{-2}).$$

Now, we use the following computation, see the proof of Lemma 5.51,

$$\begin{aligned} e'_4(\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}') &= -\mathrm{tr}\,\chi'^2\mathrm{tr}\,\underline{\chi}' + 2\mathrm{tr}\,\chi'\rho' + 2\mathrm{tr}\,\underline{\chi}'\mathrm{div}'\xi' + 2\mathrm{tr}\,\chi'\mathrm{div}'\underline{\eta}' \\ &\quad + \mathrm{tr}\,\underline{\chi}'\Big(2\xi'\cdot(\underline{\eta}' + 3\zeta') - |\widehat{\chi}'|^2\Big) + \mathrm{tr}\,\chi'\Big(2|\underline{\eta}'|^2 - \widehat{\chi}'\cdot\underline{\widehat{\chi}}'\Big). \end{aligned}$$

In view of (3.78), (3.79) and (3.80), this yields

$$e'_4(\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}') = -\mathrm{tr}\,\chi'^2\mathrm{tr}\,\underline{\chi}' + \frac{4}{r}\rho' - \frac{4}{r}\mathrm{div}'\xi' + \frac{4}{r}\mathrm{div}'\underline{\eta}' - \frac{2}{r}\widehat{\chi}'\cdot\underline{\widehat{\chi}}' + O(r^{-5}).$$

Also, since $\lambda = 1 + O(r^{-2})$ and $f = O(r^{-1})$, we have

$$\begin{aligned} e'_4(\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}') &= \lambda\left(e_4 + f\cdot\nabla + \frac{1}{4}|f|^2 e_3\right)(\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}') \\ &= e_4(\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}') + O(r^{-2})e_4(\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}') + O(r^{-1})\nabla(\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}') \end{aligned}$$

$$
\begin{aligned}
&+O(r^{-2})e_3(\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}')\\
=\;& e_4(\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}')+O(r^{-5})
\end{aligned}
$$

and hence

$$
e_4(\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}') = -\mathrm{tr}\,\chi'^2\mathrm{tr}\,\underline{\chi}'+\frac{4}{r}\rho'-\frac{4}{r}\mathrm{div}\,'\xi'+\frac{4}{r}\mathrm{div}\,'\underline{\eta}'-\frac{2}{r}\widehat{\chi}'\cdot\underline{\widehat{\chi}}'+O(r^{-5}).
$$

Integrating on S, and integrating the divergences by parts, we obtain

$$
\int_S\Big(e_4(\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}')+(\mathrm{tr}\,\chi')^2\mathrm{tr}\,\underline{\chi}'\Big) = \int_S\Big(\frac{4}{r}\rho'-\frac{2}{r}\widehat{\chi}'\cdot\underline{\widehat{\chi}}'\Big)+O(r^{-3}).
$$

We deduce from the above

$$
e_4(m_H) = \frac{m_H}{r}+\frac{1}{8\pi r}\sqrt{\frac{|S|}{4\pi}}\int_S\Big(\rho'-\frac{1}{2}\widehat{\chi}'\cdot\underline{\widehat{\chi}}'\Big)+O(r^{-2}).
$$

Now, in view of the Gauss equation, and using Gauss-Bonnet and the definition of the Hawking mass, we have

$$
\begin{aligned}
\int_S\Big(\rho'-\frac{1}{2}\widehat{\chi}'\cdot\underline{\widehat{\chi}}'\Big) &= \int_S\Big(-K-\frac{1}{4}\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}'\Big)=-4\pi-\frac{1}{4}\int_S\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}'\\
&= -\frac{8\pi m_H}{\sqrt{\frac{|S|}{4\pi}}}
\end{aligned}
$$

and hence

$$
|e_4(m_H)| \lesssim r^{-2}. \tag{3.83}
$$

Since r^{-2} is integrable, we infer the existence of a limit to m_H as $r\to+\infty$ along the leaves of the u-foliation of ${}^{(ext)}\mathcal{M}$

$$
M_B(u) = \lim_{r\to+\infty}m_H(u,r)\text{ for all }1\leq u<+\infty,
$$

where $M_B(u)$ is the so-called Bondi mass.

Next, using the equation for $\nabla_4\widehat{\chi}$ in Proposition 2.3 and that ${}^{(a)}\mathrm{tr}\chi'={}^{(a)}\mathrm{tr}\underline{\chi}'=0$ in the integrable frame (e_4',e_3',e_1',e_2'), we have the following null structure equation in ${}^{(ext)}\mathcal{M}$

$$
\nabla_4'\underline{\widehat{\chi}}' = -\frac{1}{2}(\mathrm{tr}\,\underline{\chi}'\widehat{\chi}'+\mathrm{tr}\,\chi'\underline{\widehat{\chi}}')+\nabla'\widehat{\otimes}\underline{\eta}'+2\omega'\underline{\widehat{\chi}}'+\xi'\widehat{\otimes}\underline{\xi}'+\underline{\eta}'\widehat{\otimes}\underline{\eta}'.
$$

In view of $\mathfrak{N}^{(Dec)}_{k_{small}} \lesssim \epsilon_0$ and (3.78) (3.79), we deduce

$$|\nabla_4(r\widehat{\underline{\chi}}')| \lesssim \frac{1}{r^2}.$$

Since r^{-2} is integrable, we infer the existence of a limit to $r\widehat{\underline{\chi}}'$ as $r \to +\infty$ along the leaves of the u-foliation of ${}^{(ext)}\mathcal{M}$

$$\underline{\Theta}(u,\cdot) = \lim_{r\to+\infty} r\widehat{\underline{\chi}}'(r,u,\cdot) \text{ for all } 1 \leq u < +\infty.$$

On the other hand, in view of $\mathfrak{N}^{(Dec)}_{k_{small}} \lesssim \epsilon_0$ and (3.78) again,

$$r|\widehat{\underline{\chi}}'| \lesssim \frac{\epsilon_0}{u^{1+\delta_{dec}}}, \qquad \text{on} \quad {}^{(ext)}\mathcal{M}.$$

We infer that

$$|\underline{\Theta}(u,\cdot)| \lesssim \frac{\epsilon_0}{u^{1+\delta_{dec}}} \text{ for all } 1 \leq u < +\infty.$$

3.8.2.4. A Bondi mass formula We use the following computation, see the proof of Lemma 5.51,

$$\begin{aligned} e_3'(\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}') = & -\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}'^2 + 2\mathrm{tr}\,\underline{\chi}'\rho' + 2\mathrm{tr}\,\chi'\mathrm{div}\,'\underline{\xi}' + 2\mathrm{tr}\,\underline{\chi}'\mathrm{div}\,'\eta' \\ & +\mathrm{tr}\,\chi'\left(2\underline{\xi}'\cdot(\eta'-3\zeta') - |\widehat{\underline{\chi}}'|^2\right) + \mathrm{tr}\,\underline{\chi}'\left(2|\eta'|^2 - \widehat{\underline{\chi}}'\cdot\widehat{\chi}'\right). \end{aligned}$$

Together with (3.78) (3.79) (3.80) and the estimates $\mathfrak{N}^{(Dec)}_{k_{small}} \lesssim \epsilon_0$, we deduce

$$\left| e_3'(m_H) + \frac{r}{32\pi}\int_S \mathrm{tr}\,\chi'|\widehat{\underline{\chi}}'|^2 \right| \lesssim \frac{1}{r}$$

and hence

$$\left| e_3'(m_H) + \frac{1}{4|S|}\int_S |r\widehat{\underline{\chi}}'|^2 \right| \lesssim \frac{1}{r}.$$

Letting $r \to +\infty$ along the leaves of the u-foliation of ${}^{(ext)}\mathcal{M}$, and using that the spheres at null infinity are round, we infer in view of the definition of M_B and $\underline{\Theta}$

$$e_3'(M_B)(u) = -\frac{1}{4}\int_{\mathbb{S}^2} |\underline{\Theta}|^2(u,\cdot) \text{ for all } 1 \leq u < +\infty.$$

Since

$$e_3'(u) = e_3(u) + O(r^{-1}) = 2 + \widecheck{e_3(u)} + O(r^{-1})$$

and e_3' is orthogonal to the spheres foliating $\mathcal{I}^+$, we infer $e_3' = (2 + \widecheck{e_3(u)})\partial_u$. Thus, we obtain the following Bondi mass type formula

$$\partial_u M_B(u) = -\frac{1}{8(1+\frac{1}{2}\widecheck{e_3(u)})}\int_{\mathbb{S}^2}\underline{\Theta}^2(u,\cdot) \text{ for all } 1 \leq u < +\infty, \tag{3.84}$$

with $\widecheck{e_3(u)}$ satisfying (3.76).

3.8.2.5. Final Bondi mass In view of the estimate

$$|\underline{\Theta}(u,\cdot)| \quad \lesssim \quad \frac{\epsilon_0}{u^{1+\delta_{dec}}} \text{ for all } 1 \leq u < +\infty,$$

and the control for $\widecheck{e_3(u)}$ in (3.76), we infer that

$$|\partial_u M_B(u)| \quad \lesssim \quad \frac{\epsilon_0^2}{u^{2+2\delta_{dec}}} \text{ for all } 1 \leq u < +\infty.$$

In particular, since $u^{-2-2\delta_{dec}}$ is integrable, the limit along $\mathcal{I}^+$ exists

$$M_B(+\infty) = \lim_{u\to+\infty} M_B(u)$$

and is the so-called final Bondi mass.

Also, recall (3.82)

$$m_H \quad = \quad m_\infty\Big(1 + O(r^{-1}) + O(\epsilon_0 u^{-\frac{1}{2}-\delta_{dec}})\Big).$$

Fixing u and letting $r \to +\infty$, we infer on $\mathcal{I}^+$, in view of the definition of the Bondi mass,

$$M_B(u) \quad = \quad m_\infty\Big(1 + O(\epsilon_0 u^{-\frac{1}{2}-\delta_{dec}})\Big).$$

Then, letting $u \to +\infty$ along $\mathcal{I}^+$, and in view of the definition of the final Bondi mass, we infer

$$m_\infty = M_B(+\infty),$$

i.e. the final mass m_∞ coincides with the final Bondi mass.

3.8.3. The final angular momentum a_∞ We exhibit below a geometric quantity converging to the final angular momentum a_∞ along $\mathcal{I}^+$ as $u \to +\infty$. Relying on the frame transformation formulas of Proposition 2.12 and the above control of $(f, \underline{f}, \lambda)$ in (3.77), we have

$$\mathrm{curl}'\beta' = \mathrm{curl}\,\beta - \frac{3m_\infty}{r^3}\mathrm{curl}\,(f) + r^{-3}\Gamma_g + O(r^{-6}).$$

Using again (3.77) for f, this yields

$$\mathrm{curl}'\beta' = \mathrm{curl}\,\beta - \frac{3m_\infty}{r^3}\mathrm{curl}\,(\nabla u) + r^{-3}\Gamma_g + O(r^{-6}).$$

We have

$$\begin{aligned}\mathrm{curl}\,(\nabla u) &= \in_{ab}\nabla_a\nabla_b u = \in_{ab}\left(\mathbf{D}_a\mathbf{D}_b u - \frac{1}{2}\chi_{ab}e_3(u) - \frac{1}{2}\underline{\chi}_{ab}e_4(u)\right)\\ &= -\frac{1}{2}{}^{(a)}\mathrm{tr}\chi e_3(u) - \frac{1}{2}{}^{(a)}\mathrm{tr}\underline{\chi}e_4(u)\end{aligned}$$

which together with the fact that $e_4(u) = 0$ implies

$$\mathrm{curl}'\beta' = \mathrm{curl}\,\beta + \frac{3m_\infty}{2r^3}{}^{(a)}\mathrm{tr}\chi e_3(u) + r^{-3}\Gamma_g + O(r^{-6}).$$

Recall, in view of Definition 2.66 and the definitions of Γ_g, Γ_b,

$$\begin{aligned}\mathrm{tr}X &= \mathrm{tr}\,\chi - i\,{}^{(a)}\mathrm{tr}\chi = \frac{2}{q} + \Gamma_g = \frac{2(r - i\cos\theta)}{r^2 + a^2} + \Gamma_g,\\ e_3(u) &= \frac{2(r^2 + a^2)}{|q|^2} + r\Gamma_b.\end{aligned}$$

Hence

$$\mathrm{curl}'\beta' = \mathrm{curl}\,\beta + \frac{6a_\infty m_\infty\cos\theta}{r^5} + r^{-3}\Gamma_g + O(r^{-6}).$$

For a scalar function h on ${}^{(ext)}\mathcal{M}$, we introduce, in Section 2.6.1,

$$(h)_{\ell=1} := \frac{1}{|S|}\left(\int_S J^{(0)}, \int_S J^{(+)}, \int_S J^{(-)}\right),$$

where

$$J^{(0)} = \cos\theta, \qquad J^{(+)} = \sin\theta\cos\varphi, \qquad J^{(-)} = \sin\theta\sin\varphi.$$

Now, in view of Lemma 6.34, we have on ${}^{(ext)}\mathcal{M}$

$$\int_S J^{(p)} J^{(q)} \;=\; \frac{4\pi}{3} r^2 \delta_{pq} + O\left(1 + \epsilon_0 r u^{-\frac{1}{2}-\delta_{dec}}\right),$$

from which we infer

$$(3.85)\qquad \begin{aligned} r^5(\text{curl}\,'\beta')_{\ell=1,0} &= r^5(\text{curl}\,\beta)_{\ell=1,0} + 2a_\infty m_\infty + r^2\Gamma_g + O(r^{-1}),\\ r^5(\text{curl}\,'\beta')_{\ell=1,\pm} &= r^5(\text{curl}\,\beta)_{\ell=1,\pm} + r^2\Gamma_g + O(r^{-1}). \end{aligned}$$

We next appeal to the first identity (6.38) of Proposition 6.43 according to which, in view of the estimate $\mathfrak{N}^{(Dec)}_{k_{small}} \lesssim \epsilon_0$ of the main theorem,

$$\nabla_4\left(\int_{S(u,r)} \frac{rJ^{(0)}}{\Sigma}[\overline{D}]_{ren}\left(r^4[B]_{ren}\right)\right) = O\left(\frac{\epsilon_0}{r^{\frac{3}{2}}u^{\frac{1}{2}+\delta_{dec}}}\right)$$

and

$$\begin{aligned} &\nabla_4\left(\int_{S(u,r)} \frac{rJ^{(\pm)}}{\Sigma}[\overline{D}]_{ren}\left(r^4[B]_{ren}\right)\right)\\ &\mp\frac{a}{r^2}\int_{S(u,r)} \frac{rJ^{(\mp)}}{\Sigma}[\overline{D}]_{ren}\left(r^4[B]_{ren}\right) \;=\; O\left(\frac{\epsilon_0}{r^{\frac{3}{2}}u^{\frac{1}{2}+\delta_{dec}}}\right), \end{aligned}$$

where

$$[B]_{ren} := B - \frac{3a}{2}\overline{\check{P}}\mathfrak{J} - \frac{a}{4}\overline{\mathfrak{J}}\cdot A, \qquad [\overline{D}]_{ren} := \overline{\mathcal{D}}\cdot - \frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_4 - \frac{a}{2}\overline{\mathfrak{J}},$$

are renormalized quantities. Recalling that $\Sigma^2 = (r^2+a^2)|q|^2 + 2mra^2(\sin\theta)^2$, in (2.59), and the estimates for A, B, P and Γ_g provided by the main theorem, we deduce

$$\nabla_4\left(r^3\int_S\left(\overline{\mathcal{D}}\cdot B + r^{-3}\mathfrak{d}^{\le 1}\Gamma_g\right)J^{(p)}\right) \;=\; O\left(\frac{\epsilon_0}{r^{\frac{3}{2}}u^{\frac{1}{2}+\delta_{dec}}}\right).$$

Integrating forward from $\mathcal{T}$, using again $\mathfrak{N}^{(Dec)}_{k_{small}} \lesssim \epsilon_0$, we obtain

$$\sup_{{}^{(ext)}\mathcal{M}} u^{\frac{1}{2}+\delta_{dec}} r^3\left|\int_S(\overline{\mathcal{D}}\cdot B)J^{(p)}\right| \;\lesssim\; \epsilon_0.$$

Since $\overline{\mathcal{D}}\cdot B = (\nabla - i\,{}^*\nabla)\cdot(\beta + i\,{}^*\beta) = 2\text{div}\,\beta + 2i\text{curl}\,\beta$, this yields

$$\sup_{{}^{(ext)}\mathcal{M}} u^{\frac{1}{2}+\delta_{dec}} r^5|(\text{curl}\,\beta)_{\ell=1}| \;\lesssim\; \epsilon_0$$

and hence, back to (3.85),

$$
\begin{aligned}
r^5(\mathrm{curl}\,'\beta')_{\ell=1,0} &= 2a_\infty m_\infty + O\left(\frac{1}{r} + \frac{\epsilon_0}{u^{\frac{1}{2}+\delta_{dec}}}\right),\\
r^5(\mathrm{curl}\,'\beta')_{\ell=1,\pm} &= O\left(\frac{1}{r} + \frac{\epsilon_0}{u^{\frac{1}{2}+\delta_{dec}}}\right).
\end{aligned}
\tag{3.86}
$$

To derive a limit for $r^5(\mathrm{curl}\,'\beta')_{\ell=1,0}$ as $r\to\infty$, at constant u, we need to estimate the quantity $e_4(r^5(\mathrm{curl}\,'\beta')_{\ell=1})$. We start with the following Bianchi identity for the integrable frame, in view of Proposition 2.3 with ${}^{(a)}\mathrm{tr}\chi = 0$,

$$
\nabla_4'\beta' + (2\mathrm{tr}\,\chi' + 2\omega')\beta' \;=\; \mathrm{div}\,'\alpha' + \alpha'\cdot(2\zeta' + \underline{\eta}') + 3\rho'\xi' + 3\ {}^*\rho'\ {}^*\xi'.
$$

Differentiating with $\mathrm{curl}\,'$, this yields

$$
\begin{aligned}
& e_4'(\mathrm{curl}\,'\beta') - [\nabla_4', \mathrm{curl}\,']\beta' + 2\mathrm{tr}\,\chi'\mathrm{curl}\,'\beta'\\
=\;& \mathrm{curl}\,'\mathrm{div}\,'\alpha' - 2\nabla'\mathrm{tr}\,\chi'\cdot\ {}^*\beta' - \mathrm{curl}\,'(\omega'\beta')\\
& +\mathrm{curl}\,'\Big(\alpha'\cdot(2\zeta' + \underline{\eta}') + 3\rho'\xi'\Big).
\end{aligned}
$$

By standard commutation formulas (see [28] or [17])

$$
\begin{aligned}
[\nabla_4', \mathrm{curl}\,']\beta' \;=\;& -\frac{1}{2}\mathrm{tr}\,\chi'\mathrm{curl}\,'\beta' - \widehat{\chi}'\cdot\nabla'\ {}^*\beta' + (\underline{\eta}' + \zeta')\cdot\nabla_4'\ {}^*\beta'\\
& +\xi'\cdot\nabla_3'\ {}^*\beta' + \xi'\cdot\underline{\chi}'\cdot\beta' + \underline{\eta}'\cdot\chi'\cdot\beta',
\end{aligned}
$$

we infer, together with (3.78) (3.79) (3.80),

$$
\begin{aligned}
e_4'(\mathrm{curl}\,'\beta') + \frac{5}{2}\mathrm{tr}\,\chi'\mathrm{curl}\,'\beta' \;=\;& \mathrm{curl}\,'\mathrm{div}\,'\alpha' + \xi'\cdot\nabla_3'\ {}^*\beta' + 3\mathrm{curl}\,'(\rho'\xi')\\
& +O\Big(r^{-7} + \epsilon_0 r^{-\frac{13}{2}}\Big).
\end{aligned}
\tag{3.87}
$$

The estimate for $\xi' = O\big(r^{-3}\big) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b$ provided by (3.79) is not enough to derive a suitable estimate for the term involving $3\mathrm{curl}\,'\big(\rho'\xi'\big)$ on the RHS of (3.87). To derive a sharper estimate for ξ', we use the frame transformation formula of Proposition 2.12 for ξ' and the control (3.77) for $(f, \underline{f}, \lambda)$. We obtain

$$
\begin{aligned}
\xi' \;&=\; \frac{1}{2}\nabla_4 f + \frac{1}{4}\mathrm{tr}\,\chi f + O(r^{-3}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b\\
&=\; \frac{1}{2r}\nabla_4(rf) + O(r^{-3}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b
\end{aligned}
$$

where we have kept the explicit form of the a priori problematic term. We now show that $\nabla_4(rf)$ behaves better than expected from (3.77). In view of the explicit formula for f

$$f = -\frac{4}{e_3(u)+\sqrt{(e_3(u))^2+4|\nabla u|^2 e_3(r)}}\nabla u$$

in (2.12), we have[94]

$$\begin{aligned}\nabla_4(rf) &= -4\big(1+r\Gamma_b+O(r^{-2})\big)\Big(\nabla_4(r\nabla u)-2e_4(e_3(u))r\nabla u\Big)+O(r^{-4})\\ &\quad +r^{-3}\Gamma_b.\end{aligned}$$

The desired gain comes from the following transport equations in e_4 of Proposition 2.21

$$\nabla_4\mathcal{D}u+\frac{1}{2}\mathrm{tr}X\mathcal{D}u = -\frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}u, \qquad e_4(e_3(u)) = -\Re\Big((Z+H)\cdot\overline{\mathcal{D}}u\Big),$$

from which

$$\nabla_4(r\nabla u) = O(r^{-2})+\Gamma_g, \qquad e_4(e_3(u)) = O(r^{-2})+r^{-1}\Gamma_b.$$

Therefore

$$\nabla_4(rf) = O(r^{-2})+\Gamma_g,$$

i.e. $\nabla_4(rf)$ behaves indeed better than expected from (3.77). Plugging in the above formula for ξ', we obtain the following improvement of (3.79) for ξ'

$$\xi' = O(r^{-3})+r^{-1}\Gamma_g+r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b. \tag{3.88}$$

Plugging in (3.87), and using (3.80) for ρ', and the Bianchi identity for $\nabla_3'\beta'$ (see Proposition 2.3 with ${}^{(a)}\mathrm{tr}\chi=0$), we deduce

$$e_4'(\mathrm{curl}'\beta')+\frac{5}{2}\mathrm{tr}\,\chi'\mathrm{curl}'\beta' = \mathrm{curl}'\mathrm{div}'\alpha'+O\Big(r^{-7}+\epsilon_0 r^{-\frac{13}{2}}\Big).$$

On the other hand, using the fact that $f=\Gamma_b+O(r^{-1})$ and $\lambda = 1+O(r^{-2})+r^{-1}\Gamma_b$ in view of (3.77), we have

$$e_4'(r^3)-\frac{3}{2}r^3\mathrm{tr}\,\chi' = 3r^2\lambda\left(e_4+f\cdot\nabla+\frac{1}{4}|f|^2e_3\right)r-\frac{3}{2}r^3\mathrm{tr}\,\chi'$$

[94] Note that $e_3(u)=2+r\Gamma_b$, $e_3(r)=-1+r\Gamma_b+O(r^{-2})$ and $\nabla u=\Gamma_b+O(r^{-1})$.

$$
\begin{aligned}
&= 3r^2\lambda - \frac{3}{2}r^3 \mathrm{tr}\,\chi' + O(1) + r\Gamma_b = -\frac{3}{2}r^3\left(\mathrm{tr}\,\chi' - \frac{2}{r}\right)\\
&\quad + O(1) + r\Gamma_b\\
&= r^3\mathfrak{d}^{\leq 1}\Gamma_g + O(1) + r\Gamma_b,
\end{aligned}
$$

where we used (3.78) for $\mathrm{tr}\,\chi'$. Using also the control for β' in (3.80), i.e. $\beta' = \beta + O(r^{-4}) + r^{-2}\Gamma_g$, we deduce

$$
e'_4(r^3\mathrm{curl}\,'\beta') + \mathrm{tr}\,\chi' r^3\mathrm{curl}\,'\beta' = r^3\mathrm{curl}\,'\mathrm{div}\,'\alpha' + O\Big(r^{-4} + \epsilon_0 r^{-\frac{7}{2}}\Big). \tag{3.89}
$$

Next we appeal to Lemma 3.40

$$
e_4\left(\int_{S(u,r)} h\right) = \int_{S(u,r)}\Big(e_4(h) + \mathrm{tr}\,\chi' h + f\cdot\nabla' h\Big) + O(r^{-1})h.
$$

Since $e_4(J^{(p)}) = 0$ for $p = 0, +, -$, we infer

$$
\begin{aligned}
e_4\left(\int_S r^3\mathrm{curl}\,'\beta' J^{(p)}\right) &= \int_S\Big(e_4(r^3\mathrm{curl}\,'\beta') + \mathrm{tr}\,\chi' r^3\mathrm{curl}\,'\beta'\Big)J^{(p)}\\
&\quad + O(r^3)\,\not{\mathfrak{d}}'^{\leq 1}\mathrm{curl}\,'\beta'.
\end{aligned}
$$

Since

$$
\begin{aligned}
e'_4(r^3\mathrm{curl}\,'\beta') &= \lambda\left(e_4 + f\cdot\nabla + \frac{1}{4}|f|^2\right)(r^3\mathrm{curl}\,'\beta') = e_4(r^3\mathrm{curl}\,'\beta')\\
&\quad + O(r)\mathfrak{d}^{\leq 1}\mathrm{curl}\,'\beta',
\end{aligned}
$$

we deduce

$$
\begin{aligned}
e_4\left(\int_S r^3\mathrm{curl}\,'\beta' J^{(p)}\right) &= \int_S e'_4(r^3\mathrm{curl}\,'\beta') + \mathrm{tr}\,\chi'(r^3\mathrm{curl}\,'\beta')\\
&\quad + O(r^3)\mathfrak{d}^{\leq 1}\mathrm{curl}\,'\beta'.
\end{aligned}
$$

Thus, in view of (3.89),

$$
\begin{aligned}
e_4\left(\int_S r^3\mathrm{curl}\,'\beta' J^{(p)}\right) &= r^3\int_S \mathrm{curl}\,'\mathrm{div}\,'\alpha' J^{(p)} + O(r^3)\mathfrak{d}^{\leq 1}\mathrm{curl}\,'\beta'\\
&\quad + O\Big(r^{-2} + \epsilon_0 r^{-\frac{3}{2}}\Big).
\end{aligned}
$$

Together with the control for $\beta' = \beta + O(r^{-4}) + r^{-2}\Gamma_g$ in (3.80) and recalling the definition of $\not{d}_2$, $\not{d}_1$

$$e_4\left(\int_S r^3 \text{curl}'\beta' J^{(p)}\right) \quad = \quad r^3\int_S \not{d}'_1\,\not{d}'_2\alpha' \cdot (0, J^{(p)}) + O\Big(r^{-2} + \epsilon_0 r^{-\frac{3}{2}}\Big).$$

Integrating by parts we deduce

$$e_4\left(\int_S r^3 \text{curl}'\beta' J^{(p)}\right) \quad = \quad r^3\int_S \alpha' \cdot \not{d}'^*_2\not{d}'^*_1(0, J^{(p)}) + O\Big(r^{-2} + \epsilon_0 r^{-\frac{3}{2}}\Big).$$

Now, according to Proposition 6.35, we have

$$|\not{d}'^*_2\not{d}'^*_1(0, J^{(p)})| \lesssim \frac{\epsilon_0}{r^3 u^{\frac{1}{2}+\delta_{dec}}} + \frac{1}{r^4}.$$

Together with the control for α' in (3.80) we deduce

$$e_4\left(\int_S r^3 \text{curl}'\beta' J^{(p)}\right) \quad = \quad O\Big(r^{-2} + \epsilon_0 r^{-\frac{3}{2}}\Big).$$

Together with the definition of $(\text{curl}'\beta')_{\ell=1}$, we finally obtain the following control for $e_4((\text{curl}'\beta')_{\ell=1})$

$$e_4\left((\text{curl}'\beta')_{\ell=1}\right) \quad = \quad O\Big(r^{-2} + \epsilon_0 r^{-\frac{3}{2}}\Big).$$

Since $r^{-3/2}$ is integrable, we infer from the above control of $e_4((\text{curl}'\beta')_{\ell=1})$ the existence of a limit to $(\text{curl}'\beta')_{\ell=1}$ as $r \to +\infty$ along the leaves of the u-foliation of ${}^{(ext)}\mathcal{M}$, for all $1 \le u < +\infty$,

$$\mathcal{J}_{\ell=1,p}(u) := \lim_{r\to+\infty}(\text{curl}'\beta')_{\ell=1,p}(u, r), \qquad p = 0, +, -. \tag{3.90}$$

Fixing u and letting $r \to +\infty$ in (3.86), we infer on $\mathcal{I}^+$, in view of the definition of $\mathcal{J}_{\ell=1,p}(u)$,

$$\begin{aligned}\mathcal{J}_{\ell=1,0}(u) &= 2a_\infty m_\infty + O\left(\frac{\epsilon_0}{u^{\frac{1}{2}+\delta_{dec}}}\right),\\ \mathcal{J}_{\ell=1,\pm}(u) &= O\left(\frac{\epsilon_0}{u^{\frac{1}{2}+\delta_{dec}}}\right).\end{aligned}$$

Then, letting $u \to +\infty$ along $\mathcal{I}^+$, we deduce that $\mathcal{J}_{\ell=1,p}(u)$ admit limits, which we denote by $\mathcal{J}_{\ell=1,p}(+\infty)$, i.e.

$$\mathcal{J}_{\ell=1,p}(+\infty) := \lim_{u\to+\infty}\mathcal{J}_{1,p}(u), \qquad p = 0, +, -,$$

and that these limits satisfy

$$a_\infty = \frac{1}{2m_\infty}\mathcal{J}_{\ell=1,0}(+\infty), \qquad \mathcal{J}_{\ell=1,\pm}(+\infty) \;\; = 0.$$

3.8.4. Other conclusions

3.8.4.1. Coordinates systems on ${}^{(ext)}\mathcal{M}$ and ${}^{(int)}\mathcal{M}$ In view of Proposition 4.1, ${}^{(ext)}\mathcal{M}$ is covered by three regular coordinates patches:

- in the (u, r, θ, φ) coordinates system, we have, for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$,

$$\mathbf{g} \;\; = \;\; \mathbf{g}_{a_\infty, m_\infty} + \Big(du, dr, rd\theta, r\sin\theta d\varphi\Big)^2 O\left(\frac{\epsilon_0}{u^{1+\delta_{dec}}}\right),$$

- in the (u, r, x^1, x^2) coordinates system, with $x^1 = J^{(+)}$ and $x^2 = J^{(-)}$, we have, for $0 \leq \theta < \frac{\pi}{3}$ and for $\frac{2\pi}{3} < \theta \leq \pi$,

$$\mathbf{g} \;\; = \;\; \mathbf{g}_{a_\infty, m_\infty} + \Big(du, dr, rdx^1, rdx^2\Big)^2 O\left(\frac{\epsilon_0}{u^{1+\delta_{dec}}}\right),$$

where in each case, $\mathbf{g}_{a_\infty, m_\infty}$ denotes the Kerr metric expressed in the corresponding coordinates system of Kerr, see Lemma 2.35 and Lemma 2.49.

Also, in view of Proposition 4.2, ${}^{(int)}\mathcal{M}$ is covered by three regular coordinates patches:

- in the $(\underline{u}, r, \theta, \varphi)$ coordinates system, we have, for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$,

$$\mathbf{g} \;\; = \;\; \mathbf{g}_{a_\infty, m_\infty} + \Big(d\underline{u}, dr, rd\theta, r\sin\theta d\varphi\Big)^2 O\left(\frac{\epsilon_0}{\underline{u}^{1+\delta_{dec}}}\right),$$

- in the $(\underline{u}, r, x^1, x^2)$ coordinates system, with $x^1 = J^{(+)}$ and $x^2 = J^{(-)}$, we have, for $0 \leq \theta < \frac{\pi}{3}$ and for $\frac{2\pi}{3} < \theta \leq \pi$,

$$\mathbf{g} \;\; = \;\; \mathbf{g}_{a_\infty, m_\infty} + \Big(d\underline{u}, dr, rdx^1, rdx^2\Big)^2 O\left(\frac{\epsilon_0}{\underline{u}^{1+\delta_{dec}}}\right),$$

where in each case, $\mathbf{g}_{a_\infty, m_\infty}$ denotes the Kerr metric expressed in the corresponding coordinates system of Kerr, i.e. the analog for ingoing PG structures of Lemma 2.35 and Lemma 2.49.

3.8.4.2. Asymptotic of the future event horizon Let

$$r_{\pm,\infty} \quad := \quad m_\infty \pm \sqrt{m_\infty^2 - a_\infty^2}.$$

We show below that $\mathcal{H}_+$ is located in the following region of ${}^{(int)}\mathcal{M}$

$$r_{+,\infty}\left(1-\frac{\sqrt{\epsilon_0}}{\underline{u}^{1+\delta_{dec}}}\right) \leq r \leq r_{+,\infty}\left(1+\frac{\sqrt{\epsilon_0}}{\underline{u}^{1+\frac{\delta_{dec}}{2}}}\right) \tag{3.91}$$
$$\text{on } \mathcal{H}_+ \text{ for any } 1 \leq \underline{u} < +\infty.$$

We consider first the lower bound. Let us denote by (e_4, e_3, e_1, e_2) and $(\underline{u}, r)$ the null frame and scalar functions associated to our ingoing PG structure of ${}^{(int)}\mathcal{M}$. The estimates $\mathfrak{N}^{(Dec)}_{k_{small}} \lesssim \epsilon_0$ imply

$$\sup_{{}^{(int)}\mathcal{M}} \underline{u}^{1+\delta_{dec}}\left(|e_3(r)+1| + \left|e_4(r) - \frac{\Delta}{|q|^2}\right|\right) \quad \lesssim \quad \epsilon_0.$$

In particular, we have for all $r \leq r_{+,\infty}\left(1-\frac{\sqrt{\epsilon_0}}{\underline{u}^{1+\delta_{dec}}}\right)$

$$\begin{aligned}
\Delta &= r^2 - 2m_\infty r + a_\infty^2 = (r - r_{+,\infty})(r - r_{-,\infty})\\
&\leq -\frac{\sqrt{\epsilon_0}}{2\underline{u}^{1+\delta_{dec}}} r_{+,\infty}\left(r_{+,\infty} - r_{-,\infty} - \frac{\sqrt{\epsilon_0}}{2\underline{u}^{1+\delta_{dec}}}\right)\\
&\lesssim -\frac{\sqrt{\epsilon_0}}{\underline{u}^{1+\delta_{dec}}} < 0
\end{aligned}$$

and hence

$$\begin{aligned}
&e_3(r) \leq -\frac{1}{2} < 0, \qquad e_4(r) \lesssim -\frac{\sqrt{\epsilon_0}}{\underline{u}^{1+\delta_{dec}}} < 0\\
&\text{on} \quad {}^{(int)}\mathcal{M}\left(r \leq r_{+,\infty}\left(1 - \frac{\sqrt{\epsilon_0}}{\underline{u}^{1+\delta_{dec}}}\right)\right).
\end{aligned}$$

Consider now $\gamma(s)$ any future directed null geodesic emanating from a point of the region ${}^{(int)}\mathcal{M}\left(r \leq r_{+,\infty}\left(1-\frac{\sqrt{\epsilon_0}}{\underline{u}^{1+\delta_{dec}}}\right)\right)$. $\dot{\gamma}$ being a null vector, there exists at any point of $\gamma(s)$ in ${}^{(int)}\mathcal{M}$ a scalar λ and a 1-form f such that

$$\dot{\gamma} \quad = \quad \lambda\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right),$$

where $\lambda > 0$ since $\dot{\gamma}$ is future directed. Since $\nabla(r) = 0$, we infer

$$\frac{dr}{ds} = \mathbf{D}_{\dot{\gamma}} r \quad = \quad \lambda\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right) r = \lambda\left(e_4(r) + \frac{1}{4}|f|^2 e_3(r)\right).$$

Since $e_3(r)<0$ and $e_4(r)<0$ in ${}^{(int)}\mathcal{M}\left(r\leq r_{+,\infty}\left(1-\frac{\sqrt{\epsilon_0}}{\underline{u}^{1+\delta_{dec}}}\right)\right)$ in view of the above, and since $\lambda>0$ and $|f|^2\geq 0$, we deduce that r decreases along $\gamma(s)$ so that $\gamma(s)$ either stays in ${}^{(int)}\mathcal{M}\left(r\leq r_{+,\infty}\left(1-\frac{\sqrt{\epsilon_0}}{\underline{u}^{1+\delta_{dec}}}\right)\right)$ for all s, or exits $\mathcal{M}$ through $r=r_+(1-\delta_{\mathcal{H}})$. Thus, ${}^{(int)}\mathcal{M}\left(r\leq r_{+,\infty}\left(1-\frac{\sqrt{\epsilon_0}}{\underline{u}^{1+\delta_{dec}}}\right)\right)$ lies strictly inside the black hole and hence, $\mathcal{H}_+$ must lie in $r\geq r_{+,\infty}\left(1-\frac{\sqrt{\epsilon_0}}{\underline{u}^{1+\delta_{dec}}}\right)$ which concludes the lower bound.

Next, we focus on proving the upper bound. We need to show that any 2-sphere

$$S(\underline{u}_1):=S\left(\underline{u}_1, r=r_{+,\infty}\left(1+\frac{\sqrt{\epsilon_0}}{\underline{u}_1^{1+\frac{\delta_{dec}}{2}}}\right)\right),\quad 1\leq\underline{u}_1<+\infty, \tag{3.92}$$

is in the past of $\mathcal{I}^+$. Since ${}^{(ext)}\mathcal{M}$ lies in the past of $\mathcal{I}^+$, is suffices to show that from any point of $S(\underline{u}_1)$ there exists a future directed null geodesic reaching ${}^{(ext)}\mathcal{M}$ in finite time. We will in fact show that the future directed null geodesics from $S(\underline{u}_1)$ with initial speed e_4 reach ${}^{(ext)}\mathcal{M}$ in finite time. Assume, by contradiction, that there exists a null geodesic from $S(\underline{u}_1)$ with initial speed e_4, denoted by γ, that does not reach ${}^{(ext)}\mathcal{M}$ in finite time. Let e_4' be the geodesic generator of γ. In view of Lemma 2.10 on general null frame transformation, and denoting by (e_4,e_3,e_1,e_2) the null frame[95] of ${}^{(int)}\mathcal{M}$, we look for e_4' under the form

$$e_4' \quad = \quad \lambda\left(e_4+f^ae_a+\frac{1}{4}|f|^2e_3\right).$$

Also, let

$$F:=f+i\,{}^*f.$$

Then, the fact that e_4' is geodesic implies the following transport equations along γ for F and λ in view of Corollary 2.14

$$\begin{aligned}\nabla_{\lambda^{-1}e_4'}F+\frac{1}{2}\mathrm{tr}XF+2\omega F &= -2\Xi-\widehat{\chi}\cdot F+E_1(f,\Gamma),\\ \lambda^{-1}\nabla_4'(\log\lambda) &= 2\omega+f\cdot(\zeta-\underline{\eta})+E_2(f,\Gamma),\end{aligned}$$

[95]Recall that we assume by contradiction that γ does not reach ${}^{(ext)}\mathcal{M}$ and hence stays in ${}^{(int)}\mathcal{M}$.

where $E_1(f,\Gamma)$ and $E_2(f,\Gamma)$ contain expressions of the type $O(\Gamma f^2)$ with no derivatives and Γ denotes the Ricci coefficients w.r.t. the original null frame (e_3,e_4,e_1,e_2) of ${}^{(int)}\mathcal{M}$.

We then proceed as follows

1. First, since e_4 is the initial speed of γ on $S(\underline{u}_1)$, f and λ satisfy
$$f=0,\quad \lambda=1 \text{ on } \gamma\cap S(\underline{u}_1).$$

2. Then, we initiate a continuity argument by assuming for some
$$\underline{u}_1<\underline{u}_2<\underline{u}_1+\left(\frac{\underline{u}_1}{\epsilon_0}\right)^{\frac{\delta_{dec}}{2}}$$
that we have
$$\begin{aligned}|f|\leq\frac{\sqrt{\epsilon_0}}{\underline{u}_1^{\frac{1}{2}+\delta_{dec}}},\quad \frac{\Delta}{|q|^2}\geq\frac{c_\infty\sqrt{\epsilon_0}}{2\underline{u}_1^{1+\frac{\delta_{dec}}{2}}},\\ 0<\lambda<+\infty \text{ on } \gamma(\underline{u}_1,\underline{u}_2)\cap {}^{(int)}\mathcal{M}\end{aligned}\tag{3.93}$$
where $\gamma(\underline{u}_1,\underline{u}_2)$ denotes the portion of γ in $\underline{u}_1\leq\underline{u}\leq\underline{u}_2$ and where the strictly positive constant c_∞ is given by
$$c_\infty:=\frac{r_{+,\infty}(r_{+,\infty}-r_{-,\infty})}{(r_{+,\infty})^2+a_\infty^2}.$$

3. We have
$$\begin{aligned}\lambda^{-1}e_4'(\underline{u}) &= e_4(\underline{u})+f\cdot\nabla(\underline{u})+\frac{1}{4}|f|^2e_3(\underline{u})\\ &= \frac{2(r^2+a^2)}{|q|^2}+\widetilde{e_4(\underline{u})}+f\cdot\nabla(\underline{u}).\end{aligned}$$
Relying on our control of the ingoing geodesic foliation of ${}^{(int)}\mathcal{M}$, the above assumption for f and the transport equation for F, we obtain on $\gamma(\underline{u}_1,\underline{u}_2)\cap {}^{(int)}\mathcal{M}$
$$\begin{aligned}\sup_{\gamma(\underline{u}_1,\underline{u}_2)\cap {}^{(int)}\mathcal{M}}|f| &\lesssim \frac{\epsilon_0}{\underline{u}_1^{1+\delta_{dec}}}(\underline{u}_2-\underline{u}_1)\\ &\lesssim \frac{\epsilon_0^{1-\frac{\delta_{dec}}{2}}}{\underline{u}_1^{1+\frac{\delta_{dec}}{2}}}\end{aligned}$$

which improves our assumption in (3.93) on f.

4. We have in view of the control of f

$$\begin{aligned}\lambda^{-1}e_4'(r) &= e_4(r)+\frac{1}{4}|f|^2e_3(r) = \frac{\Delta}{|q|^2}+O\left(\frac{\epsilon_0}{\underline{u}_1^{1+\delta_{dec}}}\right),\\ \lambda^{-1}e_4'(\cos\theta) &= e_4(\cos\theta)+f\cdot\nabla(\cos\theta) = O\left(\frac{\sqrt{\epsilon_0}}{\underline{u}_1^{\frac{1}{2}+\delta_{dec}}}\right).\end{aligned}$$

This yields

$$\begin{aligned}&\lambda^{-1}e_4'\left(\log\left(\frac{\Delta}{|q|^2}\right)\right)\\ &= \lambda^{-1}e_4'(r)\partial_r\log\left(\frac{\Delta}{|q|^2}\right)-\lambda^{-1}e_4'(\cos\theta)\partial_{\cos\theta}\log(|q|^2)\\ &= \partial_r\left(\frac{\Delta}{|q|^2}\right)+\frac{|q|^2}{\Delta}O\left(\frac{\epsilon_0}{\underline{u}_1^{1+\delta_{dec}}}\right)+O\left(\frac{\sqrt{\epsilon_0}}{\underline{u}_1^{\frac{1}{2}+\delta_{dec}}}\right).\end{aligned}$$

Thanks to our assumption on the lower bound of $\frac{\Delta}{|q|^2}$, we infer

$$\lambda^{-1}e_4'\left(\log\left(\frac{\Delta}{|q|^2}\right)\right) = \partial_r\left(\frac{\Delta}{|q|^2}\right)(1+O(\sqrt{\epsilon_0}))$$

and since we are in ${}^{(int)}\mathcal{M}$, $r\geq r_0$ and hence

$$\lambda^{-1}e_4'\left(\log\left(\frac{\Delta}{|q|^2}\right)\right) \geq \frac{m_0}{2r_0^2}.$$

Integrating from $\underline{u}=\underline{u}_1$, we deduce

$$\frac{\Delta}{|q|^2} \geq (1+O(\sqrt{\epsilon_0}))\frac{c_\infty\sqrt{\epsilon_0}}{\underline{u}_1^{1+\frac{\delta_{dec}}{2}}}\exp\left(\frac{m_0}{2r_0^2}(\underline{u}-\underline{u}_1)\right)$$

which is an improvement of our assumption in (3.93) on $\frac{\Delta}{|q|^2}$.

5. In view of the control of f and of the ingoing geodesic foliation of ${}^{(int)}\mathcal{M}$, we rewrite the transport equation for λ as

$$\begin{aligned}\lambda^{-1}\nabla_4'(\log\lambda) &= 2\omega+f\cdot(\zeta-\underline{\eta})+E_2(f,\Gamma)\\ &= -\partial_r\left(\frac{\Delta}{|q|^2}\right)+O\left(\frac{\epsilon_0^{1-\frac{\delta_{dec}}{2}}}{\underline{u}_1^{1+\frac{\delta_{dec}}{2}}}\right).\end{aligned}$$

Since we have obtained above the other hand

$$\lambda^{-1}e_4'\left(\log\left(\frac{\Delta}{|q|^2}\right)\right) \quad = \quad \partial_r\left(\frac{\Delta}{|q|^2}\right)(1+O(\sqrt{\epsilon_0}))$$

we immediately infer

$$\lambda^{-1}e_4'\left(\log\left(\lambda\left(\frac{\Delta}{|q|^2}\right)^2\right)\right) > 0, \quad \lambda^{-1}e_4'\left(\log\left(\lambda\sqrt{\frac{\Delta}{|q|^2}}\right)\right) < 0.$$

Integrating from $\underline{u} = \underline{u}_1$, this yields

$$\begin{aligned}&\left((1+O(\sqrt{\epsilon_0}))\frac{c_\infty\sqrt{\epsilon_0}}{\underline{u}_1^{1+\frac{\delta_{dec}}{2}}}\right)^2\left(\frac{\Delta}{|q|^2}\right)^{-2}\\ \leq \quad & \lambda \leq \left((1+O(\sqrt{\epsilon_0}))\frac{c_\infty\sqrt{\epsilon_0}}{\underline{u}_1^{1+\frac{\delta_{dec}}{2}}}\right)^{\frac{1}{2}}\left(\frac{\Delta}{|q|^2}\right)^{-\frac{1}{2}}.\end{aligned}$$

Since $\frac{\Delta}{|q|^2}$ has an explicit lower bounded in view of our previous estimate, as well as an explicit upper bound since we are in ${}^{(int)}\mathcal{M}$, this yields an improvement of our assumptions in (3.93) for λ.

6. Since we have improved all our bootstrap assumptions (3.93), we infer by a continuity argument the following bound

$$\begin{aligned}&\frac{\Delta}{|q|^2} \geq (1+O(\sqrt{\epsilon_0}))\frac{c_\infty\sqrt{\epsilon_0}}{\underline{u}_1^{1+\frac{\delta_{dec}}{2}}}\exp\left(\frac{m_0}{2r_0^2}(\underline{u}-\underline{u}_1)\right)\\ &\text{on } \gamma\left(\underline{u}_1, \underline{u}_1 + \left(\frac{\underline{u}_1}{\epsilon_0}\right)^{\frac{\delta_{dec}}{2}}\right) \cap {}^{(int)}\mathcal{M}.\end{aligned}$$

Now, in this $\underline{u}$ interval, we may choose

$$\underline{u}_3 := \underline{u}_1 + \frac{2r_0^2}{m_0}\left(1+\frac{\delta_{dec}}{2}\right)\log\left(\frac{\underline{u}_1}{\epsilon_0}\right)$$

for which we have $\frac{\Delta}{|q|^2} \geq 1$. This is a contradiction since $\frac{\Delta}{|q|^2} < 1$ in ${}^{(int)}\mathcal{M}$ (and even in $\mathcal{M}$). Thus, we deduce that γ reaches ${}^{(ext)}\mathcal{M}$ before $\underline{u} = \underline{u}_3$, a contradiction to our assumption on γ. This concludes the proof of (3.91).

3.9. Structure of the rest of the paper

The rest of this paper is devoted to the proof of Theorem M0–M8. Note that the following results will be proved in separate papers:

- Theorem M1 concerning decay estimates for $\mathfrak{q}$ and A, and Theorem M2 concerning decay estimates for $\underline{A}$ are proved in Part II of [28].
- The control of the curvature components in Theorem M8, concerning top order boundedness estimates, is derived in Part III of [28].

In this paper, we prove the remaining results, i.e. Theorem M0 and Theorems M3–M7, as well as[96] Theorem M8. More precisely:

1. We prove first consequences of the bootstrap assumptions, i.e. the control of coordinates systems and of the global frame of Section 3.6, in Chapter 4.
2. Theorem M3 is proved in Chapter 5.
3. Theorem M4 is proved in Chapter 6.
4. Theorem M5 is proved in Chapter 7.
5. Theorems M0, M6 and M7 are proved in Chapter 8.
6. Theorem M8 is proved in Chapter 9 assuming the control of the curvature components in [28].

Remark 3.41. *Note that Theorem M0 should be proved first but has been postponed to chapter 8 for convenience as its proofs has a similar flavor to the ones of Theorems M6 and M7. In particular, while its proof is in chapter 8, it relies only on the bootstrap assumptions and on the assumptions on the initial data layer.*

[96] Assuming the control of top order derivatives of the curvature components in Part III of [28].

4. FIRST CONSEQUENCES OF THE BOOTSTRAP ASSUMPTIONS

In this chapter, we derive first consequences of our bootstrap assumptions on decay and boundedness. We start with the control of coordinates systems on ${}^{(ext)}\mathcal{M}$ and ${}^{(int)}\mathcal{M}$ in Section 4.1. Then, we prove Proposition 3.26 on the construction of a second frame of ${}^{(ext)}\mathcal{M}$ in Section 4.2. Finally, we prove Proposition 3.33 on the construction of a global frame on $\mathcal{M}$ in section 4.5.

4.1. Control of coordinates systems

In this section, we show that the metric in ${}^{(ext)}\mathcal{M}$ and ${}^{(int)}\mathcal{M}$ is close to Kerr in suitable coordinates systems by relying on the bootstrap assumptions.

Proposition 4.1. *${}^{(ext)}\mathcal{M}$ is covered by three regular coordinates patches:*

- *in the (u, r, θ, φ) coordinates system, we have, for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$,*

$$\mathbf{g} \;=\; \mathbf{g}_{a,m} + \Big(du, dr, rd\theta, r\sin\theta d\varphi\Big)^2 O\left(\frac{\epsilon}{u^{1+\delta_{dec}}}\right),$$

- *in the (u, r, x^1, x^2) coordinates system, with $x^1 = J^{(+)}$ and $x^2 = J^{(-)}$, we have, for $0 \le \theta < \frac{\pi}{3}$ and for $\frac{2\pi}{3} < \theta \le \pi$,*

$$\mathbf{g} \;=\; \mathbf{g}_{a,m} + \Big(du, dr, rdx^1, rdx^2\Big)^2 O\left(\frac{\epsilon}{u^{1+\delta_{dec}}}\right),$$

where in each case, $\mathbf{g}_{a,m}$ denotes the Kerr metric expressed in the corresponding coordinates system of Kerr, see Lemma 2.35 and Lemma 2.49.

Proposition 4.2. *${}^{(int)}\mathcal{M}$ is covered by three regular coordinates patches:*

- *in the $(\underline{u}, r, \theta, \varphi)$ coordinates system, we have, for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$,*

$$\mathbf{g} \;=\; \mathbf{g}_{a,m} + \Big(d\underline{u}, dr, rd\theta, r\sin\theta d\varphi\Big)^2 O\left(\frac{\epsilon}{\underline{u}^{1+\delta_{dec}}}\right),$$

- *in the $(\underline{u}, r, x^1, x^2)$ coordinates system, with $x^1 = J^{(+)}$ and $x^2 = J^{(-)}$, we have, for $0 \le \theta < \frac{\pi}{3}$ and for $\frac{2\pi}{3} < \theta \le \pi$,*

$$\mathbf{g} \;=\; \mathbf{g}_{a,m} + \Big(d\underline{u}, dr, rdx^1, rdx^2\Big)^2 O\left(\frac{\epsilon}{\underline{u}^{1+\delta_{dec}}}\right),$$

where in each case, $\mathbf{g}_{a,m}$ denotes the Kerr metric expressed in the corresponding coordinates system of Kerr, i.e. the analog for ingoing PG structures of Lemma 2.35 and Lemma 2.49.

The proof of Proposition 4.1 and Proposition 4.2 being similar, we focus in the rest of this section on the proof of Proposition 4.1.

It will be in fact easier to control first the coefficients of the inverse metric. To this end, we rely on the following simple lemma.

Lemma 4.3. *In a coordinates system (x^α), we have*

$$\mathbf{g}^{\alpha\beta} = -\frac{1}{2}e_4(x^\alpha)e_3(x^\beta) - \frac{1}{2}e_3(x^\alpha)e_4(x^\beta) + \nabla(x^\alpha)\cdot\nabla(x^\beta).$$

Proof. For a scalar function h, we have

$$\mathbf{D}h = \mathbf{g}^{\alpha\beta}e_\alpha(h)e_\beta = -\frac{1}{2}e_4(h)e_3 - \frac{1}{2}e_3(h)e_4 + \nabla(h).$$

We infer

$$\mathbf{g}(\mathbf{D}x^\alpha, \mathbf{D}x^\beta) = -\frac{1}{2}e_4(x^\alpha)e_3(x^\beta) - \frac{1}{2}e_3(x^\alpha)e_4(x^\beta) + \nabla(x^\alpha)\cdot\nabla(x^\beta).$$

Since

$$\mathbf{g}(\mathbf{D}x^\alpha, \mathbf{D}x^\beta) = \mathbf{g}\left(\mathbf{g}^{\alpha\mu}\partial_{x^\mu}, \mathbf{g}^{\beta\nu}\partial_{x^\nu}\right) = \mathbf{g}^{\alpha\mu}\mathbf{g}^{\beta\nu}\mathbf{g}_{\mu\nu} = \mathbf{g}^{\alpha\beta},$$

we deduce

$$\mathbf{g}^{\alpha\beta} = -\frac{1}{2}e_4(x^\alpha)e_3(x^\beta) - \frac{1}{2}e_3(x^\alpha)e_4(x^\beta) + \nabla(x^\alpha)\cdot\nabla(x^\beta)$$

as stated. □

Lemma 4.4. *We have in ${}^{(ext)}\mathcal{M}$*

$$\mathbf{g}^{rr} = -e_3(r), \qquad \mathbf{g}^{r\alpha} = -\frac{1}{2}e_3(x^\alpha) \quad \textit{for} \quad x^\alpha = u,\, \theta,\, \varphi,\, x^1,\, x^2,$$

and

$$\mathbf{g}^{\alpha\beta} = \nabla(x^\alpha)\cdot\nabla(x^\beta) \quad \textit{for} \quad x^\alpha,\, x^\beta = u,\, \theta,\, \varphi,\, x^1,\, x^2.$$

Proof. Recall that (u, r, θ, φ) and $J^{(\pm)}$ verify on ${}^{(ext)}\mathcal{M}$

$$e_4(r) = 1, \quad \nabla(r) = 0, \quad e_4(u) = e_4(\theta) = e_4(\varphi) = e_4(J^{(+)}) = e_4(J^{(-)}) = 0.$$

In view of Lemma 4.3, we infer

$$\mathbf{g}^{rr} = -e_3(r), \qquad \mathbf{g}^{r\alpha} = -\frac{1}{2}e_3(x^\alpha) \quad \text{for} \quad x^\alpha = u,\, \theta,\, \varphi,\, x^1,\, x^2,$$

and

$$\mathbf{g}^{\alpha\beta} \quad = \quad \nabla(x^\alpha)\cdot\nabla(x^\beta) \quad \text{for} \quad x^\alpha,\, x^\beta = u,\, \theta,\, \varphi,\, x^1,\, x^2,$$

as stated. □

Corollary 4.5. *In ${}^{(ext)}\mathcal{M}$, we have*

$$\begin{aligned}
&\mathbf{g}^{rr} = \frac{\Delta}{|q|^2} + r\Gamma_b, \quad \mathbf{g}^{ru} = -\frac{r^2+a^2}{|q|^2} + r\Gamma_b, \quad \mathbf{g}^{rx^1} = \frac{ax^2}{|q|^2} + \Gamma_b,\\
&\mathbf{g}^{rx^2} = -\frac{ax^1}{|q|^2} + \Gamma_b, \quad \mathbf{g}^{r\theta} = (\sin\theta)^{-1}\Gamma_b, \quad \mathbf{g}^{r\varphi} = -\frac{a}{|q|^2} + (\sin\theta)^{-2}\Gamma_b,\\
&\mathbf{g}^{uu} = \frac{a^2(\sin\theta)^2}{|q|^2} + r^{-1}\Gamma_b, \quad \mathbf{g}^{ux^1} = -\frac{ax^2}{|q|^2} + r^{-1}\Gamma_b,\\
&\mathbf{g}^{ux^2} = \frac{ax^1}{|q|^2} + r^{-1}\Gamma_b, \quad \mathbf{g}^{u\theta} = r^{-1}(\sin\theta)^{-1}\Gamma_b,\\
&\mathbf{g}^{u\varphi} = \frac{a}{|q|^2} + r^{-1}(\sin\theta)^{-2}\Gamma_b, \quad \mathbf{g}^{x^1x^1} = \frac{(\cos\theta\cos\varphi)^2 + (\sin\varphi)^2}{|q|^2} + r^{-1}\Gamma_b,\\
&\mathbf{g}^{x^1x^2} = \frac{((\cos\theta)^2-1)\sin\varphi\cos\varphi}{|q|^2} + r^{-1}\Gamma_b,\\
&\mathbf{g}^{x^2x^2} = \frac{(\cos\theta\sin\varphi)^2 + (\cos\varphi)^2}{|q|^2} + r^{-1}\Gamma_b, \quad \mathbf{g}^{\theta\theta} = \frac{1}{|q|^2} + r^{-1}(\sin\theta)^{-2}\Gamma_b,\\
&\mathbf{g}^{\theta\varphi} = r^{-1}(\sin\theta)^{-3}\Gamma_b, \quad \mathbf{g}^{\varphi\varphi} = \frac{1}{|q|^2(\sin\theta)^2} + r^{-1}(\sin\theta)^{-4}\Gamma_b.
\end{aligned}$$

Proof. Recall the notations

$$\widetilde{e_3(r)} = e_3(r) + \frac{\Delta}{|q|^2}, \quad \widetilde{\nabla} u = \nabla u - a\Re(\mathfrak{J}), \quad \widetilde{e_3(u)} = e_3(u) - \frac{2(r^2+a^2)}{|q|^2},$$

and

$$\begin{aligned}
&\widetilde{\nabla J^{(0)}} = \nabla J^{(0)} + \Im(\mathfrak{J}), \qquad \widetilde{\nabla J^{(\pm)}} = \nabla J^{(\pm)} - \Re(\mathfrak{J}_\pm),\\
&\widetilde{e_3(J^{(+)})} = e_3(J^{(\pm)}) \pm \frac{2a}{|q|^2}J^{(\mp)}.
\end{aligned}$$

Also, recall that

$$\begin{array}{ll}\widecheck{e_3(r)},\ \widecheck{e_3(u)} \in r\Gamma_b, & \widecheck{\nabla} u \in \Gamma_b,\\ e_3(J^{(0)}),\ \widecheck{e_3(J^{(\pm)})} \in \Gamma_b, & \widecheck{\nabla J^{(p)}} \in \Gamma_b,\ p = 0, +, -.\end{array}$$

We infer

$$\begin{aligned}&e_3(r) = -\frac{\Delta}{|q|^2} + r\Gamma_b, \quad e_3(u) = \frac{2(r^2+a^2)}{|q|^2} + r\Gamma_b, \quad \nabla u = a\Re(\mathfrak{J}) + \Gamma_b,\\ &\nabla J^{(0)} = -\Im(\mathfrak{J}) + \Gamma_b, \quad \nabla J^{(\pm)} = \Re(\mathfrak{J}_\pm) + \Gamma_b,\\ &e_3(J^{(0)}) = \Gamma_b, \quad e_3(J^{(\pm)}) = \mp\frac{2a}{|q|^2} J^{(\mp)} + \Gamma_b.\end{aligned}$$

Also, we observe

$$\begin{aligned}&\sin\theta\nabla(\theta) = -\nabla(J^{(0)}), \qquad \sin\theta e_3(\theta) = -e_3(J^{(0)}),\\ &(\sin\theta)^2\nabla(\varphi) = -J^{(-)}\nabla(J^{(+)}) + J^{(+)}\nabla(J^{(-)}),\\ &(\sin\theta)^2 e_3(\varphi) = -J^{(-)}e_3(J^{(+)}) + J^{(+)}\nabla(J^{(-)}),\end{aligned}$$

so that we have in view of the above

$$\begin{aligned}&\sin\theta\nabla(\theta) = \Im(\mathfrak{J}) + \Gamma_b, \quad (\sin\theta)^2\nabla(\varphi) = -J^{(-)}\Re(\mathfrak{J}_+) + J^{(+)}\Re(\mathfrak{J}_-) + \Gamma_b,\\ &\sin\theta e_3(\theta) = \Gamma_b, \quad (\sin\theta)^2 e_3(\varphi) = \frac{2a(\sin\theta)^2}{|q|^2} + \Gamma_b.\end{aligned}$$

In view of Lemma 4.4, the proof of the corollary follows then easily. $\square$

We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1. In view of Corollary 4.5 for the inverse metric coefficients in ${}^{(ext)}\mathcal{M}$, and Lemma 2.35 for the inverse metric coefficients of the Kerr metric, we have in the (u, r, θ, φ) coordinates system of ${}^{(ext)}\mathcal{M}$

$$\mathbf{g} \;=\; \mathbf{g}_{a,m} + \Big(du, dr, rd\theta, r\sin\theta d\varphi\Big)^2 (\sin\theta)^{-2} r\Gamma_b$$

and the conclusion follows from the control of Γ_b and the fact that $\sin\theta > \frac{\sqrt{2}}{2}$ in the range $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$.

Similarly, in view of Corollary 4.5 for the inverse metric coefficients in ${}^{(ext)}\mathcal{M}$, and Lemma 2.49 for the inverse metric coefficients of the Kerr metric, we have in the (u, r, x^1, x^2) coordinates system of ${}^{(ext)}\mathcal{M}$

$$\mathbf{g} \;=\; \mathbf{g}_{a,m} + \Big(du, dr, rdx^1, rdx^2\Big)^2 (\cos\theta)^{-2} r\Gamma_b$$

and the conclusion follows from the control of Γ_b and the fact that $\cos\theta > \frac{1}{2}$ in the range $0 \leq \theta < \frac{\pi}{3}$ and $\cos\theta < -\frac{1}{2}$ in the range $\frac{2\pi}{3} < \theta \leq \pi$. This concludes the proof of the Proposition 4.1. $\square$

4.2. Proof of Proposition 3.26

Let (e_4, e_3, e_1, e_2) be the outgoing PG frame of ${}^{(ext)}\mathcal{M}$. We will exhibit another frame (e_4', e_3', e_1', e_2') of ${}^{(ext)}\mathcal{M}$ provided by

$$\begin{aligned} e_4' &= e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3, \\ e_a' &= e_a + \frac{1}{2} f_a e_3, \quad a = 1, 2, \\ e_3' &= e_3, \end{aligned} \tag{4.1}$$

where f is such that

$$f = 0 \text{ on } S_*, \qquad \check{\eta}' = 0 \text{ on } \Sigma_*, \qquad \xi' = 0 \text{ on } {}^{(ext)}\mathcal{M}. \tag{4.2}$$

The desired estimates for the Ricci coefficients and curvature components with respect to the new frame (e_4', e_3', e_1', e_2') of ${}^{(ext)}\mathcal{M}$ will be obtained in the region ${}^{(ext)}\mathcal{M}(r \geq u^{\frac{1}{2}})$ using:

- the change of frame formulas of Proposition 2.12, applied to the change of frame from (e_4, e_3, e_1, e_2) to (e_4', e_3', e_1', e_2'),
- the estimates for f on ${}^{(ext)}\mathcal{M}$, and the fact that $\underline{f} = 0$ and $\lambda = 1$ in the null frame transformation (4.1),
- the estimates for the Ricci coefficients and curvature components with respect to the outgoing PG frame (e_4, e_3, e_1, e_2) of ${}^{(ext)}\mathcal{M}$ provided by the bootstrap assumptions on decay and boundedness.

Now, as it turns out, the frame (e_4', e_3', e_1', e_2') does not satisfy the desired estimates in the region ${}^{(ext)}\mathcal{M}(r \leq u^{\frac{1}{2}})$. We will thus introduce a third frame $(e_4'', e_3'', e_1'', e_2'')$ on ${}^{(ext)}\mathcal{M}$, agreeing with (e_4, e_3, e_1, e_2) on ${}^{(ext)}\mathcal{M}(r \leq u^{\frac{1}{2}})$, and with (e_4', e_3', e_1', e_2') on ${}^{(ext)}\mathcal{M}(r \geq u^{\frac{1}{2}})$, and satisfying all desired properties of Proposition 3.26. In Steps 1–3 below, we study the properties of (e_4', e_3', e_1', e_2').

We then introduce the frame $(e''_4, e''_3, e''_1, e''_2)$ in Step 4, and conclude, in Steps 4–6, the proof of Proposition 3.26.

Step 1. We start by deriving an equation for f on ${}^{(ext)}\mathcal{M}$. In view of the condition $\xi' = 0$ on ${}^{(ext)}\mathcal{M}$, see (4.2), in view of $\xi = \omega = 0$ satisfied by the outgoing PG structure of ${}^{(ext)}\mathcal{M}$, and in view of Corollary 2.14, we have

$$\nabla'_4 F + \frac{1}{2}\mathrm{tr}X F \;=\; -\widehat{\chi}\cdot F + E_1(f,\Gamma) \;\text{ on }\; {}^{(ext)}\mathcal{M}, \tag{4.3}$$

where $E_1(f,\Gamma)$ contains expressions of the type $O(\Gamma f^2)$ with no derivatives, and where

$$F := f + i\,{}^*f.$$

We also derive an equation for f on Σ_*. In view of the change of frame formulas of Proposition 2.12 in the particular case where $\lambda = 1$ and $\underline{f} = 0$, we have on ${}^{(ext)}\mathcal{M}$

$$\nabla'_3 F \;=\; 2H' - 2H + 2\underline{\omega}F + \mathrm{Err}[\nabla_3 F],$$

where the lower order term $\mathrm{Err}[\nabla_3 F]$ contains expressions of the type $O(f^2\Gamma_b)$ with no derivatives. Now, we have[97]

$$H = \frac{aq}{|q|^2}\mathfrak{J} + \check{H}, \qquad H' = \frac{aq}{|q|^2}\mathfrak{J} + \check{H}',$$

and hence

$$H' - H \;=\; \check{H}' - \check{H},$$

so that, together with the condition $\check{H}' = 0$ on Σ_*, see (4.2), we infer

$$\nabla'_3 F \;=\; -2\check{H} + 2\underline{\omega}F + \mathrm{Err}[\nabla_3 F] \quad \text{on} \quad \Sigma_*. \tag{4.4}$$

Now, since $u + r$ is constant on Σ_*, the following vectorfield

$$\nu'_{\Sigma_*} := e'_3 + b'e'_4, \qquad b' := -\frac{e'_3(u+r)}{e'_4(u+r)},$$

[97]Recall that for the second frame, the Ricci coefficients and curvature components are also linearized using the scalar function r and θ and the complex 1-form $\mathfrak{J}$ attached to the principal frame of ${}^{(ext)}\mathcal{M}$.

is tangent to Σ_*. We compute in view of the above

$$\begin{aligned}\nabla'_{\nu'_{\Sigma_*}} F &= \nabla'_3 F + b'\nabla'_4 F\\ &= -2\check{H} + 2\underline{\omega}F + \mathrm{Err}[\nabla_3 F] + b'\left(-\frac{1}{2}\mathrm{tr}XF - \widehat{\chi}\cdot F + E_1(f,\Gamma)\right).\end{aligned}$$

Using (4.1), as well as

$$e_4(r) = 1, \qquad e_4(u) = 0, \qquad \nabla(r) = 0,$$

we have

$$\begin{aligned}b' &= -\frac{e'_3(u+r)}{e'_4(u+r)} = -\frac{e_3(u+r)}{\left(e_4 + f\cdot\nabla + \frac{1}{4}|f|^2 e_3\right)(u+r)}\\ &= -\frac{e_3(u)+e_3(r)}{1 + f\cdot\nabla(u) + \frac{1}{4}|f|^2(e_3(u)+e_3(r))}.\end{aligned}$$

Recalling the following linearizations

$$\widecheck{e_3(r)} = e_3(r) + \frac{\Delta}{|q|^2}, \qquad \widecheck{\mathcal{D}}u = \mathcal{D}u - a\mathfrak{J}, \qquad \widecheck{e_3(u)} = e_3(u) - \frac{2(r^2+a^2)}{|q|^2},$$

we deduce

$$b' = -\frac{1 + \frac{2mr + a^2(\sin\theta)^2}{|q|^2} + \widecheck{e_3(r)} + \widecheck{e_3(u)}}{1 + f\cdot\nabla(u) + \frac{1}{4}|f|^2(e_3(u)+e_3(r))}$$

and hence

$$\begin{aligned}\nabla'_{\nu'_{\Sigma_*}} F &= -2\check{H} + 2\underline{\omega}F + \mathrm{Err}[\nabla_3 F]\\ &\quad -\frac{1 + \frac{2mr + a^2(\sin\theta)^2}{|q|^2} + \widecheck{e_3(r)} + \widecheck{e_3(u)}}{1 + f\cdot\nabla(u) + \frac{1}{4}|f|^2(e_3(u)+e_3(r))}\\ &\quad \times\left(-\frac{1}{2}\mathrm{tr}XF - \widehat{\chi}\cdot F + E_1(f,\Gamma)\right) \text{ on } \Sigma_*.\end{aligned} \tag{4.5}$$

Step 2. Next, we estimate f on Σ_*. In view of (3.46) and (3.51), we have

$$\delta_{dec}(k_{large} - k_{small}) \geq \frac{1}{2}\delta_{dec}k_{large} - \delta_{dec} \gg 1,$$

and we may thus assume from now on

$$\frac{\delta_{dec}}{3}(k_{large}-k_{small})\geq 130.$$

Also, we introduce a small constant $\delta_0>0$ satisfying

$$\delta_0=\frac{130}{k_{large}-k_{small}}\leq\frac{\delta_{dec}}{3}.$$

Note in particular, from the bootstrap assumptions on decay and boundedness for the outgoing PG frame (e_4,e_3,e_1,e_2) of ${}^{(ext)}\mathcal{M}$ that

$$\begin{aligned}
|\mathfrak{d}^{\leq k_{small}+130}\Gamma_g| &\lesssim |\mathfrak{d}^{\leq k_{small}}\Gamma_g|^{1-\frac{130}{k_{large}-k_{small}}}|\mathfrak{d}^{\leq k_{large}}\Gamma_g|^{\frac{130}{k_{large}-k_{small}}}\\
&\lesssim \min\left[\frac{\epsilon}{r^2}\left(\frac{1}{u^{\frac{1}{2}+\delta_{dec}}}\right)^{1-\frac{130}{k_{large}-k_{small}}},\right.\\
&\qquad\left.\frac{\epsilon}{r}\left(\frac{1}{u^{1+\delta_{dec}}}\right)^{1-\frac{130}{k_{large}-k_{small}}}\right],\\
|\mathfrak{d}^{\leq k_{small}+130}\Gamma_b| &\lesssim |\mathfrak{d}^{\leq k_{small}}\Gamma_b|^{1-\frac{130}{k_{large}-k_{small}}}|\mathfrak{d}^{\leq k_{large}}\Gamma_b|^{\frac{130}{k_{large}-k_{small}}}\\
&\lesssim \frac{\epsilon}{r}\left(\frac{1}{u^{1+\delta_{dec}}}\right)^{1-\frac{130}{k_{large}-k_{small}}}.
\end{aligned}$$

Since we have, in view of the definition of δ_0,

$$\begin{aligned}
\frac{130}{k_{large}-k_{small}}(1+\delta_{dec}) &= (1+\delta_{dec})\,\delta_0\leq 2\delta_0,\\
\frac{130}{k_{large}-k_{small}}\left(\frac{1}{2}+\delta_{dec}\right) &= \left(\frac{1}{2}+\delta_{dec}\right)\delta_0\leq\delta_0,
\end{aligned}$$

we infer

$$\begin{aligned}
&\sup_{{}^{(ext)}\mathcal{M}}\left(r^2u^{\frac{1}{2}+\delta_{dec}-2\delta_0}+ru^{1+\delta_{dec}-2\delta_0}\right)|\mathfrak{d}^{\leq k_{small}+130}\Gamma_g|\\
&+\sup_{{}^{(ext)}\mathcal{M}} ru^{1+\delta_{dec}-2\delta_0}|\mathfrak{d}^{\leq k_{small}+130}\Gamma_b| \lesssim \epsilon.
\end{aligned}\tag{4.6}$$

Next, we assume the following local bootstrap assumption for f on Σ_*

$$|\mathfrak{d}^{\leq k_{small}+130}f|\leq\frac{\sqrt{\epsilon}}{ru^{\frac{1}{2}+\delta_{dec}-2\delta_0}}\quad\text{on } u_1\leq u\leq u_*\tag{4.7}$$

where

$$1 \leq u_1 < u_*.$$

Since $f = 0$ on S_* in view of (4.2), (4.7) holds for u_1 close enough to u_*, and our goal is to prove that we may in fact choose $u_1 = 1$ and replace $\sqrt{\epsilon}$ with ϵ in (4.7).

In view of the bootstrap assumptions on boundedness for the Ricci coefficients and curvature components with respect to the outgoing principal frame (e_4, e_3, e_1, e_2) of ${}^{(ext)}\mathcal{M}$, (4.5) yields

$$\nabla'_{\nu'_{\Sigma_*}} F = -2\check{H} + h, \qquad |\mathfrak{d}^k h| \lesssim r^{-1}(|\mathfrak{d}^{\leq k} f| + |\mathfrak{d}^{\leq k} f|^4) \text{ for } k \leq k_{large}.$$

To differentiate this transport equation, we introduce the derivation

$$\widetilde{\nabla} := \nabla' - \frac{\nabla'(u+r)}{e'_4(u+r)}\nabla'_4$$

so that $(\nabla'_{\nu'_{\Sigma_*}}, \widetilde{\nabla})$ span all tangential derivatives to Σ_*. Note that

$$\begin{aligned}\widetilde{\nabla} &= \nabla' - \frac{\nabla(u+r) + \frac{1}{2} f e_3(u+r)}{e_4(u+r) + f\cdot\nabla(u+r) + \frac{1}{4}|f|^2 e_3(u+r)}\nabla'_4\\ &= \nabla' - \frac{\nabla(u) + \frac{1}{2} f e_3(u+r)}{1 + f\cdot\nabla(u) + \frac{1}{4}|f|^2 e_3(u+r)}\nabla'_4.\end{aligned}$$

We introduce the following weighted derivatives on Σ_*:

$$\widetilde{\not{\mathfrak{d}}} := r\widetilde{\nabla}, \qquad \widetilde{\mathfrak{d}} = (\nabla'_{\nu'_{\Sigma_*}}, \widetilde{\not{\mathfrak{d}}}).$$

Using commutator identities, using also (4.3) and (4.4), and in view of (4.7), we infer

$$|\nabla'_{\nu'_{\Sigma_*}} \widetilde{\not{\mathfrak{d}}}^k F| \lesssim |\mathfrak{d}^{\leq k}\check{H}| + \frac{\sqrt{\epsilon}}{r^2 u^{\frac{1}{2}+\delta_{dec}-2\delta_0}} \text{ for } k \leq k_{small} + 130, \ u_1 \leq u \leq u_*.$$

Since $f = 0$ on S_* in view of (4.2), and since ν'_{Σ_*} is tangent to Σ_*, we deduce on Σ_*, integrating along the integral curve of ν'_{Σ_*}

$$|\widetilde{\not{\mathfrak{d}}}^k F| \lesssim \int_u^{u_*} |\mathfrak{d}^{\leq k}\check{H}| + \frac{\sqrt{\epsilon}}{u^{\frac{1}{2}+\delta_{dec}-2\delta_0}} \int_u^{u_*} \frac{1}{\nu'_{\Sigma_*}(u')r^2}$$
$$\text{for } k \leq k_{small} + 130, \ u_1 \leq u \leq u_*.$$

Since

$$\begin{aligned}
& \nu'_{\Sigma_*}(u) \\
= \;& e'_3(u) + b' e'_4(u) \\
= \;& e_3(u) - \frac{1 + \frac{2mr + a^2(\sin\theta)^2}{|q|^2} + \widecheck{e_3(r)} + \widecheck{e_3(u)}}{1 + f\cdot\nabla(u) + \frac{1}{4}|f|^2(e_3(u) + e_3(r))}\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right)u \\
= \;& \frac{2(r^2+a^2)}{|q|^2} + \widecheck{e_3(u)} \\
& - \frac{1 + \frac{2mr + a^2(\sin\theta)^2}{|q|^2} + \widecheck{e_3(r)} + \widecheck{e_3(u)}}{1 + f\cdot\nabla(u) + \frac{1}{4}|f|^2(e_3(u) + e_3(r))}\left(f\cdot\nabla(u) + \frac{1}{4}|f|^2 e_3(u)\right)
\end{aligned}$$

we have

$$\nu'_{\Sigma_*}(u) \;=\; 2 + O\left(\frac{1}{r} + \epsilon\right)$$

and hence, we have on Σ_*

$$|\widetilde{\mathfrak{d}}^k F| \lesssim \int_u^{u_*} |\mathfrak{d}^{\leq k}\check{H}| + \frac{\sqrt{\epsilon}}{u^{\frac{1}{2}+\delta_{dec}-2\delta_0}}\int_u^{u_*}\frac{1}{r^2} \text{ for } k \leq k_{small} + 130, \; u_1 \leq u \leq u_*.$$

Together with the behavior (3.50) of r on Σ_*, we infer

$$|\widetilde{\mathfrak{d}}^k F| \;\lesssim\; \int_u^{u_*} |\mathfrak{d}^{\leq k}\check{H}| + \frac{\epsilon}{r u^{\frac{1}{2}+\delta_{dec}-2\delta_0}} \text{ for } k \leq k_{small} + 130, \; u_1 \leq u \leq u_*.$$

Next, we estimate $\check{H}$. We have by interpolation

$$\|\mathfrak{d}^{\leq k_{small}+132}\check{H}\|_{L^2(S)} \;\lesssim\; \|\mathfrak{d}^{\leq k_{small}}\check{H}\|_{L^2(S)}^{1-\frac{132}{k_{large}-k_{small}}} \|\mathfrak{d}^{\leq k_{large}}\check{H}\|_{L^2(S)}^{\frac{132}{k_{large}-k_{small}}},$$

and hence, using $\delta_0 > 0$, we have

$$\begin{aligned}
& \int_{\Sigma_*(\geq u)} |\mathfrak{d}^{\leq k_{small}+132}\check{H}| \\
\lesssim \;& \left(\int_{\Sigma_*(\geq u)} u'^{1+\delta_0}|\mathfrak{d}^{\leq k_{small}+132}\check{H}|^2\right)^{\frac{1}{2}} \\
\lesssim \;& \frac{1}{u^{\frac{1}{2}+\delta_{dec}-2\delta_0}}\left(\int_{\Sigma_*} u'^{2+2\delta_{dec}}|\mathfrak{d}^{\leq k_{small}}\check{H}|^2\right)^{\frac{1}{2}-\frac{132}{2(k_{large}-k_{small})}}
\end{aligned}$$

$$\times \left(\int_{\Sigma_*} |\mathfrak{d}^{\leq k_{large}} \check{H}|^2\right)^{\frac{132}{2(k_{large}-k_{small})}},$$

where we have used the fact, in view of the definition of δ_0, that

$$\frac{132}{k_{large}-k_{small}}(1+\delta_{dec})+\frac{\delta_0}{2} = \left(\left(1+\frac{2}{130}\right)(1+\delta_{dec})+\frac{1}{2}\right)\delta_0 \leq 2\delta_0$$

and

$$\frac{1}{2}+\delta_{dec}-2\delta_0 \geq \frac{1}{2}+\delta_{dec}-\frac{2}{3}\delta_{dec} = \frac{1}{2}+\frac{\delta_{dec}}{3} > 0.$$

Now, recall from the bootstrap assumptions on decay and boundedness for $\check{H}$ on ${}^{(ext)}\mathcal{M}$ that we have

$$\int_{\Sigma_*} u^{2+2\delta_{dec}}|\mathfrak{d}^{\leq k_{small}}\check{H}|^2+\int_{\Sigma_*}|\mathfrak{d}^{\leq k_{large}}\check{H}|^2 \leq \epsilon^2.$$

We deduce

$$\int_{\Sigma_*(\geq u)} |\mathfrak{d}^{\leq k_{small}+132}\check{H}| \lesssim \frac{\epsilon}{u^{\frac{1}{2}+\delta_{dec}-2\delta_0}}.$$

Together with the Sobolev embedding on the 2-spheres S foliating Σ_*, as well as the behavior (3.50) of r on Σ_*, we infer

$$\int_u^{u_*} |\mathfrak{d}^{\leq k_{small}+130}\check{H}| \lesssim \frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}-2\delta_0}}.$$

Plugging in the above estimate for F, we infer on Σ_*

$$|\widetilde{\mathfrak{F}}^k F| \lesssim \frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}-2\delta_0}} \text{ for } k\leq k_{small}+130,\ u_1\leq u\leq u_*.$$

Together with (4.3) and (4.4), we recover e_4 and e_3 derivatives to deduce

$$|\mathfrak{d}^k F| \lesssim \frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}-2\delta_0}} \text{ for } k\leq k_{small}+130,\ u_1\leq u\leq u_*$$

and hence, since $F=f+i\,{}^*f$,

$$|\mathfrak{d}^k f| \lesssim \frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}-2\delta_0}} \text{ for } k\leq k_{small}+130,\ u_1\leq u\leq u_*.$$

This is an improvement of the bootstrap assumption (4.7). Thus, we may choose $u_1 = 1$, and f satisfies the following estimate

$$|\mathfrak{d}^k f| \lesssim \frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}-2\delta_0}} \text{ for } k \leq k_{small} + 130 \text{ on } \Sigma_*.$$

Together with (4.4), as well as the behavior (3.50) of r on Σ_*, and the control of $\check{H}$ provided by (4.6), we infer

$$\begin{aligned} |\mathfrak{d}^{k-1}\nabla_3' f| &\lesssim |\mathfrak{d}^{k-1}\check{H}| + \frac{\epsilon}{r^2} \\ &\lesssim \frac{\epsilon}{ru^{1+\delta_{dec}-2\delta_0}} \text{ for } k \leq k_{small} + 130 \text{ on } \Sigma_*. \end{aligned}$$

Collecting the two above estimates, we obtain

$$|\mathfrak{d}^k f| \lesssim \frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}-2\delta_0}}, \quad |\mathfrak{d}^{k-1}\nabla_3' f| \lesssim \frac{\epsilon}{ru^{1+\delta_{dec}-2\delta_0}} \text{ for } k \leq k_{small} + 130 \text{ on } \Sigma_*. \tag{4.8}$$

Step 3. Next, we estimate f on ${}^{(ext)}\mathcal{M}$. We assume the following local bootstrap assumption

$$|\mathfrak{d}^{\leq k_{small}+130} f| \leq \frac{\sqrt{\epsilon}}{ru^{\frac{1}{2}+\delta_{dec}-2\delta_0}} \quad \text{on } r \geq r_1, \tag{4.9}$$

where $r_1 \geq r_0$. In view of the control of f on Σ_* provided by (4.8), (4.9) holds for r_1 sufficiently large, and our goal is to prove that we may in fact choose $r_1 = r_0$ and replace $\sqrt{\epsilon}$ with ϵ in (4.9).

From Corollary 2.14, we may rewrite (4.3) as

$$\nabla_4'(qF) = E_4(f, \Gamma) \text{ on } {}^{(ext)}\mathcal{M},$$

where

$$E_4(f,\Gamma) = -\frac{1}{2}q\widecheck{\mathrm{tr}X}F - q\widehat{\chi}\cdot F + qE_1(f,\Gamma) + f\cdot\nabla(q)F + \frac{1}{4}|f|^2 e_3(q)F.$$

In view of the estimate (4.6) for the Ricci coefficients and curvature components with respect to the outgoing principal frame (e_4, e_3, e_1, e_2) of ${}^{(ext)}\mathcal{M}$, and in view of the form of $E_1(f,\Gamma)$, we have

$$\left|\mathfrak{d}^k E_4(f,\Gamma)\right| \lesssim \epsilon r^{-1}u^{-\frac{1}{2}}|\mathfrak{d}^{\leq k} f| + |\mathfrak{d}^{\leq k} f|^2 + |\mathfrak{d}^{\leq k} f|^4 \text{ for } k \leq k_{small} + 130.$$

Using commutator identities, and in view of (4.9), we infer[98]

$$\left|\nabla_4'(\not{\mathfrak{d}}, \not{\mathcal{L}}_{\mathbf{T}})^k(qF)\right| \quad \leq \quad \frac{\epsilon}{r^2u^{1+\delta_{dec}-2\delta_0}} \text{ for } k\leq k_{small}+130, \ r\geq r_1.$$

Integrating backwards from Σ_* where we have (4.8), we deduce

$$|(\not{\mathfrak{d}}, \not{\mathcal{L}}_{\mathbf{T}})^kf| \quad \leq \quad \frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}-2\delta_0}} \quad \text{for } k\leq k_{small}+130, \ r\geq r_1.$$

Together with (4.3), we recover the e_4 derivatives and obtain

$$|\mathfrak{d}^kf| \quad \leq \quad \frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}-2\delta_0}} \quad \text{for } k\leq k_{small}+130, \ r\geq r_1.$$

This is an improvement of the bootstrap assumption (4.9). Thus, we may choose $r_1=r_0$, and we have

$$|\mathfrak{d}^kf| \quad \lesssim \quad \frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}-2\delta_0}} \quad \text{for } k\leq k_{small}+130 \text{ on } {}^{(ext)}\mathcal{M}.$$

Also, commuting once (4.3) with e_3', using the schematic commutator identity

$$[\nabla_3', \nabla_4'] \quad = \quad 2\underline{\omega}'\nabla_4' - 2\omega'\nabla_3' + 2(\eta'-\underline{\eta}')\cdot\nabla' + \Big(\underline{\xi}'\xi', \, \eta'\underline{\eta}', \, {}^*\rho'\Big),$$

and proceeding as above to integrate backward from Σ_* where $\nabla_3'f$ is under control from (4.8), we also obtain

$$|\mathfrak{d}^{k-1}\nabla_3'f| \lesssim \frac{\epsilon}{ru^{1+\delta_{dec}-2\delta_0}} + \frac{\epsilon}{r^2u^{\frac{1}{2}+\delta_{dec}-2\delta_0}} \quad \text{for } k\leq k_{small}+130 \text{ on } {}^{(ext)}\mathcal{M}.$$

Collecting the two above estimates, we obtain in the region $r\geq u^{\frac{1}{2}}$

(4.10)
$$\begin{aligned}
|\mathfrak{d}^kf| &\lesssim \frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}-2\delta_0}}, \quad \text{for } k\leq k_{small}+130 \text{ on } {}^{(ext)}\mathcal{M}(r\geq u^{\frac{1}{2}}),\\
|\mathfrak{d}^{k-1}\nabla_3'f| &\lesssim \frac{\epsilon}{ru^{1+\delta_{dec}-2\delta_0}} \quad \text{for } k\leq k_{small}+130 \text{ on } {}^{(ext)}\mathcal{M}(r\geq u^{\frac{1}{2}}).
\end{aligned}$$

---

[98]Note that we have

$$\delta_{dec}-2\delta_0 \geq \delta_{dec} - \frac{2}{3}\delta_{dec} \geq \frac{\delta_{dec}}{3} > 0.$$

Also, since $f=0$ on S_* in view of (4.2), and in view of the transport equation (4.3) for F, and the fact that $F=f+i\,{}^*f$, f satisfies the following addition property

$$f=0 \quad \text{on} \quad \{u=u_*\}. \tag{4.11}$$

Step 4. We now introduce a third frame $(e_4'', e_3'', e_1'', e_2'')$ on ${}^{(ext)}\mathcal{M}$ given by

$$\begin{aligned} e_4'' &= e_4 + f'^b e_b + \frac{1}{4}|f'|^2 e_3, \\ e_a'' &= e_a + \frac{1}{2} f'_a e_3, \quad a=1,2, \\ e_3'' &= e_3, \end{aligned} \tag{4.12}$$

where the horizontal 1-form f' is defined by

$$f' := \psi\left(\frac{u^{\frac{1}{2}}}{r}\right) f, \tag{4.13}$$

with ψ a smooth function on $\mathbb{R}$ such that $\psi=1$ for $s\le\frac{1}{2}$ and $\psi=0$ for $s\ge 1$. Note in particular that $(e_4'', e_3'', e_1'', e_2'')$ coincides with (e_4, e_3, e_1, e_2) in ${}^{(ext)}\mathcal{M}(r\le u^{\frac{1}{2}})$.

We estimate f'. Note that

$$\begin{aligned} re_4\left(\log\left(\frac{u^{\frac{1}{2}}}{r}\right)\right) &= -1, \\ r\nabla\left(\log\left(\frac{u^{\frac{1}{2}}}{r}\right)\right) &= \frac{r}{2u}\nabla(u) = \frac{ar}{2u}\Re(\mathfrak{J}) + \frac{r}{2u}\Gamma_b, \\ e_3\left(\log\left(\frac{u^{\frac{1}{2}}}{r}\right)\right) &= \frac{1}{2}\frac{e_3(u)}{u} + \frac{e_3(r)}{r} = \frac{r^2+a^2}{u|q|^2} - \frac{\Delta}{r|q|^2} + \frac{r}{2u}\Gamma_b + \Gamma_b. \end{aligned}$$

Thus, in view of the bootstrap assumptions on boundedness, and since $\frac{1}{2}\le\frac{u^{\frac{1}{2}}}{r}\le 1$ on the support of ψ', we infer, for $0\le k\le k_{large}$,

$$\left|\mathfrak{d}^k\left(\psi\left(\frac{u^{\frac{1}{2}}}{r}\right)\right)\right| \lesssim 1, \qquad \left|\mathfrak{d}^{k-1}e_3\left(\psi\left(\frac{u^{\frac{1}{2}}}{r}\right)\right)\right| \lesssim \frac{1}{r}.$$

Together with the estimates (4.10) for f on ${}^{(ext)}\mathcal{M}(r\ge u^{\frac{1}{2}})$, the property (4.11) for f, and the definition (4.13) for f', we infer

(4.14)
$$
\begin{aligned}
f'=0, \quad \text{on } {}^{(ext)}\mathcal{M}\left(r\leq u^{\frac{1}{2}}\right) \quad \text{and on} \quad \{u=u_*\},\\
|\mathfrak{d}^k f'| \lesssim \frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}-2\delta_0}}, \text{ for } k\leq k_{small}+130 \text{ on } {}^{(ext)}\mathcal{M}\left(r\geq u^{\frac{1}{2}}\right),\\
|\mathfrak{d}^{k-1}\nabla_3'' f'| \lesssim \frac{\epsilon}{ru^{1+\delta_{dec}-2\delta_0}} \quad \text{for } k\leq k_{small}+130 \text{ on } {}^{(ext)}\mathcal{M}\left(r\geq u^{\frac{1}{2}}\right),
\end{aligned}
$$

which are the desired estimates for f'.

Step 5. In view of Proposition 2.12 applied to our particular case, i.e. a triplet $(f', \underline{f}', \lambda')$ with $\underline{f}'=0$ and $\lambda'=1$, and the fact that (e_4, e_3, e_1, e_2) is an outgoing PG frame, we have

$$
\begin{aligned}
\xi'' &= \frac{1}{2}\nabla_4'' f' + \frac{1}{4}(\mathrm{tr}\,\chi f' - {}^{(a)}\mathrm{tr}\chi\, {}^*f') + \frac{1}{2}f'\cdot\widehat{\chi} + \frac{1}{4}|f'|^2\eta + \frac{1}{2}(f'\cdot\zeta)\, f'\\
&\quad -\frac{1}{4}|f'|^2\underline{\eta} + \text{ l.o.t.},\\
\underline{\xi}'' &= \underline{\xi},\\
\underline{\eta}'' &= -\zeta + \frac{1}{4}(\mathrm{tr}\,\underline{\chi} f' - {}^{(a)}\mathrm{tr}\underline{\chi}\, {}^*f') + \frac{1}{2}f'\cdot\widehat{\underline{\chi}} + \frac{|f'|^2}{4}\underline{\xi},\\
\eta'' &= \eta + \frac{1}{2}\nabla_3 f' - \underline{\omega} f' - \frac{1}{4}|f'|^2\underline{\xi},\\
\zeta'' &= \zeta - \frac{1}{4}(\mathrm{tr}\,\underline{\chi} f' + {}^{(a)}\mathrm{tr}\underline{\chi}\, {}^*f') - \underline{\omega} f' - \frac{1}{2}f'\cdot\widehat{\underline{\chi}} - \frac{1}{2}(f'\cdot\underline{\xi})f',
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}\,\underline{\chi}'' &= \mathrm{tr}\,\underline{\chi} + f'\cdot\underline{\xi},\\
{}^{(a)}\mathrm{tr}\underline{\chi}'' &= {}^{(a)}\mathrm{tr}\underline{\chi} + f'\wedge\underline{\xi},\\
\widehat{\underline{\chi}}'' &= \widehat{\underline{\chi}} + \frac{1}{2}f'\widehat{\otimes}\underline{\xi},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}\,\chi'' &= \mathrm{tr}\,\chi + \mathrm{div}'' f' + f'\cdot(\zeta+\eta) - \underline{\omega}|f'|^2 - \frac{|f'|^2}{4}\left(\mathrm{tr}\,\underline{\chi} + f'\cdot\underline{\xi}\right),\\
{}^{(a)}\mathrm{tr}\chi'' &= {}^{(a)}\mathrm{tr}\chi + \mathrm{curl}'' f' + f'\wedge(\zeta-\eta) - \frac{|f'|^2}{4}\left({}^{(a)}\mathrm{tr}\underline{\chi} + f'\wedge\underline{\xi}\right),\\
\widehat{\chi}'' &= \widehat{\chi} + \frac{1}{2}\nabla''\widehat{\otimes} f' + \frac{1}{2}f'\widehat{\otimes}(\zeta+\eta) - \frac{1}{2}\underline{\omega} f'\widehat{\otimes} f' - \frac{|f'|^2}{8}\left(\widehat{\underline{\chi}} + f'\widehat{\otimes}\underline{\xi}\right),
\end{aligned}
$$

$$
\begin{aligned}
\underline{\omega}'' &= \underline{\omega} + \frac{1}{2}f'\cdot\underline{\xi},\\
\omega'' &= \zeta\cdot f' - \frac{1}{2}|f'|^2\underline{\omega} - \frac{1}{2}f'\cdot f'\cdot\underline{\chi} - \frac{|f'|^2}{4}\underline{\xi}\cdot f',
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha'' &= \alpha + (f'\widehat{\otimes}\beta - {}^*f'\widehat{\otimes}{}^*\beta) + (f'\widehat{\otimes}f' - \frac{1}{2}{}^*f'\widehat{\otimes}{}^*f')\rho + \frac{3}{2}(f'\widehat{\otimes}{}^*f')\,{}^*\rho \\
&\quad + \text{l.o.t.}, \\
\underline{\alpha}'' &= \underline{\alpha}, \\
\beta'' &= \beta + \frac{3}{2}(f'\rho + {}^*f'\,{}^*\rho) + \text{ l.o.t.}, \\
\underline{\beta}'' &= \underline{\beta} + \frac{1}{2}\underline{\alpha}\cdot f' + \text{ l.o.t.}, \\
\rho'' &= \rho - f'\cdot\underline{\beta} + \text{ l.o.t.}, \\
{}^*\rho'' &= {}^*\rho - f'\wedge\underline{\beta} + \text{ l.o.t.},
\end{aligned}
$$

where the lower order terms denoted by l.o.t. are linear with respect to the Ricci coefficients and curvature components of the outgoing PG structure of ${}^{(ext)}\mathcal{M}$, and quadratic or higher order in f', and do not contain derivatives of the latter. Also, we have

$$
\begin{aligned}
e_4'(r) &= 1 + \frac{1}{4}|f'|^2 e_3(r), \\
e_4'(u) &= f'\cdot\nabla(u) + \frac{1}{4}|f'|^2 e_3(u), \\
e_4'(J^{(0)}) &= f'\cdot\nabla(J^{(0)}) + \frac{1}{4}|f'|^2 e_3(J^{(0)}), \\
\nabla'(r) &= \frac{1}{2}e_3(r)f', \\
\nabla'(u) &= \nabla(u) + \frac{1}{2}e_3(u)f', \\
\nabla'(J^{(0)}) &= \nabla(J^{(0)}) + \frac{1}{2}e_3(J^{(0)})f', \\
\nabla_4'\mathfrak{J} &= -\frac{1}{q}\mathfrak{J} + f'\cdot\nabla\mathfrak{J} + \frac{1}{4}|f'|^2\nabla_3\mathfrak{J}, \\
\nabla_a'\mathfrak{J} &= \nabla_a\mathfrak{J} + \frac{1}{2}f_a\nabla_3\mathfrak{J}, \;\; a = 1, 2,
\end{aligned}
$$

while the derivatives of r, u, $J^{(0)}$ and $\mathfrak{J}$ in the e_3 direction are the same as for the PG structure of ${}^{(ext)}\mathcal{M}$ since $e_3'' = e_3$. Together with the estimates (4.14) for f' on ${}^{(ext)}\mathcal{M}$, and the estimates for the PG structure of ${}^{(ext)}\mathcal{M}$ provided by[99] (4.6), we immediately infer

[99]We need additional estimates for $\nabla_3\Gamma_g$, α, β, $\nabla_3\alpha$ and $\nabla_3\beta$ compared to (4.6).

$$\max_{0\leq k\leq k_{small}+129}\sup_{{}^{(ext)}\mathcal{M}}\Bigg\{\Big(r^2u^{\frac{1}{2}+\delta_{dec}-2\delta_0}+ru^{1+\delta_{dec}-2\delta_0}\Big)|\mathfrak{d}^k(\Gamma''_g\setminus\{\check{H}''\})|$$
$$+ru^{1+\delta_{dec}-2\delta_0}|\mathfrak{d}^k\Gamma''_b|$$
$$+r^2u^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^{k-1}\nabla''_3\left(\Gamma''_g\setminus\{\check{H}''\}\right)\right|$$
$$+r^{\frac{7}{2}+\frac{\delta_B}{2}}\Big(|\mathfrak{d}^kA''|+|\mathfrak{d}^kB''|\Big)$$
$$+\left(r^{\frac{9}{2}+\frac{\delta_B}{2}}+r^4u^{\frac{1}{2}+\delta_{dec}-2\delta_0}\right)|\mathfrak{d}^{k-1}\nabla''_3A''|$$
$$+r^4u^{\frac{1}{2}+\delta_{dec}-2\delta_0}|\mathfrak{d}^{k-1}\nabla''_3B''|\Bigg\}\lesssim\epsilon, \tag{4.15}$$

where, according to Definition 3.25, Γ''_g, Γ''_b are defined as follows:

- the linearized quantities for the frame $(e''_1, e''_2, e''_3, e''_4)$ are defined in the same way as Definition 2.66 for the outgoing PG frame of ${}^{(ext)}\mathcal{M}$, with respect to the coordinates (r,θ) and the complex 1-form $\mathfrak{J}$ of the PG structure[100],
- in addition, we introduce the following linearized quantities which are trivial for an outgoing PG structure[101]

$$\widecheck{\underline{H}}'' = \underline{H}'' + \frac{a\overline{q}}{|q|^2}\mathfrak{J}, \qquad \widecheck{e''_4(r)} = e''_4(r) - 1, \qquad \widecheck{\nabla''_4\mathfrak{J}} = \nabla''_4\mathfrak{J} + \frac{1}{q}\mathfrak{J},$$

- the notation Γ''_b is the one of Definition 2.67, except that $\check{H}''$ does not belong to Γ''_b,
- the notation Γ'_g is given by

$$\Gamma''_g = \Gamma''_{g,1}\cup\Gamma''_{g,2},$$

where $\Gamma''_{g,1}$ is the one of Definition 2.67, and where $\Gamma''_{g,2}$ is given by[102]

They are easily obtained in the same way, i.e by interpolation between the bootstrap assumptions on decay and boundedness for the outgoing PG frame (e_4, e_3, e_1, e_2) of ${}^{(ext)}\mathcal{M}$.

[100]Thus, for example, $\widecheck{\mathrm{tr}X}'' = \mathrm{tr}X'' - \frac{2}{q}$, $\check{H}'' = H'' - \frac{aq}{|q|^2}\mathfrak{J}$, $\widecheck{e''_3(r)} = e''_3(r) - \frac{\Delta}{|q|^2}$, $\widecheck{\mathcal{D}''J^{(0)}} = \mathcal{D}''J^{(0)} - i\mathfrak{J}$, $\widecheck{\nabla''_3\mathfrak{J}} = \nabla''_3\mathfrak{J} - \frac{\Delta q}{|q|^4}\mathfrak{J}$,...

[101]In fact $\underline{H}$ satisfies $\underline{H} = -Z$.

[102]Note that all quantities in $\Gamma''_{g,2}$ vanish identically in the case of an outgoing PG structure except $\check{H}''$ and $\widecheck{\underline{H}}''$.

$$\Gamma''_{g,2} = \Big\{\omega'',\ \Xi'',\ \widecheck{\underline{H}}'',\ \widecheck{H}'',\ \widetilde{e''_4(r)},\ e''_4(u),\ e''_4(J^{(0)}),\ r^{-1}\nabla''(r),\ \widetilde{\nabla''_4\mathfrak{J}}\Big\}.$$

Furthermore, recall that $\xi = 0$, and that $\xi' = 0$ by the construction of f, see (4.2). Since we have, by the choice of f', $\xi'' = \xi$ on ${}^{(ext)}\mathcal{M}(r \leq u^{\frac{1}{2}})$ and $\xi'' = \xi'$ on ${}^{(ext)}\mathcal{M}(r \geq 2u^{\frac{1}{2}})$, we infer

$$\xi'' = 0 \quad \text{on} \quad {}^{(ext)}\mathcal{M} \setminus \left\{\frac{r}{2} \leq u^{\frac{1}{2}} \leq r\right\}.$$

Together with the fact that $\xi'' \in \Gamma''_g \setminus \{\widecheck{H}''\}$ and (4.15), this implies in particular, for $k \leq k_{small} + 129$,

$$\begin{aligned}
|\mathfrak{d}^k \xi''| &\lesssim \frac{\epsilon}{r^2 u^{\frac{1}{2}+\delta_{dec}-2\delta_0}} \mathbf{1}_{\frac{r}{2}\leq u^{\frac{1}{2}} \leq r} \lesssim \frac{\epsilon}{r^{3+2(\delta_{dec}-2\delta_0)}},\\
|\mathfrak{d}^{k-1}\nabla''_3\xi''| &\lesssim \frac{\epsilon}{r^2 u^{1+\delta_{dec}-2\delta_0}} \mathbf{1}_{\frac{r}{2}\leq u^{\frac{1}{2}} \leq r} \lesssim \frac{\epsilon}{r^3 u^{\frac{1}{2}+\delta_{dec}-2\delta_0}},
\end{aligned}$$

and thus, in addition to the estimates for ξ'' and $\nabla_3\xi''$ provided by (4.15), we have

$$\begin{aligned}
\max_{0\leq k\leq k_{small}+129} \sup_{{}^{(ext)}\mathcal{M}} \Big[& r^{3+2(\delta_{dec}-2\delta_0)}|\mathfrak{d}^k\xi''| \\
& + r^3 u^{\frac{1}{2}+\delta_{dec}-2\delta_0}|\mathfrak{d}^{k-1}\nabla''_3\xi''|\Big] \lesssim \epsilon.
\end{aligned} \tag{4.16}$$

Finally, since we have, by the choice of f', $\eta'' = \eta$ on ${}^{(ext)}\mathcal{M}(r \leq u^{\frac{1}{2}})$, and since $\widecheck{H} \in \Gamma_b$ satisfies (4.6), we infer, on ${}^{(ext)}\mathcal{M}(r \leq u^{\frac{1}{2}})$,

$$\max_{0\leq k\leq k_{small}+130} \sup_{{}^{(ext)}\mathcal{M}(r\leq u^{\frac{1}{2}})} r u^{1+\delta_{dec}-2\delta_0}|\mathfrak{d}^k\widecheck{H}''| \lesssim \epsilon. \tag{4.17}$$

Step 6. Notice that (4.17) yields the desired estimate for $\widecheck{H}''$ in ${}^{(ext)}\mathcal{M}(r \leq u^{\frac{1}{2}})$. We now focus on estimating $\widecheck{H}''$ in the region ${}^{(ext)}\mathcal{M}(r \geq u^{\frac{1}{2}})$. Proceeding as for the other Ricci coefficients would yield for $\widecheck{H}''$ the same behavior than $\widecheck{H}$ and hence a loss of r^{-1} compared to the desired estimate. Instead, we rely on the following null structure equations in Proposition 2.8

$$\begin{aligned}
& \nabla''_4 Z'' + \frac{1}{2}\mathrm{tr}X''(Z'' - \underline{H}'') - 2\omega''(Z'' + \underline{H}'') \\
= \ & 2\mathcal{D}''\omega'' + \frac{1}{2}\widehat{X}''\cdot(-\overline{Z}'' + \overline{\underline{H}}'') - B'' \\
& -\frac{1}{2}\mathrm{tr}\underline{X}''\Xi'' - 2\underline{\omega}''\Xi'' - \frac{1}{2}\overline{\Xi}''\cdot\widehat{\underline{X}}'',
\end{aligned}$$

$$
\begin{aligned}
& \nabla_4'' H'' + \frac{1}{2}\overline{\mathrm{tr} X}''(H'' - \underline{H}'') \\
= \quad & -\frac{1}{2}\widehat{X}'' \cdot (\overline{H}'' - \overline{\underline{H}}'') - B'' + \nabla_3''\Xi'' - 4\underline{\omega}''\Xi''.
\end{aligned}
$$

This yields

$$
\begin{aligned}
& \nabla_4''(H'' - Z'') + \frac{1}{2}\overline{\mathrm{tr} X}''(H'' - Z'') + \frac{1}{2}\widehat{X}'' \cdot (\overline{H'' - Z''}) \\
= \quad & -2\mathcal{D}''\omega'' + \frac{1}{2}(\mathrm{tr} X'' - \overline{\mathrm{tr} X}'')(Z'' - \underline{H}'') - 2\omega''(Z'' + \underline{H}'') \\
& +\nabla_3''\Xi'' - 4\underline{\omega}''\Xi'' + \frac{1}{2}\mathrm{tr}\underline{X}''\Xi'' + 2\underline{\omega}''\Xi'' + \frac{1}{2}\overline{\Xi}'' \cdot \widehat{\underline{X}}''.
\end{aligned}
$$

Since we have

$$
H'' = \frac{aq}{|q|^2}\mathfrak{J} + \widecheck{H}'', \qquad Z'' = \frac{a\overline{q}}{|q|^2}\mathfrak{J} + \widecheck{Z}'', \qquad \underline{H}'' = -\frac{a\overline{q}}{|q|^2}\mathfrak{J} + \widecheck{\underline{H}}'',
$$

we deduce

$$
\begin{aligned}
& \nabla_4''\left(\widecheck{H}'' - \widecheck{Z}''\right) + \frac{1}{2}\overline{\mathrm{tr} X}''\left(\widecheck{H}'' - \widecheck{Z}''\right) \\
= \quad & -\nabla_4''\left(\frac{a(q - \overline{q})}{|q|^2}\mathfrak{J}\right) - \frac{1}{\overline{q}}\left(\frac{a(q - \overline{q})}{|q|^2}\mathfrak{J}\right) + \frac{1}{2}\left(\frac{2}{q} - \frac{2}{\overline{q}}\right)\left(\frac{2a\overline{q}}{|q|^2}\mathfrak{J}\right) \\
& -2\mathcal{D}''\omega'' - \frac{1}{2}\widehat{X}'' \cdot (\overline{H'' - Z''}) + \frac{1}{2}(\mathrm{tr} X'' - \overline{\mathrm{tr} X}'')\left(\widecheck{Z}'' - \widecheck{\underline{H}}''\right) \\
& -2\omega''(Z'' + \underline{H}'') - \frac{1}{2}\overline{\widecheck{\mathrm{tr} X}}''\left(\frac{a(q - \overline{q})}{|q|^2}\mathfrak{J}\right) + \frac{1}{2}(\widecheck{\mathrm{tr} X}'' - \overline{\widecheck{\mathrm{tr} X}}'')\left(\frac{2a\overline{q}}{|q|^2}\mathfrak{J}\right) \\
& +\nabla_3''\Xi'' - 4\underline{\omega}''\Xi'' + \frac{1}{2}\mathrm{tr}\underline{X}''\Xi'' + 2\underline{\omega}''\Xi'' + \frac{1}{2}\overline{\Xi}'' \cdot \widehat{\underline{X}}''.
\end{aligned}
$$

Also, using $e_4(q) = 1$ and $\nabla_4\mathfrak{J} = -q^{-1}\mathfrak{J}$, we have

$$
\begin{aligned}
& \nabla_4''\left(\frac{a(q - \overline{q})}{|q|^2}\mathfrak{J}\right) \\
= \quad & \left(\nabla_4 + f \cdot \nabla + \frac{|f|^2}{4}\nabla_3\right)\left(\frac{a(q - \overline{q})}{|q|^2}\mathfrak{J}\right) \\
= \quad & \frac{a(q - \overline{q})}{|q|^2}\nabla_4\mathfrak{J} + e_4\left(\frac{a(q - \overline{q})}{|q|^2}\right)\mathfrak{J} + \left(f \cdot \nabla + \frac{|f|^2}{4}\nabla_3\right)\left(\frac{a(q - \overline{q})}{|q|^2}\mathfrak{J}\right) \\
= \quad & -\frac{1}{q}\frac{a(q - \overline{q})}{|q|^2}\mathfrak{J} + a\left(-\frac{1}{\overline{q}^2} + \frac{1}{q^2}\right)\mathfrak{J} + \left(f \cdot \nabla + \frac{|f|^2}{4}\nabla_3\right)\left(\frac{a(q - \overline{q})}{|q|^2}\mathfrak{J}\right)
\end{aligned}
$$

and hence

$$
\begin{aligned}
& \nabla_4''\left(\check{H}''-\check{Z}''\right)+\frac{1}{2}\overline{\mathrm{tr}X}''\left(\check{H}''-\check{Z}''\right)\\
= \; & -2\mathcal{D}''\omega''-\frac{1}{2}\widehat{X}''\cdot(\overline{H''-Z''})+\frac{1}{2}(\mathrm{tr}X''-\overline{\mathrm{tr}X}'')\left(\check{Z}''-\check{\underline{H}}''\right)\\
& -2\omega''(Z''+\underline{H}'')-\frac{1}{2}\overline{\widecheck{\mathrm{tr}X}}''\left(\frac{a(q-\overline{q})}{|q|^2}\mathfrak{J}\right)+\frac{1}{2}(\widecheck{\mathrm{tr}X}''-\overline{\widecheck{\mathrm{tr}X}}'')\left(\frac{2a\overline{q}}{|q|^2}\mathfrak{J}\right)\\
& -\left(f\cdot\nabla+\frac{|f|^2}{4}\nabla_3\right)\left(\frac{a(q-\overline{q})}{|q|^2}\mathfrak{J}\right)+\nabla_3''\Xi''-4\underline{\omega}''\Xi''\\
& +\frac{1}{2}\mathrm{tr}\underline{X}''\Xi''+2\underline{\omega}''\Xi''+\frac{1}{2}\overline{\Xi}''\cdot\widehat{\underline{X}}''.
\end{aligned}
$$

Next,

- we commute with $\mathfrak{d}\!\!/''$ and $\mathcal{L}\!\!\!/''_{\mathbf{T}}$, and we rely on the corresponding commutator identities,
- we use the above equation for $\nabla_4''(\check{H}''-\check{Z}'')$ to recover the e_4'' derivatives,
- we rely on the estimates (4.15), as well as the estimate (4.16) for ξ'',

which allows us to derive, for $k\leq k_{small}+129$,

$$
\begin{aligned}
& \left|\nabla_4''(\mathfrak{d}^k(\check{H}''-\check{Z}''))+\frac{1}{2}\overline{\mathrm{tr}X}''\mathfrak{d}^k(\check{H}''-\check{Z}'')\right|\\
\lesssim \; & \frac{\epsilon}{r^3u^{\frac{1}{2}+\delta_{dec}-2\delta_0}}+\frac{\epsilon}{r^2}|\mathfrak{d}^{\leq k}(\check{H}''-\check{Z}'')|.
\end{aligned}
$$

Since we have, by the choice of f', $\eta''=\eta'$ on ${}^{(ext)}\mathcal{M}(r\geq 2u^{\frac{1}{2}})$, and since $\check{H}'=0$ on Σ_*, see (4.2), we have $\check{H}''=0$ on Σ_*. Thus, integrating backwards from Σ_*, and using the control $\check{Z}''$ provided by (4.15), we infer

$$
\begin{aligned}
& \max_{0\leq k\leq k_{small}+129}\sup_{{}^{(ext)}\mathcal{M}} r^2u^{\frac{1}{2}+\delta_{dec}-2\delta_0}|\mathfrak{d}^k\check{H}''|\\
\lesssim \; & \epsilon+\max_{0\leq k\leq k_{small}+129}\sup_{{}^{(ext)}\mathcal{M}} r^2u^{\frac{1}{2}+\delta_{dec}-2\delta_0}|\mathfrak{d}^k\check{Z}''|\\
\lesssim \; & \epsilon.
\end{aligned}
$$

Also, commuting first the equation for $\nabla_4''(\check{H}''-\check{Z}'')$ with ∇_3'', using the schematic commutator identity

$$
[\nabla_3'',\nabla_4''] \;=\; 2\underline{\omega}''\nabla_4''-2\omega''\nabla_3''+2(\eta''-\underline{\eta}'')\cdot\nabla''+\left(\underline{\xi}''\xi'',\,\eta''\underline{\eta}'',\;{}^*\rho''\right),
$$

and proceeding as above to integrate backward from Σ_*, we also obtain

$$\begin{aligned}
&\max_{0\leq k\leq k_{small}+129}\sup_{^{(ext)}\mathcal{M}} r^2u^{1+\delta_{dec}-2\delta_0}|\mathfrak{d}^{k-1}\nabla_3''\check{H}''|\\
\lesssim\;& \epsilon+\max_{0\leq k\leq k_{small}+129}\sup_{^{(ext)}\mathcal{M}} r^2u^{1+\delta_{dec}-2\delta_0}|\mathfrak{d}^{k-1}\nabla_3''\check{Z}''|\\
\lesssim\;& \epsilon.
\end{aligned}$$

In view of the control of $\check{H}''$ on ${}^{(ext)}\mathcal{M}(r\leq u^{\frac{1}{2}})$ provided by (4.17), this yields

$$\begin{aligned}
\max_{0\leq k\leq k_{small}+129}\sup_{^{(ext)}\mathcal{M}} \Big\{\Big(r^2u^{\frac{1}{2}+\delta_{dec}-2\delta_0}+ru^{1+\delta_{dec}-2\delta_0}\Big)|\mathfrak{d}^k\check{H}''|\\
+r^2u^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^{k-1}\nabla_3''\check{H}''\right|\Big\} \;\lesssim\; \epsilon.
\end{aligned}$$

Thus, together with (4.15), we infer

$$\begin{aligned}
\max_{0\leq k\leq k_{small}+129}\sup_{^{(ext)}\mathcal{M}} \quad & \Bigg\{\Big(r^2u^{\frac{1}{2}+\delta_{dec}-2\delta_0}+ru^{1+\delta_{dec}-2\delta_0}\Big)|\mathfrak{d}^k\Gamma_g''|\\
&+ru^{1+\delta_{dec}-2\delta_0}|\mathfrak{d}^k\Gamma_b''|+r^2u^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^{k-1}\nabla_3''\Gamma_g''\right|\\
&+r^{\frac{7}{2}+\frac{\delta_B}{2}}\Big(|\mathfrak{d}^kA''|+|\mathfrak{d}^kB''|\Big)\\
&+\left(r^{\frac{9}{2}+\frac{\delta_B}{2}}+r^4u^{\frac{1}{2}+\delta_{dec}-2\delta_0}\right)|\mathfrak{d}^{k-1}\nabla_3''A''|\\
&+r^4u^{\frac{1}{2}+\delta_{dec}-2\delta_0}|\mathfrak{d}^{k-1}\nabla_3''B''|\Bigg\}\lesssim\epsilon.
\end{aligned}$$

Together with the control of f' provided by (4.14) and the control of ξ'' provided by (4.16), this concludes the proof of Proposition 3.26.

4.3. Proof of Proposition 3.29

Let (e_4,e_3,e_1,e_2) be the outgoing PG frame of ${}^{(ext)}\mathcal{M}$. We will exhibit another frame (e_4',e_3',e_1',e_2') of ${}^{(ext)}\mathcal{M}$ provided by

$$\begin{aligned}
e_4' &= e_4,\\
e_a' &= e_a+\frac{1}{2}\underline{f}_ae_4,\quad a=1,2,\\
e_3' &= e_3+\underline{f}^be_b+\frac{1}{4}|\underline{f}|^2e_4,
\end{aligned}\tag{4.18}$$

where $\underline{f}$ is such that

$$\underline{f} = 0 \text{ on } \Sigma_*, \qquad \widecheck{\underline{H}}' = 0 \text{ on } {}^{(ext)}\mathcal{M}. \tag{4.19}$$

Step 1. We start by deriving an equation for $\underline{f}$ on ${}^{(ext)}\mathcal{M}$. In view of (4.18), we have

$$\begin{aligned} 2\underline{\eta}'_a &= \mathbf{g}(\mathbf{D}_{e'_4}e'_3, e'_a) = \mathbf{g}\left(\mathbf{D}_{e_4}\left(e_3 + \underline{f}^b e_b + \frac{1}{4}|\underline{f}|^2 e_4\right), e_a + \frac{1}{2}\underline{f}_a e_4\right) \\ &= 2\underline{\eta}_a - 2\omega \underline{f}_a + e_4(\underline{f}_a) + \mathbf{g}(\mathbf{D}_4 e_b, e_a)\underline{f}_b + \frac{1}{2}|\underline{f}|^2\xi_a. \end{aligned}$$

Since $\omega = \xi = 0$ and $\underline{\eta} = -\zeta$ in the outgoing PG structure of ${}^{(ext)}\mathcal{M}$, we infer

$$\underline{\eta}' = -\zeta + \frac{1}{2}\nabla_4 \underline{f}.$$

This yields

$$\underline{H}' = -Z + \frac{1}{2}\nabla_4 \underline{F}, \qquad \underline{F} := \underline{f} + i\,{}^*\underline{f}.$$

Now, we have[103]

$$\underline{H}' = -\frac{a\overline{q}}{|q|^2}\mathfrak{J} + \widecheck{\underline{H}}', \qquad Z = \frac{a\overline{q}}{|q|^2}\mathfrak{J} + \widecheck{Z},$$

and hence

$$\underline{H}' + Z = \widecheck{\underline{H}}' + \widecheck{Z},$$

so that, together with the condition $\widecheck{\underline{H}}' = 0$ on ${}^{(ext)}\mathcal{M}$, see (4.19), we infer

$$\nabla_4 \underline{F} = 2\widecheck{Z}, \quad \text{on } {}^{(ext)}\mathcal{M}. \tag{4.20}$$

Step 2. Next, we estimate $\underline{f}$ on ${}^{(ext)}\mathcal{M}$. We assume the following local bootstrap assumption

$$|\mathfrak{d}^{\leq k_{small}+130}\underline{f}| \leq \frac{\sqrt{\epsilon}}{ru^{\frac{1}{2}+\delta_{dec}-2\delta_0} + r^{\delta_0}u^{1+\delta_{dec}-\frac{5\delta_0}{2}}} \quad \text{on } r \geq r_1, \tag{4.21}$$

[103] Recall that for the third frame of ${}^{(ext)}\mathcal{M}$, the Ricci coefficients and curvature components are also linearized using the scalar function r and θ and the complex 1-form $\mathfrak{J}$ attached to the principal frame of ${}^{(ext)}\mathcal{M}$.

where $r_1 \geq r_0$. Since $\underline{f} = 0$ on Σ_*, (4.21) holds for r_1 sufficiently large, and our goal is to prove that we may in fact choose $r_1 = r_0$ and replace $\sqrt{\epsilon}$ with ϵ in (4.21).

In view of the control of Γ_g in (4.6), we have

$$\sup_{{}^{(ext)}\mathcal{M}} \left(r^2 u^{\frac{1}{2}+\delta_{dec}-2\delta_0} + r u^{1+\delta_{dec}-2\delta_0}\right)|\mathfrak{d}^{\leq k_{small}+130}\check{Z}| \lesssim \epsilon$$

and hence, by interpolation,

$$\sup_{{}^{(ext)}\mathcal{M}} \left(r^2 u^{\frac{1}{2}+\delta_{dec}-2\delta_0} + r^{1+\delta_0} u^{1+\delta_{dec}-\frac{5\delta_0}{2}}\right)|\mathfrak{d}^{\leq k_{small}+130}\check{Z}| \lesssim \epsilon.$$

Using commutator identities, and in view of (4.9), we infer[104]

$$\left|\nabla_4'(\not{\mathfrak{d}}, \not{\mathcal{L}}_{\mathbf{T}})^k \underline{F}\right| \leq \frac{\epsilon}{r^2 u^{1+\delta_{dec}-2\delta_0} + r^{1+\delta_0} u^{1+\delta_{dec}-\frac{5\delta_0}{2}}}$$
$$\text{for } k \leq k_{small}+130, \ r \geq r_1.$$

Integrating backwards from Σ_* where $\underline{f} = 0$, we deduce

$$|(\not{\mathfrak{d}}, \not{\mathcal{L}}_{\mathbf{T}})^k \underline{f}| \leq \frac{\epsilon}{r u^{\frac{1}{2}+\delta_{dec}-2\delta_0} + r^{\delta_0} u^{1+\delta_{dec}-\frac{5\delta_0}{2}}} \quad \text{for } k \leq k_{small}+130, \ r \geq r_1.$$

Together with (4.20), we recover the e_4 derivatives and obtain

$$|\mathfrak{d}^k \underline{f}| \leq \frac{\epsilon}{r u^{\frac{1}{2}+\delta_{dec}-2\delta_0} + r^{\delta_0} u^{1+\delta_{dec}-\frac{5\delta_0}{2}}} \quad \text{for } k \leq k_{small}+130, \ r \geq r_1.$$

This is an improvement of the bootstrap assumption (4.9). Thus, we may choose $r_1 = r_0$, and we have

$$|\mathfrak{d}^k \underline{f}| \lesssim \frac{\epsilon}{r u^{\frac{1}{2}+\delta_{dec}-2\delta_0} + r^{\delta_0} u^{1+\delta_{dec}-\frac{5\delta_0}{2}}} \quad \text{for } k \leq k_{small}+130 \text{ on } {}^{(ext)}\mathcal{M}.$$

Also, commuting once (4.20) with e_3', using the schematic commutator identity

$$[\nabla_3', \nabla_4'] = 2\underline{\omega}'\nabla_4' - 2\omega'\nabla_3' + 2(\eta' - \underline{\eta}')\cdot\nabla' + \left(\underline{\xi}'\xi', \ \eta'\underline{\eta}', \ {}^*\rho'\right),$$

[104]Note that we have

$$\delta_{dec} - \frac{5\delta_0}{2}\delta_{dec} - \frac{5}{6}\delta_{dec} \geq \frac{\delta_{dec}}{6} > 0.$$

and the fact that $\nabla_3\check{Z} \in r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b$, and proceeding as above to integrate backward from Σ_* where $\nabla'_3\underline{f}$ is under control using the fact that $\underline{f}=0$ on Σ_* as well as (4.20), we also obtain

$$|\mathfrak{d}^{k-1}\nabla'_3\underline{f}| \lesssim \frac{\epsilon}{ru^{1+\delta_{dec}-2\delta_0}} \quad \text{for } k\leq k_{small}+130 \text{ on } {}^{(ext)}\mathcal{M}.$$

Collecting the two above estimates, we obtain

(4.22)
$$\begin{aligned}
|\mathfrak{d}^k\underline{f}| &\lesssim \frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}-2\delta_0}+u^{1+\delta_{dec}-\frac{5\delta_0}{2}}}, \quad \text{for } k\leq k_{small}+130 \text{ on } {}^{(ext)}\mathcal{M},\\
|\mathfrak{d}^{k-1}\nabla'_3\underline{f}| &\lesssim \frac{\epsilon}{ru^{1+\delta_{dec}-2\delta_0}} \quad \text{for } k\leq k_{small}+130 \text{ on } {}^{(ext)}\mathcal{M}.
\end{aligned}$$

Step 3. In view of Proposition 2.12 applied to our particular case, i.e. a triplet $(f', \underline{f}', \lambda')$ with $f'=0$ and $\lambda'=1$, and the fact that (e_4, e_3, e_1, e_2) is an outgoing PG frame, we have

$$\begin{aligned}
\xi' &= 0,\\
\underline{\xi}' &= \underline{\xi}+\frac{1}{2}\nabla'_3\underline{f}+\underline{\omega}\,\underline{f}+\frac{1}{4}\mathrm{tr}\,\underline{\chi}\,\underline{f}-\frac{1}{4}{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\underline{f}+\frac{1}{2}\underline{f}\cdot\widehat{\underline{\chi}}-\frac{1}{2}(\underline{f}\cdot\zeta)\underline{f}\\
&\quad +\frac{1}{4}|\underline{f}|^2\eta-\frac{1}{4}|\underline{f}|^2\eta'+\text{ l.o.t.},\\
\zeta' &= \zeta+\omega\underline{f}+\frac{1}{4}\underline{f}\mathrm{tr}\,\chi+\frac{1}{4}\,{}^*\underline{f}\,{}^{(a)}\mathrm{tr}\chi+\frac{1}{2}\lambda^{-1}\underline{f}\cdot\widehat{\chi}'+\text{ l.o.t.},\\
\eta' &= \eta+\frac{1}{4}\underline{f}\mathrm{tr}\,\chi-\frac{1}{4}\,{}^*\underline{f}\,{}^{(a)}\mathrm{tr}\chi+\frac{1}{2}\underline{f}\cdot\widehat{\chi}+\text{ l.o.t.},
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}\,\chi' &= \mathrm{tr}\,\chi,\\
{}^{(a)}\mathrm{tr}\chi' &= {}^{(a)}\mathrm{tr}\chi,\\
\widehat{\chi}' &= \widehat{\chi},
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}\,\underline{\chi}' &= \mathrm{tr}\,\underline{\chi}+\mathrm{div}'\underline{f}+\underline{f}\cdot\underline{\eta}-\underline{f}\cdot\zeta-\frac{1}{4}|\underline{f}|^2\lambda^{-1}\mathrm{tr}\,\chi'+\text{ l.o.t.},\\
{}^{(a)}\mathrm{tr}\underline{\chi}' &= {}^{(a)}\mathrm{tr}\underline{\chi}+\mathrm{curl}'\underline{f}+\underline{f}\wedge\underline{\eta}-\zeta\wedge\underline{f}-\frac{1}{4}|\underline{f}|^2\lambda^{-1}\,{}^{(a)}\mathrm{tr}\chi'+\text{ l.o.t.},\\
\widehat{\underline{\chi}}' &= \widehat{\underline{\chi}}+\nabla'\widehat{\otimes}\underline{f}+\underline{f}\widehat{\otimes}\underline{\eta}-\underline{f}\widehat{\otimes}\zeta-\frac{1}{4}|\underline{f}|^2\lambda^{-1}\widehat{\chi}'+\text{ l.o.t.},
\end{aligned}$$

$$\begin{aligned}
\omega' &= 0,\\
\underline{\omega}' &= \underline{\omega}-\frac{1}{2}\underline{f}\cdot\zeta-\frac{1}{2}\underline{f}\cdot\eta+-\frac{1}{8}|\underline{f}|^2\mathrm{tr}\,\chi+\text{ l.o.t.}
\end{aligned}$$

and

$$
\begin{array}{rcl}
\alpha' & = & \alpha,\\
\underline{\alpha}' & = & \underline{\alpha} - (\underline{f}\widehat{\otimes}\underline{\beta} - {}^*\underline{f}\widehat{\otimes}{}^*\underline{\beta}) + (\underline{f}\widehat{\otimes}\underline{f} - \frac{1}{2}{}^*\underline{f}\widehat{\otimes}{}^*\underline{f})\rho + \frac{3}{2}(\underline{f}\widehat{\otimes}{}^*\underline{f})\,{}^*\rho\\
&& + \text{ l.o.t.},\\
\beta' & = & \beta + \frac{1}{2}\alpha\cdot\underline{f} + \text{ l.o.t.},\\
\underline{\beta}' & = & \underline{\beta} - \frac{3}{2}(\underline{f}\rho + {}^*\underline{f}\,{}^*\rho) + \text{ l.o.t.},\\
\rho' & = & \rho + \underline{f}\cdot\beta + \text{ l.o.t.},\\
{}^*\rho' & = & {}^*\rho - \underline{f}\cdot{}^*\beta + \text{ l.o.t.}
\end{array}
$$

where the lower order terms denoted by l.o.t. are linear with respect to the Ricci coefficients and curvature components of the outgoing PG structure of ${}^{(ext)}\mathcal{M}$, and quadratic or higher order in $\underline{f}$, and do not contain derivatives of the latter. Also, we have

$$
\begin{array}{rcl}
e_3'(r) & = & e_3(r) + \frac{1}{4}|\underline{f}|^2,\\
e_3'(u) & = & e_3(u) + \underline{f}\cdot\nabla(u),\\
e_3'(J^{(0)}) & = & e_3(J^{(0)}) + \underline{f}\cdot\nabla((J^{(0)})),
\end{array}
$$

$$
\begin{array}{rcl}
\nabla'(r) & = & \frac{1}{2}\underline{f},\\
\nabla'(u) & = & \nabla(u),\\
\nabla'(J^{(0)}) & = & \nabla(J^{(0)}),
\end{array}
$$

$$
\begin{array}{rcl}
\nabla_3'\mathfrak{J} & = & \nabla_3'\mathfrak{J} + \underline{f}\cdot\nabla\mathfrak{J} - \frac{1}{4q}|\underline{f}|^2\mathfrak{J},\\
\nabla_a'\mathfrak{J} & = & \nabla_a\mathfrak{J} - \frac{1}{2q}\underline{f}_a\mathfrak{J},\;\; a=1,2,
\end{array}
$$

while the derivatives of r, u, $J^{(0)}$ and $\mathfrak{J}$ in the e_4 direction are the same as for the PG structure of ${}^{(ext)}\mathcal{M}$ since $e_4' = e_4$. Together with the estimates (4.22) for $\underline{f}$ on ${}^{(ext)}\mathcal{M}$, and the estimates for the PG structure of ${}^{(ext)}\mathcal{M}$ provided by[105] (4.6), we immediately infer

[105] We need additional estimates for $\nabla_3\Gamma_g$, α, β, $\nabla_3\alpha$ and $\nabla_3\beta$ compared to (4.6). They are easily obtained in the same way, i.e by interpolation between the bootstrap assumptions on decay and boundedness for the outgoing PG frame (e_4, e_3, e_1, e_2) of ${}^{(ext)}\mathcal{M}$.

$$\max_{0\leq k\leq k_{small}+129}\sup_{{}^{(ext)}\mathcal{M}}\Bigg\{\Big(r^2u^{\frac{1}{2}+\delta_{dec}-2\delta_0}+ru^{1+\delta_{dec}-\frac{5}{2}\delta_0}\Big)|\mathfrak{d}^k\Gamma_g'|$$
$$+ru^{1+\delta_{dec}-2\delta_0}|\mathfrak{d}^k\Gamma_b'|+r^2u^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^{k-1}\nabla_3'\Gamma_g'\right|$$
$$+r^{\frac{7}{2}+\frac{\delta_B}{2}}\Big(|\mathfrak{d}^kA'|+|\mathfrak{d}^kB'|\Big)$$
$$+\left(r^{\frac{9}{2}+\frac{\delta_B}{2}}+r^4u^{\frac{1}{2}+\delta_{dec}-2\delta_0}\right)|\mathfrak{d}^{k-1}\nabla_3'A'|$$
$$+r^4u^{\frac{1}{2}+\delta_{dec}-2\delta_0}|\mathfrak{d}^{k-1}\nabla_3'B'|\Bigg\}\lesssim\epsilon.$$

Finally, we have also obtained $\underline{\widecheck{H}}' = 0$, $\xi' = 0$ and $\omega' = 0$. This concludes the proof of proof of Proposition 3.29.

4.4. A second null frame covering ${}^{(top)}\mathcal{M}$

In order to construct the global frames of Propositions 3.33 and 3.35, respectively in section 4.5 and 4.6, we will need to cover ${}^{(top)}\mathcal{M}$ with a regular null frame that fully covers the following region

$$(4.23)\qquad {}^{(top)}\mathcal{M}\cup\big({}^{(int)}\mathcal{M}\cap\{\underline{u}\geq u_*-1\}\big)\cup\big({}^{(ext)}\mathcal{M}\cap\{u\geq u_*-1\}\big).$$

The ingoing PG frame of ${}^{(top)}\mathcal{M}$ is a priori a natural candidate, but it does unfortunately not fully cover the region[106] (4.23). We thus need to construct another structure that fully covers the region (4.23). This is the goal of this section where we construct in fact an ingoing PT structure[107] that fully covers the region (4.23).

4.4.1. Definition of an ingoing PT structure covering ${}^{(top)}\mathcal{M}$ While the ingoing PG structure of ${}^{(top)}\mathcal{M}$ is initialized on $\{u = u_*\}$, see section 3.2.5, we will initialize below the ingoing PT structure covering ${}^{(top)}\mathcal{M}$ on $\{u = u_* - 4\}$ to ensure that it fully covers the region (4.23).

To ease the notations, we denote with primes quantities associated to this ingoing PT structure. In particular, we consider the corresponding null frame (e_3', e_3', e_1', e_2'), scalar functions $(r', \underline{u}', \theta')$ and complex 1-form $\mathfrak{J}'$. The following holds:

[106] Indeed, even after being extended beyond $u = u_*$, it does not cover part of a neighborhood of S_* in ${}^{(ext)}\mathcal{M}\cap\{u\geq u_*-1\}$.

[107] It turns out to be simpler to ensure the property $\xi\in r^{-1}\Gamma_g$ stated in Proposition 4.8 with an ingoing PT structure than with an ingoing PG structure due to the simpler initialization on a hypersurface.

1. e'_3 is null ingoing geodesic.
2. We have

$$e'_3(r') = -1, \qquad e'_3(\theta') = 0, \qquad \nabla'_3(\overline{q'}\mathfrak{J}') = 0, \qquad q' = r' + ai\cos(\theta').$$

3. We have

$$H' = \frac{aq'}{|q'|^2}\mathfrak{J}'.$$

4. $\underline{u}'$ satisfies

$$e'_3(\underline{u}') = 0.$$

5. We denote by ${}^{(top)}\mathcal{M}'$ the region covered by this ingoing PT structure covering ${}^{(top)}\mathcal{M}$. ${}^{(top)}\mathcal{M}'$ is given by

$${}^{(top)}\mathcal{M}' = \{\underline{u}' \geq u_* - 4\} \setminus {}^{(ext)}\mathcal{M}(u \leq u_* - 4).$$

6. This ingoing PT structure is initialized on $\{u = u_* - 4\}$ as it will be made precise below.

4.4.2. Initialization of the ingoing PT structure covering ${}^{(top)}\mathcal{M}$ The second ingoing PG structure covering ${}^{(top)}\mathcal{M}$ is initialized from the outgoing PG structure of ${}^{(ext)}\mathcal{M}$ on $\{u = u_* - 4\}$ as follows:

1. The scalar functions (r', θ') and the complex 1-form $\mathfrak{J}'$ are prescribed on $\{u = u_* - 4\}$ as follows

$$r' = {}^{(ext)}r, \qquad \theta' = {}^{(ext)}\theta, \qquad \mathfrak{J}' = {}^{(ext)}\mathfrak{J}. \tag{4.24}$$

2. Moreover, the scalar function $\underline{u}'$ is prescribed on $\{u = u_* - 4\}$ as follows

$$\underline{u}' = u + 2\int_{r_0}^{{}^{(ext)}r} \frac{\tilde{r}^2 + a^2}{\tilde{r}^2 - 2m\tilde{r} + a^2} d\tilde{r}. \tag{4.25}$$

In particular, the 2-spheres $S(\underline{u}', r')$ coincide on $\{u = u_* - 4\}$ with $S(u, {}^{(ext)}r)$.

3. Finally, the null frame (e'_3, e'_4, e'_1, e'_2) is prescribed on $\{u = u_* - 4\}$ by the transformation formulas

$$\begin{aligned} e'_4 &= \lambda\, {}^{(ext)}e_4, \\ e'_a &= {}^{(ext)}e_a, \qquad a = 1, 2, \\ e'_3 &= \lambda^{-1}\, {}^{(ext)}e_3, \end{aligned} \tag{4.26}$$

where

$$\lambda = {}^{(ext)}\lambda = \frac{{}^{(ext)}\Delta}{|{}^{(ext)}q|^2}.$$

Remark 4.6. *Since we have* ${}^{(top)}\mathcal{M}' = \{\underline{u}' \geq u_* - 4\} \setminus {}^{(ext)}\mathcal{M}(u \leq u_* - 4)$, *and in view of the initialization of* $\underline{u}'$ *on* $\{u = u_* - 4\}$, *we infer that* ${}^{(top)}\mathcal{M}'$ *verifies*

$$(4.27) \qquad {}^{(top)}\mathcal{M} \cup \big({}^{(int)}\mathcal{M} \cap \{\underline{u} \geq u_* - 1\}\big) \cup \big({}^{(ext)}\mathcal{M} \cap \{u \geq u_* - 1\}\big) \subset {}^{(top)}\mathcal{M}'.$$

Remark 4.7. *Along a level set of* $\underline{u}'$ *in* ${}^{(top)}\mathcal{M}'$, *denoting* $r'_+(\underline{u}')$ *the maximal value of* r', *i.e. the one on* $\{u = u_* - 4\}$, *and* $r'_-(\underline{u}')$ *the minimal value of* r', *i.e. the one on* ${}^{(top)}\Sigma$, *we have* $0 < r'_+(\underline{u}') - r'_-(\underline{u}') \lesssim 1$, *see the analog statement in* (3.7) *for the ingoing PG structure of* ${}^{(top)}\mathcal{M}$. *In particular, the integration along* e'_3 *is always finite in* ${}^{(top)}\mathcal{M}'$.

4.4.3. Properties of the ingoing PT structure covering ${}^{(top)}\mathcal{M}$

Proposition 4.8. *Let* $\delta_0 > 0$ *be the small constant which satisfies* (3.61) *and let* $\delta'_{dec} = \delta_{dec} - 2\delta_0$. *Also, let the ingoing PT structure introduced in section 4.4.1 and initialized on* $\{u = u_* - 4\}$ *from the outgoing PG structure of* ${}^{(ext)}\mathcal{M}$ *in section 4.4.2. For convenience, we denote with primes the quantities associated to this ingoing PT structure. Then:*

1. *The scalar functions* $(r', \underline{u}', \theta')$ *and the complex 1-form* $\mathfrak{J}'$ *verify, for* $k \leq k_{small} + 130$,

$$(4.28) \qquad \sup_{{}^{(ext)}\mathcal{M}(u \geq u_* - 4)} u^{1+\delta'_{dec}}\Big(r^2\left|\mathfrak{d}^k(\mathfrak{J}' - {}^{(ext)}\mathfrak{J})\right| + r\left|\mathfrak{d}^k\big(\cos(\theta') - \cos({}^{(ext)}\theta)\big)\right| + \left|\mathfrak{d}^k(r' - {}^{(ext)}r, \underline{u}' - {}^{(ext)}\underline{u})\right|\Big)$$
$$+ \sup_{{}^{(int)}\mathcal{M}(\underline{u} \geq u_* - 1)} \underline{u}^{1+\delta'_{dec}}\Big|\mathfrak{d}^k\big(r' - {}^{(int)}r, \underline{u}' - \underline{u}, \cos(\theta') - \cos({}^{(int)}\theta), \mathfrak{J}' - {}^{(int)}\mathfrak{J}\big)\Big|$$
$$+ \sup_{{}^{(top)}\mathcal{M}(r \geq r_0)} ({}^{(top)}u)^{1+\delta'_{dec}}\Big(r^2\left|\mathfrak{d}^k(\mathfrak{J}' - {}^{(top)}\mathfrak{J})\right|$$

$$+r\Big|\mathfrak{d}^k\big(\cos(\theta')-\cos({}^{(top)}\theta)\big)\Big|$$
$$+\Big|\mathfrak{d}^k\big(r'-{}^{(top)}r,\underline{u}'-\underline{u}\big)\Big|\Big)$$
$$+\sup_{{}^{(top)}\mathcal{M}(r\le r_0)}\underline{u}^{1+\delta'_{dec}}\Big|\mathfrak{d}^k\big(r'-{}^{(top)}r,\underline{u}'-\underline{u},\cos(\theta')-\cos({}^{(top)}\theta),$$
$$\mathfrak{J}'-{}^{(top)}\mathfrak{J}\big)\Big|\lesssim\epsilon,$$

where we have introduced the following notation

$$(4.29)\qquad {}^{(ext)}\underline{u}:=u+2\int_{r_0}^{{}^{(ext)}r}\frac{\tilde{r}^2+a^2}{\tilde{r}^2-2m\tilde{r}+a^2}d\tilde{r}.$$

2. *Let:*
   - $(f,\underline{f},\lambda)$ *denote the change of frame coefficients from the outgoing PG frame of* ${}^{(ext)}\mathcal{M}$ *to the ingoing PT frame* (e'_4,e'_3,e'_1,e'_2),
   - $(\tilde{f},\tilde{\underline{f}},\tilde{\lambda})$ *denote the change of frame coefficients from the ingoing PG frame of* ${}^{(int)}\mathcal{M}$ *to the ingoing PT frame* (e'_4,e'_3,e'_1,e'_2),
   - $(\tilde{\tilde{f}},\tilde{\tilde{\underline{f}}},\tilde{\tilde{\lambda}})$ *denote the change of frame coefficients from the ingoing PG frame of* ${}^{(top)}\mathcal{M}$ *to the ingoing PT frame* (e'_4,e'_3,e'_1,e'_2).

   Then, we have, for $k\le k_{small}+130$,

$$(4.30)\qquad \sup_{{}^{(ext)}\mathcal{M}(u\ge u_*-4)}ru^{1+\delta'_{dec}}\left|\mathfrak{d}^k\left(f,\underline{f},\log\left(\frac{|{}^{(ext)}q|^2}{{}^{(ext)}\Delta}\lambda\right)\right)\right|$$
$$+\sup_{{}^{(int)}\mathcal{M}(\underline{u}\ge u_*-1)}\underline{u}^{1+\delta'_{dec}}\Big|\mathfrak{d}^k\big(\tilde{f},\tilde{\underline{f}},\tilde{\lambda}-1\big)\Big|$$
$$+\sup_{{}^{(top)}\mathcal{M}(r\ge r_0)}r\big({}^{(top)}u\big)^{1+\delta'_{dec}}\Big|\mathfrak{d}^k\big(\tilde{\tilde{f}},\tilde{\tilde{\underline{f}}},\tilde{\tilde{\lambda}}-1\big)\Big|$$
$$+\sup_{{}^{(top)}\mathcal{M}(r\le r_0)}\underline{u}^{1+\delta'_{dec}}\Big|\mathfrak{d}^k\big(\tilde{\tilde{f}},\tilde{\tilde{\underline{f}}},\tilde{\tilde{\lambda}}-1\big)\Big|\lesssim\epsilon.$$

3. $(\Gamma'_b,\Gamma'_g,A',B')$ *verify, for* $k\le k_{small}+129$,

$$(4.31)\qquad \sup_{{}^{(top)}\mathcal{M}'(r\ge r_0)}\Big(r\big({}^{(top)}u'\big)^{1+\delta'_{dec}}|\mathfrak{d}^k\Gamma'_b|\Big)$$
$$+\sup_{{}^{(top)}\mathcal{M}'(r\ge r_0)}\Big(\big(r^2\big({}^{(top)}u'\big)^{\frac{1}{2}+\delta'_{dec}}+r\big({}^{(top)}u'\big)^{1+\delta'_{dec}}\big)|\mathfrak{d}^k\Gamma'_g|\Big)$$
$$+\sup_{{}^{(top)}\mathcal{M}'(r\ge r_0)}\Big(r^2\big({}^{(top)}u'\big)^{1+\delta'_{dec}}|\mathfrak{d}^{k-1}\nabla'_3\Gamma'_g|+r^{\frac{7}{2}+\delta'_{dec}}|\mathfrak{d}^k(A',B')|\Big)$$

$$+\sup_{^{(top)}\mathcal{M}'(r\geq r_0)}\Big(r^4({}^{(top)}u')^{\frac{1}{2}+\delta'_{dec}}+r^{\frac{9}{2}+\delta'_{dec}}\Big)|\mathfrak{d}^{\leq k-1}\nabla'_3A'|$$
$$+\sup_{^{(top)}\mathcal{M}'(r\geq r_0)}r^4({}^{(top)}u')^{\frac{1}{2}+\delta'_{dec}}|\mathfrak{d}^{k-1}\nabla'_3B'|$$
$$+\sup_{^{(top)}\mathcal{M}'(r\leq r_0)}(\underline{u}')^{1+\delta'_{dec}}|\mathfrak{d}^k(\Gamma'_b,\Gamma'_g)|\lesssim\epsilon,$$

where the quantities (Γ'_b,Γ'_g) are given by Definition 9.26 with respect to the above ingoing PT structure, and where ${}^{(top)}u'$ is given by

$${}^{(top)}u' \quad := \quad \underline{u}'-2\int_{r_0}^{r'}\frac{\tilde{r}^2+a^2}{\tilde{r}^2-2m\tilde{r}+a^2}d\tilde{r}.$$

4. *Moreover, the following additional property holds on ${}^{(top)}\mathcal{M}'$*

$$\xi'\in r^{-1}\Gamma'_g. \tag{4.32}$$

4.4.4. Proof of Proposition 4.8 The proof proceeds in several steps.

Step 1. First, we focus on estimates in ${}^{(ext)}\mathcal{M}(u\geq u_*-4)$. For convenience, we denote in Step 1 to Step 5:

- without primes quantities associated to the outgoing PG structure of ${}^{(ext)}\mathcal{M}$,
- with primes quantities associated to the ingoing PT structure introduced in section 4.4.1 and initialized on $\{u=u_*-4\}$ in section 4.4.2.

Also, we denote by $(f,\underline{f},\lambda)$ the change of frame coefficients from the outgoing PG frame of ${}^{(ext)}\mathcal{M}$ to (e'_3,e'_4,e'_1,e'_2).

In view of the initialization (4.24) (4.25) (4.26) of the ingoing PT structure introduced in section 4.4.1, we have on $\{u=u_*-4\}$

$$f=\underline{f}=0,\ \ \lambda=\frac{\Delta}{|q|^2},\ \ r'=r,\ \ \underline{u}'=\underline{u},\ \ \theta'=\theta,\ \ \mathfrak{J}'=\mathfrak{J}, \tag{4.33}$$

where, in Step 1 to Step 5, $\underline{u}$ denotes ${}^{(ext)}\underline{u}$, i.e.

$$\underline{u}=u+2\int_{r_0}^{r}\frac{\tilde{r}^2+a^2}{\tilde{r}^2-2m\tilde{r}+a^2}d\tilde{r}$$

in view of (4.29). In order to control $(f,\underline{f},\lambda)$, we introduce the following auxiliary transformation

$$
\begin{aligned}
e_3' =&\lambda'\left(e_3+(\underline{f}')^b e_b+\frac{1}{4}|\underline{f}'|^2 e_3\right),\\
e_a' =&\left(\delta_a^b+\frac{1}{2}(f')_a(\underline{f}')^b\right)e_b+\frac{1}{2}(f')_a e_3+\left(\frac{1}{2}(\underline{f}')_a+\frac{1}{8}|\underline{f}'|^2(f')_a\right)e_4,\\
e_4' =&(\lambda')^{-1}\Bigg(\left(1+\frac{1}{2}f'\cdot\underline{f}'+\frac{1}{16}|f'|^2|\underline{f}'|^2\right)e_4+\left((f')^b+\frac{1}{4}|f'|^2(\underline{f}')^b\right)e_b\\
&+\frac{1}{4}|f'|^2 e_3\Bigg),
\end{aligned}
$$

where $\lambda'>0$ is a scalar and $(f',\underline{f}')$ are 1-forms. Since

$$
\begin{aligned}
\mathbf{g}(e_a',e_3) &= -\underline{f}_a=-\left((\underline{f}')_a+\frac{1}{4}|\underline{f}'|^2(f')_a\right),\\
\mathbf{g}(e_4',e_3) &= -2\lambda=-2(\lambda')^{-1}\left(1+\frac{1}{2}f'\cdot\underline{f}'+\frac{1}{16}|f'|^2|\underline{f}'|^2\right),\\
\mathbf{g}(e_4',e_a) &= f_a=(f')_a+\frac{1}{4}|f'|^2(\underline{f}')_a,
\end{aligned}
$$

we infer

$$
\begin{aligned}
\underline{f} &= \underline{f}'+\frac{1}{4}|\underline{f}'|^2 f',\\
\lambda &= (\lambda')^{-1}\left(1+\frac{1}{2}f'\cdot\underline{f}'+\frac{1}{16}|f'|^2|\underline{f}'|^2\right),\\
f &= f'+\frac{1}{4}|f'|^2\underline{f}',
\end{aligned}
\tag{4.34}
$$

so that it suffices to control $(f',\underline{f}',\lambda')$ in order to control $(f,\underline{f},\lambda)$. Note also that (4.33) and (4.34) imply

$$
f'=\underline{f}'=0,\quad \lambda'=\frac{|q|^2}{\Delta}\quad \text{on}\quad \{r=r_0\}. \tag{4.35}
$$

Now, let

$$
F':=f'+i\,{}^*f',\qquad \underline{F}':=\underline{f}'+i\,{}^*\underline{f}'.
$$

By exchanging the role of e_3 and e_4, we have the following analog of the two first transport equations of Corollary 2.14

$$
\nabla_{(\lambda')^{-1}e_3'}\underline{F}'+\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\underline{F}'+2\underline{\omega}\,\underline{F}' = -2\underline{\Xi}-\widehat{\underline{\chi}}\cdot\underline{F}'+E_1(\underline{f}',\Gamma),
$$

$$(\lambda')^{-1}\nabla'_3(\log(\lambda')) \quad = \quad 2\underline{\omega} - \underline{f}' \cdot (\zeta+\eta) + E_2(\underline{f}', \Gamma),$$

where $E_1(\underline{f}', \Gamma)$ and $E_2(\underline{f}', \Gamma)$ contain expressions of the type $O(\Gamma(\underline{f}')^2)$ with no derivatives. Also, the transformation formula of Proposition 2.12 for η' implies

$$H' \quad = H + \frac{1}{2}(\lambda')^{-1}\nabla'_3 F' - \underline{\omega} F' + O(r^{-1})\underline{F}' + \Gamma_g \cdot \underline{F}' + E_3(f', \underline{f}', \Gamma),$$

where $E_3(f', \underline{f}', \Gamma)$ contains expressions of the type $O(\Gamma(f', \underline{f}')^2)$ with no derivatives. Since we have $H' = \frac{aq'}{|q'|^2}\mathfrak{J}'$ in an ingoing PT structure, we infer

$$\begin{aligned}(\lambda')^{-1}\nabla'_3 F' - 2\underline{\omega} F' \quad = \quad & -2\check{H} + \frac{2aq'}{|q'|^2}\mathfrak{J}' - \frac{2aq}{|q|^2}\mathfrak{J} + O(r^{-1})\underline{F}' + \Gamma_g \cdot \underline{F}' \\ & + E_3(f, \underline{f}, \Gamma)\end{aligned}$$

and hence

(4.36)
$$\begin{aligned}\nabla_{(\lambda')^{-1}e'_3}\underline{F}' + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\underline{F}' + 2\underline{\omega}\,\underline{F}' =& \Gamma_b + \Gamma_b \cdot \underline{F}' + E_1(\underline{f}', \Gamma), \\ (\lambda')^{-1}\nabla'_3 F' - 2\underline{\omega} F' =& \frac{2aq'}{|q'|^2}\mathfrak{J}' - \frac{2aq}{|q|^2}\mathfrak{J} + \Gamma_b + O(r^{-1})\underline{F}' + \Gamma_g \cdot \underline{F}' \\ & + E_3(f', \underline{f}', \Gamma), \\ (\lambda')^{-1}e'_3(\log(\lambda')) =& 2\underline{\omega} + O(r^{-2})\underline{F}' + \Gamma_b \cdot \underline{F}' + E_2(\underline{f}', \Gamma).\end{aligned}$$

In order to control $(f', \underline{f}', \lambda')$ from (4.36), we will rely in particular on the following consequence of the bootstrap assumptions for the outgoing PG structure of ${}^{(top)}\mathcal{M}$

$$\begin{aligned}&\sup_{{}^{(ext)}\mathcal{M}}\left(r^2u^{\frac{1}{2}+\delta_{dec}-2\delta_0} + ru^{1+\delta_{dec}-2\delta_0}\right)|\mathfrak{d}^{\leq k_{small}+130}\Gamma_g| \\ & \quad + \sup_{{}^{(ext)}\mathcal{M}} ru^{1+\delta_{dec}-2\delta_0}|\mathfrak{d}^{\leq k_{small}+130}\Gamma_b| \quad \lesssim \quad \epsilon,\end{aligned} \tag{4.37}$$

see (4.6). We make the following local bootstrap assumption on ${}^{(ext)}\mathcal{M}(u_* - 4 \leq u \leq u_1)$, for $k \leq k_{small} + 130$,

$$u^{1+\delta'_{dec}}\left(r\left|\mathfrak{d}^k\left(f', \underline{f}', \log\left(\lambda'\frac{|q|^2}{\Delta}\right)\right)\right| + |\mathfrak{d}^k(r'-r)|\right.$$

$$+r|\mathfrak{d}^k(\cos(\theta')-\cos\theta)|+r^2|\mathfrak{d}^k(\mathfrak{J}'-\mathfrak{J})|\Bigg) \leq \sqrt{\epsilon}, \tag{4.38}$$

where $u_*-4<u_1\leq u_*$. In view of (4.33) and (4.35), (4.38) holds true provided u_1 is chosen sufficiently close to u_*-4. From now on, we assume the local bootstrap assumption (4.38).

Step 2. In this step, we improve (4.38) for $\underline{f}'$. To this end, we rely on the first equation of (4.36), i.e.

$$\nabla_{(\lambda')^{-1}e_3'}\underline{F}'+\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\underline{F}'+2\underline{\omega}\,\underline{F}' = \Gamma_b+\Gamma_b\cdot\underline{F}'+E_1(\underline{f}',\Gamma).$$

In view of the control for (Γ_b,Γ_g) provided by (4.37), the local bootstrap assumptions (4.38) on ${}^{(ext)}\mathcal{M}(u_*-4\leq u\leq u_1)$, and since $\underline{f}'=0$ on $\{u=u_*-4\}$ in view of (4.35), we easily infer

$$\sup_{{}^{(ext)}\mathcal{M}(u_*-4\leq u\leq u_1)} ru^{1+\delta'_{dec}}\left|\underline{f}'\right| \lesssim \epsilon,$$

where we recall that the integration of transport equations along e_3' in ${}^{(top)}\mathcal{M}'$ takes place on finite regions in r' in view of Remark 4.7. Also, commuting first with $\not{\mathfrak{d}}'$ to recover angular derivatives, then with e_4', using the transport equation to recover e_3' derivatives, and proceeding as above, we obtain the following control of higher order derivatives, for $k\leq k_{small}+130$,

$$\sup_{{}^{(ext)}\mathcal{M}(u_*-4\leq u\leq u_1)} ru^{1+\delta'_{dec}}\left|\mathfrak{d}^k\underline{f}'\right| \lesssim \epsilon, \tag{4.39}$$

which improves the local bootstrap assumption (4.38) for $\underline{f}'$.

Step 3. In this step, we improve (4.38) for λ'. To this end, we rely on the third equation of (4.36), i.e.

$$(\lambda')^{-1}e_3'(\log(\lambda')) = 2\underline{\omega}+O(r^{-2})\underline{F}'+\Gamma_b\cdot\underline{F}'+E_2(\underline{f}',\Gamma).$$

We have

$$(\lambda')^{-1}\nabla_3'\left(\log\left(\frac{\Delta}{|q|^2}\lambda'\right)\right) = (\lambda')^{-1}\nabla_3'(\log(\lambda'))+\frac{|q|^2}{\Delta}(\lambda')^{-1}\nabla_3'\left(\frac{\Delta}{|q|^2}\right)$$

which together with the above equation for $(\lambda')^{-1}\nabla_3'(\log\lambda)$ implies

$$(\lambda')^{-1}\nabla_3'\left(\log\left(\frac{\Delta|q|^2{}'}{\lambda}\right)\right) = 2\underline{\omega}+\frac{|q|^2}{\Delta}(\lambda')^{-1}\nabla_3'\left(\frac{\Delta}{|q|^2}\right)+O(r^{-2})\underline{F}'$$

$$+\Gamma_b\cdot\underline{F}'+E_2(\underline{f}',\Gamma).$$

Also, we have by the transformation formula for e_3'

$$\begin{aligned}(\lambda')^{-1}\nabla_3'\left(\frac{\Delta}{|q|^2}\right) &= \left(e_3+\underline{f}'\cdot\nabla+\frac{1}{4}|\underline{f}'|^2e_3\right)\left(\frac{\Delta}{|q|^2}\right)\\ &= e_3\left(\frac{\Delta}{|q|^2}\right)+\left(\underline{f}'\cdot\nabla+\frac{1}{4}|\underline{f}'|^2e_3\right)\left(\frac{\Delta}{|q|^2}\right)\end{aligned}$$

and hence

$$\begin{aligned}(\lambda')^{-1}\nabla_3'\left(\log\left(\frac{\Delta}{|q|^2}\lambda'\right)\right) &= 2\underline{\omega}+\frac{|q|^2}{\Delta}e_3\left(\frac{\Delta}{|q|^2}\right)+O(r^{-2})\underline{F}'+\Gamma_b\cdot\underline{F}'\\ &\quad +E_2(\underline{f}',\Gamma)+\frac{|q|^2}{\Delta}\left(\underline{f}'\cdot\nabla+\frac{1}{4}|\underline{f}'|^2e_3\right)\left(\frac{\Delta}{|q|^2}\right).\end{aligned}$$

Note that we have for the frame of ${}^{(ext)}\mathcal{M}$

$$2\underline{\omega}+\frac{|q|^2}{\Delta}e_3\left(\frac{\Delta}{|q|^2}\right) = 2\underline{\breve{\omega}}+r\Delta^{-1}\Gamma_b=\Gamma_b.$$

We deduce

$$\begin{aligned}(\lambda')^{-1}\nabla_3'\left(\log\left(\frac{\Delta}{|q|^2}\lambda'\right)\right) &= \Gamma_b+O(r^{-2})\underline{F}'+\Gamma_b\cdot\underline{F}'+E_2(\underline{f}',\Gamma)\\ &\quad +\frac{|q|^2}{\Delta}\left(\underline{f}'\cdot\nabla+\frac{1}{4}|\underline{f}'|^2e_3\right)\left(\frac{\Delta}{|q|^2}\right).\end{aligned}$$

In view of the control for (Γ_b,Γ_g) provided by (4.37), the local bootstrap assumptions (4.38) on ${}^{(ext)}\mathcal{M}(u_*-4\leq u\leq u_1)$, the control for $\underline{f}'$ in (4.39), and since $\lambda'=\frac{|q|^2}{\Delta}$ on $\{u=u_*-4\}$ in view of (4.35), we easily infer

$$\sup_{{}^{(ext)}\mathcal{M}(u_*-4\leq u\leq u_1)} ru^{1+\delta'_{dec}}\left|\log\left(\frac{\Delta}{|q|^2}\lambda'\right)\right| \lesssim \epsilon,$$

where we recall that the integration of transport equations along e_3' in ${}^{(top)}\mathcal{M}'$ takes place on finite regions in r' in view of Remark 4.7. Also, commuting first with $\not{\mathfrak{d}}'$ to recover angular derivatives, then with e_4', using the transport equation to recover e_3' derivatives, and proceeding as above, we obtain the following control of higher order derivatives, for $k\leq k_{small}+130$,

$$\sup_{{}^{(ext)}\mathcal{M}(u_*-4\leq u\leq u_1)} ru^{1+\delta'_{dec}}\left|\mathfrak{d}^k\log\left(\frac{\Delta}{|q|^2}\lambda'\right)\right| \lesssim \epsilon, \tag{4.40}$$

which improves the local bootstrap assumption (4.38) for λ'.

Note that (4.39), (4.40), (4.34) and (4.38) imply, for $k \leq k_{small} + 130$,

$$\sup_{{}^{(ext)}\mathcal{M}(u_*-4\leq u\leq u_1)} r u^{1+\delta'_{dec}} \left|\mathfrak{d}^k \left(\underline{f}, \log\left(\frac{|q|^2}{\Delta}\lambda\right)\right)\right| \lesssim \epsilon. \tag{4.41}$$

Step 4. In this step, we improve (4.38) for $r' - r$, $\underline{u}' - \underline{u}$, $\cos(\theta') - \cos\theta$ and $\mathfrak{J}' - \mathfrak{J}$, where $\underline{u}$ denotes ${}^{(ext)}\underline{u}$, i.e.

$$\underline{u} = u + 2\int_{r_0}^{r} \frac{\tilde{r}^2 + a^2}{\tilde{r}^2 - 2m\tilde{r} + a^2} d\tilde{r}$$

in view of (4.29). The ingoing PT structure on ${}^{(top)}\mathcal{M}'$ satisfies in particular

$$e_3'(r') = -1, \qquad e_3'(\underline{u}') = 0, \qquad e_3'(\theta') = 0, \qquad \nabla_3'\mathfrak{J}' = \frac{1}{\overline{q}'}\mathfrak{J}'.$$

Together with the relation between e_3' and e_3, see (2.6), we infer

$$\begin{aligned}
e_3'(r'-r) &= -1 - e_3'(r) = -1 - \lambda^{-1}e_3(r) + O(1)\underline{f}\cdot(f,\underline{f}) + r\Gamma_g\cdot\underline{f}\\
&= -1 + \lambda^{-1}\frac{\Delta}{|q|^2} - \lambda^{-1}\widecheck{e_3(r)} + O(1)\underline{f}\cdot(f,\underline{f}) + r\Gamma_g\cdot\underline{f}\\
&= -\lambda^{-1}\left(\lambda - \frac{\Delta}{|q|^2}\right) + r\Gamma_b + O(1)\underline{f}\cdot(f,\underline{f}) + r\Gamma_g\cdot\underline{f},\\
e_3'(\underline{u}'-\underline{u}) &= -e_3'(\underline{u}) = -\lambda^{-1}e_3(\underline{u}) + O(r^{-1})\underline{f} + O(r^{-1})\underline{f}\cdot(f,\underline{f})\\
&= -\lambda^{-1}e_3\left(u + \int_{r_0}^{r}\frac{\tilde{r}^2+a^2}{\tilde{r}^2-2m\tilde{r}+a^2}d\tilde{r}\right) + O(r^{-1})\underline{f}\\
&\quad +O(r^{-1})\underline{f}\cdot(f,\underline{f})\\
&= -\lambda^{-1}e_3(u) - \lambda^{-1}e_3(r)\frac{r^2+a^2}{\Delta} + O(r^{-1})\underline{f} + O(r^{-1})\underline{f}\cdot(f,\underline{f})\\
&= -\lambda^{-1}\left(\widecheck{e_3(u)} + \widecheck{e_3(r)}\frac{r^2+a^2}{\Delta}\right) + O(r^{-1})\underline{f} + O(r^{-1})\underline{f}\cdot(f,\underline{f})\\
&= r\Gamma_b + O(r^{-1})\underline{f} + O(r^{-1})\underline{f}\cdot(f,\underline{f}),\\
e_3'(\cos(\theta')-\cos(\theta)) &= -e_3'(\cos(\theta)) = -\lambda^{-1}e_3(\cos\theta) + O(r^{-1})\underline{f}\\
&\quad +\Gamma_b\cdot\underline{f}\cdot(f,\underline{f})\\
&= \Gamma_b + O(r^{-1})\underline{f} + \Gamma_b\cdot\underline{f}\cdot(f,\underline{f}),
\end{aligned}$$

and

$$\begin{aligned}
\nabla'_3(\mathfrak{J}'-\mathfrak{J})+\frac{1}{\overline{q'}}(\mathfrak{J}'-\mathfrak{J}) &= -\nabla'_3\mathfrak{J}-\frac{1}{\overline{q'}}\mathfrak{J}\\
&= -\lambda^{-1}\nabla_3\mathfrak{J}-\frac{1}{\overline{q'}}\mathfrak{J}+O(r^{-2})\underline{f}+\Gamma_b\cdot(f,\underline{f})\\
&\quad +O(r^{-2})\underline{f}\cdot(f,\underline{f})\\
&= \lambda^{-1}\frac{\Delta}{|q|^2}\frac{1}{q}\mathfrak{J}-\lambda^{-1}\widetilde{\nabla_3\mathfrak{J}}-\frac{1}{\overline{q'}}\mathfrak{J}+O(r^{-2})\underline{f}\\
&\quad +\Gamma_b\cdot(f,\underline{f})+O(r^{-2})\underline{f}\cdot(f,\underline{f})\\
&= r^{-1}\Gamma_b+O(r^{-2})\lambda^{-1}\left(\lambda-\frac{\Delta}{|q|^2}\right)\\
&\quad +O(r^{-3})(r'-r)+O(r^{-3})(\cos(\theta')-\cos\theta)\\
&\quad +O(r^{-2})\underline{f}+\Gamma_b\cdot(f,\underline{f})+O(r^{-2})\underline{f}\cdot(f,\underline{f}).
\end{aligned}$$

Collecting the above, we have obtained the following transport equations

$$\begin{aligned}
e'_3(r'-r) &= -\lambda^{-1}\left(\lambda-\frac{\Delta}{|q|^2}\right)+r\Gamma_b+O(1)\underline{f}\cdot(f,\underline{f})\\
&\quad +r\Gamma_g\cdot\underline{f},\\
e'_3(\underline{u}'-\underline{u}) &= r\Gamma_b+O(r^{-1})\underline{f}+O(r^{-1})\underline{f}\cdot(f,\underline{f}),\\
e'_3(\cos(\theta')-\cos(\theta)) &= \Gamma_b+O(r^{-1})\underline{f}+\Gamma_b\cdot\underline{f}\cdot(f,\underline{f}),\\
\nabla'_3(\mathfrak{J}'-\mathfrak{J})+\frac{1}{\overline{q'}}(\mathfrak{J}'-\mathfrak{J}) &= r^{-1}\Gamma_b+O(r^{-2})\lambda^{-1}\left(\lambda-\frac{\Delta}{|q|^2}\right)\\
&\quad +O(r^{-3})(r'-r)+O(r^{-3})(\cos(\theta')-\cos\theta)\\
&\quad +O(r^{-2})\underline{f}+\Gamma_b\cdot(f,\underline{f})+O(r^{-2})\underline{f}\cdot(f,\underline{f}).
\end{aligned}$$

Integrating these transport equations from $\{u=u_*-4\}$ where (4.33) holds, relying on the control for (Γ_b,Γ_g) provided by (4.37), on the local bootstrap assumptions (4.38) on ${}^{(ext)}\mathcal{M}(u_*-4\leq u\leq u_1)$, and on the control for $(\underline{f},\lambda)$ in (4.41), we easily infer, for $k\leq k_{small}+130$,

$$\sup_{{}^{(ext)}\mathcal{M}(u_*-4\leq u\leq u_1)} u^{1+\delta'_{dec}}\Big(|\mathfrak{d}^k(r'-r,\underline{u}'-\underline{u})|+r|\mathfrak{d}^k(\cos(\theta')-\cos\theta)| +r^2|\mathfrak{d}^k(\mathfrak{J}'-\mathfrak{J})|\Big) \lesssim \epsilon, \tag{4.42}$$

where we first control $r'-r$, $\underline{u}'-\underline{u}$ and $\cos(\theta')-\cos(\theta)$, and then plug it in the RHS of the transport equation for $\mathfrak{J}'-\mathfrak{J}$. This improves the local bootstrap assumption (4.38) for $r'-r$, $\cos(\theta')-\cos\theta$ and $\mathfrak{J}'-\mathfrak{J}$.

Step 5. We finally improve (4.38) for f'. In view of the second equation of (4.36), we have

$$\begin{aligned}(\lambda')^{-1}\nabla_3'F' - 2\underline{\omega}F' &= O(r^{-3})(r'-r) + O(r^{-3})(\cos(\theta')-\cos(\theta))\\ &+O(r^{-1})(\mathfrak{J}'-\mathfrak{J}) + \Gamma_b + O(r^{-1})\underline{F}' + \Gamma_g\cdot\underline{F}'\\ &+E_3(f,\underline{f},\Gamma).\end{aligned}$$

Integrating this transport equation from $\{u=u_*-4\}$ where $f=0$ in view of (4.33), relying on the control for (Γ_b,Γ_g) provided by (4.37), on the local bootstrap assumptions (4.38) on ${}^{(ext)}\mathcal{M}(u_*-4\leq u\leq u_1)$, and on the control for $(\underline{f}',\lambda',r-r',\cos(\theta')-\cos\theta,\mathfrak{J}'-\mathfrak{J})$ in (4.39), (4.40) and (4.42), we easily infer, for $k\leq k_{small}+130$,

$$\sup_{{}^{(ext)}\mathcal{M}(u_*-4\leq u\leq u_1)} ru^{1+\delta'_{dec}}|\mathfrak{d}^k f'| \lesssim \epsilon.$$

Together with (4.39), (4.40) and (4.42), this implies, for $k\leq k_{small}+130$,

$$\begin{aligned}\sup_{{}^{(ext)}\mathcal{M}(u_*-4\leq u\leq u_1)} u^{1+\delta'_{dec}}\Bigg(r\left|\mathfrak{d}^k\left(f',\underline{f}',\log\left(\frac{\Delta}{|q|^2}\lambda'\right)\right)\right| + |\mathfrak{d}^k(r'-r)|\\ +r|\mathfrak{d}^k(\cos(\theta')-\cos\theta)| + r^2|\mathfrak{d}^k(\mathfrak{J}'-\mathfrak{J})|\Bigg) \lesssim \epsilon,\end{aligned}$$

which improves the local bootstrap assumptions (4.38). Thus, we may choose $u_1=u_*$ and we obtain, for $k\leq k_{small}+130$,

$$\begin{aligned}\sup_{{}^{(ext)}\mathcal{M}(u\geq u_*-4)} u^{1+\delta'_{dec}}\Bigg(r\left|\mathfrak{d}^k\left(f',\underline{f}',\log\left(\frac{\Delta}{|q|^2}\lambda'\right)\right)\right| + |\mathfrak{d}^k(r'-r)|\\ +r|\mathfrak{d}^k(\cos(\theta')-\cos\theta)| + r^2|\mathfrak{d}^k(\mathfrak{J}'-\mathfrak{J})|\Bigg) \lesssim \epsilon.\end{aligned}$$

Together with (4.34), and the control of $\underline{u}'-\underline{u}$ derived in Step 4, we infer

(4.43)
$$\begin{aligned}\sup_{{}^{(ext)}\mathcal{M}(u\geq u_*-4)} u^{1+\delta'_{dec}}\Bigg(r\left|\mathfrak{d}^k\left(f,\underline{f},\log\left(\frac{|q|^2}{\Delta}\lambda\right)\right)\right| + |\mathfrak{d}^k(r'-r,\underline{u}'-\underline{u})|\\ +r|\mathfrak{d}^k(\cos(\theta')-\cos\theta)| + r^2|\mathfrak{d}^k(\mathfrak{J}'-\mathfrak{J})|\Bigg) \lesssim \epsilon,\end{aligned}$$

which are the stated estimates for $(f, \underline{f}, \lambda)$ and $(r', \underline{u}', \cos(\theta'), \mathfrak{J}')$ on the region ${}^{(ext)}\mathcal{M}(u \geq u_* - 4)$.

Finally, the transformation formulas of Proposition 2.12, the control for Γ_b, Γ_g, A and B provided by (4.37) and the control for $(f, \underline{f}, \lambda)$ and $(r', \cos(\theta'), \mathfrak{J}')$ on ${}^{(ext)}\mathcal{M}(u \geq u_* - 4)$ provided by (4.43) immediately yields, for $k \leq k_{small} + 129$,

$$\begin{aligned}\sup_{{}^{(ext)}\mathcal{M}(u\geq u_*-4)}&\left(ru^{1+\delta'_{dec}}\left|\mathfrak{d}^k\Gamma'_b\right| + \left(ru^{1+\delta'_{dec}} + r^2u^{\frac{1}{2}+\delta'_{dec}}\right)\left|\mathfrak{d}^k\Gamma'_g\right|\right)\\ &+ \sup_{{}^{(ext)}\mathcal{M}(u\geq u_*-4)}\left(r^2u^{1+\delta'_{dec}}\left|\mathfrak{d}^{k-1}\nabla'_3\Gamma'_g\right| + r^{\frac{7}{2}+\delta_B}\left|\mathfrak{d}^k(A', B')\right|\right)\\ &\qquad + \sup_{{}^{(ext)}\mathcal{M}(u\geq u_*-4)}\left(r^4u^{\frac{1}{2}+\delta'_{dec}}\left|\mathfrak{d}^{k-1}\nabla'_3B'\right|\right)\\ &\quad + \sup_{{}^{(ext)}\mathcal{M}(u\geq u_*-4)}\left(r^{\frac{9}{2}+\delta_B} + r^4u^{\frac{1}{2}+\delta'_{dec}}\right)\left|\mathfrak{d}^{k-1}\nabla'_3A'\right|\right) \lesssim \epsilon,\end{aligned}$$

which is the stated control of $(\Gamma'_b, \Gamma'_g, A', B')$ on ${}^{(ext)}\mathcal{M}(u \geq u_* - 4)$. This concludes the proof of the part of Proposition 4.8 on ${}^{(ext)}\mathcal{M}(u \geq u_* - 4)$.

Step 6. Next, we consider the control on ${}^{(int)}\mathcal{M}(\underline{u} \geq u_* - 1)$ and on ${}^{(top)}\mathcal{M}$. We start with ${}^{(int)}\mathcal{M}(\underline{u} \geq u_* - 1)$. In view of the control (4.43) in ${}^{(ext)}\mathcal{M}(u \geq u_* - 4)$, and in view of the initialization of the ingoing PG structure of ${}^{(int)}\mathcal{M}$ from the ingoing PG structure of ${}^{(ext)}\mathcal{M}$, see section 3.2.5, we have

(4.44)
$$\begin{aligned}\sup_{\mathcal{T}\cap\{\underline{u}\geq u_*-4\}}&\underline{u}^{1+\delta'_{dec}}\left|\mathfrak{d}^k\left(r' - {}^{(int)}r, \underline{u}' - \underline{u}, \cos(\theta') - \cos({}^{(int)}\theta), \mathfrak{J}' - {}^{(int)}\mathfrak{J}\right)\right|\\ &+ \sup_{\mathcal{T}\cap\{\underline{u}\geq u_*-4\}}\underline{u}^{1+\delta'_{dec}}\left|\mathfrak{d}^k(\tilde{f}, \underline{\tilde{f}}, \tilde{\lambda} - 1)\right| \lesssim \epsilon,\end{aligned}$$

where we recall that $\mathcal{T} = \{r = r_0\} = {}^{(int)}\mathcal{M} \cap {}^{(ext)}\mathcal{M}$, $\underline{u} = u$ on $\mathcal{T}$ and $(\tilde{f}, \underline{\tilde{f}}, \tilde{\lambda})$ denote the change of frame coefficients from the ingoing PG frame of ${}^{(int)}\mathcal{M}$ to the ingoing PT frame (e'_4, e'_3, e'_1, e'_2). Starting from the control provided by (4.44), and proceeding as in Step 1 to Step 5, we propagate (4.44) to ${}^{(int)}\mathcal{M}(\underline{u}' \geq u_* - 4)$ using transport equations in e'_3, and we obtain the following analog of (4.43)

$$\begin{aligned}\sup_{{}^{(int)}\mathcal{M}(\underline{u}\geq u_*-1)}&\underline{u}^{1+\delta'_{dec}}\Big|\mathfrak{d}^k\Big(r' - {}^{(int)}r, \underline{u}' - \underline{u}, \cos(\theta') - \cos({}^{(int)}\theta),\\ &\mathfrak{J}' - {}^{(int)}\mathfrak{J}\Big)\Big| + \sup_{{}^{(int)}\mathcal{M}(\underline{u}\geq u_*-1)}\underline{u}^{1+\delta'_{dec}}\left|\mathfrak{d}^k(\tilde{f}, \underline{\tilde{f}}, \tilde{\lambda} - 1)\right| \lesssim \epsilon,\end{aligned}$$

where we used in particular the fact that ${}^{(int)}\mathcal{M}(\underline{u}\geq u_*-1)\subset {}^{(int)}\mathcal{M}(\underline{u}'\geq u_*-4)$. This is the stated control of $(r'-{}^{(int)}r, \underline{u}'-\underline{u}, \cos(\theta')-\cos({}^{(int)}\theta), \mathfrak{J}'-{}^{(int)}\mathfrak{J})$ and $(\tilde{f}, \tilde{\underline{f}}, \tilde{\lambda}-1)$ on ${}^{(int)}\mathcal{M}(\underline{u}\geq u_*-1)$. Also, proceeding as in Step 5, we deduce, using in particular the transformation formulas of Proposition 2.12, the stated estimates for (Γ_b', Γ_g') on ${}^{(int)}\mathcal{M}(\underline{u}\geq u_*-1)$, which concludes the proof of the part of Proposition 4.8 on ${}^{(int)}\mathcal{M}(\underline{u}\geq u_*-1)$.

Finally, we consider ${}^{(top)}\mathcal{M}$. In view of the control (4.43) in ${}^{(ext)}\mathcal{M}(u\geq u_*-4)$, and in view of the initialization of the ingoing PG structure of ${}^{(top)}\mathcal{M}$ from the ingoing PG structure of ${}^{(ext)}\mathcal{M}$, see section 3.2.5, we obtain the analog of (4.44) on $\{u=u_*\}$ where we recall that ${}^{(top)}\mathcal{M}\cap{}^{(ext)}\mathcal{M}=\{u=u_*\}$. We then propagate to ${}^{(top)}\mathcal{M}$ using transport equations in e_3' and conclude as above. This concludes the proof of the part of Proposition 4.8 on ${}^{(top)}\mathcal{M}$.

Step 7. It remains to prove that $\xi'\in r^{-1}\Gamma_g'$ on ${}^{(top)}\mathcal{M}'$. We rely on the following linearized null structure equation for ingoing PT structures

$$\begin{aligned}\nabla_3'\Xi' &= O(r^{-1})\widecheck{\underline{H}}' + O(r^{-2})\widecheck{\mathrm{tr}X}' + O(r^{-2})\widehat{X}' + B' + O(r^{-1})\widetilde{\nabla_4'\mathfrak{J}'}\\ &\quad + O(r^{-3})\widetilde{e_4'(r')} + O(r^{-3})e_4'(\cos(\theta')) + \Gamma_b'\cdot\Gamma_g',\end{aligned}$$

see Proposition 9.27. We infer

$$\nabla_3'\xi' \quad = \quad r^{-1}\Gamma_g' + \Gamma_b'\cdot\Gamma_g'.$$

Also, since $f=\underline{f}=0$ on $\{u=u_*-4\}$ in view of (4.33), since $e_4'=\lambda^{(ext)}e_4$ is tangent to $\{u=u_*\}$, and since ${}^{(ext)}\xi=0$ in the outgoing PG frame of ${}^{(ext)}\mathcal{M}$, we infer

$$\xi'=0 \quad \text{on} \quad \{u=u_*-4\}.$$

Thus, integrating the above transport equation for ξ' from $\{u=u_*-4\}$ where $\xi'=0$, and using the control for (Γ_b', Γ_g') derived in Steps 5 and 6, we infer

$$\sup_{{}^{(top)}\mathcal{M}'}\left(r^2u^{1+\delta_{dec}'}+r^3u^{\frac{1}{2}+\delta_{dec}'}\right)|\xi'| \quad \lesssim \quad \epsilon,$$

where we recall that the integration of transport equations along e_3' in ${}^{(top)}\mathcal{M}'$ takes place on finite regions in r' in view of Remark 4.7. Also, commuting first with $\not{\mathfrak{d}}'$ to recover angular derivatives, then with e_4', using the transport equation to recover e_3' derivatives, and proceeding as above, we obtain the following control of higher order derivatives, for $k\leq k_{small}+130$,

$$\sup_{{}^{(top)}\mathcal{M}'}\left(r^2u^{1+\delta_{dec}'}+r^3u^{\frac{1}{2}+\delta_{dec}'}\right)\left|\mathfrak{d}^k\xi'\right| \quad \lesssim \quad \epsilon,$$

which is indeed consistent with $\xi' \in r^{-1}\Gamma'_g$ on ${}^{(top)}\mathcal{M}'$. This concludes the proof of Proposition 4.8.

4.5. Proof of Proposition 3.33

Recall the small constant $\delta_0 > 0$ introduced in the proof of Proposition 3.26

$$\delta_0 = \frac{130}{k_{large} - k_{small}} \leq \frac{\delta_{dec}}{3}. \tag{4.45}$$

In order to produce a global frame on $\mathcal{M}$, we will proceed in several steps. First, we extend the ingoing PG structure of ${}^{(int)}\mathcal{M}$ slightly inside ${}^{(ext)}\mathcal{M}$.

Lemma 4.9. *We may extend the ingoing PG structure of ${}^{(int)}\mathcal{M}$ into the region*

$$\mathcal{R}_{(1)} := {}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1) \cap \{\underline{u} \leq u_*\}. \tag{4.46}$$

Furthermore:

- *we have*

$$\max_{0\leq k\leq k_{small}+128} \sup_{\mathcal{R}_{(1)}} u^{1+\delta_{dec}-2\delta_0} \left|\mathfrak{d}^k({}^{(int)}\Gamma_g, {}^{(int)}\Gamma_b)\right| \lesssim \epsilon,$$

- *we have*

$$\max_{0\leq k\leq k_{small}+129} \sup_{\mathcal{R}_{(1)}} u^{1+\delta_{dec}-2\delta_0} \left|\mathfrak{d}^k \left(f, \underline{f}, \log\left(\frac{|{}^{(ext)}q|^2}{{}^{(ext)}\Delta}\lambda\right)\right)\right| \lesssim \epsilon,$$

  where $(f, \underline{f}, \lambda)$ denotes the change of frame coefficients from the outgoing PG frame of ${}^{(ext)}\mathcal{M}$ to the ingoing PG frame of ${}^{(int)}\mathcal{M}$,
- *we have*

$$\max_{0\leq k\leq k_{small}+129} \sup_{\mathcal{R}_{(1)}\cap {}^{(top)}\mathcal{M}'} u^{1+\delta_{dec}-2\delta_0} \left|\mathfrak{d}^k \left(f, \underline{f}, \log\lambda\right)\right| \lesssim \epsilon,$$

  where $(f, \underline{f}, \lambda)$ denotes the change of frame coefficients from the ingoing PG frame of ${}^{(int)}\mathcal{M}$ to the ingoing PT frame of Proposition 4.8 on the region ${}^{(top)}\mathcal{M}'$,
- *we have*

$$\max_{0\leq k\leq k_{small}+129} \sup_{\mathcal{R}_{(1)}} u^{1+\delta_{dec}-2\delta_0} \left(\left|\mathfrak{d}^k \left({}^{(int)}r - {}^{(ext)}r\right)\right|\right.$$

$$+\left|\mathfrak{d}^k\left(\cos({}^{(int)}\theta)-\cos({}^{(ext)}\theta)\right)\right|+\left|\mathfrak{d}^k\left({}^{(int)}\mathfrak{J}-{}^{(ext)}\mathfrak{J}\right)\right|\Big) \lesssim \epsilon,$$

and

$$\max_{0\leq k\leq k_{small}+129}\sup_{\mathcal{R}_{(1)}\cap{}^{(top)}\mathcal{M}'} u^{1+\delta_{dec}-2\delta_0}\Big(\left|\mathfrak{d}^k\left({}^{(int)}r-r',\underline{u}-\underline{u}'\right)\right|$$
$$+\left|\mathfrak{d}^k\left(\cos({}^{(int)}\theta)-\cos(\theta')\right)\right|+\left|\mathfrak{d}^k\left({}^{(int)}\mathfrak{J}-\mathfrak{J}'\right)\right|\Big) \lesssim \epsilon,$$

where $(r',\underline{u}',\theta',\mathfrak{J}')$ *are associated with the ingoing PT structure of Proposition 4.8.*

Proof. See Section 4.5.1. □

Remark 4.10. *Along level hypersurfaces of* $\underline{u}$*, in the region* $r\sim r_0$ *on* ${}^{(ext)}\mathcal{M}$*, we have*

$$\frac{du}{dr}=\frac{e_3(u)}{e_3(r)}=-2+O(r_0^{-1}).$$

In particular, since $\underline{u}=u$ *on* $\mathcal{T}=\{r=r_0\}$*, we infer*

$$u\geq(u_*-1)-2+O(r_0^{-1})>u_*-\frac{7}{2}$$
$$\text{on}\quad {}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1)\cap\{u_*-1\leq\underline{u}\leq u_*\}$$

and hence

$${}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1)\cap\{u_*-1\leq\underline{u}\leq u_*\}\subset{}^{(ext)}\mathcal{M}(u\geq u_*-7/2).$$

We now glue the ingoing PG frame of ${}^{(int)}\mathcal{M}$, extended slightly into ${}^{(ext)}\mathcal{M}$ in Lemma 4.9, to the ingoing PT frame of Proposition 4.8 in the matching region

$$\text{Match}_1 := {}^{(int)}\mathcal{M}(\underline{u}\geq u_*-1)\cup\Big({}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1)\cap\{u_*-1\leq\underline{u}\leq u_*\}\Big). \tag{4.47}$$

Lemma 4.11. *We denote with primes quantities associated to the ingoing PT structure of Proposition 4.8. There exists a frame* $(e''_4,e''_3,e''_1,e''_2)$ *on*

$${}^{(int)}\mathcal{M}\cup{}^{(top)}\mathcal{M}\cup{}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1)\cup{}^{(ext)}\mathcal{M}(u\geq u_*-1),$$

as well as a pair of scalar functions $(r'',J''^{(0)})$*, and a complex 1-form* $\mathfrak{J}''$*, such that:*

(a) In

$$ {}^{(top)}\mathcal{M}\cup\Big(\big({}^{(ext)}\mathcal{M}(u\geq u_*-1)\cup{}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1)\big)\setminus\{\underline{u}\leq u_*\}\Big), $$

we have

$$ (e''_4, e''_3, e''_1, e''_2) = (e'_4, e'_3, e'_1, e'_2), $$

as well as $r''=r'$, $J''^{(0)}=\cos(\theta')$, *and* $\mathfrak{J}''=\mathfrak{J}'$.

(b) In

$$ \Big({}^{(int)}\mathcal{M}\cup{}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1)\Big)\cap\{\underline{u}\leq u_*-1\}, $$

we have

$$ (e''_4, e''_3, e''_1, e''_2) = ({}^{(int)}e_4, {}^{(int)}e_3, {}^{(int)}e_1, {}^{(int)}e_2), $$

as well as $r''={}^{(int)}r$, $J''^{(0)}=\cos({}^{(int)}\theta)$, *and* $\mathfrak{J}''={}^{(int)}\mathfrak{J}$, *where we recall that the ingoing PG structure of* ${}^{(int)}\mathcal{M}$ *has been extended slightly into* ${}^{(ext)}\mathcal{M}$ *in Lemma 4.9.*

(c) In the matching region, we have

$$ \max_{0\leq k\leq k_{small}+128}\sup_{Match_1}\underline{u}^{1+\delta_{dec}-2\delta_0}\Big|\mathfrak{d}^k(\Gamma''_g,\Gamma''_b)\Big| \lesssim \epsilon, $$

where the Ricci coefficients and curvature components are the one associated to the frame $(e''_4, e''_3, e''_1, e''_2)$.

(d) In the matching region, we also have

$$ \max_{0\leq k\leq k_{small}+129}\sup_{Match_1}\underline{u}^{1+\delta_{dec}-2\delta_0}\Big|\mathfrak{d}^k(f,\underline{f},\log\lambda)\Big| \lesssim \epsilon, $$

where $(f,\underline{f},\lambda)$ *denotes*

- *either the change of frame coefficients from the frame* (e'_4, e'_3, e'_1, e'_2) *to* $(e''_4, e''_3, e''_1, e''_2)$,
- *or the one from* $({}^{(int)}e_4, {}^{(int)}e_3, {}^{(int)}e_1, {}^{(int)}e_2)$ *to* $(e''_4, e''_3, e''_1, e''_2)$.

(e) In the matching region, we have

$$ \max_{0\leq k\leq k_{small}+129}\sup_{Match_1}\underline{u}^{1+\delta_{dec}-2\delta_0}\Big|\mathfrak{d}^k\Big(r''-{}^{(int)}r,\ J''^{(0)}-\cos({}^{(int)}\theta),\ \mathfrak{J}''-{}^{(int)}\mathfrak{J}\Big)\Big| \lesssim \epsilon. $$

Proof. See Section 4.5.2. □

Finally, we glue a renormalization of the second frame of ${}^{(ext)}\mathcal{M}$ constructed in Proposition 3.26 to the frame of Lemma 4.11 in the matching region

$$\text{Match}_2 \quad := \quad {}^{(ext)}\mathcal{M}(u \geq u_* - 1) \cup {}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1). \tag{4.48}$$

Remark 4.12. *Note that the frame of Lemma 4.11 is defined on*

$${}^{(int)}\mathcal{M} \cup {}^{(top)}\mathcal{M} \cup \mathit{Match}_2.$$

In particular, the second frame of ${}^{(ext)}\mathcal{M}$ constructed in Proposition 3.26 and the frame of Lemma 4.11 are both defined in Match$_2$.

Lemma 4.13. *Let $({}^{(ext)}e'_4, {}^{(ext)}e'_3, {}^{(ext)}e'_1, {}^{(ext)}e'_2)$ be the second frame of ${}^{(ext)}\mathcal{M}$ constructed in Proposition 3.26, and let $(e''_4, e''_3, e''_1, e''_2)$ be the frame of Lemma 4.11. There exists a global null frame $({}^{(glo)}e_4, {}^{(glo)}e_3, {}^{(glo)}e_1, {}^{(glo)}e_2)$ defined on $\mathcal{M}$, as well as a pair of scalar functions $({}^{(glo)}r, {}^{(glo)}J^{(0)})$, and a complex 1-form ${}^{(glo)}\mathfrak{J}$, such that:*

(a) In ${}^{(ext)}\mathcal{M} \setminus \mathit{Match}_2$, we have

$$\begin{aligned} &({}^{(glo)}e_4, {}^{(glo)}e_3, {}^{(glo)}e_1, {}^{(glo)}e_2) \\ = \; &({}^{(ext)}\lambda {}^{(ext)}e'_4, {}^{(ext)}\lambda^{-1} {}^{(ext)}e'_3, {}^{(ext)}e'_1, {}^{(ext)}e'_2) \end{aligned}$$

where ${}^{(ext)}\lambda := \frac{{}^{(ext)}\Delta}{|{}^{(ext)}q|^2}$, as well as ${}^{(glo)}r = {}^{(ext)}r$, ${}^{(glo)}J^{(0)} = \cos({}^{(ext)}\theta)$, and ${}^{(glo)}\mathfrak{J} = {}^{(ext)}\mathfrak{J}$.

(b) In ${}^{(int)}\mathcal{M} \cup {}^{(top)}\mathcal{M}$, we have

$$({}^{(glo)}e_4, {}^{(glo)}e_3, {}^{(glo)}e_1, {}^{(glo)}e_2) = (e''_4, e''_3, e''_1, e''_2),$$

as well as ${}^{(glo)}r = r''$, ${}^{(glo)}J^{(0)} = J''^{(0)}$, and ${}^{(glo)}\mathfrak{J} = \mathfrak{J}''$.

(c) In the matching region, we have, for $k \leq k_{small} + 128$,

$$\begin{aligned} &\sup_{\mathit{Match}_2} \left(r u^{1+\delta'_{dec}} \left| \mathfrak{d}^k {}^{(glo)}\Gamma_b \right| + \left(r u^{1+\delta'_{dec}} + r^2 u^{\frac{1}{2}+\delta'_{dec}} \right) \left| \mathfrak{d}^k {}^{(glo)}\Gamma_g \right| \right) \\ &+ \sup_{\mathit{Match}_2} \left(r^2 u^{1+\delta'_{dec}} \left| \mathfrak{d}^{k-1} \nabla_{{}^{(glo)}e_3} {}^{(glo)}\Gamma_g \right| + r^{\frac{7}{2}+\delta_B} |\mathfrak{d}^k ({}^{(glo)}A, {}^{(glo)}B)| \right) \\ &\qquad + \sup_{\mathit{Match}_2} r^4 u^{\frac{1}{2}+\delta'_{dec}} \left| \mathfrak{d}^{k-1} \nabla_{{}^{(glo)}e_3} {}^{(glo)}B \right| \\ &\qquad + \sup_{\mathit{Match}_2} \left(r^{\frac{9}{2}+\delta_B} + r^4 u^{\frac{1}{2}+\delta'_{dec}} \right) \left| \mathfrak{d}^{k-1} \nabla_{{}^{(glo)}e_3} {}^{(glo)}A \right| \lesssim \epsilon. \end{aligned}$$

(d) *In the matching region, we have*

$$\max_{0\le k\le k_{small}+129}\sup_{Match_2} ru^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^k(f,\underline{f},\log\lambda)\right| \lesssim \epsilon,$$

where $(f,\underline{f},\lambda)$ *denotes*

- *either the change of frame coefficients from* $(e''_4,e''_3,e''_1,e''_2)$ *to the global frame* $({}^{(glo)}e_4,{}^{(glo)}e_3,{}^{(glo)}e_1,{}^{(glo)}e_2)$,
- *or the one from* $({}^{(ext)}\lambda{}^{(ext)}e'_4,{}^{(ext)}\lambda^{-1}{}^{(ext)}e'_3,{}^{(ext)}e'_1,{}^{(ext)}e'_2)$ *to the global frame* $({}^{(glo)}e_4,{}^{(glo)}e_3,{}^{(glo)}e_1,{}^{(glo)}e_2)$.

(e) *In the matching region, we have*

$$\max_{0\le k\le k_{small}+129}\sup_{Match_2} u^{1+\delta_{dec}-2\delta_0}\Bigg(\left|\mathfrak{d}^k\left({}^{(glo)}r-r''\right)\right| +r\left|\mathfrak{d}^k\left({}^{(glo)}J^{(0)}-J''^{(0)}\right)\right|+r^2\left|\mathfrak{d}^k\left({}^{(glo)}\mathfrak{J}-\mathfrak{J}''\right)\right|\Bigg)\lesssim\epsilon$$

and

$$\max_{0\le k\le k_{small}+129}\sup_{Match_2} u^{1+\delta_{dec}-2\delta_0}\Bigg(\left|\mathfrak{d}^k\left({}^{(glo)}r-{}^{(ext)}r\right)\right| +r\left|\mathfrak{d}^k\left({}^{(glo)}J^{(0)}-\cos({}^{(ext)}\theta)\right)\right|+r^2\left|\mathfrak{d}^k\left({}^{(glo)}\mathfrak{J}-{}^{(ext)}\mathfrak{J}\right)\right|\Bigg)\lesssim\epsilon.$$

Proof. See Section 4.5.3. □

We are now ready to prove Proposition 3.33. The global frame of Proposition 3.33 is the one constructed in Lemma 4.13. First, note from the definition of Match_1 in (4.47) and Match_2 in (4.48) that

$$\mathrm{Match}=(\mathrm{Match}_1\cap{}^{(int)}\mathcal{M})\cup\mathrm{Match}_2,$$

where Match is the matching region of Definition 3.31 appearing in the statement of Proposition 3.33. Next, note that property (c) of Proposition 3.33 follows from property (b) of Lemma 4.13, together with property (a) of Lemma 4.11 and the third property of Proposition 4.8. Also, properties (a) and (b) of Proposition 3.33 follow from property (a) and (b) of Lemma 4.13, together with property (b) of Lemma 4.11. Furthermore, properties (d), (e) and (f) of Proposition 3.33 follow from properties (c), (d) and (e) of Lemma 4.13, as well as property (c), (d) and (e) of Lemma 4.11. Finally, property (g) of Proposition 3.33 follows from the fifth property of Proposition 3.26 for the region ${}^{(ext)}\mathcal{M}(u\le u_*-1)$, and from the fourth property of Proposition 4.8 for ${}^{(top)}\mathcal{M}(r\ge r_0)$. This concludes the proof of Proposition 3.33.

4.5.1. Proof of Lemma 4.9 To simplify the notations, in this section, we denote

- by (e_4, e_3, e_1, e_2) the outgoing PG frame of ${}^{(ext)}\mathcal{M}$,
- by (u, r, θ) and by $\mathfrak{J}$ respectively the triplet of scalar functions and the complex 1-form associated to the outgoing PG structure of ${}^{(ext)}\mathcal{M}$,
- by (e'_4, e'_3, e'_1, e'_2) the ingoing PG frame of ${}^{(int)}\mathcal{M}$ slightly extended into ${}^{(ext)}\mathcal{M}$,
- by $(\underline{u}, r', \theta')$ and by $\mathfrak{J}'$ respectively the triplet of scalar functions and the complex 1-form associated to the ingoing PG structure of ${}^{(int)}\mathcal{M}$ slightly extended into ${}^{(ext)}\mathcal{M}$,
- by $(f, \underline{f}, \lambda)$ the change of frame from (e_4, e_3, e_1, e_2) to (e'_4, e'_3, e'_1, e'_2).

Recall from Section 3.2.5 that we have, in view of the initialization of the ingoing PG frame of ${}^{(int)}\mathcal{M}$ on $\mathcal{T} = \{r = r_0\}$ from the outgoing PG frame of ${}^{(ext)}\mathcal{M}$,

$$\underline{u} = u, \quad r' = r, \quad \theta' = \theta, \quad f = \underline{f} = 0, \quad \lambda = \frac{\Delta}{|q|^2} \quad \text{on} \quad \{r = r_0\}. \tag{4.49}$$

In order to control $(f, \underline{f}, \lambda)$, we introduce, as in the proof of Proposition 4.8, the following auxiliary transformation

$$\begin{aligned}
e'_3 =& \lambda' \left(e_3 + (\underline{f}')^b e_b + \frac{1}{4} |\underline{f}'|^2 e_3 \right), \\
e'_a =& \left(\delta_a^b + \frac{1}{2} (f')_a (\underline{f}')^b \right) e_b + \frac{1}{2} (f')_a e_3 + \left(\frac{1}{2} (\underline{f}')_a + \frac{1}{8} |\underline{f}'|^2 (f')_a \right) e_4, \\
e'_4 =& (\lambda')^{-1} \Bigg(\left(1 + \frac{1}{2} f' \cdot \underline{f}' + \frac{1}{16} |f'|^2 |\underline{f}'|^2 \right) e_4 + \left((f')^b + \frac{1}{4} |f'|^2 (\underline{f}')^b \right) e_b \\
& + \frac{1}{4} |f'|^2 e_3 \Bigg),
\end{aligned}$$

where $\lambda' > 0$ is a scalar and $(f', \underline{f}')$ are 1-forms. In view of (4.34), it suffices to control $(f', \underline{f}', \lambda')$ in order to control $(f, \underline{f}, \lambda)$. Note also that (4.49) and (4.34) imply

$$f' = \underline{f}' = 0, \quad \lambda' = \frac{|q|^2}{\Delta} \quad \text{on} \quad \{r = r_0\}. \tag{4.50}$$

Now, let

$$F' := f' + i\, {}^*f', \qquad \underline{F}' := \underline{f}' + i\, {}^*\underline{f}'.$$

By exchanging the role of e_3 and e_4, we have the following analog of the transport equations of Corollary 2.14

$$\begin{aligned}
\nabla_{(\lambda')^{-1}e_3'}\underline{F}' + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\underline{F}' + 2\underline{\omega}\,\underline{F}' &= -2\underline{\Xi} - \widehat{\underline{\chi}}\cdot\underline{F}' + E_1(\underline{f}',\Gamma),\\
(\lambda')^{-1}\nabla_3'(\log(\lambda')) &= 2\underline{\omega} - \underline{f}'\cdot(\zeta+\eta) + E_2(\underline{f}',\Gamma),\\
\nabla_{(\lambda')^{-1}e_3'}F' + \frac{1}{2}\mathrm{tr}\underline{X}F' &= -2(H-Z) - 2\mathcal{D}'(\log(\lambda')) + 2\omega F'\\
&\quad +E_3(\nabla'^{\leq 1}\underline{f}', f', \Gamma, \lambda\underline{\chi}'),
\end{aligned}$$

where $E_1(f,\Gamma)$ and $E_2(f,\Gamma)$ contain expressions of the type $O(\Gamma(\underline{f}')^2)$ with no derivatives, and $E_3(f,\underline{f},\Gamma)$ contains expressions of the type $f'\nabla'\underline{f}' + O(\Gamma(f,\underline{f})^2)$.

Since the transport equations for $\underline{F}'$ and λ' are the same as the ones in Step 1 of the proof of Proposition 4.8, we may proceed as in Step 2 and Step 3 of the proof of Proposition 4.8 and obtain

$$\max_{k\leq k_{small}+130}\ \sup_{{}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1)\cap\{\underline{u}\leq u_*\}} u^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^k\left(\underline{f}', \log\left(\frac{\Delta}{|q|^2}\lambda'\right)\right)\right| \lesssim \epsilon.$$

Next, we estimate f'. We have

$$\mathcal{D}'(\log(\lambda')) = \mathcal{D}'\left(\log\left(\frac{\Delta}{|q|^2}\lambda'\right)\right) + \mathcal{D}'\left(\log\left(\frac{|q|^2}{\Delta}\right)\right).$$

Using the transformation formula for e_a', we infer

$$\begin{aligned}
\mathcal{D}'(\log(\lambda')) &= \mathcal{D}'\left(\log\left(\frac{\Delta}{|q|^2}\lambda'\right)\right) + \mathcal{D}\left(\log\left(\frac{|q|^2}{\Delta}\right)\right)\\
&\quad + \left(F'\underline{f}'\cdot\nabla + F'e_3 + \left(\frac{1}{2}\underline{F}' + \frac{1}{8}|\underline{f}'|^2F'\right)e_4\right)\left(\log\left(\frac{|q|^2}{\Delta}\right)\right).
\end{aligned}$$

Together with the above transport equation for F', we deduce

$$\begin{aligned}
&\nabla_{(\lambda')^{-1}e_3'}F' + \frac{1}{2}\mathrm{tr}\underline{X}F'\\
= &-2\left(H - Z + \mathcal{D}\left(\log\left(\frac{|q|^2}{\Delta}\right)\right)\right) - 2\mathcal{D}'\left(\log\left(\frac{\Delta}{|q|^2}\lambda'\right)\right) + 2\omega\underline{F}'\\
&+E_3(\nabla'^{\leq 1}\underline{f}', f', \Gamma, \lambda\underline{\chi}')
\end{aligned}$$

$$-2\left(F'\underline{f}'\cdot\nabla+F'e_3+\left(\frac{1}{2}\underline{F}'+\frac{1}{8}|\underline{f}'|^2F'\right)e_4\right)\left(\log\left(\frac{|q|^2}{\Delta}\right)\right).$$

Note that we have for the frame of ${}^{(ext)}\mathcal{M}$, recalling that $\mathcal{D}(r)=0$ and hence $\mathcal{D}(\Delta)=0$,

$$H-Z+\mathcal{D}\left(\log\left(\frac{|q|^2}{\Delta}\right)\right) = \frac{aq}{|q|^2}\mathfrak{J}+\check{H}-\frac{a\overline{q}}{|q|^2}\mathfrak{J}-\check{Z}-\frac{\mathcal{D}(|q|^2)}{|q|^2}=\Gamma_b$$

and hence

$$\begin{aligned}&\nabla_{(\lambda')^{-1}e_3'}F'+\frac{1}{2}\mathrm{tr}\underline{X}F'\\ = \;&\Gamma_b-2\mathcal{D}'\left(\log\left(\frac{\Delta}{|q|^2}\lambda'\right)\right)+2\omega\underline{F}'+E_3(\nabla'^{\leq 1}\underline{f}',f',\Gamma,\lambda\underline{\chi}')\\ &-2\left(F'\underline{f}'\cdot\nabla+F'e_3+\left(\frac{1}{2}\underline{F}'+\frac{1}{8}|\underline{f}'|^2F'\right)e_4\right)\left(\log\left(\frac{|q|^2}{\Delta}\right)\right).\end{aligned}$$

Integrating this transport equation from $\{r=r_0\}$, where $F'=0$ in view of (4.50), to the region ${}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1)\cap\{\underline{u}\leq u_*\}$, using the above control for $\underline{f}'$ and λ', the control (4.6) for the outgoing PG structure of ${}^{(ext)}\mathcal{M}$, and the above mentioned structure of the error term $E_3(\nabla'^{\leq 1}\underline{f}',f',\Gamma,\lambda\underline{\chi}'$, we easily deduce, since $F'=f'+i\,{}^*f'$,

$$\max_{k\leq k_{small}+129}\sup_{{}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1)\cap\{\underline{u}\leq u_*\}}u^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^kf'\right| \lesssim \epsilon.$$

Together with the above control of $\underline{f}'$ and λ', we infer

$$\max_{k\leq k_{small}+129}\sup_{{}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1)\cap\{\underline{u}\leq u_*\}}u^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^k\left(f',\underline{f}',\frac{\Delta}{|q|^2}\lambda'\right)\right| \lesssim \epsilon.$$

In view of (4.34), we infer

$$\max_{k\leq k_{small}+129}\sup_{{}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1)\cap\{\underline{u}\leq u_*\}}u^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^k\left(f,\underline{f},\frac{|q|^2}{\Delta}\lambda\right)\right| \lesssim \epsilon,$$

which is the desired control of $(f,\underline{f},\lambda)$ in ${}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1)\cap\{\underline{u}\leq u_*\}$.

Next, since $e_3'(r)=1$, $e_3'(\cos\theta)=0$ and $\nabla_3'\mathfrak{J}'=\frac{1}{q'}\mathfrak{J}'$, we may proceed as in Step 4 of the proof of Proposition 4.8 and obtain

$$\begin{aligned}
e_3'(r'-r) &= -\lambda^{-1}\left(\lambda-\frac{\Delta}{|q|^2}\right)+r\Gamma_b+O(1)\underline{f}\cdot(f,\underline{f})\\
&\quad +r\Gamma_g\cdot\underline{f},\\
e_3'(\cos(\theta')-\cos(\theta)) &= \Gamma_b+O(r^{-1})\underline{f}+\Gamma_b\cdot\underline{f}\cdot(f,\underline{f}),\\
\nabla_3'(\mathfrak{J}'-\mathfrak{J})+\frac{1}{\overline{q'}}(\mathfrak{J}'-\mathfrak{J}) &= r^{-1}\Gamma_b+O(r^{-2})\lambda^{-1}\left(\lambda-\frac{\Delta}{|q|^2}\right)\\
&\quad +O(r^{-3})(r'-r)+O(r^{-3})(\cos(\theta')-\cos\theta)\\
&\quad +O(r^{-2})\underline{f}+\Gamma_b\cdot(f,\underline{f})+O(r^{-2})\underline{f}\cdot(f,\underline{f}).
\end{aligned}$$

Integrating these transport equations from $\{r=r_0\}$, where $r'=r$, $\cos(\theta')=\cos(\theta)$, $q'=q$ and $\mathfrak{J}'=\mathfrak{J}$ in view of (4.49), to the region ${}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1)\cap\{\underline{u}\leq u_*\}$, using the above control for $(f,\underline{f},\lambda)$ and the control (4.6) for the outgoing PG structure of ${}^{(ext)}\mathcal{M}$, we easily deduce

$$\begin{aligned}
\max_{k\leq k_{small}+129}\sup_{{}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1)\cap\{\underline{u}\leq u_*\}} & u^{1+\delta_{dec}-2\delta_0}\\
&\times\left|\mathfrak{d}^k\left(r'-r,\,\cos(\theta')-\cos(\theta),\,\mathfrak{J}'-\mathfrak{J}\right)\right| \lesssim \epsilon
\end{aligned}$$

as stated.

Also, we consider the control of (Γ_g',Γ_b'). The change of frame formulas of Proposition 2.12, the above control of the change of frame coefficients $(f,\underline{f},\lambda)$, the control (4.6) for the outgoing PG structure of ${}^{(ext)}\mathcal{M}$, and the above control of $r'-r$, $\cos(\theta')-\cos(\theta)$ and $\mathfrak{J}'-\mathfrak{J}$ imply

$$\max_{k\leq k_{small}+128}\sup_{{}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1)\cap\{\underline{u}\leq u_*\}} u^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^k\left(\Gamma_b',\Gamma_g'\right)\right| \lesssim \epsilon$$

as stated.

So far, we have proved all statements concerning comparisons between the slight extension of the ingoing PG structure of ${}^{(int)}\mathcal{M}$ into ${}^{(ext)}\mathcal{M}$ and the outgoing PG structure of ${}^{(ext)}\mathcal{M}$. Then, the statements concerning comparisons between the slight extension of the frame of ${}^{(int)}\mathcal{M}$ into ${}^{(ext)}\mathcal{M}$ and the ingoing PT structure of Proposition 4.8 follow immediately from the above comparisons and the comparison between the ingoing PT structure of Proposition 4.8 and the outgoing PG structure of ${}^{(ext)}\mathcal{M}$ done in that proposition. This concludes the proof of Lemma 4.9.

4.5.2. Proof of Lemma 4.11 To simplify the notations, in this section, we denote

- by (e_4, e_3, e_1, e_2) the ingoing PG frame of ${}^{(int)}\mathcal{M}$ slightly extended into ${}^{(ext)}\mathcal{M}$ in Lemma 4.9,
- by $(\underline{u}, r, \theta)$ and by $\mathfrak{J}$ respectively the triplet of scalar functions and the complex 1-form associated to the ingoing PG structure of ${}^{(int)}\mathcal{M}$,
- by (e'_4, e'_3, e'_1, e'_2) the ingoing PT frame of Proposition 4.8 which exists on ${}^{(top)}\mathcal{M}'$,
- by $(\underline{u}', r', \theta')$ and by $\mathfrak{J}'$ respectively the triplet of scalar functions and the complex 1-form associated to the ingoing PT structure of Proposition 4.8,
- by $(f, \underline{f}, \lambda)$ the change of frame coefficients from the frame (e_4, e_3, e_1, e_2) to (e'_4, e'_3, e'_1, e'_2).

Recall the definition of the matching region in (4.47)

$$\begin{aligned}\mathrm{Match}_1 &= {}^{(int)}\mathcal{M}(\underline{u} \geq u_* - 1)\\ &\quad \cup\Big({}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1) \cap \{u_* - 1 \leq \underline{u} \leq u_*\}\Big).\end{aligned}$$

Note that

$$\mathrm{Match}_1 \subset {}^{(int)}\mathcal{M} \cup \Big({}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1) \cap \{\underline{u} \leq u_*\}\Big),$$

and that, in view of the definition of ${}^{(top)}\mathcal{M}'$, Remark 4.10 and the control of $\underline{u}' - \underline{u}$ in Proposition 4.8, we also have

$$\mathrm{Match}_1 \subset {}^{(top)}\mathcal{M}'.$$

In particular, the above mentioned frames, scalars and complex 1-forms exist on Match_1. Let ψ be a smooth cut-off function of $\underline{u}$ such that $\psi = 0$ for $\underline{u} \leq u_* - 1$ and $\psi = 1$ for $\underline{u} \geq u_*$. Then, we define the null frame $(e''_4, e''_3, e''_1, e''_2)$ on

$${}^{(int)}\mathcal{M} \cup {}^{(top)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1) \cup {}^{(ext)}\mathcal{M}(u \geq u_* - 1),$$

and the quantities $(r'', J''^{(0)}, \mathfrak{J}'')$ as follows:

- In

  $${}^{(top)}\mathcal{M} \cup \Big(\big({}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1) \cup {}^{(ext)}\mathcal{M}(u \geq u_* - 1)\big) \setminus \{\underline{u} \leq u_*\}\Big),$$

  we have

  $$(e''_4, e''_3, e''_1, e''_2) = (e'_4, e'_3, e'_1, e'_2), \quad r'' = r', \quad J''^{(0)} = \cos(\theta'), \quad \mathfrak{J}'' = \mathfrak{J}'.$$

- In

$$\Big({}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}({}^{(ext)}r \le r_0+1)\Big) \cap \{\underline{u} \le u_* - 1\},$$

  we have

$$(e''_4, e''_3, e''_1, e''_2) = (e_4, e_3, e_1, e_2), \quad r'' = r, \quad J''^{(0)} = \cos(\theta), \quad \mathfrak{J}'' = \mathfrak{J}.$$

- In the matching region Match_1, the frame $(e''_4, e''_3, e''_1, e''_2)$ is defined from (e_4, e_3, e_1, e_2) using the change of frame coefficients $(f', \underline{f}', \lambda')$ with

$$f' = \psi(\underline{u})f, \qquad \underline{f}' = \psi(\underline{u})\underline{f}, \qquad \lambda' = 1 - \psi(\underline{u}) + \psi(\underline{u})\lambda,$$

  where we recall that $(f, \underline{f}, \lambda)$ denotes the coefficients corresponding to the change of frame from (e_4, e_3, e_1, e_2) to (e'_4, e'_3, e'_1, e'_2).
- In the matching region Match_1, r'', $J''^{(0)}$ and $\mathfrak{J}''$ are defined by

$$\begin{aligned} & r'' = \psi(\underline{u})r' + (1-\psi(\underline{u}))r, \quad J''^{(0)} = \psi(\underline{u})\cos(\theta') + (1-\psi(\underline{u}))\cos\theta, \\ & \mathfrak{J}'' = \psi(\underline{u})\mathfrak{J}' + (1-\psi(\underline{u}))\mathfrak{J}. \end{aligned}$$

In view of the above definitions, properties (a) and (b) of Lemma 4.11 are immediate. Also, using the definition of $(f', \underline{f}', \lambda')$ and the control of $(f, \underline{f}, \lambda)$ provided in $\text{Match}_1 \cap {}^{(int)}\mathcal{M}$ by Proposition 4.8 and in $\text{Match}_1 \cap {}^{(ext)}\mathcal{M}$ by Lemma 4.9, we have

$$\max_{k \le k_{small}+129} \sup_{\text{Match}_1} u^{1+\delta_{dec}-2\delta_0} \Big|(f', \underline{f}', \log(\lambda'))\Big| \quad \lesssim \quad \epsilon.$$

Also, if $(f'', \underline{f}'', \lambda'')$ denotes the coefficients of the change of frame from (e'_4, e'_3, e'_1, e'_2) to $(e''_4, e''_3, e''_1, e''_2)$, we easily obtain from the above control of $(f', \underline{f}', \lambda')$ and $(f, \underline{f}, \lambda)$

$$\max_{k \le k_{small}+129} \sup_{\text{Match}_1} u^{1+\delta_{dec}-2\delta_0} \Big|(f'', \underline{f}'', \log(\lambda''))\Big| \quad \lesssim \quad \epsilon,$$

which concludes the proof of property (d) of Lemma 4.11.

Finally, in view of the definition of r'', $J''^{(0)}$ and $\mathfrak{J}''$ in Match_1, and the control of $r'-r$, $\cos(\theta') - \cos\theta$ and $\mathfrak{J}' - \mathfrak{J}$ provided in $\text{Match}_1 \cap {}^{(int)}\mathcal{M}$ by Proposition 4.8 and in $\text{Match}_1 \cap {}^{(ext)}\mathcal{M}$ by Lemma 4.9, we have

$$\max_{k \le k_{small}+129} \sup_{\text{Match}_1} u^{1+\delta_{dec}-2\delta_0} \left|\mathfrak{d}^k \left(r''-r,\ J''^{(0)} - \cos(\theta),\ \mathfrak{J}''-\mathfrak{J}\right)\right| \quad \lesssim \quad \epsilon.$$

Together with the change of frame formulas of Proposition 2.12, the above control of the change of frame coefficients $(f', \underline{f}', \lambda')$, and the bootstrap assumptions on decay and boundedness for the ingoing PG structure of ${}^{(int)}\mathcal{M}$, we infer

$$\max_{k\leq k_{small}+128}\sup_{\text{Match}_1} u^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^k\left(\Gamma''_g, \Gamma''_b\right)\right| \lesssim \epsilon$$

which is property (c). This concludes the proof of Lemma 4.11.

4.5.3. Proof of Lemma 4.13 To simplify the notations, in this section, we denote

- by (e_4, e_3, e_1, e_2) the frame of Lemma 4.11,
- by $(r, J^{(0)})$ and by $\mathfrak{J}$ respectively the pair of scalar functions and the complex 1-form of Lemma 4.11,
- by (e'_4, e'_3, e'_1, e'_2) the second frame of ${}^{(ext)}\mathcal{M}$ constructed in Proposition 3.26,
- by (u, r', θ') and by $\mathfrak{J}'$ respectively the triplet of scalar functions and the complex 1-form associated to the outgoing PG structure of ${}^{(ext)}\mathcal{M}$.

Recall the definition of the matching region in (4.48)

$$\text{Match}_2 \quad = \quad {}^{(ext)}\mathcal{M}(u\geq u_*-1)\cup {}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1).$$

Note that the above mentioned frames, scalars and complex 1-forms exist on Match_2. Also, in view of the change of frame formulas of Proposition 2.12, and the control of (e_4, e_3, e_1, e_2) provided by Proposition 4.8, Lemma 4.9 and Lemma 4.11, we have, for $k\leq k_{small}+128$,

$$\begin{aligned}\sup_{\text{Match}_2}\Big(ru^{1+\delta'_{dec}}\left|\mathfrak{d}^k\Gamma_b\right| + \left(ru^{1+\delta'_{dec}}+r^2u^{\frac{1}{2}+\delta'_{dec}}\right)\left|\mathfrak{d}^k\Gamma_g\right|\Big)\\ +\sup_{\text{Match}_2}\left(r^2u^{1+\delta'_{dec}}\left|\mathfrak{d}^{k-1}\nabla_3\Gamma_g\right| + r^{\frac{7}{2}+\delta_B}|\mathfrak{d}^k(A,B)|\right)\\ +\sup_{\text{Match}_2} r^4u^{\frac{1}{2}+\delta'_{dec}}\left|\mathfrak{d}^{k-1}\nabla_3 B\right| + \sup_{\text{Match}_2}\left(r^{\frac{9}{2}+\delta_B}+r^4u^{\frac{1}{2}+\delta'_{dec}}\right)\left|\mathfrak{d}^{k-1}\nabla_3 A\right| \lesssim \epsilon.\end{aligned}$$

We define the following null frame $(e''_4, e''_3, e''_1, e''_2)$ on ${}^{(ext)}\mathcal{M}$

$$e''_4 = \frac{\Delta'}{|q'|^2}e'_4, \quad e''_3 = \frac{|q'|^2}{\Delta'}e'_3, \quad e'_a = e_a, \;\; a=1,2.$$

We define the linearized quantities (Γ''_g, Γ''_b) using (r', θ') and $\mathfrak{J}'$, with the ingoing normalization. Let also $(f, \underline{f}, \lambda)$ denote the coefficients corresponding to the change of frame from (e_4, e_3, e_1, e_2) to $(e''_4, e''_3, e''_1, e''_2)$. In view of the control of the change of frame coefficients in Proposition 3.26, Proposition 4.8, Lemma 4.9 and Lemma 4.11, we have, for $k \leq k_{small} + 129$,

$$\sup_{\mathrm{Match}_2} \Big(\big(ru^{1+\delta_{dec}-2\delta_0} + r^2 u^{\frac{1}{2}+\delta_{dec}-2\delta_0} \big) \left| \mathfrak{d}^k \left(f, \underline{f}, \lambda - 1 \right) \right| \\ + r^2 u^{1+\delta_{dec}-2\delta_0} \left| \mathfrak{d}^{k-1} \nabla_3 \left(f, \underline{f}, \lambda - 1 \right) \right| \Big) \lesssim \epsilon.$$

Let ψ be a smooth cut-off function of (r', u) such that $\psi = 0$ on ${}^{(ext)}\mathcal{M} \setminus \mathrm{Match}_2$, $\psi = 1$ for ${}^{(int)}\mathcal{M} \cup {}^{(top)}\mathcal{M}$, and such that ψ only depends on u for $r' \geq r_0 + 1$. Then, we define the global null frame $(e'''_4, e'''_3, e'''_1, e'''_2)$ of $\mathcal{M}$ and the quantities $(r''', J'''^{(0)}, \mathfrak{J}''')$ as follows:

- In ${}^{(ext)}\mathcal{M} \setminus \mathrm{Match}_2$, we have

$$(e'''_4, e'''_3, e'''_1, e'''_2) = (e''_4, e''_3, e''_1, e''_2), \quad r''' = r', \quad J'''^{(0)} = \cos(\theta'), \quad \mathfrak{J}''' = \mathfrak{J}'.$$

- In ${}^{(int)}\mathcal{M} \cup {}^{(top)}\mathcal{M}$, we have

$$(e'''_4, e'''_3, e'''_1, e'''_2) = (e_4, e_3, e_1, e_2), \quad r''' = r, \quad J'''^{(0)} = J^{(0)}, \quad \mathfrak{J}''' = \mathfrak{J}.$$

- In the matching region Match_2, the frame $(e'''_4, e'''_3, e'''_1, e'''_2)$ is defined from (e_4, e_3, e_1, e_2) using the change of frame coefficients $(f', \underline{f}', \lambda')$ with

$$f' = \psi(r', u) f, \qquad \underline{f}' = \psi(r', u)\underline{f}, \qquad \lambda' = 1 - \psi(r', u) + \psi(r', u)\lambda,$$

  where we recall that $(f, \underline{f}, \lambda)$ denotes the coefficients corresponding to the change of frame from (e_4, e_3, e_1, e_2) to $(e''_4, e''_3, e''_1, e''_2)$.
- In the matching region Match_1, r''', $J'''^{(0)}$ and $\mathfrak{J}'''$ are defined by

$$\begin{aligned} r''' &= \psi(r', u) r' + (1 - \psi(r', u)) r, \\ J'''^{(0)} &= \psi(r', u) \cos(\theta') + (1 - \psi(r', u)) J^{(0)}, \\ \mathfrak{J}''' &= \psi(r', u) \mathfrak{J}' + (1 - \psi(r', u)) \mathfrak{J}. \end{aligned}$$

In view of the above definitions, properties (a) and (b) of Lemma 4.11 are immediate. Also, using the definition of $(f', \underline{f}', \lambda')$, the fact that ψ only depends on[108] u for $r' \geq r_0 + 1$ and the above control of $(f, \underline{f}, \lambda)$, we have,

[108]Note in particular that we have $|\mathfrak{d}(\psi)| \lesssim |\mathfrak{d}(u)| \lesssim 1$ for $r' \geq r_0 + 1$.

for $k\leq k_{small}+129$,

$$\sup_{\text{Match}_2}\Big(\big(ru^{1+\delta_{dec}-2\delta_0}+r^2u^{\frac{1}{2}+\delta_{dec}-2\delta_0}\big)\left|\mathfrak{d}^k\left(f',\underline{f}',\lambda'-1\right)\right| \\ +r^2u^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^{k-1}\nabla_3\left(f',\underline{f}',\lambda'-1\right)\right|\Big) \lesssim \epsilon.$$

Also, if $(f'',\underline{f}'',\lambda'')$ denotes the coefficients of the change of frame from (e_4',e_3',e_1',e_2') to $(e_4'',e_3'',e_1'',e_2'')$, we easily obtain from the above control of $(f',\underline{f}',\lambda')$ and $(f,\underline{f},\lambda)$, for $k\leq k_{small}+129$,

$$\sup_{\text{Match}_2}\Big(\big(ru^{1+\delta_{dec}-2\delta_0}+r^2u^{\frac{1}{2}+\delta_{dec}-2\delta_0}\big)\left|\mathfrak{d}^k\left(f'',\underline{f}'',\lambda''-1\right)\right| \\ +r^2u^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^{k-1}\nabla_3\left(f'',\underline{f}'',\lambda''-1\right)\right|\Big) \lesssim \epsilon$$

which concludes the proof of property (d) of Lemma 4.13.

Next, in view of the definition of r''', $J'''^{(0)}$ and $\mathfrak{J}'''$ in Match_1, and the control of $r'-r$, $\cos(\theta')-J^{(0)}$ and $\mathfrak{J}'-\mathfrak{J}$ of Lemma 4.9 and Lemma 4.11, we have

$$\max_{k\leq k_{small}+129}\sup_{\text{Match}_1} u^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^k\left(r'''-r,\ J'''^{(0)}-J^{(0)},\ \mathfrak{J}'''-\mathfrak{J}\right)\right| \lesssim \epsilon$$

which concludes the proof of property (e) of Lemma 4.13. Together with the change of frame formulas of Proposition 2.12, the above control of the change of frame coefficients $(f',\underline{f}',\lambda')$, and the above control for (Γ_b,Γ_g,A,B), we infer, for $k\leq k_{small}+128$,

$$\sup_{\text{Match}_2}\Big(ru^{1+\delta'_{dec}}\left|\mathfrak{d}^k\Gamma_b'''\right|+\big(ru^{1+\delta'_{dec}}+r^2u^{\frac{1}{2}+\delta'_{dec}}\big)\left|\mathfrak{d}^k\Gamma_g'''\right|\Big) \\ +\sup_{\text{Match}_2}\Big(r^2u^{1+\delta'_{dec}}\left|\mathfrak{d}^{k-1}\nabla_3'''\Gamma_g'''\right|+r^{\frac{7}{2}+\delta_B}|\mathfrak{d}^k(A''',B''')|\Big) \\ +\sup_{\text{Match}_2}r^4u^{\frac{1}{2}+\delta'_{dec}}\left|\mathfrak{d}^{k-1}\nabla_3'''B'''\right| \\ +\sup_{\text{Match}_2}\big(r^{\frac{9}{2}+\delta_B}+r^4u^{\frac{1}{2}+\delta'_{dec}}\big)\left|\mathfrak{d}^{k-1}\nabla_3'''A'''\right| \lesssim \epsilon$$

which is property (c) of Lemma 4.13. This concludes the proof of Lemma 4.13.

4.6. Proof of Proposition 3.35

We glue a renormalization of the third frame of ${}^{(ext)}\mathcal{M}$ constructed in Proposition 3.29 to the frame of Lemma 4.11 in the matching region Match_2 given by (4.48), noticing in view of Remark 4.12 that both frames are defined in Match_2.

Lemma 4.14. *Let $({}^{(ext)}e'_4, {}^{(ext)}e'_3, {}^{(ext)}e'_1, {}^{(ext)}e'_2)$ be the third frame of ${}^{(ext)}\mathcal{M}$ constructed in Proposition 3.29, and let $(e''_4, e''_3, e''_1, e''_2)$ be the frame of Lemma 4.11. There exists a global null frame $({}^{(glo')}e_4, {}^{(glo')}e_3, {}^{(glo')}e_1, {}^{(glo')}e_2)$ defined on $\mathcal{M}$, as well as a pair of scalar functions $({}^{(glo')}r, {}^{(glo')}J^{(0)})$, and a complex 1-form ${}^{(glo')}\mathfrak{J}$, such that:*

(a) In ${}^{(ext)}\mathcal{M}\setminus Match_2$, we have

$$\begin{aligned}&({}^{(glo')}e_4, {}^{(glo')}e_3, {}^{(glo')}e_1, {}^{(glo')}e_2)\\ =\;& ({}^{(ext)}\lambda{}^{(ext)}e'_4, {}^{(ext)}\lambda^{-1}{}^{(ext)}e'_3, {}^{(ext)}e'_1, {}^{(ext)}e'_2)\end{aligned}$$

where ${}^{(ext)}\lambda := \frac{{}^{(ext)}\Delta}{|{}^{(ext)}q|^2}$, as well as ${}^{(glo')}r = {}^{(ext)}r$, ${}^{(glo')}J^{(0)} = \cos({}^{(ext)}\theta)$, and ${}^{(glo')}\mathfrak{J} = {}^{(ext)}\mathfrak{J}$.

(b) In ${}^{(int)}\mathcal{M}\cup{}^{(top)}\mathcal{M}$, we have

$$({}^{(glo')}e_4, {}^{(glo')}e_3, {}^{(glo')}e_1, {}^{(glo')}e_2) = (e''_4, e''_3, e''_1, e''_2),$$

as well as ${}^{(glo')}r = r''$, ${}^{(glo')}J^{(0)} = J''^{(0)}$, and ${}^{(glo')}\mathfrak{J} = \mathfrak{J}''$.

(c) In the matching region, we have, for $k\le k_{small}+128$,

$$\begin{aligned}&\sup_{Match_2}\Big(ru^{1+\delta'_{dec}}\left|\mathfrak{d}^k{}^{(glo')}\Gamma_b\right| + \big(ru^{1+\delta'_{dec}} + r^2u^{\frac{1}{2}+\delta'_{dec}}\big)\left|\mathfrak{d}^k{}^{(glo')}\Gamma_g\right|\Big)\\ &\qquad + \sup_{Match_2}\Big(r^2u^{1+\delta'_{dec}}\left|\mathfrak{d}^{k-1}\nabla_{{}^{(glo')}e_3}{}^{(glo')}\Gamma_g\right|\\ &\qquad\qquad + r^{\frac{7}{2}+\delta_B}\left|\mathfrak{d}^k({}^{(glo')}A, {}^{(glo')}B)\right|\Big)\\ &\qquad + \sup_{Match_2} r^4u^{\frac{1}{2}+\delta'_{dec}}\left|\mathfrak{d}^{k-1}\nabla_{{}^{(glo')}e_3}{}^{(glo')}B\right|\\ &\qquad + \sup_{Match_2}\big(r^{\frac{9}{2}+\delta_B} + r^4u^{\frac{1}{2}+\delta'_{dec}}\big)\left|\mathfrak{d}^{k-1}\nabla_{{}^{(glo')}e_3}{}^{(glo')}A\right| \;\lesssim\; \epsilon.\end{aligned}$$

(d) In the matching region, we have

$$\max_{0\le k\le k_{small}+129}\;\sup_{Match_2\cap{}^{(ext)}\mathcal{M}} ru^{1+\delta_{dec}-2\delta_0}\left|\mathfrak{d}^k(f, \underline{f}, \log\lambda)\right| \;\lesssim\; \epsilon,$$

where $(f, \underline{f}, \lambda)$ denotes

- *either the change of frame coefficients from $(e''_4, e''_3, e''_1, e''_2)$ to the global frame $({}^{(glo')}e_4, {}^{(glo')}e_3, {}^{(glo')}e_1, {}^{(glo')}e_2)$,*

– *or the one from* $({}^{(ext)}\lambda^{(ext)}e_4', {}^{(ext)}\lambda^{-1(ext)}e_3', {}^{(ext)}e_1', {}^{(ext)}e_2')$ *to the global frame* $({}^{(glo')}e_4, {}^{(glo')}e_3, {}^{(glo')}e_1, {}^{(glo')}e_2)$.

(e) *In the matching region, we have*

$$\max_{0\leq k\leq k_{small}+129}\sup_{Match_2} u^{1+\delta_{dec}-2\delta_0}\Big(\Big|\mathfrak{d}^k\Big({}^{(glo')}r - r''\Big)\Big| + r\Big|\mathfrak{d}^k\Big({}^{(glo')}J^{(0)} - J''^{(0)}\Big)\Big| + r^2\Big|\mathfrak{d}^k\Big({}^{(glo')}\mathfrak{J} - \mathfrak{J}''\Big)\Big|\Big) \lesssim \epsilon$$

and

$$\max_{0\leq k\leq k_{small}+129}\sup_{Match_2} u^{1+\delta_{dec}-2\delta_0}\Big(\Big|\mathfrak{d}^k\Big({}^{(glo')}r - {}^{(ext)}r\Big)\Big| + r\Big|\mathfrak{d}^k\Big({}^{(glo')}J^{(0)} - \cos({}^{(ext)}\theta)\Big)\Big| + r^2\Big|\mathfrak{d}^k\Big({}^{(glo')}\mathfrak{J} - {}^{(ext)}\mathfrak{J}\Big)\Big|\Big) \lesssim \epsilon.$$

Proof. The proof of Lemma 4.14 is completely analogous to the one Lemma 4.13. □

We are now ready to prove Proposition 3.35. The global frame of Proposition 3.35 is the one constructed in Lemma 4.14. The proof of properties (a) to (c) of Proposition 3.35 is completely analogous to the one of properties (a) to (f) of Proposition 3.33. Finally, property (d) of Proposition 3.35 follows from the fourth property of Proposition 3.29. This concludes the proof of Proposition 3.35.

5. DECAY ESTIMATES ON THE LAST SLICE (THEOREM M3)

The goal of this chapter is to prove Theorem M3, i.e. to improve our bootstrap assumptions on decay for the integrable frame of Σ_*. This will be achieved in Section 5.5, see Proposition 5.53. We will then use these improvements to derive decay estimates for the PG frame of ${}^{(ext)}\mathcal{M}$ on Σ_* in Section 5.7, see Proposition 5.77.

In order to count the number of derivatives under control in this chapter, we introduce for convenience the following notation

$$k_* \quad := \quad k_{small} + 80. \tag{5.1}$$

5.1. Geometric setting on Σ_*

We recall the properties of the defining boundary Σ_* of our GCM admissible spacetime introduced in Section 3.2.3. To start with Σ_* is equipped, in view of Section 3.2.3, with a frame (e_1, e_2, e_3, e_4) and a function r such that $(\Sigma_*, r, (e_1, e_2, e_3, e_4))$ is a framed hypersurface, see Definition 2.55. Also, Σ_* comes equipped with function u which verifies, for a constant[109] c_*,

$$u = c_* - r. \tag{5.2}$$

Recall that we have also imposed the transversality conditions on Σ_* (see (3.27))

$$\xi = 0, \qquad \omega = 0, \qquad \underline{\eta} = -\zeta, \qquad e_4(r) = 1, \qquad e_4(u) = 0, \tag{5.3}$$

which allows us to make sense of all the Ricci coefficients in the frame of Σ_*, i.e.,

$$\widehat{\chi}, \quad \kappa = \operatorname{tr}\chi, \quad \eta, \quad \zeta, \quad \underline{\eta}, \quad \xi, \quad \underline{\widehat{\chi}}, \quad \underline{\kappa} = \operatorname{tr}\underline{\chi}, \quad \underline{\xi},$$

as well as make sense of all first order derivatives of r and u on Σ_*.

For convenience, we introduce the following notations

$$y := e_3(r), \qquad z := e_3(u). \tag{5.4}$$

[109] Recall that c_* is fixed such that $c_* = 1 + r(S_1)$ where S_1 is the only sphere of Σ_* intersecting the curve of the south poles, see Section 3.2.5.

We thus have, using the transversality condition for $e_4(r)$ and $e_4(u)$, and the fact that (e_1, e_2) is adapted to the r-foliation on Σ_*,

$$\nabla(r) = \nabla(u) = 0, \quad e_4(u) = 0, \quad e_4(r) = 1, \quad e_3(r) = y, \quad e_3(u) = z.$$

Also, recall that

$$\nu = e_3 + b_* e_4 \tag{5.5}$$

denotes the vectorfield tangent to Σ_*, orthogonal to the foliation and normalized by the condition $\mathbf{g}(\nu, e_4) = -2$.

5.1.1. Effective uniformization of almost round 2-spheres In this section, we recall some of the basic results on effective uniformization from [41] that will be useful in this chapter, as well as in chapter 8.

Definition 5.1. *A 2 dimensional, closed, Riemannian surface (S, g^S) is said to be almost round if its Gauss curvature K^S verifies, for a sufficiently small $\epsilon > 0$,*

$$\left|K^S - \frac{1}{(r^S)^2}\right| \leq \frac{\epsilon}{(r^S)^2}, \tag{5.6}$$

where r^S is the area radius of S.

The following theorem is Corollary 3.8 in [41].

Theorem 5.2 (Effective uniformization)**.** *Given an almost round sphere (S, g^S) as above there exists, up to isometries*[110] *of $\mathbb{S}^2$, a unique diffeomorphism $\Phi : \mathbb{S}^2 \to S$ and a unique conformal factor u such that*

$$\begin{aligned} \Phi^{\#}(g^S) &= (r^S)^2 e^{2u} \gamma_{\mathbb{S}^2}, \\ \int_{\mathbb{S}^2} e^{2u} x^i &= 0, \qquad i = 1, 2, 3. \end{aligned} \tag{5.7}$$

Moreover, the size of the conformal factor u is small with respect to the parameter ϵ, i.e.

$$\|u\|_{L^\infty(\mathbb{S}^2)} \lesssim \epsilon.$$

Theorem 5.2 is used to define a canonical $\ell = 1$ basis on S, see Definition 3.10 in [41].

[110] In other words, all the solutions are of the form $(\Phi \circ O, u \circ O)$ for $O \in O(3)$.

Definition 5.3 (Basis of canonical $\ell = 1$ modes on S)**.** *Let (Φ, u) be the unique, up to isometries of $\mathbb{S}^2$, uniformization pair given by Theorem 5.2. We define the basis of canonical $\ell = 1$ modes on S by*

$$J^S \quad := \quad J^{\mathbb{S}^2} \circ \Phi^{-1}, \tag{5.8}$$

where $J^{\mathbb{S}^2}$ denotes the $\ell = 1$ spherical harmonics.

The main properties of this basis are given in Lemma 3.12 in [41] which we review below.

Lemma 5.4. *Let J^S denote the basis of canonical $\ell = 1$ modes on S of Definition 5.3. Then*

$$\begin{aligned}\Delta_S J^{(p,S)} &= -\frac{2}{(r^S)^2} J^{(p,S)} + \frac{2}{(r^S)^2}(1 - e^{-2v}) J^{(p,S)},\\ \int_S J^{(p,S)} J^{(q,S)} da_g &= \frac{4\pi}{3}(r^S)^2 \delta_{pq} + \int_S J^{(p,S)} J^{(q,S)}(1 - e^{-2v}) da_{g^S},\\ \int_S J^{(p,S)} da_g &= 0,\end{aligned} \tag{5.9}$$

with Δ^S the Laplace-Beltrami of the metric g^S and $v := u \circ \Phi^{-1}$. Moreover we have

$$\begin{aligned}\Delta_S J^{(p,S)} &= \left(-\frac{2}{(r^S)^2} + O\left(\frac{\epsilon}{(r^S)^2}\right)\right) J^{(p,S)},\\ \int_S J^{(p,S)} J^{(q,S)} da_g &= \frac{4\pi}{3}(r^S)^2 \delta_{pq} + O(\epsilon (r^S)^2),\end{aligned} \tag{5.10}$$

where $\epsilon > 0$ is the smallness constant appearing in (5.6).

The following proposition is Proposition 4.15 in [41].

Proposition 5.5. *Let (S, g^S) be an almost round sphere, i.e. verifying (5.6). Consider an approximate uniformization pair $(\widetilde{\Phi}, \widetilde{u})$, where $\widetilde{\Phi} : \mathbb{S}^2 \to S$ is a smooth diffeomorphism and $\widetilde{u}$ a smooth scalar function on $\mathbb{S}^2$ such that the following are verified, for δ such that $0 < \delta \le \epsilon$ and for $s \ge 2$,*

$$\left\| \widetilde{\Phi}^\#(g^S) - (r^S)^2 e^{2\widetilde{u}} \gamma_0 \right\|_{H^s(\mathbb{S}^2)} \quad \le \quad (r^S)^2 \delta, \qquad \|\widetilde{u}\|_{H^2(\mathbb{S}^2)} \le \epsilon. \tag{5.11}$$

Assume in addition that the scalar functions $\widetilde{J}^{(p)} := J^{(p,\mathbb{S}^2)} \circ \widetilde{\Phi}^{-1}$, $p \in \{0, +, -\}$, satisfy

$$\left| \int_S \widetilde{J}^{(p)} \right| \le (r^S)^2 \delta. \tag{5.12}$$

Then we can choose[111] *the uniformization pair* (Φ, u) *in Theorem 5.2 such that*[112]

$$(5.13) \qquad \begin{aligned} &(r^S)^{-1}\|\widetilde{u}\circ(\widetilde{\Phi})^{-1} - u\circ\Phi^{-1}\|_{\mathfrak{h}_s(S)} \\ &+ \max_{p=0,+,-}(r^S)^{-1}\|\widetilde{J}^{(p)} - J^{(p,S)}\|_{\mathfrak{h}_{s+1}(S)} \lesssim \delta. \end{aligned}$$

We recall the following definition from [41] on the calibration of uniformization maps between almost round spheres S_1, S_2 and diffeomorphisms $\Psi : S_1 \to S_2$.

Definition 5.6. *On* $\mathbb{S}^2$ *we fix*[113] *a point* N *and a unit vector* v *in the tangent space* $T_N\mathbb{S}^2$. *Given* $\Psi : S_1 \to S_2$, *we say that the effective uniformization maps* $\Phi_1 : \mathbb{S}^2 \to S_1$, $\Phi_2 : \mathbb{S}^2 \to S_2$ *are calibrated relative to* Ψ *if the map* $\widehat{\Psi} := (\Phi_2)^{-1}\circ\Psi\circ\Phi_1 : \mathbb{S}^2 \to \mathbb{S}^2$ *is such that:*

1. *The map* $\widehat{\Psi}$ *fixes the point* N, *i.e.* $\widehat{\Psi}(N) = N$.
2. *The tangent map* $\widehat{\Psi}_\#$ *fixes the direction of* v, *i.e.* $\widehat{\Psi}_\#(v) = a_{1,2}v$ *where* $a_{1,2} > 0$.
3. *The tangent map* $\widehat{\Psi}_\#$ *preserves the orientation of* $T_N\mathbb{S}^2$.

The following theorem is Corollary 4.11 in [41].

Theorem 5.7. *Let* $\Psi : S_1 \to S_2$ *be a given diffeomorphism and assume the following, for* δ *such that* $0 < \delta \le \epsilon$,

1. *The surfaces* S_1, S_2 *are close to each other, i.e. for some* $s \ge 0$,

$$(5.14) \qquad \left\|g^{S_1} - \Psi^\#(g^{S_2})\right\|_{\mathfrak{h}_{4+s}(S^1)} \le (r^{S_1})^3\delta.$$

2. *The maps* Φ_1, Φ_2 *are calibrated according to Definition 5.6.*

Then

$$(5.15) \qquad \|\widehat{\Psi} - I\|_{L^\infty(\mathbb{S}^2)} + \|\widehat{\Psi} - I\|_{H^1(\mathbb{S}^2)} \lesssim \delta,$$

and the conformal factors u_1, u_2 *verify*

$$(5.16) \qquad \left\|u_1 - \widehat{\Psi}^\# u_2\right\|_{L^\infty(\mathbb{S}^2)} \lesssim \delta.$$

[111] Recall that the pair (Φ, u) is unique up to isometries of $\mathbb{S}^2$.

[112] We refer to (5.59) for the definition of the standard weighted Sobolev spaces $\mathfrak{h}_k(S)$.

[113] In particular, one can choose $N = (0, 0, 1)$ and $v = (1, 0, 0)$.

5.1.2. The GCM conditions on Σ_* The framed hypersurface $(\Sigma_*, r, (e_1, e_2, e_3, e_4))$ introduced in Section 3.2.3 terminates in a future boundary S_* on which the given function r is constant, i.e. S_* is a leaf of the r-foliation of Σ_*. On S_* there exist coordinates (θ, φ) such that

1. The induced metric g on S_* takes the form

$$g = r^2 e^{2\phi}\Big((d\theta)^2 + \sin^2\theta (d\varphi)^2\Big). \tag{5.17}$$

2. The functions

$$J^{(0)} := \cos\theta, \quad J^{(-)} := \sin\theta \sin\varphi, \quad J^{(+)} := \sin\theta\cos\varphi, \tag{5.18}$$

verify the balanced conditions

$$\int_{S_*} J^{(p)} = 0, \qquad p = 0, +, -. \tag{5.19}$$

Once (θ, φ) are chosen on S_* we extend them to Σ_* by setting

$$\nu(\theta) = \nu(\varphi) = 0. \tag{5.20}$$

We extend the $J^{(p)}$ functions to Σ_* by setting

$$\nu(J^{(p)}) = 0, \quad p = 0, +, -. \tag{5.21}$$

We also impose the following transversality conditions on Σ_* for (θ, φ) and $J^{(p)}$

$$e_4(\theta) = 0, \qquad e_4(\varphi) = 0, \qquad e_4(J^{(p)}) = 0, \quad p = 0, +, -. \tag{5.22}$$

Recall the following, see Definition 2.61

Definition 5.8. *Given a scalar function f on any sphere S of the r-foliation, its $\ell = 1$ modes are given by*

$$(f)_{\ell=1} = \left\{ \frac{1}{|S|}\int_S f J^{(p)}, \, p = +, 0, - \right\}.$$

Remark 5.9. *This definition is such that the $\ell = 1$ modes have the same scaling in r as the corresponding quantity.*

Endowed with these canonical coordinates and $J^{(p)}$ basis, Σ_* is an admissible GCM hypersurface, i.e.,

1. On S_* we have

$$\kappa = \frac{2}{r}, \qquad \underline{\kappa} = -\frac{2\Upsilon}{r}, \tag{5.23}$$

$$(\mathrm{div}\,\beta)_{\ell=1} = 0, \tag{5.24}$$

as well as

$$\int_{S_*} J^{(+)}\mathrm{curl}\,\beta = 0, \qquad \int_{S_*} J^{(-)}\mathrm{curl}\,\beta = 0. \tag{5.25}$$

2. On any sphere of the r-foliation of Σ_*, we have

$$\begin{aligned} \kappa &= \frac{2}{r}, \\ \underline{\kappa} &= -\frac{2\Upsilon}{r} + \underline{C}_0 + \sum_{p=0,+,-} \underline{C}_p J^{(p)}, \\ \mu &= \frac{2m}{r^3} + M_0 + \sum_{p=0,+,-} M_p J^{(p)}, \end{aligned} \tag{5.26}$$

where $\underline{C}_0$, $\underline{C}_p$, M_0, M_p are scalar functions on Σ_*, constant on the leaves of the foliation. Also

$$(\mathrm{div}\,\eta)_{\ell=1} = (\mathrm{div}\,\underline{\xi})_{\ell=1} = 0, \qquad \overline{b_*} = -1 - \frac{2m}{r}, \tag{5.27}$$

where $\overline{b_*}$ denotes the average of b_* on the spheres foliating Σ_*.

3. The mass m is constant on Σ_* and chosen to be the Hawking mass of S_*, i.e.,

$$\frac{2m}{r} = 1 + \frac{1}{16\pi}\int_{S_*} \kappa\underline{\kappa}. \tag{5.28}$$

4. The angular momentum is constant on Σ_* and chosen as

$$a \;:=\; \frac{r^3}{8\pi m}\int_{S_*} J^{(0)}\mathrm{curl}\,\beta. \tag{5.29}$$

5. Let r_*, u_* denote the values of r and u on S_*. The function r is monotonically decreasing on Σ_* and the following dominance condition is verified

$$r_* \;=\; \delta_* \epsilon_0^{-1} {u_*}^{1+\delta_{dec}} \tag{5.30}$$

which implies in particular on Σ_*

$$r \geq r_* = \delta_* \epsilon_0^{-1} {u_*}^{1+\delta_{dec}} \geq \delta_* \epsilon_0^{-1} u^{1+\delta_{dec}}.$$

5.1.3. Main equations in the frame of Σ_* Recall that the following transversality conditions hold on Σ_*, see (5.3),

$$\xi = 0, \qquad \omega = 0, \qquad \underline{\eta} = -\zeta. \tag{5.31}$$

Proposition 5.10. *We have on Σ_*, for the Ricci coefficients and curvature components associated to the GCM frame of Σ_*, the following*[114]*:*

1. *The Ricci coefficients verify the equations*

$$\begin{aligned}
\nabla_4 tr\chi &= -\frac{1}{2} tr\chi^2 - |\widehat{\chi}|^2, \\
\nabla_4 \widehat{\chi} + tr\chi\, \widehat{\chi} &= -\alpha, \\
\nabla_4 tr\underline{\chi} + \frac{1}{2} tr\chi tr\underline{\chi} &= -2 div\zeta + 2|\zeta|^2 + 2\rho - \widehat{\chi}\cdot\underline{\widehat{\chi}}, \\
\nabla_4 \underline{\widehat{\chi}} + \frac{1}{2}(tr\underline{\chi}\widehat{\chi} + tr\chi\underline{\widehat{\chi}}) &= -\nabla\widehat{\otimes}\zeta + \zeta\widehat{\otimes}\zeta, \\
\nabla_3 tr\chi + \frac{1}{2} tr\underline{\chi} tr\chi &= 2 div\eta + 2\underline{\omega} tr\chi + 2|\eta|^2 + 2\rho - \underline{\widehat{\chi}}\cdot\widehat{\chi}, \\
\nabla_3 tr\underline{\chi} + \frac{1}{2} tr\underline{\chi}^2 + 2\underline{\omega} tr\underline{\chi} &= 2 div\underline{\xi} + 2\underline{\xi}\cdot(\eta - 3\zeta) - |\underline{\widehat{\chi}}|^2, \\
\nabla_3 \widehat{\chi} + \frac{1}{2}(tr\chi\underline{\widehat{\chi}} + tr\underline{\chi}\widehat{\chi}) - 2\underline{\omega}\widehat{\chi} &= \nabla\widehat{\otimes}\eta + \eta\widehat{\otimes}\eta, \\
\nabla_3 \underline{\widehat{\chi}} + tr\underline{\chi}\, \underline{\widehat{\chi}} + 2\underline{\omega}\underline{\widehat{\chi}} &= \nabla\widehat{\otimes}\underline{\xi} + \underline{\xi}\widehat{\otimes}(\eta - 3\zeta) - \underline{\alpha},
\end{aligned}$$

$$\begin{aligned}
\nabla_4 \zeta + tr\chi\zeta &= -2\widehat{\chi}\cdot\zeta - \beta, \\
\nabla_3 \zeta + 2\nabla\underline{\omega} &= -\underline{\widehat{\chi}}\cdot(\zeta + \eta) - \frac{1}{2} tr\underline{\chi}(\zeta + \eta) + 2\underline{\omega}(\zeta - \eta) \\
&\quad + \widehat{\chi}\cdot\underline{\xi} + \frac{1}{2} tr\chi\,\underline{\xi} - \underline{\beta}, \\
div\widehat{\chi} + \zeta\cdot\widehat{\chi} &= \frac{1}{2}\nabla tr\chi + \frac{1}{2} tr\chi\zeta - \beta, \\
div\underline{\widehat{\chi}} - \zeta\cdot\underline{\widehat{\chi}} &= \frac{1}{2}\nabla tr\underline{\chi} - \frac{1}{2} tr\underline{\chi}\zeta + \underline{\beta}, \\
K &= -\rho - \frac{1}{4} tr\chi tr\underline{\chi} + \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}},
\end{aligned}$$

[114] We only provide the equations relevant for the control of the GCM frame of Σ_*.

$$
\begin{aligned}
curl\zeta &= -\frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}} + {}^*\rho,\\
curl\eta &= \frac{1}{2}\widehat{\underline{\chi}}\wedge\widehat{\chi} + {}^*\rho,\\
curl\underline{\xi} &= \underline{\xi}\wedge(\eta-\zeta).
\end{aligned}
$$

2. The curvature components verify the equations

$$
\begin{aligned}
\nabla_3\alpha - \nabla\widehat{\otimes}\beta &= -\frac{1}{2}tr\underline{\chi}\alpha + 4\underline{\omega}\alpha + (\zeta+4\eta)\widehat{\otimes}\beta\\
&\quad -3(\rho\widehat{\chi} + {}^*\rho\,{}^*\widehat{\chi}),\\
\nabla_4\beta - div\,\alpha &= -2tr\chi\beta + \alpha\cdot\zeta,\\
\nabla_3\beta - (\nabla\rho + {}^*\nabla\,{}^*\rho) &= -tr\underline{\chi}\beta + 2\underline{\omega}\,\beta + 2\underline{\beta}\cdot\widehat{\chi} + 3(\rho\eta + {}^*\rho\,{}^*\eta)\\
&\quad +\alpha\cdot\underline{\xi},\\
\nabla_4\rho - div\,\beta &= -\frac{3}{2}tr\chi\rho - \zeta\cdot\beta - \frac{1}{2}\widehat{\underline{\chi}}\cdot\alpha,\\
\nabla_4\,{}^*\rho + curl\,\beta &= -\frac{3}{2}tr\chi\,{}^*\rho + \zeta\cdot{}^*\beta + \frac{1}{2}\widehat{\underline{\chi}}\cdot{}^*\alpha,\\
\nabla_3\rho + div\,\underline{\beta} &= -\frac{3}{2}tr\underline{\chi}\rho - (2\eta-\zeta)\cdot\underline{\beta} + 2\underline{\xi}\cdot\beta - \frac{1}{2}\widehat{\chi}\cdot\underline{\alpha},\\
\nabla_3\,{}^*\rho + curl\,\underline{\beta} &= -\frac{3}{2}tr\underline{\chi}\,{}^*\rho - (2\eta-\zeta)\cdot{}^*\underline{\beta} - 2\underline{\xi}\cdot{}^*\beta\\
&\quad -\frac{1}{2}\widehat{\chi}\cdot{}^*\underline{\alpha}.
\end{aligned}
$$

Proof. In view of the transversality conditions (5.3) and the fact that (e_1, e_2) are tangent to the 2-spheres of the r-foliation of Σ_*, we have on Σ_*

$$
{}^{(a)}\mathrm{tr}\chi = {}^{(a)}\mathrm{tr}\underline{\chi} = 0, \qquad \xi = 0, \quad \underline{\omega} = 0, \quad \underline{\eta} + \zeta = 0.
$$

The proof follows then by plugging these identities in Propositions 2.3 and 2.4. ∎

Definition 5.11. *The mass aspect function μ is defined on Σ by*

$$
\mu \;:=\; -div\,\zeta - \rho + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}}.
$$

Lemma 5.12. *The following relations hold true for $y = e_3(r)$ and $z = e_3(u)$:*

$$
\nabla y = -\underline{\xi} + (\zeta-\eta)y, \qquad \nabla z = (\zeta-\eta)z. \tag{5.32}
$$

Also, we have

$$b_* = -y - z. \tag{5.33}$$

Proof. We make use of $[e_a, e_3] = (\zeta_a - \eta_a)e_3 - \underline{\xi}_a e_4$ for $a = 1, 2$, which we apply to r. Using $\nabla(r) = 0$ and $e_4(r) = 1$, we infer

$$\nabla(e_3(r)) \quad = \quad (\zeta - \eta)e_3(r) - \underline{\xi}.$$

Similarly, using $\nabla(u) = 0$ and $e_4(u) = 0$, we have

$$\nabla(e_3(u)) \quad = \quad (\zeta - \eta)e_3(u).$$

Since $y = e_3(r)$ and $z = e_3(u)$, this concludes the proof of the first identities.

Next, since $u + r = c_*$ on Σ_* and ν is tangent to Σ_*, we have

$$0 \quad = \quad \nu(u+r) = e_3(u) + e_3(r) + b_* e_4(u) + b_* e_4(r) = y + z + b_*$$

and hence $b_* = -y - z$ as stated. This concludes the proof of the lemma. □

5.1.4. Linearized quantities and main quantitative assumptions Recall the notation $\kappa = \operatorname{tr}\chi$, $\underline{\kappa} = \operatorname{tr}\underline{\chi}$ and $y = e_3(r)$, $z = e_3(u)$. Also, recall the following linearized quantities introduced in Definition 3.6:

$$\begin{array}{llllll}
\check{\kappa} & := & \operatorname{tr}\chi - \dfrac{2}{r}, & \check{\underline{\kappa}} & := & \operatorname{tr}\underline{\chi} + \dfrac{2\Upsilon}{r}, \\
\check{\underline{\omega}} & := & \underline{\omega} - \dfrac{m}{r^2}, & \check{\rho} & := & \rho + \dfrac{2m}{r^3}, \\
\check{y} & := & y + \Upsilon, & \check{z} & := & z - 2, \\
\check{b}_* & := & b_* + 1 + \dfrac{2m}{r}, & \check{\mu} & := & \mu - \dfrac{2m}{r^3},
\end{array}$$

where $\Upsilon = 1 - \frac{2m}{r}$.

We denote by Γ_g, Γ_b the sets of linearized quantities[115] below.

- The set Γ_g

$$\Gamma_g := \Big\{\check{\kappa},\ \widehat{\chi},\ \zeta,\ \check{\underline{\kappa}},\ r\alpha,\ r\beta,\ r\check{\rho},\ r\ {}^*\rho,\ r\check{\mu}\Big\}. \tag{5.34}$$

[115] Note that these were denoted by Γ_g^*, Γ_b^* in Definition 3.7.

- The set Γ_b

$$\Gamma_b = \Big\{\eta,\ \widehat{\underline{\chi}},\ \underline{\check{\omega}},\ \underline{\xi},\ r\underline{\beta},\ \underline{\alpha},\ r^{-1}\check{\underline{y}},\ r^{-1}\check{\underline{z}},\ r^{-1}\check{b}_*\Big\}. \tag{5.35}$$

Definition 5.13. *We make use of the following norms on $S = S(u) \subset \Sigma_*$,*

$$\begin{aligned} \|f\|_\infty(u) &:= \|f\|_{L^\infty(S(u))}, & \|f\|_2(u) &:= \|f\|_{L^2(S(u))}, \\ \|f\|_{\infty,k}(u) &:= \sum_{i=0}^k \|\mathfrak{d}_*^i f\|_\infty(u), & \|f\|_{2,k}(u) &:= \sum_{i=0}^k \|\mathfrak{d}_*^i f\|_2(u). \end{aligned} \tag{5.36}$$

Throughout this chapter we rely on the following assumptions.

Ref 1. According to our bootstrap assumptions BA-D on decay, and BA-B on r-weighted sup norms, we have on Σ_*

1. For $0 \le k \le k_{small}$,

$$\begin{aligned} \|\Gamma_g\|_{\infty,k} &\le \epsilon r^{-2} u^{-\frac{1}{2}-\delta_{dec}}, \\ \|\nabla_\nu \Gamma_g\|_{\infty,k-1} &\le \epsilon r^{-2} u^{-1-\delta_{dec}}, \\ \|\Gamma_b\|_{\infty,k} &\le \epsilon r^{-1} u^{-1-\delta_{dec}}. \end{aligned} \tag{5.37}$$

2. For $0 \le k \le k_{large}$,

$$\|\Gamma_g\|_{\infty,k} \le \epsilon r^{-2}, \qquad \|\Gamma_b\|_{\infty,k} \le \epsilon r^{-1}. \tag{5.38}$$

Ref 2. Recall from (5.1) that $k_* = k_{small} + 80$ in this chapter. The following estimates on Σ_* are obtained in Theorems M1 and M2:

1. The quantity $\mathfrak{q}$ satisfies, in view of Theorem M1, on Σ_*, for all $0 \le k \le k_*$,

$$\begin{aligned} \|\mathfrak{q}\|_{\infty,k} &\lesssim \epsilon_0 r^{-1} u^{-\frac{1}{2}-\delta_{extra}}, \\ \|\nabla_3 \mathfrak{q}\|_{\infty,k-1} &\lesssim \epsilon_0 r^{-1} u^{-1-\delta_{extra}}, \end{aligned} \tag{5.39}$$

as well as

$$\int_{\Sigma_*(u,u_*)} u^{2+2\delta_{dec}} |\nabla_3 \mathfrak{d}^k \mathfrak{q}|^2 \lesssim \epsilon_0^2. \tag{5.40}$$

2. According to Theorem M2, $\underline{\alpha}$ satisfies, for all $0 \le k \le k_*$,

$$\int_{\Sigma_*(u,u_*)} u^{2+2\delta_{dec}} |\mathfrak{d}_*^k \underline{\alpha}|^2 \lesssim \epsilon_0^2. \tag{5.41}$$

3. According to Theorem M1, α satisfies on Σ_*, for all $0\leq k\leq k_*$,

$$
\begin{aligned}
\|\alpha\|_{\infty,k} &\lesssim \epsilon_0 r^{-\frac{7}{2}-\delta_{extra}},\\
\left\|\nabla_3\alpha+(\tilde{f}\widehat{\otimes}\beta-{}^*\tilde{f}\widehat{\otimes}{}^*\beta)\right\|_{\infty,k-1} &\lesssim \epsilon_0 r^{-\frac{9}{2}-\delta_{extra}},
\end{aligned}
\tag{5.42}
$$

where the 1-form $\tilde{f}$ satisfies, for all $0\leq k\leq k_*-1$,

$$\|\tilde{f}\|_{\infty,k} \lesssim \epsilon r^{-1}. \tag{5.43}$$

Remark 5.14. *Recall from the statement of Theorem M1 that* $\mathfrak{q}$ *satisfying* (5.39) (5.40) *is expressed in the global frame of Proposition 3.33. Also, Theorem M1 implies in fact on* Σ_**, for all* $0\leq k\leq k_*$,

$$
\begin{aligned}
\|\alpha'\|_{\infty,k} &\lesssim \epsilon_0 r^{-\frac{7}{2}-\delta_{extra}},\\
\|\nabla_{e_3'}\alpha'\|_{\infty,k-1} &\lesssim \epsilon_0 r^{-\frac{9}{2}-\delta_{extra}},
\end{aligned}
\tag{5.44}
$$

where the quantities with prime are expressed in the global frame of Proposition 3.33. To infer (5.42) *from* (5.44)*, notice first that the change of frame coefficients* $(f,\underline{f},\lambda)$ *from the frame of* Σ_* *to the global frame of Proposition 3.33 satisfy, for all* $0\leq k\leq k_*$,

$$
\begin{aligned}
& f=f^{(1)}+f^{(2)},\quad \|f^{(2)}\|_{\infty,k}+\|\lambda-1\|_{\infty,k}\lesssim \epsilon r^{-1},\quad \|\underline{f}\|_{\infty,k}\lesssim r^{-1},\\
& (f^{(1)})_1=0,\quad (f^{(1)})_2=\frac{a\sin\theta}{r},\quad on\quad S_*,\quad \nabla_\nu(rf^{(1)})=0\quad on\quad \Sigma_*,
\end{aligned}
\tag{5.45}
$$

in view of property (f) of Proposition 3.33, (3.63) *and* (3.11)–(3.12)*. Then, we choose* $\tilde{f}=\nabla_3(f^{(2)})$ *which satisfies* (5.43)*, and* (5.42) *follows immediately from* (5.44) (5.45)*, the following change of frame formula of Proposition 2.12*

$$
\begin{aligned}
\lambda^{-2}\alpha' &= \alpha+(f\widehat{\otimes}\beta-{}^*f\widehat{\otimes}{}^*\beta)+\left(f\widehat{\otimes}f-\frac{1}{2}{}^*f\widehat{\otimes}{}^*f\right)\rho+\frac{3}{2}(f\widehat{\otimes}{}^*f)\,{}^*\rho\\
&\quad + l.o.t.,
\end{aligned}
$$

and the dominant condition (5.30) *for* r *on* Σ_*[116]*. Finally, Theorem M2 implies in fact on* Σ_**, for all* $0\leq k\leq k_*$,

$$\int_{\Sigma_*(u,u_*)} u^{2+2\delta_{dec}}|\mathfrak{d}_*^k\underline{\alpha}''|^2 \lesssim \epsilon_0^2. \tag{5.46}$$

[116]Notice in particular that we have on Σ_*

$$\nabla_3(f^{(1)})=r^{-1}\nabla_\nu(rf^{(1)})-r^{-1}\nu(r)f^{(1)}-b\nabla_4(f^{(1)})=-r^{-1}\nu(r)f^{(1)}-b\nabla_4(f^{(1)})$$

where $\underline{\alpha}''$ is expressed in the global frame of Proposition 3.35. Now, the change of frame formula of Proposition 2.12 for $\underline{\alpha}$ and the control of the change of frame coefficients from the frame of Σ_* to the global frame of Proposition 3.35 following from property (c) of Proposition 3.35, (3.68) and (3.11)–(3.12), implies immediately

$$|\mathfrak{d}^k\underline{\alpha}| \lesssim |\mathfrak{d}^k\underline{\alpha}''| + r^{-1}|\mathfrak{d}^{\leq k}\underline{\beta}| + \cdots \lesssim |\mathfrak{d}^k\underline{\alpha}''| + r^{-3}.$$

Together with (5.46) and the dominant condition (5.30) for r on Σ_*, we obtain (5.41).

We conclude this section with an interpolation lemma.

Lemma 5.15. *We have for*[117] $k \leq k_*$

$$\begin{aligned} \|\Gamma_g\|_{\infty,k} &\lesssim \epsilon r^{-2}u^{-\frac{1}{2}-\frac{\delta_{dec}}{2}},\\ \|\nabla_\nu\Gamma_g\|_{\infty,k-1} &\lesssim \epsilon r^{-2}u^{-1-\frac{\delta_{dec}}{2}},\\ \|\Gamma_b\|_{\infty,k} &\lesssim \epsilon r^{-1}u^{-1-\frac{\delta_{dec}}{2}}. \end{aligned} \tag{5.47}$$

Proof. Since $k_{small} < k_* < k_{large}$, we have, interpolating between (5.37) and (5.38), for $k \leq k_*$,

$$\begin{aligned} \|\Gamma_g\|_{\infty,k} &\lesssim \epsilon r^{-2}\left(u^{-\frac{1}{2}-\delta_{dec}}\right)^{1-\frac{k_*-k_{small}}{k_{large}-k_{small}}},\\ \|\nabla_\nu\Gamma_g\|_{\infty,k-1} &\lesssim \epsilon r^{-2}\left(u^{-1-\delta_{dec}}\right)^{1-\frac{k_*-k_{small}}{k_{large}-k_{small}}},\\ \|\Gamma_b\|_{\infty,k} &\lesssim \epsilon r^{-1}\left(u^{-1-\delta_{dec}}\right)^{1-\frac{k_*-k_{small}}{k_{large}-k_{small}}}. \end{aligned}$$

Now, since k_* satisfies[118]

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which together with (5.30) and the definition of $f^{(1)}$ in (5.45) yields

$$|\nabla_3(f^{(1)}\widehat{\otimes}\beta - {}^*f^{(1)}\widehat{\otimes}{}^*\beta)| \lesssim r^{-1}|\nabla_3\beta| + r^{-2}|\beta| \lesssim \epsilon r^{-5} \lesssim \epsilon_0 r^{-\frac{9}{2}-\delta_{extra}}.$$

[117] Recall that $k_* = k_{small} + 80$ throughout this chapter, see (5.1).
[118] Note that we have in view of (3.46) and (3.51)

$$\delta_{dec}(k_{large} - k_{small}) \geq \frac{1}{2}\delta_{dec}k_{large} - \delta_{dec} \gg 1,$$

and we may thus assume $\delta_{dec}(k_{large} - k_{small}) \geq 240$ so that, in view of $k_* = k_{small} + 80$, we have indeed $3(k_* - k_{small}) = 240 \leq \delta_{dec}(k_{large} - k_{small})$.

$$k_* \leq k_{small} + \frac{\delta_{dec}}{3}(k_{large} - k_{small}),$$

we infer

$$(2+2\delta_{dec})\frac{k_* - k_{small}}{k_{large} - k_{small}} \leq \delta_{dec}$$

and hence

$$\begin{aligned}
&\left(-\frac{1}{2} - \delta_{dec}\right)\left(1 - \frac{k_* - k_{small}}{k_{large} - k_{small}}\right)\\
= \quad & -\frac{1}{2} - \frac{\delta_{dec}}{2} - \frac{1}{2}\left(\delta_{dec} - (1+2\delta_{dec})\frac{k_* - k_{small}}{k_{large} - k_{small}}\right) \leq -\frac{1}{2} - \frac{\delta_{dec}}{2}
\end{aligned}$$

as well as

$$\begin{aligned}
&(-1 - \delta_{dec})\left(1 - \frac{k_* - k_{small}}{k_{large} - k_{small}}\right)\\
= \quad & -1 - \frac{\delta_{dec}}{2} - \frac{1}{2}\left(\delta_{dec} - (2+2\delta_{dec})\frac{k_* - k_{small}}{k_{large} - k_{small}}\right) \leq -1 - \frac{\delta_{dec}}{2}.
\end{aligned}$$

This yields

$$\begin{aligned}
\|\Gamma_g\|_{\infty,k} &\lesssim \epsilon r^{-2} u^{-\frac{1}{2} - \frac{\delta_{dec}}{2}},\\
\|\nabla_\nu \Gamma_g\|_{\infty,k-1} &\lesssim \epsilon r^{-2} u^{-1-\frac{\delta_{dec}}{2}},\\
\|\Gamma_b\|_{\infty,k} &\lesssim \epsilon r^{-1} u^{-1-\frac{\delta_{dec}}{2}},
\end{aligned} \tag{5.48}$$

as stated. □

5.1.5. Hodge operators We recall the following Hodge operators acting on 2-surfaces S (see [17] chapter 2 and [39]):

Definition 5.16. *We define the following Hodge operators:*

1. *The operator $\not{d}_1$ takes any 1-form f into the pair of scalars $(div\, f, curl\, f)$.*
2. *The operator $\not{d}_2$ takes any symmetric traceless 2-tensor f into the 1-form $div\, f$.*
3. *The operator $\not{d}_1^*$ takes any pair of scalars $(h, {}^*h)$ into the 1-form $-\nabla h + {}^*\nabla\, {}^*h$.*

4. The operator $\not{d}_2^*$ *takes any 1-form* f *into the symmetric traceless 2-tensor* $-\frac{1}{2}\nabla\widehat{\otimes}f$.

One can easily check that $\not{d}_k^*$ is the formal adjoint on $L^2(S)$ of $\not{d}_k$ for $k = 1, 2$. Moreover,

$$\begin{aligned} \not{d}_1^* \not{d}_1 &= -\Delta_1 + K, & \not{d}_1 \not{d}_1^* &= -\Delta_0, \\ \not{d}_2^* \not{d}_2 &= -\frac{1}{2}\Delta_2 + K, & \not{d}_2 \not{d}_2^* &= -\frac{1}{2}(\Delta_1 + K). \end{aligned} \tag{5.49}$$

Using (5.49) one can prove the following (see Chapter 2 in [17]).

Proposition 5.17. *Let* (S, g) *be a compact manifold with Gauss curvature* K*. We have,*

i.) *The following identity holds for 1-forms* f

$$\int_S (|\nabla f|^2 + K|f|^2) = \int_S |\not{d}_1 f|^2.$$

ii.) *The following identity holds for symmetric traceless 2-tensors* f

$$\int_S (|\nabla f|^2 + 2K|f|^2) = 2\int_S |\not{d}_2 f|^2.$$

iii.) *The following identity holds for scalars* f

$$\int_S |\nabla f|^2 = \int_S |\not{d}_1^* f|^2.$$

iv.) *The following identity holds for 1-forms* f

$$\int_S (|\nabla f|^2 - K|f|^2) = 2\int_S |\not{d}_2^* f|^2.$$

5.1.6. Basic equations for the linearized quantities

Proposition 5.18. *The following equations hold true on* Σ_**:*

1. The linearized null structure equations are given by

$$\begin{aligned} \nabla_4 \check{\kappa} &= \Gamma_g \cdot \Gamma_g, \\ \nabla_4 \widehat{\chi} + \frac{2}{r}\widehat{\chi} &= -\alpha, \\ \nabla_4 \zeta + \frac{2}{r}\zeta &= -\beta + \Gamma_g \cdot \Gamma_g, \end{aligned}$$

$$\nabla_4\underline{\check{\kappa}} + \frac{1}{r}\underline{\check{\kappa}} = -2div\zeta + 2\check{\rho} + \Gamma_b\cdot\Gamma_g,$$
$$\nabla_4\underline{\widehat{\chi}} + \frac{1}{r}\underline{\widehat{\chi}} = \frac{\Upsilon}{r}\widehat{\chi} - \nabla\widehat{\otimes}\zeta + \Gamma_b\cdot\Gamma_g,$$

and

$$\nabla_3\check{\kappa} = 2div\eta + 2\check{\rho} - \frac{1}{r}\underline{\check{\kappa}} + \frac{4}{r}\underline{\check{\omega}} + \frac{2}{r^2}\check{y} + \Gamma_b\cdot\Gamma_b,$$
$$\nabla_3\underline{\check{\kappa}} - \frac{2\Upsilon}{r}\underline{\check{\kappa}} = 2div\underline{\xi} + \frac{4\Upsilon}{r}\underline{\check{\omega}} - \frac{2m}{r^2}\underline{\check{\kappa}} - \left(\frac{2}{r^2} - \frac{8m}{r^3}\right)\check{y} + \Gamma_b\cdot\Gamma_b,$$
$$\nabla_3\underline{\widehat{\chi}} - \frac{2\Upsilon}{r}\underline{\widehat{\chi}} = -\underline{\alpha} - \frac{2m}{r^2}\underline{\widehat{\chi}} + \nabla\widehat{\otimes}\underline{\xi} + \Gamma_b\cdot\Gamma_b,$$
$$\nabla_3\zeta - \frac{\Upsilon}{r}\zeta = -\underline{\beta} - 2\nabla\underline{\check{\omega}} + \frac{\Upsilon}{r}(\eta+\zeta) + \frac{1}{r}\underline{\xi} + \frac{2m}{r^2}(\zeta-\eta) + \Gamma_b\cdot\Gamma_b,$$
$$\nabla_3\widehat{\chi} - \frac{\Upsilon}{r}\widehat{\chi} = \nabla\widehat{\otimes}\eta - \frac{1}{r}\underline{\widehat{\chi}} + \frac{2m}{r^2}\widehat{\chi} + \Gamma_b\cdot\Gamma_b.$$

Also,

$$\begin{aligned} div\widehat{\chi} &= \frac{1}{r}\zeta - \beta + \Gamma_g\cdot\Gamma_g,\\ div\underline{\widehat{\chi}} &= \frac{1}{2}\nabla\underline{\check{\kappa}} + \frac{\Upsilon}{r}\zeta + \underline{\beta} + \Gamma_b\cdot\Gamma_g,\end{aligned}$$

$$\begin{aligned} curl\zeta &= {}^*\rho + \Gamma_b\cdot\Gamma_g,\\ curl\eta &= {}^*\rho + \Gamma_b\cdot\Gamma_g,\\ curl\underline{\xi} &= \Gamma_b\cdot\Gamma_b,\end{aligned}$$

and

$$\begin{aligned} \check{K} &= -\frac{1}{2r}\underline{\check{\kappa}} - \check{\rho} + \Gamma_b\cdot\Gamma_g,\\ \check{\mu} &= -div\zeta - \check{\rho} + \Gamma_b\cdot\Gamma_g.\end{aligned}$$

2. *The linearized Bianchi identities are given by*

$$\nabla_3\alpha - \frac{\Upsilon}{r}\alpha = \nabla\widehat{\otimes}\beta + \frac{4m}{r^2}\alpha + \frac{6m}{r^3}\widehat{\chi} + \Gamma_b\cdot(\alpha,\beta) + r^{-1}\Gamma_g\cdot\Gamma_g,$$
$$\nabla_4\beta + \frac{4}{r}\beta = -div\alpha + r^{-1}\Gamma_g\cdot\Gamma_g,$$
$$\nabla_3\beta - \frac{2\Upsilon}{r}\beta = (\nabla\rho + {}^*\nabla\,{}^*\rho) + \frac{2m}{r^2}\beta - \frac{6m}{r^3}\eta + r^{-1}\Gamma_b\cdot\Gamma_g,$$

$$\nabla_4\check{\rho} + \frac{3}{r}\check{\rho} = div\,\beta + r^{-1}\Gamma_b\cdot\Gamma_g,$$
$$\nabla_3\check{\rho} - \frac{3\Upsilon}{r}\check{\rho} = -div\,\underline{\beta} + \frac{3m}{r^3}\underline{\check{\kappa}} - \frac{6m}{r^4}\check{y} - \frac{1}{2}\widehat{\chi}\cdot\underline{\alpha} + r^{-1}\Gamma_b\cdot\Gamma_b,$$
$$\nabla_4\ {}^*\rho + \frac{3}{r}\ {}^*\rho = -curl\,\beta + r^{-1}\Gamma_b\cdot\Gamma_g,$$
$$\nabla_3\ {}^*\rho - \frac{3\Upsilon}{r}\ {}^*\rho = -curl\,\underline{\beta} - \frac{1}{2}\widehat{\chi}\cdot\ {}^*\underline{\alpha} + r^{-1}\Gamma_b\cdot\Gamma_b.$$

Proof. The proof follows immediately from Proposition 5.10, the definition of the linearized quantities, the definition of Γ_g and Γ_b, the fact that $y = e_3(r)$, and the GCM condition $\underline{\check{\kappa}} = 0$ on Σ_*. □

5.1.7. Commutation lemmas We start with the following lemma.

Lemma 5.19. *For any tensor on S, the following commutation formulas hold true*

$$[\nabla_3,\nabla_a]f = -\frac{1}{2}tr\,\underline{\chi}\nabla_a f + (\eta_a-\zeta_a)\nabla_3 f - \underline{\widehat{\chi}}_{ab}\nabla_b f + \underline{\xi}_a\nabla_4 f + (\underline{F}[f])_a,$$
$$[\nabla_4,\nabla_a]f = -\frac{1}{2}tr\,\chi\nabla_a f + (\underline{\eta}_a+\zeta_a)\nabla_4 f - \widehat{\chi}_{ab}\nabla_b f + (F[f])_a,$$

where the tensors $F[f]$ and $\underline{F}[f]$ have the following schematic form

$$F[f] = (\beta, \chi\cdot\underline{\eta}, \underline{\chi}\cdot\xi)\cdot f, \qquad \underline{F}[f] = (\underline{\beta}, \underline{\chi}\cdot\eta, \chi\cdot\underline{\xi})\cdot f.$$

Proof. See Lemma 7.3.3 in [17]. □

Lemma 5.20. *The following commutation formulas hold true for any tensor f on $S\subset\Sigma_*$:*

1. *We have*

$$[\nabla_3,\nabla]f = \frac{\Upsilon}{r}\nabla f + \Gamma_b\cdot\nabla_3 f + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}f,$$
$$[\nabla_4,\nabla]f = -\frac{1}{r}\nabla f + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}f.$$

2. *We have*

$$[\nabla_3,\Delta]f = \frac{2\Upsilon}{r}\Delta f + r^{-1}\,\not{\mathfrak{d}}^{\leq 1}\Big(\Gamma_b\cdot\nabla_3 f + r^{-1}\Gamma_b\cdot\mathfrak{d}f\Big),$$
$$[\nabla_4,\Delta]f = -\frac{2}{r}\Delta f + r^{-1}\,\not{\mathfrak{d}}^{\leq 1}\Big(\Gamma_g\cdot\mathfrak{d}f\Big).$$

3. *We have*

$$[\nabla_\nu, \nabla]f = \frac{2}{r}\nabla f + \Gamma_b \cdot \nabla_\nu f + r^{-1}\Gamma_b \cdot \mathfrak{d}^{\leq 1} f.$$

4. *We have*

$$[\nabla_\nu, \Delta]f = \frac{4}{r}\Delta f + r^{-1}\not{\mathfrak{d}}^{\leq 1}\Big(\Gamma_b \cdot \nabla_\nu f + r^{-1}\Gamma_b \cdot \mathfrak{d} f\Big).$$

Proof. The proof follows from Lemma 5.19, the definitions of Γ_g, Γ_b, the transversality conditions $\xi = 0$ and $\underline{\eta} = -\zeta$ on Σ_*, and $\nu = e_3 + b_* e_4$. □

Corollary 5.21. *The following commutation formulas hold true for any tensor f on $S \subset \Sigma_*$:*

$$\begin{aligned}
&[\nabla_3, r\nabla]f = r\Gamma_b \cdot \nabla_3 f + \Gamma_b \cdot \mathfrak{d}^{\leq 1} f,\\
&[\nabla_4, r\nabla]f = \Gamma_g \cdot \mathfrak{d}^{\leq 1} f,\\
&[\nabla_3, r^2\Delta]f = r\not{\mathfrak{d}}^{\leq 1}\Big(\Gamma_b \cdot \nabla_3 f + r^{-1}\Gamma_b \cdot \mathfrak{d} f\Big),\\
&[\nabla_4, r^2\Delta]f = r\not{\mathfrak{d}}^{\leq 1}\Big(\Gamma_g \cdot \mathfrak{d} f\Big),
\end{aligned} \tag{5.50}$$

and

$$\begin{aligned}
&[\nabla_\nu, r\nabla]f = r\Gamma_b \cdot \nabla_\nu f + \Gamma_b \cdot \mathfrak{d}^{\leq 1} f,\\
&[\nabla_\nu, r^2\Delta]f = r\not{\mathfrak{d}}^{\leq 1}\Big(\Gamma_b \cdot \nabla_\nu f\Big) + \not{\mathfrak{d}}^{\leq 1}\Big(\Gamma_b \cdot \mathfrak{d} f\Big).
\end{aligned} \tag{5.51}$$

Proof. The proof follows from Lemma 5.20, the transversality condition $e_4(r) = 1$ on Σ_*, and the fact that $e_3(r) = -\Upsilon + r\Gamma_b$ and $\nu(r) = -2 + r\Gamma_b$. □

5.1.8. Additional equations

Proposition 5.22. *We have, schematically,*

$$\begin{aligned}
2\nabla\check{\underline{\omega}} - \frac{1}{r}\underline{\xi} &= -\nabla_3\zeta - \underline{\beta} + \frac{1}{r}\eta + r^{-1}\Gamma_g + \Gamma_b \cdot \Gamma_b,\\
2\,\not{d}_2\,\not{d}_2^*\eta &= -\nabla_3\nabla\check{\kappa} - \frac{2}{r}\nabla_3\zeta - \frac{2}{r}\underline{\beta} + r^{-2}\not{\mathfrak{d}}^{\leq 1}\Gamma_g + r^{-1}\not{\mathfrak{d}}^{\leq 1}(\Gamma_b \cdot \Gamma_b),\\
2\,\not{d}_2\,\not{d}_2^*\underline{\xi} &= -\nabla_3\nabla\check{\underline{\kappa}} - \frac{2}{r}\nabla_3\zeta - \frac{2}{r}\underline{\beta} + r^{-2}\not{\mathfrak{d}}^{\leq 1}\Gamma_g + r^{-1}\not{\mathfrak{d}}^{\leq 1}(\Gamma_b \cdot \Gamma_b).
\end{aligned}$$

Proof. See Section B.1 in the appendix. □

The following corollary of Proposition 5.22 will be very useful.

Proposition 5.23. *The following identities hold true on Σ_*:*

$$2\,{\not{d}}_2^*{\not{d}}_1^*{\not{d}}_1\,{\not{d}}_2\,{\not{d}}_2^*\eta = -\,{\not{d}}_2^*{\not{d}}_1^*{\not{d}}_1\nabla_3\nabla\check{\kappa} + \frac{2}{r}\nabla_3\,{\not{d}}_2^*{\not{d}}_1^*\check{\mu} - \frac{4}{r}\,{\not{d}}_2^*{\not{d}}_1^* div\,\underline{\beta} + r^{-5}\,\mathfrak{d}^{\leq 4}\Gamma_g + r^{-4}\,\mathfrak{d}^{\leq 4}(\Gamma_b\cdot\Gamma_b), \tag{5.52}$$

$$2\,{\not{d}}_2^*{\not{d}}_1^*{\not{d}}_1\,{\not{d}}_2\,{\not{d}}_2^*\underline{\xi} = \nabla_3\left({\not{d}}_2^*{\not{d}}_2 + \frac{2}{r^2}\right){\not{d}}_2^*{\not{d}}_1^*\check{\underline{\kappa}} + \frac{2}{r}\nabla_3\,{\not{d}}_2^*{\not{d}}_1^*\check{\mu} - \frac{4}{r}\,{\not{d}}_2^*{\not{d}}_1^* div\,\underline{\beta} + r^{-5}\,\mathfrak{d}^{\leq 4}\Gamma_g + r^{-4}\,\mathfrak{d}^{\leq 4}(\Gamma_b\cdot\Gamma_b), \tag{5.53}$$

where by convention, for any scalar f,

$$ {\not{d}}_1^* f := {\not{d}}_1^*(f,0) = -\nabla f.$$

Proof. See Section B.2 in the appendix. □

Corollary 5.24. *The following identities hold true on Σ_*.*

$$2\,{\not{d}}_2^*{\not{d}}_1^*{\not{d}}_1\,{\not{d}}_2\,{\not{d}}_2^*\eta = -\,{\not{d}}_2^*{\not{d}}_1^*{\not{d}}_1\nabla_\nu\nabla\check{\kappa} + \frac{2}{r}\nabla_\nu\,{\not{d}}_2^*{\not{d}}_1^*\check{\mu} - \frac{4}{r}\,{\not{d}}_2^*{\not{d}}_1^* div\,\underline{\beta} + r^{-5}\,\mathfrak{d}^{\leq 5}\Gamma_g + r^{-4}\,\mathfrak{d}^{\leq 4}(\Gamma_b\cdot\Gamma_b), \tag{5.54}$$

$$2\,{\not{d}}_2^*{\not{d}}_1^*{\not{d}}_1\,{\not{d}}_2\,{\not{d}}_2^*\underline{\xi} = \nabla_\nu\left({\not{d}}_2^*{\not{d}}_2 + \frac{2}{r^2}\right){\not{d}}_2^*{\not{d}}_1^*\check{\underline{\kappa}} + \frac{2}{r}\nabla_\nu\,{\not{d}}_2^*{\not{d}}_1^*\check{\mu} - \frac{4}{r}\,{\not{d}}_2^*{\not{d}}_1^* div\,\underline{\beta} + r^{-5}\,\mathfrak{d}^{\leq 5}\Gamma_g + r^{-4}\,\mathfrak{d}^{\leq 4}(\Gamma_b\cdot\Gamma_b). \tag{5.55}$$

Proof. We have in view of Proposition 5.18

$$\nabla_4\check{\kappa}\in\Gamma_g\cdot\Gamma_g,\qquad \nabla_4\check{\underline{\kappa}}\in r^{-1}\,\mathfrak{d}^{\leq 1}\Gamma_g,\qquad \nabla_4\check{\mu}\in r^{-2}\,\mathfrak{d}^{\leq 1}\Gamma_g,$$

which together with $b_* = -1-\frac{2m}{r}+r\Gamma_b$ yields

$${\not{d}}_2^*{\not{d}}_1^*{\not{d}}_1 b_*\nabla_4\nabla\check{\kappa}\in r^{-5}\,\mathfrak{d}^{\leq 5}\Gamma_g,\qquad \frac{b_*}{r}\nabla_4\,{\not{d}}_2^*{\not{d}}_1^*\check{\mu}\in r^{-5}\,\mathfrak{d}^{\leq 5}\Gamma_g,$$
$$b_*\nabla_4\left({\not{d}}_2^*{\not{d}}_2+\frac{2}{r^2}\right){\not{d}}_2^*{\not{d}}_1^*\check{\underline{\kappa}}\in r^{-5}\,\mathfrak{d}^{\leq 5}\Gamma_g.$$

Since $\nu = e_3 + b_* e_4$, the proof of (5.54) and (5.55) follows then immediately from Proposition 5.23. □

5.1.9. Additional renormalized equations on Σ_*

Lemma 5.25. *We have along* Σ_*

$$\begin{aligned}
\nabla_\nu\left(\Delta\underline{\check{\kappa}}+\frac{2\Upsilon}{r}div\,\zeta\right) &= O(r^{-1})\Delta\underline{\check{\kappa}}+2\Delta\, div\,\underline{\xi}+O(r^{-2})\Delta\check{y}+O(r^{-2})\,div\,\zeta\\
&\quad +O(r^{-1})\,div\,\underline{\beta}+O(r^{-2})\,div\,\eta+O(r^{-2})\,div\,\underline{\xi}\\
&\quad +2\left(1+O(r^{-1})\right)\Delta\, div\,\zeta-2\left(1+O(r^{-1})\right)\Delta\check{\rho}\\
&\quad +O(r^{-1})\,div\,\beta+r^{-2}\,\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_b),\\
\nabla_\nu\, div\,\beta &= O(r^{-1})\,div\,\beta+\Delta\rho+(1+O(r^{-1}))\,div\,div\,\alpha\\
&\quad +O(r^{-3})\,div\,\eta+r^{-2}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g),\\
\nabla_\nu\, curl\,\beta &= \frac{8}{r}(1+O(r^{-1}))\,curl\,\beta-\Delta\ {}^*\rho\\
&\quad +(1+O(r^{-1}))\,curl\,div\,\alpha+O(r^{-3})\ {}^*\rho\\
&\quad +r^{-2}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g),\\
\nabla_\nu\left(\check{\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right) &= -div\,\underline{\beta}-(1+O(r^{-1}))\,div\,\beta+O(r^{-1})\check{\rho}+O(r^{-3})\underline{\check{\kappa}}\\
&\quad +O(r^{-4})\check{y}+r^{-1}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b),
\end{aligned}\tag{5.56}$$

where the notation $O(r^a)$, for $a\in\mathbb{R}$, denotes an explicit function of r which is bounded by r^a as $r\to+\infty$.

Proof. See Section B.3 in the appendix. □

5.1.10. Equations involving $\mathfrak{q}$

Proposition 5.26. *Let $O(r^a)$ denote, for $a\in\mathbb{R}$, a function of $(r,\cos\theta)$ bounded by r^a as $r\to+\infty$. In the frame of Σ_*, the following holds:*

1. *We have*

$$\begin{aligned}
\Re(\mathfrak{q}) &= r^4\,\not{d}_2^*\not{d}_1^*(-\rho,\ {}^*\rho)+O(r^{-2})\\
&\quad +\mathfrak{d}^{\leq 2}\Gamma_b+r^2\,\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g).
\end{aligned}\tag{5.57}$$

2. *We have*

$$\begin{aligned}
\Re(\nabla_3(r\mathfrak{q})) &= r^5\,\not{d}_2^*\not{d}_1^*(div\,\underline{\beta},-curl\,\underline{\beta})+O(r)\,\mathfrak{d}^{\leq 3}\underline{\alpha}\\
&\quad +O(r^{-2})+r\,\mathfrak{d}^{\leq 3}\Gamma_g+r^3\,\mathfrak{d}^{\leq 3}(\Gamma_b\cdot\Gamma_g).
\end{aligned}\tag{5.58}$$

Proof. See Section B.5 in the appendix. □

5.1.11. Hodge elliptic systems For a tensor f on S, we define the following standard weighted Sobolev norms for any integer $k \geq 0$

$$\|f\|_{\mathfrak{h}_k(S)} := \sum_{j=0}^{k} \|\not{\mathfrak{d}}^j f\|_{L^2(S)}. \tag{5.59}$$

We record below the following well known coercive properties of the operators $\not{d}_1$, $\not{d}_2$, $\not{d}_1^*$, see Chapter 2 in [17].

Lemma 5.27. *For any sphere $S = S(u) \subset \Sigma_*$ we have, for all $k \leq k_{large}$:*

1. *If f is a 1-form*
$$\|f\|_{\mathfrak{h}_{k+1}(S)} \lesssim r\|\not{d}_1 f\|_{\mathfrak{h}_k(S)}. \tag{5.60}$$
2. *If f is a symmetric traceless 2-tensor*
$$\|v\|_{\mathfrak{h}_{k+1}(S)} \lesssim r\|\not{d}_2 v\|_{\mathfrak{h}_k(S)}. \tag{5.61}$$
3. *If $(h,\ {}^*h)$ is a pair of scalars*
$$\|(h-\overline{h},\ {}^*h-\overline{{}^*h})\|_{\mathfrak{h}_{k+1}(S)} \lesssim r\|\not{d}_1^*(h,\ {}^*h)\|_{\mathfrak{h}_k(S)}. \tag{5.62}$$

Proof. Recall from Proposition 5.18 the linearized Gauss equation

$$K = -\frac{1}{2r}\check{\underline{\kappa}} - \check{\rho} + \Gamma_b \cdot \Gamma_g.$$

In view of the control (5.38) for Γ_g and Γ_b, we have

$$\left\|\not{\mathfrak{d}}^{\leq k_{large}}\left(K - \frac{1}{r^2}\right)\right\|_{L^\infty(S)} \lesssim \frac{\epsilon}{r^3}.$$

Together with Proposition 5.17, and a Poincaré inequality for $\not{d}_1^*$, this immediately yields the case $k = 0$. The case $k \geq 1$ then follows by the above control of K and elliptic regularity. □

The operator $\not{d}_2^*$ is not coercive but satisfies instead the following estimates.

Lemma 5.28. *On a fixed sphere $S = S(u) \subset \Sigma_*$, we have for any 1-form f and all $k \leq k_{large}$*

$$\|f\|_{\mathfrak{h}_{k+1}(S)} \lesssim r\|\not{d}_2^* f\|_{\mathfrak{h}_k(S)} + r^2|(\not{d}_1 f)_{\ell=1}|. \tag{5.63}$$

Note also the straightforward inequality

$$|(\not{d}_1 f)_{\ell=1}| \lesssim r^{-1}\|\not{d}_1 f\|_{L^2(S)}. \tag{5.64}$$

Proof. The case $k=0$ is proved in Lemma 2.19 of [40]. The higher derivative estimates follow by elliptic regularity and the above control of K. □

Corollary 5.29. *On a fixed sphere $S=S(u)\subset\Sigma_*$, we have for any pair of scalars $(h,\ {}^*h)$ and all $k\leq k_{large}-1$*

$$\begin{aligned}\|(h-\overline{h},\ {}^*h-\overline{{}^*h})\|_{\mathfrak{h}_{k+2}(S)} &\lesssim r^2\|\not{d}_2^*\,\not{d}_1^*(h,\ {}^*h)\|_{\mathfrak{h}_k(S)}+r^3|(\Delta h)_{\ell=1}|\\ &\quad +r^3|(\Delta\ {}^*h)_{\ell=1}|.\end{aligned}$$

Proof. Applying Lemma 5.28 with $f=\not{d}_1^*(h,\ {}^*h)$ yields the control of $\not{d}_1^*(h,\ {}^*h)$. The control of $(h-\overline{h},\ {}^*h-\overline{{}^*h})$ follows then from applying Lemma 5.27. □

5.2. Preliminary estimates on Σ_*

5.2.1. Behavior of r on Σ_* The following lemma shows that r is comparable to $r_*=r(S_*)$ on Σ_*.

Lemma 5.30. *The function r satisfies on Σ_**

$$r_*\leq r\leq r_*\left(1+O(\epsilon_0)\right). \tag{5.65}$$

Proof. Recall that $u(S_*)=u_*$, $u(S_1)=1$ and $1\leq u\leq u_*$ along Σ_*. Since r is a decreasing function of u on Σ_*, we infer

$$r_*\leq r\leq r(S_1)\quad\text{on}\quad\Sigma_*.$$

To conclude, it thus suffices to prove that $r(S_1)\leq r_*(1+O(\epsilon_0))$. To this end, recall first (5.2), i.e., $u+r=c_*$ along Σ_*, so that

$$r(S_*)+u_*=c_*=r(S_1)+1$$

and hence

$$r(S_1)=r_*+u_*-1=r_*\left(1+\frac{u_*-1}{r_*}\right)\leq r_*\left(1+\frac{u_*}{r_*}\right).$$

Together with the dominant condition on r (5.30) on Σ_*, i.e., $r_*=\frac{\delta_*}{\epsilon_0}u_*^{1+\delta_{dec}}$, we infer $r(S_1)\leq r_*(1+O(\epsilon_0))$ which concludes the proof of the lemma. □

5.2.2. Transport lemmas along Σ_*

Lemma 5.31. *For any scalar function h on Σ_*, we have the formula*

$$\nu\left(\int_S h\right) = z\int_S \frac{1}{z}\Big(\nu(h)+(\underline{\kappa}+b_*\kappa)h\Big). \tag{5.66}$$

In particular[119]

$$\nu(r) = \frac{rz}{2}\overline{z^{-1}(\underline{\kappa}+b_*\kappa)} \tag{5.67}$$

where for a scalar function f, $\overline{f}$ denotes the average of f with respect to the spheres of the foliation of Σ_*.

Proof. Recall that on Σ_* the coordinates θ,φ where defined s.t. $\nu(\theta)=\nu(\varphi)=0$. Moreover, in view of the transversality condition $e_4(u)=0$ on Σ_*, we have $\nu(u)=e_3(u)$. Thus $\nu(u)=z$ and hence $\nu=z\partial_u$ in the (u,θ,φ) coordinates system. We infer

$$\begin{aligned}\nu\left(\int_S h\right) &= z\partial_u\left(\int_S h\right) = z\int_S\Big(\partial_u h+g^{ab}g(\mathbf{D}_a\partial_u,e_b)\Big)\\ &= z\int_S\frac{1}{z}\Big(\nu(h)+g^{ab}g(\mathbf{D}_a\nu,e_b)\Big)\\ &= z\int_S\frac{1}{z}\Big(\nu(h)+g^{ab}g(\mathbf{D}_a(e_3+b_*e_4),e_b)\Big)\\ &= z\int_S\frac{1}{z}\Big(\nu(h)+(\underline{\kappa}+b_*\kappa)h\Big)\end{aligned}$$

which yields the first identity. The second identity follows then by choosing $h=1$ in the first identity. □

Corollary 5.32. *For any scalar function h on Σ_*, we have*

$$\nu\left(\int_S h\right) = \int_S\nu(h)-\frac{4}{r}\int_S h+r^3\Gamma_b\nu(h)+r^2\Gamma_b h \tag{5.68}$$

and

$$\nu(r) = -2+r\Gamma_b. \tag{5.69}$$

In particular, we have

$$\nu(\overline{h}) = \overline{\nu(h)}+r\Gamma_b\nu(h)+\Gamma_b h, \tag{5.70}$$

[119]Recall that r denotes the area radius of GCM spheres along Σ_*.

where $\overline{h}$ and $\overline{\nu(h)}$ denote respectively the average of h and $\nu(h)$ on the spheres of Σ_*.

Proof. Since we have

$$\kappa = \frac{2}{r} + \Gamma_g, \qquad \underline{\kappa} = -\frac{2\Upsilon}{r} + \Gamma_g, \qquad b_* = -1 - \frac{2m}{r} + r\Gamma_b,$$

we infer

$$\underline{\kappa} + b_*\kappa \;=\; -\frac{4}{r} + \Gamma_b. \tag{5.71}$$

The proof follows then easily from Lemma 5.31, (5.71) and the fact that $z = 2 + r\Gamma_b$. □

We now control transport equation in ν along Σ_*.

Lemma 5.33. *Let n and m be two integers, and let f and h be two scalar functions on Σ_*. Assume that f satisfies along Σ_**

$$\nu(f) - \frac{n}{r}f \;=\; h + \Gamma_b f. \tag{5.72}$$

Then, we have for all $1 \le u \le u_*$

$$\begin{aligned} \|r^m f\|_{L^\infty(S(u))} \;&\lesssim\; r_*^m\|f\|_{L^\infty(S_*)} \\ &\quad + \int_u^{u_*}\Big(\|r^m h\|_{L^\infty(S(u'))} + \|r^m\Gamma_b f\|_{L^\infty(S(u'))}\Big)du' \end{aligned} \tag{5.73}$$

where the inequality is uniform in u.

Proof. We rewrite the transport equation for f as

$$\begin{aligned} \nu(r^{\frac{n}{2}}f) \;&=\; r^{\frac{n}{2}}\left(\nu(f) + \frac{n}{2}\frac{\nu(r)}{r}f\right) = r^{\frac{n}{2}}\left(\frac{n}{r}f + h + \Gamma_b f + \frac{n}{2}\frac{-2+r\Gamma_b}{r}f\right) \\ &=\; r^{\frac{n}{2}}\left(h + \Gamma_b f\right) \end{aligned}$$

where we have used $\nu(r) = -2 + r\Gamma_b$. Integrating from S_*, we deduce for all $1 \le u \le u_*$

$$\|r^{\frac{n}{2}}f\|_{L^\infty(S(u))} \lesssim r_*^{\frac{n}{2}}\|f\|_{L^\infty(S_*)} + \int_u^{u_*}\Big(\|r^{\frac{n}{2}}h\|_{L^\infty(S(u'))} + \|r^{\frac{n}{2}}\Gamma_b f\|_{L^\infty(S(u'))}\Big)du'$$

where the inequality is uniform in u. Multiplying this estimate by $r_*^{m-\frac{n}{2}}$, and since r is comparable to r_* on Σ_* in view of (5.65), we infer the stated estimate. □

Corollary 5.34. *Let n and m be two integers, let $s \geq 0$ be a positive real number, and let f and h be two scalar functions on Σ_*. Assume that f satisfies along Σ_**

$$\nu(f) - \frac{n}{r} f = h + \Gamma_b f. \tag{5.74}$$

Assume also that there exists a constant $C > 0$ such that

$$\sup_{1 \leq u \leq u_*} u^s \left(r_*^m \|f\|_{L^\infty(S_*)} + \int_u^{u_*} \|r^m h\|_{L^\infty(S(u'))} \right) \leq C. \tag{5.75}$$

Then, we have

$$\sup_{\Sigma_*} r^m u^s |f| \lesssim C. \tag{5.76}$$

Proof. In view of Lemma 5.33, there exists a constant $C_0 > 0$, uniform in $1 \leq u \leq u_*$, such that for all $1 \leq u \leq u_*$

$$\begin{aligned} \|r^m f\|_{L^\infty(S(u))} \leq & C_0 r_*^m \|f\|_{L^\infty(S_*)} \\ & + C_0 \int_u^{u_*} \Big(\|r^m h\|_{L^\infty(S(u'))} + \|r^m \Gamma_b f\|_{L^\infty(S(u'))} \Big) du' \end{aligned}$$

and hence, in view of the assumptions of the corollary, and the assumption on Γ_b, we infer

$$u^s \|r^m f\|_{L^\infty(S(u))} \leq C_0 C + u^s \epsilon \int_u^{u_*} \frac{\|r^m f\|_{L^\infty(S(u'))}}{u'^{1+\delta_{dec}}} du'$$

and the proof easily follows by bootstrap from $u = u_*$ and the fact that $s \geq 0$. □

5.2.3. Control of ϕ and the $\ell = 1$ basis $J^{(p)}$ on S_* In this section, we will rely on the following control of the Gauss curvature

$$\sup_{S_*} \left| \not{\mathfrak{d}}^{\leq k_*} \left(K - \frac{1}{r^2} \right) \right| \lesssim \frac{\epsilon}{r^3 u^{\frac{1}{2} + \frac{\delta_{dec}}{2}}}, \tag{5.77}$$

which follows immediately from the fact that $\check{K} \in r^{-1}\Gamma_g$ in view of the linearized Gauss equation of Proposition 5.18, and the control of Lemma 5.15 for Γ_g.

Recall that ϕ is the conformal factor of the metric g on S_*, see (5.17). We have the following lemma.

Lemma 5.35. *We have on S_**

$$\|\,\not{\mathfrak{d}}^{\leq k_*}\phi\|_{L^\infty(S_*)} \lesssim \frac{\epsilon}{ru^{\frac{1}{2}+\frac{\delta_{dec}}{2}}}. \tag{5.78}$$

Proof. In view of

- the control (5.77) of K on S_*,
- the fact that g is conformal to the metric of $\mathbb{S}^2$ with conformal factor $r^2e^{2\phi}$, see (5.17),
- and the balanced condition (5.19) for the $\ell = 1$ modes $J^{(p)}$,

we are in position to apply Corollary 3.8 in [41] (restated here as Theorem 5.2) which yields[120]

$$\|\phi\|_{\mathfrak{h}_{k_*+2}(S_*)} \lesssim \frac{\epsilon}{ru^{\frac{1}{2}+\frac{\delta_{dec}}{2}}}.$$

Together with Sobolev, this concludes the proof of the lemma. □

Lemma 5.36. *We have on S_**

$$\begin{aligned}
\int_{S_*} J^{(p)} &= 0,\\
\int_{S_*} J^{(p)}J^{(q)} &= \frac{4\pi}{3}r^2\delta_{pq} + O\left(\epsilon r u^{-\frac{1}{2}-\delta_{dec}}\right),\\
\left\|\not{\mathfrak{d}}^{\leq k_*}\left(\Delta J^{(p)} + \frac{2}{r^2}J^{(p)}\right)\right\|_{L^\infty(S_*)} &= O\left(\epsilon r^{-3}u^{-\frac{1}{2}-\frac{\delta_{dec}}{2}}\right).
\end{aligned} \tag{5.79}$$

Proof. The first identity in (5.79) is the balanced condition (5.19), so we focus on the two other identities. In view of Lemma 3.12 in [41] (restated here as Lemma 5.4), we have[121]

$$\begin{aligned}
\int_{S_*} J^{(p)}J^{(q)} &= \frac{4\pi}{3}r^2\delta_{pq} + \int_{S_*} J^{(p)}J^{(q)}(1-e^{-2\phi}),\\
\Delta J^{(p)} + \frac{2}{r^2}J^{(p)} &= \frac{2}{r^2}(1-e^{-2\phi})J^{(p)},
\end{aligned}$$

and the last two identities in (5.79) follow from the control (5.78) of ϕ. □

[120] Note that the vanishing integrals in (5.7) are equivalent to the balanced condition (5.19).

[121] In view of (5.17)–(5.19), $J^{(p)}$ is a canonical basis of $\ell = 1$ mode of S_* in the terminology of Definition 3.10 in [41] (restated here as Definition 5.3), so Lemma 3.12 in [41] applies.

5.2.4. Properties of the $\ell = 1$ basis $J^{(p)}$ on Σ_* We are ready to derive the basic properties of the $\ell = 1$ basis on Σ_*.

Lemma 5.37. *The functions $J^{(p)}$ verify the following properties*

1. *We have on Σ_**

$$\begin{aligned} \int_S J^{(p)} &= O\left(\epsilon r u^{-\delta_{dec}}\right), \\ \int_S J^{(p)} J^{(q)} &= \frac{4\pi}{3} r^2 \delta_{pq} + O\left(\epsilon r u^{-\delta_{dec}}\right). \end{aligned} \tag{5.80}$$

2. *We have on Σ_**

$$\nabla_\nu \Big[(r^2 \Delta + 2) J^{(p)} \Big] = O(\not{\mathfrak{d}}^{\leq 1} \Gamma_b). \tag{5.81}$$

3. *For any $k \leq k_* - 1$, we have on Σ_**

$$\left| \mathfrak{d}_*^k \left(\Delta + \frac{2}{r^2} \right) J^{(p)} \right| \lesssim \epsilon r^{-3} u^{-\frac{\delta_{dec}}{2}}.$$

4. *We have for any $k \leq k_* - 3$ on Σ_**

$$\left| \mathfrak{d}_*^k \, \not{d}_2^* \not{d}_1^* J^{(p)} \right| \lesssim \epsilon r^{-3} u^{-\frac{\delta_{dec}}{2}},$$

where by $\not{d}_1^* J^{(p)}$, we mean either $\not{d}_1^*(J^{(p)}, 0)$ or $\not{d}_1^*(0, J^{(p)})$.

Proof. We proceed in steps as follows.

Step 1. Since $\nu(J^{(p)}) = 0$ and $J^{(p)} = O(1)$, we have, in view of Corollary 5.32

$$\begin{aligned} \nu\left(\int_S J^{(p)} \right) &= -\frac{4}{r} \int_S J^{(p)} + r^2 \Gamma_b, \\ \nu\left(\int_S J^{(p)} J^{(q)} \right) &= -\frac{4}{r} \int_S J^{(p)} J^{(q)} + r^2 \Gamma_b. \end{aligned}$$

Since $\nu(r) = -2 + r\Gamma_b$, we infer

$$\begin{aligned} \nu\left(r^{-2} \int_S J^{(p)} \right) &= \Gamma_b, \\ \nu\left(r^{-2} \int_S J^{(p)} J^{(q)} - \frac{4\pi}{3} \delta_{pq} \right) &= \Gamma_b. \end{aligned}$$

Applying Corollary 5.34 to the above transport equations with the choices $s=\delta_{dec}>0$, $n=0$ and $m=1$, and using the control on S_* provided by (5.79) and the control of Γ_b on Σ_*, we infer (5.80).

Step 2. Using the commutation formulas of Corollary 5.21 we have, since $\nu(J^{(p)})=0$,

$$\begin{aligned}\nabla_\nu\Big[(r^2\Delta+2)J^{(p)}\Big] &= \big[\nabla_\nu, r^2\Delta\big]J^{(p)}\\ &= r\,\not{\mathfrak{d}}^{\leq 1}\Big(\Gamma_b\cdot\nabla_\nu J^{(p)}\Big)+\not{\mathfrak{d}}^{\leq 1}\Big(\Gamma_b\cdot\mathfrak{d}J^{(p)}\Big)\\ &= \not{\mathfrak{d}}^{\leq 1}\Big(\Gamma_b\cdot\mathfrak{d}J^{(p)}\Big)=\not{\mathfrak{d}}^{\leq 1}\Gamma_b\end{aligned}$$

as stated.

Step 3. Commuting the identity in Step 2 with $\not{\mathfrak{d}}^k$, and using Corollary 5.21, we have

$$\nabla_\nu\Big[\not{\mathfrak{d}}^k(r^2\Delta+2)J^{(p)}\Big] = \not{\mathfrak{d}}^{\leq k+1}\Gamma_b.$$

Applying Corollary 5.34 to the above transport equations with the choices $s=\frac{\delta_{dec}}{2}>0$, $n=0$ and $m=1$, and using the control on S_* provided by (5.79) and the control of Γ_b on Σ_*, we infer, for $k\leq k_*-1$,

$$\left|\not{\mathfrak{d}}^k\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right|\lesssim \epsilon r^{-3}u^{-\frac{\delta_{dec}}{2}}.$$

Together with the transport equation of Step 2 and the control of Γ_b, we infer, for any $k\leq k_*-1$,

$$\left|\mathfrak{d}_*^k\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right|\lesssim \epsilon r^{-3}u^{-\frac{\delta_{dec}}{2}}$$

as stated.

Step 4. We introduce the notation $F:=\not{d}_2^*\not{d}_1^*(J^{(p)})$, where by $\not{d}_1^*J^{(p)}$, we mean either $\not{d}_1^*(J^{(p)},0)$ or $\not{d}_1^*(0,J^{(p)})$. We have

$$\not{d}_1\not{d}_2F = \not{d}_1\not{d}_2\not{d}_2^*\not{d}_1^*(J^{(p)}).$$

Using the identity $2\not{d}_2\not{d}_2^*=\not{d}_1^*\not{d}_1-2K$, we deduce[122]

$$2\not{d}_1\not{d}_2F = \not{d}_1(\not{d}_1^*\not{d}_1-2K)\not{d}_1^*(J^{(p)})$$

[122] In fact $\Delta J^{(p)}$ should be replaced by either $(\Delta J^{(p)},0)$ or $(0,\Delta J^{(p)})$ depending whether we consider $\not{d}_1^*(J^{(p)},0)$ or $\not{d}_1^*(0,J^{(p)})$.

$$\begin{aligned}
&= \not{d}_1 \not{d}_1^* \not{d}_1 \not{d}_1^* J^{(p)} - 2K \not{d}_1 \not{d}_1^*(J^{(p)}) + (\nabla K, {}^*\nabla K) \cdot \not{d}_1^*(J^{(p)}) \\
&= \Delta^2 J^{(p)} + 2K\Delta J^{(p)} + (\nabla K, {}^*\nabla K) \cdot \not{d}_1^*(J^{(p)}).
\end{aligned}$$

Therefore, since $\check{K} = K - r^{-2} \in r^{-1}\Gamma_g$, we have

$$\begin{aligned}
2\not{d}_1 \not{d}_2 F &= \Delta^2 J^{(p)} + \frac{2}{r^2}\Delta J^{(p)} + 2\check{K}\Delta J^{(p)} + (\nabla \check{K}, {}^*\nabla \check{K}) \cdot \not{d}_1^*(J^{(p)}) \\
&= \Delta\left(\Delta + \frac{2}{r^2}\right) J^{(p)} + r^{-3}\not{\mathfrak{d}}^{\leq 1}\Gamma_g.
\end{aligned}$$

In view of the coercivity of the operators $\not{d}_1$ and $\not{d}_2$, see Lemma 5.27, and in view of the definition of F, we deduce for $k \geq 2$

$$\|\not{d}_2^* \not{d}_1^* J^{(p)}\|_{\mathfrak{h}_k(S)} \lesssim \left\|\left(\Delta + \frac{2}{r^2}\right) J^{(p)}\right\|_{\mathfrak{h}_k(S)} + r^{-1}\|\Gamma_g\|_{\mathfrak{h}_k(S)}.$$

In view of the estimate proved in Step 3 and the control of Γ_g, we infer

$$\|\not{d}_2^* \not{d}_1^* J^{(p)}\|_{\mathfrak{h}_{k_*-1}(S)} \lesssim \epsilon r^{-2} u^{-\frac{\delta_{dec}}{2}}.$$

Using Sobolev, we deduce, for $k \leq k_* - 3$,

$$\left|\not{\mathfrak{d}}^k \not{d}_2^* \not{d}_1^* J^{(p)}\right| \lesssim \epsilon r^{-3} u^{-\frac{\delta_{dec}}{2}}.$$

Together with the above transport equation along ν and the control of Γ_b, we infer, for any $k \leq k_* - 3$,

$$\left|\mathfrak{d}_*^k \not{d}_2^* \not{d}_1^* J^{(p)}\right| \lesssim \epsilon r^{-3} u^{-\frac{\delta_{dec}}{2}},$$

as stated. This concludes the proof of Lemma 5.37. □

We state below two corollaries of Lemma 5.37.

Corollary 5.38. *On a fixed sphere $S = S(u) \subset \Sigma_*$, we have for any pair of scalars $(h, {}^*h)$ and all $k \leq k_{large} - 1$*

$$\begin{aligned}
\|(h, {}^*h)\|_{\mathfrak{h}_{k+2}(S)} &\lesssim r^2\|\not{d}_2^* \not{d}_1^*(h, {}^*h)\|_{\mathfrak{h}_k(S)} + r|(h)_{\ell=1}| + r|({}^*h)_{\ell=1}| \\
&\quad + r|\overline{h}| + r|\overline{{}^*h}|.
\end{aligned}$$

Proof. In view of Corollary 5.29, we have

$$\begin{aligned}
\|(h - \overline{h}, {}^*h - \overline{{}^*h})\|_{\mathfrak{h}_{k+2}(S)} &\lesssim r^2\|\not{d}_2^* \not{d}_1^*(h, {}^*h)\|_{\mathfrak{h}_k(S)} + r^3|(\Delta h)_{\ell=1}| \\
&\quad + r^3|(\Delta\, {}^*h)_{\ell=1}|.
\end{aligned}$$

Since

$$(\Delta h)_{\ell=1} = (\Delta(h-\overline{h}))_{\ell=1} = -\frac{2}{r^2}(h-\overline{h})_{\ell=1} + \left(\left(\Delta+\frac{2}{r^2}\right)(h-\overline{h})\right)_{\ell=1},$$

and similarly for ${}^*h$, we infer

$$\begin{aligned}
& \|(h-\overline{h},\ {}^*h-\overline{{}^*h})\|_{\mathfrak{h}_{k+2}(S)} \\
\lesssim\ & r^2\|\,\not{d}_2^*\,\not{d}_1^*(h,\ {}^*h)\|_{\mathfrak{h}_k(S)} + r\big|(h-\overline{h})_{\ell=1}\big| + r\big|({}^*h-\overline{{}^*h})_{\ell=1}\big| \\
& + r^3\left|\left(\left(\Delta+\frac{2}{r^2}\right)(h-\overline{h})\right)_{\ell=1}\right| + r^3\left|\left(\left(\Delta+\frac{2}{r^2}\right)({}^*h-\overline{{}^*h})\right)_{\ell=1}\right|.
\end{aligned}$$

Now, using integration by parts and Lemma 5.37, we have, for $p=0,+,-$,

$$\begin{aligned}
\left|\int_S\left(\Delta+\frac{2}{r^2}\right)(h-\overline{h})J^{(p)}\right| &= \left|\int_S(h-\overline{h})\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right| \\
&\lesssim \|h-\overline{h}\|_{L^2(S)}r\left\|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right\|_{L^\infty(S)} \\
&\lesssim \frac{\epsilon}{r^2}\|h-\overline{h}\|_{L^2(S)}
\end{aligned}$$

and hence

$$\left|\left(\left(\Delta+\frac{2}{r^2}\right)(h-\overline{h})\right)_{\ell=1}\right| \lesssim \frac{\epsilon}{r^4}\|h-\overline{h}\|_{L^2(S)}.$$

We infer

$$\begin{aligned}
\|(h-\overline{h},\ {}^*h-\overline{{}^*h})\|_{\mathfrak{h}_{k+2}(S)} \lesssim\ & r^2\|\,\not{d}_2^*\,\not{d}_1^*(h,\ {}^*h)\|_{\mathfrak{h}_k(S)} + r\big|(h-\overline{h})_{\ell=1}\big| \\
& + r\big|({}^*h-\overline{{}^*h})_{\ell=1}\big| + \frac{\epsilon}{r}\|h-\overline{h}\|_{L^2(S)} \\
& + \frac{\epsilon}{r}\|\,{}^*h-\overline{{}^*h}\|_{L^2(S)}.
\end{aligned}$$

For $\epsilon>0$ small enough, we may absorb the last terms on the RHS and deduce

$$\begin{aligned}
\|(h-\overline{h},\ {}^*h-\overline{{}^*h})\|_{\mathfrak{h}_{k+2}(S)} \lesssim\ & r^2\|\,\not{d}_2^*\,\not{d}_1^*(h,\ {}^*h)\|_{\mathfrak{h}_k(S)} + r\big|(h-\overline{h})_{\ell=1}\big| \\
& + r\big|({}^*h-\overline{{}^*h})_{\ell=1}\big|.
\end{aligned}$$

In particular, we infer

$$\begin{aligned}
\|(h,\ {}^*h)\|_{\mathfrak{h}_{k+2}(S)} \lesssim\ & r^2\|\,\not{d}_2^*\,\not{d}_1^*(h,\ {}^*h)\|_{\mathfrak{h}_k(S)} + r\big|(h)_{\ell=1}\big| + r\big|({}^*h)_{\ell=1}\big| \\
& + r|\overline{h}| + r|\overline{{}^*h}|
\end{aligned}$$

as desired. □

Corollary 5.39. *The scalar functions* $\underline{C}_0$, $\underline{C}_p$, M_0, M_p *on* Σ_* *verify*

$$\underline{C}_0 \in \Gamma_g, \qquad \underline{C}_p \in \Gamma_g, \qquad M_0 \in r^{-1}\Gamma_g, \qquad M_p \in r^{-1}\Gamma_g. \tag{5.82}$$

Proof. Recall that we have on Σ_* in view of our GCM conditions

$$\check{\underline{\kappa}} = \underline{C}_0 + \sum_p \underline{C}_p J^{(p)}$$

and hence

$$\underline{C}_0 + \sum_p \underline{C}_p J^{(p)} = \Gamma_g.$$

Integrating this identity on S, as well as multiplying it by $J^{(q)}$ and integrating is also on S, we obtain, after dividing by $|S|$,

$$\begin{aligned}
\underline{C}_0 &= \Gamma_g + O(r^{-2})\sum_p \left(\int_S J^{(p)}\right)\underline{C}_p,\\
\underline{C}_q &= \Gamma_g + O(r^{-2})\left(\int_S J^{(q)}\right)\underline{C}_0\\
&\quad +O(r^{-2})\sum_p\left(\int_S J^{(p)}J^{(q)} - \frac{4\pi}{3}r^2\delta_{pq}\right)\underline{C}_p, \; q=0,+,-.
\end{aligned}$$

Together with Lemma 5.37, we deduce

$$\begin{aligned}
\underline{C}_0 &= \Gamma_g + O(\epsilon)\sum_p \underline{C}_p,\\
\underline{C}_q &= \Gamma_g + O(\epsilon)\underline{C}_0 + O(\epsilon)\sum_p \underline{C}_p, \quad q=0,+,-,
\end{aligned}$$

which yields the desired result for $\underline{C}_0$ and $\underline{C}_p$.

Next, recall that we have on Σ_* in view of our GCM conditions

$$\check{\mu} = M_0 + \sum_p M_p J^{(p)}$$

and hence

$$M_0 + \sum_p M_p J^{(p)} = r^{-1}\Gamma_g.$$

The proof for M_0 and M_p follows then the same line as the one for $\underline{C}_0$ and $\underline{C}_p$. This concludes the proof of the corollary. □

Finally, we state below a corollary of Corollary 5.39 and Corollary 5.24.

Corollary 5.40. *The following identities hold true on Σ_*.*

$$
\begin{aligned}
2\,{\not{d}}_2^*{\not{d}}_1^*{\not{d}}_1\,{\not{d}}_2\,{\not{d}}_2^*\eta &= -\frac{4}{r}\,{\not{d}}_2^*{\not{d}}_1^* div\,\underline{\beta} + r^{-5}\,\mathfrak{d}^{\leq 5}\Gamma_g + r^{-4}\,\mathfrak{d}^{\leq 4}(\Gamma_b\cdot\Gamma_b)\\
&\quad + \sum_{p=0,+,-} r^{-1}\nabla_\nu\Big(r^{-1}\Gamma_g\,{\not{d}}_2^*{\not{d}}_1^*(J^{(p)})\Big),\\
2\,{\not{d}}_2^*{\not{d}}_1^*{\not{d}}_1\,{\not{d}}_2\,{\not{d}}_2^*\underline{\xi} &= -\frac{4}{r}\,{\not{d}}_2^*{\not{d}}_1^* div\,\underline{\beta} + r^{-5}\,\mathfrak{d}^{\leq 5}\Gamma_g + r^{-4}\,\mathfrak{d}^{\leq 4}(\Gamma_b\cdot\Gamma_b)\\
&\quad + \sum_{p=0,+,-}\nabla_\nu\Big(r^{-2}\Gamma_g\,\mathfrak{d}^{\leq 2}\,{\not{d}}_2^*{\not{d}}_1^*(J^{(p)})\Big)\\
&\quad + \sum_{p=0,+,-} r^{-1}\nabla_\nu\Big(r^{-1}\Gamma_g\,{\not{d}}_2^*{\not{d}}_1^*(J^{(p)})\Big).
\end{aligned}
\tag{5.83}
$$

Proof. In view of the definition of the linearized quantities $\check{\kappa}$, $\check{\underline{\kappa}}$ and $\check{\mu}$, we may rewrite the GCM conditions (5.26) as follows

$$
\check{\kappa}=0,\qquad \check{\underline{\kappa}} = \underline{C}_0 + \sum_{p=0,+,-}\underline{C}_p J^{(p)},\qquad \check{\mu} = M_0 + \sum_{p=0,+,-} M_p J^{(p)}.
$$

Since the scalar functions $\underline{C}_0$, $\underline{C}_p$, M_0, and M_p are constant on the leaves of the r-foliation of Σ_*, and since ν is tangent to Σ_*, we infer

$$
\begin{aligned}
{\not{d}}_2^*{\not{d}}_1^*{\not{d}}_1\nabla_\nu\nabla\check{\kappa} &= 0,\\
\nabla_\nu\left({\not{d}}_2^*{\not{d}}_2+\frac{2}{r^2}\right){\not{d}}_2^*{\not{d}}_1^*\check{\underline{\kappa}} &= \nabla_\nu\left({\not{d}}_2^*{\not{d}}_2+\frac{2}{r^2}\right){\not{d}}_2^*{\not{d}}_1^*\left(\underline{C}_0+\sum_{p=0,+,-}\underline{C}_p J^{(p)}\right)\\
&= \sum_{p=0,+,-}\nabla_\nu\left(\underline{C}_p\left({\not{d}}_2^*{\not{d}}_2+\frac{2}{r^2}\right){\not{d}}_2^*{\not{d}}_1^*(J^{(p)})\right),\\
\nabla_\nu\,{\not{d}}_2^*{\not{d}}_1^*\check{\mu} &= \nabla_\nu\,{\not{d}}_2^*{\not{d}}_1^*\left(M_0+\sum_{p=0,+,-}M_p J^{(p)}\right)\\
&= \sum_{p=0,+,-}\nabla_\nu\Big(M_p\,{\not{d}}_2^*{\not{d}}_1^*(J^{(p)})\Big).
\end{aligned}
$$

Since we have, in view of Corollary 5.39, $\underline{C}_p\in\Gamma_g$ and $M_p\in r^{-1}\Gamma_g$, we infer

$$
\begin{aligned}
{\not{d}}_2^*{\not{d}}_1^*{\not{d}}_1\nabla_\nu\nabla\check{\kappa} &= 0,\\
\nabla_\nu\left({\not{d}}_2^*{\not{d}}_2+\frac{2}{r^2}\right){\not{d}}_2^*{\not{d}}_1^*\check{\underline{\kappa}} &= \sum_{p=0,+,-}\nabla_\nu\left(r^{-2}\Gamma_g\,\mathfrak{d}^{\leq 2}\,{\not{d}}_2^*{\not{d}}_1^*(J^{(p)})\right),
\end{aligned}
$$

$$\nabla_\nu \,{\mathfrak{d}\mkern-9mu /}_2^*{\mathfrak{d}\mkern-9mu /}_1^*\check{\mu} \;=\; \sum_{p=0,+,-}\nabla_\nu\Big(r^{-1}\Gamma_g\,{\mathfrak{d}\mkern-9mu /}_2^*{\mathfrak{d}\mkern-9mu /}_1^*(J^{(p)})\Big).$$

Plugging these identities in (5.54) and (5.55), we obtain

$$\begin{aligned}
2\,{\mathfrak{d}\mkern-9mu /}_2^*{\mathfrak{d}\mkern-9mu /}_1^*{\mathfrak{d}\mkern-9mu /}_1\,{\mathfrak{d}\mkern-9mu /}_2\,{\mathfrak{d}\mkern-9mu /}_2^*\eta &= -\frac{4}{r}\,{\mathfrak{d}\mkern-9mu /}_2^*{\mathfrak{d}\mkern-9mu /}_1^*\mathrm{div}\,\underline{\beta}+r^{-5}\,\mathfrak{d}^{\leq 5}\Gamma_g+r^{-4}\,\mathfrak{d}^{\leq 4}(\Gamma_b\cdot\Gamma_b)\\
&\quad+\sum_{p=0,+,-}r^{-1}\nabla_\nu\Big(r^{-1}\Gamma_g\,{\mathfrak{d}\mkern-9mu /}_2^*{\mathfrak{d}\mkern-9mu /}_1^*(J^{(p)})\Big),\\
2\,{\mathfrak{d}\mkern-9mu /}_2^*{\mathfrak{d}\mkern-9mu /}_1^*{\mathfrak{d}\mkern-9mu /}_1\,{\mathfrak{d}\mkern-9mu /}_2\,{\mathfrak{d}\mkern-9mu /}_2^*\underline{\xi} &= -\frac{4}{r}\,{\mathfrak{d}\mkern-9mu /}_2^*{\mathfrak{d}\mkern-9mu /}_1^*\mathrm{div}\,\underline{\beta}+r^{-5}\,\mathfrak{d}^{\leq 5}\Gamma_g+r^{-4}\,\mathfrak{d}^{\leq 4}(\Gamma_b\cdot\Gamma_b)\\
&\quad+\sum_{p=0,+,-}\nabla_\nu\left(r^{-2}\Gamma_g\,\mathfrak{d}^{\leq 2}\,{\mathfrak{d}\mkern-9mu /}_2^*{\mathfrak{d}\mkern-9mu /}_1^*(J^{(p)})\right)\\
&\quad+\sum_{p=0,+,-}r^{-1}\nabla_\nu\Big(r^{-1}\Gamma_g\,{\mathfrak{d}\mkern-9mu /}_2^*{\mathfrak{d}\mkern-9mu /}_1^*(J^{(p)})\Big),
\end{aligned}$$

as stated. □

5.2.5. Propagation equations along Σ_* for some $\ell = 1$ modes We have the following corollary of Lemma 5.25.

Corollary 5.41. *We have along Σ_*, for $p = 0,+,-$,*

$$\begin{aligned}
&\nu\left(\int_S\left(\Delta\check{\underline{\kappa}}+\frac{2\Upsilon}{r}div\,\zeta\right)J^{(p)}\right)\\
&=O(r^{-3})\int_S\check{\underline{\kappa}}J^{(p)}+O(r^{-2})\int_S div\,\zeta J^{(p)}+O(r^{-1})\int_S div\,\underline{\beta}J^{(p)}\\
&\quad+O(r^{-2})\int_S\check{\rho}J^{(p)}+O(r^{-1})\int_S div\,\beta J^{(p)}+r\left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right|\mathfrak{d}^{\leq 1}\Gamma_b\\
&\quad+\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_b),
\end{aligned}\tag{5.84}$$

$$\begin{aligned}
\nu\left(\int_S div\,\beta J^{(p)}\right) &= O(r^{-1})\int_S div\,\beta J^{(p)}+O(r^{-2})\int_S\check{\rho}J^{(p)}\\
&\quad+r\left(\left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right|+\left|{\mathfrak{d}\mkern-9mu /}_2^*{\mathfrak{d}\mkern-9mu /}_1^*J^{(p)}\right|\right)\Gamma_g+\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g),
\end{aligned}\tag{5.85}$$

$$\begin{aligned}
\nu\left(\int_S curl\,\beta J^{(p)}\right) &= \frac{4}{r}(1+O(r^{-1}))\int_S curl\,\beta J^{(p)}\\
&\quad+\frac{2}{r^2}(1+O(r^{-1}))\int_S {}^*\rho J^{(p)}
\end{aligned}$$

$$
+ r\left(\left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right| + \left|{\not{d}}_2^*{\not{d}}_1^* J^{(p)}\right|\right)\Gamma_g
$$
$$
+ \mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g), \tag{5.86}
$$

and

$$
\begin{aligned}
\nu\left(\int_S\left(\check{\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)J^{(p)}\right) = & -\int_S div\,\underline{\beta}J^{(p)} - (1+O(r^{-1}))\int_S div\,\beta J^{(p)}\\
& + O(r^{-1})\int_S\left(\check{\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)J^{(p)}\\
& + O(r^{-3})\int_S\underline{\check{\kappa}}J^{(p)} + O(r^{-2})\int_S div\,\zeta J^{(p)}\\
& + r\left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right|\Gamma_b + r\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b),
\end{aligned} \tag{5.87}
$$

where by ${\not{d}}_1^* J^{(p)}$*, we mean* ${\not{d}}_1^*(J^{(p)},0)$ *or* ${\not{d}}_1^*(0,J^{(p)})$*.*

Proof. See Section B.4. □

5.3. Control of the flux of some quantities on Σ_*

The goal of this section is to establish the following.

Proposition 5.42. *The following estimate holds true for all* $k\leq k_*-7$

$$
\int_{\Sigma_*} u^{2+2\delta_{dec}}\left|\mathfrak{d}_*^k\Gamma_b\right|^2 \lesssim \epsilon_0^2. \tag{5.88}
$$

We also have, for $k\leq k_*-10$*,*

$$
\|\Gamma_b\|_{\infty,k} \lesssim \epsilon_0 r^{-1}u^{-1-\delta_{dec}}. \tag{5.89}
$$

Proof. Note that (5.89) follows immediately from (5.88) using the trace theorem and Sobolev. We thus concentrate our attention on deriving (5.88).

Step 1. We first prove the corresponding estimates for $\underline{\beta}$ away from its $\ell=1$ mode. More precisely we prove the following.

Lemma 5.43. *The following estimates hold true for all* $k\leq k_*-3$

$$
\int_{\Sigma_*} r^2u^{2+2\delta_{dec}}\left|\mathfrak{d}_*^k\underline{\beta}\right|^2 \lesssim \int_{\Sigma_*} r^4u^{2+2\delta_{dec}}\left|\left({\not{d}}_1\nabla_\nu^{\leq k}\underline{\beta}\right)_{\ell=1}\right|^2 + \epsilon_0^2. \tag{5.90}
$$

Proof. Recall the identity (5.58)

$$\begin{aligned}\Re(\nabla_3(r\mathfrak{q})) &= r^5 \,{\not d}_2^* {\not d}_1^*(\operatorname{div}\underline{\beta}, -\operatorname{curl}\underline{\beta}) + O(r)\,\not{\mathfrak{d}}^{\leq 3}\underline{\alpha} + O(r^{-2}) + r\,\not{\mathfrak{d}}^{\leq 3}\Gamma_g \\ &\quad + r^3\,\not{\mathfrak{d}}^{\leq 3}(\Gamma_b\cdot\Gamma_g).\end{aligned}$$

Together with the control of Γ_b and Γ_g provided by **Ref 1** and Lemma 5.15, we infer, for $k \leq k_* - 3$,

$$\begin{aligned}r^5|\mathfrak{d}_*^k \,{\not d}_2^* {\not d}_1^*(\operatorname{div}\underline{\beta}, -\operatorname{curl}\underline{\beta})| &\lesssim |\mathfrak{d}_*^{\leq k_*-3}\nabla_3(r\mathfrak{q})| + r|\mathfrak{d}_*^{\leq k_*}\underline{\alpha}| + \frac{1}{r^2} + \frac{\epsilon}{ru^{\frac{1}{2}+\frac{\delta_{dec}}{2}}} \\ &\quad + \frac{\epsilon^2}{u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}} \\ &\lesssim |\mathfrak{d}_*^{\leq k_*-3}\nabla_3(r\mathfrak{q})| + r|\mathfrak{d}_*^{\leq k_*}\underline{\alpha}| + \frac{\epsilon_0}{u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}}\end{aligned}$$

where we used in the last inequality the dominance condition (5.30) on r on Σ_*. Also, using the fact that $e_3(r) = -\Upsilon + O(r)$ and **Ref 2** for $\mathfrak{q}$, we have

$$\begin{aligned}|\mathfrak{d}_*^{\leq k_*-3}\nabla_3(r\mathfrak{q})| &\lesssim r|\mathfrak{d}_*^{\leq k_*-3}\nabla_3\mathfrak{q}| + |\mathfrak{d}_*^{\leq k_*-3}\mathfrak{q}| \lesssim r|\mathfrak{d}_*^{\leq k_*-3}\nabla_3\mathfrak{q}| + \frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}}} \\ &\lesssim r|\mathfrak{d}_*^{\leq k_*-3}\nabla_3\mathfrak{q}| + \frac{\epsilon_0}{u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}}\end{aligned}$$

where we used again the dominance condition (5.30) on r on Σ_*. We infer

$$r^5|\mathfrak{d}_*^k \,{\not d}_2^* {\not d}_1^*(\operatorname{div}\underline{\beta}, -\operatorname{curl}\underline{\beta})| \lesssim r|\mathfrak{d}_*^{\leq k_*-3}\nabla_3\mathfrak{q}| + r|\mathfrak{d}_*^{\leq k_*}\underline{\alpha}| + \frac{\epsilon_0}{u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}}.$$

Dividing by r, squaring, and integrating on Σ_*, we deduce

$$\begin{aligned}&\int_{\Sigma_*} r^8 u^{2+2\delta_{dec}}|\mathfrak{d}_*^k \,{\not d}_2^* {\not d}_1^*(\operatorname{div}\underline{\beta}, -\operatorname{curl}\underline{\beta})|^2 \\ &\lesssim \int_{\Sigma_*} u^{2+2\delta_{dec}}|\mathfrak{d}_*^{\leq k_*-3}\nabla_3\mathfrak{q}|^2 + \int_{\Sigma_*} u^{2+2\delta_{dec}}|\mathfrak{d}_*^{\leq k_*}\underline{\alpha}|^2 + \int_{\Sigma_*}\frac{\epsilon_0^2}{r^2u^{1+\delta_{dec}}}\end{aligned}$$

and hence, since $\delta_{dec} > 0$ and using the control of $\underline{\alpha}$ and $\nabla_3\mathfrak{q}$ provided by **Ref 2**, we deduce, for $k \leq k_* - 3$,

$$\int_{\Sigma_*} r^8 u^{2+2\delta_{dec}}|\mathfrak{d}_*^k \,{\not d}_2^* {\not d}_1^*(\operatorname{div}\underline{\beta}, -\operatorname{curl}\underline{\beta})|^2 \lesssim \epsilon_0^2.$$

Taking into account the commutator Lemma 5.20, this yields, for $k\leq k_*-3$,

$$\int_{\Sigma_*} r^8u^{2+2\delta_{dec}}|\,{\not d}_2^*{\not d}_1^*(\mathrm{div}\,,-\mathrm{curl}\,)\,\mathfrak{d}_*^k\underline{\beta}|^2 \quad\lesssim\quad \epsilon_0^2.$$

Using the Hodge estimates of Lemma 5.27 for ${\not d}_1^*$ and ${\not d}_1$, and the one of Lemma 5.28 for ${\not d}_2^*$, we infer, for $k\leq k_*-3$,

$$\int_{\Sigma_*} r^2u^{2+2\delta_{dec}}\big|\mathfrak{d}_*^k\underline{\beta}\big|^2 \lesssim \int_{\Sigma_*} r^4u^{2+2\delta_{dec}}\Big|\big({\not d}_1\nabla_\nu^{\leq k}\underline{\beta}\big)_{\ell=1}\Big|^2+\epsilon_0^2$$

as stated. This concludes the proof of Lemma 5.43. □

Step 2. We next derive the following non-sharp, preliminary, estimate for the $\ell=1$ mode of ${\not d}_1\underline{\beta}$

$$\Big|\big({\not d}_1\nabla_\nu^k\underline{\beta}\big)_{\ell=1}\Big| \quad\lesssim\quad \frac{\epsilon_0}{r^3u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}}, \qquad k\leq k_*-3. \tag{5.91}$$

To this end, we use the following consequence of the Codazzi equation for $\widehat{\underline{\chi}}$

$$ {\not d}_2\widehat{\underline{\chi}} \quad=\quad \underline{\beta}+r^{-1}\,{\not\mathfrak{d}}^{\leq 1}\Gamma_g+\Gamma_b\cdot\Gamma_g.$$

Differentiating w.r.t. $\nu^k\,{\not d}_1$ for $k\leq k_*-3$, we infer

$$\nu^k\,{\not d}_1\,{\not d}_2\widehat{\underline{\chi}} \quad=\quad \nu^k\,{\not d}_1\underline{\beta}+r^{-2}\mathfrak{d}_*^{\leq k_*-1}\Gamma_g+r^{-1}\mathfrak{d}_*^{\leq k_*-2}(\Gamma_b\cdot\Gamma_g).$$

Taking into account the commutator Lemma 5.20, this yields, for $k\leq k_*-3$,

$${\not d}_1\,{\not d}_2\nabla_\nu^k\widehat{\underline{\chi}} \quad=\quad {\not d}_1\nabla_\nu^k\underline{\beta}+r^{-2}\mathfrak{d}_*^{\leq k_*-1}\Gamma_g+r^{-1}\mathfrak{d}_*^{\leq k_*-2}(\Gamma_b\cdot\Gamma_g).$$

Projecting on the $\ell=1$ modes, this yields for $k\leq k_*-3$

$$({\not d}_1\,{\not d}_2\nabla_\nu^k\widehat{\underline{\chi}})_{\ell=1} \quad=\quad ({\not d}_1\nabla_\nu^k\underline{\beta})_{\ell=1}+r^{-2}\mathfrak{d}_*^{\leq k_*-1}\Gamma_g+r^{-1}\mathfrak{d}_*^{\leq k_*-2}(\Gamma_b\cdot\Gamma_g).$$

Together with the control of Γ_b and Γ_g provided by **Ref 1** and Lemma 5.15, we infer, for $k\leq k_*-3$,

$$|({\not d}_1\nabla_\nu^k\underline{\beta})_{\ell=1}| \quad\lesssim\quad |({\not d}_1\,{\not d}_2\nabla_\nu^k\widehat{\underline{\chi}})_{\ell=1}|+\frac{\epsilon}{r^4u^{\frac{1}{2}+\frac{\delta_{dec}}{2}}}$$

which together with the dominance condition (5.30) on r on Σ_* implies, for $k\leq k_*-3$,

$$|({\not d}_1\nabla_\nu^k\underline{\beta})_{\ell=1}| \quad\lesssim\quad |({\not d}_1\,{\not d}_2\nabla_\nu^k\widehat{\underline{\chi}})_{\ell=1}|+\frac{\epsilon_0}{r^3u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}}.$$

Next, we estimate $(\not{d}_1\not{d}_2\nabla_\nu^k\underline{\widehat{\chi}})_{\ell=1}$. We have

$$(\not{d}_1\not{d}_2\nabla_\nu^k\underline{\widehat{\chi}})_{\ell=1,p} = \frac{1}{|S|}\int_S \not{d}_1\not{d}_2\nabla_\nu^k\underline{\widehat{\chi}}J^{(p)} = \frac{1}{|S|}\int_S \nabla_\nu^k\underline{\widehat{\chi}}\cdot \not{d}_2^*\not{d}_1^*J^{(p)}$$

and hence

$$|(\not{d}_1\not{d}_2\nabla_\nu^k\underline{\widehat{\chi}})_{\ell=1}| \lesssim |\not{d}_2^*\not{d}_1^*J^{(p)}||\mathfrak{d}_*^k\Gamma_b|.$$

Together with the control of Γ_b provided by Lemma 5.15, and the control of $\not{d}_2^*\not{d}_1^*J^{(p)}$ provided by Lemma 5.37 we infer, for $k\le k_*-3$,

$$|(\not{d}_1\not{d}_2\nabla_\nu^k\underline{\widehat{\chi}})_{\ell=1}| \lesssim \frac{\epsilon}{r^3u^{\frac{\delta_{dec}}{2}}}\frac{\epsilon}{ru^{1+\frac{\delta_{dec}}{2}}}\lesssim\frac{\epsilon_0}{r^4u^{1+\delta_{dec}}}\lesssim\frac{\epsilon_0}{r^3u^{2+2\delta_{dec}}}.$$

Plugging in the above, we deduce, for $k\le k_*-3$,

$$|(\not{d}_1\nabla_\nu^k\underline{\beta})_{\ell=1}| \lesssim |(\not{d}_1\not{d}_2\nabla_\nu^k\underline{\widehat{\chi}})_{\ell=1}| + \frac{\epsilon_0}{r^3u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}}\lesssim\frac{\epsilon_0}{r^3u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}}$$

which concludes the proof of (5.91).

Step 3. Using the estimate (5.90) of Step 1 and the estimate (5.91) of Step 2, we obtain, for $k\le k_*-3$,

$$\begin{aligned}\int_{\Sigma_*} r^2u^{2+2\delta_{dec}}\left|\mathfrak{d}_*^k\underline{\beta}\right|^2 &\lesssim \int_{\Sigma_*} r^4u^{2+2\delta_{dec}}\left|\left(\not{d}_1\nabla_\nu^{\le k}\underline{\beta}\right)_{\ell=1}\right|^2+\epsilon_0^2\\ &\lesssim \epsilon_0^2+\epsilon_0^2\int_1^{u_*}u^{-1-\delta_{dec}}\end{aligned}$$

and hence

$$\int_{\Sigma_*} r^2u^{2+2\delta_{dec}}|\mathfrak{d}_*^k\underline{\beta}|^2 \lesssim \epsilon_0^2,\qquad k\le k_*-3, \tag{5.92}$$

which is the desired estimate for $\underline{\beta}$.

Step 4. We now prove the desired estimate for $\underline{\widehat{\chi}}$, i.e.

$$\int_{\Sigma_*} u^{2+2\delta_{dec}}|\mathfrak{d}_*^k\underline{\widehat{\chi}}|^2\lesssim\epsilon_0^2,\qquad k\le k_*-3. \tag{5.93}$$

Proof of (5.93). One starts with the the following consequence of the Codazzi equation for $\underline{\widehat{\chi}}$

$$\not{d}_2\underline{\widehat{\chi}} = \underline{\beta}+r^{-1}\not{\mathfrak{d}}^{\le 1}\Gamma_g+\Gamma_b\cdot\Gamma_g.$$

Differentiating w.r.t. $\not{d}_2^*$, we infer

$$\not{d}_2^*\not{d}_2\widehat{\underline{\chi}} \quad = \quad \not{d}_2^*\underline{\beta} + r^{-2}\,\not{\mathfrak{d}}^{\leq 2}\Gamma_g + r^{-1}\,\not{\mathfrak{d}}^{\leq 1}\left(\Gamma_b\cdot\Gamma_g\right).$$

Taking into account the commutator Lemma 5.20, this yields, for $k\leq k_*-3$,

$$\not{d}_2^*\not{d}_2\mathfrak{d}_*^k\widehat{\underline{\chi}} \quad = \quad \not{d}_2^*(\mathfrak{d}_*^k\underline{\beta}) + r^{-2}\,\not{\mathfrak{d}}^{\leq k_*-1}\Gamma_g + r^{-1}\,\not{\mathfrak{d}}^{\leq k_*-2}\left(\Gamma_b\cdot\Gamma_g\right).$$

Together with the control of Γ_b and Γ_g provided by **Ref 1** and Lemma 5.15, we infer, for $k\leq k_*-3$,

$$\begin{aligned}
|\not{d}_2^*\not{d}_2\mathfrak{d}_*^k\widehat{\underline{\chi}}| &\lesssim |\not{d}_2^*(\mathfrak{d}_*^k\underline{\beta})| + \frac{\epsilon}{r^4u^{\frac{1}{2}+\frac{\delta_{dec}}{2}}}\\
&\lesssim |\not{d}_2^*(\mathfrak{d}_*^k\underline{\beta})| + \frac{\epsilon_0}{r^3u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}},
\end{aligned}$$

where we used again the dominance condition (5.30) on r on Σ_*. Squaring and multiplying by $r^4u^{2+2\delta_{dec}}$, we infer

$$r^4u^{2+2\delta_{dec}}|\not{d}_2^*\not{d}_2\mathfrak{d}_*^k\widehat{\underline{\chi}}|^2 \quad\lesssim\quad r^4u^{2+2\delta_{dec}}|\not{d}_2^*(\mathfrak{d}_*^k\underline{\beta})|^2 + \frac{\epsilon_0^2}{r^2u^{1+\delta_{dec}}},$$

which yields after integration on Σ_* implies, for $k\leq k_*-3$,

$$\begin{aligned}
\int_{\Sigma_*} r^4u^{2+2\delta_{dec}}|\not{d}_2^*\not{d}_2\mathfrak{d}_*^k\widehat{\underline{\chi}}|^2 &\lesssim \int_{\Sigma_*} r^4u^{2+2\delta_{dec}}|\not{d}_2^*(\mathfrak{d}_*^k\underline{\beta})|^2 + \int_{\Sigma_*}\frac{\epsilon_0^2}{r^2u^{1+\delta_{dec}}}\\
&\lesssim \epsilon_0^2 + \int_{\Sigma_*} r^4u^{2+2\delta_{dec}}|\not{d}_2^*(\mathfrak{d}_*^k\underline{\beta})|^2.
\end{aligned}$$

Hence, together with (5.90), we infer

$$\int_{\Sigma_*} r^4u^{2+2\delta_{dec}}|\not{d}_2^*\not{d}_2\mathfrak{d}_*^k\widehat{\underline{\chi}}|^2 \quad\lesssim\quad \epsilon_0^2.$$

In view of the coercivity of $\not{d}_2^*\not{d}_2$, see (5.49), we deduce

$$\int_{\Sigma_*} r^4u^{2+2\delta_{dec}}|\mathfrak{d}_*^k\widehat{\underline{\chi}}|^2 \quad\lesssim\quad \epsilon_0^2, \qquad k\leq k_*-3, \tag{5.94}$$

which is the stated estimate (5.93). □

Step 5. Next, we establish the estimates for η and $\underline{\xi}$ in Proposition 5.42. To this end, we first estimate $\not{d}_2^*\eta$ and $\not{d}_2^*\underline{\xi}$.

Lemma 5.44. *We have for $k \le k_* - 6$*

$$\int_{\Sigma_*} r^2 u^{2+2\delta_{dec}} \Big(| \, {\not{d}}_2^* \mathfrak{d}_*^k \eta|^2 + | \, {\not{d}}_2^* \mathfrak{d}_*^k \underline{\xi}|^2 \Big) \;\; \lesssim \;\; \epsilon_0^2. \tag{5.95}$$

Proof. Recall from Corollary 5.40 that the following identities hold true on Σ_*

$$\begin{aligned}
2 \, {\not{d}}_2^* {\not{d}}_1^* {\not{d}}_1 \, {\not{d}}_2 \, {\not{d}}_2^* \eta &= -\frac{4}{r} \, {\not{d}}_2^* {\not{d}}_1^* \mathrm{div} \, \underline{\beta} + r^{-5} \, \not{\mathfrak{d}}^{\le 5} \Gamma_g + r^{-4} \, \not{\mathfrak{d}}^{\le 4} (\Gamma_b \cdot \Gamma_b) \\
&\quad + \sum_{p=0,+,-} r^{-1} \nabla_\nu \Big(r^{-1} \Gamma_g \, {\not{d}}_2^* {\not{d}}_1^* (J^{(p)}) \Big), \\
2 \, {\not{d}}_2^* {\not{d}}_1^* {\not{d}}_1 \, {\not{d}}_2 \, {\not{d}}_2^* \underline{\xi} &= -\frac{4}{r} \, {\not{d}}_2^* {\not{d}}_1^* \mathrm{div} \, \underline{\beta} + r^{-5} \, \not{\mathfrak{d}}^{\le 5} \Gamma_g + r^{-4} \, \not{\mathfrak{d}}^{\le 4} (\Gamma_b \cdot \Gamma_b) \\
&\quad + \sum_{p=0,+,-} \nabla_\nu \Big(r^{-2} \Gamma_g \, \not{\mathfrak{d}}^{\le 2} \, {\not{d}}_2^* {\not{d}}_1^* (J^{(p)}) \Big) \\
&\quad + \sum_{p=0,+,-} r^{-1} \nabla_\nu \Big(r^{-1} \Gamma_g \, {\not{d}}_2^* {\not{d}}_1^* (J^{(p)}) \Big).
\end{aligned}$$

Taking into account the commutator Lemma 5.20, this yields, for $k \le k_* - 6$,

$$\begin{aligned}
2 \, {\not{d}}_2^* {\not{d}}_1^* {\not{d}}_1 \, {\not{d}}_2 \, {\not{d}}_2^* \mathfrak{d}_*^k \eta &= r^{-4} \mathfrak{d}_*^{\le k_* - 3} \underline{\beta} + r^{-5} \mathfrak{d}_*^{\le k_*} \Gamma_g + r^{-4} \mathfrak{d}_*^{\le k_*} (\Gamma_b \cdot \Gamma_b) \\
&\quad + \sum_{p=0,+,-} r^{-1} \mathfrak{d}_*^{\le k_* - 3} \Big(r^{-1} \Gamma_g \, {\not{d}}_2^* {\not{d}}_1^* (J^{(p)}) \Big), \\
2 \, {\not{d}}_2^* {\not{d}}_1^* {\not{d}}_1 \, {\not{d}}_2 \, {\not{d}}_2^* \mathfrak{d}_*^k \underline{\xi} &= r^{-4} \mathfrak{d}_*^{\le k_* - 3} \underline{\beta} + r^{-5} \mathfrak{d}_*^{\le k_*} \Gamma_g + r^{-4} \mathfrak{d}_*^{\le k_*} (\Gamma_b \cdot \Gamma_b) \\
&\quad + \sum_{p=0,+,-} r^{-1} \mathfrak{d}_*^{\le k_* - 3} \Big(r^{-1} \Gamma_g \, {\not{d}}_2^* {\not{d}}_1^* (J^{(p)}) \Big).
\end{aligned}$$

Together with the control of Γ_b and Γ_g provided by **Ref 1** and Lemma 5.15, and the control of ${\not{d}}_2^* {\not{d}}_1^* (J^{(p)})$ provided by Lemma 5.37, we infer, for $k \le k_* - 6$,

$$\begin{aligned}
| \, {\not{d}}_2^* {\not{d}}_1^* {\not{d}}_1 \, {\not{d}}_2 \, {\not{d}}_2^* \mathfrak{d}_*^k \eta| + | \, {\not{d}}_2^* {\not{d}}_1^* {\not{d}}_1 \, {\not{d}}_2 \, {\not{d}}_2^* \mathfrak{d}_*^k \underline{\xi}| \;\; &\lesssim \;\; r^{-4} |\mathfrak{d}_*^{\le k_* - 3} \underline{\beta}| + \frac{\epsilon}{r^7 u^{\frac{1}{2} + \frac{\delta_{dec}}{2}}} \\
&\quad + \frac{\epsilon^2}{r^6 u^{2 + \frac{3\delta_{dec}}{2}}} \\
&\lesssim \;\; r^{-4} |\mathfrak{d}_*^{\le k_* - 3} \underline{\beta}| + \frac{\epsilon_0}{r^6 u^{\frac{3}{2} + \frac{3\delta_{dec}}{2}}}
\end{aligned}$$

where we used again the dominance condition (5.30) on r on Σ_*. Squaring and multiplying by $r^{10} u^{2+2\delta_{dec}}$, we infer

$$
\begin{aligned}
& r^{10}u^{2+2\delta_{dec}}\Big(|\,\not{d}_2^*\not{d}_1^*\not{d}_1\not{d}_2\not{d}_2^*\mathfrak{d}_*^k\eta|^2+|\,\not{d}_2^*\not{d}_1^*\not{d}_1\not{d}_2\not{d}_2^*\mathfrak{d}_*^k\underline{\xi}|^2\Big)\\
\lesssim\ & r^2u^{2+2\delta_{dec}}|\mathfrak{d}_*^{\leq k_*-3}\underline{\beta}|^2+\frac{\epsilon_0^2}{r^2u^{1+\delta_{dec}}}
\end{aligned}
$$

which yields after integration on Σ_* implies, for $k\leq k_*-6$,

$$
\begin{aligned}
& \int_{\Sigma_*}r^{10}u^{2+2\delta_{dec}}\Big(|\,\not{d}_2^*\not{d}_1^*\not{d}_1\not{d}_2\not{d}_2^*\mathfrak{d}_*^k\eta|^2+|\,\not{d}_2^*\not{d}_1^*\not{d}_1\not{d}_2\not{d}_2^*\mathfrak{d}_*^k\underline{\xi}|^2\Big)\\
\lesssim\ & \int_{\Sigma_*}r^2u^{2+2\delta_{dec}}|\mathfrak{d}_*^{\leq k_*-3}\underline{\beta}|^2+\int_{\Sigma_*}\frac{\epsilon_0^2}{r^2u^{1+\delta_{dec}}}\\
\lesssim\ & \epsilon_0^2+\int_{\Sigma_*}r^2u^{2+2\delta_{dec}}|\mathfrak{d}_*^{\leq k_*-3}\underline{\beta}|^2.
\end{aligned}
$$

Hence, together with (5.92), we deduce, for $k\leq k_*-6$,

$$
\int_{\Sigma_*}r^{10}u^{2+2\delta_{dec}}\Big(|\,\not{d}_2^*\not{d}_1^*\not{d}_1\not{d}_2\not{d}_2^*\mathfrak{d}_*^k\eta|^2+|\,\not{d}_2^*\not{d}_1^*\not{d}_1\not{d}_2\not{d}_2^*\mathfrak{d}_*^k\underline{\xi}|^2\Big)\ \lesssim\ \epsilon_0^2.
$$

Since, $\not{d}_1^*\not{d}_1=\not{d}_2\not{d}_2^*+2K$, we have

$$
\begin{aligned}
\not{d}_2^*\not{d}_1^*\not{d}_1\not{d}_2 &= \not{d}_2^*(\not{d}_2\not{d}_2^*+2K)\not{d}_2=\not{d}_2^*\not{d}_2\left(\not{d}_2^*\not{d}_2+\frac{2}{r^2}\right)+2\not{d}_2^*\check{K}\not{d}_2\\
&= \not{d}_2^*\not{d}_2\left(\not{d}_2^*\not{d}_2+\frac{2}{r^2}\right)+2r^{-3}\not{\mathfrak{d}}\Gamma_g\not{\mathfrak{d}}
\end{aligned}
$$

and in view of the control of Γ_g and the coercivity of $\not{d}_2^*\not{d}_2$ (see (5.49)), we obtain, for $k\leq k_*-6$,

$$
\int_{\Sigma_*}r^2u^{2+2\delta_{dec}}\Big(|\,\not{d}_2^*\mathfrak{d}_*^k\eta|^2+|\,\not{d}_2^*\mathfrak{d}_*^k\underline{\xi}|^2\Big)\ \lesssim\ \epsilon_0^2,
$$

which is the stated estimate (5.95). This completes the proof of Lemma 5.44. □

Step 6. In this step, we derive the desired estimates for η and $\underline{\xi}$, i.e. we show

$$
\int_{\Sigma_*}u^{2+2\delta_{dec}}\Big(|\mathfrak{d}_*^k\eta|^2+|\mathfrak{d}_*^k\underline{\xi}|^2\Big)\ \lesssim\ \epsilon_0^2,\qquad k\leq k_*-6. \tag{5.96}
$$

To this end, we apply Lemma 5.28 which yields

$$
\begin{aligned}
\|\mathfrak{d}_*^k\eta\|_{L^2(S)}+\|\mathfrak{d}_*^k\underline{\xi}\|_{L^2(S)}\ \lesssim\ & r\|\,\not{d}_2^*\mathfrak{d}_*^k\eta\|_{L^2(S)}+r\|\,\not{d}_2^*\mathfrak{d}_*^k\underline{\xi}\|_{L^2(S)}\\
& +r^2\left|(\not{d}_1\nabla_\nu^k\eta)_{\ell=1}\right|+r^2\left|(\not{d}_1\nabla_\nu^k\underline{\xi})_{\ell=1}\right|.
\end{aligned}
$$

Squaring, multiplying by $u^{2+2\delta_{dec}}$ and integrating in u, we infer

$$\begin{aligned}&\int_{\Sigma_*} u^{2+2\delta_{dec}}\Big(|\mathfrak{d}_*^k\eta|^2+|\mathfrak{d}_*^k\underline{\xi}|^2\Big)\\ \lesssim\;&\int_{\Sigma_*} r^2u^{2+2\delta_{dec}}\Big(|\,\not{d}_2^*\mathfrak{d}_*^k\eta|^2+|\,\not{d}_2^*\mathfrak{d}_*^k\underline{\xi}|^2\Big)\\ &+\int_{\Sigma_*} r^2u^{2+2\delta_{dec}}\left(\Big|(\,\not{d}_1\nabla_\nu^k\eta)_{\ell=1}\Big|^2+\Big|(\,\not{d}_1\nabla_\nu^k\underline{\xi})_{\ell=1}\Big|^2\right)\end{aligned}$$

which together with the estimate (5.95) of Step 5 implies, for $k\le k_*-6$,

$$\begin{aligned}&\int_{\Sigma_*} u^{2+2\delta_{dec}}\Big(|\mathfrak{d}_*^k\eta|^2+|\mathfrak{d}_*^k\underline{\xi}|^2\Big)\\ \lesssim\;&\epsilon_0^2+\int_{\Sigma_*} r^2u^{2+2\delta_{dec}}\left(\Big|(\,\not{d}_1\nabla_\nu^k\eta)_{\ell=1}\Big|^2+\Big|(\,\not{d}_1\nabla_\nu^k\underline{\xi})_{\ell=1}\Big|^2\right).\end{aligned}$$

Now, assume that we have

$$\Big|(\,\not{d}_1\nabla_\nu^k\eta)_{\ell=1}\Big|+\Big|(\,\not{d}_1\nabla_\nu^k\underline{\xi})_{\ell=1}\Big| \;\lesssim\; \frac{\epsilon_0}{r^2u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}},\qquad k\le k_*. \tag{5.97}$$

Then, we infer, for $k\le k_*-6$,

$$\int_{\Sigma_*} u^{2+2\delta_{dec}}\Big(|\mathfrak{d}_*^k\eta|^2+|\mathfrak{d}_*^k\underline{\xi}|^2\Big)\;\lesssim\;\epsilon_0^2+\int_{\Sigma_*}\frac{\epsilon_0^2}{r^2u^{1+\delta_{dec}}}\lesssim\epsilon_0^2$$

which is the desired estimate (5.96).

In view of the above, to complete the proof of (5.96), it suffices to prove (5.97). In view of the commutator Lemma 5.20, we have

$$\begin{aligned}\not{d}_1\nabla_\nu^k\eta &= \nu^k\,\not{d}_1\eta+r^{-2}\,\not{\mathfrak{d}}_*^{\le k}\Gamma_b+\,\not{\mathfrak{d}}_*^{\le k}(\Gamma_b\cdot\Gamma_b),\\ \not{d}_1\nabla_\nu^k\underline{\xi} &= \nu^k\,\not{d}_1\underline{\xi}+r^{-2}\,\not{\mathfrak{d}}_*^{\le k}\Gamma_b+\,\not{\mathfrak{d}}_*^{\le k}(\Gamma_b\cdot\Gamma_b).\end{aligned}$$

Using $\not{d}_1=(\mathrm{div}\,,\mathrm{curl}\,)$ and following consequence of the null structure equations

$$\mathrm{curl}\,\eta=r^{-1}\Gamma_g+\Gamma_b\cdot\Gamma_g,\qquad \mathrm{curl}\,\underline{\xi}=\Gamma_b\cdot\Gamma_b,$$

we infer

$$\begin{aligned}\not{d}_1\nabla_\nu^k\eta &= \nu^k\mathrm{div}\,\eta+r^{-1}\,\not{\mathfrak{d}}_*^{\le k}\Gamma_g+\,\not{\mathfrak{d}}_*^{\le k}(\Gamma_b\cdot\Gamma_b),\\ \not{d}_1\nabla_\nu^k\underline{\xi} &= \nu^k\mathrm{div}\,\underline{\xi}+r^{-2}\,\not{\mathfrak{d}}_*^{\le k}\Gamma_b+\,\not{\mathfrak{d}}_*^{\le k}(\Gamma_b\cdot\Gamma_b),\end{aligned}$$

which together with the control of Γ_b provided by **Ref 1** and Lemma 5.15 implies, for $k\leq k_*$,

$$\begin{aligned}
&|(\not{d}_1\nabla_\nu^k\eta)_{\ell=1}|+|(\not{d}_1\nabla_\nu^k\underline{\xi})_{\ell=1}|\\
\lesssim\;& |(\nu^k\mathrm{div}\,\eta)_{\ell=1}|+|(\nu^k\mathrm{div}\,\underline{\xi})_{\ell=1}|+\frac{\epsilon}{r^3u^{\frac{1}{2}+\frac{\delta_{dec}}{2}}}+\frac{\epsilon^2}{r^2u^{2+\frac{3\delta_{dec}}{2}}}\\
\lesssim\;& |(\nu^k\mathrm{div}\,\eta)_{\ell=1}|+|(\nu^k\mathrm{div}\,\underline{\xi})_{\ell=1}|+\frac{\epsilon_0}{r^2u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}}
\end{aligned}$$

where we used the dominance condition (5.30) on r on Σ_*. Also, since $\nu(J^{(p)})=0$ and $\nu(r)=-2+r\Gamma_b$, we have in view of Corollary 5.32

$$\begin{aligned}
\nu^k\left(\frac{1}{|S|}\int_S \mathrm{div}\,\eta J^{(p)}\right)&=\frac{1}{|S|}\int_S\nu^k(\mathrm{div}\,\eta)J^{(p)}+r^{-2}\mathfrak{d}_*^{\leq k_*-1}\Gamma_b+\mathfrak{d}_*^{\leq k_*}(\Gamma_b\cdot\Gamma_b),\\
\nu^k\left(\frac{1}{|S|}\int_S \mathrm{div}\,\underline{\xi} J^{(p)}\right)&=\frac{1}{|S|}\int_S\nu^k(\mathrm{div}\,\underline{\xi})J^{(p)}+r^{-2}\mathfrak{d}_*^{\leq k_*-1}\Gamma_b+\mathfrak{d}_*^{\leq k_*}(\Gamma_b\cdot\Gamma_b).
\end{aligned}$$

Now, recall the GCM conditions

$$(\mathrm{div}\,\eta)_{\ell=1}=(\mathrm{div}\,\underline{\xi})_{\ell=1}=0.$$

Since ν is tangent to Σ_*, we infer

$$\nu^k\left(\frac{1}{|S|}\int_S \mathrm{div}\,\eta J^{(p)}\right)=0,\qquad \nu^k\left(\frac{1}{|S|}\int_S \mathrm{div}\,\underline{\xi} J^{(p)}\right)=0,\qquad p=0,+,-,$$

and plugging in the above

$$\begin{aligned}
(\nu^k\mathrm{div}\,\eta)_{\ell=1}&=r^{-2}\mathfrak{d}_*^{\leq k_*-1}\Gamma_b+\mathfrak{d}_*^{\leq k_*}(\Gamma_b\cdot\Gamma_b),\\
(\nu^k\mathrm{div}\,\underline{\xi})_{\ell=1}&=r^{-2}\mathfrak{d}_*^{\leq k_*-1}\Gamma_b+\mathfrak{d}_*^{\leq k_*}(\Gamma_b\cdot\Gamma_b).
\end{aligned}$$

Together with the control of Γ_b provided by **Ref 1** and Lemma 5.15, we infer, for $k\leq k_*$,

$$|(\nu^k\mathrm{div}\,\eta)_{\ell=1}|+|(\nu^k\mathrm{div}\,\underline{\xi})_{\ell=1}|\lesssim\frac{\epsilon}{r^3u^{1+\frac{\delta_{dec}}{2}}}+\frac{\epsilon^2}{r^2u^{2+\frac{3\delta_{dec}}{2}}}\lesssim\frac{\epsilon_0}{r^2u^{2+\frac{3\delta_{dec}}{2}}}$$

where we used the dominance condition (5.30) on r on Σ_*. Recalling the above estimate, for $k\leq k_*$,

$$|(\not{d}_1\nabla_\nu^k\eta)_{\ell=1}|+|(\not{d}_1\nabla_\nu^k\underline{\xi})_{\ell=1}|\lesssim|(\nu^k\mathrm{div}\,\eta)_{\ell=1}|+|(\nu^k\mathrm{div}\,\underline{\xi})_{\ell=1}|+\frac{\epsilon_0}{r^2u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}},$$

we deduce, for $k \le k_*$,

$$|(\not{d}_1 \nabla_\nu^k \eta)_{\ell=1}| + |(\not{d}_1 \nabla_\nu^k \underline{\xi})_{\ell=1}| \lesssim \frac{\epsilon_0}{r^2 u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}}$$

which is the desired estimate (5.97). This concludes the proof of (5.96).

Step 7. In this step, we derive the desired estimates for $y = e_3(r)$, $z = e_3(u)$ and b_*, i.e. we show

$$\int_{\Sigma_*} r^{-2} u^{2+2\delta_{dec}} \left(\left|\mathfrak{d}_*^k(\check{y})\right|^2 + \left|\mathfrak{d}_*^k(\check{z})\right|^2 + \left|\mathfrak{d}_*^k(\check{b}_*)\right|^2 \right) \lesssim \epsilon_0^2, \quad k \le k_* - 7. \tag{5.98}$$

To this end, we make use of the equations (see Lemma 5.12)

$$\nabla \check{y} = -\underline{\xi} + (\zeta - \eta) y = -\underline{\xi} + \eta + \Gamma_g, \qquad \nabla \check{z} = (\zeta - \eta) z = -2\eta + \Gamma_g.$$

In view of the commutator Lemma 5.20, we infer

$$\begin{aligned}
\nabla \mathfrak{d}_*^k \check{y} &= -\mathfrak{d}_*^k \underline{\xi} + \mathfrak{d}_*^k \eta + \mathfrak{d}_*^{\le k} \Gamma_g + r \mathfrak{d}_*^{\le k} (\Gamma_b \cdot \Gamma_b), \\
\nabla \mathfrak{d}_*^k \check{z} &= -2\mathfrak{d}_*^k \eta + \mathfrak{d}_*^{\le k} \Gamma_g + r \mathfrak{d}_*^{\le k} (\Gamma_b \cdot \Gamma_b).
\end{aligned}$$

Using the control of Γ_b and Γ_g provided by **Ref 1** and Lemma 5.15, and the dominance condition (5.30) on r on Σ_*, we obtain

$$\begin{aligned}
|\nabla \mathfrak{d}_*^k \check{y}| + |\nabla \mathfrak{d}_*^k \check{z}| &\lesssim |\mathfrak{d}_*^k \underline{\xi}| + |\mathfrak{d}_*^k \eta| + \frac{\epsilon}{r^2 u^{\frac{1}{2}+\frac{\delta_{dec}}{2}}} + \frac{\epsilon^2}{r u^{2+\frac{3\delta_{dec}}{2}}} \\
&\lesssim |\mathfrak{d}_*^k \underline{\xi}| + |\mathfrak{d}_*^k \eta| + \frac{\epsilon_0}{r u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}}.
\end{aligned}$$

Squaring, multiplying by $u^{2+2\delta_{dec}}$ and integrating on Σ_*, we infer, for $k \le k_* - 6$,

$$\int_{\Sigma_*} u^{2+2\delta_{dec}} \Big(|\nabla \mathfrak{d}_*^k \check{y}|^2 + |\nabla \mathfrak{d}_*^k \check{z}|^2 \Big) \lesssim \epsilon_0^2 + \int_{\Sigma_*} u^{2+2\delta_{dec}} \Big(|\mathfrak{d}_*^k \eta|^2 + |\mathfrak{d}_*^k \underline{\xi}|^2 \Big).$$

Together with (5.96), this yields, for $k \le k_* - 6$,

$$\int_{\Sigma_*} u^{2+2\delta_{dec}} \Big(|\nabla \mathfrak{d}_*^k \check{y}|^2 + |\nabla \mathfrak{d}_*^k \check{z}|^2 \Big) \lesssim \epsilon_0^2.$$

Since $b_* = -y - z$ in view of Lemma 5.12, we deduce, for $k \le k_* - 6$,

$$\int_{\Sigma_*} u^{2+2\delta_{dec}} \Big(|\nabla \mathfrak{d}_*^k \check{y}|^2 + |\nabla \mathfrak{d}_*^k \check{z}|^2 + |\nabla \mathfrak{d}_*^k \check{b}_*|^2 \Big) \lesssim \epsilon_0^2. \tag{5.99}$$

In view of (5.99), it remains to estimate the averages $\overline{\nu^k(\check{y})}$, $\overline{\nu^k(\check{z})}$ and $\overline{\nu^k(\check{b}_*)}$. We start with b_*. In view of the definition of $\check{b}_*$ and (5.27), we have

$$\overline{\check{b}_*} = 0,$$

where $\overline{\check{b}_*}$ denotes the average of $\check{b}_*$ on the spheres foliating Σ_*. Since ν is tangent along Σ_*, we infer, for any k,

$$\nu^k\left(\overline{\check{b}_*}\right) = 0.$$

Together with Corollary 5.32, we deduce, for any k,

$$\overline{\nu^k(\check{b}_*)} \quad = \quad \mathfrak{d}^{\leq k}(r^2\Gamma_b \cdot \Gamma_b).$$

Using the control of Γ_b provided by **Ref 1** and Lemma 5.15, this yields, for $k \leq k_*$,

$$|\overline{\nu^k(\check{b}_*)}| \quad \lesssim \quad \frac{\epsilon^2}{u^{2+\frac{3\delta_{dec}}{2}}} \lesssim \frac{\epsilon_0}{u^{2+\frac{3\delta_{dec}}{2}}}.$$

In particular, we infer, for $k \leq k_*$,

$$\int_{\Sigma_*} r^{-2}u^{2+2\delta_{dec}}\left|\overline{\nu^k(\check{b}_*)}\right|^2 \quad \lesssim \quad \epsilon_0^2. \tag{5.100}$$

Next, we estimate the average $\overline{\nu^k(\check{y})}$. Recall from Lemma 5.31 the following identity

$$\nu(r) = \frac{rz}{2}\overline{z^{-1}(\underline{\kappa} + b_*\kappa)}.$$

Using the transversality condition $e_4(r) = 1$ on Σ_*, the fact that $\nu = e_3 + b_*e_4$, and the GCM condition $\kappa = 2/r$, we infer

$$\begin{aligned} y + b_* \quad &= \quad \frac{rz}{2}\overline{z^{-1}\left(\underline{\kappa} + \frac{2}{r}b_*\right)} = \frac{r(2+\check{z})}{2}\overline{\frac{1}{2+\check{z}}\left(-\frac{2\Upsilon}{r} + \frac{2}{r}b_*\right)} + r\Gamma_g \\ &= \quad \left(1 + \frac{1}{2}\check{z}\right)\overline{\left(1 - \frac{1}{2}\check{z}\right)(-\Upsilon + b_*)} + r\Gamma_g + r^2\Gamma_b \cdot \Gamma_b \end{aligned}$$

and hence

$$\check{y} \quad = \quad -\frac{1}{2}\left(\check{z} - \overline{\check{z}}\right) - \left(\check{b}_* - \overline{\check{b}_*}\right) + r\Gamma_g + r^2\Gamma_b \cdot \Gamma_b.$$

Taking the average, we infer

$$\overline{\check{y}} = r\Gamma_g + r^2\Gamma_b \cdot \Gamma_b.$$

Together with Corollary 5.32, we deduce

$$\overline{\nu^k(\check{y})} = r\mathfrak{d}_*^{\leq k}\Gamma_g + r^2\mathfrak{d}_*^{\leq k}(\Gamma_b \cdot \Gamma_b).$$

Using the control of Γ_b and Γ_g provided by **Ref 1** and Lemma 5.15, and the dominance condition (5.30) on r on Σ_*, we deduce, for $k \leq k_*$,

$$|\overline{\nu^k(\check{y})}| \lesssim \frac{\epsilon}{ru^{\frac{1}{2}+\frac{\delta_{dec}}{2}}} + \frac{\epsilon^2}{u^{2+\frac{3\delta_{dec}}{2}}} \lesssim \frac{\epsilon_0}{u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}}.$$

Squaring, multiplying by $r^{-2}u^{2+2\delta_{dec}}$ and integrating on Σ_*, we infer, for $k \leq k_*$,

$$\int_{\Sigma_*} r^{-2}u^{2+2\delta_{dec}} \left|\overline{\nu^k(\check{y})}\right|^2 \lesssim \epsilon_0^2.$$

Together with (5.100) and the fact that $z = -y - b_*$ in view of Lemma 5.12, we deduce, for $k \leq k_*$,

$$\int_{\Sigma_*} r^{-2}u^{2+2\delta_{dec}} \left(\left|\overline{\nu^k(\check{y})}\right|^2 + \left|\overline{\nu^k(\check{z})}\right|^2 + \left|\overline{\nu^k(\check{b}_*)}\right|^2\right) \lesssim \epsilon_0^2.$$

Together with (5.99), and using a Poincaré inequality, we infer, for $k \leq k_* - 7$,

$$\int_{\Sigma_*} r^{-2}u^{2+2\delta_{dec}} \left(\left|\mathfrak{d}_*^k(\check{y})\right|^2 + \left|\mathfrak{d}_*^k(\check{z})\right|^2 + \left|\mathfrak{d}_*^k(\check{b}_*)\right|^2\right) \lesssim \epsilon_0^2$$

which is the desired estimate (5.98).

Step 8. In this final step, we derive the desired estimate for $\underline{\check{\omega}}$, i.e. we show

$$\int_{\Sigma_*} u^{2+2\delta_{dec}} \left|\mathfrak{d}_*^k(\underline{\check{\omega}})\right|^2 \lesssim \epsilon_0^2, \quad k \leq k_* - 7. \tag{5.101}$$

We start with the equations

$$\begin{aligned} \nabla_4\check{\kappa} &= \Gamma_g \cdot \Gamma_g, \\ \nabla_3\check{\kappa} &= 2\mathrm{div}\,\eta + \frac{4}{r}\underline{\check{\omega}} + \frac{2}{r^2}\check{y} + r^{-1}\Gamma_g + \Gamma_b \cdot \Gamma_b, \end{aligned}$$

which together with the fact that $\nu = e_3 + b_* e_4$ yield

$$\nabla_\nu \check{\kappa} \quad = \quad 2\text{div}\,\eta + \frac{4}{r}\underline{\check{\omega}} + \frac{2}{r^2}\check{y} + r^{-1}\Gamma_g + \Gamma_b \cdot \Gamma_b.$$

Since our GCM assumption for κ on Σ_* implies $\check{\kappa} = 0$, and since ν is tangent to Σ_*, we infer

$$0 \quad = \quad 2\text{div}\,\eta + \frac{4}{r}\underline{\check{\omega}} + \frac{2}{r^2}\check{y} + r^{-1}\Gamma_g + \Gamma_b \cdot \Gamma_b,$$

and hence

$$\underline{\check{\omega}} \quad = \quad -\frac{r}{2}\text{div}\,\eta - \frac{1}{2r}\check{y} + \Gamma_g + r\Gamma_b \cdot \Gamma_b.$$

Using the control of Γ_b and Γ_g provided by **Ref 1** and Lemma 5.15, and the dominance condition (5.30) on r on Σ_*, we deduce, for $k \leq k_*$,

$$\begin{aligned} |\mathfrak{d}_*^k \underline{\check{\omega}}| &\lesssim |\mathfrak{d}_*^{k+1}\eta| + r^{-1}|\mathfrak{d}_*^k \check{y}| + \frac{\epsilon}{r^2 u^{\frac{1}{2}+\frac{\delta_{dec}}{2}}} + \frac{\epsilon^2}{r u^{2+\frac{3\delta_{dec}}{2}}} \\ &\lesssim |\mathfrak{d}_*^{k+1}\eta| + r^{-1}|\mathfrak{d}_*^k \check{y}| + \frac{\epsilon_0}{r u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}}. \end{aligned}$$

Squaring, multiplying by $u^{2+2\delta_{dec}}$, and integrating on Σ_*, we infer

$$\int_{\Sigma_*} u^{2+2\delta_{dec}} |\mathfrak{d}_*^k \underline{\check{\omega}}|^2 \quad \lesssim \quad \int_{\Sigma_*} u^{2+2\delta_{dec}} |\mathfrak{d}_*^{k+1}\eta|^2 + \int_{\Sigma_*} r^{-2} u^{2+2\delta_{dec}} |\mathfrak{d}_*^k \check{y}|^2 + \epsilon_0^2.$$

Together with (5.96) for η and (5.98) for $\check{y}$, we infer, for $k \leq k_* - 7$,

$$\int_{\Sigma_*} u^{2+2\delta_{dec}} |\mathfrak{d}_*^k \underline{\check{\omega}}|^2 \quad \lesssim \quad \epsilon_0^2$$

which is the desired estimate for $\underline{\check{\omega}}$. Together with the estimates of Step 1 to Step 7, we deduce, for $k \leq k_* - 7$,

$$\int_{\Sigma_*} u^{2+2\delta_{dec}} |\mathfrak{d}_*^k \Gamma_b|^2 \quad \lesssim \quad \epsilon_0^2$$

which is the desired estimate (5.88). This concludes the proof of Proposition 5.42. □

As a corollary of the above we derive the following improved version of Lemma 5.37.

Corollary 5.45. *The functions $J^{(p)}$ verify the following properties*

1. *We have on Σ_**

$$\int_S J^{(p)} = O\left(\epsilon r u^{-\frac{1}{2}-\frac{\delta_{dec}}{2}}\right),$$
$$\int_S J^{(p)} J^{(q)} = \frac{4\pi}{3} r^2 \delta_{pq} + O\left(\epsilon r u^{-\frac{1}{2}-\frac{\delta_{dec}}{2}}\right).$$

2. *For any $k \leq k_* - 10$, we have on Σ_**

$$\left|\mathfrak{d}_*^k \left(\Delta + \frac{2}{r^2}\right) J^{(p)}\right| \lesssim \epsilon r^{-3} u^{-\frac{1}{2}-\frac{\delta_{dec}}{2}}.$$

3. *We have for any $k \leq k_* - 10$ on Σ_**

$$\left|\mathfrak{d}_*^k \,{\not{d}}_2^* {\not{d}}_1^* J^{(p)}\right| \lesssim \epsilon r^{-3} u^{-\frac{1}{2}-\frac{\delta_{dec}}{2}},$$

where by ${\not{d}}_1^* J^{(p)}$, we mean either ${\not{d}}_1^*(J^{(p)}, 0)$ or ${\not{d}}_1^*(0, J^{(p)})$.

Proof. The proof follows exactly the same lines as the proof of Lemma 5.37 by replacing the pointwise estimate of Γ_b by the improved flux estimates of Proposition 5.42 for Γ_b. □

5.4. Estimates for $\ell = 0$ and $\ell = 1$ modes on Σ_*

5.4.1. Estimates for some $\ell = 1$ modes on S_* We start with estimates for the $\ell = 1$ modes on S_*. Recall that on S_* we have in particular the following, see Section 5.1.2,

$$\kappa = \frac{2}{r}, \qquad \underline{\kappa} = -\frac{2\Upsilon}{r},$$
$$(\mathrm{div}\,\beta)_{\ell=1} = 0, \qquad (\mathrm{curl}\,\beta)_{\ell=1,\pm} = 0, \qquad (\mathrm{curl}\,\beta)_{\ell=1,0} = \frac{2am}{r^5}.$$

Lemma 5.46. *The Gauss curvature K of S_* verifies*

$$|(\check{K})_{\ell=1}| \lesssim \frac{\epsilon_0}{r^3 u^{2+2\delta_{dec}}}. \tag{5.102}$$

Proof. Recall that the metric on S_* is given by

$$g_{S_*} = e^{2\phi} r^2 \Big((d\theta)^2 + (\sin\theta)^2 (d\varphi)^2\Big).$$

The Gauss curvature thus satisfies

$$K = -\Delta\phi + \frac{e^{-2\phi}}{r^2}$$

and hence

$$K - \frac{1}{r^2} = -\left(\Delta + \frac{2}{r^2}\right)\phi + \frac{h(\phi)}{r^2}, \qquad h(\phi) = e^{-2\phi} - 1 + 2\phi.$$

Integrating by parts,

$$\begin{aligned}\int_{S_*}\left(K - \frac{1}{r^2}\right)J^{(p)} &= -\int_{S_*}\left(\Delta + \frac{2}{r^2}\right)\phi J^{(p)} + \int_{S_*}\frac{h(\phi)}{r^2}J^{(p)} \\ &= -\int_{S_*}\phi\left(\Delta + \frac{2}{r^2}\right)J^{(p)} + \int_{S_*}\frac{h(\phi)}{r^2}J^{(p)}.\end{aligned}$$

Using the control of ϕ of Lemma 5.35, i.e. $\phi = O(\epsilon r^{-1}u^{-\frac{1}{2}-\frac{\delta_{dec}}{2}})$ and the estimate for $(\Delta + 2/r^2)J^{(p)}$ in Corollary 5.45, we easily deduce that

$$\left|\int_{S_*}\left(K - \frac{1}{r^2}\right)J^{(p)}\right| \lesssim \frac{\epsilon}{r^2 u^{1+\delta_{dec}}}.$$

Using the dominance condition (5.30) on r on Σ_*, we infer

$$\left|\int_{S_*}\left(K - \frac{1}{r^2}\right)J^{(p)}\right| \lesssim \frac{\epsilon_0}{r u^{2+2\delta_{dec}}},$$

i.e. recalling the definition of $\ell = 1$ modes,

$$|(\check{K})_{\ell=1}| \lesssim \frac{\epsilon_0}{r^3 u^{2+2\delta_{dec}}}$$

as stated. ☐

Lemma 5.47. *The following holds on S_**

$$\left|\left(\check{\rho} - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)_{\ell=1}\right| \lesssim \epsilon_0 r^{-3}u^{-2-2\delta_{dec}}. \tag{5.103}$$

Proof. Recall that our GCM conditions on S_* imply in particular $\check{\kappa} = \underline{\check{\kappa}} = 0$ on S_*. This implies that the Gauss equation on S_* takes the form

$$\check{K} = -\check{\rho} + \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}$$

and hence

$$\left(\check{\rho} - \frac{1}{2}\widehat{\chi} \cdot \underline{\widehat{\chi}}\right)_{\ell=1} \quad = \quad -\left(\check{K}\right)_{\ell=1}.$$

Hence, in view of Lemma 5.46, we obtain

$$\left|\left(\check{\rho} - \frac{1}{2}\widehat{\chi} \cdot \underline{\widehat{\chi}}\right)_{\ell=1}\right| \quad \lesssim \quad \epsilon_0 r^{-3} u^{-2-2\delta_{dec}}$$

as stated. □

5.4.2. Estimates for the $\ell = 1$ modes on Σ_*

Proposition 5.48 (Control of $\ell = 1$ modes)**.** *The following estimates hold on* Σ_**.*

$$\begin{aligned}
|(div\,\beta)_{\ell=1}| + |(curl\,\beta)_{\ell=1,\pm}| + \left|(curl\,\beta)_{\ell=1,0} - \frac{2am}{r^5}\right| &\lesssim \epsilon_0 r^{-5} u^{-1-\delta_{dec}},\\
|(\nu\, div\,\beta)_{\ell=1}| + |(\nu\, curl\,\beta)_{\ell=1}| &\lesssim \epsilon_0 r^{-5} u^{-1-\delta_{dec}},\\
|(div\,\zeta)_{\ell=1}| + |(curl\,\zeta)_{\ell=1,\pm}| + \left|(curl\,\zeta)_{\ell=1,0} - \frac{2am}{r^4}\right| &\lesssim \epsilon_0 r^{-4} u^{-1-\delta_{dec}},\\
|(\check{\underline{\kappa}})_{\ell=1}| &\lesssim \epsilon_0 r^{-2} u^{-2-2\delta_{dec}},\\
\left|\left(\check{\rho} - \frac{1}{2}\widehat{\chi} \cdot \underline{\widehat{\chi}}\right)_{\ell=1}\right| &\lesssim \epsilon_0 r^{-3} u^{-2-2\delta_{dec}},\\
\left|\left({}^*\rho - \frac{1}{2}\widehat{\chi} \wedge \underline{\widehat{\chi}}\right)_{\ell=1,\pm}\right| + \left|\left({}^*\rho - \frac{1}{2}\widehat{\chi} \wedge \underline{\widehat{\chi}}\right)_{\ell=1,0} - \frac{2am}{r^4}\right| &\lesssim \epsilon_0 r^{-3} u^{-2-2\delta_{dec}},\\
|(\check{\mu})_{\ell=1}| &\lesssim \epsilon_0 r^{-3} u^{-2-2\delta_{dec}}.
\end{aligned}$$

Remark 5.49. *Note that the results are consistent with strong peeling.*

Proof. We make the following local bootstrap assumption on Σ_*

$$\begin{aligned}
|(\mathrm{div}\,\beta)_{\ell=1}| + |(\mathrm{curl}\,\beta)_{\ell=1,\pm}| + \left|(\mathrm{curl}\,\beta)_{\ell=1,0} - \frac{2am}{r^5}\right| &\leq \epsilon r^{-5} u^{-1-\delta_{dec}},\\
|(\check{\underline{\kappa}})_{\ell=1}| &\leq \epsilon r^{-2} u^{-2-\delta_{dec}}.
\end{aligned} \tag{5.104}$$

Step 1. We start with the control of $(\not{d}_1\zeta)_{\ell=1}$. Recall the following consequence of the Codazzi equation for $\widehat{\chi}$

$$\not{d}_2\widehat{\chi} \quad = \quad \frac{1}{r}\zeta - \beta + \Gamma_g \cdot \Gamma_g.$$

Differentiating w.r.t. $\not{d}_1$, we infer

$$\not{d}_1 \not{d}_2 \widehat{\chi} = \frac{1}{r} \not{d}_1 \zeta - \not{d}_1 \beta + r^{-1} \not{\mathfrak{d}}^{\leq 1}(\Gamma_g \cdot \Gamma_g).$$

Projecting on the $\ell = 1$ modes, this yields

$$(\not{d}_1 \not{d}_2 \widehat{\chi})_{\ell=1} = \frac{1}{r}(\not{d}_1 \zeta)_{\ell=1} - (\not{d}_1 \beta)_{\ell=1} + r^{-1} \not{\mathfrak{d}}^{\leq 1}(\Gamma_g \cdot \Gamma_g).$$

Next, we estimate $(\not{d}_1 \not{d}_2 \widehat{\chi})_{\ell=1}$. We have

$$(\not{d}_1 \not{d}_2 \widehat{\chi})_{\ell=1,p} = \frac{1}{|S|} \int_S \not{d}_1 \not{d}_2 \widehat{\chi} J^{(p)} = \frac{1}{|S|} \int_S \widehat{\chi} \cdot \not{d}_2^* \not{d}_1^* J^{(p)}$$

and hence

$$|(\not{d}_1 \not{d}_2 \widehat{\chi})_{\ell=1}| \lesssim |\not{d}_2^* \not{d}_1^* J^{(p)}||\Gamma_g|.$$

We deduce

$$(\not{d}_1 \zeta)_{\ell=1} = r(\not{d}_1 \beta)_{\ell=1} + r|\not{d}_2^* \not{d}_1^* J^{(p)}|\Gamma_g + \not{\mathfrak{d}}^{\leq 1}(\Gamma_g \cdot \Gamma_g)$$

and thus

$$\begin{aligned} |(\mathrm{div}\, \zeta)_{\ell=1}| + |(\mathrm{curl}\, \zeta)_{\ell=1,\pm}| &\lesssim r|(\mathrm{div}\, \beta)_{\ell=1}| + r|(\mathrm{curl}\, \beta)_{\ell=1,\pm}| \\ &\quad + r|\not{d}_2^* \not{d}_1^* J^{(p)}|\Gamma_g + \not{\mathfrak{d}}^{\leq 1}(\Gamma_g \cdot \Gamma_g), \\ \left|(\mathrm{curl}\, \zeta)_{\ell=1,0} - \frac{2am}{r^4}\right| &\lesssim r\left|(\mathrm{curl}\, \beta)_{\ell=1,0} - \frac{2am}{r^5}\right| \\ &\quad + r|\not{d}_2^* \not{d}_1^* J^{(p)}|\Gamma_g + \not{\mathfrak{d}}^{\leq 1}(\Gamma_g \cdot \Gamma_g). \end{aligned}$$

Using the control of Γ_g provided by **Ref 1**, the control of $\not{d}_2^* \not{d}_1^* J^{(p)}$ provided by Corollary 5.45, and the local bootstrap assumption (5.104) on $(\not{d}_1 \beta)_{\ell=1}$, we deduce

(5.105)
$$|(\mathrm{div}\, \zeta)_{\ell=1}| + |(\mathrm{curl}\, \zeta)_{\ell=1,\pm}| + \left|(\mathrm{curl}\, \zeta)_{\ell=1,0} - \frac{2am}{r^4}\right| \lesssim \frac{\epsilon}{r^4 u^{1+\delta_{dec}}}.$$

Step 2. Next, we consider the control of $(\mathrm{div}\, \underline{\beta})_{\ell=1}$. Recall the following consequence of the Codazzi equation for $\underline{\widehat{\chi}}$

$$\mathrm{div}\, \underline{\widehat{\chi}} = \frac{1}{2} \nabla \underline{\check{\kappa}} + \frac{\Upsilon}{r} \zeta + \underline{\beta} + \Gamma_b \cdot \Gamma_g.$$

Differentiating w.r.t. div , we infer

$$\mathrm{div}\, \not{d}_2\underline{\widehat{\chi}} \;=\; \frac{1}{2}\Delta\underline{\check{\kappa}} + \frac{\Upsilon}{r}\mathrm{div}\,\zeta + \mathrm{div}\,\underline{\beta} + r^{-1}\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_g).$$

Projecting on the $\ell=1$ modes, this yields

$$(\mathrm{div}\, \not{d}_2\underline{\widehat{\chi}})_{\ell=1} \;=\; \frac{1}{2}(\Delta\underline{\check{\kappa}})_{\ell=1} + \frac{\Upsilon}{r}(\mathrm{div}\,\zeta)_{\ell=1} + (\mathrm{div}\,\underline{\beta})_{\ell=1} + r^{-1}\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_g).$$

As in Step 1, we have

$$|(\mathrm{div}\, \not{d}_2\underline{\widehat{\chi}})_{\ell=1}| \;\lesssim\; |\not{d}_2^*\not{d}_1^*J^{(p)}||\Gamma_b|.$$

Also, we have

$$(\Delta\underline{\check{\kappa}})_{\ell=1,p} = \frac{1}{|S|}\int_S \Delta\underline{\check{\kappa}}J^{(p)} = -\frac{2}{r^2}\frac{1}{|S|}\int_S\underline{\check{\kappa}}J^{(p)} + \frac{1}{|S|}\int_S\underline{\check{\kappa}}\left(\Delta+\frac{2}{r^2}\right)J^{(p)}$$

and hence

$$|(\Delta\underline{\check{\kappa}})_{\ell=1}| \;\lesssim\; r^{-2}|(\underline{\check{\kappa}})_{\ell=1}| + \left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right||\Gamma_g|.$$

We deduce

$$\begin{aligned}|(\mathrm{div}\,\underline{\beta})_{\ell=1}| \;&\lesssim\; r^{-2}|(\underline{\check{\kappa}})_{\ell=1}| + r^{-1}|(\mathrm{div}\,\zeta)_{\ell=1}| + \left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right||\Gamma_g| \\ &\quad + |\not{d}_2^*\not{d}_1^*J^{(p)}||\Gamma_b| + r^{-1}|\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_g)|.\end{aligned}$$

Together with the local bootstrap assumption (5.104) on $(\underline{\check{\kappa}})_{\ell=1}$, the control of $(\mathrm{div}\,\zeta)_{\ell=1}$ in (5.105), the control of $\not{d}_2^*\not{d}_1^*J^{(p)}$ and $(\Delta+2/r^2)J^{(p)}$ provided by Corollary 5.45, and the control of Γ_g provided by **Ref 1**, we obtain

$$|(\mathrm{div}\,\underline{\beta})_{\ell=1}| \;\lesssim\; \frac{\epsilon}{r^4u^{2+\delta_{dec}}} + \frac{\epsilon}{r^5u^{1+\delta_{dec}}} + \frac{\epsilon}{r^3u^{\frac{1}{2}+\frac{\delta_{dec}}{2}}}|\not{\mathfrak{d}}^{\leq 1}\Gamma_b|.$$

Using the dominance condition (5.30) on r on Σ_*, we infer

$$|(\mathrm{div}\,\underline{\beta})_{\ell=1}| \;\lesssim\; \frac{\epsilon_0}{r^3u^{3+2\delta_{dec}}} + \frac{\epsilon_0}{r^2u^{\frac{3}{2}+\frac{3\delta_{dec}}{2}}}|\not{\mathfrak{d}}^{\leq 1}\Gamma_b|.$$

By integration in u, and using Sobolev, we deduce

$$\int_u^{u_*} r^3|(\mathrm{div}\,\underline{\beta})_{\ell=1}| \;\lesssim\; \frac{\epsilon_0}{u^{2+2\delta_{dec}}} + \epsilon_0\int_u^{u_*} u^{-\frac{3}{2}-\frac{3\delta_{dec}}{2}}\|\not{\mathfrak{d}}^{\leq 3}\Gamma_b\|_{L^2(S)}$$

$$\lesssim \frac{\epsilon_0}{u^{2+2\delta_{dec}}} + \frac{\epsilon_0}{u^{2+2\delta_{dec}}} \left(\int_{\Sigma_*} u^{2+2\delta_{dec}} |\not{\mathfrak{d}}^{\leq 3} \Gamma_b|^2 \right)^{\frac{1}{2}}.$$

Hence, in view of Proposition 5.42, we obtain

$$\int_u^{u_*} r^3 |(\operatorname{div} \underline{\beta})_{\ell=1}| \lesssim \frac{\epsilon_0}{u^{2+2\delta_{dec}}}. \tag{5.106}$$

Step 3. We provide the estimate for $(\check{\rho} - \frac{1}{2}\widehat{\chi} \cdot \widehat{\underline{\chi}})_{\ell=1}$. Recall from Corollary 5.41 that we have along Σ_*, for $p = 0, +, -$,

$$\nu \left(\int_S \left(\check{\rho} - \frac{1}{2}\widehat{\chi} \cdot \widehat{\underline{\chi}} \right) J^{(p)} \right) = O(r^{-1}) \int_S \left(\check{\rho} - \frac{1}{2}\widehat{\chi} \cdot \widehat{\underline{\chi}} \right) J^{(p)} + h_1,$$

where the scalar function h_1 is given by

$$\begin{aligned} h_1 &= -\int_S \operatorname{div} \underline{\beta} J^{(p)} - (1 + O(r^{-1})) \int_S \operatorname{div} \beta J^{(p)} + O(r^{-3}) \int_S \underline{\check{\kappa}} J^{(p)} \\ &\quad + O(r^{-2}) \int_S \operatorname{div} \zeta J^{(p)} + r \left| \left(\Delta + \frac{2}{r^2} \right) J^{(p)} \right| \Gamma_b + r \not{\mathfrak{d}}^{\leq 1} (\Gamma_b \cdot \Gamma_b). \end{aligned}$$

In view of the local bootstrap assumption (5.104) on $(\underline{\check{\kappa}})_{\ell=1}$ and $(\operatorname{div} \beta)_{\ell=1}$, the control of $(\operatorname{div} \zeta)_{\ell=1}$ in (5.105), the control of $(\Delta + 2/r^2) J^{(p)}$ provided by Corollary 5.45, and the control of Γ_g provided by **Ref 1**, we obtain

$$|h_1| \lesssim r^2 |(\operatorname{div} \underline{\beta})_{\ell=1}| + \frac{\epsilon}{r^3 u^{1+\delta_{dec}}} + r |\not{\mathfrak{d}}^{\leq 1} (\Gamma_b \cdot \Gamma_b)|.$$

Using the dominance condition (5.30) on r on Σ_*, we infer

$$|h_1| \lesssim r^2 |(\operatorname{div} \underline{\beta})_{\ell=1}| + r |\not{\mathfrak{d}}^{\leq 1} (\Gamma_b \cdot \Gamma_b)| + \frac{\epsilon_0}{r u^{3+3\delta_{dec}}}.$$

By integration in u, and using Sobolev, we deduce

$$\begin{aligned} \int_u^{u_*} r|h_1| &\lesssim \frac{\epsilon_0}{u^{2+3\delta_{dec}}} + \int_u^{u_*} r^3 |(\operatorname{div} \underline{\beta})_{\ell=1}| + \int_u^{u_*} \|\not{\mathfrak{d}}^{\leq 3} \Gamma_b\|^2_{L^2(S)} \\ &\lesssim \frac{\epsilon_0}{u^{2+3\delta_{dec}}} + \int_u^{u_*} r^3 |(\operatorname{div} \underline{\beta})_{\ell=1}| \\ &\quad + \frac{\epsilon_0}{u^{2+2\delta_{dec}}} \left(\int_{\Sigma_*} u^{2+2\delta_{dec}} |\not{\mathfrak{d}}^{\leq 3} \Gamma_b|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Together with the control of $(\operatorname{div} \underline{\beta})_{\ell=1}$ in (5.106) and the control of Γ_b in

Proposition 5.42, we obtain

$$\int_u^{u_*} r|h_1| \lesssim \frac{\epsilon_0}{u^{2+2\delta_{dec}}}.$$

Since

$$\nu\left(\int_S \left(\check{\rho} - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right) J^{(p)}\right) = O(r^{-1})\int_S \left(\check{\rho} - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right) J^{(p)} + h_1,$$

we may thus apply Corollary 5.34 which implies

$$ru^{2+2\delta_{dec}}\left|\int_S \left(\check{\rho} - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right) J^{(p)}\right| \lesssim r_* u_*^{2+2\delta_{dec}}\left|\int_{S_*} \left(\check{\rho} - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right) J^{(p)}\right| + \epsilon_0.$$

Together with the control of $(\check{\rho} - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}})_{\ell=1}$ on S_* provided by Lemma 5.47, we infer

$$ru^{2+2\delta_{dec}}\left|\int_S \left(\check{\rho} - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right) J^{(p)}\right| \lesssim \epsilon_0$$

and hence

$$\left|\left(\check{\rho} - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)_{\ell=1}\right| \lesssim \frac{\epsilon_0}{r^3 u^{2+2\delta_{dec}}}. \tag{5.107}$$

Step 4. We provide the estimate for $(\underline{\check{\kappa}})_{\ell=1}$. Recall from Corollary 5.41 that we have along Σ_*, for $p = 0, +, -$,

$$\nu\left(\int_S \left(\Delta\underline{\check{\kappa}} + \frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right) J^{(p)}\right) = h_2$$

where the scalar function h_2 is given by

$$\begin{aligned} h_2 &= O(r^{-3})\int_S \underline{\check{\kappa}} J^{(p)} + O(r^{-2})\int_S \mathrm{div}\,\zeta J^{(p)} + O(r^{-1})\int_S \mathrm{div}\,\underline{\beta} J^{(p)} \\ &\quad + O(r^{-2})\int_S \left(\check{\rho} - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right) J^{(p)} + O(r^{-1})\int_S \mathrm{div}\,\beta J^{(p)} \\ &\quad + r\left|\left(\Delta + \frac{2}{r^2}\right) J^{(p)}\right| \not{\mathfrak{d}}^{\leq 1}\Gamma_b + \not{\mathfrak{d}}^{\leq 2}(\Gamma_b\cdot\Gamma_b). \end{aligned}$$

In view of the local bootstrap assumption (5.104) on $(\underline{\check{\kappa}})_{\ell=1}$ and $(\mathrm{div}\,\beta)_{\ell=1}$, the control of $(\mathrm{div}\,\zeta)_{\ell=1}$ in (5.105), and the control of $(\Delta + 2/r^2)J^{(p)}$ provided by Corollary 5.45, we obtain

$$\begin{aligned}|h_2| &\lesssim r|(\mathrm{div}\,\underline{\beta})_{\ell=1}| + \frac{\epsilon}{r^3u^{2+\delta_{dec}}} + \frac{\epsilon}{r^4u^{1+\delta_{dec}}} + \frac{\epsilon}{r^2u^{\frac{1}{2}+\frac{\delta_{dec}}{2}}}|\Gamma_b| \\ &\quad +|\not{\mathfrak{d}}^{\leq 2}(\Gamma_b\cdot\Gamma_b)| \\ &\lesssim r|(\mathrm{div}\,\underline{\beta})_{\ell=1}| + \frac{\epsilon}{r^3u^{2+\delta_{dec}}} + \frac{\epsilon}{r^4u^{1+\delta_{dec}}} + |\not{\mathfrak{d}}^{\leq 2}(\Gamma_b\cdot\Gamma_b)|.\end{aligned}$$

Using the dominance condition (5.30) on r on Σ_*, we infer

$$|h_2| \lesssim r|(\mathrm{div}\,\underline{\beta})_{\ell=1}| + \frac{\epsilon_0}{r^2u^{3+2\delta_{dec}}} + |\not{\mathfrak{d}}^{\leq 2}(\Gamma_b\cdot\Gamma_b)|.$$

By integration in u, and using Sobolev, we deduce

$$\begin{aligned}\int_u^{u_*} r^2|h_2| &\lesssim \frac{\epsilon_0}{u^{2+2\delta_{dec}}} + \int_u^{u_*} r^3|(\mathrm{div}\,\underline{\beta})_{\ell=1}| + \int_u^{u_*}\|\not{\mathfrak{d}}^{\leq 4}\Gamma_b\|^2_{L^2(S)} \\ &\lesssim \frac{\epsilon_0}{u^{2+2\delta_{dec}}} + \int_u^{u_*} r^3|(\mathrm{div}\,\underline{\beta})_{\ell=1}| \\ &\quad + \frac{1}{u^{2+2\delta_{dec}}}\left(\int_{\Sigma_*} u^{2+2\delta_{dec}}|\not{\mathfrak{d}}^{\leq 4}\Gamma_b|^2\right).\end{aligned}$$

Together with the control of $(\mathrm{div}\,\underline{\beta})_{\ell=1}$ in (5.106) and the control of Γ_b in Proposition 5.42, we obtain

$$\int_u^{u_*} r^2|h_2| \lesssim \frac{\epsilon_0}{u^{2+2\delta_{dec}}}.$$

Since

$$\nu\left(\int_S\left(\Delta\underline{\check{\kappa}} + \frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right)J^{(p)}\right) = h_2,$$

we may thus apply Corollary 5.34 which implies

$$\begin{aligned}& r^2u^{2+2\delta_{dec}}\left|\int_S\left(\Delta\underline{\check{\kappa}} + \frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right)J^{(p)}\right| \\ &\lesssim r_*^2u_*^{2+2\delta_{dec}}\left|\int_{S_*}\left(\Delta\underline{\check{\kappa}} + \frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right)J^{(p)}\right| + \epsilon_0.\end{aligned}$$

Since $\underline{\check{\kappa}} = 0$ on S_*, and using the control of $(\mathrm{div}\,\zeta)_{\ell=1}$ in (5.105), we infer

$$r^2u^{2+2\delta_{dec}}\left|\int_S \Delta\underline{\check{\kappa}}J^{(p)}\right| \lesssim \epsilon_0 + \frac{\epsilon u^{1+\delta_{dec}}}{r}$$

and hence

$$u^{2+2\delta_{dec}}\left|\int_S \check{\underline{\kappa}} J^{(p)}\right| \lesssim \epsilon_0 + \frac{\epsilon u^{1+\delta_{dec}}}{r} + r^4 u^{2+2\delta_{dec}}\left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right||\Gamma_g|.$$

Together with the control of $(\Delta + 2/r^2)J^{(p)}$ provided by Corollary 5.45, the control of Γ_g provided by **Ref 1**, and the dominance condition (5.30) on r on Σ_*, we infer

$$u^{2+2\delta_{dec}}\left|\int_S \check{\underline{\kappa}} J^{(p)}\right| \lesssim \epsilon_0 + \frac{\epsilon u^{1+\delta_{dec}}}{r} \lesssim \epsilon_0$$

and hence

$$|(\check{\underline{\kappa}})_{\ell=1}| \lesssim \frac{\epsilon_0}{r^2 u^{2+2\delta_{dec}}}. \tag{5.108}$$

Step 5. Next, we estimate $({}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}})_{\ell=1}$. Recall that we have

$$\mathrm{curl}\,\zeta = {}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}}.$$

We infer from the control of $(\mathrm{curl}\,\zeta)_{\ell=1}$ in (5.105)

$$\left|\left({}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}}\right)_{\ell=1,\pm}\right| + \left|\left({}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}}\right)_{\ell=1,0} - \frac{2am}{r^4}\right| \lesssim \frac{\epsilon}{r^4 u^{1+\delta_{dec}}}.$$

Together with the dominance condition (5.30) on r on Σ_*, we infer

$$\left|\left({}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}}\right)_{\ell=1,\pm}\right| + \left|\left({}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}}\right)_{\ell=1,0} - \frac{2am}{r^4}\right| \lesssim \frac{\epsilon_0}{r^3 u^{2+2\delta_{dec}}}. \tag{5.109}$$

Step 6. We provide the estimate for $(\mathrm{div}\,\beta)_{\ell=1}$. Recall from Corollary 5.41 that we have along Σ_*, for $p = 0, +, -$,

$$\nu\left(\int_S \mathrm{div}\,\beta J^{(p)}\right) = O(r^{-1})\int_S \mathrm{div}\,\beta J^{(p)} + h_3$$

where the scalar function h_3 is given by

$$h_3 = O(r^{-2})\int_S\left(\check{\rho} - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)J^{(p)} + r\left(\left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right| + \left|\not{d}_2^*\not{d}_1^*J^{(p)}\right|\right)\Gamma_g$$

$$+\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_g).$$

In view of the control of $(\check{\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}})_{\ell=1}$ in (5.107), the control of $(\Delta+2/r^2)J^{(p)}$ and $\not{d}_2^*\not{d}_1^*J^{(p)}$ provided by Corollary 5.45, and the control of Γ_g provided by **Ref 1**, we obtain

$$|h_3| \lesssim \frac{\epsilon_0}{r^3u^{2+2\delta_{dec}}}+\frac{\epsilon^2}{r^4u^{1+\frac{3\delta_{dec}}{2}}}+\frac{\epsilon}{r^2u^{\frac{1}{2}+\delta_{dec}}}|\not{\mathfrak{d}}^{\leq 1}\Gamma_b|.$$

Using the dominance condition (5.30) on r on Σ_*, we infer

$$|h_3| \lesssim \frac{\epsilon_0}{r^3u^{2+2\delta_{dec}}}+\frac{\epsilon}{r^2u^{\frac{1}{2}+\delta_{dec}}}|\not{\mathfrak{d}}^{\leq 1}\Gamma_b|.$$

By integration in u, and using Sobolev, we obtain

$$\begin{aligned}\int_u^{u_*}r^3|h_3| &\lesssim \frac{\epsilon_0}{u^{1+\delta_{dec}}}+\int_u^{u_*}\frac{\epsilon}{u'^{\frac{1}{2}+\delta_{dec}}}\|\not{\mathfrak{d}}^{\leq 3}\Gamma_b\|_{L^2(S)}\\ &\lesssim \frac{\epsilon_0}{u^{1+\delta_{dec}}}+\frac{\epsilon_0}{u^{1+2\delta_{dec}}}\left(\int_{\Sigma_*}u^{2+2\delta_{dec}}|\not{\mathfrak{d}}^{\leq 3}\Gamma_b|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Together with the control of Γ_b in Proposition 5.42, we infer

$$\int_u^{u_*}r^3|h_3| \lesssim \frac{\epsilon_0}{u^{1+\delta_{dec}}}.$$

Since

$$\nu\left(\int_S \mathrm{div}\,\beta J^{(p)}\right) = O(r^{-1})\int_S \mathrm{div}\,\beta J^{(p)}+h_3,$$

we may thus apply Corollary 5.34 which implies, together with the fact that $(\mathrm{div}\,\beta)_{\ell=1}=0$ on S_*,

$$r^3u^{1+\delta_{dec}}\left|\int_S \mathrm{div}\,\beta J^{(p)}\right| \lesssim \epsilon_0,$$

and hence

$$|(\mathrm{div}\,\beta)_{\ell=1}| \lesssim \frac{\epsilon_0}{r^5u^{1+\delta_{dec}}}. \tag{5.110}$$

Step 7. We provide the estimate for $(\mathrm{curl}\,\beta)_{\ell=1}$. Recall from Corollary 5.41 that we have along Σ_*, for $p=0,+,-$,

$$\nu\left(\int_S \mathrm{curl}\,\beta J^{(p)}\right) = \frac{4}{r}(1+O(r^{-1}))\int_S \mathrm{curl}\,\beta J^{(p)}$$

$$+\frac{2}{r^2}(1+O(r^{-1}))\int_S\left({}^*\rho-\frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}}\right)J^{(p)}$$
$$+r\left(\left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right|+\left|\not{d}_2^*\not{d}_1^*J^{(p)}\right|\right)\Gamma_g+\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_g).$$

In the case $p=\pm$, since we have $(\mathrm{curl}\,\beta)_{\ell=1,\pm}=0$ on S_*, using (5.109) to control $({}^*\rho-\frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}})_{\ell=1,\pm}$, and arguing exactly as for the control of $(\mathrm{div}\,\beta)_{\ell=1}$ in Step 6, we obtain the following analog of (5.110)

$$\left|(\mathrm{curl}\,\beta)_{\ell=1,\pm}\right| \lesssim \frac{\epsilon_0}{r^5u^{1+\delta_{dec}}}. \tag{5.111}$$

Next, we focus on the case $p=0$. We rewrite the above transport equation in this particular case

$$\begin{aligned}\nu\left(\int_S\mathrm{curl}\,\beta J^{(0)}\right) &= \frac{4}{r}(1+O(r^{-1}))\int_S\mathrm{curl}\,\beta J^{(0)}\\ &\quad+\frac{2}{r^2}(1+O(r^{-1}))\int_S\left({}^*\rho-\frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}}\right)J^{(0)}\\ &\quad+r\left(\left|\left(\Delta+\frac{2}{r^2}\right)J^{(0)}\right|+\left|\not{d}_2^*\not{d}_1^*J^{(0)}\right|\right)\Gamma_g+\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_g).\end{aligned}$$

Since $\nu(r)=-2+r\Gamma_b$, we have

$$\begin{aligned}\nu\left(r^3\int_S\mathrm{curl}\,\beta J^{(0)}\right) &= r^3\nu\left(\int_S\mathrm{curl}\,\beta J^{(0)}\right)+3r^2\nu(r)\int_S\mathrm{curl}\,\beta J^{(0)}\\ &= r^3\nu\left(\int_S\mathrm{curl}\,\beta J^{(0)}\right)-6r^2\int_S\mathrm{curl}\,\beta J^{(0)}\\ &\quad+r^5\Gamma_b(\mathrm{curl}\,\beta)_{\ell=1,0},\end{aligned}$$

and hence

$$\begin{aligned}&\nu\left(r^3\int_S\mathrm{curl}\,\beta J^{(0)}\right)\\ =\;& -\frac{2}{r}r^3(1+O(r^{-1}))\int_S\mathrm{curl}\,\beta J^{(0)}\\ &+2r(1+O(r^{-1}))\int_S\left({}^*\rho-\frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}}\right)J^{(0)}+r^5\Gamma_b(\mathrm{curl}\,\beta)_{\ell=1,0}\\ &+r^4\left(\left|\left(\Delta+\frac{2}{r^2}\right)J^{(0)}\right|+\left|\not{d}_2^*\not{d}_1^*J^{(0)}\right|\right)\Gamma_g+r^3\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_g)\end{aligned}$$

which we rewrite

$$\nu\left(r^3\int_S \mathrm{curl}\,\beta J^{(0)} - 8\pi am\right) = -\frac{2}{r}\left(r^3\int_S \mathrm{curl}\,\beta J^{(0)} - 8\pi am\right) + h_4$$

where the scalar function h_4 is given by

$$\begin{aligned} h_4 &= O(r^3)\left(\left({}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}}\right)_{\ell=1,0} - \frac{2am}{r^4}\right) + O(r^3)(\mathrm{curl}\,\beta J^{(0)})_{\ell=1,0} \\ &\quad + O(r^2)\left({}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}}\right)_{\ell=1,0} + r^5\Gamma_b(\mathrm{curl}\,\beta)_{\ell=1,0} \\ &\quad + r^4\left(\left|\left(\Delta + \frac{2}{r^2}\right)J^{(0)}\right| + \left|\not{d}_2^*\not{d}_1^* J^{(0)}\right|\right)\Gamma_g + r^3\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_g). \end{aligned}$$

In view of the control of $({}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}})_{\ell=1}$ in (5.109), the local bootstrap assumption (5.104) on $(\mathrm{curl}\,\beta)_{\ell=1,0}$, the control of $(\Delta+2/r^2)J^{(p)}$ and $\not{d}_2^*\not{d}_1^* J^{(p)}$ provided by Corollary 5.45, and the control of Γ_g provided by **Ref 1**, we obtain

$$|h_4| \lesssim \frac{1}{r^2} + \frac{\epsilon_0}{u^{2+2\delta_{dec}}} + \frac{\epsilon}{ru^{1+\delta_{dec}}} + \frac{r\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}|\not{\mathfrak{d}}^{\leq 1}\Gamma_b|.$$

Using the dominance condition (5.30) on r on Σ_*, we infer

$$|h_4| \lesssim \frac{\epsilon_0}{u^{2+2\delta_{dec}}} + \frac{r\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}|\not{\mathfrak{d}}^{\leq 1}\Gamma_b|.$$

By integration in u, and using Sobolev, we obtain

$$\begin{aligned} \int_u^{u_*}|h_4| &\lesssim \frac{\epsilon_0}{u^{1+\delta_{dec}}} + \int_u^{u_*}\frac{\epsilon}{u'^{\frac{1}{2}+\delta_{dec}}}\|\not{\mathfrak{d}}^{\leq 3}\Gamma_b\|_{L^2(S)} \\ &\lesssim \frac{\epsilon_0}{u^{1+\delta_{dec}}} + \frac{\epsilon}{u^{1+2\delta_{dec}}}\left(\int_{\Sigma_*} u^{2+2\delta_{dec}}|\not{\mathfrak{d}}^{\leq 3}\Gamma_b|^2\right)^{\frac{1}{2}}. \end{aligned}$$

Together with the control of Γ_b in Proposition 5.42, we infer

$$\int_u^{u_*}|h_4| \lesssim \frac{\epsilon_0}{u^{1+\delta_{dec}}}.$$

Since

$$\nu\left(r^3\int_S \mathrm{curl}\,\beta J^{(0)} - 8\pi am\right) = -\frac{2}{r}\left(r^3\int_S \mathrm{curl}\,\beta J^{(0)} - 8\pi am\right) + h_4,$$

we may thus apply Corollary 5.34 which implies, together with the fact that there holds $(\mathrm{curl}\,\beta)_{\ell=1,0} = \frac{2am}{r^5}$ on S_*,

$$r^3 u^{1+\delta_{dec}} \left| \int_S \mathrm{curl}\, \beta J^{(0)} - \frac{8\pi am}{r^3} \right| \lesssim \epsilon_0,$$

and hence, together with (5.111), we have obtained

$$\left|(\mathrm{curl}\, \beta)_{\ell=1,\pm}\right| + \left|(\mathrm{curl}\, \beta)_{\ell=1,0} - \frac{2am}{r^5}\right| \lesssim \frac{\epsilon_0}{r^5 u^{1+\delta_{dec}}}. \tag{5.112}$$

Remark 5.50. *Note that (5.108) for $(\check{\underline{\kappa}})_{\ell=1}$, (5.110) for $(div\, \beta)_{\ell=1}$, and (5.112) for $(curl\, \zeta)_{\ell=1}$, improve the local bootstrap assumptions (5.104).*

Step 8. We have by the definition of the mass aspect function μ

$$\check{\mu} = -\mathrm{div}\, \zeta - \left(\check{\rho} - \frac{1}{2}\widehat{\chi} \cdot \widehat{\underline{\chi}}\right)$$

and hence

$$(\check{\mu})_{\ell=1} = -(\mathrm{div}\, \zeta)_{\ell=1} - \left(\check{\rho} - \frac{1}{2}\widehat{\chi} \cdot \widehat{\underline{\chi}}\right)_{\ell=1}.$$

Together with the estimates (5.105) for $(\mathrm{div}\, \zeta)_{\ell=1}$ and (5.107) for $(\check{\rho} - \frac{1}{2}\widehat{\chi} \cdot \widehat{\underline{\chi}})_{\ell=1}$, we infer

$$|(\check{\mu})_{\ell=1}| \lesssim \frac{\epsilon_0}{r^3 u^{2+2\delta_{dec}}} + \frac{\epsilon}{r^4 u^{1+\delta_{dec}}}.$$

Using the dominance condition (5.30) on r on Σ_*, we deduce

$$|(\check{\mu})_{\ell=1}| \lesssim \frac{\epsilon_0}{r^3 u^{2+2\delta_{dec}}}. \tag{5.113}$$

Step 9. It remains to derive estimates for $(\nu \mathrm{div}\, \beta)_{\ell=1}$ and $(\nu \mathrm{curl}\, \beta)_{\ell=1}$. In view of Lemma 5.25, we have along Σ_*

$$\begin{aligned}
\nu \mathrm{div}\, \beta &= O(r^{-1})\mathrm{div}\, \beta + \Delta\rho + (1 + O(r^{-1}))\mathrm{div}\,\mathrm{div}\, \alpha \\
&\quad + O(r^{-3})\mathrm{div}\, \eta + r^{-2}\, \mathfrak{d}^{\leq 1}(\Gamma_b \cdot \Gamma_g), \\
\nu \mathrm{curl}\, \beta &= \frac{8}{r}(1 + O(r^{-1}))\mathrm{curl}\, \beta - \Delta\ {}^*\rho + (1 + O(r^{-1}))\mathrm{curl}\,\mathrm{div}\, \alpha \\
&\quad + O(r^{-3})\ {}^*\rho + r^{-2}\, \mathfrak{d}^{\leq 1}(\Gamma_b \cdot \Gamma_g),
\end{aligned}$$

where the notation $O(r^a)$, for $a \in \mathbb{R}$, denotes an explicit function of r which is bounded by r^a as $r \to +\infty$. We infer

$$
\begin{aligned}
(\nu \mathrm{div}\, \beta)_{\ell=1} &= O(r^{-1})(\mathrm{div}\, \beta)_{\ell=1} + (\Delta\rho)_{\ell=1} + (1+O(r^{-1}))(\mathrm{div}\,\mathrm{div}\,\alpha)_{\ell=1}\\
&+ O(r^{-3})(\mathrm{div}\,\eta)_{\ell=1} + r^{-2}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g),\\
(\nu \mathrm{curl}\, \beta)_{\ell=1} &= \frac{8}{r}(1+O(r^{-1}))(\mathrm{curl}\,\beta)_{\ell=1} - (\Delta\ {}^*\rho)_{\ell=1}\\
&+ (1+O(r^{-1}))(\mathrm{curl}\,\mathrm{div}\,\alpha)_{\ell=1} + O(r^{-3})(\ {}^*\rho)_{\ell=1}\\
&+ r^{-2}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g).
\end{aligned}
$$

Using the fact that $(\mathrm{div}\,\eta)_{\ell=1} = 0$, and integrating by parts Δ, $\mathrm{div}\,\mathrm{div}$ and $\mathrm{curl}\,\mathrm{div}$, we obtain

$$
\begin{aligned}
& |(\nu \mathrm{div}\,\beta)_{\ell=1}| + |(\nu \mathrm{curl}\,\beta)_{\ell=1}|\\
\lesssim\ & r^{-1}|(\mathrm{div}\,\beta)_{\ell=1}| + r^{-1}|(\mathrm{curl}\,\beta)_{\ell=1}| + r^{-2}\left|\left(\check{\rho} - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)_{\ell=1}\right|\\
& + r^{-2}\left|\left(\ {}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}}\right)_{\ell=1}\right|\\
& + r^{-1}\left(\left|\left(\Delta + \frac{2}{r^2}\right)J^{(p)}\right| + \left|\,{\not{d}}_2^*{\not{d}}_1^* J^{(p)}\right|\right)|\Gamma_g| + r^{-2}|\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g)|.
\end{aligned}
$$

Together with the above estimates, we infer

$$
|(\nu \mathrm{div}\,\beta)_{\ell=1}| + |(\nu \mathrm{curl}\,\beta)_{\ell=1}| \lesssim \frac{1}{r^6} + \frac{\epsilon_0}{r^5 u^{\frac{3}{2}+2\delta_{dec}}}.
$$

Using the dominance condition (5.30) on r on Σ_*, we deduce

$$
|(\nu \mathrm{div}\,\beta)_{\ell=1}| + |(\nu \mathrm{curl}\,\beta)_{\ell=1}| \lesssim \frac{\epsilon_0}{r^5 u^{1+\delta_{dec}}}. \tag{5.114}
$$

This concludes the proof of Proposition 5.48. □

5.4.3. Estimates for $\ell = 0$ modes on Σ_* In this section, we control the average (i.e. the $\ell = 0$ mode) of $\check{\underline{\kappa}}$, $\check{\rho}$, ${}^*\rho$ and $\check{\mu}$. Recall the definition of the Hawking mass

$$
\frac{2m_H}{r} = 1 + \frac{1}{16\pi}\int_S \kappa\underline{\kappa}.
$$

In order to control $\ell = 0$ modes on Σ_*, we will need in particular to compare the Hawking mass m_H with the constant m. To this end, we will rely on the following lemma.

Lemma 5.51. *We have*

$$\overline{\rho} = -\frac{2m_H}{r^3} + \Gamma_b \cdot \Gamma_g \tag{5.115}$$

and

$$\nu(m_H) \quad = \quad r^2 \not{\mathfrak{d}}^{\leq 1}(\Gamma_b \cdot \Gamma_b). \tag{5.116}$$

Proof. We start with the identity for the average of ρ. Recall the Gauss equation

$$K \quad = \quad -\rho - \frac{1}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + \frac{1}{2}\widehat{\chi} \cdot \underline{\widehat{\chi}}.$$

Integrating on S, and using the definition of the Hawking mass m_H, we obtain

$$\int_S K \quad = \quad -\int_S \rho - 4\pi\left(\frac{2m_H}{r} - 1\right) + \frac{1}{2}\int_S \widehat{\chi} \cdot \underline{\widehat{\chi}}.$$

Since from Gauss Bonnet we have

$$\int_S K \quad = \quad 4\pi,$$

we infer

$$\int_S \rho \quad = \quad -\frac{8\pi m_H}{r} + \frac{1}{2}\int_S \widehat{\chi} \cdot \underline{\widehat{\chi}}$$

and hence

$$\overline{\rho} = -\frac{2m_H}{r^3} + \Gamma_b \cdot \Gamma_g$$

as stated. Note that this implies

$$\begin{aligned}
\rho + \frac{2m_H}{r^3} \quad &= \quad \left(\overline{\rho} + \frac{2m_H}{r^3}\right) + \rho - \overline{\rho} = \left(\overline{\rho} + \frac{2m_H}{r^3}\right) + \left(\rho + \frac{2m}{r^3}\right) - \overline{\rho + \frac{2m}{r^3}} \\
&= \quad r^{-1}\Gamma_g + \Gamma_b \cdot \Gamma_g = r^{-1}\Gamma_g
\end{aligned}$$

so that $\rho + \frac{2m_H}{r^3} \in r^{-1}\Gamma_g$.

Next, we focus on the identity for $\nu(m_H)$. From the null structure equations, we have on Σ_*

$$e_3(\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}) \quad = \quad \mathrm{tr}\,\chi\left(-\frac{1}{2}\mathrm{tr}\,\underline{\chi}^2 - 2\underline{\omega}\mathrm{tr}\,\underline{\chi} + 2\mathrm{div}\,\underline{\xi} + 2\underline{\xi}\cdot(\eta - 3\zeta) - |\underline{\widehat{\chi}}|^2\right)$$

$$
\begin{aligned}
& +\operatorname{tr}\underline{\chi}\left(-\frac{1}{2}\operatorname{tr}\underline{\chi}\operatorname{tr}\chi+2\underline{\omega}\operatorname{tr}\chi+2\operatorname{div}\eta+2|\eta|^2+2\rho-\widehat{\underline{\chi}}\cdot\widehat{\chi}\right)\\
= & -\operatorname{tr}\chi\operatorname{tr}\underline{\chi}^2+2\operatorname{tr}\underline{\chi}\rho+2\operatorname{tr}\chi\operatorname{div}\underline{\xi}+2\operatorname{tr}\underline{\chi}\operatorname{div}\eta\\
& +\operatorname{tr}\chi\left(2\underline{\xi}\cdot(\eta-3\zeta)-|\widehat{\underline{\chi}}|^2\right)+\operatorname{tr}\underline{\chi}\left(2|\eta|^2-\widehat{\underline{\chi}}\cdot\widehat{\chi}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
e_4(\operatorname{tr}\chi\operatorname{tr}\underline{\chi}) = & \operatorname{tr}\chi\left(-\frac{1}{2}\operatorname{tr}\chi\operatorname{tr}\underline{\chi}-2\operatorname{div}\zeta+2|\zeta|^2+2\rho-\widehat{\chi}\cdot\widehat{\underline{\chi}}\right)\\
& +\operatorname{tr}\underline{\chi}\left(-\frac{1}{2}\operatorname{tr}\chi^2-|\widehat{\chi}|^2\right)\\
= & -\operatorname{tr}\chi^2\operatorname{tr}\underline{\chi}+2\operatorname{tr}\chi\rho-2\operatorname{tr}\chi\operatorname{div}\zeta+\operatorname{tr}\chi\left(2|\zeta|^2-\widehat{\chi}\cdot\widehat{\underline{\chi}}\right)\\
& -\operatorname{tr}\underline{\chi}|\widehat{\chi}|^2.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
\nu(\operatorname{tr}\chi\operatorname{tr}\underline{\chi}) = & e_3(\operatorname{tr}\chi\operatorname{tr}\underline{\chi})+b_*e_4(\operatorname{tr}\chi\operatorname{tr}\underline{\chi})\\
= & -\operatorname{tr}\chi\operatorname{tr}\underline{\chi}(\operatorname{tr}\underline{\chi}+b_*\operatorname{tr}\chi)+2(\operatorname{tr}\underline{\chi}+b_*\operatorname{tr}\chi)\rho+2\operatorname{tr}\chi\operatorname{div}\underline{\xi}\\
& +2\operatorname{tr}\underline{\chi}\operatorname{div}\eta+\operatorname{tr}\chi\left(2\underline{\xi}\cdot(\eta-3\zeta)-|\widehat{\underline{\chi}}|^2\right)\\
& +\operatorname{tr}\underline{\chi}\left(2|\eta|^2-\widehat{\underline{\chi}}\cdot\widehat{\chi}\right)\\
& +b_*\Big\{-2\operatorname{tr}\chi\operatorname{div}\zeta+\operatorname{tr}\chi\left(2|\zeta|^2-\widehat{\chi}\cdot\widehat{\underline{\chi}}\right)-\operatorname{tr}\underline{\chi}|\widehat{\chi}|^2\Big\},
\end{aligned}
$$

which we rewrite

$$
\begin{aligned}
\nu(\operatorname{tr}\chi\operatorname{tr}\underline{\chi})+\operatorname{tr}\chi\operatorname{tr}\underline{\chi}(\operatorname{tr}\underline{\chi}+b_*\operatorname{tr}\chi) = & 2(\operatorname{tr}\underline{\chi}+b_*\operatorname{tr}\chi)\rho+\frac{4}{r}\operatorname{div}\underline{\xi}\\
& -\frac{4\Upsilon}{r}\operatorname{div}\eta+\frac{4\left(1+\frac{2m}{r}\right)}{r}\operatorname{div}\zeta\\
& +r^{-1}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_b\cdot\Gamma_b).
\end{aligned}
$$

Together with Lemma 5.31, we infer

$$
\begin{aligned}
\nu\left(\int_S\operatorname{tr}\chi\operatorname{tr}\underline{\chi}\right) = & z\int_S\frac{1}{z}\Big(\nu(\operatorname{tr}\chi\operatorname{tr}\underline{\chi})+(\underline{\kappa}+b_*\kappa)\operatorname{tr}\chi\operatorname{tr}\underline{\chi}\Big)\\
= & z\int_S\frac{1}{z}\Big\{2(\operatorname{tr}\underline{\chi}+b_*\operatorname{tr}\chi)\rho+\frac{4}{r}\operatorname{div}\underline{\xi}-\frac{4\Upsilon}{r}\operatorname{div}\eta\\
& +\frac{4\left(1+\frac{2m}{r}\right)}{r}\operatorname{div}\zeta+r^{-1}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_b\cdot\Gamma_b)\Big\}.
\end{aligned}
$$

Integrating by parts the divergences, we deduce

$$\nu\left(\int_S \mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi}\right) = 2z\int_S \frac{1}{z}(\mathrm{tr}\,\underline{\chi} + b_* \mathrm{tr}\,\chi)\rho + r\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

Thus, in view of the definition of the Hawking mass,

$$\frac{2m_H}{r} = 1 + \frac{1}{16\pi}\int_S \mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi},$$

we infer

$$\nu\left(\frac{2m_H}{r}\right) = \frac{z}{8\pi}\int_S \frac{1}{z}(\mathrm{tr}\,\underline{\chi} + b_* \mathrm{tr}\,\chi)\rho + r\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

On the other hand, we have

$$\nu\left(\frac{2m_H}{r}\right) = \frac{2\nu(m_H)}{r} - \frac{2m_H\nu(r)}{r^2}$$

and hence, using again Lemma 5.31,

$$\nu\left(\frac{2m_H}{r}\right) = \frac{2\nu(m_H)}{r} - \frac{2m_H}{r^2}\frac{z}{8\pi r}\int_S \frac{1}{z}(\mathrm{tr}\,\underline{\chi} + b_* \mathrm{tr}\,\chi)$$

which yields

$$\nu(m_H) = \frac{rz}{16\pi}\int_S \frac{1}{z}(\mathrm{tr}\,\underline{\chi} + b_* \mathrm{tr}\,\chi)\left(\rho + \frac{2m_H}{r^3}\right) + r^2\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

Recalling from above that we have $\rho + \frac{2m_H}{r^3} \in r^{-1}\Gamma_g$, we infer

$$\nu(m_H) = -\frac{1}{4\pi}\int_S \left(\rho + \frac{2m_H}{r^3}\right) + r^2\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

Together with the above control of the average of $\rho + \frac{2m_H}{r^3}$, we deduce

$$\nu(m_H) = r^2\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b)$$

as desired. This concludes the proof of Lemma 5.51. □

Proposition 5.52 (Control of $\ell = 0$ modes on Σ_*)**.** *We have on* Σ_*

$$\sup_{\Sigma_*} u^{1+2\delta_{dec}}\left(|m_H - m| + r^2\left|\overline{\underline{\check{\kappa}}}\right| + r^3\left|\overline{\check{\rho}}\right| + r^3\left|\overline{\check{\mu}}\right| + r^3\left|\overline{{}^*\rho}\right|\right) \lesssim \epsilon_0. \tag{5.117}$$

Proof. Recall from Lemma 5.51 that we have

$$\nu(m_H) \quad = \quad r^2 \mathfrak{d}^{\leq 1}(\Gamma_b \cdot \Gamma_b).$$

Together with the control of Γ_b provided by **Ref 1**, and since m is a constant, we infer

$$|\nu(m_H - m)| \quad \lesssim \quad \frac{\epsilon^2}{u^{2+2\delta_{dec}}} \lesssim \frac{\epsilon_0}{u^{2+2\delta_{dec}}}.$$

Also, recall that by definition of m, we have $m = m_H$ on S_*. Integrating from S_*, we deduce on Σ_*

$$|m_H - m| \quad \lesssim \quad \frac{\epsilon_0}{u^{1+2\delta_{dec}}}$$

as desired.

Next, recall from Lemma 5.51 that we have

$$\overline{\rho} = -\frac{2m_H}{r^3} + \Gamma_b \cdot \Gamma_g.$$

Together with the above control for $m_H - m$, and the control of Γ_b and Γ_g provided by **Ref 1**, we deduce

$$\left|\overline{\check{\rho}}\right| = \left|\overline{\rho} + \frac{2m}{r^3}\right| \quad \lesssim \quad \frac{\epsilon_0}{r^3 u^{1+2\delta_{dec}}}$$

as desired.

Next, taking the average of

$$\mathrm{curl}\,\zeta \quad = \quad {}^*\rho - \frac{1}{2}\widehat{\chi} \wedge \underline{\widehat{\chi}},$$

we infer

$$\overline{{}^*\rho} \quad = \quad \frac{1}{2}\overline{\widehat{\chi} \wedge \underline{\widehat{\chi}}} = \Gamma_b \cdot \Gamma_g,$$

and the conclusion follows from the control of Γ_b and Γ_g provided by **Ref 1**.

Next, using the definition of the Hawking mass and the GCM condition for κ on Σ_*, we have

$$\frac{2m_H}{r} \quad = \quad 1 + \frac{1}{16\pi}\int_S \kappa\underline{\kappa} = 1 + \frac{1}{8\pi r}\int_S \underline{\kappa} = 1 + \frac{r}{2}\overline{\underline{\kappa}}$$

and hence

$$\underline{\overline{\kappa}} = -\frac{2}{r}\left(1-\frac{2m_H}{r}\right) = -\frac{2\Upsilon}{r}+\frac{4}{r^2}(m_H-m).$$

Together with the above control for $m_H - m$, we deduce

$$\left|\underline{\overline{\kappa}}+\frac{2\Upsilon}{r}\right| \lesssim \frac{\epsilon_0}{r^2u^{1+2\delta_{dec}}}$$

as desired.

Finally, we consider $\overline{\mu}$. We have by definition of μ, and in view of Gauss equation,

$$\begin{aligned}\mu &= -\mathrm{div}\,\zeta-\rho+\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\\ &= -\mathrm{div}\,\zeta+K+\frac{1}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}.\end{aligned}$$

Integrating, and using integration by parts, we obtain

$$\int_S\mu = \int_S K+\frac{1}{4}\int_S\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}.$$

Using Gauss Bonnet and the definition of the Hawking mass, we deduce

$$\begin{aligned}\int_S\mu &= 4\pi+4\pi\left(\frac{2m_H}{r}-1\right)=\frac{8\pi m_H}{r}\\ &= \frac{8\pi m}{r}+\frac{8\pi(m_H-m)}{r}.\end{aligned}$$

Together with the above control for $m_H - m$, we deduce

$$\left|\overline{\mu}-\frac{2m}{r^3}\right| \lesssim \frac{\epsilon_0}{r^3u^{1+2\delta_{dec}}}$$

as desired. This concludes the proof of the proposition. □

5.5. Proof of Theorem M3

We are now ready to prove Theorem M3.

Proposition 5.53. *We have along Σ_*, for all $k \leq k_* - 12$,*

$$
\begin{aligned}
\big|\mathfrak{d}_*^{\le k}\Gamma_b\big| &\lesssim \epsilon_0 r^{-1}u^{-1-\delta_{dec}},\\
\big|\mathfrak{d}_*^{\le k}\Gamma_g\big| &\lesssim \epsilon_0 r^{-2}u^{-\frac{1}{2}-\delta_{dec}},\\
\big|\mathfrak{d}_*^{\le k-1}\nabla_\nu\Gamma_g\big| &\lesssim \epsilon_0 r^{-2}u^{-1-\delta_{dec}}.
\end{aligned}
\tag{5.118}
$$

Moreover, for all $k\le k_*-12$,

$$
\begin{aligned}
\big|\mathfrak{d}_*^{\le k}\check{\kappa}\big| &\lesssim \epsilon_0 r^{-2}u^{-1-\delta_{dec}},\\
\big|\mathfrak{d}_*^{\le k}\check{\mu}\big| &\lesssim \epsilon_0 r^{-3}u^{-1-\delta_{dec}},\\
\big|\mathfrak{d}_*^{\le k-1}\nabla_\nu\alpha\big| &\lesssim \epsilon_0 r^{-\frac{9}{2}-\delta_{extra}},\\
\big|\mathfrak{d}_*^{\le k-1}\nabla_\nu\beta\big| &\lesssim \epsilon_0 r^{-4}u^{-\frac{1}{2}-\delta_{dec}}.
\end{aligned}
\tag{5.119}
$$

Remark 5.54. *Note that (5.118) yields the proof of Theorem M3. Indeed, in view of the definition of the decay norm ${}^*\mathfrak{D}_k$ in Section 3.3.1, (5.118) can be rewritten as ${}^*\mathfrak{D}_k\lesssim\epsilon_0$ for $k\le k_*-12$. Thus, since $k_*=k_{small}+80$ in view of (5.1), Proposition 5.53 yields in particular ${}^*\mathfrak{D}_{k_{small}+60}\lesssim\epsilon_0$ and thus concludes the proof of Theorem M3.*

Proof. Note that the estimates for Γ_b have already been established in Proposition 5.42. Note also that $\check{\kappa}=0$ in view of our GCM conditions, and that the estimate for α[123] has already been established in (5.42). Thus, it only remains to control the following quantities

$$
\check{\underline{\kappa}},\quad \widehat{\chi},\quad \zeta,\quad \check{\rho},\quad {}^*\rho,\quad \check{\mu},\quad \beta.
$$

We control these quantities as follows, starting first with estimates for angular derivatives.

Step 1. We start with $\not{\mathfrak{d}}^k\check{\underline{\kappa}}$ and $\not{\mathfrak{d}}^k\check{\mu}$. Recall from our GCM conditions that we have on Σ_*

$$
\check{\underline{\kappa}}=\underline{C}_0+\sum_p\underline{C}_pJ^{(p)},\qquad \check{\mu}=M_0+\sum_pM_pJ^{(p)}.
$$

Differentiating w.r.t. $\not{d}_2^*\not{d}_1^*$, and recalling that $\underline{C}_0$, $\underline{C}_p$, M_0 and M_p are constant on the spheres S, we infer

$$
\not{d}_2^*\not{d}_1^*\check{\underline{\kappa}}=\sum_p\underline{C}_p\not{d}_2^*\not{d}_1^*J^{(p)},\qquad \not{d}_2^*\not{d}_1^*\check{\mu}=\sum_pM_p\not{d}_2^*\not{d}_1^*J^{(p)},
$$

[123] But not the one for $\nabla_\nu\alpha$ in (5.119) which is in fact derived in Step 4.

which yields, for $k \geq 0$,

$$\begin{aligned} \| \not{d}_2^* \not{d}_1^* \check{\underline{\kappa}} \|_{\mathfrak{h}_k(S)} &\lesssim r \sum_p |\underline{C}_p| \| \not{\mathfrak{d}}^{\leq k} \not{d}_2^* \not{d}_1^* J^{(p)} \|_{L^\infty(S)}, \\ \| \not{d}_2^* \not{d}_1^* \check{\mu} \|_{\mathfrak{h}_k(S)} &\lesssim r \sum_p |M_p| \| \not{\mathfrak{d}}^{\leq k} \not{d}_2^* \not{d}_1^* J^{(p)} \|_{L^\infty(S)}. \end{aligned}$$

Together with Corollary 5.38, we infer

$$\begin{aligned} \|\check{\underline{\kappa}}\|_{\mathfrak{h}_{k+2}(S)} &\lesssim r^3 \sum_p |\underline{C}_p| \| \not{\mathfrak{d}}^{\leq k} \not{d}_2^* \not{d}_1^* J^{(p)} \|_{L^\infty(S)} + r|(\check{\underline{\kappa}})_{\ell=1}| + r|\overline{\check{\underline{\kappa}}}|, \\ \|\check{\mu}\|_{\mathfrak{h}_{k+2}(S)} &\lesssim r^3 \sum_p |M_p| \| \not{\mathfrak{d}}^{\leq k} \not{d}_2^* \not{d}_1^* J^{(p)} \|_{L^\infty(S)} + r|(\check{\mu})_{\ell=1}| + r|\overline{\check{\mu}}|. \end{aligned}$$

In view of the control of the $\ell = 1$ mode of $\check{\underline{\kappa}}$ and $\check{\mu}$ in Proposition 5.48, the control of the average of $\check{\underline{\kappa}}$ and $\check{\mu}$ in Proposition 5.52, the fact that $\underline{C}_p \in \Gamma_g$ and $M_p \in r^{-1}\Gamma_g$ in view of Corollary 5.39, the control of Γ_g provided by **Ref 1**, and the control of $\not{d}_2^* \not{d}_1^* J^{(p)}$ provided by Corollary 5.45, we deduce, for $k \leq k_* - 10$,

$$\|\check{\underline{\kappa}}\|_{\mathfrak{h}_{k+2}(S)} \lesssim \frac{\epsilon_0}{r u^{1+\delta_{dec}}}, \qquad \|\check{\mu}\|_{\mathfrak{h}_{k+2}(S)} \lesssim \frac{\epsilon_0}{r^2 u^{1+\delta_{dec}}}.$$

Together with Sobolev, this implies, for $k \leq k_* - 10$,

$$|\not{\mathfrak{d}}^k \check{\underline{\kappa}}| \lesssim \frac{\epsilon_0}{r^2 u^{1+\delta_{dec}}}, \qquad |\not{\mathfrak{d}}^k \check{\mu}| \lesssim \frac{\epsilon_0}{r^3 u^{1+\delta_{dec}}}. \tag{5.120}$$

Step 2. Next, we focus on $\not{\mathfrak{d}}^k \check{\rho}$ and $\not{\mathfrak{d}}^k \ {}^*\rho$. Recall from Proposition 5.26 that we have

$$\Re(\mathfrak{q}) = r^4 \not{d}_2^* \not{d}_1^* (-\rho, \ {}^*\rho) + O(r^{-2}) + \not{\mathfrak{d}}^{\leq 2}\Gamma_b + r^2 \not{\mathfrak{d}}^{\leq 2}(\Gamma_b \cdot \Gamma_g).$$

We infer, for $k \geq 0$,

$$\begin{aligned} \| \not{d}_2^* \not{d}_1^* (-\check{\rho}, \ {}^*\rho) \|_{\mathfrak{h}_k(S)} \lesssim\ & r^{-3} \| \not{\mathfrak{d}}^{\leq k} \mathfrak{q} \|_{L^\infty(S)} + r^{-5} + r^{-3} \| \not{\mathfrak{d}}^{\leq k+2} \Gamma_b \|_{L^\infty(S)} \\ & + r^{-1} \| \not{\mathfrak{d}}^{\leq k+2} (\Gamma_b \cdot \Gamma_g) \|_{L^\infty(S)}. \end{aligned}$$

In view of **Ref 2** for $\mathfrak{q}$, the control of Γ_b established in Proposition 5.42, and the control of Γ_g provided by **Ref 1**, we obtain, for $k \leq k_* - 12$,

$$\| \not{d}_2^* \not{d}_1^* (-\check{\rho}, \ {}^*\rho) \|_{\mathfrak{h}_k(S)} \lesssim \frac{1}{r^5} + \frac{\epsilon_0}{r^4 u^{\frac{1}{2}+\delta_{dec}}}.$$

Together with the dominance condition (5.30) on r on Σ_*, this yields, for $k\leq k_*-12$,

$$\|\,{\not{d}}_2^*{\not{d}}_1^*(-\check{\rho},\ {}^*\rho)\|_{\mathfrak{h}_k(S)} \lesssim \frac{\epsilon_0}{r^4u^{\frac{1}{2}+\delta_{dec}}}.$$

In view Corollary 5.38, we deduce, for $k\leq k_*-12$,

$$\|(-\check{\rho},\ {}^*\rho)\|_{\mathfrak{h}_{k+2}(S)} \lesssim \frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta_{dec}}}+r|(\check{\rho})_{\ell=1}|+r|({}^*\rho)_{\ell=1}|+r|\overline{\check{\rho}}|+r|\overline{{}^*\rho}|.$$

In view of the control of the $\ell=1$ mode of $\check{\rho}$ and ${}^*\rho$ in Proposition 5.48, and the control of the average of $\check{\rho}$ and ${}^*\rho$ in Proposition 5.52, we infer, for $k\leq k_*-12$,

$$\|\check{\rho}\|_{\mathfrak{h}_{k+2}(S)}\lesssim\frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta_{dec}}},\qquad \|\,{}^*\rho\|_{\mathfrak{h}_{k+2}(S)}\lesssim\frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta_{dec}}}.$$

Together with Sobolev, this implies, for $k\leq k_*-12$,

$$|\,{\not{\mathfrak{d}}}^k\check{\rho}|\lesssim\frac{\epsilon_0}{r^3u^{\frac{1}{2}+\delta_{dec}}},\qquad |\,{\not{\mathfrak{d}}}^k\ {}^*\rho|\lesssim\frac{\epsilon_0}{r^3u^{\frac{1}{2}+\delta_{dec}}}. \tag{5.121}$$

Step 3. Next, we focus on ${\not{\mathfrak{d}}}^k\zeta$. From the definition of μ, and the null structure equation for $\mathrm{curl}\,\zeta$, we have

$$\begin{aligned}{\not{d}}_1\zeta &= \left(-\check{\mu}-\check{\rho}+\frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}},\ {}^*\rho-\frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}}\right)\\ &= (-\check{\mu}-\check{\rho},\ {}^*\rho)+\Gamma_b\cdot\Gamma_g.\end{aligned}$$

In view of Lemma 5.27, we infer, for $k\geq 0$,

$$\|\zeta\|_{\mathfrak{h}_{k+1}(S)} \lesssim r\|\check{\mu}\|_{\mathfrak{h}_k(S)}+r\|\check{\rho}\|_{\mathfrak{h}_k(S)}+r\|\,{}^*\rho\|_{\mathfrak{h}_k(S)}+r^2\|\,{\not{\mathfrak{d}}}^{\leq k}(\Gamma_b\cdot\Gamma_g)\|_{L^\infty(S)}.$$

Together with the control of $\check{\mu}$ derived in Step 1, the control of $(\check{\rho},\ {}^*\rho)$ derived in Step 2, the control of Γ_b established in Proposition 5.42, and the control of Γ_g provided by **Ref 1**, we obtain, for $k\leq k_*-10$,

$$\|\zeta\|_{\mathfrak{h}_{k+1}(S)} \lesssim \frac{\epsilon_0}{ru^{\frac{1}{2}+\delta_{dec}}}.$$

Together with Sobolev, this implies, for $k\leq k_*-11$,

$$|\,{\not{\mathfrak{d}}}^k\zeta|\lesssim\frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta_{dec}}}. \tag{5.122}$$

Step 4. Next, we focus on $\not{\mathfrak{d}}^k\beta$ and $\not{\mathfrak{d}}^k\nabla_\nu\alpha$. Recall the following consequence of Bianchi

$$\not{d}_2^*\beta = \nabla_3\alpha + O(r^{-1})\alpha + O(r^{-3})\Gamma_g + \Gamma_b(\alpha,\beta) + r^{-1}\Gamma_g\cdot\Gamma_g.$$

In view of Lemma 5.28, we infer, for $k\geq 0$,

$$\begin{aligned}\|\beta\|_{\mathfrak{h}_{k+1}(S)} \lesssim\;& r\|\not{\mathfrak{d}}^{\leq k}\nabla_3\alpha\|_{L^2(S)} + r\|\not{\mathfrak{d}}^{\leq k}\alpha\|_{L^\infty(S)} + r^{-1}\|\not{\mathfrak{d}}^{\leq k}\Gamma_g\|_{L^\infty(S)}\\ &+r\|\not{\mathfrak{d}}^{\leq k}(\Gamma_b\cdot(\alpha,\beta))\|_{L^2(S)} + r\|\not{\mathfrak{d}}^{\leq k}(\Gamma_g\cdot\Gamma_g)\|_{L^\infty(S)}\\ &+r^2\big|(\not{d}_1\beta)_{\ell=1}\big|,\end{aligned}$$

which we rewrite in the following form, for $k\leq k_*-1$,

$$\begin{aligned}\|\beta\|_{\mathfrak{h}_{k+1}(S)} \lesssim\;& r^2\left\|\not{\mathfrak{d}}^{\leq k}\left(\nabla_3\alpha + \left(\tilde{f}\widehat{\otimes}\beta - {}^*\tilde{f}\widehat{\otimes}\,{}^*\beta\right)\right)\right\|_{L^\infty(S)} + \epsilon\|\beta\|_{\mathfrak{h}_k(S)}\\ &+r\|\not{\mathfrak{d}}^{\leq k}\alpha\|_{L^\infty(S)} + r^{-1}\|\not{\mathfrak{d}}^{\leq k}\Gamma_g\|_{L^\infty(S)}\\ &+r\|\not{\mathfrak{d}}^{\leq k}(\Gamma_g\cdot\Gamma_g)\|_{L^\infty(S)} + r^2\big|(\not{d}_1\beta)_{\ell=1}\big|,\end{aligned}$$

where $\tilde{f}$ is the 1-form introduced in (5.42), and where we used the fact that $\tilde{f}$ satisfies (5.43). Together with the control of α and $\nabla_3\alpha$ provided by **Ref 2**, the control of Γ_b established in Proposition 5.42, the control of Γ_g provided by Lemma 5.15, and the control of $(\not{d}_1\beta)_{\ell=1}$ in Proposition 5.48, we obtain, for $k\leq k_*-1$,

$$\|\beta\|_{\mathfrak{h}_{k+1}(S)} \lesssim \epsilon\|\beta\|_{\mathfrak{h}_k(S)} + \frac{\epsilon_0}{r^{\frac{5}{2}+\delta_{extra}}} + \frac{1}{r^3}.$$

Absorbing the first term on the RHS for $\epsilon>0$ small enough, and in view of the dominance condition (5.30) for r on Σ_*, this yields, for $k\leq k_*-1$,

$$\|\beta\|_{\mathfrak{h}_{k+1}(S)} \lesssim \frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta_{dec}}}.$$

Together with Sobolev, this implies, for $k\leq k_*-2$,

$$|\not{\mathfrak{d}}^k\beta| \lesssim \frac{\epsilon_0}{r^3u^{\frac{1}{2}+\delta_{dec}}}. \tag{5.123}$$

We also obtain the sharper estimate

$$|\not{\mathfrak{d}}^k\beta| \lesssim \frac{\epsilon_0}{r^{\frac{7}{2}+\delta_{extra}}} + \frac{1}{r^4},$$

and plugging back in (5.42), and using (5.43) as well as the dominance condition (5.30) for r on Σ_*, we moreover obtain, for $k \leq k_* - 2$,

$$|\mathfrak{d}^k \nabla_\nu \alpha| \lesssim \frac{\epsilon_0}{r^{\frac{9}{2}+\delta_{extra}}} + \frac{\epsilon}{r^5} \lesssim \frac{\epsilon_0}{r^{\frac{9}{2}+\delta_{extra}}}. \tag{5.124}$$

Step 5. Next, we focus on $\mathfrak{d}^k \widehat{\chi}$. Recall the following consequence of Codazzi (see Proposition 5.18)

$$\not{d}_2 \widehat{\chi} \;=\; \frac{1}{r}\zeta - \beta + \Gamma_g \cdot \Gamma_g.$$

In view of Lemma 5.27, we infer, for $k \geq 0$,

$$\|\widehat{\chi}\|_{\mathfrak{h}_{k+1}(S)} \;\lesssim\; \|\zeta\|_{\mathfrak{h}_k(S)} + r\|\beta\|_{\mathfrak{h}_k(S)} + r^2\|\mathfrak{d}^{\leq k}(\Gamma_g \cdot \Gamma_g)\|_{L^\infty(S)}.$$

Together with the control of ζ derived in Step 3, the control of β derived in Step 4, and the control of Γ_g provided by **Ref 1**, we obtain, for $k \leq k_* - 10$,

$$\|\widehat{\chi}\|_{\mathfrak{h}_{k+1}(S)} \;\lesssim\; \frac{\epsilon_0}{ru^{\frac{1}{2}+\delta_{dec}}}.$$

Together with Sobolev, this implies, for $k \leq k_* - 11$,

$$|\mathfrak{d}^k \widehat{\chi}| \lesssim \frac{\epsilon_0}{r^2 u^{\frac{1}{2}+\delta_{dec}}}. \tag{5.125}$$

Step 6. In view of Steps 1–5, and the fact that $\check{\kappa} = 0$ in view of our GCM conditions, and that the estimate for α has already been established in Theorem M2, we have obtained, for $k \leq k_* - 12$,

$$|\mathfrak{d}^k \Gamma_g| \lesssim \frac{\epsilon_0}{r^2 u^{\frac{1}{2}+\delta_{dec}}}, \quad |\mathfrak{d}^{\leq k}\check{\kappa}| \lesssim \frac{\epsilon_0}{r^2 u^{1+\delta_{dec}}}, \quad |\mathfrak{d}^{\leq k}\check{\mu}| \lesssim \frac{\epsilon_0}{r^3 u^{1+\delta_{dec}}}. \tag{5.126}$$

Step 7. Next, we estimate $\nabla_\nu \Gamma_g$. From the null structure equations and Bianchi identities, one observes that all quantities in Γ_g verify schematically[124]

$$\nabla_\nu \Gamma_g \;=\; r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b + r^{-1}\Gamma_g + \Gamma_b \cdot \Gamma_b.$$

Together with the control of Γ_b established in Proposition 5.42, and the control of Γ_g provided by **Ref 1**, we infer, for $k \leq k_* - 10$,

[124] This follows by combining the equations for $\nabla_3 \Gamma_g$ and $\nabla_4 \Gamma_g$ of Proposition 5.18, using the fact that $\nu = e_3 + b_* e_4$ and the definition (5.34) of Γ_g.

$$|\mathfrak{d}_*^{\leq k-1}\nabla_\nu\Gamma_g| \lesssim \frac{\epsilon_0}{r^2u^{1+\delta_{dec}}}+\frac{1}{r^3}.$$

In view of the dominance condition (5.30) on r on Σ_*, this yields, for $k\leq k_*-10$,

$$|\mathfrak{d}_*^{\leq k-1}\nabla_\nu\Gamma_g| \lesssim \frac{\epsilon_0}{r^2u^{1+\delta_{dec}}}.$$

Together with the estimates of Step 6, and the control of Γ_b established in Proposition 5.42, we deduce, for $k\leq k_*-12$,

$$\begin{aligned}|\mathfrak{d}_*^{\leq k}\Gamma_b| &\lesssim \epsilon_0 r^{-1}u^{-1-\delta_{dec}}, & |\mathfrak{d}_*^{\leq k}\Gamma_g| &\lesssim \epsilon_0 r^{-2}u^{-\frac{1}{2}-\delta_{dec}},\\ |\mathfrak{d}_*^{\leq k-1}\nabla_\nu\Gamma_g| &\lesssim \epsilon_0 r^{-2}u^{-1-\delta_{dec}}, & |\mathfrak{d}_*^{\leq k}\check{\kappa}| &\lesssim \epsilon_0 r^{-2}u^{-1-\delta_{dec}},\\ |\mathfrak{d}_*^{\leq k}\check{\mu}| &\lesssim \epsilon_0 r^{-3}u^{-1-\delta_{dec}}. \end{aligned} \tag{5.127}$$

Step 8. We conclude the proof with an estimate for $\nabla_\nu\beta$. In view of the Bianchi identities for $\nabla_3\beta$ and $\nabla_4\beta$, and using the fact that $\nu=e_3+b_*e_4$, we have

$$\nabla_\nu\beta = r^{-2}\,\mathfrak{\not{d}}^{\leq 1}\Gamma_g.$$

Together with the estimate for Γ_g derived in Step 7, we infer, for $k\leq k_*-12$,

$$|\mathfrak{d}_*^{\leq k-1}\nabla_\nu\beta| \lesssim \epsilon_0 r^{-4}u^{-\frac{1}{2}-\delta_{dec}}. \tag{5.128}$$

This concludes the proof of Proposition 5.53. ☐

We conclude this section with the following non-sharp corollary of Proposition 5.53, Proposition 5.48 and Corollary 5.41 that will be useful in Section 8.5.

Corollary 5.55. *We have along* Σ_*

$$\begin{aligned}|\nu((div\,\beta)_{\ell=1})|+|\nu((curl\,\beta)_{\ell=1,\pm})|&\\ +\left|\nu\left((curl\,\beta)_{\ell=1,0}-\frac{2am}{r^5}\right)\right| &\lesssim \frac{\epsilon_0}{r^5u^{1+\delta_{dec}}},\\ |\nu((\check{\underline{\kappa}})_{\ell=1})| &\lesssim \frac{\epsilon_0^2}{r^2u^{2+2\delta_{dec}}}+\frac{\epsilon_0}{r^3u^{1+\delta_{dec}}}.\end{aligned}$$

Proof. Recall the following identities derived in Corollary 5.41 along Σ_*, for $p=0,+,-$,

$$\nu\left(\int_S\left(\Delta\check{\underline{\kappa}}+\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right)J^{(p)}\right)$$

$$\begin{aligned} &= O(r^{-3})\int_S \underline{\check{\kappa}} J^{(p)} + O(r^{-2})\int_S \operatorname{div}\zeta J^{(p)} + O(r^{-1})\int_S \operatorname{div}\underline{\beta} J^{(p)} \\ &\quad + O(r^{-2})\int_S \check{\rho} J^{(p)} + O(r^{-1})\int_S \operatorname{div}\beta J^{(p)} + r\left|\left(\Delta + \frac{2}{r^2}\right) J^{(p)}\right| \mathfrak{d}^{\leq 1}\Gamma_b \\ &\quad + \mathfrak{d}^{\leq 2}(\Gamma_b \cdot \Gamma_b), \end{aligned}$$

$$\begin{aligned} \nu\left(\int_S \operatorname{div}\beta J^{(p)}\right) &= O(r^{-1})\int_S \operatorname{div}\beta J^{(p)} + O(r^{-2})\int_S \check{\rho} J^{(p)} \\ &\quad + r\left(\left|\left(\Delta + \frac{2}{r^2}\right) J^{(p)}\right| + \left|{d\!\!/}_2^* {d\!\!/}_1^* J^{(p)}\right|\right)\Gamma_g + \mathfrak{d}^{\leq 1}(\Gamma_b \cdot \Gamma_g), \end{aligned}$$

and

$$\begin{aligned} \nu\left(\int_S \operatorname{curl}\beta J^{(p)}\right) &= \frac{4}{r}(1 + O(r^{-1}))\int_S \operatorname{curl}\beta J^{(p)} \\ &\quad + \frac{2}{r^2}(1 + O(r^{-1}))\int_S {}^*\rho J^{(p)} \\ &\quad + r\left(\left|\left(\Delta + \frac{2}{r^2}\right) J^{(p)}\right| + \left|{d\!\!/}_2^* {d\!\!/}_1^* J^{(p)}\right|\right)\Gamma_g + \mathfrak{d}^{\leq 1}(\Gamma_b \cdot \Gamma_g). \end{aligned}$$

The estimates then easily follow from the control of the $\ell = 1$ modes in Proposition 5.48, the control of Γ_g and Γ_b provided by Proposition 5.53, the control of $(\Delta + \frac{2}{r^2})J^{(p)}$ and ${d\!\!/}_2^* {d\!\!/}_1^* J^{(p)}$ provided by Corollary 5.45 (with ϵ_0 smallness constant instead of ϵ thanks to Proposition 5.53), and the dominance condition for r on Σ_*. □

5.6. Control of $J^{(p)}$ and $\mathfrak{J}$ on Σ_*

Recall that the induced metric g on S_* takes the form

$$g = r^2 e^{2\phi}\Big((d\theta)^2 + \sin^2\theta (d\varphi)^2\Big).$$

For the constructions in Definition 5.56 below, we will rely on a special orthonormal basis (e_1, e_2) of the tangent space of S_* given by

$$e_1 = \frac{1}{re^{\phi}}\partial_\theta, \qquad e_2 = \frac{1}{r\sin\theta e^{\phi}}\partial_\varphi, \quad \text{on} \quad S_*. \tag{5.129}$$

To control the regularity of the basis of $\ell = 1$ modes $J^{(p)}$, $p = 0, +, -$, we introduce the following 1-forms.

Definition 5.56. *Let f_0, f_+ and f_- be the 1-forms defined on S_* by:*

$$(f_0)_1 = 0, \qquad (f_0)_2 = \sin\theta, \qquad (f_+)_1 = \cos\theta\cos\varphi, \qquad (f_+)_2 = -\sin\varphi,$$
$$(f_-)_1 = \cos\theta\sin\varphi, \qquad (f_-)_2 = \cos\varphi, \quad \textit{on} \quad S_*,$$

in the orthonormal basis (e_1, e_2) of S_* given by (5.129), and extended to Σ_* by:

$$\nabla_\nu f_0 = 0, \qquad \nabla_\nu f_+ = 0, \qquad \nabla_\nu f_- = 0.$$

This allows us to renormalize $\nabla(J^{(p)})$, $p = 0, +, -$ on Σ_* as follows.

Definition 5.57. *We introduce the notations*

$$\widetilde{\nabla J^{(0)}} := \nabla J^{(0)} + \frac{1}{r}\,{}^*f_0, \quad \widetilde{\nabla J^{(+)}} := \nabla J^{(+)} - \frac{1}{r}f_+, \quad \widetilde{\nabla J^{(-)}} := \nabla J^{(-)} - \frac{1}{r}f_-.$$

We also introduce the following renormalization for angular derivatives of f_0 and $f_\pm$.

Definition 5.58. *We introduce the notations*

$$\widetilde{curl\,(f_0)} := curl\,(f_0) - \frac{2}{r}\cos\theta, \qquad \widetilde{div\,(f_\pm)} := div\,(f_\pm) + \frac{2}{r}J^{(\pm)}.$$

Finally, note that the complex horizontal 1-form $\mathfrak{J}$ introduced in Definition 3.8 verifies

$$\mathfrak{J} = \frac{1}{|q|}\left(f_0 + i\,{}^*f_0\right) \quad \text{on} \quad \Sigma_*. \tag{5.130}$$

We also introduce the following two complex horizontal 1-forms $\mathfrak{J}_\pm$ given by

$$\mathfrak{J}_\pm := \frac{1}{|q|}\left(f_\pm + i\,{}^*f_\pm\right) \quad \text{on} \quad \Sigma_*, \tag{5.131}$$

as well as the following renormalizations

$$\begin{aligned} \widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}} &:= \overline{\mathcal{D}}\cdot\mathfrak{J} - \frac{4i(r^2+a^2)\cos\theta}{|q|^4}, \\ \widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm} &:= \overline{\mathcal{D}}\cdot\mathfrak{J}_\pm + \frac{4r^2}{|q|^4}J^{(\pm)} + \frac{2ia^2\cos\theta}{|q|^4}J^{(\mp)}. \end{aligned} \tag{5.132}$$

The goal of this section is to prove the following proposition.

Proposition 5.59. *We have on Σ_*, for all $k \leq k_* - 12$*

$$\left|\mathfrak{d}_*^k\left[\widetilde{\nabla J^{(p)}}\right]\right| \lesssim \frac{\epsilon_0}{r^2 u^{\frac{1}{2}+\delta_{dec}}},$$

and on Σ_*, for all $k \leq k_* - 13$

$$\left|\mathfrak{d}_*^k\left[div(f_0),\ \widetilde{curl(f_0)},\ \nabla\widehat{\otimes} f_0,\ \nabla f_0 - \frac{1}{r}\cos\theta \in\right]\right| + \left|\mathfrak{d}_*^k\left[\widetilde{div(f_\pm)},\ curl(f_\pm),\ \nabla\widehat{\otimes} f_\pm,\ \nabla f_\pm + \frac{1}{r}J^{(\pm)}\delta\right]\right| \lesssim \frac{\epsilon_0}{r^2 u^{\frac{1}{2}+\delta_{dec}}},$$

as well as

$$\left|\mathfrak{d}_*^k\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}\right| + \left|\mathfrak{d}_*^k\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm}\right| \lesssim \frac{\epsilon_0}{r^3 u^{\frac{1}{2}+\delta_{dec}}}.$$

5.6.1. Control on S_* First, we derive the following corollary of Lemma 5.35 and Lemma 5.36.

Corollary 5.60. *The following holds on S_*:*

$$\|\not{\mathfrak{d}}^{\leq k_*-12}\phi\|_{L^\infty(S_*)} \lesssim \frac{\epsilon_0}{r u^{\frac{1}{2}+\delta_{dec}}}, \tag{5.133}$$

and

$$\begin{aligned} \int_{S_*} J^{(p)} &= 0,\\ \int_{S_*} J^{(p)}J^{(q)} &= \frac{4\pi}{3}r^2\delta_{pq} + O\left(\epsilon_0 r u^{-\frac{1}{2}-\delta_{dec}}\right),\\ \left\|\not{\mathfrak{d}}^{\leq 2}\left(\Delta + \frac{2}{r^2}\right)J^{(p)}\right\|_{L^\infty(S_*)} &= O\left(\epsilon_0 r^{-3} u^{-\frac{1}{2}-\delta_{dec}}\right). \end{aligned} \tag{5.134}$$

Proof. In view of the Gauss equation, we have $\check{K} \in r^{-1}\Gamma_g$. Together with the estimate for Γ_g in Proposition 5.53, we infer

$$\sup_{S_*}\left|\not{\mathfrak{d}}^{\leq k_*-12}\left(K - \frac{1}{r^2}\right)\right| \lesssim \frac{\epsilon_0}{r^3 u^{\frac{1}{2}+\delta_{dec}}}.$$

The proof follows then immediately from the one of Lemma 5.35 and Lemma 5.36 upon replacing the estimate (5.77) for $\check{K}$ with the above one. □

The following lemma provides identities for first order derivatives of $J^{(p)}$, $p = 0, +, -$.

Lemma 5.61. *We have on S_**

$$\widetilde{\nabla J^{(0)}} = -\frac{1}{r}(e^{-\phi}-1)\,{}^*f_0, \quad \widetilde{\nabla J^{(+)}} = \frac{1}{r}(e^{-\phi}-1)f_+, \tag{5.135}$$
$$\widetilde{\nabla J^{(-)}} = \frac{1}{r}(e^{-\phi}-1)f_-.$$

Proof. Let the orthonormal basis (e_1, e_2) of S_* be given by (5.129). We have

$$\begin{aligned}
e_1(J^{(0)}) &= \frac{1}{re^{\phi}}\partial_\theta(\cos\theta) = -\frac{1}{re^{\phi}}\sin\theta,\\
e_2(J^{(0)}) &= \frac{1}{r\sin\theta e^{\phi}}\partial_\varphi(\cos\theta) = 0,\\
e_1(J^{(+)}) &= \frac{1}{re^{\phi}}\partial_\theta(\sin\theta\cos\varphi) = \frac{1}{re^{\phi}}\cos\theta\cos\varphi,\\
e_2(J^{(+)}) &= \frac{1}{r\sin\theta e^{\phi}}\partial_\varphi(\sin\theta\cos\varphi) = -\frac{1}{re^{\phi}}\sin\varphi,\\
e_1(J^{(-)}) &= \frac{1}{re^{\phi}}\partial_\theta(\sin\theta\sin\varphi) = \frac{1}{re^{\phi}}\cos\theta\sin\varphi,\\
e_2(J^{(-)}) &= \frac{1}{r\sin\theta e^{\phi}}\partial_\varphi(\sin\theta\sin\varphi) = \frac{1}{re^{\phi}}\cos\varphi.
\end{aligned}$$

Together with the definition of f_0 and $f_\pm$, we infer

$$\nabla J^{(0)} = -\frac{1}{re^{\phi}}\,{}^*f_0, \qquad \nabla J^{(+)} = \frac{1}{re^{\phi}}f_+, \qquad \nabla J^{(-)} = \frac{1}{re^{\phi}}f_-.$$

In view of Definition 5.57, this concludes the proof of the lemma. □

The following lemma provides identities for first order derivatives of f_0, f_+ and f_-.

Lemma 5.62. *We have on S_**

$$\begin{aligned}
&\mathrm{div}\,(f_0) = f_0\cdot\nabla\phi, && \mathrm{curl}\,(f_0) = \frac{2}{re^{\phi}}\cos\theta - f_0\wedge\nabla\phi,\\
&\mathrm{div}\,(f_+) = -\frac{2}{re^{\phi}}J^{(+)} + f_+\cdot\nabla\phi, && \mathrm{curl}\,(f_+) = f_+\wedge\nabla\phi,\\
&\mathrm{div}\,(f_-) = -\frac{2}{re^{\phi}}J^{(-)} + f_-\cdot\nabla\phi, && \mathrm{curl}\,(f_-) = -f_-\wedge\nabla\phi,
\end{aligned}$$

and

$$\nabla\widehat{\otimes} f_0 = \begin{pmatrix} f_0\cdot\nabla\phi & f_0\wedge\nabla\phi \\ f_0\wedge\nabla\phi & -f_0\cdot\nabla\phi \end{pmatrix},$$

$$
\nabla\widehat{\otimes}f_+ = \begin{pmatrix} (f_+)_2\nabla_2\phi - (f_+)_1\nabla_1\phi & -(f_+)_1\nabla_2\phi - (f_+)_2\nabla_1\phi \\ -(f_+)_1\nabla_2\phi - (f_+)_2\nabla_1\phi & -(f_+)_2\nabla_2\phi + (f_+)_1\nabla_1\phi \end{pmatrix},
$$
$$
\nabla\widehat{\otimes}f_- = \begin{pmatrix} (f_-)_2\nabla_2\phi - (f_-)_1\nabla_1\phi & -(f_-)_1\nabla_2\phi - (f_-)_2\nabla_1\phi \\ -(f_-)_1\nabla_2\phi - (f_-)_2\nabla_1\phi & -(f_-)_2\nabla_2\phi + (f_-)_1\nabla_1\phi \end{pmatrix}.
$$

In particular, in view of Definition 5.58, we have

$$
\begin{aligned}
\widecheck{curl\,(f_0)} &= \frac{2}{r}\cos\theta(e^{-\phi}-1) - f_0\wedge\nabla\phi,\\
\widecheck{div\,(f_\pm)} &= -\frac{2}{r}J^{(\pm)}(e^{-\phi}-1) + f_\pm\cdot\nabla\phi.
\end{aligned}
$$

Proof. See Section B.6 in the Appendix. □

Lemma 5.63. *On S_*, there holds, for $k\le k_*-12$,*

$$
\left|\not{\mathfrak{d}}^k\Big(\widecheck{\nabla J^{(0)}},\ \widecheck{\nabla J^{(+)}},\ \widecheck{\nabla J^{(-)}}\Big)\right| \lesssim \frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta_{dec}}}.
$$

Also, we have on S_*, for $k\le k_*-13$,

$$
\begin{aligned}
\left|\not{\mathfrak{d}}^k\Big(div\,(f_0),\ \widecheck{curl\,(f_0)},\ \nabla\widehat{\otimes}f_0\Big)\right| &\lesssim \frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta_{dec}}},\\
\left|\not{\mathfrak{d}}^k\Big(\widecheck{div\,(f_\pm)},\ curl\,(f_\pm),\ \nabla\widehat{\otimes}f_\pm\Big)\right| &\lesssim \frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta_{dec}}}.
\end{aligned}
$$

In particular, we have on S_*, for $k\le k_*-13$,

$$
\left|\not{\mathfrak{d}}^k\Big(\nabla f_0 - \frac{1}{r}\cos\theta\in,\ \nabla f_\pm + \frac{1}{r}J^{(\pm)}\delta\Big)\right| \lesssim \frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta_{dec}}}.
$$

Proof. The proof follows immediately from the identities of Lemma 5.61 and Lemma 5.62 together with the control of ϕ provided by (5.133). □

5.6.2. Proof of Proposition 5.59

We start with the following lemma.

Lemma 5.64. *We have on Σ_**

$$
\begin{aligned}
|f_+|^2 &= (\cos\theta)^2(\cos\varphi)^2 + (\sin\varphi)^2, \quad |f_-|^2 = (\cos\theta)^2(\sin\varphi)^2 + (\cos\varphi)^2,\\
|f_0|^2 &= (\sin\theta)^2, \quad f_+\cdot f_0 = -J^{(-)}, \quad f_-\cdot f_0 = J^{(+)},\\
f_+\cdot f_- &= -(\sin\theta)^2\cos\varphi\sin\varphi.
\end{aligned}
$$

Proof. Since $\nabla_\nu f_\pm = 0$, $\nabla_\nu f_0 = 0$, $\nu(J^{(\pm)}) = 0$, and $\nu(\theta) = \nu(\varphi) = 0$ on Σ_*, it suffices to prove these identities on S_* which follows immediately from the definition of $f_\pm$, f_0 and $J^{(\pm)}$ on S_*. □

The next lemma relates angular derivatives of $\mathfrak{J}$ and $\mathfrak{J}_\pm$ with the ones of f_0 and $f_\pm$.

Lemma 5.65. *We have on* Σ_*

$$\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}} = O(r^{-4}) + \frac{2}{|q|}div(f_0) - \frac{2a^2\cos\theta}{|q|^3}f_0\cdot\widetilde{\nabla J^{(0)}} + i\left(\frac{2}{|q|}\widetilde{curl}(f_0) - \frac{2a^2\cos\theta}{|q|^3}f_0\cdot{}^*\widetilde{\nabla J^{(0)}}\right), \tag{5.136}$$

$$\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm} = O(r^{-4}) + \frac{2}{|q|}\widetilde{div(f_\pm)} - \frac{2a^2\cos\theta}{|q|^3}f_\pm\cdot\widetilde{\nabla J^{(0)}} + i\left(\frac{2}{|q|}curl(f_\pm) - \frac{2a^2\cos\theta}{|q|^3}f_\pm\cdot{}^*\widetilde{\nabla J^{(0)}}\right), \tag{5.137}$$

where $O(r^a)$ *denotes, for* $a\in\mathbb{R}$, *a function of* $(r,\cos\theta)$ *bounded by* r^a *as* $r\to+\infty$.

Proof. See Section B.7 in the Appendix. □

The following lemma provides a transport equation for $\nabla J^{(p)}$, ∇f_0 and $\nabla f_\pm$ along Σ_*.

Lemma 5.66. *Assume the following transversality conditions on* Σ_*

$$\nu(\theta) = 0, \qquad \nu(\varphi) = 0, \qquad \nabla_4 f_0 = 0, \qquad \nabla_4 f_+ = 0, \qquad \nabla_4 f_- = 0.$$

Then, we have on Σ_*

$$\nabla_\nu\Big[r\nabla f_0 - \cos\theta \in\Big] = \Gamma_b\cdot \not{\mathfrak{d}}^{\le 1}f_0, \qquad \nabla_\nu\Big[r\nabla f_\pm + J^{(\pm)}\delta\Big] = \Gamma_b\cdot\not{\mathfrak{d}}^{\le 1}f_\pm,$$
$$\nabla_\nu\Big[r\widetilde{\nabla J^{(p)}}\Big] = \Gamma_b\cdot\not{\mathfrak{d}}^{\le 1}J^{(p)}, \quad p = 0,+,-.$$

Proof. Recall from Corollary 5.21 the following commutation formula

$$[\nabla_\nu, r\nabla]f \quad = r\Gamma_b\cdot\nabla_\nu f + \Gamma_b\cdot\mathfrak{d}^{\le 1}f.$$

Applying it to f_0, $f_\pm$, and using the fact that $\nabla_\nu(f_0,f_\pm) = 0$ and $\nabla_4(f_0,f_\pm) = 0$, we infer

$$\nabla_\nu(r\nabla f_0) = \Gamma_b\cdot\not{\mathfrak{d}}^{\le 1}f_0, \qquad \nabla_\nu(r\nabla f_\pm) = \Gamma_b\cdot\not{\mathfrak{d}}^{\le 1}f_\pm,$$

$$\nabla_\nu(r\nabla J^{(p)}) = \Gamma_b\cdot \not{\mathfrak{d}}^{\leq 1}J^{(p)}, \quad p=0,+,-,$$

which yields, since $\nu(\theta)=\nu(\varphi)=0$,

$$\begin{aligned}
&\nabla_\nu\Big[r\nabla f_0-\cos\theta \in\Big] = \Gamma_b\cdot \not{\mathfrak{d}}^{\leq 1}f_0, \qquad \nabla_\nu\Big[r\nabla f_\pm + J^{(\pm)}\delta\Big] = \Gamma_b\cdot \not{\mathfrak{d}}^{\leq 1}f_\pm,\\
&\nabla_\nu\Big[r\widetilde{\nabla J^{(p)}}\Big] = \Gamma_b\cdot \not{\mathfrak{d}}^{\leq 1}J^{(p)}, \quad p=0,+,-,
\end{aligned}$$

as stated. This concludes the proof of the lemma. □

We are now ready to prove Proposition 5.59.

Proof of Proposition 5.59. Recall from Lemma 5.66 that we have on Σ_*

$$\begin{aligned}
&\nabla_\nu\Big[r\nabla f_0-\cos\theta \in\Big] = \Gamma_b\cdot \not{\mathfrak{d}}^{\leq 1}f_0, \qquad \nabla_\nu\Big[r\nabla f_\pm + J^{(\pm)}\delta\Big] = \Gamma_b\cdot \not{\mathfrak{d}}^{\leq 1}f_\pm,\\
&\nabla_\nu\Big[r\widetilde{\nabla J^{(p)}}\Big] = \Gamma_b\cdot \not{\mathfrak{d}}^{\leq 1}J^{(p)}, \quad p=0,+,-.
\end{aligned}$$

Using the commutation formula from Corollary 5.21

$$[\nabla_\nu, r\nabla]f \quad = r\Gamma_b\cdot\nabla_\nu f+\Gamma_b\cdot\mathfrak{d}^{\leq 1}f,$$

we infer[125]

$$\begin{aligned}
&\nabla_\nu \not{\mathfrak{d}}^k\Big[r\nabla f_0-\cos\theta \in\Big] = \not{\mathfrak{d}}^{\leq k}(\Gamma_b\cdot\not{\mathfrak{d}}^{\leq 1}f_0) + \not{\mathfrak{d}}^{\leq k}\left(\Gamma_b\cdot\Big[r\nabla f_0-\cos\theta\in\Big]\right),\\
&\nabla_\nu \not{\mathfrak{d}}^k\Big[r\nabla f_\pm+J^{(\pm)}\delta\Big] = \not{\mathfrak{d}}^{\leq k}(\Gamma_b\cdot\not{\mathfrak{d}}^{\leq 1}f_\pm) + \not{\mathfrak{d}}^{\leq k}\left(\Gamma_b\cdot\Big[r\nabla f_\pm+J^{(\pm)}\delta\Big]\right),\\
&\nabla_\nu \not{\mathfrak{d}}^k\Big[r\widetilde{\nabla J^{(p)}}\Big] = \not{\mathfrak{d}}^{\leq k}(\Gamma_b\cdot\not{\mathfrak{d}}^{\leq 1}J^{(p)}), \quad p=0,+,-.
\end{aligned}$$

In view of Corollary 5.34, we infer on Σ_*, for all $k\leq k_*-12$

$$r\left|\not{\mathfrak{d}}^k\Big[r\widetilde{\nabla J^{(p)}}\Big]\right| \quad \lesssim \quad \left|r\not{\mathfrak{d}}^k\Big[r\widetilde{\nabla J^{(p)}}\Big]\right|_{L^\infty(S_*)} + \int_u^{u_*} r|\not{\mathfrak{d}}^{\leq k}\Gamma_b|$$

and on Σ_*, for all $k\leq k_*-13$

$$\begin{aligned}
r\left|\not{\mathfrak{d}}^k\Big[r\nabla f_0-\cos\theta\in\Big]\right| \quad &\lesssim \quad \left|r\not{\mathfrak{d}}^k\Big[r\nabla f_0-\cos\theta\in\Big]\right|_{L^\infty(S_*)} + \int_u^{u_*} r|\not{\mathfrak{d}}^{\leq k}\Gamma_b|,\\
r\left|\not{\mathfrak{d}}^k\Big[r\nabla f_\pm+J^{(\pm)}\delta\Big]\right| \quad &\lesssim \quad \left|r\not{\mathfrak{d}}^k\Big[r\nabla f_\pm+J^{(\pm)}\delta\Big]\right|_{L^\infty(S_*)} + \int_u^{u_*} r|\not{\mathfrak{d}}^{\leq k}\Gamma_b|.
\end{aligned}$$

[125] For $k>1$, we neglect all higher order terms involving $(r\Gamma_b)^j\Gamma_b$, $1\leq j\leq k-1$, since they satisfy at least the same estimates as the corresponding ones involving Γ_b.

Now, in view of Sobolev and Proposition 5.42, we have on Σ_*, for $k\le k_*-12$,

$$\begin{aligned}\int_u^{u_*} r|\mathfrak{d}^{\le k}\Gamma_b| &\lesssim \frac{1}{u^{\frac12+\delta_{dec}}}\left(\int_u^{u_*} r^2 u'^{2+2\delta_{dec}}|\mathfrak{d}^{\le k_*-12}\Gamma_b|^2\right)^{\frac12}\\ &\lesssim \frac{1}{u^{\frac12+\delta_{dec}}}\left(\int_u^{u_*} u'^{2+2\delta_{dec}}\|\mathfrak{d}^{\le k_*-10}\Gamma_b\|^2_{L^2(S)}\right)^{\frac12}\\ &\lesssim \frac{1}{u^{\frac12+\delta_{dec}}}\left(\int_{\Sigma_*} u^{2+2\delta_{dec}}|\mathfrak{d}^{\le k_*-10}\Gamma_b|^2\right)^{\frac12}\\ &\lesssim \frac{\epsilon_0}{u^{\frac12+\delta_{dec}}}.\end{aligned}$$

Together with the control on S_* provided by Lemma 5.63, we infer on Σ_*, for all $k\le k_*-12$

$$\left|\mathfrak{d}^k\left[\widetilde{\nabla J^{(p)}}\right]\right| \lesssim \frac{\epsilon_0}{r^2u^{\frac12+\delta_{dec}}},$$

and on Σ_*, for all $k\le k_*-13$

$$\left|\mathfrak{d}^k\left[\nabla f_0-\frac{1}{r}\cos\theta \in\right]\right|+\left|\mathfrak{d}^k\left[\nabla f_\pm+\frac{1}{r}J^{(\pm)}\delta\right]\right| \lesssim \frac{\epsilon_0}{r^2u^{\frac12+\delta_{dec}}}.$$

Together with Lemma 5.66 and the control for Γ_b in Proposition 5.42, we deduce on Σ_*, for all $k\le k_*-12$

$$\left|\mathfrak{d}_*^k\left[\widetilde{\nabla J^{(p)}}\right]\right| \lesssim \frac{\epsilon_0}{r^2u^{\frac12+\delta_{dec}}},$$

and on Σ_*, for all $k\le k_*-13$

$$\left|\mathfrak{d}_*^k\left[\nabla f_0-\frac{1}{r}\cos\theta \in\right]\right|+\left|\mathfrak{d}_*^k\left[\nabla f_\pm+\frac{1}{r}J^{(\pm)}\delta\right]\right| \lesssim \frac{\epsilon_0}{r^2u^{\frac12+\delta_{dec}}}.$$

This implies in particular, on Σ_*, for all $k\le k_*-13$,

$$\begin{aligned}&\left|\mathfrak{d}_*^k\mathrm{div}\,(f_0)\right|+\left|\mathfrak{d}_*^k\widetilde{\mathrm{curl}\,(f_0)}\right|+\left|\mathfrak{d}_*^k\nabla\widehat{\otimes} f_0\right|\\ &+\left|\mathfrak{d}_*^k\widetilde{\mathrm{div}\,(f_\pm)}\right|+\left|\mathfrak{d}_*^k\mathrm{curl}\,(f_\pm)\right|+\left|\mathfrak{d}_*^k\nabla\widehat{\otimes} f_\pm\right| \lesssim \frac{\epsilon_0}{r^2u^{\frac12+\delta_{dec}}}.\end{aligned}$$

Finally, recalling from Lemma 5.65 the following identities on Σ_*

$$\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}} = O(r^{-4})+\frac{2}{|q|}\mathrm{div}\,(f_0)-\frac{2a^2\cos\theta}{|q|^3}f_0\cdot\widetilde{\nabla J^{(0)}}$$

$$
\begin{aligned}
&+i\left(\frac{2}{|q|}\widetilde{\mathrm{curl}\,(f_0)}-\frac{2a^2\cos\theta}{|q|^3}f_0\cdot\widetilde{{}^*\nabla J^{(0)}}\right),\\
\widetilde{\mathcal{D}\cdot\mathfrak{J}_\pm} \;=\;& O(r^{-4})+\frac{2}{|q|}\widetilde{\mathrm{div}\,(f_\pm)}-\frac{2a^2\cos\theta}{|q|^3}f_\pm\cdot\widetilde{\nabla J^{(0)}}\\
&+i\left(\frac{2}{|q|}\mathrm{curl}\,(f_\pm)-\frac{2a^2\cos\theta}{|q|^3}f_\pm\cdot\widetilde{{}^*\nabla J^{(0)}}\right),
\end{aligned}
$$

where $O(r^a)$ denotes, for $a\in\mathbb{R}$, a function of $(r,\cos\theta)$ bounded by r^a as $r\to+\infty$, we immediately infer from the above that there holds on Σ_*, for all $k\leq k_*-13$,

$$
\left|\mathfrak{d}_*^k\widetilde{\mathcal{D}\cdot\mathfrak{J}}\right|+\left|\mathfrak{d}_*^k\widetilde{\mathcal{D}\cdot\mathfrak{J}_\pm}\right| \;\lesssim\; \frac{\epsilon_0}{r^3u^{\frac12+\delta_{dec}}}+\frac{1}{r^4}.
$$

In view of the dominance condition (5.30) on r on Σ_*, this yields, for $k\leq k_*-13$,

$$
\left|\mathfrak{d}_*^k\widetilde{\mathcal{D}\cdot\mathfrak{J}}\right|+\left|\mathfrak{d}_*^k\widetilde{\mathcal{D}\cdot\mathfrak{J}_\pm}\right| \;\lesssim\; \frac{\epsilon_0}{r^3u^{\frac12+\delta_{dec}}}
$$

which concludes the proof of Proposition 5.59. □

5.6.3. A additional estimate for β on Σ_* This section is devoted to a decay estimate for β, see Corollary 5.70. We start with the following two lemmas.

Lemma 5.67. *We have on Σ_*, for $k\leq k_*-13$,*

$$
\begin{aligned}
\left|\not{\mathfrak{d}}^k\nabla\widehat{\otimes}\left(\beta-\frac{3am}{r^4}f_0\right)\right| &\lesssim |\not{\mathfrak{d}}^k\nabla\widehat{\otimes}\beta|+\frac{\epsilon_0}{r^6u^{\frac12+\delta_{dec}}},\\
\left|\mathrm{div}\left(\beta-\frac{3am}{r^4}f_0\right)-\mathrm{div}\,\beta\right| &\lesssim \frac{\epsilon_0}{r^6u^{\frac12+\delta_{dec}}},\\
\left|\mathrm{curl}\left(\beta-\frac{3am}{r^4}f_0\right)-\left(\mathrm{curl}\,\beta-\frac{6amJ^{(0)}}{r^5}\right)\right| &\lesssim \frac{\epsilon_0}{r^6u^{\frac12+\delta_{dec}}}.
\end{aligned}
$$

Proof. The proof follows immediately from the control of f_0 provided by Proposition 5.59 and the definition of $J^{(0)}$ and $\widetilde{\mathrm{curl}\,(f_0)}$. □

Lemma 5.68. *The functions $J^{(p)}$ verify the following properties on Σ_**

$$
\int_S J^{(p)}=O\left(\epsilon_0 r u^{-\frac12-\delta_{dec}}\right),
$$

$$\int_S J^{(p)} J^{(q)} = \frac{4\pi}{3} r^2 \delta_{pq} + O\left(\epsilon_0 r u^{-\frac{1}{2}-\delta_{dec}}\right),$$
$$\left|\left(\Delta + \frac{2}{r^2}\right) J^{(p)}\right| \lesssim \epsilon_0 r^{-3} u^{-\frac{1}{2}-\delta_{dec}},$$

and

$$\left|\not{d}_2^* \not{d}_1^* J^{(p)}\right| \lesssim \epsilon_0 r^{-3} u^{-\frac{1}{2}-\delta_{dec}},$$

where by $\not{d}_1^* J^{(p)}$*, we mean either* $\not{d}_1^*(J^{(p)}, 0)$ *or* $\not{d}_1^*(0, J^{(p)})$*.*

Proof. The proof follows exactly the same lines as the proof of Corollary 5.45 by replacing the control on S_* of Lemma 5.36 by the improved control on S_* provided by Corollary 5.60. □

The following lemma controls the $\ell = 1$ modes of $J^{(0)}$.

Lemma 5.69. *We have on* Σ_*

$$\left|\left(J^{(0)}\right)_{\ell=1,0} - \frac{1}{3}\right| + \left|\left(J^{(0)}\right)_{\ell=1,\pm}\right| \quad \lesssim \quad \frac{\epsilon_0}{r u^{\frac{1}{2}+\delta_{dec}}}.$$

Proof. We have, by definition of the $\ell = 1$ modes on Σ_*, for $p = 0, +, -$,

$$\left(J^{(0)}\right)_{\ell=1,p} \quad = \quad \frac{1}{|S|} \int_S J^{(0)} J^{(p)}$$

and hence

$$\left(J^{(0)}\right)_{\ell=1,p} - \frac{1}{3}\delta_{p0} \quad = \quad \frac{1}{4\pi r^2}\left(\int_S J^{(0)} J^{(p)} - \frac{4\pi}{3} r^2 \delta_{p0}\right).$$

The proof follows then from Lemma 5.68. □

Corollary 5.70. *We have on* Σ_*

$$\left|\left[\not{d}_1\left(\beta - \frac{3am}{r^4} f_0\right)\right]_{\ell=1}\right| \quad \lesssim \quad \frac{\epsilon_0}{r^5 u^{1+\delta_{dec}}},$$

and, for $k \le k_* - 14$*,*

$$\sup_{\Sigma_*} r^{\frac{7}{2}+\delta_{extra}} \left|\not{\mathfrak{d}}^{\le k}\left(\beta - \frac{3am\sin\theta}{r^4} f_0\right)\right| \quad \lesssim \quad \epsilon_0.$$

Proof. We start with the first estimate. In view of the definition of $\not{d}_1$ and Lemma 5.67, we have

$$\left|\left[\not{d}_1\left(\beta-\frac{3am}{r^4}f_0\right)\right]_{\ell=1}\right| \lesssim |(\mathrm{div}\,\beta)_{\ell=1}|+\left|(\mathrm{curl}\,\beta)_{\ell=1}-\frac{6am(J^{(0)})_{\ell=1}}{r^5}\right| +\frac{\epsilon_0}{r^6u^{\frac{1}{2}+\delta_{dec}}}.$$

Together with Lemma 5.69, we infer

$$\left|\left[\not{d}_1\left(\beta-\frac{3am}{r^4}f_0\right)\right]_{\ell=1}\right| \lesssim |(\mathrm{div}\,\beta)_{\ell=1}|+|(\mathrm{curl}\,\beta)_{\ell=1,\pm}| +\left|(\mathrm{curl}\,\beta)_{\ell=1,0}-\frac{2am}{r^5}\right|+\frac{\epsilon_0}{r^6u^{\frac{1}{2}+\delta_{dec}}}.$$

In view of Proposition 5.48, we infer

$$\left|\left[\not{d}_1\left(\beta-\frac{3am}{r^4}f_0\right)\right]_{\ell=1}\right| \lesssim \frac{\epsilon_0}{r^5u^{1+\delta_{dec}}}+\frac{\epsilon_0}{r^6u^{\frac{1}{2}+\delta_{dec}}}$$

and hence, using the dominance of r on Σ_*,

$$\left|\left[\not{d}_1\left(\beta-\frac{3am}{r^4}f_0\right)\right]_{\ell=1}\right| \lesssim \frac{\epsilon_0}{r^5u^{1+\delta_{dec}}}$$

as stated.

Next, we focus on the second estimate. In view of Lemma 5.28, we have

$$\left\|\beta-\frac{3am}{r^4}f_0\right\|_{\mathfrak{h}_{k_*-12}(S)} \lesssim r\left\|\not{d}_2^*\left(\beta-\frac{3am}{r^4}f_0\right)\right\|_{\mathfrak{h}_{k_*-13}(S)} +r^2\left|\left(\not{d}_1\left(\beta-\frac{3am}{r^4}f_0\right)\right)_{\ell=1}\right|.$$

In view of the above estimate, we deduce

$$\left\|\not{\mathfrak{d}}^{k_*-14}\left(\beta-\frac{3am}{r^4}f_0\right)\right\|_{L^\infty(S)} \lesssim r\left\|\not{\mathfrak{d}}^{\leq k_*-13}\nabla\widehat{\otimes}\beta\right\|_{L^\infty(S)}+\frac{\epsilon_0}{r^4u^{1+\delta_{dec}}}.$$

It remains to control $\nabla\widehat{\otimes}\beta$. We have the following consequence of Bianchi

$$\nabla\widehat{\otimes}\beta = \nabla_3\alpha-\frac{\Upsilon}{r}\alpha+r^{-3}\Gamma_g+\Gamma_b\cdot(\alpha,\beta)+r^{-1}\Gamma_g\cdot\Gamma_g.$$

Together with the control of α in (5.42), the control for $\nabla_3\alpha$ in (5.119), and the control for Γ_g and Γ_b in **Ref 1**, we infer

$$\left\|\not{\mathfrak{d}}^{\leq k_*-13}\nabla\widehat{\otimes}\beta\right\|_{L^\infty(S)} \lesssim \frac{\epsilon_0}{r^{\frac{9}{2}+\delta_{extra}}}$$

and hence

$$\left\| \mathfrak{d}^{k_*-14}\left(\beta - \frac{3am}{r^4} f_0\right)\right\|_{L^\infty(S)} \lesssim \frac{\epsilon_0}{r^{\frac{7}{2}+\delta_{extra}}} + \frac{\epsilon_0}{r^4 u^{1+\delta_{dec}}} \lesssim \frac{\epsilon_0}{r^{\frac{7}{2}+\delta_{extra}}}$$

as stated. This concludes the proof of the corollary. □

5.6.4. An estimate for high order derivatives of $J^{(p)}$ and $\mathfrak{J}$ In this section, we derive the following proposition on the control of k_{large} derivatives of $J^{(p)}$, f_0, $f_\pm$ and $\mathfrak{J}$.

Proposition 5.71. *We have on Σ_*, for all $k \leq k_{large}$*

$$\left|\mathfrak{d}_*^k\left[\widetilde{\nabla J^{(p)}}\right]\right| \lesssim \frac{\epsilon}{r},$$

and on Σ_*, for all $k \leq k_{large} - 1$

$$\left|\mathfrak{d}_*^k\left[div(f_0),\ \widetilde{curl(f_0)},\ \nabla\widehat{\otimes} f_0,\ \nabla f_0 - \frac{1}{r}\cos\theta \in\right]\right| + \left|\mathfrak{d}_*^k\left[\widetilde{div(f_\pm)},\ curl(f_\pm),\ \nabla\widehat{\otimes} f_\pm,\ \nabla f_\pm + \frac{1}{r}J^{(\pm)}\delta\right]\right| \lesssim \frac{\epsilon}{r},$$

as well as

$$\left|\mathfrak{d}_*^k\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}\right| + \left|\mathfrak{d}_*^k\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm}\right| \lesssim \frac{\epsilon}{r^2}.$$

Proof. In view of the Gauss equation, we have $\check{K} \in r^{-1}\Gamma_g$. Together with the estimate for Γ_g in **Ref 1**, we infer

$$\sup_{S_*}\left|\mathfrak{d}^{\leq k_{large}}\left(K - \frac{1}{r^2}\right)\right| \lesssim \frac{\epsilon}{r^3}.$$

Arguing as in Corollary 5.60, we infer

$$\sup_{S_*}\left|\mathfrak{d}^{\leq k_{large}}\phi\right| \lesssim \frac{\epsilon}{r}.$$

Then, arguing as in Lemma 5.63, we deduce, for $k \leq k_{large}$,

$$\sup_{S_*}\left|\mathfrak{d}^k\left(\widetilde{\nabla J^{(0)}},\ \widetilde{\nabla J^{(+)}},\ \widetilde{\nabla J^{(-)}}\right)\right| \lesssim \frac{\epsilon}{r^2},$$

and for $k\leq k_{large}-1$,

$$\begin{aligned}
\sup_{S_*}\left|\not{\mathfrak{d}}^k\Big(\mathrm{div}\,(f_0),\,\widetilde{\mathrm{curl}\,(f_0)},\,\nabla\widehat{\otimes}f_0\Big)\right| &\lesssim \frac{\epsilon}{r^2},\\
\sup_{S_*}\left|\not{\mathfrak{d}}^k\Big(\widetilde{\mathrm{div}\,(f_\pm)},\,\mathrm{curl}\,(f_\pm),\,\nabla\widehat{\otimes}f_\pm\Big)\right| &\lesssim \frac{\epsilon}{r^2},\\
\sup_{S_*}\left|\not{\mathfrak{d}}^k\Big(\nabla f_0-\frac{1}{r}\cos\theta\in,\;\nabla f_\pm+\frac{1}{r}J^{(\pm)}\delta\Big)\right| &\lesssim \frac{\epsilon}{r^2}.
\end{aligned}$$

Next, recall from the proof of Proposition 5.59 (recall also Footnote 125)

$$\begin{aligned}
\nabla_\nu\not{\mathfrak{d}}^k\Big[r\nabla f_0-\cos\theta\in\Big] &= \not{\mathfrak{d}}^{\leq k}(\Gamma_b\cdot\not{\mathfrak{d}}^{\leq 1}f_0)+\not{\mathfrak{d}}^{\leq k}\left(\Gamma_b\cdot\Big[r\nabla f_0-\cos\theta\in\Big]\right),\\
\nabla_\nu\not{\mathfrak{d}}^k\Big[r\nabla f_\pm+J^{(\pm)}\delta\Big] &= \not{\mathfrak{d}}^{\leq k}(\Gamma_b\cdot\not{\mathfrak{d}}^{\leq 1}f_\pm)+\not{\mathfrak{d}}^{\leq k}\left(\Gamma_b\cdot\Big[r\nabla f_\pm+J^{(\pm)}\delta\Big]\right),\\
\nabla_\nu\not{\mathfrak{d}}^k\Big[r\widetilde{\nabla J^{(p)}}\Big] &= \not{\mathfrak{d}}^{\leq k}(\Gamma_b\cdot\not{\mathfrak{d}}^{\leq 1}J^{(p)}),\quad p=0,+,-.
\end{aligned}$$

In view of Corollary 5.34, we infer on Σ_*, for all $k\leq k_{large}$

$$r\left|\not{\mathfrak{d}}^k\Big[r\widetilde{\nabla J^{(p)}}\Big]\right| \;\lesssim\; \left|r\not{\mathfrak{d}}^k\Big[r\widetilde{\nabla J^{(p)}}\Big]\right|_{L^\infty(S_*)}+\int_u^{u_*}r|\not{\mathfrak{d}}^{\leq k}\Gamma_b|$$

and on Σ_*, for all $k\leq k_{large}-1$

$$\begin{aligned}
r\left|\not{\mathfrak{d}}^k\Big[r\nabla f_0-\cos\theta\in\Big]\right| &\lesssim \left|r\not{\mathfrak{d}}^k\Big[r\nabla f_0-\cos\theta\in\Big]\right|_{L^\infty(S_*)}+\int_u^{u_*}r|\not{\mathfrak{d}}^{\leq k}\Gamma_b|,\\
r\left|\not{\mathfrak{d}}^k\Big[r\nabla f_\pm+J^{(\pm)}\delta\Big]\right| &\lesssim \left|r\not{\mathfrak{d}}^k\Big[r\nabla f_\pm+J^{(\pm)}\delta\Big]\right|_{L^\infty(S_*)}+\int_u^{u_*}r|\not{\mathfrak{d}}^{\leq k}\Gamma_b|.
\end{aligned}$$

In view of the control of Γ_b provided by **Ref 1**, we have, for $k\leq k_{large}$,

$$\int_u^{u_*}r|\not{\mathfrak{d}}^{\leq k}\Gamma_b| \;\lesssim\; \epsilon\int_1^{u_*}du\lesssim u_*\epsilon.$$

Together with the above control on S_*, and the dominance of r on Σ_*, we obtain on Σ_*, for all $k\leq k_{large}$

$$\left|\not{\mathfrak{d}}^k\Big[r\widetilde{\nabla J^{(p)}}\Big]\right| \;\lesssim\; \frac{\epsilon u_*}{r}\lesssim\epsilon_0,$$

and on Σ_*, for all $k\leq k_{large}-1$

$$\left|\not{\mathfrak{d}}^k\Big[r\nabla f_0-\cos\theta\in\Big]\right| \;\lesssim\; \frac{\epsilon u_*}{r}\lesssim\epsilon_0,$$

$$\left|\mathfrak{\not{\partial}}^k\left[r\nabla f_\pm + J^{(\pm)}\delta\right]\right| \lesssim \frac{\epsilon u_*}{r} \lesssim \epsilon_0.$$

This implies in particular, on Σ_*, for all $k \leq k_{large} - 1$,

$$\begin{aligned}&\left|\mathfrak{d}_*^k \mathrm{div}\,(f_0)\right| + \left|\mathfrak{d}_*^k \widetilde{\mathrm{curl}\,(f_0)}\right| + \left|\mathfrak{d}_*^k \nabla\widehat{\otimes} f_0\right| \\ &+ \left|\mathfrak{d}_*^k \widetilde{\mathrm{div}\,(f_\pm)}\right| + \left|\mathfrak{d}_*^k \mathrm{curl}\,(f_\pm)\right| + \left|\mathfrak{d}_*^k \nabla\widehat{\otimes} f_\pm\right| \lesssim \frac{\epsilon_0}{r}.\end{aligned}$$

Finally, recalling from Lemma 5.65 the following identities on Σ_*

$$\begin{aligned}\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}} &= O(r^{-4}) + \frac{2}{|q|}\mathrm{div}\,(f_0) - \frac{2a^2\cos\theta}{|q|^3} f_0\cdot\widetilde{\nabla J^{(0)}} \\ &\quad + i\left(\frac{2}{|q|}\widetilde{\mathrm{curl}\,(f_0)} - \frac{2a^2\cos\theta}{|q|^3} f_0\cdot {}^*\widetilde{\nabla J^{(0)}}\right), \\ \widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm} &= O(r^{-4}) + \frac{2}{|q|}\widetilde{\mathrm{div}\,(f_\pm)} - \frac{2a^2\cos\theta}{|q|^3} f_\pm\cdot\widetilde{\nabla J^{(0)}} \\ &\quad + i\left(\frac{2}{|q|}\mathrm{curl}\,(f_\pm) - \frac{2a^2\cos\theta}{|q|^3} f_\pm\cdot {}^*\widetilde{\nabla J^{(0)}}\right),\end{aligned}$$

where $O(r^a)$ denotes, for $a \in \mathbb{R}$, a function of $(r, \cos\theta)$ bounded by r^a as $r \to +\infty$, we immediately infer from the above that there holds on Σ_*, for all $k \leq k_{large} - 1$,

$$\left|\mathfrak{d}_*^k \widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}\right| + \left|\mathfrak{d}_*^k \widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm}\right| \lesssim \frac{\epsilon}{r^2}$$

which concludes the proof of Proposition 5.71. □

Proposition 5.59 and Proposition 5.71 motivate the following definition.

Definition 5.72. *We denote by $\widetilde{\Gamma}_b$ the set of linearized quantities below*

$$\begin{aligned}\widetilde{\Gamma}_b &:= \Gamma_b \cup \left\{\widetilde{\nabla J^{(p)}}\right\}, \\ \mathfrak{d}_*^k\widetilde{\Gamma}_b &:= \mathfrak{d}_*^k\Gamma_b \cup \left\{\mathfrak{d}_*^k\widetilde{\nabla J^{(p)}}\right\} \cup \mathfrak{d}_*^{k-1}\widetilde{\Gamma}_{b,1}, \quad \text{for} \quad k \geq 1, \\ \widetilde{\Gamma}_{b,1} &:= \left\{div\,(f_0),\ \widetilde{curl\,(f_0)},\ \nabla\widehat{\otimes} f_0,\ \widetilde{div\,(f_\pm)},\ curl\,(f_\pm),\ \nabla\widehat{\otimes} f_\pm,\ r\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}},\right. \\ &\qquad\qquad \left. r\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm}\right\}.\end{aligned}$$

Corollary 5.73. *We have on Σ_*, for $k \leq k_* - 12$,*

$$|\mathfrak{d}_*^k \widetilde{\Gamma}_b| \lesssim \frac{\epsilon_0}{ru^{1+\delta_{dec}}},$$

and, for $k \leq k_{large}$

$$|\mathfrak{d}_*^k \widetilde{\Gamma}_b| \lesssim \frac{\epsilon}{r}.$$

Proof. This is an immediate consequence of **Ref 1** and Proposition 5.42 for Γ_b, and of Proposition 5.59 and Proposition 5.71 for the rest of $\widetilde{\Gamma}_b$. □

Remark 5.74. *In view of Corollary 5.73, $\widetilde{\Gamma}_b$ enjoys the same estimates as Γ_b. Note that the estimates of angular derivatives of $J^{(p)}$, f_0, $f_\pm$ and $\mathfrak{J}$ are*

- *consistent with Γ_g for $k \leq k_* - 12$ derivatives in view of Proposition 5.59,*
- *consistent with Γ_b for $k \leq k_{large}$ derivatives in view of Proposition 5.71.*

We do not need the better decay properties and simply treat these angular derivatives as Γ_b, which justifies their inclusion in $\widetilde{\Gamma}_b$.

5.7. Decay estimates for the PG frame on Σ_*

In this section, we use the decay estimates derived for the integrable frame of Σ_* in Proposition 5.53, and the estimates of Section 5.6, to derive decay estimates for the PG frame of ${}^{(ext)}\mathcal{M}$. This is a prerequisite to the improvement of the bootstrap assumptions on decay on ${}^{(ext)}\mathcal{M}$ of Chapter 6.

5.7.1. Initialization of the PG frame on Σ_*

Let (e_3, e_4, e_1, e_2) denote the null frame of Σ_*, and let (e_3', e_4', e_1', e_2') denote the PG frame of ${}^{(ext)}\mathcal{M}$. Then, (e_3', e_4', e_1', e_2') is initialized on Σ_* by

$$\begin{aligned}
e_4' &= e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3, \\
e_a' &= \left(\delta_a^b + \frac{1}{2}\underline{f}_a f^b\right) e_b + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right) e_3, \\
e_3' &= \left(1 + \frac{1}{2}f \cdot \underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right) e_3 + \left(\underline{f}^b + \frac{1}{4}|\underline{f}|^2 f^b\right) e_b \\
&\quad + \frac{1}{4}|\underline{f}|^2 e_4,
\end{aligned} \tag{5.138}$$

where

$$f = \frac{a}{r} f_0, \qquad \underline{f} = -\frac{(\nu(r) - b_*)}{1 - \frac{1}{4} b_* |f|^2} f, \tag{5.139}$$

with the 1-form f_0 being defined in Definition 5.56. All quantities with primes denote in Section 5.7 the ones corresponding to the PG frame of ${}^{(ext)}\mathcal{M}$. Furthermore, the coordinates $(r', u', \theta', \varphi')$ associated to the PG frame of ${}^{(ext)}\mathcal{M}$ are initialized on Σ_* as follows

$$r' = r, \qquad u' = u, \qquad \theta' = \theta, \qquad \varphi' = \varphi, \quad \text{on} \quad {}^{(ext)}\mathcal{M}. \tag{5.140}$$

Also, note that the complex horizontal 1-form $\mathfrak{J}$ introduced in Definition 3.8 verifies

$$\begin{aligned} \mathfrak{J} &= \frac{1}{|q|} (f_0 + i\, {}^* f_0) \quad \text{on} \quad \Sigma_*, \\ \nabla'_4 \mathfrak{J} &= -\frac{1}{q} \mathfrak{J} \quad \text{on} \quad {}^{(ext)}\mathcal{M}. \end{aligned} \tag{5.141}$$

We also introduce the following two complex horizontal 1-forms $\mathfrak{J}_\pm$ given by

$$\begin{aligned} \mathfrak{J}_\pm &= \frac{1}{|q|} (f_\pm + i\, {}^* f_\pm) \quad \text{on} \quad \Sigma_*, \\ \nabla'_4 \mathfrak{J}_\pm &= -\frac{1}{q} \mathfrak{J}_\pm \quad \text{on} \quad {}^{(ext)}\mathcal{M}. \end{aligned} \tag{5.142}$$

Remark 5.75. *Recall that the complex 1-form $\mathfrak{J}$ is needed to linearize Z, H, $\mathcal{D}(\cos\theta)$, and $\mathcal{D}(u)$, and the complex 1-form $\mathfrak{J}_\pm$ is needed to linearize $\mathcal{D}(J^{(\pm)})$.*

Also, note that the transformation formulas involve all Ricci coefficients of the foliation of Σ_*. We thus need to prescribe transversality conditions for the Ricci coefficients not defined on Σ_*, i.e. ξ, ω and $\underline{\eta}$. We recall that we choose them to be compatible with an outgoing geodesic foliation initialized on Σ_*, i.e.

$$\xi = 0, \qquad \omega = 0, \qquad \underline{\eta} = -\zeta. \tag{5.143}$$

Recall the definition of Γ'_b and Γ'_g in the PG frame of ${}^{(ext)}\mathcal{M}$.

Definition 5.76. *Recall Definition 2.66 for the definition of the linearized quantities in an outgoing PG frame. The set of all linearized quantities is of the form $\Gamma_g \cup \Gamma_b$ with Γ_g, Γ_b defined as follows.*

1. *The set* Γ_g *with*

$$\Gamma'_g = \left\{ \widetilde{trX}', \quad \widehat{X}', \quad \check{Z}', \quad \widetilde{tr\underline{X}}', \quad r\check{P}', \quad rB', \quad rA' \right\}.$$

2. *The set* $\Gamma'_b = \Gamma'_{b,1} \cup \Gamma'_{b,2} \cup \Gamma'_{b,3}$ *with*

$$\Gamma'_{b,1} = \left\{ \check{H}', \quad \widehat{\underline{X}}', \quad \underline{\check{\omega}}', \quad \underline{\Xi}', \quad r\underline{B}', \quad \underline{A}' \right\},$$

$$\Gamma'_{b,2} = \left\{ r^{-1}\widetilde{e'_3(r)}, \quad \widetilde{\mathcal{D}'(\cos\theta)}, \quad e'_3(\cos\theta), \quad \widetilde{\mathcal{D}'u}, \quad r^{-1}\widetilde{e'_3(u)}, \right.$$
$$\left. \widetilde{\mathcal{D}'(J^{(+)})}, \quad \widetilde{\mathcal{D}'(J^{(-)})}, \quad \widetilde{e'_3(J^{(+)})}, \quad \widetilde{e'_3(J^{(-)})} \right\},$$

$$\Gamma_{b,3} = \left\{ r\,\widetilde{\overline{\mathcal{D}'}\cdot\mathfrak{J}}, \quad r\,\mathcal{D}'\widehat{\otimes}\mathfrak{J}, \quad r\,\widetilde{\nabla'_3\mathfrak{J}}, \quad r\,\widetilde{\overline{\mathcal{D}'}\cdot\mathfrak{J}_\pm}, \quad r\,\mathcal{D}'\widehat{\otimes}\mathfrak{J}_\pm, \quad r\,\widetilde{\nabla'_3\mathfrak{J}_\pm} \right\}.$$

The goal of this section is to prove the following proposition concerning the control of the PG frame on Σ_*.

Proposition 5.77. *We have on* Σ_**, for* $k \leq k_* - 15$*,*

$$\sup_{\Sigma_*}\left(ru^{1+\delta_{dec}}|\mathfrak{d}^k\Gamma'_b| + r^2u^{\frac{1}{2}+\delta_{dec}}|\mathfrak{d}^k\Gamma'_g| + r^2u^{1+\delta_{dec}}|\mathfrak{d}^{k-1}\nabla_3\Gamma'_g|\right) \lesssim \epsilon_0,$$
$$\sup_{\Sigma_*}\left(r^2u^{1+\delta_{dec}}|\mathfrak{d}^k\widetilde{trX}'| + r^3u^{1+\delta_{dec}}\left|\mathfrak{d}^k\left(\overline{\mathcal{D}'}\cdot\check{Z}' + 2\overline{\check{P}'}\right)\right|\right) \lesssim \epsilon_0,$$
$$\sup_{\Sigma_*}\left(r^{\frac{7}{2}+\delta_{extra}}|\mathfrak{d}^kB'| + r^4u^{\frac{1}{2}+\delta_{dec}}|\mathfrak{d}^{k-1}\nabla_3B'|\right) \lesssim \epsilon_0,$$
$$\sup_{\Sigma_*} r^5u^{1+\delta_{dec}}\left|\left[\left(\overline{\mathcal{D}'}\cdot - \frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_{e'_4} - \frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_{e'_3}\right)\cdot\left(B' - \frac{3a}{2}\overline{\check{P}'}\mathfrak{J} - \frac{a}{4}\overline{\mathfrak{J}}\cdot A'\right)\right]_{\ell=1}\right|$$
$$\lesssim \epsilon_0,$$

and

$$\sup_{\Sigma_*} r^5u^{1+\delta_{dec}}\left|\left[\overline{\mathcal{D}}\cdot\not{\mathcal{L}}_{\mathbf{T}'}B'\right]_{\ell=1}\right| \lesssim \epsilon_0.$$

The proof of Proposition 5.77 is done in Section 5.7.5, relying on the estimates of Sections 5.7.2, 5.7.3 and 5.7.4.

5.7.2. First decay estimates for the PG frame on Σ_*

Lemma 5.78. *We have*

$$e'_4(r') = 1, \qquad e'_4(u') = e'_4(\theta') = e'_4(\varphi') = 0, \qquad \nabla'(r') = 0,$$

and

$$\begin{aligned}
&\sup_{\Sigma_*} r u^{1+\delta_{dec}} \Big(\Big|\mathfrak{d}^{\leq k_*-12} \widetilde{\mathcal{D}'(\cos\theta')}\Big| + \Big|\mathfrak{d}^{\leq k_*-12} \widetilde{\mathcal{D}'(u')}\Big| \\
&\qquad\qquad + \Big|\mathfrak{d}^{\leq k_*-12} \widetilde{\mathcal{D}'(J'^{(\pm)})}\Big| \Big) \\
&+ \sup_{\Sigma_*} r u^{1+\delta_{dec}} \Big(\Big|\mathfrak{d}^{\leq k_*-12} e'_3(\cos\theta')\Big| + \Big|\mathfrak{d}^{\leq k_*-12} \widetilde{e'_3(J'^{(\pm)})}\Big| \Big) \\
&+ \sup_{\Sigma_*} u^{1+\delta_{dec}} \Big(\Big|\mathfrak{d}^{\leq k_*-12} \widetilde{e'_3(r')}\Big| + \Big|\mathfrak{d}^{\leq k_*-12} \widetilde{e'_3(u')}\Big| \Big) \lesssim \epsilon_0.
\end{aligned}$$

Proof. The identities $e'_4(r') = 1$, and $e'_4(u') = e'_4(\theta') = e'_4(\varphi') = 0$ are true in ${}^{(ext)}\mathcal{M}$ (and hence on Σ_*) by definition. In particular, since $\nabla(r') = \nabla(u') = 0$ on Σ_*, this implies[126]

$$\begin{aligned}
1 &= e_4(r') + \frac{1}{4}|f|^2 e_3(r'), \\
0 &= e_4(u') + \frac{1}{4}|f|^2 e_3(u'), \\
0 &= e_4(\cos\theta') + f\cdot\nabla(\cos\theta) + \frac{1}{4}|f|^2 e_3(\cos\theta'), \\
0 &= e_4(J'^{(\pm)}) + f\cdot\nabla(J^{(\pm)}) + \frac{1}{4}|f|^2 e_3(J'^{(\pm)}).
\end{aligned}$$

Since $e_3 = \nu - b_* e_4$, and since $\nu(r') = \nu(r)$, $\nu(u') = \nu(u)$, $\nu(\cos\theta') = 0$ and $\nu(J'^{(\pm)}) = 0$, we infer, using also $\nu(u) = -\nu(r)$, $f = \frac{a}{r} f_0$, and $|f|^2 = \frac{a^2(\sin\theta)^2}{r^2}$,

$$\begin{aligned}
e_4(r') &= \frac{1 - \frac{\nu(r)a^2(\sin\theta)^2}{4r^2}}{1 - \frac{b_* a^2(\sin\theta)^2}{4r^2}}, \\
e_4(u') &= \frac{\frac{\nu(r)a^2(\sin\theta)^2}{4r^2}}{1 - \frac{b_* a^2(\sin\theta)^2}{4r^2}},
\end{aligned}$$

[126]Since $r' = r$, $u' = u$, $\nabla(r) = 0$ and $\nabla(u) = 0$ on Σ_*, and since ∇ is tangent to Σ_*, we have indeed $\nabla(r') = 0$ and $\nabla(u') = 0$, $\nabla(\cos\theta') = \nabla(\cos\theta)$ and $\nabla(J'^{(\pm)}) = \nabla(J^{(\pm)})$ on Σ_*.

$$\begin{aligned} e_4(\cos\theta') &= -\frac{\frac{a}{r}f_0\cdot\nabla(\cos\theta)}{1-\frac{b_*a^2(\sin\theta)^2}{4r^2}},\\ e_4(J'^{(\pm)}) &= -\frac{\frac{a}{r}f_0\cdot\nabla(J^{(\pm)})}{1-\frac{b_*a^2(\sin\theta)^2}{4r^2}}. \end{aligned}$$

Since we have

$$\nu(r)=-2+\Gamma_b,\quad b_*=-1-\frac{2m}{r}+r\Gamma_b,\quad \nabla(\cos\theta)=-\frac{1}{r}\,{}^*f_0+\widetilde{\nabla J^{(0)}},$$
$$\nabla J^{(\pm)}=\frac{1}{r}f_\pm+\widetilde{\nabla J^{(\pm)}},$$

we infer, in view of Definition 5.72 for $\widetilde{\Gamma}_b$,

$$\begin{aligned} e_4(r') &= 1+O(r^{-2})+r^{-1}\widetilde{\Gamma}_b,\\ e_4(u') &= O(r^{-2})+r^{-1}\widetilde{\Gamma}_b,\\ e_4(\cos\theta') &= r^{-1}\widetilde{\Gamma}_b,\\ e_4(J'^{(\pm)}) &= O(r^{-2})+r^{-1}\widetilde{\Gamma}_b. \end{aligned}$$

Also, in view of the change of the definition (5.138) of the frame of ${}^{(ext)}\mathcal{M}$, and using again $\nabla(r')=0$ on Σ_*, we have

$$\begin{aligned} \nabla'(r') &= \frac{1}{2}\underline{f}e_4(r')+\left(\frac{1}{2}f+\frac{1}{8}|f|^2\underline{f}\right)e_3(r')\\ &= \frac{1}{2}\left(e_4(r')+\frac{1}{4}|f|^2e_3(r')\right)\underline{f}+\frac{1}{2}fe_3(r'). \end{aligned}$$

Together with (5.139), we infer

$$\nabla'(r) = \frac{1}{2}\left[-\frac{(\nu(r)-b_*)}{1-\frac{1}{4}b_*|f|^2}\left(e_4(r')+\frac{1}{4}|f|^2e_3(r')\right)+e_3(r')\right]f.$$

In view of the above, we have

$$e_4(r')+\frac{1}{4}|f|^2e_3(r')=1,\qquad e_3(r')=\nu(r)-b_*e_4(r'),$$

and hence

$$\nabla'(r) = \frac{1}{2}\left[-\frac{(\nu(r)-b_*)}{1-\frac{1}{4}b_*|f|^2}+\nu(r)-b_*e_4(r')\right]f.$$

Using again the above, we have

$$e_4(r') = \frac{1-\frac{\nu(r)|f|^2}{4}}{1-\frac{b_*|f|^2}{4}}$$

and hence

$$\nabla'(r) = \frac{1}{2}\left[-\frac{(\nu(r)-b_*)}{1-\frac{1}{4}b_*|f|^2}+\nu(r)-b_*\frac{1-\frac{\nu(r)|f|^2}{4}}{1-\frac{b_*|f|^2}{4}}\right]f=0$$

as stated.

Next, we focus on deriving the stated estimates. First, using the above identities for $e_4(r')$, $e_4(u')$, $e_4(\cos\theta')$ and $e_4(J'^{(\pm)})$, we have

$$\begin{aligned}
e_3(r') &= \nu(r')-b_*e_4(r') = \nu(r)-b_*\Big(1+O(r^{-2})+r^{-1}\widetilde{\Gamma}_b\Big)\\
&= -\Upsilon+O(r^{-2})+r^{-1}\widetilde{\Gamma}_b,\\
e_3(u') &= \nu(u')-b_*e_4(u') = \nu(u)-b_*\Big(O(r^{-2})+r^{-1}\widetilde{\Gamma}_b\Big)\\
&= 2+O(r^{-2})+r^{-1}\widetilde{\Gamma}_b,\\
e_3(\cos\theta') &= \nu(\cos\theta')-b_*e_4(\cos\theta') = \nu(\cos\theta)-b_*r^{-1}\widetilde{\Gamma}_b = r^{-1}\widetilde{\Gamma}_b,\\
e_3(J'^{(\pm)}) &= \nu(J'^{(\pm)})-b_*e_4(J'^{(\pm)}) = \nu(J^{(\pm)})-b_*\Big(O(r^{-2})+r^{-1}\widetilde{\Gamma}_b\Big)\\
&= O(r^{-2})+r^{-1}\widetilde{\Gamma}_b.
\end{aligned}$$

Together with (5.139) and the above identities for the e_4 derivatives, we infer

$$\begin{aligned}
\widetilde{e_3'(r')} &= O(r^{-2})+r^{-1}\widetilde{\Gamma}_b,\\
\widetilde{e_3'(u')} &= O(r^{-2})+r^{-1}\widetilde{\Gamma}_b,\\
e_3'(\cos\theta') &= r^{-1}\widetilde{\Gamma}_b,\\
e_3'(J'^{(\pm)}) &= O(r^{-2})+r^{-1}\widetilde{\Gamma}_b.
\end{aligned}$$

Together with the dominance condition (5.30) on r on Σ_*, and the estimates for $\widetilde{\Gamma}_b$ of Corollary 5.73, we obtain

$$\begin{aligned}
&\sup_{\Sigma_*} ru^{1+\delta_{dec}}\Big(\Big|\mathfrak{d}^{\leq k_*-12}e_3'(\cos\theta')\Big|+\Big|\mathfrak{d}^{\leq k_*-12}e_3'\widetilde{(J'^{(\pm)})}\Big|\Big)\\
&\quad+\sup_{\Sigma_*} u^{1+\delta_{dec}}\Big(\Big|\mathfrak{d}^{\leq k_*-12}\widetilde{e_3'(r')}\Big|+\Big|\mathfrak{d}^{\leq k_*-12}\widetilde{e_3'(u')}\Big|\Big) \lesssim \epsilon_0.
\end{aligned}$$

Finally, we control the angular derivatives. We have in view of (5.139) and the above identities for the e_4 derivatives and e_3 derivatives

$$\begin{aligned}
\nabla'(u') &= O(r^{-1})e_4(u') + \frac{1}{2}f\Big(1+O(r^{-2})\Big)e_3(u')\\
&= \frac{a}{r}f_0 + O(r^{-3}) + r^{-2}\widetilde{\Gamma}_b,\\
\nabla'(\cos\theta') &= \Big(1+O(r^{-2})\Big)\nabla(\cos\theta) + O(r^{-1})e_3(\cos\theta') + O(r^{-1})e_4(\cos\theta')\\
&= -\frac{1}{r}\,{}^*f_0 + \widetilde{\Gamma}_b,\\
\nabla'(J'^{(\pm)}) &= \Big(1+O(r^{-2})\Big)\nabla(J^{(\pm)}) + O(r^{-1})e_3(J'^{(\pm)}) + O(r^{-1})e_4(J'^{(\pm)})\\
&= \frac{1}{r}f_\pm + \widetilde{\Gamma}_b + O(r^{-3}),
\end{aligned}$$

and hence

$$\begin{aligned}
\widetilde{\nabla'(u')} &= O(r^{-3}) + r^{-2}\widetilde{\Gamma}_b,\\
\widetilde{\nabla'(\cos\theta')} &= O(r^{-3}) + \widetilde{\Gamma}_b,\\
\widetilde{\nabla'(J'^{(\pm)})} &= O(r^{-3}) + \widetilde{\Gamma}_b.
\end{aligned}$$

Together with the dominance condition (5.30) on r on Σ_*, and the estimates for $\widetilde{\Gamma}_b$ of Corollary 5.73, we obtain

$$\sup_{\Sigma_*} ru^{1+\delta_{dec}}\left(\left|\mathfrak{d}^{\leq k_*-12}\widetilde{\mathcal{D}'(\cos\theta')}\right| + \left|\mathfrak{d}^{\leq k_*-12}\widetilde{\mathcal{D}'(u')}\right| + \left|\mathfrak{d}^{\leq k_*-12}\widetilde{\mathcal{D}'(J'^{(\pm)})}\right|\right)$$
$$\lesssim \epsilon_0.$$

This concludes the proof of the lemma. □

Lemma 5.79. *Consider the change of frame coefficients $(f', \underline{f}', \lambda')$ from the PG frame (e_3', e_4', e_1', e_2') of ${}^{(ext)}\mathcal{M}$ to the frame (e_3, e_4, e_1, e_2) on Σ_*, i.e.*

$$\begin{aligned}
e_4 &= \lambda'\left(e_4' + f_b'e_b' + \frac{1}{4}|f'|^2e_3'\right),\\
e_a &= \left(\delta_a^b + \frac{1}{2}\underline{f}_a'f'^b\right)e_b' + \frac{1}{2}\underline{f}_a'e_4' + \left(\frac{1}{2}f_a' + \frac{1}{8}|f'|^2\underline{f}_a'\right)e_3',\\
e_3 &= (\lambda')^{-1}\Bigg(\left(1+\frac{1}{2}f'\cdot\underline{f}' + \frac{1}{16}|f'|^2|\underline{f}'|^2\right)e_3' + \left(\underline{f}'^b + \frac{1}{4}|\underline{f}'|^2f'^b\right)e_b'\\
&\quad + \frac{1}{4}|\underline{f}'|^2e_4'\Bigg).
\end{aligned} \tag{5.144}$$

Then, we have

$$\lambda' = 1 + O(r^{-2}) + r^{-1}\Gamma_b,$$
$$f' = -\frac{a}{r}\Big(1 + O(r^{-2}) + r^{-1}\Gamma_b\Big) f_0,$$
$$\underline{f}' = -\frac{a\Upsilon}{r}\Big(1 + O(r^{-2}) + r^{-1}\Gamma_b\Big) f_0.$$

Proof. Recall that $(f, \underline{f}, \lambda)$, with $\lambda = 1$ and $(f, \underline{f})$ given by (5.139), are the change of frame coefficients from the null frame (e_3, e_4, e_1, e_2) to the null frame (e_3', e_4', e_1', e_2'), so that $(f', \underline{f}', \lambda')$ correspond to the inverse transformation of the one of $(f, \underline{f}, \lambda)$. Thus, according to (2.8), $(f', \underline{f}', \lambda')$ is related to the transition coefficients $(f, \underline{f}, \lambda)$ by

$$\lambda' = \lambda^{-1}\left(1 + \frac{1}{2} f \cdot \underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right),$$
$$f_a' = -\frac{\lambda}{1 + \frac{1}{2} f \cdot \underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2}\left(f_a + \frac{1}{4}|f|^2\underline{f}_a\right),$$
$$\underline{f}_a' = -\lambda^{-1}\left(\underline{f}_a + \frac{1}{4}|\underline{f}|^2 f_a\right).$$

Since $(f, \underline{f})$ are given by (5.139), and since $|f_0|^2 = (\sin\theta)^2$, $\nu(r) = -2 + r\Gamma_b$ and $b_* = -1 - \frac{2m}{r} + r\Gamma_b$, we have

$$
\begin{aligned}
f &= \frac{a}{r} f_0,\\
\underline{f} &= -\frac{(\nu(r) - b_*)}{1 - \frac{1}{4} b_* \frac{a^2(\sin\theta)^2}{r^2}} \frac{a}{r} f_0 = \frac{a\Upsilon}{r}\Big(1 + O(r^{-2}) + r\Gamma_b\Big) f_0,\\
f \cdot \underline{f} &= -\frac{(\nu(r) - b_*)}{1 - \frac{1}{4} b_* \frac{a^2(\sin\theta)^2}{r^2}} \frac{a^2(\sin\theta)^2}{r^2} = \frac{a^2(\sin\theta)^2\Upsilon}{r^2}\Big(1 + O(r^{-2}) + r\Gamma_b\Big),\\
|f|^2 &= \frac{a^2(\sin\theta)^2}{r^2},\\
|\underline{f}|^2 &= \frac{(\nu(r) - b_*)^2}{\Big(1 - \frac{1}{4} b_* \frac{a^2(\sin\theta)^2}{r^2}\Big)^2} \frac{a^2(\sin\theta)^2}{r^2} = \frac{a^2(\sin\theta)^2\Upsilon^2}{r^2}\Big(1 + O(r^{-2}) + r\Gamma_b\Big),
\end{aligned}
$$

and hence, using also $\lambda = 1$, we infer

$$\lambda' = 1 + O(r^{-2}) + r^{-1}\Gamma_b,$$
$$f' = -\frac{a}{r}\Big(1 + O(r^{-2}) + r^{-1}\Gamma_b\Big) f_0,$$
$$\underline{f}' = -\frac{a\Upsilon}{r}\Big(1 + O(r^{-2}) + r^{-1}\Gamma_b\Big) f_0,$$

as stated. ∎

Lemma 5.80. *We have*

$$\nabla\Big(O(r^{-j})\Big) = O(r^{-j-1})+O(r^{-j})\widetilde{\Gamma}_b,$$
$$\nu\Big(O(r^{-j})\Big) = O(r^{-j-1})+O(r^{-j})\widetilde{\Gamma}_b,$$

where the notation $\widetilde{\Gamma}_b$ has been introduced in Definition 5.72.

Proof. Recall that by $O(r^{-j})$, we mean a function $h(r,\cos\theta)$ such that

$$|(r\partial_r)^k(\partial_{\cos\theta})^l h|\lesssim \frac{C_{k,l}}{r^j}, \quad \text{as} \quad r\to+\infty.$$

The proof is then an immediate consequence of the fact that $\nu(r)=-2+r\Gamma_b$ and $\nabla(\cos\theta)=-\frac{1}{r}\,{}^*f_0+\widetilde{\Gamma}_b$. □

Lemma 5.81. *We have*

$$\begin{aligned}
\mathrm{div}\,(f') &= r^{-1}\mathfrak{d}_*^{\leq 1}\widetilde{\Gamma}_b+O\left(\frac{1}{r^3}\right),\\
\mathrm{curl}\,(f') &= -\frac{2a\cos\theta}{r^2}+r^{-1}\mathfrak{d}_*^{\leq 1}\widetilde{\Gamma}_b+O\left(\frac{1}{r^3}\right),\\
\nabla\widehat{\otimes} f' &= r^{-1}\mathfrak{d}_*^{\leq 1}\widetilde{\Gamma}_b+O\left(\frac{1}{r^3}\right),\\
\mathrm{div}\,(\underline{f}') &= r^{-1}\mathfrak{d}_*^{\leq 1}\widetilde{\Gamma}_b+O\left(\frac{1}{r^3}\right),\\
\mathrm{curl}\,(\underline{f}') &= -\frac{2a\cos\theta\Upsilon}{r^2}+r^{-1}\mathfrak{d}_*^{\leq 1}\widetilde{\Gamma}_b+O\left(\frac{1}{r^3}\right),\\
\nabla\widehat{\otimes}(\underline{f}') &= r^{-1}\mathfrak{d}_*^{\leq 1}\widetilde{\Gamma}_b+O\left(\frac{1}{r^3}\right),\\
\nabla\lambda' &= O(r^{-3})+r^{-2}\mathfrak{d}_*^{\leq 1}\widetilde{\Gamma}_b.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\nabla_\nu f' &= \frac{2}{r}f'+r^{-1}\Gamma_b+O\left(\frac{1}{r^3}\right),\\
\nabla_\nu \underline{f}' &= \frac{2}{r}\underline{f}'+r^{-1}\mathfrak{d}_*^{\leq 1}\Gamma_b+O\left(\frac{1}{r^3}\right),\\
\nu(\lambda') &= O(r^{-3})+r^{-1}\mathfrak{d}_*^{\leq 1}\widetilde{\Gamma}_b,
\end{aligned}$$

where the notation $\widetilde{\Gamma}_b$ has been introduced in Definition 5.72.

Proof. Recall that

$$\lambda' = 1 + O(r^{-2}) + r^{-1}\Gamma_b,$$
$$f' = -\frac{a}{r}\Big(1 + O(r^{-2}) + r^{-1}\Gamma_b\Big) f_0,$$
$$\underline{f}' = -\frac{a\Upsilon}{r}\Big(1 + O(r^{-2}) + r^{-1}\Gamma_b\Big) f_0.$$

The proof follows then immediately from the definition of $\widetilde{\Gamma}_b$, Lemma 5.80, and the fact that $\nu(r) = -2 + r\Gamma_b$ and $\nabla_\nu f_0 = 0$. □

5.7.3. Decay estimates for the PG frame on Σ_* We start with the control of the Ricci coefficients of the PG frame of ${}^{(ext)}\mathcal{M}$ on Σ_*.

Lemma 5.82. *We have on Σ_*, for $k \leq k_* - 13$,*

$$\begin{aligned}
\left|\mathfrak{d}_*^k\Big(\widehat{\underline{\chi}}', \underline{\xi}', \widecheck{\underline{\omega}}', \widecheck{\underline{\eta}}'\Big)\right| &\lesssim \frac{\epsilon_0}{ru^{1+\delta_{dec}}},\\
\left|\mathfrak{d}_*^k\Big(\widehat{\chi}', \widecheck{tr\underline{\chi}}', \widecheck{{}^{(a)}tr\underline{\chi}}', \widecheck{\zeta}'\Big)\right| &\lesssim \frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta_{dec}}},\\
\left|\mathfrak{d}_*^{k-1}\nabla_\nu\Big(\widehat{\chi}', \widecheck{tr\underline{\chi}}', \widecheck{{}^{(a)}tr\underline{\chi}}', \widecheck{\zeta}'\Big)\right| &\lesssim \frac{\epsilon_0}{r^2u^{1+\delta_{dec}}},\\
\left|\mathfrak{d}_*^k\Big(\widecheck{tr\chi}', \widecheck{{}^{(a)}tr\chi}'\Big)\right| &\lesssim \frac{\epsilon_0}{r^2u^{1+\delta_{dec}}}.
\end{aligned} \tag{5.145}$$

Proof. We consider the frame transformation from the frame (e_1', e_2', e_3', e_4') of ${}^{(ext)}\mathcal{M}$ to the frame (e_1, e_2, e_3, e_4) of Σ_*, with corresponding change of frame coefficients $(f', \underline{f}', \lambda')$. Using the transformation formulas of Proposition 2.12, Lemma 5.79 on the control of $(f', \underline{f}', \lambda')$, and the fact that ${}^{(a)}\mathrm{tr}\chi = {}^{(a)}\mathrm{tr}\underline{\chi} = 0$, $\xi' = 0$, $\omega' = 0$, and $\underline{\eta}' = -\zeta'$, we have

$$\begin{aligned}
\mathrm{tr}\,\chi &= \mathrm{tr}\,\chi' + \mathrm{div}\,(f') + O(r^{-3}) + O(r^{-1})\widecheck{\eta}' + r^{-1}\Gamma_g' + r^{-2}\Gamma_b,\\
0 &= {}^{(a)}\mathrm{tr}\chi' + \mathrm{curl}\,(f') + O(r^{-3}) + O(r^{-1})\widecheck{\eta}' + r^{-1}\Gamma_g' + r^{-2}\Gamma_b,\\
\widehat{\chi} &= \widehat{\chi}' + \nabla\widehat{\otimes} f' + O(r^{-3}) + O(r^{-1})\widecheck{\eta}' + r^{-1}\Gamma_g' + r^{-2}\Gamma_b,\\
\mathrm{tr}\,\underline{\chi} &= \mathrm{tr}\,\underline{\chi}' + \mathrm{div}\,(\underline{f}') + O(r^{-3}) + O(r^{-1})\underline{\xi}' + r^{-1}\Gamma_g' + r^{-2}\Gamma_b,\\
0 &= {}^{(a)}\mathrm{tr}\underline{\chi}' + \mathrm{curl}\,(\underline{f}') + O(r^{-3}) + O(r^{-1})\underline{\xi}' + r^{-1}\Gamma_g' + r^{-2}\Gamma_b,\\
\widehat{\underline{\chi}} &= \widehat{\underline{\chi}}' + \nabla\widehat{\otimes}\underline{f}' + O(r^{-3}) + r^{-1}\Gamma_b' + r^{-2}\Gamma_b,
\end{aligned}$$

and

$$\zeta = \zeta' - \nabla(\log\lambda') - \frac{1}{4}\mathrm{tr}\,\underline{\chi}' f' + \frac{1}{4}\underline{f}'\mathrm{tr}\,\chi' + \frac{1}{4}\underline{f}'\mathrm{div}\,(f') + \frac{1}{4}\,{}^*\underline{f}'\mathrm{curl}\,(f')$$

$$+O(r^{-1})(\underline{\check{\omega}}', \underline{\widehat{\chi}}') + r^{-1}\Gamma_g' + O(r^{-3}) + r^{-1}\Gamma_b' + r^{-2}\Gamma_b + r^{-1}\Gamma_g.$$

Together with Lemma 5.81 on the control of first order derivatives of $(f', \underline{f}', \lambda')$, and Corollary 2.51 on the asymptotic for r large of the Kerr values of the PG frame, we infer

$$\begin{aligned}
\widecheck{\mathrm{tr}\chi} &= \widecheck{\mathrm{tr}\chi}' + O(r^{-3}) + O(r^{-1})\check{\eta}' + r^{-1}\Gamma_g' + r^{-1}\mathfrak{d}_*^{\le 1}\widetilde{\Gamma}_b,\\
0 &= \widecheck{{}^{(a)}\mathrm{tr}\chi}' + O(r^{-3}) + O(r^{-1})\check{\eta}' + r^{-1}\Gamma_g' + r^{-1}\mathfrak{d}_*^{\le 1}\widetilde{\Gamma}_b,\\
\widehat{\chi} &= \widehat{\chi}' + O(r^{-3}) + O(r^{-1})\check{\eta}' + r^{-1}\Gamma_g' + r^{-1}\mathfrak{d}_*^{\le 1}\widetilde{\Gamma}_b,\\
\widecheck{\mathrm{tr}\underline{\chi}} &= \widecheck{\mathrm{tr}\underline{\chi}}' + O(r^{-3}) + O(r^{-1})\underline{\xi}' + r^{-1}\Gamma_g' + r^{-1}\mathfrak{d}_*^{\le 1}\widetilde{\Gamma}_b,\\
0 &= \widecheck{{}^{(a)}\mathrm{tr}\underline{\chi}}' + O(r^{-3}) + O(r^{-1})\underline{\xi}' + r^{-1}\Gamma_g' + r^{-1}\mathfrak{d}_*^{\le 1}\widetilde{\Gamma}_b,\\
\underline{\widehat{\chi}} &= \underline{\widehat{\chi}}' + O(r^{-3}) + r^{-1}\Gamma_b' + r^{-1}\mathfrak{d}_*^{\le 1}\widetilde{\Gamma}_b,
\end{aligned}$$

and

$$\zeta = \check{\zeta}' + O(r^{-3}) + O(r^{-1})(\underline{\check{\omega}}', \underline{\widehat{\chi}}') + r^{-1}\Gamma_g' + r^{-1}\mathfrak{d}_*^{\le 1}\widetilde{\Gamma}_b.$$

Next, we use again the transformation formulas of Proposition 2.12, summing the one of $\underline{\xi}$ with the one for $\underline{\eta}$ multiplied by $\lambda^2 b_*$, summing the one of $\underline{\omega}$ with the one for ω multiplied by $-\lambda^2 b_*$, and summing the one for η with the one for ξ multiplied by $\lambda^2 b_*$. Proceeding as above, and using in addition the fact that $\nu = e_3 + b_* e_4$, and the transversality conditions $\xi = 0$, $\omega = 0$ and $\underline{\eta} = -\zeta$ for the frame of Σ_*, we infer

$$\begin{aligned}
\underline{\xi} - b_*\zeta &= \underline{\xi}' - b_*\zeta' + \frac{1}{2}\nabla_\nu \underline{f}' + \frac{1}{4}\mathrm{tr}\,\underline{\chi}'\,\underline{f}' + \frac{b_*}{4}\mathrm{tr}\,\underline{\chi}' f' + O(r^{-3}) + r^{-1}\Gamma_b'\\
&\quad + r^{-2}\Gamma_b,\\
\underline{\omega} &= \underline{\omega}' + \frac{1}{2}\nu(\log\lambda') + O(r^{-3}) + r^{-1}\Gamma_b' + r^{-2}\Gamma_b,\\
\eta &= \eta' + \frac{1}{2}\nabla_\nu f' + \frac{1}{4}\underline{f}'\mathrm{tr}\,\chi' + \frac{b_*}{4}\mathrm{tr}\,\chi' f' + O(r^{-3}) + r^{-1}\Gamma_b' + r^{-2}\Gamma_b.
\end{aligned}$$

Together with Lemma 5.81 on the control of first order derivatives of $(f', \underline{f}', \lambda')$, and Corollary 2.51 on the asymptotic for r large of the Kerr values of the PG frame, we deduce

$$\begin{aligned}
\underline{\xi} - b_*\zeta &= \underline{\xi}' - \check{\zeta}' + O(r^{-3}) + r^{-1}\Gamma_b' + r^{-1}\mathfrak{d}_*^{\le 1}\widetilde{\Gamma}_b,\\
\underline{\check{\omega}} &= \underline{\check{\omega}}' + O(r^{-3}) + r^{-1}\Gamma_b' + r^{-1}\mathfrak{d}_*^{\le 1}\widetilde{\Gamma}_b,
\end{aligned}$$

$$\eta = \check{\eta}' + O(r^{-3}) + r^{-1}\Gamma_b' + r^{-1}\mathfrak{d}_*^{\leq 1}\widetilde{\Gamma}_b.$$

In view of the above, we infer

$$\Big(\widehat{\underline{\chi}}', \underline{\xi}', \check{\underline{\omega}}', \check{\eta}'\Big) = O(r^{-3}) + \Gamma_g' + r^{-1}\Gamma_b' + \mathfrak{d}_*^{\leq 1}\widetilde{\Gamma}_b.$$

Together with the bootstrap assumptions for Γ_g' and Γ_b', and Corollary 5.73 on the control of $\widetilde{\Gamma}_b$, this yields on Σ_*, for $k \leq k_* - 13$,

$$\Big|\mathfrak{d}_*^k\Big(\widehat{\underline{\chi}}', \underline{\xi}', \check{\underline{\omega}}', \check{\eta}'\Big)\Big| \lesssim \frac{\epsilon_0}{ru^{1+\delta_{dec}}} + \frac{\epsilon}{r^2} + \frac{1}{r^3},$$

and hence, together with the dominance condition (5.30) on r on Σ_*, we obtain, for $k \leq k_* - 13$,

$$\Big|\mathfrak{d}_*^k\Big(\widehat{\underline{\chi}}', \underline{\xi}', \check{\underline{\omega}}', \check{\eta}'\Big)\Big| \lesssim \frac{\epsilon_0}{ru^{1+\delta_{dec}}}.$$

Next, using again the above, we have

$$\begin{aligned}
\Big(\widehat{\chi}', \widecheck{\mathrm{tr}\underline{\chi}}', \widecheck{{}^{(a)}\mathrm{tr}\underline{\chi}}', \check{\zeta}'\Big) &= O(r^{-3}) + \Gamma_g + O(r^{-1})\Big(\widehat{\underline{\chi}}', \underline{\xi}', \check{\underline{\omega}}', \check{\eta}'\Big) + r^{-1}\Gamma_g' \\
&\quad + r^{-1}\mathfrak{d}_*^{\leq 1}\widetilde{\Gamma}_b, \\
\Big(\widecheck{\mathrm{tr}\chi}', \widecheck{{}^{(a)}\mathrm{tr}\chi}'\Big) &= O(r^{-3}) + \widecheck{\mathrm{tr}\chi} + O(r^{-1})\Big(\widehat{\underline{\chi}}', \underline{\xi}', \check{\underline{\omega}}', \check{\eta}'\Big) + r^{-1}\Gamma_g' \\
&\quad + r^{-1}\mathfrak{d}_*^{\leq 1}\widetilde{\Gamma}_b.
\end{aligned}$$

Together with the above control for $\widehat{\underline{\chi}}'$, $\underline{\xi}'$, $\check{\underline{\omega}}'$, and $\check{\eta}'$, the control for Γ_g and $\widecheck{\mathrm{tr}\chi}$ of Proposition 5.53, the bootstrap assumptions for Γ_g', and Corollary 5.73 on the control of $\widetilde{\Gamma}_b$, this yields on Σ_*, for $k \leq k_* - 13$,

$$\begin{aligned}
\Big|\mathfrak{d}_*^k\Big(\widehat{\chi}', \widecheck{\mathrm{tr}\underline{\chi}}', \widecheck{{}^{(a)}\mathrm{tr}\underline{\chi}}', \check{\zeta}'\Big)\Big| &\lesssim \frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta_{dec}}} + \frac{1}{r^3}, \\
\Big|\mathfrak{d}_*^{k-1}\nabla_\nu\Big(\widehat{\chi}', \widecheck{\mathrm{tr}\underline{\chi}}', \widecheck{{}^{(a)}\mathrm{tr}\underline{\chi}}', \check{\zeta}'\Big)\Big| &\lesssim \frac{\epsilon_0}{r^2u^{1+\delta_{dec}}} + \frac{1}{r^3}, \\
\Big|\mathfrak{d}_*^k\Big(\widecheck{\mathrm{tr}\chi}', \widecheck{{}^{(a)}\mathrm{tr}\chi}'\Big)\Big| &\lesssim \frac{\epsilon_0}{r^2u^{1+\delta_{dec}}} + \frac{1}{r^3},
\end{aligned}$$

and hence, together with the dominance condition (5.30) on r on Σ_*, we obtain, for $k \leq k_* - 13$,

$$\Big|\mathfrak{d}_*^k\Big(\widehat{\chi}', \widecheck{\mathrm{tr}\underline{\chi}}', \widecheck{{}^{(a)}\mathrm{tr}\underline{\chi}}', \check{\zeta}'\Big)\Big| \lesssim \frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta_{dec}}},$$

$$\begin{aligned}\left|\mathfrak{d}_*^{k-1}\nabla_\nu\Big(\widehat{\chi}', \widecheck{\mathrm{tr}\underline{\chi}}', \widecheck{{}^{(a)}\mathrm{tr}\underline{\chi}}', \widecheck{\zeta}'\Big)\right| &\lesssim \frac{\epsilon_0}{r^2u^{1+\delta_{dec}}},\\ \left|\mathfrak{d}_*^{k}\Big(\widecheck{\mathrm{tr}\chi}', \widecheck{{}^{(a)}\mathrm{tr}\chi}'\Big)\right| &\lesssim \frac{\epsilon_0}{r^2u^{1+\delta_{dec}}}.\end{aligned}$$

This concludes the proof of the lemma. □

Lemma 5.83. *We have on Σ_*, for $k\leq k_*-14$,*

$$\begin{aligned}& r^{\frac{7}{2}+\delta_{extra}}\left|\mathfrak{d}^k\beta'\right| + r^4u^{\frac{1}{2}+\delta_{dec}}\left|\mathfrak{d}^{k-1}\nabla_\nu\beta'\right| + r^3u^{\frac{1}{2}+\delta_{dec}}\left|\mathfrak{d}^k\widecheck{P}'\right|\\ &\qquad +r^3u^{1+\delta_{dec}}\left|\mathfrak{d}^{k-1}\nabla_\nu\widecheck{P}'\right| + r^2u^{1+\delta_{dec}}\left|\mathfrak{d}^k\underline{\beta}'\right| \lesssim \epsilon_0,\end{aligned}$$

and

$$\left|\mathfrak{d}_*^k\left(\beta' - \left(\beta - \frac{3am}{r^4}f_0\right) - \frac{3a}{2r}\left(\widecheck{\rho}'f_0 + \widecheck{{}^*\rho}'\,{}^*f_0\right) - \frac{a}{2r}f_0\cdot\alpha'\right)\right| \lesssim \frac{\epsilon_0}{r^4u^{1+\delta_{dec}}}.$$

Proof. We consider the frame transformation from the frame (e_1', e_2', e_3', e_4') of ${}^{(ext)}\mathcal{M}$ to the frame (e_1, e_2, e_3, e_4) of Σ_*, with corresponding change of frame coefficients $(f', \underline{f}', \lambda')$. Using the transformation formulas of Proposition 2.12, and Lemma 5.79 on the control of $(f', \underline{f}', \lambda')$, we have[127]

$$\begin{aligned}\beta &= \beta' + \frac{3}{2}\left(-\frac{a}{r}f_0\left(-\frac{2m}{r^3}+\widecheck{\rho}'\right) - \frac{a}{r}\,{}^*f_0\,\widecheck{{}^*\rho}'\right) - \frac{a}{2r}f_0\cdot\alpha' + O(r^{-2})\underline{\beta}'\\ &\quad +O(r^{-3})\underline{\alpha}' + O(r^{-3})\Gamma_g' + O(r^{-5}) + r^{-3}\Gamma_b,\\ \underline{\beta} &= \underline{\beta}' + O(r^{-1})\underline{\alpha}' + O(r^{-4}) + r^{-2}\Gamma_g' + r^{-3}\Gamma_b,\\ \rho &= \rho' + O(r^{-4}) + r^{-2}\Gamma_g' + O(r^{-1})\underline{\beta}' + O(r^{-2})\underline{\alpha}',\\ {}^*\rho &= {}^*\rho' + O(r^{-4}) + r^{-2}\Gamma_g' + O(r^{-1})\underline{\beta}' + O(r^{-2})\underline{\alpha}'.\end{aligned}$$

In particular, we have

$$\underline{\beta}' = O(r^{-1})\underline{\alpha}' + O(r^{-4}) + r^{-2}\Gamma_g' + r^{-1}\Gamma_b,$$

which together with the control of Theorem M2 for $\underline{\alpha}'$, the control of Proposition 5.42 for Γ_b, and the bootstrap assumptions for Γ_g', this yields on Σ_*,

[127] Note that the term $O(r^{-3})\underline{\alpha}'$ appearing in the change of frame formula for β claimed here is better by one power of r than what is consistent with Proposition 2.12. This gain of one power of r can easily be checked by signature considerations.

for $k \leq k_* - 10$,

$$|\mathfrak{d}_*^k \underline{\beta}'| \lesssim \frac{\epsilon_0}{r^2 u^{1+\delta_{dec}}} + \frac{1}{r^4},$$

and hence, together with the dominance condition (5.30) on r on Σ_*, we obtain, for $k \leq k_* - 10$,

$$|\mathfrak{d}_*^k \underline{\beta}'| \lesssim \frac{\epsilon_0}{r^2 u^{1+\delta_{dec}}}.$$

Next, we rewrite the above identities for ρ and ${}^*\rho$ as

$$\begin{aligned}
\check{\rho}' &= O(r^{-4}) + r^{-2}\Gamma'_g + O(r^{-1})\underline{\beta}' + O(r^{-2})\underline{\alpha}' + r^{-1}\Gamma_g,\\
\widecheck{{}^*\rho}' &= O(r^{-4}) + r^{-2}\Gamma'_g + O(r^{-1})\underline{\beta}' + O(r^{-2})\underline{\alpha}' + r^{-1}\Gamma_g,
\end{aligned}$$

which together with the above control of $\underline{\beta}'$, the control of Theorem M2 for $\underline{\alpha}'$, the control for Γ_g of Proposition 5.53, and the bootstrap assumptions for Γ'_g, this yields on Σ_*, for $k \leq k_* - 10$,

$$\begin{aligned}
|\mathfrak{d}_*^k(\check{\rho}', \widecheck{{}^*\rho}')| &\lesssim \frac{\epsilon_0}{r^3 u^{\frac{1}{2}+\delta_{dec}}} + \frac{1}{r^4},\\
|\mathfrak{d}_*^{k-1}\nabla_\nu(\check{\rho}', \widecheck{{}^*\rho}')| &\lesssim \frac{\epsilon_0}{r^3 u^{1+\delta_{dec}}} + \frac{1}{r^4},
\end{aligned}$$

and hence, together with the dominance condition (5.30) on r on Σ_*, we obtain, for $k \leq k_* - 12$,

$$\begin{aligned}
|\mathfrak{d}_*^k(\check{\rho}', \widecheck{{}^*\rho}')| &\lesssim \frac{\epsilon_0}{r^3 u^{\frac{1}{2}+\delta_{dec}}},\\
|\mathfrak{d}_*^{k-1}\nabla_\nu(\check{\rho}', \widecheck{{}^*\rho}')| &\lesssim \frac{\epsilon_0}{r^3 u^{1+\delta_{dec}}}.
\end{aligned}$$

Next, we rewrite the above identity for β as[128]

$$\begin{aligned}
\beta' = {} & \left(\beta - \frac{3am}{r^4} f_0\right) + \frac{3a}{2r}\left(\check{\rho}' f_0 + \widecheck{{}^*\rho}'\, {}^*f_0\right) + \frac{a}{2r} f_0 \cdot \alpha' + O(r^{-2})\underline{\beta}'\\
& + O(r^{-3})\underline{\alpha}' + O(r^{-3})\Gamma'_g + O(r^{-5}) + r^{-3}\Gamma_b,
\end{aligned}$$

which together with the above control of $\underline{\beta}'$, the control of Theorem M2 for $\underline{\alpha}'$, the control for Γ_g of Proposition 5.53, and the bootstrap assumptions for

[128] See Footnote 127 for the gain of one power of r in front of $\underline{\alpha}'$.

Γ'_g, this yields on Σ_*, for $k \leq k_* - 12$,

$$\begin{aligned}&\left|\mathfrak{d}_*^k\left(\beta' - \left(\beta - \frac{3am}{r^4}f_0\right) - \frac{3a}{2r}\left(\check{\rho}' f_0 + \check{{}^*\rho}'\, {}^*f_0\right) - \frac{a}{2r}f_0\cdot\alpha'\right)\right| \\ \lesssim\ & \frac{\epsilon_0}{r^4 u^{1+\delta_{dec}}} + \frac{1}{r^5},\end{aligned}$$

and hence, together with the dominance condition (5.30) on r on Σ_*, we obtain, for $k \leq k_* - 12$,

$$\left|\mathfrak{d}_*^k\left(\beta' - \left(\beta - \frac{3am}{r^4}f_0\right) - \frac{3a}{2r}\left(\check{\rho}' f_0 + \check{{}^*\rho}'\, {}^*f_0\right) - \frac{a}{2r}f_0\cdot\alpha'\right)\right| \lesssim \frac{\epsilon_0}{r^4 u^{1+\delta_{dec}}}.$$

In particular, we have, for $k \leq k_* - 12$,

$$\begin{aligned}\left|\mathfrak{d}_*^k\beta'\right| &\lesssim \left|\mathfrak{d}_*^k\left(\beta - \frac{3am}{r^4}f_0\right)\right| + r^{-1}\left|\mathfrak{d}_*^k\left(\check{\rho}', \check{{}^*\rho}', \alpha'\right)\right| + \frac{\epsilon_0}{r^4 u^{1+\delta_{dec}}},\\ \left|\mathfrak{d}_*^{k-1}\nabla_\nu\beta'\right| &\lesssim \left|\mathfrak{d}_*^{k-1}\nabla_\nu\left(\beta - \frac{3am}{r^4}f_0\right)\right| + r^{-1}\left|\mathfrak{d}_*^k\left(\check{\rho}', \check{{}^*\rho}', \alpha'\right)\right| + \frac{\epsilon_0}{r^4 u^{1+\delta_{dec}}}\\ &\lesssim \left|\mathfrak{d}_*^{k-1}\nabla_\nu\beta\right| + r^{-1}\left|\mathfrak{d}_*^k\left(\check{\rho}', \check{{}^*\rho}', \alpha'\right)\right| + \frac{1}{r^5} + \frac{\epsilon_0}{r^4 u^{1+\delta_{dec}}},\end{aligned}$$

where we also used the fact that $\nabla_\nu f_0 = 0$, $\nu(r) = -2 + r\Gamma_b$, and the estimates **Ref 1** for Γ_b. Together with the estimate of Corollary 5.70 for $\beta - \frac{3am\sin\theta}{r^4}f_0$, the estimates of Proposition 5.53 for $\nabla_\nu\beta$, the above estimates for $\check{\rho}'$ and $\check{{}^*\rho}'$, the control of Theorem M1 for α', and the dominance condition (5.30) on r on Σ_*, we infer, for $k \leq k_* - 14$,

$$\begin{aligned}\left|\mathfrak{d}_*^k\beta'\right| &\lesssim \frac{\epsilon_0}{r^{\frac{7}{2}+\delta_{extra}}},\\ \left|\mathfrak{d}_*^{k-1}\nabla_\nu\beta'\right| &\lesssim \frac{\epsilon_0}{r^4 u^{\frac{1}{2}+\delta_{dec}}},\end{aligned}$$

which concludes the proof of the lemma. □

5.7.4. Additional decay estimates on Σ_* In this section, we prove the remaining estimates of Proposition 5.77. We start with the following lemma.

Lemma 5.84. *We have*

$$\begin{aligned}&\sup_{\Sigma_*} r^5 u^{1+\delta_{dec}}\left|\left[\left(\overline{\mathcal{D}'}\cdot - \frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_{e'_4} - \frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_{e'_3}\right)\cdot\left(B' - \frac{3a}{2}\overline{\check{P}'}\mathfrak{J} - \frac{a}{4}\overline{\mathfrak{J}}\cdot A'\right)\right]_{\ell=1}\right|\\ &\lesssim \epsilon_0.\end{aligned}$$

Proof. Recall from Lemma 5.83 that we have on Σ_*, for $k \leq k_* - 14$,

$$\left|\mathfrak{d}_*^k\left(\beta' - \left(\beta - \frac{3am}{r^4}f_0\right) - \frac{3a}{2r}\left(\widecheck{\rho}'f_0 + \widecheck{{}^*\rho}'\,{}^*f_0\right) - \frac{a}{2r}f_0\cdot\alpha'\right)\right| \lesssim \frac{\epsilon_0}{r^4u^{1+\delta_{dec}}},$$

and hence, in particular,

$$\left|\not{d}_1\left(\beta' - \frac{3a}{2r}\left(\widecheck{\rho}'f_0 + \widecheck{{}^*\rho}'\,{}^*f_0\right) - \frac{a}{2r}f_0\cdot\alpha'\right) - \not{d}_1\left(\beta - \frac{3am}{r^4}f_0\right)\right| \lesssim \frac{\epsilon_0}{r^5u^{1+\delta_{dec}}}.$$

Taking the $\ell = 1$ mode, we infer

$$\begin{aligned}
&\left|\left[\not{d}_1\left(\beta' - \frac{3a}{2r}\left(\widecheck{\rho}'f_0 + \widecheck{{}^*\rho}'\,{}^*f_0\right) - \frac{a}{2r}f_0\cdot\alpha'\right)\right]_{\ell=1}\right|\\
\lesssim\ &\left|\left[\not{d}_1\left(\beta - \frac{3am}{r^4}f_0\right)\right]_{\ell=1}\right| + \frac{\epsilon_0}{r^5u^{1+\delta_{dec}}}.
\end{aligned}$$

Together with Corollary 5.70, we deduce

$$\left|\left[\not{d}_1\left(\beta' - \frac{3a}{2r}\left(\widecheck{\rho}'f_0 + \widecheck{{}^*\rho}'\,{}^*f_0\right) - \frac{a}{2r}f_0\cdot\alpha'\right)\right]_{\ell=1}\right| \lesssim \frac{\epsilon_0}{r^5u^{1+\delta_{dec}}}.$$

Next, in view of the definition of $\mathfrak{J}$ in terms of f_0, and since $|q| = r(1 + O(r^{-2}))$, we have

$$\begin{aligned}
&\left(B' - \frac{3a}{2}\overline{\widecheck{P}'}\mathfrak{J} - \frac{a}{4}\overline{\mathfrak{J}}\cdot A'\right)_1\\
=\ &\left(\beta' - \frac{3a}{2r}\left(\widecheck{\rho}'f_0 + \widecheck{{}^*\rho}'\,{}^*f_0\right) - \frac{a}{2r}\alpha'\cdot f_0\right)_1\\
&+i\left(\beta' - \frac{3a}{2r}\left(\widecheck{\rho}'f_0 + \widecheck{{}^*\rho}'\,{}^*f_0\right) - \frac{a}{2r}\alpha'\cdot f_0\right)_2 + O(r^{-2})(\widecheck{P}', A'),
\end{aligned}$$

and hence

$$\left(B' - \frac{3a}{2}\overline{\widecheck{P}}\mathfrak{J} - \frac{a}{4}\overline{\mathfrak{J}}\cdot A\right) = w + i\,{}^*w + O(r^{-2})(\widecheck{P}', A'),$$

with

$$w := \beta' - \frac{3a}{2r}\left(\widecheck{\rho}'f_0 + \widecheck{{}^*\rho}'\,{}^*f_0\right) - \frac{a}{2r}\alpha'\cdot f_0.$$

Then, we have

$$\overline{\mathcal{D}}\cdot\left(B' - \frac{3a}{2}\overline{\widecheck{P}'}\mathfrak{J} - \frac{a}{4}\overline{\mathfrak{J}}\cdot A'\right)$$

$$
\begin{aligned}
&= \overline{\mathcal{D}}\cdot(w+i\,{}^*w)+O(r^{-3})\,\mathfrak{d}\mkern-9mu /^{\leq 1}(\check{P}',A')\\
&= (\nabla-i\,{}^*\nabla)\cdot(w+i\,{}^*w)+O(r^{-3})\,\mathfrak{d}\mkern-9mu /^{\leq 1}(\check{P}',A')\\
&= 2\mathrm{div}\,(w)+2i\mathrm{curl}\,(w)+O(r^{-3})\,\mathfrak{d}\mkern-9mu /^{\leq 1}(\check{P}',A')
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\sup_{\Sigma_*} r^5u^{1+\delta_{dec}}\left|\left[\overline{\mathcal{D}}\cdot\left(B'-\frac{3a}{2r}\overline{\check{P}'}\mathfrak{J}-\frac{a}{4r}\overline{\mathfrak{J}}\cdot A'\right)\right]_{\ell=1}\right|\\
\lesssim\;&\sup_{\Sigma_*} r^5u^{1+\delta_{dec}}|(\,\mathfrak{d}\mkern-9mu /_1 w)_{\ell=1}|+\sup_{\Sigma_*} r^2u^{1+\delta_{dec}}|\,\mathfrak{d}\mkern-9mu /^{\leq 1}(\check{P}',A')|.
\end{aligned}
$$

In view of the definition of w, we infer from the above, using also the improved estimates for $\check{P}'$ of Lemma 5.83 and the control of Theorem M1 for A',

$$
\sup_{\Sigma_*} r^5u^{1+\delta_{dec}}\left|\left[\overline{\mathcal{D}}\cdot\left(B'-\frac{3a}{2}\overline{\check{P}'}\mathfrak{J}-\frac{a}{4}\overline{\mathfrak{J}}\cdot A'\right)\right]_{\ell=1}\right| \;\lesssim\; \epsilon_0.
$$

Next, using the decomposition

$$
\mathcal{D}'-\frac{a}{2}\mathfrak{J}\nabla_{e_4'}-\frac{a}{2}\mathfrak{J}\nabla_{e_3'}=(1+O(r^{-2}))\mathcal{D}+O(r^{-2})\nabla_{e_4'}+O(r^{-2})\nabla_{e_3'}+O(r^{-3}),
$$

we have

$$
\begin{aligned}
&\Bigg|\left(\overline{\mathcal{D}'}\cdot-\frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_{e_4'}-\frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_{e_3'}\right)\cdot\left(B'-\frac{3a}{2}\overline{\check{P}'}\mathfrak{J}-\frac{a}{4}\overline{\mathfrak{J}}\cdot A'\right)\\
&\qquad -\overline{\mathcal{D}}\cdot\left(B'-\frac{3a}{2}\overline{\check{P}'}\mathfrak{J}-\frac{a}{4}\overline{\mathfrak{J}}\cdot A'\right)\Bigg| \lesssim r^{-4}|\mathfrak{d}^{\leq 1}\Gamma_g'|+r^{-2}|\nabla_\nu B'|.
\end{aligned}
$$

Together with the bootstrap assumptions on Γ_g' and the estimate for $\nabla_\nu B'$ in Lemma 5.83, we infer on Σ_*

$$
\begin{aligned}
&\Bigg|\left(\overline{\mathcal{D}'}\cdot-\frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_{e_4'}-\frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_{e_3'}\right)\cdot\left(B'-\frac{3a}{2}\overline{\check{P}'}\mathfrak{J}-\frac{a}{4}\overline{\mathfrak{J}}\cdot A'\right)\\
&\qquad -\overline{\mathcal{D}}\cdot\left(B'-\frac{3a}{2}\overline{\check{P}'}\mathfrak{J}-\frac{a}{4}\overline{\mathfrak{J}}\cdot A'\right)\Bigg| \lesssim \frac{1}{r^6},
\end{aligned}
$$

and hence, together with the dominance in r condition on Σ_*, we infer

$$
\Bigg|\left(\overline{\mathcal{D}'}\cdot-\frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_{e_4'}-\frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_{e_3'}\right)\cdot\left(B'-\frac{3a}{2}\overline{\check{P}'}\mathfrak{J}-\frac{a}{4}\overline{\mathfrak{J}}\cdot A'\right)
$$

$$-\overline{\mathcal{D}}\cdot\left(B'-\frac{3a}{2}\overline{\check{P}'}\mathfrak{J}-\frac{a}{4}\overline{\mathfrak{J}}\cdot A'\right)\Bigg| \lesssim \frac{\epsilon_0}{r^5u^{1+\delta_{dec}}}.$$

Together with the above, we infer

$$\sup_{\Sigma_*} r^5u^{1+\delta_{dec}}\left|\left[\left(\overline{\mathcal{D}'}\cdot-\frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_{e'_4}-\frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_{e'_3}\right)\cdot\left(B'-\frac{3a}{2}\overline{\check{P}'}\mathfrak{J}-\frac{a}{4}\overline{\mathfrak{J}}\cdot A'\right)\right]_{\ell=1}\right| \lesssim \epsilon_0$$

as desired. $\square$

Lemma 5.85. *We have*

$$\sup_{\Sigma_*} r^5u^{1+\delta_{dec}}\left|\left[\overline{\mathcal{D}}\cdot \not{\mathcal{L}}_{\mathbf{T}'}B'\right]_{\ell=1}\right| \lesssim \epsilon_0.$$

Proof. We have

$$(\not{\mathcal{L}}_{\mathbf{T}'}B')_{ab} = (\nabla_{\mathbf{T}'}B')_{ab}+\mathbf{g}(\mathbf{D}_{e'_a}\mathbf{T},e'_c)B'_{cb}+\mathbf{g}(\mathbf{D}_{e'_b}\mathbf{T},e'_c)B'_{ac}.$$

Now, recall that $k_{ab}=\mathbf{g}(\mathbf{D}_{e'_a}\mathbf{T},e'_b)$ verifies

$$k_{ab}=O(r^{-3})+\Gamma_b.$$

We infer

$$\not{\mathcal{L}}_{\mathbf{T}'}B' = \nabla_{\mathbf{T}'}B'+O(r^{-3})B'+\Gamma_bB'.$$

This yields

$$2\not{\mathcal{L}}_{\mathbf{T}'}B' = \nabla_{e'_3}B'+\nabla_{e'_4}B'+O(r^{-2})\mathfrak{d}^{\leq 1}B'+\Gamma_bB'.$$

Using the Bianchi identities, we infer

$$\begin{aligned}
2\not{\mathcal{L}}_{\mathbf{T}'}B' &= \mathcal{D}'\overline{\check{P}'}+\frac{2}{r}B'+\frac{1}{2}\overline{\mathcal{D}}'\cdot A'-\frac{4}{r}B'\\
&\quad +O(r^{-2})\mathfrak{d}^{\leq 1}B'+O(r^{-2})A'+r^{-3}\Gamma'_b+r^{-1}\Gamma'_g\cdot\Gamma'_b\\
&= \mathcal{D}'\overline{\check{P}'}+\frac{2}{r}B'+\frac{1}{2}\overline{\mathcal{D}}'\cdot A'-\frac{4}{r}B'+r^{-3}\mathfrak{d}^{\leq 1}(\Gamma'_b)+r^{-1}\Gamma'_g\cdot\Gamma'_b.
\end{aligned}$$

Hence, using also the decomposition

$$\mathcal{D}' = (1+O(r^{-2}))\mathcal{D}+O(r^{-1})\nabla_{e'_4}+O(r^{-1})\nabla_{e'_3}+O(r^{-3}),$$

and relying again on Bianchi identities to include the terms $O(r^{-1})\nabla_{e'_3}\check{P}'$ and $O(r^{-1})\nabla_{e'_3}A'$ in $r^{-3}\mathfrak{d}^{\leq 1}(\Gamma'_b)$, we obtain

$$\begin{aligned}
2\not{\mathcal{L}}_{\mathbf{T}'}B' &= \mathcal{D}\overline{\check{P}'} + \frac{2}{r}B' + \frac{1}{2}\overline{\mathcal{D}}\cdot A' - \frac{4}{r}B' + r^{-3}\mathfrak{d}^{\leq 1}(\Gamma'_b) + r^{-1}\Gamma'_g\cdot\Gamma'_b\\
&= \mathcal{D}\overline{\check{P}'} - \frac{2}{r}B' + \frac{1}{2}\overline{\mathcal{D}}\cdot A' + r^{-3}\mathfrak{d}^{\leq 1}(\Gamma'_b) + r^{-1}\Gamma'_g\cdot\Gamma'_b.
\end{aligned}$$

This yields

$$\begin{aligned}
2\not{\mathcal{L}}_{\mathbf{T}'}B' &= \mathcal{D}\overline{\check{P}'} - \frac{2}{r}\left(B' - \frac{3a}{2r}\overline{\check{P}'}\mathfrak{J} - \frac{a}{4r}\overline{\mathfrak{J}}\cdot A'\right) + \frac{1}{2}\overline{\mathcal{D}}\cdot A'\\
&\quad + r^{-3}\mathfrak{d}^{\leq 1}(\Gamma'_b) + r^{-1}\Gamma'_g\cdot\Gamma'_b.
\end{aligned}$$

We infer

$$\begin{aligned}
\overline{\mathcal{D}}\cdot\not{\mathcal{L}}_{\mathbf{T}'}B' &= \frac{1}{2}\overline{\mathcal{D}}\cdot\mathcal{D}\overline{\check{P}'} - \frac{1}{r}\overline{\mathcal{D}}\cdot\left(B' - \frac{3a}{2r}\overline{\check{P}'}\mathfrak{J} - \frac{a}{4r}\overline{\mathfrak{J}}\cdot A'\right) + \frac{1}{4}\overline{\mathcal{D}}\cdot\overline{\mathcal{D}}\cdot A'\\
&\quad + r^{-4}\mathfrak{d}^{\leq 2}(\Gamma'_b) + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma'_g\cdot\Gamma'_b)\\
&= \Delta\overline{\check{P}'} - \frac{1}{r}\overline{\mathcal{D}}\cdot\left(B' - \frac{3a}{2r}\overline{\check{P}'}\mathfrak{J} - \frac{a}{4r}\overline{\mathfrak{J}}\cdot A'\right) + \frac{1}{4}\overline{\mathcal{D}}\cdot\overline{\mathcal{D}}\cdot A'\\
&\quad + r^{-4}\mathfrak{d}^{\leq 2}(\Gamma'_b) + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma'_g\cdot\Gamma'_b).
\end{aligned}$$

Recalling

$$\begin{aligned}
\check{\rho}' &= \check{\rho} + r^{-2}\Gamma_b + O\left(\frac{1}{r^4}\right),\\
\widecheck{{}^*\rho}' &= {}^*\rho + r^{-2}\Gamma_b + O\left(\frac{1}{r^4}\right),
\end{aligned}$$

we obtain

$$\begin{aligned}
\overline{\mathcal{D}}\cdot\not{\mathcal{L}}_{\mathbf{T}'}B' &= \Delta(\check{\rho} - i\ {}^*\rho) - \frac{1}{r}\overline{\mathcal{D}}\cdot\left(B' - \frac{3a}{2r}\overline{\check{P}'}\mathfrak{J} - \frac{a}{4r}\overline{\mathfrak{J}}\cdot A'\right) + \frac{1}{4}\overline{\mathcal{D}}\cdot\overline{\mathcal{D}}\cdot A'\\
&\quad + r^{-4}\mathfrak{d}^{\leq 2}(\Gamma'_b) + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma'_g\cdot\Gamma'_b).
\end{aligned}$$

We deduce

$$\begin{aligned}
\left[\overline{\mathcal{D}}\cdot\not{\mathcal{L}}_{\mathbf{T}'}B'\right]_{\ell=1} &= [\Delta\check{\rho}]_{\ell=1} - i\,[\Delta\ {}^*\rho]_{\ell=1} + \frac{1}{4}\left[\overline{\mathcal{D}}\cdot\overline{\mathcal{D}}\cdot A'\right]_{\ell=1}\\
&\quad - \frac{1}{r}\left[\overline{\mathcal{D}}\cdot\left(B' - \frac{3a}{2r}\overline{\check{P}'}\mathfrak{J} - \frac{a}{4r}\overline{\mathfrak{J}}\cdot A'\right)\right]_{\ell=1}
\end{aligned}$$

$$+r^{-4}\mathfrak{d}^{\leq 2}(\Gamma'_b) + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma'_g \cdot \Gamma'_b).$$

Since, for a scalar f,

$$\begin{aligned} {}^*(\overline{\mathcal{D}} \cdot \overline{\mathcal{D}} \cdot) f &= 2\,{}^*(\operatorname{div}\operatorname{div} + i\operatorname{curl}\operatorname{div})f \\ &= 2\,\not{d}_2^*\not{d}_1^*(f,0) + 2i\,\not{d}_2^*\not{d}_1^*(0,f), \end{aligned}$$

we infer, by integration by parts of $\overline{\mathcal{D}} \cdot \overline{\mathcal{D}} \cdot$,

$$\Big[\overline{\mathcal{D}} \cdot \overline{\mathcal{D}} \cdot A'\Big]_{\ell=1} \lesssim |A'||\not{d}_2^*\not{d}_1^* J^{(p)}|$$

and hence

$$\begin{aligned} & \left|\Big[\overline{\mathcal{D}} \cdot \not{\mathcal{L}}_{\mathbf{T}'} B'\Big]_{\ell=1}\right| \\ \lesssim\; & \frac{1}{r}\left|\left[\overline{\mathcal{D}} \cdot \left(B' - \frac{3a}{2r}\overline{\check{P}'}\mathfrak{J} - \frac{a}{4r}\overline{\mathfrak{J}} \cdot A'\right)\right]_{\ell=1}\right| + \frac{2}{r^2}|[\check{\rho}]_{\ell=1}| + \frac{2}{r^2}|[\,{}^*\rho]_{\ell=1}| \\ & + \left|\left[\left(\Delta + \frac{2}{r^2}\right)\check{\rho}\right]_{\ell=1}\right| + \left|\left[\left(\Delta + \frac{2}{r^2}\right)\,{}^*\rho\right]_{\ell=1}\right| + |A'||\not{d}_2^*\not{d}_1^* J^{(p)}| \\ & + r^{-4}|\mathfrak{d}^{\leq 2}(\Gamma'_b)| + r^{-2}|\mathfrak{d}^{\leq 1}(\Gamma'_g \cdot \Gamma'_b)|. \end{aligned}$$

The conclusion then follows from Proposition 5.48 for the control of the $\ell = 1$ mode of $\check{\rho}$ and ${}^*\rho$, the control of Theorem M1 for A', the control of $\not{d}_2^*\not{d}_1^* J^{(p)}$ and $(\Delta + \frac{2}{r^2})J^{(p)}$ in Lemma 5.68, the above improved control of Γ'_b, and the one of Corollary 5.84 for

$$\left[\overline{\mathcal{D}} \cdot \left(B' - \frac{3a}{2r}\overline{\check{P}'}\mathfrak{J} - \frac{a}{4r}\overline{\mathfrak{J}} \cdot A'\right)\right]_{\ell=1}.$$

This concludes the proof of the corollary. □

Lemma 5.86. *We have, for $k \leq k_* - 14$,*

$$\sup_{\Sigma_*} r^3 u^{1+\delta_{dec}} \left|\mathfrak{d}_*^k\left(\overline{\mathcal{D}'} \cdot \check{Z}' + 2\overline{\check{P}'}\right)\right| \lesssim \epsilon_0.$$

Proof. Using the decomposition

$$\mathcal{D}' = (1 + O(r^{-2}))\mathcal{D} + O(r^{-1})\nabla_{e'_4} + O(r^{-1})\nabla_{e'_3} + O(r^{-3}),$$

we have, using also $e'_3 = \nu + r^{-1}\mathfrak{d}$,

$$\overline{\mathcal{D}'} \cdot \check{Z}' = \overline{\mathcal{D}} \cdot \check{Z}' + O(r^{-1})\nabla_\nu \check{Z}' + r^{-2}\mathfrak{d}^{\leq}\Gamma'_g$$

and hence

$$\overline{\mathcal{D}'}\cdot\check{Z}'+2\overline{\check{P}'} = \overline{\mathcal{D}}\cdot\check{Z}'+2\overline{\check{P}'}+O(r^{-1})\nabla_\nu\check{Z}'+r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g'.$$

Recalling from the above the transformation formulas

$$\begin{aligned}\check{Z}' &= Z+O(r^{-1})(\underline{\check{\omega}}',\underline{\widehat{\chi}}')+r^{-1}\Gamma_g'+O(r^{-3})+r^{-1}\mathfrak{d}_*^{\leq 1}\widetilde{\Gamma_b},\\ \check{P}' &= \check{\rho}+i\ {}^*\rho+O(r^{-4})+r^{-2}\Gamma_g'+O(r^{-1})\underline{\beta}'+O(r^{-2})\underline{\alpha}'.\end{aligned}$$

We infer

$$\begin{aligned}\overline{\mathcal{D}'}\cdot\check{Z}'+2\overline{\check{P}'} &= \overline{\mathcal{D}}\cdot Z+2(\check{\rho}-i\ {}^*\rho)+O(r^{-4})+O(r^{-2})\,\not{\mathfrak{d}}^{\leq 1}(\underline{\check{\omega}}',\underline{\widehat{\chi}}')\\ &\quad +O(r^{-1})\underline{\beta}'+O(r^{-2})\underline{\alpha}'+r^{-2}\mathfrak{d}_*^{\leq 2}\widetilde{\Gamma_b}+O(r^{-1})\nabla_\nu\check{Z}'\\ &\quad +r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g'.\end{aligned}$$

Together with the control of the Ricci coefficients of the PG frame on Σ_* provided by Lemma 5.82, the control of the curvature components of the PG frame on Σ_* provided by Lemma 5.83, the bootstrap assumptions on Γ_g', and the control of $\widetilde{\Gamma_b}$ provided by Corollary 5.73, we obtain on Σ_*, for $k\leq k_*-14$,

$$\left|\mathfrak{d}_*^k\left(\overline{\mathcal{D}'}\cdot\check{Z}'+2\overline{\check{P}'}\right)\right| \lesssim \left|\mathfrak{d}_*^k\left(\overline{\mathcal{D}}\cdot Z+2(\check{\rho}-i\ {}^*\rho)\right)\right|+\frac{\epsilon_0}{r^3u^{1+\delta_{dec}}}+\frac{1}{r^4},$$

and hence, in view of the dominance in r condition for r on Σ_*, this yields, for $k\leq k_*-14$,

$$\left|\mathfrak{d}_*^k\left(\overline{\mathcal{D}'}\cdot\check{Z}'+2\overline{\check{P}'}\right)\right| \lesssim \left|\mathfrak{d}_*^k\left(\overline{\mathcal{D}}\cdot Z+2(\check{\rho}-i\ {}^*\rho)\right)\right|+\frac{\epsilon_0}{r^3u^{1+\delta_{dec}}}.$$

Next, we compute

$$\begin{aligned}\overline{\mathcal{D}}\cdot Z+2(\check{\rho}-i\ {}^*\rho) &= (\nabla-i\ {}^*\nabla)\cdot(\zeta+i\ {}^*\zeta)+2(\check{\rho}-i\ {}^*\rho)\\ &= 2(\operatorname{div}\zeta+\check{\rho})+2i(\operatorname{curl}\zeta-\ {}^*\rho).\end{aligned}$$

Together with the definition of μ and the null structure equation for $\operatorname{curl}\zeta$, we infer

$$\overline{\mathcal{D}}\cdot Z+2(\check{\rho}-i\ {}^*\rho) = 2\left(-\check{\mu}+\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)-i\widehat{\chi}\wedge\underline{\widehat{\chi}}=-2\check{\mu}+\Gamma_b\cdot\Gamma_g.$$

We infer on Σ_*, for $k\leq k_*-14$,

$$\left|\mathfrak{d}_*^k\left(\overline{\mathcal{D}'}\cdot\check{Z}'+2\overline{\check{P}'}\right)\right| \lesssim \left|\mathfrak{d}_*^k\check{\mu}\right|+\left|\mathfrak{d}_*^k(\Gamma_b\cdot\Gamma_g)\right|+\frac{\epsilon_0}{r^3u^{1+\delta_{dec}}}.$$

Together with the control for $\check{\mu}$, Γ_b and Γ_g of Proposition 5.53, we deduce on Σ_*, for $k \leq k_* - 14$,

$$\left|\mathfrak{d}_*^k \left(\overline{\mathcal{D}'} \cdot \check{Z}' + 2\overline{\check{P}'}\right)\right| \lesssim \frac{\epsilon_0}{r^3 u^{1+\delta_{dec}}}.$$

This concludes the proof of the lemma. □

Lemma 5.87. *We have*

$$\sup_{\Sigma_*} r u^{\frac{1}{2}+\delta_{dec}} \Big(\left|(r^2\Delta' + 2)J^{(p)}\right| + \left|r^2 \, {\mathcal{D}'}_2^* {\mathcal{D}'}_1^*(J^{(p)}, 0)\right| + \left|r^2 \, {\mathcal{D}'}_2^* {\mathcal{D}'}_1^*(0, J^{(p)})\right| \Big) \lesssim \epsilon_0.$$

Proof. Since $\nu(J^{(p)}) = 0$, and since $J^{(p)}$ is extended to ${}^{(ext)}\mathcal{M}$ by $e_4'(J^{(p)}) = 0$, we easily infer from (5.138) the following estimate on Σ_*

$$\begin{aligned} & \left|(r^2\Delta' + 2)J^{(p)}\right| + \left|r^2 \, {\mathcal{D}'}_2^* {\mathcal{D}'}_1^*(J^{(p)}, 0)\right| + \left|r^2 \, {\mathcal{D}'}_2^* {\mathcal{D}'}_1^*(0, J^{(p)})\right| \\ \lesssim \; & \left|(r^2\Delta + 2)J^{(p)}\right| + \left|r^2 \, \mathcal{D}_2^* \mathcal{D}_1^*(J^{(p)}, 0)\right| + \left|r^2 \, \mathcal{D}_2^* \mathcal{D}_1^*(0, J^{(p)})\right| + r^{-2}|\mathfrak{d}^{\leq 2} J^{(p)}|. \end{aligned}$$

In view of the control of $\mathcal{D}_2^* \mathcal{D}_1^* J^{(p)}$ and $(\Delta + \frac{2}{r^2})J^{(p)}$ in Lemma 5.68, we infer on Σ_*

$$\left|(r^2\Delta' + 2)J^{(p)}\right| + \left|r^2 \, {\mathcal{D}'}_2^* {\mathcal{D}'}_1^*(J^{(p)}, 0)\right| + \left|r^2 \, {\mathcal{D}'}_2^* {\mathcal{D}'}_1^*(0, J^{(p)})\right| \lesssim \frac{\epsilon_0}{r u^{\frac{1}{2}+\delta_{dec}}} + \frac{1}{r^2}.$$

Together with the dominance condition on r on Σ_*, we infer

$$\left|(r^2\Delta' + 2)J^{(p)}\right| + \left|r^2 \, {\mathcal{D}'}_2^* {\mathcal{D}'}_1^*(J^{(p)}, 0)\right| + \left|r^2 \, {\mathcal{D}'}_2^* {\mathcal{D}'}_1^*(0, J^{(p)})\right| \lesssim \frac{\epsilon_0}{r u^{\frac{1}{2}+\delta_{dec}}}$$

as stated. □

5.7.5. Proof of Proposition 5.77 The results of Sections 5.7.2, 5.7.3 and 5.7.4 imply the proof of Proposition 5.77 with e_3' replaced by ν and $\mathfrak{d}$ replaced by $\mathfrak{d}_*$. The extension from $\mathfrak{d}_*$ to $\mathfrak{d}$ follows immediately from the null structure equations and Bianchi identities expressing derivatives in the e_4' direction in terms of angular derivatives. Finally, since

$$\nu = e_3 + b_* e_4 = e_3' + r^{-1}\mathfrak{d},$$

we may replace ν by e_3' in the corresponding estimates. This concludes the proof of Proposition 5.77.

6. DECAY ESTIMATES ON THE REGION ${}^{(ext)}\mathcal{M}$ (THEOREM M4)

The goal of this chapter is to prove Theorem M4 by extending the decay estimates on Σ_* derived in Section 5.7 to the full spacetime region ${}^{(ext)}\mathcal{M}$. The main result is stated in Proposition 6.49. The estimates on Σ_*, derived in Section 5.7, are summarized in Proposition 6.48.

The results proved in this chapter rely only on the main bootstrap assumptions on ${}^{(ext)}\mathcal{M}$, the estimates for the extreme curvature component A derived in Theorem M1, and the estimates on the last slice proved in Section 5.7.

In order to count the number of derivatives under control in this chapter, we introduce for convenience the following notation

$$k_* \quad := \quad k_{small} + 60. \tag{6.1}$$

6.1. Preliminaries

6.1.1. The PG structure of ${}^{(ext)}\mathcal{M}$ Throughout this chapter we work with the spacetime region ${}^{(ext)}\mathcal{M}$, terminating in the GCM last slice Σ_*, as discussed in Section 3.2. For the convenience of the reader we recall below some the main facts concerning ${}^{(ext)}\mathcal{M}$.

1. The PG structure of ${}^{(ext)}\mathcal{M}$, given by $\{r, (e_3, e_4), \mathcal{H}\}$ together with adapted PG coordinates (u, θ, φ), is such that

$$\xi = 0, \quad \omega = 0, \quad \underline{\eta} + \zeta = 0, \tag{6.2}$$

and

$$e_4(r) = 1, \quad e_4(u) = e_4(\theta) = e_4(\varphi) = 0, \quad \nabla(r) = 0. \tag{6.3}$$

2. In ${}^{(ext)}\mathcal{M}$, we have $1 \leq u \leq u_*$ and $r \geq r_0$, with r_0 sufficiently large.
3. The timelike hypersurface $\mathcal{T} = \{r = r_0\}$ is a boundary of ${}^{(ext)}\mathcal{M}$.
4. The constants (a, m) are the ones associated to S_* according to Definition 2.59, see Section 3.2.4.
5. ${}^{(ext)}\mathcal{M}$ comes equipped, see Section 2.6.1, with a basis of $\ell = 1$ modes $J^{(p)}$ with $p = 0, +, -$ verifying

$$e_4(J^{(p)}) = 0. \tag{6.4}$$

6. ${}^{(ext)}\mathcal{M}$ comes equipped with complex, anti-selfadjoint[129] 1-form $\mathfrak{J}$, $\mathfrak{J}_\pm$ satisfying, see Section 2.6.2 and Section 3.3.2

$$
\begin{aligned}
\nabla_4\mathfrak{J} &= -\frac{1}{q}\mathfrak{J}, & \nabla_4\mathfrak{J}_\pm &= -\frac{1}{q}\mathfrak{J}_\pm,\\
\Re(\mathfrak{J}_+)\cdot\Re(\mathfrak{J}) &= -\frac{1}{|q|^2}J^{(-)}, & \Re(\mathfrak{J}_-)\cdot\Re(\mathfrak{J}) &= \frac{1}{|q|^2}J^{(+)},
\end{aligned}
\tag{6.5}
$$

as well as

$$
\begin{aligned}
\mathfrak{J}\cdot\overline{\mathfrak{J}} &= \frac{2(\sin\theta)^2}{|q|^2},\\
\mathfrak{J}_+\cdot\overline{\mathfrak{J}_+} &= \frac{2(\cos\theta)^2(\cos\varphi)^2+2(\sin\varphi)^2}{|q|^2},\\
\mathfrak{J}_-\cdot\overline{\mathfrak{J}_-} &= \frac{2(\cos\theta)^2(\sin\varphi)^2+2(\cos\varphi)^2}{|q|^2}.
\end{aligned}
\tag{6.6}
$$

6.1.2. Linearized quantities and definition of Γ_g and Γ_b We recall the linearized quantities obtained by subtracting their Kerr(a,m) values, see Definition 2.66:

1. Linearization of Ricci and curvature coefficients.

$$
\begin{aligned}
\widetilde{\mathrm{tr}X} &:= \mathrm{tr}X - \frac{2}{q}, & \widetilde{\mathrm{tr}\underline{X}} &:= \mathrm{tr}\underline{X} + \frac{2q\Delta}{|q|^4},\\
\check{Z} &:= Z - \frac{a\overline{q}}{|q|^2}\mathfrak{J}, & \check{H} &:= H - \frac{aq}{|q|^2}\mathfrak{J},\\
\check{\underline{\omega}} &:= \underline{\omega} - \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right), & \check{P} &:= P + \frac{2m}{q^3}.
\end{aligned}
$$

2. Linearization of derivatives of r, q, u.

$$
\begin{aligned}
\widetilde{\mathcal{D}q} &:= \mathcal{D}q + a\mathfrak{J}, & \widetilde{\mathcal{D}\overline{q}} &:= \mathcal{D}\overline{q} - a\mathfrak{J},\\
\widetilde{e_3(r)} &:= e_3(r) + \frac{\Delta}{|q|^2}, & &\\
\widetilde{\mathcal{D}u} &:= \mathcal{D}u - a\mathfrak{J}, & \widetilde{e_3(u)} &:= e_3(u) - \frac{2(r^2+a^2)}{|q|^2}.
\end{aligned}
$$

3. Linearization for $\mathfrak{J}$ and $\mathfrak{J}_\pm$.

[129] i.e. ${}^*\mathfrak{J} = -i\mathfrak{J}$, ${}^*\mathfrak{J}_\pm = -i\mathfrak{J}_\pm$.

$$\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}} := \overline{\mathcal{D}}\cdot\mathfrak{J} - \frac{4i(r^2+a^2)\cos\theta}{|q|^4}, \qquad \widetilde{\nabla_3\mathfrak{J}} := \nabla_3\mathfrak{J} - \frac{\Delta q}{|q|^4}\mathfrak{J},$$
$$\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm} := \overline{\mathcal{D}}\cdot\mathfrak{J}_\pm + \frac{4}{r^2}J^{(\pm)} \pm \frac{4ia^2\cos\theta}{|q|^4}J^{(\mp)},$$
$$\widetilde{\nabla_3\mathfrak{J}_\pm} := \nabla_3\mathfrak{J}_\pm - \frac{\Delta q}{|q|^4}\mathfrak{J}_\pm \pm \frac{2a}{|q|^2}\mathfrak{J}_\mp.$$

4. Linearization for $J^{(p)}$.

$$\widetilde{\mathcal{D}J^{(0)}} := \mathcal{D}J^{(0)} - i\mathfrak{J}, \qquad \widetilde{\mathcal{D}(J^{(\pm)})} := \mathcal{D}(J^{(\pm)}) - \mathfrak{J}_\pm,$$
$$\widetilde{e_3(J^{(+)})} := e_3(J^{(+)}) + \frac{2a}{|q|^2}J^{(-)}, \tag{6.7}$$
$$\widetilde{e_3(J^{(-)})} := e_3(J^{(-)}) - \frac{2a}{|q|^2}J^{(+)}.$$

We also recall the sets Γ_g, Γ_b, see Definition 2.67:

1. The set Γ_g with

$$\Gamma_g = \Big\{\widetilde{\mathrm{tr}X}, \quad \widehat{X}, \quad \check{Z}, \quad \widetilde{\mathrm{tr}\underline{X}}, \quad r\check{P}, \quad rB, \quad rA\Big\}.$$

2. The set $\Gamma_b = \Gamma_{b,1}\cup\Gamma_{b,2}\cup\Gamma_{b,3}\cup\Gamma_{b,4}$ with

$$\Gamma_{b,1} = \Big\{\check{H}, \quad \underline{\widehat{X}}, \quad \underline{\check{\omega}}, \quad \underline{\Xi}, \quad r\underline{B}, \quad \underline{A}\Big\},$$
$$\Gamma_{b,2} = \Big\{r^{-1}\widetilde{e_3(r)}, \quad \widetilde{\mathcal{D}q}, \quad \widetilde{\mathcal{D}\overline{q}}, \quad \widetilde{\mathcal{D}u}, \quad r^{-1}\widetilde{e_3(u)}\Big\},$$
$$\Gamma_{b,3} = \Big\{\widetilde{\mathcal{D}(J^{(0)})}, \quad \widetilde{\mathcal{D}(J^{(\pm)})}, \quad e_3(J^{(0)}), \quad \widetilde{e_3(J^{(\pm)})}\Big\},$$
$$\Gamma_{b,4} = \Big\{r\,\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}, \quad r\,\mathcal{D}\widehat{\otimes}\mathfrak{J}, \quad r\,\widetilde{\nabla_3\mathfrak{J}}, \quad r\,\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm}, \quad r\,\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm, \quad r\,\widetilde{\nabla_3\mathfrak{J}_\pm}\Big\}.$$

6.1.3. Main assumptions

Definition 6.1. *We make use of the following norms on $S = S(u,r) \subset {}^{(ext)}\mathcal{M}$,*

$$\|f\|_\infty(u,r) := \|f\|_{L^\infty(S(u,r))}, \qquad \|f\|_2(u,r) := \|f\|_{L^2(S(u,r))},$$
$$\|f\|_{\infty,k}(u,r) := \sum_{i=0}^k \|\mathfrak{d}^i f\|_\infty(u,r), \qquad \|f\|_{2,k}(u,r) := \sum_{i=0}^k \|\mathfrak{d}^i f\|_2(u,r).$$

We shall also make use of

$$\|f\|_{\infty,k}({}^{(ext)}\mathcal{M}) = \sup_{{}^{(ext)}\mathcal{M}} |\mathfrak{d}^{\leq k} f|.$$

Remark 6.2. *We note that the derivatives $\not{\mathfrak{d}} = (r\nabla)$ and $\mathfrak{d} = (r\nabla, r\nabla_4, \nabla_3)$ are defined with respect to the outgoing PG frame of ${}^{(ext)}\mathcal{M}$, which is not adapted to the spheres $S(u,r)$.*

Definition 6.3 (Order of magnitude notation)**.** *Throughout this chapter, we will be using the notation $O(r^{-p})$ to denote:*

1. *a scalar function depending only on (r,θ) which is smooth and such that*

$$r^p|(r\partial_r, \partial_\theta)^k O(r^{-p})| \lesssim 1 \quad \text{for} \quad k \geq 0 \quad \text{and} \quad r \geq r_0,$$

2. *a 1-form of the type $O(r^{-p+1})\mathfrak{J}$ where $O(r^{-p+1})$ denotes a scalar function as above,*
3. *a symmetric traceless 2-tensor of the type $O(r^{-p+2})\mathfrak{J}\widehat{\otimes}\mathfrak{J}$ where $O(r^{-p+2})$ denotes a scalar function as above.*

Often in the text we shall use the notation $r^{-p}U$, where U is a small quantity, instead of $O(r^{-p})U$.

We will also make use of the following notation.

Definition 6.4. *We introduce the notation $U \in r^{-p}Good_k$ for horizontal tensors U satisfying*

$$\left|\mathfrak{d}^{\leq k} U\right| \lesssim \epsilon_0 r^{-p} u^{-1-\delta_{dec}}. \tag{6.8}$$

For the benefit of the reader we state below the main assumptions which will be used throughout this chapter:

Ref 1. In view of our main bootstrap assumptions on decay and boundedness, the following estimates hold on ${}^{(ext)}\mathcal{M}$:

1. For $0 \leq k \leq k_{small}$, we have

$$\begin{aligned} \|\Gamma_g\|_{\infty,k} &\lesssim \epsilon \min\left\{r^{-2}u^{-\frac{1}{2}-\delta_{dec}},\, r^{-1}u^{-1-\delta_{dec}}\right\}, \\ \|\nabla_3\Gamma_g\|_{\infty,k-1} &\lesssim \epsilon r^{-2}u^{-1-\delta_{dec}}, \\ \|\Gamma_b\|_{\infty,k} &\lesssim \epsilon r^{-1}u^{-1-\delta_{dec}}, \end{aligned} \tag{6.9}$$

and

(6.10)
$$\|B\|_{\infty,k} \lesssim \epsilon \min\left\{r^{-3}(u+2r)^{-\frac{1}{2}-\delta_{dec}},\, r^{-2}(u+2r)^{-1-\delta_{dec}}\right\},$$
$$\|\nabla_3 B\|_{\infty,k-1} \lesssim \epsilon \min\left\{r^{-4}(u+2r)^{-\frac{1}{2}-\delta_{dec}},\, r^{-3}(u+2r)^{-1-\delta_{dec}}\right\}.$$

2. We also make the auxiliary bootstrap assumption[130] for $0 \leq k \leq k_{small}$

$$\|\widetilde{\mathrm{tr} X}\|_{\infty,k} \lesssim \epsilon r^{-2}u^{-1-\delta_{dec}}. \tag{6.11}$$

3. For $k \leq k_{large}$, we have

$$\|\Gamma_g\|_{\infty,k} \lesssim \epsilon r^{-2}, \qquad \|\Gamma_b\|_{\infty,k} \lesssim \epsilon r^{-1}. \tag{6.12}$$

Remark 6.5. *Note that we can interpolate between the estimates (6.9) for $k \leq k_{small}$ and (6.12) for $k \leq k_{large}$ to derive on ${}^{(ext)}\mathcal{M}$, for[131] all $k \leq k_*$,*

$$\begin{aligned}\|\Gamma_g\|_{\infty,k} &\lesssim \epsilon \min\left\{r^{-2}u^{-\frac{1}{2}-\frac{\delta_{dec}}{2}},\, r^{-1}u^{-1-\frac{\delta_{dec}}{2}}\right\},\\ \|\nabla_3\Gamma_g\|_{\infty,k-1} &\lesssim \epsilon r^{-2}u^{-1-\frac{\delta_{dec}}{2}},\\ \|\Gamma_b\|_{\infty,k} &\lesssim \epsilon r^{-1}u^{-1-\frac{\delta_{dec}}{2}},\end{aligned} \tag{6.13}$$

see Lemma 5.15 for the corresponding statement on Σ_*.

Remark 6.6. *In view of the assumptions* **Ref 1** *for Γ_g and Γ_b, we see that the estimates satisfied by Γ_g are stronger that those satisfied by Γ_b so that we will often replace terms of type $\Gamma_g + \Gamma_b$ by Γ_b. Similarly, we will also often replace terms of type $r^{-1}\Gamma_b + \Gamma_g$ by Γ_g, and terms of type $\nabla_3\Gamma_g + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b$ by $r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b$.*

Ref 2. According to Theorem M1, we have on ${}^{(ext)}\mathcal{M}$, for[132] all $0 \leq k \leq k_*$,

$$\begin{aligned}\|A\|_{\infty,k} &\lesssim \epsilon_0 \min\left\{r^{-3}(u+2r)^{-\frac{1}{2}-\delta_{extra}},\, r^{-2}u^{-1-\delta_{extra}}\right\},\\ \|\nabla_3 A\|_{\infty,k-1} &\lesssim \epsilon_0 \min\left\{r^{-\frac{9}{2}-\delta_{dec}},\, r^{-4}u^{-\frac{1}{2}-\delta_{extra}},\, r^{-3}u^{-1-\delta_{extra}}\right\},\\ \|\nabla_3^2 A\|_{\infty,k-1} &\lesssim \epsilon_0 \min\left\{r^{-\frac{9}{2}-\delta_{dec}}u^{-\frac{1}{2}-\delta_{extra}},\, r^{-4}u^{-1-\delta_{extra}}\right\},\end{aligned} \tag{6.14}$$

where we recall that $\delta_{extra} > \delta_{dec}$.

[130] This auxiliary bootstrap assumption will be improved in Proposition 6.50.
[131] Recall from (6.1) that $k_* = k_{small} + 60$ throughout this chapter.
[132] Recall from (6.1) that $k_* = k_{small} + 60$ throughout this chapter.

Remark 6.7. *According to Theorem M1, we have in fact on* ${}^{(ext)}\mathcal{M}$*, for all* $0 \le k \le k_*$,[133]

$$\begin{aligned}
\|A'\|_{\infty,k} &\lesssim \epsilon_0 \min\left\{r^{-3}(u+2r)^{-\frac{1}{2}-\delta_{extra}},\, r^{-2}u^{-1-\delta_{extra}}\right\},\\
\|\nabla_{e_3'}A'\|_{\infty,k-1} &\lesssim \epsilon_0 \min\left\{r^{-4}(u+2r)^{-\frac{1}{2}-\delta_{extra}},\, r^{-3}u^{-1-\delta_{extra}}\right\},\\
\|\nabla^2_{e_3'}A'\|_{\infty,k-1} &\lesssim \epsilon_0 \min\left\{r^{-5}u^{-\frac{1}{2}-\delta_{extra}},\, r^{-4}u^{-1-\delta_{extra}}\right\},
\end{aligned}$$

where the quantities with prime are expressed in the global frame of Proposition 3.33. Then, (6.14) *follows immediately from these estimates, the following change of frame formula of Proposition 2.12*

$$\begin{aligned}
\lambda^{-2}\alpha' &= \alpha + (f\widehat{\otimes}\beta - {}^*f\widehat{\otimes}{}^*\beta) + \left(f\widehat{\otimes}f - \frac{1}{2}{}^*f\widehat{\otimes}{}^*f\right)\rho + \frac{3}{2}(f\widehat{\otimes}{}^*f)\,{}^*\rho\\
&\quad + l.o.t.,
\end{aligned}$$

and the control of the change of frame coefficients $(f, \underline{f}, \lambda)$ *provided by Property (f) of Proposition 3.33 together with* (3.63)*. Note that the anomalous estimates in* (6.14) *for the maximum power of* r *in* $\nabla_3 A$ *and* $\nabla_3^2 A$ *are due to the term* $f\widehat{\otimes}\beta - {}^*f\widehat{\otimes}{}^*\beta$ *in the transformation formula.*

Remark 6.8. *Let*

$$\delta' := \frac{1}{2}(\delta_{extra} - \delta_{dec}), \tag{6.15}$$

where $\delta' > 0$ *since* $\delta_{extra} > \delta_{dec}$*. In view of Definition 6.4 and* **Ref 2***, we have*

$$A \in r^{-2-\delta'}Good_{k_*}, \qquad \nabla_3 A \in r^{-3-\delta'}Good_{k_*}, \qquad \nabla_3^2 A \in r^{-4-\delta'}Good_{k_*}.$$

6.1.4. Main equations in ${}^{(ext)}\mathcal{M}$ We recall below a subset of the null structure and Bianchi equations holding for an outgoing PG structure, see Proposition 2.19.

Proposition 6.9. *In the outgoing PG structure of* ${}^{(ext)}\mathcal{M}$*, we have*

$$\begin{aligned}
\nabla_4 trX + \frac{1}{2}(trX)^2 &= -\frac{1}{2}\widehat{X}\cdot\overline{\widehat{X}},\\
\nabla_4\widehat{X} + \Re(trX)\widehat{X} &= -A,
\end{aligned}$$

[133]The estimate for $\nabla^2_{e_3'}A'$ follows immediately from the estimate in Theorem M1 for A', $\nabla_{e_3'}A'$ and $\mathfrak{q}$, and the definition of $\mathfrak{q}$ which yields $\nabla^2_{e_3'}A' = O(r^{-4})\mathfrak{q} + O(r^{-1})\nabla_3 A' + O(r^{-2})A'$.

$$
\begin{aligned}
\nabla_4 tr\underline{X} + \frac{1}{2} trX\, tr\underline{X} &= -\mathcal{D}\cdot\overline{Z} + Z\cdot\overline{Z} + 2\overline{P} - \frac{1}{2}\widehat{X}\cdot\overline{\widehat{\underline{X}}},\\
\nabla_4\widehat{\underline{X}} + \frac{1}{2} trX\,\widehat{\underline{X}} &= -\frac{1}{2}\mathcal{D}\widehat{\otimes}Z + \frac{1}{2}Z\widehat{\otimes}Z - \frac{1}{2}\overline{tr\underline{X}}\widehat{X},\\
\nabla_4 Z + trX Z &= -\widehat{X}\cdot\overline{Z} - B,\\
\nabla_4 H + \frac{1}{2}\overline{trX}(H+Z) &= -\frac{1}{2}\widehat{X}\cdot(\overline{H}+\overline{Z}) - B,\\
\nabla_4\underline{\omega} - (2\eta+\zeta)\cdot\zeta &= \rho,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2}\overline{\mathcal{D}}\cdot\widehat{X} + \frac{1}{2}\widehat{X}\cdot\overline{Z} &= \frac{1}{2}\mathcal{D}\overline{trX} + \frac{1}{2}\overline{trX}Z - i\Im(trX)H - B,\\
\frac{1}{2}\overline{\mathcal{D}}\cdot\widehat{\underline{X}} - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{Z} &= \frac{1}{2}\mathcal{D}\overline{tr\underline{X}} - \frac{1}{2}\overline{tr\underline{X}}Z - i\Im(tr\underline{X})(-Z+\underline{\Xi}) + \underline{B}.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\nabla_3 A - \frac{1}{2}\mathcal{D}\widehat{\otimes}B &= -\frac{1}{2}tr\underline{X}A + 4\underline{\omega}A + \frac{1}{2}(Z+4H)\widehat{\otimes}B - 3\overline{P}\widehat{X},\\
\nabla_4 B - \frac{1}{2}\overline{\mathcal{D}}\cdot A &= -2\overline{trX}B + \frac{1}{2}A\cdot\overline{Z},\\
\nabla_3 B - \mathcal{D}\overline{P} &= -tr\underline{X}B + 2\underline{\omega}B + \overline{\underline{B}}\cdot\widehat{X} + 3\overline{P}H + \frac{1}{2}A\cdot\overline{\underline{\Xi}},\\
\nabla_4 P - \frac{1}{2}\mathcal{D}\cdot\overline{B} &= -\frac{3}{2}trXP - \frac{1}{2}Z\cdot\overline{B} - \frac{1}{4}\widehat{\underline{X}}\cdot\overline{A},\\
\nabla_3 P + \frac{1}{2}\overline{\mathcal{D}}\cdot\underline{B} &= -\frac{3}{2}\overline{tr\underline{X}}P - \frac{1}{2}(\overline{2H-Z})\cdot\underline{B} + \underline{\Xi}\cdot\overline{B} - \frac{1}{4}\overline{\widehat{X}}\cdot\underline{A},\\
\nabla_4\underline{B} + \mathcal{D}P &= -trX\underline{B} + \overline{B}\cdot\widehat{\underline{X}} + 3PZ,\\
\nabla_3\underline{B} + \frac{1}{2}\overline{\mathcal{D}}\cdot\underline{A} &= -2\overline{tr\underline{X}}\,\underline{B} - 2\underline{\omega}\,\underline{B} - \frac{1}{2}\underline{A}\cdot(\overline{-2Z+H}) - 3P\,\underline{\Xi},\\
\nabla_4\underline{A} + \frac{1}{2}\mathcal{D}\widehat{\otimes}\underline{B} &= -\frac{1}{2}\overline{trX}\underline{A} + \frac{5}{2}Z\widehat{\otimes}\underline{B} - 3P\widehat{\underline{X}}.
\end{aligned}
$$

We also recall the following transport equations in the e_4 direction for derivatives of the outgoing PG coordinates (r,u,θ), see Proposition 2.21.

Proposition 6.10. *The following equations hold true for the coordinates (u,r,θ) associated to an outgoing PG structure*

$$
\begin{aligned}
e_4(e_3(r)) &= -2\underline{\omega},\\
\nabla_4\mathcal{D}u + \frac{1}{2}trX\mathcal{D}u &= -\frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}u,
\end{aligned}
$$

$$\begin{aligned} e_4(e_3(u)) &= -\Re\Big((Z+H)\cdot\overline{\mathcal{D}}u\Big),\\ \nabla_4\mathcal{D}\cos\theta+\frac{1}{2}trX\mathcal{D}\cos\theta &= -\frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}\cos\theta,\\ e_4(e_3(\cos\theta)) &= -\Re\Big((Z+H)\cdot\overline{\mathcal{D}}\cos\theta\Big). \end{aligned}$$

6.1.5. Commutator formulas revisited We record below the main commutation formulas which will be used in this chapter.

6.1.5.1. Real case The following commutation formulas are an immediate adaptation of those in Lemma 2.2 to the case of an outgoing PG structure[134].

Lemma 6.11. *We have the following commutations formulas:*

1. *If f is a scalar, we have*

$$\begin{aligned} [\nabla_4,\nabla_b]f &= -\frac{1}{2}\Big(tr\chi\nabla_b f+{}^{(a)}tr\chi\,{}^*\nabla_b f\Big)-\widehat{\chi}_{bc}\nabla_c f,\\ [\nabla_4,\nabla_3]f &= -2(\zeta+\eta)\cdot\nabla f-2\underline{\omega}\nabla_4 f. \end{aligned}$$

2. *If U is a horizontal tensor, we have*

$$\begin{aligned} [\nabla_4,\nabla_b]U &= -\frac{1}{2}\Big(tr\chi\nabla_b+{}^{(a)}tr\chi\,{}^*\nabla_b\Big)U-\widehat{\chi}_{bc}\nabla_c U\\ &\quad +O(1)tr\chi\zeta U+O(r^{-2})\widehat{\chi}U+O(1)\beta U,\\ [\nabla_4,\nabla_3]U &= -2(\zeta+\eta)\cdot\nabla U-2\underline{\omega}\nabla_4 U+O(1)\eta\zeta U+O(1)\,{}^*\rho U. \end{aligned}$$

6.1.5.2. Complex case The following commutation formulas can be easily derived from the ones above, see also section 4.2 in [28].

Lemma 6.12. *The following commutation formulas hold true.*

1. *For a scalar complex function F, we have*

$$[\nabla_4,\mathcal{D}]F = -\frac{1}{2}trX\mathcal{D}F+r^{-1}\Gamma_g\cdot\mathfrak{d}\!\!/F. \tag{6.16}$$

2. *For an anti-self dual horizontal 1-form U, we have*

$$[\nabla_4,\mathcal{D}\widehat{\otimes}]U = -\frac{1}{2}trX(\mathcal{D}\widehat{\otimes}U-Z\widehat{\otimes}U)+r^{-1}\Gamma_g\cdot\mathfrak{d}\!\!/^{\leq 1}U \tag{6.17}$$

[134] That is $\xi=0$, $\omega=0$ and $\underline{\eta}+\zeta=0$.

and

$$[\nabla_4, \overline{\mathcal{D}}\cdot]U = -\frac{1}{2}\overline{trX}(\overline{\mathcal{D}}\cdot U + \overline{Z}\cdot U) + r^{-1}\Gamma_g \cdot \not{\mathfrak{d}}^{\leq 1}U. \tag{6.18}$$

3. *For an anti-self dual symmetric traceless horizontal 2-form U, we have*

$$[\nabla_4, \overline{\mathcal{D}}\cdot]U = -\frac{1}{2}\overline{trX}(\overline{\mathcal{D}}\cdot U + 2\overline{Z}\cdot U) + r^{-1}\Gamma_g \cdot \not{\mathfrak{d}}^{\leq 1}U. \tag{6.19}$$

4. *For an anti-self dual horizontal k-tensor, we have*

$$[\nabla_4, \mathcal{D}]U = -\frac{1}{2}trX\mathcal{D}U + O(1)ZU + r^{-1}\Gamma_g \cdot \not{\mathfrak{d}}^{\leq 1}U. \tag{6.20}$$

Remark 6.13. *We note that the terms denoted by Γ_g in (6.16) only contain $\widehat{X}$. Also, the terms denoted by Γ_g in (6.17)–(6.20) contain only $\widehat{X}$, $\check{Z}$ and B.*

Corollary 6.14. *If U denotes*

1. *a scalar, we have*

$$[\nabla_4, q\mathcal{D}]U = \Gamma_g \cdot \not{\mathfrak{d}}U,$$

2. *an anti-self dual horizontal 1-form, we have*

$$[\nabla_4, q\mathcal{D}\widehat{\otimes}]U = O(r^{-2})U + \Gamma_g \cdot \not{\mathfrak{d}}^{\leq 1}U,$$

3. *an anti-self dual horizontal 1-form, or an anti-self dual symmetric traceless horizontal 2-form, we have*

$$[\nabla_4, \overline{q}\,\overline{\mathcal{D}}\cdot]U = O(r^{-2})U + \Gamma_g \cdot \not{\mathfrak{d}}^{\leq 1}U,$$

4. *an anti-self dual horizontal k-tensor, we have*

$$[\nabla_4, q\mathcal{D}]U = O(r^{-2})U + \Gamma_g \cdot \not{\mathfrak{d}}^{\leq 1}U.$$

Proof. This follows immediately from Lemma 6.12 and the fact that $e_4(q) = 1$, $\mathrm{tr}X = \frac{2}{q} + \Gamma_g$ and $Z = O(r^{-2}) + \Gamma_g$. □

6.1.6. Linearized null structure equations and Bianchi identities for outgoing PG structures Recall the definition of the linearized quantities and of Γ_g and Γ_b in Section 6.1.2. We use extensively the notation $O(r^{-p})$ made in Definition 6.3 to denote lower order linear terms. The following lemma provides the linearized null structure equations and Bianchi identities in ${}^{(ext)}\mathcal{M}$.

Lemma 6.15. *The linearized null structure equations in the e_4 direction are*

$$\begin{aligned}
\nabla_4(\widecheck{trX}) + \frac{2}{q}\widecheck{trX} &= \Gamma_g \cdot \Gamma_g,\\
\nabla_4\widehat{X} + \frac{2r}{|q|^2}\widehat{X} &= -A + \Gamma_g \cdot \Gamma_g,\\
\nabla_4\check{Z} + \frac{2}{q}\check{Z} &= -\frac{aq}{|q|^2}\overline{\mathfrak{J}} \cdot \widehat{X} - B + O(r^{-2})\widecheck{trX} + \Gamma_g \cdot \Gamma_g,\\
\nabla_4\check{H} + \frac{1}{\overline{q}}\check{H} &= -\frac{1}{\overline{q}}\check{Z} - \frac{ar}{|q|^2}\overline{\mathfrak{J}} \cdot \widehat{X} - B + O(r^{-2})\widecheck{trX} + \Gamma_b \cdot \Gamma_g,\\
\nabla_4\widecheck{tr\underline{X}} + \frac{1}{q}\widecheck{tr\underline{X}} &= -\mathcal{D} \cdot \overline{\check{Z}} + 2\overline{\check{P}} + O(r^{-2})\check{Z} + O(r^{-1})\widecheck{trX}\\
&\quad + O(r^{-1})\widecheck{\mathcal{D} \cdot \overline{\mathfrak{J}}} + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} + \Gamma_b \cdot \Gamma_g,\\
\nabla_4\widehat{\underline{X}} + \frac{1}{q}\widehat{\underline{X}} &= -\frac{1}{2}\mathcal{D}\widehat{\otimes}\check{Z} + O(r^{-2})\check{Z} + O(r^{-1})\widehat{X} + O(r^{-1})\mathcal{D}\widehat{\otimes}\mathfrak{J}\\
&\quad + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} + \Gamma_b \cdot \Gamma_g,\\
\nabla_4(\check{\underline{\omega}}) &= \Re(\check{P}) + O(r^{-2})\check{Z} + O(r^{-2})\check{H} + \Gamma_b \cdot \Gamma_g,\\
\nabla_4\underline{\Xi} + \frac{1}{q}\underline{\Xi} &= O(r^{-1})\,\not{\mathfrak{d}}^{\leq 1}(\check{\underline{\omega}}) + O(r^{-2})\check{Z} + O(r^{-2})\check{H} + O(r^{-2})\widecheck{tr\underline{X}}\\
&\quad + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} + \Gamma_b \cdot \Big(\check{\underline{\omega}}, \Gamma_g\Big).
\end{aligned}$$

The linearized Codazzi for $\widehat{X}$ takes the form

$$\begin{aligned}
\frac{1}{2}\overline{\mathcal{D}} \cdot \widehat{X} &= \frac{1}{\overline{q}}\check{Z} - B + O(r^{-2})\widehat{X} + O(r^{-2})\check{H} + O(r^{-1})\,\not{\mathfrak{d}}^{\leq 1}\widecheck{trX}\\
&\quad + O(r^{-2})\widecheck{\mathcal{D}(\cos\theta)} + \Gamma_b \cdot \Gamma_g.
\end{aligned}$$

The linearized Bianchi equations for $B, P, \underline{B}$ are

$$\begin{aligned}
\nabla_4 B + \frac{4}{\overline{q}}B &= \frac{1}{2}\overline{\mathcal{D}} \cdot A + \frac{aq}{2|q|^2}\overline{\mathfrak{J}} \cdot A + \Gamma_g \cdot (B, A),\\
\nabla_4\left(\check{P}\right) - \frac{1}{2}\mathcal{D} \cdot \overline{B} &= -\frac{3}{q}\check{P} - \frac{a\overline{q}}{2|q|^2}\mathfrak{J} \cdot \overline{B} + O(r^{-3})\widecheck{trX} + r^{-1}\Gamma_g \cdot \Gamma_g + \Gamma_b \cdot A,\\
\nabla_4\underline{B} + \mathcal{D}\left(\check{P}\right) &= -\frac{2}{q}\underline{B} + O(r^{-2})\check{P} + O(r^{-3})\check{Z} + O(r^{-4})\widecheck{\mathcal{D}(\cos\theta)}\\
&\quad + r^{-1}\Gamma_b \cdot \Gamma_g.
\end{aligned}$$

Also

$$\nabla_3 B - \mathcal{D}\check{\overline{P}} = \frac{2}{r}B + O(r^{-2})B + O(r^{-2})\check{P} + O(r^{-3})\check{H} + O(r^{-4})\widetilde{\mathcal{D}(\cos\theta)} + r^{-1}\Gamma_b\cdot\Gamma_g.$$

Proof. The proof of the lemma relies on the null structure equations and Bianchi identities of Proposition 6.9, the definition of the linearized quantities and of Γ_g and Γ_b in Section 6.1.2, the notation $O(r^{-p})$ made in Definition 6.3, the fact that a and m are constants, and the following identities

$$e_4(r) = 1, \quad e_4(\theta) = 0, \quad \nabla_4\mathfrak{J} = -\frac{1}{q}\mathfrak{J}, \quad e_4(q) = 1, \quad e_4(\overline{q}) = 1, \quad \nabla(r) = 0,$$

where we used in particular the fact that $q = r + ai\cos\theta$ and $\overline{q} = r - ai\sin\theta$. See Section C.1 for the proof. ☐

6.1.7. Other linearized equations for outgoing PG structures

Lemma 6.16. *We have*

$$\begin{aligned}
e_4\left(\widetilde{e_3(r)}\right) &= -2\check{\underline{\omega}},\\
\nabla_4\widetilde{\mathcal{D}u} + \frac{1}{q}\widetilde{\mathcal{D}u} &= O(r^{-1})\widetilde{trX} + O(r^{-1})\widehat{X} + \Gamma_b\cdot\Gamma_g,\\
e_4\left(\widetilde{e_3(u)}\right) &= O(r^{-1})\check{H} + O(r^{-1})\check{Z} + O(r^{-2})\widetilde{\mathcal{D}u} + \Gamma_b\cdot\Gamma_b,\\
\nabla_4\widetilde{\mathcal{D}\cos\theta} + \frac{1}{q}\widetilde{\mathcal{D}\cos\theta} &= \frac{i}{2}\overline{\mathfrak{J}}\cdot\widehat{X} + O(r^{-1})\widetilde{trX} + \Gamma_b\cdot\Gamma_g,\\
e_4(e_3(\cos\theta)) &= O(r^{-1})\check{H} + O(r^{-1})\check{Z} + O(r^{-2})\widetilde{\mathcal{D}\cos\theta} + \Gamma_b\cdot\Gamma_b.
\end{aligned}$$

Proof. One can proceed as in Lemma 6.15 starting with the corresponding equations in Proposition 6.10. See Section C.2 for the details. ☐

Lemma 6.17. *The following equations hold for the tensors $\mathfrak{J}, \mathfrak{J}_\pm$:*

1. *We have*

$$\begin{aligned}
\nabla_4(\mathcal{D}\widehat{\otimes}\mathfrak{J}) + \frac{2}{q}\mathcal{D}\widehat{\otimes}\mathfrak{J} &= O(r^{-1})B + O(r^{-2})\widetilde{trX} + O(r^{-2})\widehat{X}\\
&\quad + O(r^{-2})\check{Z} + O(r^{-3})\widetilde{\mathcal{D}(\cos\theta)}\\
&\quad + r^{-1}\Gamma_b\cdot\Gamma_g,\\
\nabla_4(\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}) + \Re\left(\frac{2}{q}\right)\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}} &= O(r^{-1})B + O(r^{-2})\widetilde{trX} + O(r^{-2})\widehat{X}\\
&\quad + O(r^{-2})\check{Z} + O(r^{-3})\widetilde{\mathcal{D}(\cos\theta)}\\
&\quad + r^{-1}\Gamma_b\cdot\Gamma_g,
\end{aligned}$$

$$\nabla_4(\widetilde{\nabla_3\mathfrak{J}}) + \frac{1}{q}\widetilde{\nabla_3\mathfrak{J}} = O(r^{-3})\widetilde{e_3(r)} + O(r^{-3})e_3(\cos\theta) + O(r^{-2})\underline{\check{\omega}} + O(r^{-2})\check{H} + O(r^{-2})\check{Z} + O(r^{-2})\check{\nabla}\mathfrak{J} + O(r^{-1})\check{P} + r^{-1}\Gamma_b\cdot\Gamma_b.$$

2. *We also have*

$$\begin{aligned}
\nabla_4(\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm) + \frac{2}{q}\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm &= O(r^{-1})B + O(r^{-2})\widetilde{trX} + O(r^{-2})\widehat{X}\\
&\quad + O(r^{-2})\check{Z} + O(r^{-3})\mathcal{D}\widetilde{(\cos\theta)}\\
&\quad + r^{-1}\Gamma_b\cdot\Gamma_g,\\
\nabla_4(\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm}) + \Re\left(\frac{2}{q}\right)\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm} &= O(r^{-1})B + O(r^{-2})\widetilde{trX} + O(r^{-2})\widehat{X}\\
&\quad + O(r^{-2})\check{Z} + O(r^{-3})\mathcal{D}\widetilde{(\cos\theta)}\\
&\quad + r^{-1}\Gamma_b\cdot\Gamma_g,\\
\nabla_4(\widetilde{\nabla_3\mathfrak{J}_\pm}) + \frac{1}{q}\widetilde{\nabla_3\mathfrak{J}_\pm} &= O(r^{-3})\widetilde{e_3(r)} + O(r^{-3})e_3(\cos\theta)\\
&\quad + O(r^{-2})\underline{\check{\omega}} + O(r^{-2})\check{H} + O(r^{-2})\check{Z}\\
&\quad + O(r^{-2})\check{\nabla}\mathfrak{J} + O(r^{-1})\check{P} + r^{-1}\Gamma_b\cdot\Gamma_b.
\end{aligned}$$

Proof. To prove the first part of the lemma, we make use of the transport equation $\nabla_4\mathfrak{J} = -q^{-1}\mathfrak{J}$, the relations

$$\overline{\mathcal{D}}\cdot\mathfrak{J} = \frac{4i(r^2+a^2)\cos\theta}{|q|^4} + \widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}, \quad \mathfrak{J}\cdot\overline{\mathfrak{J}} = \frac{2(\sin\theta)^2}{|q|^2}, \quad \mathfrak{J} = O(r^{-1}),$$

and the commutation formulas of Lemma 6.12 (see also Remark 6.13) as follows

$$\begin{aligned}
&\nabla_4\mathcal{D}\widehat{\otimes}\mathfrak{J} = \mathcal{D}\widehat{\otimes}\nabla_4\mathfrak{J} + [\nabla_4, \mathcal{D}\widehat{\otimes}]\mathfrak{J}\\
= \;& -\mathcal{D}\widehat{\otimes}\left(\frac{1}{q}\mathfrak{J}\right) - \frac{1}{2}\mathrm{tr}X(\mathcal{D}\widehat{\otimes}\mathfrak{J} - Z\widehat{\otimes}\mathfrak{J}) + O(r^{-1})B + O(r^{-2})\widehat{X}\\
= \;& -\frac{2}{q}\mathcal{D}\widehat{\otimes}\mathfrak{J} + \frac{\mathcal{D}(q)}{q^2}\widehat{\otimes}\mathfrak{J} + \frac{1}{q}\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J} + \check{Z}\right)\widehat{\otimes}\mathfrak{J} + O(r^{-1})B + O(r^{-2})\widetilde{\mathrm{tr}X}\\
&+ O(r^{-2})\widehat{X} + r^{-1}\Gamma_b\cdot\Gamma_g\\
= \;& -\frac{2}{q}\mathcal{D}\widehat{\otimes}\mathfrak{J} + O(r^{-1})B + O(r^{-2})\widetilde{\mathrm{tr}X} + O(r^{-2})\widehat{X} + O(r^{-2})\check{Z}\\
&+ O(r^{-3})\mathcal{D}\widetilde{(\cos\theta)} + r^{-1}\Gamma_b\cdot\Gamma_g.
\end{aligned}$$

Also, in the same vein,

$$\begin{aligned}
\nabla_4\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}} &= \nabla_4\left(\overline{\mathcal{D}}\cdot\mathfrak{J}-\frac{4i(r^2+a^2)\cos\theta}{|q|^4}\right)\\
&= -\overline{\mathcal{D}}\cdot\left(\frac{1}{q}\mathfrak{J}\right)+[\nabla_4,\overline{\mathcal{D}}\cdot]\mathfrak{J}-\frac{8ir\cos\theta}{|q|^4}+\frac{8i(r^2+a^2)\cos\theta}{|q|^6}e_4(|q|^2)\\
&= -\frac{1}{q}\overline{\mathcal{D}}\cdot\mathfrak{J}+\frac{\overline{\mathcal{D}}(q)}{q^2}\cdot\mathfrak{J}-\frac{1}{2}\overline{\mathrm{tr}X}(\overline{\mathcal{D}}\cdot\mathfrak{J}+\overline{Z}\cdot\mathfrak{J})+O(r^{-1})B\\
&\quad +O(r^{-2})\widehat{X}-\frac{8ir\cos\theta}{|q|^4}+\frac{16ir(r^2+a^2)\cos\theta}{|q|^6}\\
&= -\left(\frac{1}{q}+\frac{1}{\overline{q}}\right)\overline{\mathcal{D}}\cdot\mathfrak{J}+\frac{a}{q^2}\overline{\mathfrak{J}}\cdot\mathfrak{J}-\frac{1}{\overline{q}}\frac{aq}{|q|^2}\overline{\mathfrak{J}}\cdot\mathfrak{J}-\frac{8ir\cos\theta}{|q|^4}\\
&\quad +\frac{16ir(r^2+a^2)\cos\theta}{|q|^6}+O(r^{-1})B+O(r^{-2})\widecheck{\mathrm{tr}X}+O(r^{-2})\widehat{X}\\
&\quad +O(r^{-2})\widecheck{Z}+O(r^{-3})\widecheck{\mathcal{D}(\overline{q})}+r^{-1}\Gamma_b\cdot\Gamma_g\\
&= -\Re\left(\frac{2}{q}\right)\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}+O(r^{-1})B+O(r^{-2})\widecheck{\mathrm{tr}X}+O(r^{-2})\widehat{X}+O(r^{-2})\widecheck{Z}\\
&\quad +O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)}+r^{-1}\Gamma_b\cdot\Gamma_g.
\end{aligned}$$

Finally, using the commutation formula for $[\nabla_4,\nabla_3]$ in Lemma 2.2, we have

$$\begin{aligned}
\nabla_4\widecheck{\nabla_3\mathfrak{J}} &= \nabla_4\left(\nabla_3\mathfrak{J}-\frac{\Delta q}{|q|^4}\mathfrak{J}\right)\\
&= -\nabla_3\left(\frac{1}{q}\mathfrak{J}\right)+[\nabla_4,\nabla_3]\mathfrak{J}-\partial_r\left(\frac{\Delta q}{|q|^4}\right)\mathfrak{J}+\frac{\Delta}{|q|^4}\mathfrak{J}\\
&= -\frac{1}{q}\nabla_3\mathfrak{J}+\frac{e_3(q)}{q^2}\mathfrak{J}-2\underline{\omega}\nabla_4\mathfrak{J}-2(\eta+\zeta)\cdot\nabla\mathfrak{J}-2(\zeta\cdot\mathfrak{J})\eta\\
&\quad +2(\eta\cdot\mathfrak{J})\zeta-2\,{}^*\rho\,{}^*\mathfrak{J}-\partial_r\left(\frac{\Delta q}{|q|^4}\right)\mathfrak{J}+\frac{\Delta}{|q|^4}\mathfrak{J}.
\end{aligned}$$

Continuing

$$\begin{aligned}
\nabla_4\widecheck{\nabla_3\mathfrak{J}} &= -\frac{1}{q}\widecheck{\nabla_3\mathfrak{J}}+O(r^{-3})\widecheck{e_3(r)}+O(r^{-3})e_3(\cos\theta)+O(r^{-2})\widecheck{\underline{\omega}}\\
&\quad +O(r^{-2})\widecheck{H}+O(r^{-2})\widecheck{Z}+O(r^{-2})\widecheck{\nabla}\mathfrak{J}+O(r^{-1})\widecheck{P}+r^{-1}\Gamma_b\cdot\Gamma_b.
\end{aligned}$$

This concludes the proof of the first part of the lemma. The proof of the second part is similar and left to the reader. □

Lemma 6.18. *The following equations hold true*[135]

$$\nabla_4(\widetilde{\mathcal{D}(J^{(\pm)})}) + \frac{1}{q}\widetilde{\mathcal{D}(J^{(\pm)})} = O(r^{-1})\widetilde{trX} + O(r^{-1})\widehat{X} + \Gamma_b\cdot\Gamma_g,$$
$$\nabla_4(\widetilde{\nabla_3 J^{(\pm)}}) = O(r^{-2})\widetilde{\mathcal{D}(J^{(\pm)})} + O(r^{-1})\check{Z} + O(r^{-1})\check{H} + \Gamma_b\cdot\Gamma_b.$$

Proof. Recall that, see (6.7),

$$\widetilde{\mathcal{D}(J^{(\pm)})} = \mathcal{D}(J^{(\pm)}) - \mathfrak{J}_\pm, \qquad e_3\widetilde{(J^{(\pm)})} = e_3(J^{(\pm)}) \pm \frac{2a}{|q|^2}J^{(\mp)}.$$

Starting with the equation $\nabla_4(J^{(\pm)}) = 0$ we use the commutator formulas of Lemma 6.12 (see also Remark 6.13) to derive

$$\nabla_4(\mathcal{D}J^{(p)}) = [\nabla_4, \mathcal{D}]J^{(p)} = -\frac{1}{2}\mathrm{tr}X\mathcal{D}J^{(p)} + r^{-1}\widehat{X}\cdot\mathfrak{d}J^{(p)}.$$

Hence

$$\nabla_4(\mathcal{D}J^{(p)}) + \frac{1}{2}\mathrm{tr}X\mathcal{D}J^{(p)} = r^{-1}\widehat{X}\cdot\mathfrak{d}J^{(p)}.$$

We further deduce, using $\nabla_4\mathfrak{J}_\pm + \frac{1}{q}\mathfrak{J}_\pm = 0$,

$$\begin{aligned}
\nabla_4(\widetilde{\mathcal{D}(J^{(\pm)})}) + \frac{1}{2}\mathrm{tr}X\widetilde{\mathcal{D}(J^{(\pm)})} &= \nabla_4(\mathcal{D}(J^{(\pm)}) - \mathfrak{J}_\pm) + \frac{1}{2}\mathrm{tr}X(\mathcal{D}(J^{(\pm)}) - \mathfrak{J}_\pm)\\
&= r^{-1}\widehat{X}\cdot\mathfrak{d}J^{(p)} - \left(\nabla_4\mathfrak{J}_\pm + \frac{1}{2}\mathrm{tr}X\mathfrak{J}_\pm\right)\\
&= r^{-1}\widehat{X}\cdot\mathfrak{d}J^{(p)} - \frac{1}{2}\widetilde{\mathrm{tr}X}\mathfrak{J}_\pm.
\end{aligned}$$

Thus, since $J^{(\pm)} = O(1)$ and $\mathfrak{J}_\pm = O(r^{-1})$,

$$\nabla_4(\widetilde{\mathcal{D}(J^{(\pm)})}) + \frac{1}{q}\widetilde{\mathcal{D}(J^{(\pm)})} = O(r^{-1})\widetilde{\mathrm{tr}X} + O(r^{-1})\widehat{X} + \Gamma_b\cdot\Gamma_g$$

as stated.

Similarly

$$\begin{aligned}
\nabla_4\nabla_3 J^{(\pm)} &= [\nabla_4, \nabla_3]J^{(\pm)} = -2(\zeta+\eta)\cdot\nabla J^{(\pm)} = -\Re\Big((\overline{Z+H})\cdot\mathcal{D}J^{(\pm)}\Big)\\
&= -\Re\Big((\overline{Z+H})\cdot(\widetilde{\mathcal{D}(J^{(\pm)})} + \mathfrak{J}_\pm)\Big)
\end{aligned}$$

[135]Similar equations hold for $J^{(0)} = \cos\theta$, see Lemma 6.16.

and thus

$$\nabla_4\Big(\nabla_3 J^{(\pm)} \pm \frac{2a}{|q|^2}J^{(\mp)}\Big) \;=\; -\Re\Big((\overline{Z+H})\cdot(\widetilde{\mathcal{D}(J^{(\pm)})}+\mathfrak{J}_\pm)\Big) \mp \frac{4ar}{|q|^4}J^{(\mp)}.$$

Hence

$$\nabla_4\Big(\widetilde{\nabla_3 J^{(\pm)}}\Big) \;=\; -\Re\Big((\overline{Z+H})\cdot\mathfrak{J}_\pm\Big) \mp \frac{4ar}{|q|^4}J^{(\mp)} + O(r^{-2})\widetilde{\mathcal{D}(J^{(\pm)})} + \Gamma_b\cdot\Gamma_b.$$

Now, recalling that,

$$Z = \check{Z} + \frac{a\overline{q}}{|q|^2}\mathfrak{J}, \qquad H = \check{H} + \frac{aq}{|q|^2}\mathfrak{J}, \qquad \Re(\mathfrak{J}_\pm)\cdot\Re(\mathfrak{J}) = \mp\frac{1}{|q|^2}J^{(\mp)},$$

we deduce

$$\begin{aligned}\Re\Big((\overline{Z+H})\cdot\mathfrak{J}_\pm\Big) &= \Re\Big((\overline{\check{Z}+\check{H}})\cdot\mathfrak{J}_\pm\Big) + \left(\frac{a\overline{q}}{|q|^2}+\frac{aq}{|q|^2}\right)\Re(\overline{\mathfrak{J}}\cdot\mathfrak{J}_\pm)\\ &= \frac{4ar}{|q|^2}\Re(\mathfrak{J})\cdot\Re(\mathfrak{J}_\pm) + O(r^{-1})\check{Z} + O(r^{-1})\check{H}\\ &= \mp\frac{4ar}{|q|^4}J^{(\mp)} + O(r^{-1})\check{Z} + O(r^{-1})\check{H}.\end{aligned}$$

Hence

$$\nabla_4\Big(\widetilde{\nabla_3 J^{(\pm)}}\Big) \;=\; O(r^{-2})\widetilde{\mathcal{D}(J^{(\pm)})} + O(r^{-1})\check{Z} + O(r^{-1})\check{H} + \Gamma_b\cdot\Gamma_b$$

as stated. □

6.1.8. The vectorfield T in $^{(ext)}\mathcal{M}$ We recall that in $^{(ext)}\mathcal{M}$, the vectorfield $\mathbf{T}$ was defined by, see Section 2.6.5,

$$\mathbf{T} \;:=\; \frac{1}{2}\left(e_3 + \frac{\Delta}{|q|^2}e_4 - 2a\Re(\mathfrak{J})^b e_b\right). \tag{6.21}$$

Lemma 6.19. *The following hold true.*

1. *We have*

$$\mathbf{g}(\mathbf{T},\mathbf{T}) \;=\; -1 + \frac{2mr}{|q|^2}. \tag{6.22}$$

2. *We have*

(6.23) $$\mathbf{T}(u) = 1 + \frac{1}{2}\left(\widecheck{e_3(u)} - 2a\Re(\mathfrak{J})\cdot\widecheck{\nabla u}\right), \quad \mathbf{T}(r) = \frac{1}{2}\widecheck{e_3(r)}.$$

3. We have

(6.24) $$\begin{aligned}\mathbf{T}(\cos\theta) &= \frac{1}{2}\left(e_3(\cos\theta) - 2a\Re(\mathfrak{J})\cdot\widecheck{\nabla\cos\theta}\right),\\ \mathbf{T}(J^{(\pm)}) &= \frac{1}{2}\widecheck{e_3(J^{(\pm)})} - a\Re(\mathfrak{J})\cdot\Re(\widecheck{\mathcal{D}(J^{(\pm)})}).\end{aligned}$$

In particular

$$\mathbf{T}(r)\in r\Gamma_b, \quad \mathbf{T}(u) = 1 + r\Gamma_b, \quad \mathbf{T}(\cos\theta)\in\Gamma_b, \quad \mathbf{T}(J^{(\pm)})\in\Gamma_b.$$

Proof. Equation (6.22) follows easily from $|\Re(\mathfrak{J})|^2 = \frac{(\sin\theta)^2}{|q|^2}$.

The identities in (6.23) follow easily from the relations $e_4(r) = 1$, $e_4(u) = 0$, $\nabla(r) = 0$, and the definition of the linearized quantities $\widecheck{e_3(r)}$, $\widecheck{e_3(u)}$ and $\widecheck{\nabla u}$.

To check the first identity in (6.24), we make use of $e_4(\theta) = 0$, the definition of the linearized quantity $\widecheck{\nabla\cos\theta} = \nabla\cos\theta - \Re(i\mathfrak{J})$, and $\Re(\mathfrak{J})\cdot\Im(\mathfrak{J}) = 0$, which yields

$$\mathbf{T}(\cos\theta) = \frac{1}{2}e_3(\cos\theta) - a\Re(\mathfrak{J})\cdot\nabla\cos\theta = \frac{1}{2}e_3(\cos\theta) - a\Re(\mathfrak{J})\cdot\widecheck{\nabla\cos\theta}.$$

To check the second identity in (6.24), we make use of $e_4(J^{(\pm)}) = 0$, the definition of the linearized quantities $\widecheck{e_3(J^{(\pm)})} = e_3(J^{(\pm)})\pm\frac{2a}{|q|^2}J^{(\mp)}$, $\widecheck{\mathcal{D}(J^{(\pm)})} = \mathcal{D}(J^{(\pm)}) - \mathfrak{J}_\pm$, and the identity $\Re(\mathfrak{J}_\pm)\cdot\Re(\mathfrak{J}) = \mp\frac{1}{|q|^2}J^{(\mp)}$, see (6.5). Thus

$$\begin{aligned}\mathbf{T}(J^{(\pm)}) &= \frac{1}{2}\left(e_3 + \frac{\Delta}{|q|^2}e_4 - 2a\Re(\mathfrak{J})^b e_b\right)J^{(\pm)} = \frac{1}{2}e_3(J^{(\pm)}) - a\Re(\mathfrak{J})\cdot\nabla J^{(\pm)}\\ &= \frac{1}{2}\widecheck{e_3(J^{(\pm)})} \mp \frac{a}{|q|^2}J^{(\mp)} - a\Re(\mathfrak{J})\cdot\Re(\mathcal{D}J^{(\pm)})\\ &= \frac{1}{2}\widecheck{e_3(J^{(\pm)})} \mp \frac{a}{|q|^2}J^{(\mp)} - a\Re(\mathfrak{J})\cdot\Re(\widecheck{\mathcal{D}(J^{(\pm)})} + \mathfrak{J}_\pm)\\ &= \frac{1}{2}\widecheck{e_3(J^{(\pm)})} - a\Re(\mathfrak{J})\cdot\Re(\widecheck{\mathcal{D}(J^{(\pm)})}) - a\Re(\mathfrak{J})\cdot\Re(\mathfrak{J}_\pm) \mp \frac{a}{|q|^2}J^{(\mp)}\\ &= \frac{1}{2}\widecheck{e_3(J^{(\pm)})} - a\Re(\mathfrak{J})\cdot\Re(\widecheck{\mathcal{D}(J^{(\pm)})})\end{aligned}$$

as stated. □

Remark 6.20. *In Kerr, we have* $\mathbf{T} = \partial_t$ *where* ∂_t *is the coordinate vectorfield corresponding to the Boyer-Lindquist coordinates.*

We recall below Proposition 2.70

Proposition 6.21. *We have* ${}^{(\mathbf{T})}\pi_{44} = 0$, ${}^{(\mathbf{T})}\pi_{4a} \in \Gamma_g$ *and all other components of* ${}^{(\mathbf{T})}\pi$*, relative to the frame of* ${}^{(ext)}\mathcal{M}$*, are in* Γ_b*. Moreover*

$$g^{ab\,(\mathbf{T})}\pi_{ab} \quad = \quad \Gamma_g.$$

In addition

$$\mathbf{g}(\mathbf{D}_a\mathbf{T}, e_4),\ \mathbf{g}(\mathbf{D}_4\mathbf{T}, e_a) \in \Gamma_g, \qquad \mathbf{g}(\mathbf{D}_a\mathbf{T}, e_3),\ \mathbf{g}(\mathbf{D}_3\mathbf{T}, e_a) \in \Gamma_b,$$

and

$$k_{ab} := \mathbf{g}(\mathbf{D}_a\mathbf{T}, e_b) = -\frac{2amr\cos\theta}{|q|^4} \in_{ab} +\Gamma_b.$$

6.1.9. Commutation formulas with $\not{\mathcal{L}}_{\mathbf{T}}$ We recall the following definition of projected Lie derivative, see section 2.2.8 of [28]. Given vectorfields X, Y, the projected Lie derivative $\not{\mathcal{L}}_X Y$ is given by

$$\not{\mathcal{L}}_X Y := \mathcal{L}_X Y + \frac{1}{2}\mathbf{g}(\mathcal{L}_X Y, e_3)e_4 + \frac{1}{2}\mathbf{g}(\mathcal{L}_X Y, e_4)e_3.$$

Given a horizontal covariant k-tensor U, the horizontal Lie derivative $\not{\mathcal{L}}_X U$ is defined to be the projection of $\mathcal{L}_X U$ to the horizontal space. Thus, for horizontal indices $A = a_1 \dots a_k$,

$$(\not{\mathcal{L}}_X U)_A : \quad = \quad \nabla_X U_A + \mathbf{D}_{a_1}X^b U_{b\dots a_k} + \cdots + \mathbf{D}_{a_k}X^b U_{a_1\dots b}. \tag{6.25}$$

We recall below Lemma 2.2.13 in [28].

Lemma 6.22. *The following commutation formulas hold true for a horizontal covariant k-tensor* U *and a vectorfield* X

$$\nabla_b(\not{\mathcal{L}}_X U_A) - \not{\mathcal{L}}_X(\nabla_b U_A) = \sum_{j=1}^k {}^{(X)}\not{\Gamma}_{a_j bc} U_{a_1\dots}{}^{c}{}_{\dots a_k},$$

$$\nabla_4(\not{\mathcal{L}}_X U_A) - \not{\mathcal{L}}_X(\nabla_4 U_A) + \nabla_{\not{\mathcal{L}}_X e_4} U_A = \sum_{j=1}^k {}^{(X)}\not{\Gamma}_{a_j 4c} U_{a_1\dots}{}^{c}{}_{\dots a_k},$$

$$\nabla_3(\not{\mathcal{L}}_X U_A) - \not{\mathcal{L}}_X(\nabla_3 U_A) + \nabla_{\not{\mathcal{L}}_X e_3} U_A = \sum_{j=1}^k {}^{(X)}\not{\Gamma}_{a_j 3c} U_{a_1\dots}{}^{c}{}_{\dots a_k},$$

with[136]

$$
\begin{aligned}
{}^{(X)}\not{\Gamma}_{abc} &= \frac{1}{2}(\nabla_a {}^{(X)}\pi_{bc} + \nabla_b {}^{(X)}\pi_{ac} - \nabla_c {}^{(X)}\pi_{ab}),\\
{}^{(X)}\not{\Gamma}_{a4b} &= \frac{1}{2}(\nabla_a {}^{(X)}\pi_{4b} + \nabla_4 {}^{(X)}\pi_{ab} - \nabla_b {}^{(X)}\pi_{a4}),\\
{}^{(X)}\not{\Gamma}_{a3b} &= \frac{1}{2}(\nabla_a {}^{(X)}\pi_{3b} + \nabla_3 {}^{(X)}\pi_{ab} - \nabla_b {}^{(X)}\pi_{a3}).
\end{aligned}
$$

We apply Lemma 6.22 to the case when X is the vectorfield $\mathbf{T}$.

Lemma 6.23. *The following holds true:*

1. *We have*

$$
\not{\mathcal{L}}_{\mathbf{T}} e_4 \in \Gamma_b. \tag{6.26}
$$

2. *For any horizontal k-tensor U, we have*

$$
\nabla_4(\not{\mathcal{L}}_{\mathbf{T}} U_A) - \not{\mathcal{L}}_{\mathbf{T}}(\nabla_4 U_A) = r^{-1}\Gamma_b \cdot \not{\mathfrak{d}} U + r^{-1}\mathfrak{d}\Gamma_b \cdot U. \tag{6.27}
$$

3. *For any horizontal k-tensor U, we have*

$$
\nabla(\not{\mathcal{L}}_{\mathbf{T}} U_A) - \not{\mathcal{L}}_{\mathbf{T}}(\nabla U_A) = r^{-1}\not{\mathfrak{d}}\Gamma_b \cdot U. \tag{6.28}
$$

Proof. In view of the definition of $\mathbf{T}$ in (6.21), we have

$$
\begin{aligned}
2[e_4, \mathbf{T}] &= \left[e_4, e_3 + \frac{\Delta}{|q|^2} e_4 - 2a\Re(\mathfrak{J})^b e_b\right]\\
&= [e_4, e_3] + e_4\left(\frac{\Delta}{|q|^2}\right) e_4 - 2a\nabla_4\Re(\mathfrak{J})^b e_b + 2a\Re(\mathfrak{J})^b \nabla_b e_4\\
&= -2(\zeta+\eta)_b e_b - 2\underline{\omega} e_4 + \partial_r\left(\frac{\Delta}{|q|^2}\right) e_4 + 2a\Re\left(\frac{1}{q}\mathfrak{J}\right)^b e_b\\
&\quad +2a\Re(\mathfrak{J})^b(\chi_{bc}e_c - \zeta_b e_4)\\
&= -2(\check{\zeta}+\check{\eta})_b e_b - 2\check{\underline{\omega}} e_4\\
&\quad +2a\Re(\mathfrak{J})^b\left(\frac{1}{2}\widecheck{\mathrm{tr}\chi}\delta_{bc}e_c + \frac{1}{2}\widecheck{{}^{(a)}\mathrm{tr}\chi} \in_{bc} e_c + \widehat{\chi}_{bc}e_c - \check{\zeta}_b e_4\right).
\end{aligned}
$$

Hence, since $\not{\mathcal{L}}_{\mathbf{T}}(e_4)$ is the horizontal projection of $[\mathbf{T}, e_4]$, we obtain $\not{\mathcal{L}}_T(e_4) \in \Gamma_b$ which is (6.26).

[136]Here, ${}^{(X)}\pi_{ab}$ is treated as a horizontal symmetric 2-tensor, and ${}^{(X)}\pi_{a4}$, ${}^{(X)}\pi_{a3}$, as horizontal 1-forms.

Next, in view of the form of ${}^{(X)}\not\Gamma$ in Lemma 6.22 with the particular choice $X=\mathbf{T}$, and together with Proposition 6.21, we have

$$
\begin{aligned}
{}^{(\mathbf{T})}\not\Gamma_{abc} &= \frac{1}{2}(\nabla_a {}^{(\mathbf{T})}\pi_{bc} + \nabla_b {}^{(\mathbf{T})}\pi_{ac} - \nabla_c {}^{(\mathbf{T})}\pi_{ab}) \in r^{-1}\not\mathfrak{d}\Gamma_b,\\
{}^{(\mathbf{T})}\not\Gamma_{a4b} &= \frac{1}{2}(\nabla_a {}^{(\mathbf{T})}\pi_{4b} + \nabla_4 {}^{(\mathbf{T})}\pi_{ab} - \nabla_b {}^{(\mathbf{T})}\pi_{a4}) \in r^{-1}\mathfrak{d}\Gamma_b.
\end{aligned}
$$

Using Lemma 6.22 with $X=\mathbf{T}$, we infer

$$
\begin{aligned}
\nabla_b(\not\mathcal{L}_{\mathbf{T}}U_A) - \not\mathcal{L}_{\mathbf{T}}(\nabla_b U_A) &= r^{-1}\not\mathfrak{d}\Gamma_b\cdot U,\\
\nabla_4(\not\mathcal{L}_{\mathbf{T}}U_A) - \not\mathcal{L}_{\mathbf{T}}(\nabla_4 U_A) + \nabla_{\not\mathcal{L}_{\mathbf{T}}e_4}U_A &= r^{-1}\mathfrak{d}\Gamma_b\cdot U.
\end{aligned}
$$

The first identity yields (6.28) while the second identity, together with (6.26), yields (6.27). This concludes the proof of the lemma. □

6.1.10. Relation between $\not\mathcal{L}_{\mathbf{T}}$ and ∇_3

Lemma 6.24. *If U is a horizontal k-tensor, we have*

$$
\begin{aligned}
\not\mathcal{L}_{\mathbf{T}}U_A &= \frac{1}{2}\nabla_3 U_A + \frac{1}{2}\frac{\Delta}{|q|^2}\nabla_4 U_A + O(r^{-1})\nabla U_A\\
&\quad + O(r^{-3})U + \Gamma_b\cdot U.
\end{aligned}
\tag{6.29}
$$

In particular

$$
\begin{aligned}
\not\mathcal{L}_{\mathbf{T}}U &= \frac{1}{2}\nabla_3 U + O(r^{-1})\mathfrak{d}^{\leq 1}U + \Gamma_b\cdot U,\\
\not\mathcal{L}^2_{\mathbf{T}}U &= \frac{1}{4}\nabla_3^2 U + O(r^{-1})\mathfrak{d}^{\leq 1}\nabla_3 U + O(r^{-2})\mathfrak{d}^{\leq 2}U + \mathfrak{d}^{\leq 1}(\Gamma_b\cdot U).
\end{aligned}
\tag{6.30}
$$

Proof. Recall from Proposition 6.21 that we have

$$
k_{ab} = \mathbf{g}(\mathbf{D}_a\mathbf{T}, e_b) = -\frac{2amr\cos\theta}{|q|^4}\in_{ab} + \Gamma_b.
$$

Together with (6.25), for $p=1,2$ and $A=a_1\dots a_p$, we obtain

$$
\begin{aligned}
(\not\mathcal{L}_{\mathbf{T}}U)_A &= \nabla_{\mathbf{T}}U_A + \mathbf{D}_{a_1}\mathbf{T}^b U_{b\dots a_p} + \cdots + \mathbf{D}_{a_p}\mathbf{T}^b U_{a_1\dots b}\\
&= \nabla_{\mathbf{T}}U_A - \frac{2pamr\cos\theta}{|q|^4}\,{}^*U_A + \Gamma_b\cdot U\\
&= \frac{1}{2}\nabla_3 U_A + \frac{1}{2}\frac{\Delta}{|q|^2}\nabla_4 U_A - \Re(\mathfrak{J})^b\nabla_b U_A + O(r^{-3})U + \Gamma_b\cdot U
\end{aligned}
$$

$$= \frac{1}{2}\nabla_3 U_A + \frac{1}{2}\frac{\Delta}{|q|^2}\nabla_4 U_A + O(r^{-1})\nabla U_A + O(r^{-3})U + \Gamma_b \cdot U$$

as stated. The two other identities easily follow from this one. □

In the same spirit, we have the following more precise decomposition.

Corollary 6.25. *We have*

$$\nabla_3 = 2\not{\mathcal{L}}_{\mathbf{T}} - \frac{\Delta}{|q|^2}\nabla_4 + O(r^{-1})\nabla + O(r^{-3}) + \Gamma_b. \tag{6.31}$$

Proof. Note from the proof of the previous lemma the identity

$$\not{\mathcal{L}}_{\mathbf{T}} f = \nabla_{\mathbf{T}} f + f \cdot k.$$

Since

$$2\mathbf{T} = e_3 + \frac{\Delta}{|q|^2}e_4 + O(r^{-1})\nabla, \qquad k = O(r^{-3}) + \Gamma_b,$$

we infer

$$\nabla_3 = 2\not{\mathcal{L}}_{\mathbf{T}} - \frac{\Delta}{|q|^2}\nabla_4 + O(r^{-1})\nabla + O(r^{-3}) + \Gamma_b$$

as desired. □

6.2. Properties of the spheres $S(u, r)$

6.2.1. An orthonormal frame of $S(u, r)$ The following lemma exhibits an orthonormal frame of $S(u, r)$.

Lemma 6.26. *Let (e_1, e_2) be an orthonormal basis of the horizontal structure associated to the PG structure of ${}^{(ext)}\mathcal{M}$. Then, there exists an orthonormal basis (e_1', e_2') of the tangent space of $S(u, r)$ of the following form*

$$e_a' = \left(\delta_a^b + \frac{1}{2}\underline{f}_a f^b\right)e_b + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right)e_3,$$

where the 1-forms f and $\underline{f}$ are given by

$$f = -\frac{4}{e_3(u) + \sqrt{(e_3(u))^2 + 4|\nabla u|^2 e_3(r)}}\nabla u,$$
$$\underline{f} = \frac{2e_3(r)}{\sqrt{(e_3(u))^2 + 4|\nabla u|^2 e_3(r)}}\nabla u.$$

Also,

$$f = \left(-\frac{2|q|^2}{r^2+a^2+\Sigma} + r\Gamma_b\right)\nabla u, \qquad \underline{f} = \left(-\frac{\Delta}{\Sigma} + r\Gamma_b\right)\nabla u. \tag{6.32}$$

Proof. In view of the definition of $\widecheck{e_3(u)}$, $\widecheck{\nabla} u$ and $\widecheck{e_3(r)}$, and since $\widecheck{e_3(u)} \in r\Gamma_b$, $\widecheck{e_3(r)} \in r\Gamma_b$ and $\widecheck{\nabla} u \in \Gamma_b$, we have

$$e_3(u) = 2 + O(r^{-2}) + r\Gamma_b, \qquad (e_3(u))^2 + 4|\nabla u|^2 e_3(r) = 4 + O(r^{-2}) + r\Gamma_b.$$

Thus, since $r \geq r_0$ in ${}^{(ext)}\mathcal{M}$, we infer

$$e_3(u) > 0, \qquad (e_3(u))^2 + 4|\nabla u|^2 e_3(r) > 0, \quad \text{on} \quad {}^{(ext)}\mathcal{M}.$$

Thus, we may apply Lemma 2.22 which yields the existence of an orthonormal basis (e_1', e_2') of the tangent space of $S(u, r)$ of the form

$$e_a' = \left(\delta_a^b + \frac{1}{2}\underline{f}_a f^b\right) e_b + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2} f_a + \frac{1}{8}|f|^2 \underline{f}_a\right) e_3,$$

with the 1-forms f and $\underline{f}$ given by

$$\begin{aligned} f &= -\frac{4}{e_3(u) + \sqrt{(e_3(u))^2 + 4|\nabla u|^2 e_3(r)}}\nabla u, \\ \underline{f} &= \frac{2e_3(r)}{\sqrt{(e_3(u))^2 + 4|\nabla u|^2 e_3(r)}}\nabla u, \end{aligned}$$

as stated. Then, (6.32) follows from the above form of f and $\underline{f}$, the definition of $\widecheck{e_3(u)}$, $\widecheck{\nabla} u$ and $\widecheck{e_3(r)}$, and the fact that $\widecheck{e_3(u)} \in r\Gamma_b$, $\widecheck{e_3(r)} \in r\Gamma_b$ and $\widecheck{\nabla} u \in \Gamma_b$. □

Lemma 6.27. *Let f and $\underline{f}$ be the horizontal 1-forms given by (6.32). Also, let the scalar function λ be given by (2.12), i.e.*

$$\lambda = 1 + \frac{1}{2} f \cdot \underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2.$$

Then, we have

$$f = O(r^{-1}), \qquad \underline{f} = O(r^{-1}), \qquad f = \underline{f} + O(r^{-2}) + \Gamma_b, \qquad \lambda = 1 + O(r^{-2}),$$

as well as

$$\underline{f} - \frac{\Delta}{|q|^2}\left(f + \frac{1}{4}|f|^2\underline{f}\right) \quad = \quad \Gamma_b.$$

Proof. In view of (6.32), we have

$$f = \left(-\frac{2|q|^2}{r^2+a^2+\Sigma} + r\Gamma_b\right)\nabla u, \qquad \underline{f} = \left(-\frac{\Delta}{\Sigma} + r\Gamma_b\right)\nabla u,$$

which, together with the formula for λ in terms of f and $\underline{f}$, immediately implies the first three claimed identities.

Concerning the last one, we have

$$\underline{f} - \frac{\Delta}{|q|^2}\left(f + \frac{1}{4}|f|^2\underline{f}\right) \quad = \quad (h + r\Gamma_b)\,\nabla u$$

where

$$\begin{aligned} h \quad &= \quad -\frac{\Delta}{\Sigma} + \frac{\Delta}{|q|^2}\left(\frac{2|q|^2}{r^2+a^2+\Sigma} + \frac{1}{4}\left(\frac{2|q|^2}{r^2+a^2+\Sigma}\right)^2 |\nabla u|^2\frac{\Delta}{\Sigma}\right) \\ &= \quad \frac{\Delta}{\Sigma(r^2+a^2+\Sigma)^2}\Big(-(r^2+a^2+\Sigma)^2 + 2(r^2+a^2+\Sigma)\Sigma + \Delta a^2(\sin\theta)^2\Big) \\ &= \quad \frac{\Delta}{\Sigma(r^2+a^2+\Sigma)^2}\Big(\Sigma^2 - (r^2+a^2)^2 + \Delta a^2(\sin\theta)^2\Big) = 0 \end{aligned}$$

so that

$$\underline{f} - \frac{\Delta}{|q|^2}\left(f + \frac{1}{4}|f|^2\underline{f}\right) \quad = \quad r\Gamma_b\nabla u = \Gamma_b$$

as desired. □

6.2.2. Comparison of horizontal derivatives and derivatives tangential to $S(u, r)$

Lemma 6.28. *We have*

$$\begin{aligned} \mathbf{g}(\mathbf{D}_{e'_a}e'_b, e'_c) \quad = \quad & \left(\delta_a^d + \frac{1}{2}\underline{f}_a f^d\right)\mathbf{g}\left(\mathbf{D}_{e_d}e_b, e_c\right) + \frac{1}{2}\underline{f}_a\mathbf{g}\left(\mathbf{D}_{e_4}e_b, e_c\right) \\ & + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right)\mathbf{g}\left(\mathbf{D}_{e_3}e_b, e_c\right) + \frac{1}{2}\underline{f}_c f_b e'_a(\log\lambda) \\ & - \frac{1}{2}\underline{f}_c\lambda^{-1}\chi'_{ab} + \frac{1}{2}\underline{f}_b\lambda^{-1}\chi'_{ac} + \frac{1}{2}\zeta'_a\underline{f}_c f_b - \frac{1}{4}\underline{f}_c\lambda^{-1}\chi'_{ad}\underline{f}_b f_d \end{aligned}$$

$$-\frac{1}{2}f_c\underline{\chi}_{ab}+\frac{1}{2}f_b\underline{\chi}_{ac}+Err[\mathbf{g}(\mathbf{D}_{e'_a}e'_b,e'_c)],$$

where $Err[\mathbf{g}(\mathbf{D}_{e'_a}e'_b,e'_c)]$ *contains all the terms depending on* $(f,\underline{f},\Gamma)$, *without derivative, and at least quadratic in* $(f,\underline{f})$, *and where the scalar function* λ *is given by* (2.12).

Proof. See appendix C.3. ∎

Proposition 6.29. *Let* V *be a horizontal* k*-tensor. Then, for horizontal indices* $B=b_1\cdots b_k$ *and* $B^c_{(j)}=b_1\cdots b_{j-1}c\,b_{j+1}\cdots b_k$ *we have*

$$\begin{aligned}
\nabla'_aV_B &= \nabla_aV_B+\frac{1}{2}\underline{f}_af^c\nabla_cV_B+\frac{1}{2}\underline{f}_a\nabla_4V_B+\left(\frac{1}{2}f_a+\frac{1}{8}|f|^2\underline{f}_a\right)\nabla_3V_B\\
&-\sum_{j=1}^k\Bigg\{\frac{1}{2}\underline{f}_cf_{b_j}\nabla'_a(\log\lambda)-\frac{1}{2}\underline{f}_c\lambda^{-1}\chi'_{ab_j}+\frac{1}{2}\underline{f}_{b_j}\lambda^{-1}\chi'_{ac}+\frac{1}{2}\zeta'_a\underline{f}_cf_{b_j}\\
&-\frac{1}{4}\underline{f}_c\lambda^{-1}\chi'_{ad}\underline{f}_{b_j}f_d-\frac{1}{2}f_c\underline{\chi}_{ab_j}+\frac{1}{2}f_{b_j}\underline{\chi}_{ac}\\
&+Err[\mathbf{g}(\mathbf{D}_{e'_a}e'_{b_j},e'_c)]\Bigg\}V_{B^c_{(j)}},\\
{}^*\nabla'_aV_B &= {}^*\nabla_aV_B+\frac{1}{2}\,{}^*\underline{f}_af^c\nabla_cV_B+\frac{1}{2}\,{}^*\underline{f}_a\nabla_4V_B\\
&+\left(\frac{1}{2}\,{}^*f_a+\frac{1}{8}|f|^2\,{}^*\underline{f}_a\right)\nabla_3V_B\\
&-\sum_{j=1}^k\Bigg\{\frac{1}{2}\underline{f}_cf_{b_j}\,{}^*\nabla'_a(\log\lambda)-\frac{1}{2}\underline{f}_c\lambda^{-1}\,{}^*\chi'_{ab_j}+\frac{1}{2}\underline{f}_{b_j}\lambda^{-1}\,{}^*\chi'_{ac}\\
&+\frac{1}{2}\,{}^*\zeta'_a\underline{f}_cf_{b_j}-\frac{1}{4}\underline{f}_c\lambda^{-1}\,{}^*\chi'_{ad}\underline{f}_{b_j}f_d-\frac{1}{2}f_c\,{}^*\underline{\chi}_{ab_j}+\frac{1}{2}f_{b_j}\,{}^*\underline{\chi}_{ac}\\
&+\in_{ad}\,Err[\mathbf{g}(\mathbf{D}_{e'_d}e'_{b_j},e'_c)]\Bigg\}V_{B^c_{(j)}},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}'_aV_B &= \mathcal{D}_aV_B+\frac{1}{2}\underline{F}_af^c\nabla_cV_B+\frac{1}{2}\underline{F}_a\nabla_4V_B+\left(\frac{1}{2}F_a+\frac{1}{8}|f|^2\underline{F}_a\right)\nabla_3V_B\\
&+(E[V])_{aB},
\end{aligned}$$

with

$$(E[V])_{aB} = -\sum_{j=1}^k\Bigg\{\frac{1}{2}\underline{f}_cf_{b_j}\mathcal{D}'_a(\log\lambda)-\frac{1}{2}\underline{f}_c\lambda^{-1}X'_{ab_j}+\frac{1}{2}\underline{f}_{b_j}\lambda^{-1}X'_{ac}$$

$$+\frac{1}{2}Z'_a\underline{f}_c f_{b_j} - \frac{1}{4}\underline{f}_c\lambda^{-1}X'_{ad}\underline{f}_{b_j}f_d - \frac{1}{2}f_c\underline{X}_{ab_j} + \frac{1}{2}f_{b_j}\underline{X}_{ac}\Big\}V_{B^c_{(j)}}$$
$$+\Big\{Err[\mathbf{g}(\mathbf{D}_{e'_a}e'_{b_j},e'_c)] + i\in_{ad} Err[\mathbf{g}(\mathbf{D}_{e'_d}e'_{b_j},e'_c)]\Big\}V_{B^c_{(j)}},$$

where the horizontal 1-forms f and $\underline{f}$ are given by (6.32), and the horizontal complex 1-forms F and $\underline{F}$ are given by

$$F := f + i\,{}^*f, \qquad \underline{F} := \underline{f} + i\,{}^*\underline{f}.$$

Proof. We prove the formula for $\nabla' V$. For simplicity, we do it for a 1-tensor V. We have

$$\nabla'_a V_b = e'_a(V_b) - \mathbf{g}(\mathbf{D}_{e'_a}e'_b, e'_c)V_c.$$

Since

$$e'_a = \left(\delta_a^b + \frac{1}{2}\underline{f}_a f^b\right)e_b + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right)e_3,$$

and

$$\begin{aligned}\mathbf{g}(\mathbf{D}_{e'_a}e'_b, e'_c) &= \left(\delta_a^d + \frac{1}{2}\underline{f}_a f^d\right)\mathbf{g}\left(\mathbf{D}_{e_d}e_b, e_c\right) + \frac{1}{2}\underline{f}_a\mathbf{g}\left(\mathbf{D}_{e_4}e_b, e_c\right)\\ &+\left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right)\mathbf{g}\left(\mathbf{D}_{e_3}e_b, e_c\right) - \frac{1}{2}\underline{f}_c\chi'_{ab} + \frac{1}{2}\underline{f}_b\chi'_{ac}\\ &+\frac{1}{2}\zeta'_a\underline{f}_c f_b - \frac{1}{4}\underline{f}_c\chi'_{ad}\underline{f}_b f_d - \frac{1}{2}f_c\underline{\chi}_{ab} + \frac{1}{2}f_b\underline{\chi}_{ac}\\ &+\mathrm{Err}[\mathbf{g}(\mathbf{D}_{e'_a}e'_b, e'_c)]\end{aligned}$$

we infer

$$\begin{aligned}\nabla'_a V_b &= \left[\left(\delta_a^b + \frac{1}{2}\underline{f}_a f^d\right)e_d + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right)e_3\right](V_b)\\ &-\Big\{\left(\delta_a^d + \frac{1}{2}\underline{f}_a f^d\right)\mathbf{g}\left(\mathbf{D}_{e_d}e_b, e_c\right) + \frac{1}{2}\underline{f}_a\mathbf{g}\left(\mathbf{D}_{e_4}e_b, e_c\right)\\ &+\left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right)\mathbf{g}\left(\mathbf{D}_{e_3}e_b, e_c\right) - \frac{1}{2}\underline{f}_c\chi'_{ab} + \frac{1}{2}\underline{f}_b\chi'_{ac} + \frac{1}{2}\zeta'_a\underline{f}_c f_b\\ &-\frac{1}{4}\underline{f}_c\chi'_{ad}\underline{f}_b f_d - \frac{1}{2}f_c\underline{\chi}_{ab} + \frac{1}{2}f_b\underline{\chi}_{ac} + \mathrm{Err}[\mathbf{g}(\mathbf{D}_{e'_a}e'_b, e'_c)]\Big\}V_c\end{aligned}$$

and hence

$$
\begin{aligned}
\nabla'_a V_b &= \left(\delta_a^b + \frac{1}{2}\underline{f}_a f^d\right)\nabla_d V_b + \frac{1}{2}\underline{f}_a\nabla_4 V_b + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right)\nabla_3 V_b\\
&\quad -\Bigg\{-\frac{1}{2}\underline{f}_c\chi'_{ab} + \frac{1}{2}\underline{f}_b\chi'_{ac} + \frac{1}{2}\zeta'_a\underline{f}_c f_b - \frac{1}{4}\underline{f}_c\chi'_{ad}\underline{f}_b f_d\\
&\quad -\frac{1}{2}f_c\underline{\chi}_{ab} + \frac{1}{2}f_b\underline{\chi}_{ac} + \mathrm{Err}[\mathbf{g}(\mathbf{D}_{e'_a}e'_b, e'_c)]\Bigg\}V_c
\end{aligned}
$$

as stated. The formulas for ${}^*\nabla'_a V_B$ and $\mathcal{D}'_a V_B$ are proved in the same manner. □

Corollary 6.30. *Let f and $\underline{f}$ be the horizontal 1-forms given by (6.32), and let the scalar function λ be given by (2.12). Then, recalling the notation of Definition 6.3, we have*

$$
\nabla' = \Big(1 + O(r^{-2})\Big)\nabla + O(r^{-1})\not{\mathcal{L}}_{\mathbf{T}} + O(r^{-3}) + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1}.
$$

Also, we have

$$
\nabla' = (1 + O(r^{-2}))\nabla + O(r^{-1})\nabla_4 + O(r^{-1})\nabla_3 + O(r^{-3}) + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1}.
$$

Proof. Note that the second identity is an immediate consequence of the first in view of (6.31). Thus, we focus on proving the first one. Recall from Proposition 6.29 that we have for a horizontal tensor V and indices $B = b_1\cdots b_k$

$$
\begin{aligned}
\nabla'_a V_B &= \nabla_a V_B + \frac{1}{2}\underline{f}_a f^c\nabla_c V_B + \frac{1}{2}\underline{f}_a\nabla_4 V_B + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right)\nabla_3 V_B\\
&\quad -\sum_{j=1}^{k}\Bigg\{\frac{1}{2}\underline{f}_c f_{b_j}\nabla'_a(\log\lambda) - \frac{1}{2}\underline{f}_c\lambda^{-1}\chi'_{ab_j} + \frac{1}{2}\underline{f}_{b_j}\lambda^{-1}\chi'_{ac} + \frac{1}{2}\zeta'_a\underline{f}_c f_{b_j}\\
&\quad -\frac{1}{4}\underline{f}_c\lambda^{-1}\chi'_{ad}\underline{f}_{b_j}f_d - \frac{1}{2}f_c\underline{\chi}_{ab_j} + \frac{1}{2}f_{b_j}\underline{\chi}_{ac} + \mathrm{Err}[\mathbf{g}(\mathbf{D}_{e'_a}e'_{b_j}, e'_c)]\Bigg\}V_{B^c_{(j)}}.
\end{aligned}
$$

Thus, using in particular $f = O(r^{-1})$ and $\underline{f} = O(r^{-1})$ and $\lambda = 1 + O(r^{-2})$, see Lemma 6.27,

$$
\nabla' = \Big(1 + O(r^{-2})\Big)\nabla + \frac{1}{2}\underline{f}\nabla_4 + \left(\frac{1}{2}f + \frac{1}{8}|f|^2\underline{f}\right)\nabla_3 + E + O(r^{-3}),
$$

where E is given by

$$(EV)_{aB} = -\sum_{j=1}^k \left\{ -\frac{1}{2}\underline{f}_c\lambda^{-1}\chi'_{ab_j} + \frac{1}{2}\underline{f}_{b_j}\lambda^{-1}\chi'_{ac} - \frac{1}{2}f_c\underline{\chi}_{ab_j} + \frac{1}{2}f_{b_j}\underline{\chi}_{ac}\right\} V_{B^c_{(j)}}.$$

We use $f = O(r^{-1})$ and $\underline{f} = O(r^{-1})$ and the transformation formulas for the Ricci coefficients of Proposition 2.12 for χ' to deduce

$$\lambda^{-1}\chi'_{ab} \quad = \quad \chi_{ab} + O(r^{-2}) + \Gamma_g$$

and hence, using also $f = \underline{f} + O(r^{-2}) + \Gamma_b$ in view of Lemma 6.27, we infer

$$\begin{aligned}(EV)_{aB} \quad &= \quad -\frac{1}{2}\sum_{j=1}^k \Big\{ -f_c(\chi_{ab_j} + \underline{\chi}_{ab_j}) + f_{b_j}(\chi_{ac} + \underline{\chi}_{ac}) + O(r^{-3}) \\ &\qquad + r^{-1}\Gamma_b\Big\} V_{B(j)}.\end{aligned}$$

Since $\chi_{ab} + \underline{\chi}_{ab} = O(r^{-2}) + \Gamma_b$,

$$(EV)_{aB} = -\frac{1}{2}\sum_{j=1}^k \Big\{O(r^{-3}) + r^{-1}\Gamma_b\Big\} V_{B(j)} \tag{6.33}$$

and hence

$$\nabla' = \Big(1 + O(r^{-2})\Big)\nabla + \frac{1}{2}\underline{f}\nabla_4 + \left(\frac{1}{2}f + \frac{1}{8}|f|^2\underline{f}\right)\nabla_3 + E + O(r^{-3}) + r^{-1}\Gamma_b.$$

Recalling the decomposition (6.31), i.e.

$$\nabla_3 \quad = \quad 2\not{\mathcal{L}}_{\mathbf{T}} - \frac{\Delta}{|q|^2}\nabla_4 + O(r^{-1})\nabla + O(r^{-3}) + \Gamma_b,$$

we infer, using also $f = O(r^{-1})$ and $\underline{f} = O(r^{-1})$ in view of Lemma 6.27,

$$\begin{aligned}\nabla' \quad &= \quad \Big(1 + O(r^{-2})\Big)\nabla + O(r^{-1})\not{\mathcal{L}}_{\mathbf{T}} + \frac{1}{2}\left(\underline{f} - \frac{\Delta}{|q|^2}\left(f + \frac{1}{4}|f|^2\underline{f}\right)\right)\nabla_4 \\ &\quad + O(r^{-3}) + r^{-1}\Gamma_b.\end{aligned}$$

Recalling also, from Lemma 6.27,

$$\underline{f} - \frac{\Delta}{|q|^2}\left(f + \frac{1}{4}|f|^2\underline{f}\right) \quad = \quad \Gamma_b,$$

we deduce

$$\nabla' = \Big(1+O(r^{-2})\Big)\nabla + O(r^{-1})\not{\mathcal{L}}_{\mathbf{T}} + O(r^{-3}) + r^{-1}\Gamma_b \mathfrak{d}^{\leq 1}$$

as stated. □

6.2.3. Derivatives in e_4 of integrals on 2-spheres $S(u,r)$

Lemma 6.31. *We have for a scalar function h*

$$e_4\left(\int_{S(u,r)} h\right) = \int_{S(u,r)} \Big(e_4(h) + \delta^{ab}\mathbf{g}(\mathbf{D}_{e'_a}e_4, e'_b)h\Big)$$

where

$$\begin{aligned}\delta^{ab}\mathbf{g}(\mathbf{D}_{e'_a}e_4, e'_b) &= \left(1 + f\cdot\underline{f} + \frac{1}{2}|\underline{f}|^2|f|^2\right) tr\chi + 2f\cdot\widehat{\chi}\cdot\underline{f} + |\underline{f}|^2 f\cdot\widehat{\chi}\cdot f \\ &\quad + f\cdot\eta + \frac{1}{2}(f\cdot\underline{f})(f\cdot\eta) + \frac{1}{4}|f|^2(\underline{f}\cdot\eta) + \frac{1}{8}|f|^2|\underline{f}|^2 f\cdot\eta \\ &\quad + f\cdot\zeta + \frac{1}{2}(f\cdot\underline{f})(f\cdot\zeta) + \frac{1}{4}|f|^2(\underline{f}\cdot\zeta) + \frac{1}{8}(f\cdot\zeta)|f|^2|\underline{f}|^2 \\ &\quad - \underline{\omega}\left(|f|^2 + \frac{1}{2}|f|^2(f\cdot\underline{f}) + \frac{1}{16}|f|^4|\underline{f}|^2\right).\end{aligned}$$

Proof. Recall that $e_4(r)=1$ and $e_4(u)=0$. Consider coordinates (x^1,x^2) on $S(u,r_0)$ and transport it by $e_4(x^1)=e_4(x^2)=0$. Then, $e_4=\partial_r$ and hence, for a scalar function h, we have

$$\begin{aligned}e_4\left(\int_{S(u,r)} h\right) &= \partial_r\left(\int h\sqrt{|g|}dx^1dx^2\right) \\ &= \int\left(\partial_r(h) + \frac{1}{\sqrt{|g|}}\partial_r(\sqrt{|g|})h\right)dx^1dx^2.\end{aligned}$$

Now, we have

$$\begin{aligned}\partial_r(g_{ab}) &= \partial_r\left(g\left(\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^b}\right)\right) = \mathbf{g}\left(\mathbf{D}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^b}\right) + \mathbf{g}\left(\frac{\partial}{\partial x^a},\mathbf{D}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial x^b}\right) \\ &= \mathbf{g}\left(\mathbf{D}_{\frac{\partial}{\partial x^a}}\frac{\partial}{\partial r},\frac{\partial}{\partial x^b}\right) + \mathbf{g}\left(\frac{\partial}{\partial x^a},\mathbf{D}_{\frac{\partial}{\partial x^b}}\frac{\partial}{\partial r}\right) \\ &= \mathbf{g}\left(\mathbf{D}_{\frac{\partial}{\partial x^a}}e_4,\frac{\partial}{\partial x^b}\right) + \mathbf{g}\left(\frac{\partial}{\partial x^a},\mathbf{D}_{\frac{\partial}{\partial x^b}}e_4\right).\end{aligned}$$

Since

$$\frac{1}{\sqrt{|g|}}\partial_r(\sqrt{|g|}) = \frac{1}{2}g^{ab}\partial_r g_{ab},$$

we infer

$$\frac{1}{\sqrt{|g|}}\partial_r(\sqrt{|g|}) = g^{ab}\mathbf{g}\left(\mathbf{D}_{\frac{\partial}{\partial x^a}}e_4, \frac{\partial}{\partial x^b}\right)$$

and hence, for an orthonormal basis (e'_1, e'_2) of $S(u,r)$, we have

$$\frac{1}{\sqrt{|g|}}\partial_r(\sqrt{|g|}) = \delta^{ab}\mathbf{g}(\mathbf{D}_{e'_a}e_4, e'_b).$$

Thus, we deduce

$$e_4\left(\int_{S(u,r)} h\right) = \int_{S(u,r)}\Big(e_4(h) + \delta^{ab}\mathbf{g}(\mathbf{D}_{e'_a}e_4, e'_b)h\Big).$$

Next, for simplicity, we write

$$e'_a = j_a^b e_b + k_a e_4 + l_a e_3$$

where

$$j_a^b = \delta_a^b + \frac{1}{2}\underline{f}_a f^b, \qquad k_a = \frac{1}{2}\underline{f}_a, \qquad l_a = \frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a. \tag{6.34}$$

We compute

$$\begin{aligned}
\mathbf{g}(\mathbf{D}_{e'_a}e_4, e'_b) &= \mathbf{g}\left(\mathbf{D}_{j_a^c e_c + k_a e_4 + l_a e_3}e_4, e'_b\right) = \mathbf{g}\left(\mathbf{D}_{j_a^c e_c + l_a e_3}e_4, e'_b\right)\\
&= \mathbf{g}\left(\mathbf{D}_{j_a^c e_c + l_a e_3}e_4, j_b^d e_d + k_b e_4 + l_b e_3\right)\\
&= \mathbf{g}\left(\mathbf{D}_{j_a^c e_c + l_a e_3}e_4, j_b^d e_d + l_b e_3\right)\\
&= j_a^c j_b^d \mathbf{g}\left(\mathbf{D}_{e_c}e_4, e_d\right) + l_a j_b^d \mathbf{g}\left(\mathbf{D}_{e_3}e_4, e_d\right) + j_a^c l_b \mathbf{g}\left(\mathbf{D}_{e_c}e_4, e_3\right)\\
&\quad + l_a l_b \mathbf{g}\left(\mathbf{D}_{e_3}e_4, e_3\right)\\
&= j_a^c j_b^d \chi_{cd} + 2l_a j_b^d \eta_d + 2j_a^c l_b \zeta_c - 4\underline{\omega} l_a l_b.
\end{aligned}$$

In view of (6.34) we infer

$$\mathbf{g}(\mathbf{D}_{e'_a}e_4, e'_b) = \left(\delta_a^c + \frac{1}{2}\underline{f}_a f^c\right)\left(\delta_b^d + \frac{1}{2}\underline{f}_b f^d\right)\chi_{cd}$$

$$
\begin{aligned}
&+2\left(\frac{1}{2}f_a+\frac{1}{8}|f|^2\underline{f}_a\right)\left(\delta_b^d+\frac{1}{2}\underline{f}_bf^d\right)\eta_d\\
&+2\left(\delta_a^c+\frac{1}{2}\underline{f}_af^c\right)\left(\frac{1}{2}f_b+\frac{1}{8}|f|^2\underline{f}_b\right)\zeta_c\\
&-4\underline{\omega}\left(\frac{1}{2}f_a+\frac{1}{8}|f|^2\underline{f}_a\right)\left(\frac{1}{2}f_b+\frac{1}{8}|f|^2\underline{f}_b\right).
\end{aligned}
$$

Taking the trace, this yields

$$
\begin{aligned}
\delta^{ab}\mathbf{g}(\mathbf{D}_{e_a'}e_4,e_b') \;=\;& \left(1+f\cdot\underline{f}+\frac{1}{2}|\underline{f}|^2|f|^2\right)\mathrm{tr}\,\chi+2f\cdot\widehat{\chi}\cdot\underline{f}+|\underline{f}|^2f\cdot\widehat{\chi}\cdot f\\
&+f\cdot\eta+\frac{1}{2}(f\cdot\underline{f})(f\cdot\eta)+\frac{1}{4}|f|^2(\underline{f}\cdot\eta)+\frac{1}{8}|f|^2|\underline{f}|^2f\cdot\eta\\
&+f\cdot\zeta+\frac{1}{2}(f\cdot\underline{f})(f\cdot\zeta)+\frac{1}{4}|f|^2(\underline{f}\cdot\zeta)+\frac{1}{8}(f\cdot\zeta)|f|^2|\underline{f}|^2\\
&-\underline{\omega}\left(|f|^2+\frac{1}{2}|f|^2(f\cdot\underline{f})+\frac{1}{16}|f|^4|\underline{f}|^2\right)
\end{aligned}
$$

which concludes the proof of the lemma. □

Corollary 6.32. *We have for a scalar function h*

$$
e_4\left(\int_{S(u,r)}h\right) \;=\; \int_{S(u,r)}\frac{e_4(\Sigma h)}{\Sigma}+O\left(\frac{\epsilon}{u^{1+\delta_{dec}}}\right)h.
$$

Proof. Recall that we have

$$
\begin{aligned}
\delta^{ab}\mathbf{g}(\mathbf{D}_{e_a'}e_4,e_b') \;=\;& \left(1+f\cdot\underline{f}+\frac{1}{2}|\underline{f}|^2|f|^2\right)\mathrm{tr}\,\chi+2f\cdot\widehat{\chi}\cdot\underline{f}+|\underline{f}|^2f\cdot\widehat{\chi}\cdot f\\
&+f\cdot\eta+\frac{1}{2}(f\cdot\underline{f})(f\cdot\eta)+\frac{1}{4}|f|^2(\underline{f}\cdot\eta)+\frac{1}{8}|f|^2|\underline{f}|^2f\cdot\eta\\
&+f\cdot\zeta+\frac{1}{2}(f\cdot\underline{f})(f\cdot\zeta)+\frac{1}{4}|f|^2(\underline{f}\cdot\zeta)+\frac{1}{8}(f\cdot\zeta)|f|^2|\underline{f}|^2\\
&-\underline{\omega}\left(|f|^2+\frac{1}{2}|f|^2(f\cdot\underline{f})+\frac{1}{16}|f|^4|\underline{f}|^2\right).
\end{aligned}
$$

Hence in view of the form of f and $\underline{f}$ in (6.32), and in view of the control of the outgoing PG structure of ${}^{(ext)}\mathcal{M}$ provided by **Ref 1**, including the estimate (6.11) for $\widetilde{\mathrm{tr}X}$, we infer

$$
\delta^{ab}\mathbf{g}(\mathbf{D}_{e_a'}e_4,e_b') \;=\; \nu_0(r,\theta)+O\left(\frac{\epsilon}{r^2u^{1+\delta_{dec}}}\right),
$$

where the function $\nu_0(r,\theta)$ denotes the Kerr value. While it is in principle computable from the above formula, it is easier to compute it directly in Kerr. We have in Kerr in the (θ,φ) coordinates

$$\partial_r\left(\int_S h\right) = \partial_r\left(\int_0^{2\pi}\int_0^{\pi} h\Sigma\sin\theta d\theta d\varphi\right) = \int_S\left(\partial_r(h)+\frac{\partial_r\Sigma}{\Sigma}h\right)$$

and hence

$$\nu_0(r,\theta) \quad = \quad \frac{\partial_r\Sigma}{\Sigma}.$$

We deduce in general

$$\begin{aligned} e_4\left(\int_{S(u,r)} h\right) &= \int_{S(u,r)}\left(e_4(h)+\delta^{ab}\mathbf{g}(\mathbf{D}_{e'_a}e_4,e'_b)h\right)\\ &= \int_{S(u,r)}\left(e_4(h)+\nu_0(r,\theta)h\right)+O\left(\frac{\epsilon}{u^{1+\delta_{dec}}}\right)h\\ &= \int_{S(u,r)}\left(e_4(h)+\frac{\partial_r\Sigma}{\Sigma}h\right)+O\left(\frac{\epsilon}{u^{1+\delta_{dec}}}\right)h\\ &= \int_{S(u,r)}\frac{e_4(\Sigma h)}{\Sigma}+O\left(\frac{\epsilon}{u^{1+\delta_{dec}}}\right)h, \end{aligned}$$

where we have used in particular the fact that $\Sigma=\Sigma(r,\theta)$ and $e_4(r)=1$, $e_4(\theta)=0$, so that $\partial_r\Sigma=e_4(\Sigma)$. This concludes the proof of the corollary. □

6.2.4. Definition of $\ell=1$ modes on $S(u,r)$ Recall the definition of the basis of $\ell=1$ modes $J^{(p)}$, $p=0,+,-$ in ${}^{(ext)}\mathcal{M}$, see Section 2.6.1. Relative to the PG coordinates (θ,φ) of ${}^{(ext)}\mathcal{M}$, we have

$$J^{(0)}=\cos\theta, \qquad J^{(+)}=\sin\theta\cos\varphi, \qquad J^{(-)}=\sin\theta\sin\varphi.$$

The $\ell=1$ modes of a scalar function on $S(u,r)$ are defined as follows.

Definition 6.33. *Given a scalar function f on a sphere $S=S(u,r)$, we define the $\ell=1$ modes of f to be the triplet of numbers*

$$(f)_{\ell=1} \quad = \quad \left(\frac{1}{|S|}\int_S fJ^{(0)},\frac{1}{|S|}\int_S fJ^{(+)},\frac{1}{|S|}\int_S fJ^{(-)}\right).$$

Lemma 6.34. *We have on* ${}^{(ext)}\mathcal{M}$

$$\int_S J^{(p)} = O\left(1+\epsilon r u^{-\frac{1}{2}-\delta_{dec}}\right),$$
$$\int_S J^{(p)}J^{(q)} = \frac{4\pi}{3}r^2\delta_{pq} + O\left(1+\epsilon r u^{-\frac{1}{2}-\delta_{dec}}\right).$$

Proof. Let h be a scalar function such that $e_4(h)=0$. Then, we have in view of Corollary 6.32

$$\begin{aligned} e_4\left(r^{-2}\int_S h\right) &= \int_S \frac{e_4(r^{-2}\Sigma h)}{\Sigma} + O\left(\frac{\epsilon}{r^2u^{1+\delta_{dec}}}\right)h \\ &= \int_S \frac{e_4(r^{-2}\Sigma)}{\Sigma}h + O\left(\frac{\epsilon}{r^2u^{1+\delta_{dec}}}\right)h. \end{aligned}$$

Also, since $e_4(r)=1$ and $e_4(\theta)=0$, we have

$$e_4\left(\frac{\Sigma}{r^2}\right) = \partial_r\left(\frac{\Sigma}{r^2}\right) = \partial_r\sqrt{\left(1+\frac{a^2}{r^2}\right)^2 + \frac{a^2(\sin\theta)^2\Delta}{r^4}} = O(r^{-3})$$

and hence

$$e_4\left(r^{-2}\int_S h\right) = O\left(\frac{1}{r^3} + \frac{\epsilon}{r^2u^{1+\delta_{dec}}}\right)h.$$

Applying this identity with $h=J^{(p)}$ and $h=J^{(p)}J^{(q)}$, we infer

$$\begin{aligned} e_4\left(r^{-2}\int_S J^{(p)}\right) &= O\left(\frac{1}{r^3} + \frac{\epsilon}{r^2u^{1+\delta_{dec}}}\right), \\ e_4\left(r^{-2}\int_S J^{(p)}J^{(q)}\right) &= O\left(\frac{1}{r^3} + \frac{\epsilon}{r^2u^{1+\delta_{dec}}}\right). \end{aligned}$$

Integrating from Σ_*, and together with the control on Σ_* of Lemma 5.68, we infer

$$\begin{aligned} r^{-2}\int_S J^{(p)} &= O\left(\frac{1}{r^2} + \frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}}}\right), \\ r^{-2}\int_S J^{(p)}J^{(q)} &= \frac{4\pi}{3}\delta_{pq} + O\left(\frac{1}{r^2} + \frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}}}\right), \end{aligned}$$

as stated. □

Proposition 6.35. *Let ∇' denote the covariant derivative on $S(u,r)$. Then, we have on ${}^{(ext)}\mathcal{M}$, for $p = 0, +, -$,*

$$|\mathcal{D}'\widehat{\otimes}\mathcal{D}'J^{(p)}| + |r^2\Delta' J^{(p)} + 2| \lesssim \frac{\epsilon}{r^3 u^{\frac{1}{2}+\delta_{dec}}} + \frac{1}{r^4}.$$

Proof. See Section C.4. □

6.2.5. Elliptic estimates on $S(u,r)$ We denote by (e_1', e_2') the orthonormal frame of $S = S(u,r) \subset {}^{(ext)}\mathcal{M}$ defined in section 6.2.1. We first estimate the Gauss curvature of the spheres S.

Lemma 6.36. *Let K denote the Gauss curvature of the sphere $S = S(u,r) \subset {}^{(ext)}\mathcal{M}$. Then, K satisfies*

$$\sup_{{}^{(ext)}\mathcal{M}} r^2\left|K - \frac{1}{r^2}\right| \lesssim \frac{1}{r_0^2} + \frac{\epsilon}{r_0}. \tag{6.35}$$

Remark 6.37. *In view of the above control of K, $S(u,r)$ is, for r_0 large enough, an almost round sphere in the sense of Definition 5.1.*

Proof. By Gauss equation, we have

$$K = -\rho' - \frac{1}{4}\mathrm{tr}\,\chi'\mathrm{tr}\,\underline{\chi}' + \frac{1}{2}\widehat{\chi}'\cdot\widehat{\underline{\chi}}',$$

where $\mathrm{tr}\,\chi'$, $\mathrm{tr}\,\underline{\chi}'$, $\widehat{\chi}'$, $\widehat{\underline{\chi}}'$ and ρ' correspond to the null frame (e_4', e_3', e_1', e_2') adapted to $S(u,r)$, with (e_1', e_2') the orthonormal frame of S defined in section 6.2.1. The change of frame formulas of Proposition 2.12, the control of $(f, \underline{f})$ in (6.32), and the computation in the Kerr case in Lemma 2.52 imply[137]

$$\begin{aligned}
\lambda^{-1}\mathrm{tr}\,\chi' &= \frac{2}{r} + O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_g,\\
\lambda\mathrm{tr}\,\underline{\chi}' &= -\frac{2\left(1-\frac{2m}{r}\right)}{r} + O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_g,\\
\lambda^{-1}\widehat{\chi}' &= O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_g,\\
\lambda\widehat{\underline{\chi}}' &= O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_b,\\
\rho' &= -\frac{2m}{r^3} + O(r^{-5}) + r^{-1}\Gamma_g.
\end{aligned}$$

[137]The main terms in the RHS come from the corresponding calculations in Kerr carried out in Lemma 2.52. The additional error terms $\mathfrak{d}^{\leq 1}\Gamma_g$ and $\mathfrak{d}^{\leq 1}\Gamma_b$ are due to the perturbation.

Plugging in Gauss equation, this yields

$$K = \frac{1}{r^2} + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g.$$

Together with the control of Γ_g, and the fact that $r \geq r_0$ in ${}^{(ext)}\mathcal{M}$, this concludes the proof of the lemma. □

We denote by $\not{d}_1'$, $\not{d}_2'$, $\not{d}_1'^*$, $\not{d}_2'^*$ the standard Hodge operators on S, see Definition 5.16. Since the spheres S are almost round, see Remark 6.37, and in view of the properties of the basis of $\ell = 1$ modes $J^{(p)}$, see Lemma 6.34 and Proposition 6.35, the Hodge elliptic estimates of Lemma 5.27 and Lemma 5.28 apply. We recall these results in the proposition below.

Proposition 6.38. *For any sphere $S = S(u, r) \subset {}^{(ext)}\mathcal{M}$ we have for $k \leq k_{large}$:*

1. *If f is a 1-form*

$$\|(\not{\mathfrak{d}}')^{\leq k+1} f\|_{L^2(S)} \lesssim r\|(\not{\mathfrak{d}}')^{\leq k} \not{d}_1' f\|_{L^2(S)}.$$

2. *If f is a symmetric traceless 2-tensor*

$$\|(\not{\mathfrak{d}}')^{\leq k+1} f\|_{L^2(S)} \lesssim r\|(\not{\mathfrak{d}}')^{\leq k} \not{d}_2' f\|_{L^2(S)}.$$

3. *If $(h, {}^*h)$ is a pair of scalars*

$$\|(\not{\mathfrak{d}}')^{\leq k+1}(h - \overline{h}, {}^*h - \overline{{}^*h})\|_{L^2(S)} \lesssim r\|(\not{\mathfrak{d}}')^{\leq k} \not{d}_1'^*(h, {}^*h)\|_{L^2(S)}.$$

4. *If f is a 1-form*

$$\|(\not{\mathfrak{d}}')^{\leq k+1} f\|_{L^2(S)} \lesssim r\|(\not{\mathfrak{d}}')^{\leq k} \not{d}_2'^* f\|_{L^2(S)} + r^2|(\not{d}_1' f)_{\ell=1}|.$$

We have the following corollary of Proposition 6.38.

Corollary 6.39. *For any sphere $S = S(u, r) \subset {}^{(ext)}\mathcal{M}$ we have for $k \leq k_{large}$:*

1. *If U is an anti-selfdual 1-form*

$$\|(\not{\mathfrak{d}}')^{\leq k+1} U\|_{L^2(S)} \lesssim r\|(\not{\mathfrak{d}}')^{\leq k}(\mathcal{D}'\widehat{\otimes} U)\|_{L^2(S)} + r^2\left|\left(\overline{\mathcal{D}'} \cdot U\right)_{\ell=1}\right|.$$

2. *If U is an anti-selfdual 1-form*

$$\|(\not{\mathfrak{d}}')^{\leq k+1} U\|_{L^2(S)} \lesssim r^2\|(\not{\mathfrak{d}}')^{\leq k-1}\mathcal{D}'(\overline{\mathcal{D}'} \cdot U)\|_{L^2(S)}.$$

3. If U is an anti-selfdual symmetric traceless 2-tensor

$$\|(\not{\mathfrak{d}}')^{\leq k+1} U\|_{L^2(S)} \lesssim r\|(\not{\mathfrak{d}}')^{\leq k}(\overline{\mathcal{D}'}\cdot U)\|_{L^2(S)}.$$

Proof. We start with the first identity. Since U is an anti-selfdual 1-form, $f = \Re(U)$ is a real 1-form and

$$U = f + i\,{}^*f.$$

In particular, we have

$$\mathcal{D}'\widehat{\otimes} U = 2(\nabla'\widehat{\otimes} f) + 2i\,{}^*(\nabla'\widehat{\otimes} f), \qquad \overline{\mathcal{D}'}\cdot U = 2\mathrm{div}'(f) + 2i\mathrm{curl}'(f).$$

Thus, since

$$\not{d}_2'^* = -\frac{1}{2}\nabla'\widehat{\otimes}, \qquad \not{d}_1' = (\mathrm{div}', \mathrm{curl}'),$$

the first identity follows immediately from the last estimate of Proposition 6.38.

Then, we consider the second identity. Since U is an anti-selfdual 1-form, $f = \Re(U)$ is a real 1-form and

$$U = f + i\,{}^*f.$$

In particular, we have

$$\mathcal{D}'(\overline{\mathcal{D}'}\cdot U) = 2\Big(\nabla'\mathrm{div}'(f) - {}^*\nabla'\mathrm{curl}'(f)\Big) + 2i\,{}^*\Big(\nabla'\mathrm{div}'(f) - {}^*\nabla'\mathrm{curl}'(f)\Big),$$

and hence, we infer

$$\mathcal{D}'(\overline{\mathcal{D}'}\cdot U) = -2\Big(\not{d}_1'^*\not{d}_1'(f) + i\,{}^*(\not{d}_1'^*\not{d}_1'(f))\Big).$$

Thus, the second identity follows immediately from the first and the third estimate of Proposition 6.38.

Finally, we consider the third identity. Since U is an anti-selfdual symmetric traceless 2-tensor, $f = \Re(U)$ is a real symmetric traceless 2-tensor and

$$U = f + i\,{}^*f.$$

In particular, we have

$$\overline{\mathcal{D}'}\cdot U = -2\Big(\mathrm{div}'(f) + i\mathrm{div}'({}^*f)\Big).$$

Thus, the third identity follows immediately from the first estimate of Proposition 6.38. This concludes the proof of Corollary 6.39. □

6.3. Renormalized quantities for outgoing PG structures

6.3.1. Renormalization of $\widecheck{H}$, $\widecheck{\cos\theta}$ and $\overline{\mathcal{D}}\cdot\widecheck{Z}$ We introduce the following renormalized quantities:

(6.36)
$$\begin{aligned}
[\widecheck{H}]_{ren} &:= \frac{1}{\overline{q}}\left(\overline{q}\widecheck{H} - q\widecheck{Z} + \frac{1}{3}\left(-\overline{q}^2 + |q|^2\right)B + \frac{a}{2}(q-\overline{q})\overline{\mathfrak{J}}\cdot\widehat{X}\right),\\
[\widecheck{\mathcal{D}\cos\theta}]_{ren} &:= \frac{1}{q}\left(q\widecheck{\mathcal{D}\cos\theta} + \frac{i}{2}|q|^2\overline{\mathfrak{J}}\cdot\widehat{X}\right),\\
[\widecheck{M}]_{ren} &:= \frac{1}{\overline{q}q^2}\Bigg[\overline{q}\,\overline{\mathcal{D}}\cdot\left(q^2\widecheck{Z} + \left(-\frac{a}{2}q^2 - \frac{a}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\right) + 2\overline{q}^3\,\overline{\widecheck{P}}\\
&\quad - 2aq^2\overline{\mathfrak{J}}\cdot\widecheck{Z} + \left(-\frac{1}{3}q^2\overline{q}^2 - \frac{1}{3}q\overline{q}^3 + \frac{2}{3}\overline{q}^4\right)\overline{\mathcal{D}}\cdot B\\
&\quad + a\left(q^2\overline{q} + \frac{2}{3}q\overline{q}^2 - \frac{13}{6}\overline{q}^3\right)\overline{\mathfrak{J}}\cdot B + a^2(q^2+|q|^2)\overline{\mathfrak{J}}\cdot\widehat{X}\cdot\overline{\mathfrak{J}}\Bigg].
\end{aligned}$$

Remark 6.40. *Note that, in the particular case $a=0$, $[\widecheck{M}]_{ren}$ is given by*

$$\overline{\mathcal{D}}\cdot\widecheck{Z} + 2\overline{\widecheck{P}} = 2(div(\zeta) + \widecheck{\rho}) + 2i(curl(\zeta) - {}^*\rho),$$

where we have used the fact that $\widecheck{\zeta} = \zeta$ and $\widecheck{{}^*\rho} = {}^*\rho$ when $a=0$. In particular, the real part of $[\widecheck{M}]_{ren}$ coincides in that case, modulo a factor of -2, with the linearized mass aspect function $\widecheck{\mu} = -div\,\zeta - \widecheck{\rho}$. Thus, while there is no quantity denoted by M in our work, the abuse of notation $\widecheck{M}$ should be thought as a complexified version of the linearized mass aspect function, and $[\widecheck{M}]_{ren}$ as its corresponding renormalized version.

Proposition 6.41. *We have*

$$\begin{aligned}
\nabla_4\left(\overline{q}[\widecheck{H}]_{ren}\right) &= O(r^{-1})\widecheck{trX} + O(1)\,\mathfrak{d}^{\leq 1}A + r\Gamma_b\cdot\Gamma_g,\\
\nabla_4\left(q[\widecheck{\mathcal{D}\cos\theta}]_{ren}\right) &= O(1)\widecheck{trX} + O(r)A + r\Gamma_b\cdot\Gamma_g,\\
\nabla_4\left(\overline{q}q^2[\widecheck{M}]_{ren}\right) &= O(1)\,\mathfrak{d}^{\leq 1}\widecheck{trX} + O(r)\,\mathfrak{d}^{\leq 2}A + r^2\,\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g) + r^3\Gamma_b\cdot A.
\end{aligned}$$

Proof. See Section C.5. □

6.3.2. Renormalization of the $\ell = 1$ modes of $\overline{\mathcal{D}} \cdot B$

Definition 6.42. *We introduce the following renormalized quantities*

$$
\begin{aligned}
[B]_{ren} &:= B - \frac{3a}{2}\overline{\check{P}}\mathfrak{J} - \frac{a}{4}\overline{\mathfrak{J}} \cdot A, \\
[\overline{\mathcal{D}}\cdot]_{ren} &:= \overline{\mathcal{D}} \cdot - \frac{a}{2}\overline{\mathfrak{J}} \cdot \nabla_4 - \frac{a}{2}\overline{\mathfrak{J}} \cdot \nabla_3.
\end{aligned} \tag{6.37}
$$

The goal of the section is to prove the following proposition.

Proposition 6.43. *Let $[B]_{ren}$ and $[\overline{\mathcal{D}}\cdot]_{ren}$ be given by Definition 6.42. Then, the following identities hold true*

$$
\begin{aligned}
& \nabla_4 \left(\int_{S(u,r)} \frac{rJ^{(0)}}{\Sigma} [\overline{\mathcal{D}}\cdot]_{ren} \left(r^4 [B]_{ren} \right) \right) \\
= \; & O(1)\mathfrak{d}^{\leq 1}\widehat{X} + O(r)\mathfrak{d}^{\leq 2}B + O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 B + O(r)\mathfrak{d}^{\leq 2}\check{P} \\
& + O(1)\mathfrak{d}^{\leq 1}\widecheck{trX} + O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 A + O(r)\mathfrak{d}^{\leq 2}A + r^4\mathfrak{d}^{\leq 1}(\Gamma_g \cdot (B, A)) \\
& + r^4\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \nabla_3 A) + r^2\mathfrak{d}^{\leq 2}(\Gamma_g \cdot \Gamma_g) + O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\left(A, B, r^{-1}\check{P}\right),
\end{aligned} \tag{6.38}
$$

and

$$
\begin{aligned}
& \nabla_4 \left(\int_{S(u,r)} \frac{rJ^{(\pm)}}{\Sigma} [\overline{\mathcal{D}}\cdot]_{ren} \left(r^4 [B]_{ren} \right) \right) \\
& \mp \frac{a}{r^2} \int_{S(u,r)} \frac{rJ^{(\mp)}}{\Sigma} [\overline{\mathcal{D}}\cdot]_{ren} \left(r^4 [B]_{ren} \right) \\
= \; & O(1)\mathfrak{d}^{\leq 1}\widehat{X} + O(r)\mathfrak{d}^{\leq 2}B + O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 B + O(r)\mathfrak{d}^{\leq 2}\check{P} \\
& + O(1)\mathfrak{d}^{\leq 1}\widecheck{trX} + O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 A + O(r)\mathfrak{d}^{\leq 2}A + r^4\mathfrak{d}^{\leq 1}(\Gamma_g \cdot (B, A)) \\
& + r^4\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \nabla_3 A) + r^2\mathfrak{d}^{\leq 2}(\Gamma_g \cdot \Gamma_g) + O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\left(A, B, r^{-1}\check{P}\right).
\end{aligned} \tag{6.39}
$$

The proof of Proposition 6.43 is based on the following identity.

Lemma 6.44. *The following identity holds true*

$$
\left(\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b\right)\left(r[\overline{\mathcal{D}}\cdot]_{ren}(r^4[B]_{ren})\right) = \frac{r^5}{2}\overline{\mathcal{D}}' \cdot \overline{\mathcal{D}}' \cdot \left(A - a(\mathfrak{J}\widehat{\otimes}B)\right) + Err,
$$

where the $\mathcal{D}'$ is taken with respect to the integral frame (e_1', e_2') adapted to

$S(u,r)$, see Section 6.2.1, and where the error term is given by

$$\begin{aligned}
Err &= O(1)\mathfrak{d}^{\leq 1}\widehat{X} + O(r)\mathfrak{d}^{\leq 2}B + O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 B + O(r)\mathfrak{d}^{\leq 2}\check{P}\\
&\quad +O(1)\mathfrak{d}^{\leq 1}\widetilde{trX} + O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 A + O(r)\mathfrak{d}^{\leq 2}A + r^4\mathfrak{d}^{\leq 1}(\Gamma_g\cdot(B,A))\\
&\quad +r^4\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\nabla_3 A) + r^2\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_g).
\end{aligned}$$

Proof. See Section C.6. □

We will also use the following lemma.

Lemma 6.45. *We have*

$$\begin{aligned}
\Re(\mathfrak{J})^b\nabla_b(J^{(0)}) &= r^{-1}\Gamma_b,\\
\Re(\mathfrak{J})^b\nabla_b(J^{(+)}) &= -\frac{1}{r^2}J^{(-)} + O(r^{-4}) + r^{-1}\Gamma_b,\\
\Re(\mathfrak{J})^b\nabla_b(J^{(-)}) &= \frac{1}{r^2}J^{(+)} + O(r^{-4}) + r^{-1}\Gamma_b.
\end{aligned}$$

Proof. Since $\Im(\mathfrak{J}) = {}^*\Re(\mathfrak{J})$ and $J^{(0)} = \cos\theta$, we have

$$\begin{aligned}
\Re(\mathfrak{J})^b\nabla_b(J^{(0)}) &= \Re(\mathfrak{J})\cdot\Re(\mathcal{D}\cos\theta) = \Re(\mathfrak{J})\cdot\Re(i\mathfrak{J} + \widetilde{\mathcal{D}\cos\theta})\\
&= \Re(\mathfrak{J})\cdot\Im(\mathfrak{J}) + r^{-1}\Gamma_b\\
&= \Re(\mathfrak{J})\cdot {}^*\Re(\mathfrak{J}) + r^{-1}\Gamma_b = r^{-1}\Gamma_b.
\end{aligned}$$

Also, from Definition 2.66 and formula (2.70), we have

$$\mathcal{D}(J^{(\pm)}) = \mathfrak{J}_\pm + \widetilde{\mathcal{D}J^{(\pm)}}, \qquad \widetilde{\mathcal{D}J^{(\pm)}} \in \Gamma_b,$$

where the complex 1-forms $\mathfrak{J}_+$ and $\mathfrak{J}_-$ satisfy, see (6.5),

$${}^*\mathfrak{J}_\pm = -i\mathfrak{J}_\pm, \qquad \Re(\mathfrak{J}_+)\cdot\Re(\mathfrak{J}) = -\frac{1}{|q|^2}J^{(-)}, \qquad \Re(\mathfrak{J}_-)\cdot\Re(\mathfrak{J}) = \frac{1}{|q|^2}J^{(+)}.$$

We infer

$$\begin{aligned}
\Re(\mathfrak{J})^b\nabla_b(J^{(\pm)}) &= \Re(\mathfrak{J})\cdot\Re(\mathcal{D}(J^{(\pm)})) = \Re(\mathfrak{J})\cdot\Re(\mathfrak{J}_\pm + \widetilde{\mathcal{D}J^{(\pm)}})\\
&= \Re(\mathfrak{J})\cdot\Re(\mathfrak{J}_\pm) + r^{-1}\Gamma_b,
\end{aligned}$$

and hence

$$\Re(\mathfrak{J})^b\nabla_b(J^{(+)}) = -\frac{1}{|q|^2}J^{(-)} + r^{-1}\Gamma_b = -\frac{1}{r^2}J^{(-)} + O(r^{-4}) + r^{-1}\Gamma_b,$$

$$\Re(\mathfrak{J})^b\nabla_b(J^{(-)}) = \frac{1}{|q|^2}J^{(+)} + r^{-1}\Gamma_b = \frac{1}{r^2}J^{(+)} + O(r^{-4}) + r^{-1}\Gamma_b.$$

This concludes the proof of Lemma 6.45. □

We are now ready to prove Proposition 6.43.

Proof of Proposition 6.43. According to Corollary 6.32 we have for a scalar function h

$$e_4\left(\int_{S(u,r)} h\right) = \int_{S(u,r)} \frac{e_4(\Sigma h)}{\Sigma} + O\left(\frac{\epsilon}{u^{1+\delta_{dec}}}\right) h.$$

We apply this identity with the choice $h = \frac{J^{(p)}}{\Sigma} r[\overline{\mathcal{D}}\cdot]_{ren}(r^4[B]_{ren})$. We obtain, using also $e_4(J^{(p)}) = 0$, $J^{(p)} = O(1)$, and $\Sigma \geq r^2$, and recalling the definition of $[\overline{\mathcal{D}}\cdot]_{ren}$, $[B]_{ren}$, see Definition 6.42, for $p = 0, +, -$,

$$\begin{aligned}
& e_4\left(\int_{S(u,r)} \frac{J^{(p)}}{\Sigma}\Big(r[\overline{\mathcal{D}}\cdot]_{ren}\big(r^4[B]_{ren}\big)\Big)\right) \\
= & \int_{S(u,r)} \frac{J^{(p)}}{\Sigma}\nabla_4\Big(r[\overline{\mathcal{D}}\cdot]_{ren}\big(r^4[B]_{ren}\big)\Big) \\
& +O\left(\frac{\epsilon}{u^{1+\delta_{dec}}}\right)\frac{J^{(p)}}{\Sigma} r[\overline{\mathcal{D}}\cdot]_{ren}\big(r^4[B]_{ren}\big) \\
= & \int_{S(u,r)} \frac{J^{(p)}}{\Sigma}\nabla_4\Big(r[\overline{\mathcal{D}}\cdot]_{ren}\big(r^4[B]_{ren}\big)\Big) \\
& +O\left(\frac{r^2\epsilon}{u^{1+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\Big(B, r^{-1}\check{P}, r^{-1}A\Big) \\
= & \int_{S(u,r)} \frac{J^{(p)}}{\Sigma}\Big(\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b\Big)\Big(r[\overline{\mathcal{D}}\cdot]_{ren}\big(r^4[B]_{ren}\big)\Big) \\
& +\int_{S(u,r)} \frac{J^{(p)}}{\Sigma} a\Re(\mathfrak{J})^b\nabla_b\Big(r[\overline{\mathcal{D}}\cdot]_{ren}\big(r^4[B]_{ren}\big)\Big) \\
& +O\left(\frac{r^2\epsilon}{u^{1+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\Big(B, r^{-1}\check{P}, r^{-1}A\Big).
\end{aligned}$$

Making use of Lemma 6.44, we infer, for $p = 0, +, -$,

$$e_4\left(\int_{S(u,r)} \frac{J^{(p)}}{\Sigma}\Big(r[\overline{\mathcal{D}}\cdot]_{ren}\big(r^4[B]_{ren}\big)\Big)\right)$$

$$
\begin{aligned}
&= \int_{S(u,r)} \frac{J^{(p)}}{\Sigma}\frac{r^5}{2}\overline{\mathcal{D}}'\cdot\overline{\mathcal{D}}'\cdot\Big(A - a(\mathfrak{J}\widehat{\otimes}B)\Big)\\
&\quad +a\int_{S(u,r)}\frac{J^{(p)}}{\Sigma}\Re(\mathfrak{J})^b\nabla_b\Big(r[\overline{\mathcal{D}}\cdot]_{ren}(r^4[B]_{ren})\Big)\\
&\quad +\int_{S(u,r)}\frac{J^{(p)}}{\Sigma}\mathrm{Err} + O\left(\frac{r^2\epsilon}{u^{1+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\Big(B, r^{-1}\check{P}, r^{-1}A\Big).
\end{aligned}
$$

Since $\Sigma\geq r^2$ and $J^{(p)}=O(1)$, we obtain, with error terms[138] Err of the same form as the ones in Lemma 6.44,

$$
\begin{aligned}
& e_4\left(\int_{S(u,r)}\frac{J^{(p)}}{\Sigma}\Big(r[\overline{\mathcal{D}}\cdot]_{ren}(r^4[B]_{ren})\Big)\right)\\
&= \int_{S(u,r)} J^{(p)}\frac{r^3}{2}\overline{\mathcal{D}}'\cdot\overline{\mathcal{D}}'\cdot\Big(A - a(\mathfrak{J}\widehat{\otimes}B)\Big)\\
&\quad +a\int_{S(u,r)}\frac{J^{(p)}}{\Sigma}\Re(\mathfrak{J})^b\nabla_b\Big(r[\overline{\mathcal{D}}\cdot]_{ren}(r^4[B]_{ren})\Big)\\
&\quad +O\left(\frac{r^2\epsilon}{u^{1+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\Big(B, r^{-1}\check{P}, r^{-1}A\Big) + \mathrm{Err}.
\end{aligned}
$$

Integrating by parts twice, we have

$$
\begin{aligned}
&\int_{S(u,r)} J^{(p)}\frac{r^3}{2}\overline{\mathcal{D}}'\cdot\overline{\mathcal{D}}'\cdot\Big(A - a(\mathfrak{J}\widehat{\otimes}B)\Big)\\
&= \frac{1}{2}\int_{S(u,r)}\frac{r^3}{2}\Big(A - a(\mathfrak{J}\widehat{\otimes}B)\Big)\overline{\mathcal{D}}'\widehat{\otimes}\overline{\mathcal{D}}' J^{(p)}\\
&= O(r^5)\Big(A - a(\mathfrak{J}\widehat{\otimes}B)\Big)\overline{\mathcal{D}'\widehat{\otimes}\mathcal{D}' J^{(p)}},
\end{aligned}
$$

where we used the fact that $J^{(p)}$ is real valued. Also, according to Proposition 6.35, we have, for $p=0,+,-$,

$$
|\mathcal{D}'\widehat{\otimes}\mathcal{D}' J^{(p)}| \lesssim \frac{\epsilon}{r^3 u^{\frac{1}{2}+\delta_{dec}}} + \frac{1}{r^4}.
$$

We infer

$$
\int_{S(u,r)} J^{(p)}\frac{r^3}{2}\overline{\mathcal{D}}'\cdot\overline{\mathcal{D}}'\cdot\Big(A - a(\mathfrak{J}\widehat{\otimes}B)\Big)
$$

[138]Here and below, by abuse of notations, we do not distinguish between Err and |Err|.

$$= \left(O\left(\frac{\epsilon r^2}{u^{\frac{1}{2}+\delta_{dec}}}\right) + O(r)\right)\Big(A - a(\mathfrak{J}\widehat{\otimes}B)\Big).$$

Therefore, with error term Err of the same form as the ones in Lemma 6.44,

$$\begin{aligned}
& e_4\left(\int_{S(u,r)} \frac{J^{(p)}}{\Sigma}\Big(r[\overline{\mathcal{D}}\cdot]_{ren}(r^4[B]_{ren})\Big)\right) \\
= \;& a\int_{S(u,r)} \frac{J^{(p)}}{\Sigma}\Re(\mathfrak{J})^b\nabla_b\Big(r[\overline{\mathcal{D}}\cdot]_{ren}(r^4[B]_{ren})\Big) \\
& + O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\Big(A, B, r^{-1}\check{P}\Big) + \text{Err}.
\end{aligned}$$

Next, in view of Corollary 6.30, we have the rough decomposition

$$\nabla' \;=\; (1+O(r^{-2}))\nabla + O(r^{-1})\nabla_4 + O(r^{-1})\nabla_3 + O(r^{-3}) + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1}$$

or

$$\nabla \;=\; \nabla' + O(r^{-1})\nabla_3 + O(r^{-2})\mathfrak{d}^{\leq 1} + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1}.$$

Hence, with error terms Err of the same form as the ones in Lemma 6.44,

$$\begin{aligned}
& e_4\left(\int_{S(u,r)} \frac{J^{(p)}}{\Sigma}\Big(r[\overline{\mathcal{D}}\cdot]_{ren}(r^4[B]_{ren})\Big)\right) \\
= \;& a\int_{S(u,r)} \frac{J^{(p)}}{\Sigma}\Re(\mathfrak{J})^b\nabla'_b\Big(r[\overline{\mathcal{D}}\cdot]_{ren}(r^4[B]_{ren})\Big) + \text{Err} \\
& + O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\Big(A, B, r^{-1}\check{P}\Big).
\end{aligned}$$

Integrating by parts

$$\begin{aligned}
& \int_{S(u,r)} \frac{J^{(p)}}{\Sigma}\Re(\mathfrak{J})^b\nabla'_b\Big(r[\overline{\mathcal{D}}\cdot]_{ren}(r^4[B]_{ren})\Big) \\
= \;& -\int_{S(u,r)} \text{div}'\left(\frac{J^{(p)}}{\Sigma}\Re(\mathfrak{J})\right) r[\overline{\mathcal{D}}\cdot]_{ren}(r^4[B]_{ren}).
\end{aligned}$$

Using again the decomposition

$$\nabla' \;=\; \nabla + O(r^{-1})\nabla_3 + O(r^{-2})\mathfrak{d}^{\leq 1} + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1},$$

recalling that $\nabla_3\mathfrak{J} = \widetilde{\nabla_3\mathfrak{J}} + \frac{\Delta q}{|q|^4}\mathfrak{J} = r^{-1}\Gamma_b + O(r^{-1})\mathfrak{J}$, as well as the fact that $\widetilde{e_3(J^{(+)})}$, $\widetilde{e_3(J^{(-)})} \in \Gamma_b$, we deduce,

$$\begin{aligned}
& e_4\left(\int_{S(u,r)}\frac{J^{(p)}}{\Sigma}\Big(r[\overline{\mathcal{D}}\cdot]_{ren}\big(r^4[B]_{ren}\big)\Big)\right)\\
= & -a\int_{S(u,r)}\mathrm{div}\left(\frac{J^{(p)}}{\Sigma}\Re(\mathfrak{J})\right)\Big(r[\overline{\mathcal{D}}\cdot]_{ren}\big(r^4[B]_{ren}\big)\Big) + \mathrm{Err}\\
& + O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\Big(A, B, r^{-1}\check{P}\Big).
\end{aligned}$$

Since $\mathrm{div}\,(\Re(\mathfrak{J})) = r^{-1}\Gamma_b$, we infer, with error term Err of the same form as the ones in Lemma 6.44,

$$\begin{aligned}
& e_4\left(\int_{S(u,r)}\frac{J^{(p)}}{\Sigma}\Big(r[\overline{\mathcal{D}}\cdot]_{ren}\big(r^4[B]_{ren}\big)\Big)\right)\\
= & -a\int_{S(u,r)}\frac{1}{r^2}\Re(\mathfrak{J})^b\nabla_b(J^{(p)})\Big(r[\overline{\mathcal{D}}\cdot]_{ren}\big(r^4[B]_{ren}\big)\Big) + \mathrm{Err}\\
& + O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\Big(A, B, r^{-1}\check{P}\Big).
\end{aligned}$$

Now, recall that we have according to Lemma 6.45

$$\begin{aligned}
\Re(\mathfrak{J})^b\nabla_b(J^{(0)}) &= r^{-1}\Gamma_b,\\
\Re(\mathfrak{J})^b\nabla_b(J^{(+)}) &= -\frac{1}{r^2}J^{(-)} + O(r^{-4}) + r^{-1}\Gamma_b,\\
\Re(\mathfrak{J})^b\nabla_b(J^{(-)}) &= \frac{1}{r^2}J^{(+)} + O(r^{-4}) + r^{-1}\Gamma_b.
\end{aligned}$$

Since $\Sigma = r^2 + O(1)$, we obtain, with error term Err of the same form as the ones in Lemma 6.44,

$$\begin{aligned}
& e_4\left(\int_{S(u,r)}\frac{J^{(0)}}{\Sigma}\Big(r[\overline{\mathcal{D}}\cdot]_{ren}\big(r^4[B]_{ren}\big)\Big)\right)\\
= & \ \mathrm{Err} + O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\Big(A, B, r^{-1}\check{P}\Big)
\end{aligned}$$

and

$$
\begin{aligned}
& e_4\left(\int_{S(u,r)} \frac{J^{(\pm)}}{\Sigma}\Big(r[\overline{\mathcal{D}}\cdot]_{ren}\big(r^4[B]_{ren}\big)\Big)\right)\\
= \quad & \pm\frac{a}{r^2}\int_{S(u,r)} \frac{rJ^{(\mp)}}{\Sigma}[\overline{\mathcal{D}}\cdot]_{ren}\big(r^4[B]_{ren}\big)\\
& +\mathrm{Err} + O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\Big(A, B, r^{-1}\check{P}\Big),
\end{aligned}
$$

as desired. This concludes the proof of Proposition 6.43. □

6.4. Main Estimates in $^{(ext)}\mathcal{M}$

6.4.1. Transport lemmas The transport lemmas derived in this section will be used repeatedly in the proof of Theorem M4. Recall the norms of Definition 6.1, i.e.

$$
\|f\|_\infty(u,r) := \|f\|_{L^\infty\left(S(u,r)\right)}, \qquad \|f\|_{\infty,k}(u,r) := \sum_{i=0}^k \|\mathfrak{d}^i f\|_\infty(u,r).
$$

Recall also that the weighted derivatives $\not{\mathfrak{d}} = (r\nabla)$ and $\mathfrak{d} = (r\nabla, r\nabla_4, \nabla_3)$ are defined with respect to the outgoing PG frame of $^{(ext)}\mathcal{M}$.

Lemma 6.46. *Let U and F be anti-selfdual k-tensors. Assume that U verifies one of the following equations, for a real constant c,*

$$
\nabla_4 U + \frac{c}{q}U = F \tag{6.40}
$$

or

$$
\nabla_4 U + \Re\left(\frac{c}{q}\right)U = F. \tag{6.41}
$$

In both cases we derive, for any $r_0 \leq r \leq r_*$ at fixed u, with $1 \leq u \leq u_*$,

$$
r^c\|U\|_\infty(u,r) \lesssim r_*^c\|U\|_\infty(u,r_*) + \int_r^{r_*}\lambda^c\|F\|_\infty(u,\lambda)d\lambda. \tag{6.42}
$$

Proof. Assume first that U satisfies (6.40). Since $e_4(q) = 1$, we can rewrite the equation in the form

$$
\nabla_4(q^c U) = q^c F.
$$

The desired inequality follows then immediately by integration in r, using the fact that $e_4(r) = 1$, $e_4(u) = 0$ and $r \leq |q| \leq 2r$.

Next, assume that U satisfies (6.41). Since $e_4(q)=1$ and $e_4(\overline{q})=1$, we can rewrite the equation in the form

$$\begin{aligned}\nabla_4(|q|^cU) &= |q|^c\left(-\Re\Big(\frac{c}{q}\Big)U+F\right)+c|q|^{c-1}\nabla_4(|q|)U\\ &= |q|^c\left(-\Re\Big(\frac{c}{q}\Big)U+F\right)+c|q|^{c-1}\frac{1}{2|q|}(q+\overline{q})U\\ &= |q|^c\left(-\Re\Big(\frac{c}{q}\Big)U+F\right)+c|q|^{c}\frac{1}{2|q|^2}(q+\overline{q})U\\ &= |q|^c\left(-\Re\left(\frac{c}{q}\right)U+F\right)+c|q|^c\Re\Big(\frac{1}{q}\Big)U=|q|^cF,\end{aligned}$$

i.e.

$$\nabla_4(|q|^cU) = |q|^cF.$$

We can then proceed in the same manner as in the first case. □

Proposition 6.47. *Solutions U of the equations (6.40) or (6.41) verify the following estimate, for all $k\leq k_{large}$, $r_0\leq r\leq r_*$, and $1\leq u\leq u_*$,*

$$r^c\|U\|_{\infty,k}(u,r)\lesssim r_*^c\|U\|_{\infty,k}(u,r_*)+\int_r^{r_*}\lambda^c\|F\|_{\infty,k}(u,\lambda)d\lambda. \tag{6.43}$$

Proof. The proof being similar for (6.40) and (6.41), we only treat the case where U verifies (6.40), i.e.

$$\nabla_4U+\frac{c}{q}U=F.$$

Recall from Corollary 6.14 that we have

$$[\nabla_4, q\mathcal{D}]U = O(r^{-2})U+\Gamma_g\cdot \not{\mathfrak{d}}^{\leq 1}U.$$

Also, recall from Lemma 6.23 that we have

$$[\nabla_4, \not{\mathcal{L}}_{\mathbf{T}}]U = r^{-1}\Gamma_b\cdot\not{\mathfrak{d}}U+r^{-1}\mathfrak{d}\Gamma_b\cdot U.$$

We infer, using also $\mathbf{T}(q)\in r\Gamma_b$ in view of Lemma 6.19, and $\mathcal{D}(q)=O(r^{-1})+\Gamma_b$,

$$\nabla_4((q\mathcal{D})^j\not{\mathcal{L}}_{\mathbf{T}}^lU)+\frac{c}{q}(q\mathcal{D})^j\not{\mathcal{L}}_{\mathbf{T}}^lU = (q\mathcal{D})^j\not{\mathcal{L}}_{\mathbf{T}}^lF+O(r^{-2})(q\mathcal{D},\not{\mathcal{L}}_{\mathbf{T}})^{\leq j+l}U$$

$$+r^{-1}\sum_{p=0}^{j+l}\mathfrak{d}^{\leq p}(\Gamma_b)(q\mathcal{D},\not{\mathcal{L}}_{\mathbf{T}})^{j+l-p}U. \tag{6.44}$$

Applying Lemma 6.46 to (6.44), and using the control of Γ_b, we obtain, for $k\leq k_{large}$,

$$\begin{aligned}
&\sum_{j+l\leq k} r^c\|(q\mathcal{D})^j\not{\mathcal{L}}_{\mathbf{T}}^l U\|_\infty(u,r)\\
\lesssim\ & r_*^c\|U\|_{\infty,k}(u,r_*)\\
&+\int_r^{r_*}\left(\lambda^c\|F\|_{\infty,k}(u,\lambda)+O(\lambda^{-2})\sum_{j+l\leq k}\lambda^c\|(q\mathcal{D})^j\not{\mathcal{L}}_{\mathbf{T}}^l U\|_\infty(u,\lambda)\right)d\lambda.
\end{aligned}$$

Together with Gronwall lemma, we infer, for $k\leq k_{large}$,

$$\begin{aligned}
&\sum_{j+l\leq k} r^c\|(q\mathcal{D})^j\not{\mathcal{L}}_{\mathbf{T}}^l U\|_\infty(u,r)\\
\lesssim\ & r_*^c\|U\|_{\infty,k}(u,r_*)+\int_r^{r_*}\lambda^c\|F\|_{\infty,k}(u,\lambda)d\lambda.
\end{aligned}\tag{6.45}$$

Next, multiplying (6.44) with r and differentiating it w.r.t. $(r\nabla_4)^p$, and using also the control of Γ_b, we have, for $p\geq 1$ and $j+l+p\leq k\leq k_{large}$,

$$|(r\nabla_4)^p(q\mathcal{D})^j\not{\mathcal{L}}_{\mathbf{T}}^l U)|\ \lesssim\ r|\mathfrak{d}^{\leq k-1}F|+|(r\nabla_4)^{p-1}(q\mathcal{D},\not{\mathcal{L}}_{\mathbf{T}})^{j+l}U)|.$$

Together with (6.45), we deduce by iteration, for $k\leq k_{large}$,

$$\begin{aligned}
&\sum_{j+l+p\leq k} r^c\|(r\nabla_4)^p(q\mathcal{D})^j\not{\mathcal{L}}_{\mathbf{T}}^l U\|_\infty(u,r)\\
\lesssim\ & r_*^c\|U\|_{\infty,k}(u,r_*)+r^{c+1}\|F\|_{\infty,k-1}(u,r)+\int_r^{r_*}\lambda^c\|F\|_{\infty,k}(u,\lambda)d\lambda.
\end{aligned}$$

Using

$$\begin{aligned}
r^{c+1}\|F\|_{\infty,k-1}(u,r) &\lesssim \int_r^{r_*}\lambda^c\|(\lambda\nabla_4)^{\leq 1}F\|_{\infty,k-1}(u,\lambda)d\lambda\\
&\lesssim \int_r^{r_*}\lambda^c\|F\|_{\infty,k}(u,\lambda)d\lambda,
\end{aligned}$$

we obtain, for $k\leq k_{large}$,

$$\sum_{j+l+p\leq k} r^c\|(r\nabla_4)^p(q\mathcal{D})^j\not{\mathcal{L}}_{\mathbf{T}}^l U\|_\infty(u,r)\ \lesssim\ r_*^c\|U\|_{\infty,k}(u,r_*)$$

$$+\int_r^{r_*}\lambda^c\|F\|_{\infty,k}(u,\lambda)d\lambda.$$

Since $q = r + O(1)$, and since e_3 is spanned by $(\mathbf{T}, e_4, e_1, e_2)$ in view of the definition of $\mathbf{T}$, see (6.21), we infer, for $k \leq k_{large}$,

$$r^c\|U\|_{\infty,k}(u,r) \quad \lesssim \quad r_*^c\|U\|_{\infty,k}(u,r_*) + \int_r^{r_*}\lambda^c\|F\|_{\infty,k}(u,\lambda)d\lambda$$

as desired. This concludes the proof of Proposition 6.47. □

6.4.2. Estimates for the outgoing PG structure of $^{(ext)}\mathcal{M}$ on Σ_* In this section, we recall the main estimates derived in Section 5.7 on Σ_* with respect to the outgoing PG structure of $^{(ext)}\mathcal{M}$. More precisely, we restate below Proposition 5.77. Note that in the statement of that proposition, the PG frame is denoted by prime, while the frame adapted to Σ_*, which is used in the proof, is unprimed. Since, in this chapter, we only deal with the outgoing PG frame of $^{(ext)}\mathcal{M}$, we therefore drop the primes in the statement of Proposition 5.77 which thus takes the following form.

Proposition 6.48. *We have on Σ_*, for*[139] $k \leq k_*$,

$$\sup_{\Sigma_*}\left(ru^{1+\delta_{dec}}|\mathfrak{d}^k\Gamma_b| + r^2u^{\frac{1}{2}+\delta_{dec}}|\mathfrak{d}^k\Gamma_g| + r^2u^{1+\delta_{dec}}|\mathfrak{d}^{k-1}\nabla_3\Gamma_g|\right) \lesssim \epsilon_0,$$

$$\sup_{\Sigma_*}\left(r^2u^{1+\delta_{dec}}|\mathfrak{d}^k\widetilde{trX}| + r^3u^{1+\delta_{dec}}\left|\mathfrak{d}^k\left(\overline{\mathcal{D}}\cdot\check{Z} + 2\overline{\check{P}}\right)\right| + r^4u^{\frac{1}{2}+\delta_{dec}}|\mathfrak{d}^{k-1}\nabla_3B|\right)$$
$$\lesssim \epsilon_0,$$

and

$$\sup_{\Sigma_*} r^5u^{1+\delta_{dec}}\left(\left|\left([\overline{\mathcal{D}}\cdot]_{ren}[B]_{ren}\right)_{\ell=1}\right| + \left|\left[\overline{\mathcal{D}}\cdot\not{\mathcal{L}}_{\mathbf{T}}B\right]_{\ell=1}\right|\right) \quad \lesssim \quad \epsilon_0.$$

6.4.3. Strategy of the proof of Theorem M4 Our goal in this chapter is to extend the results of Proposition 6.48 to $^{(ext)}\mathcal{M}$, i.e. to prove the following proposition which implies Theorem M4.

Proposition 6.49. *We have on $^{(ext)}\mathcal{M}$, for*[140] $k \leq k_* - 8$,

$$\sup_{^{(ext)}\mathcal{M}}\left(ru^{1+\delta_{dec}}|\mathfrak{d}^k\Gamma_b| + \left(r^2u^{\frac{1}{2}+\delta_{dec}} + ru^{1+\delta_{dec}}\right)|\mathfrak{d}^k\Gamma_g| + r^2u^{1+\delta_{dec}}|\mathfrak{d}^{k-1}\nabla_3\Gamma_g|\right)$$
$$\lesssim \epsilon_0,$$

[139] Recall from (6.1) that $k_* = k_{small} + 60$ in this chapter.
[140] Recall from (6.1) that $k_* = k_{small} + 60$ in this chapter.

and

$$\sup_{{}^{(ext)}\mathcal{M}} r^2 u^{1+\delta_{dec}} |\mathfrak{d}^k \widetilde{trX}| + \sup_{{}^{(ext)}\mathcal{M}} r^4 u^{\frac{1}{2}+\delta_{dec}} |\mathfrak{d}^{k-1} \nabla_3 B| \lesssim \epsilon_0.$$

We now describe the strategy of the proof of Proposition 6.49. To start with, we need to distinguish between the estimates for the Γ_g quantities which involve $O(r^{-2}u^{-1/2-\delta_{dec}})$ decay, and those which involve $O(r^{-1}u^{-1-\delta_{dec}})$ decay. The first are relatively easy to derive using our main linearized equations, see Lemma 6.15, the corresponding estimates on the last slice, Proposition 6.47 and the assumptions **Ref 1** and **Ref 2**. In what follows, we describe the main steps in deriving the much more subtle $O(r^{-1}u^{-1-\delta_{dec}})$ estimates for the Γ_g quantities:

1. First derive an estimate for $\widetilde{\mathrm{tr}X}$ using the Raychadhouri equation it verifies, Proposition 6.47, and its estimate on the last slice. The resulting estimates improve the stronger assumption made in **Ref 1**, see (6.11), i.e. we obtain (6.11) with ϵ being replaced by ϵ_0.
2. We observe that we are not able to estimate directly the other primary quantities $\widehat{X}, B, \check{Z}, \check{P}$. Consider for example the equation verified by $\widehat{X}$

$$\nabla_4 \widehat{X} + \Re(\mathrm{tr}X)\widehat{X} = -A.$$

This works well, with the help of Proposition 6.47, to derive an estimate of the form $O\big(\epsilon_0 r^{-2}u^{-1/2-\delta_{dec}}\big)$ but fails to provide an $O\big(\epsilon_0 r^{-1}u^{-1-\delta_{dec}}\big)$ estimate. Indeed according to **Ref 2**, we only have the estimate $A = O\big(\epsilon_0 r^{-2-\delta'}u^{-1-\delta_{dec}}\big)$, for a small constant $\delta' = \frac{1}{2}\big(\delta_{extra} - \delta_{dec}\big) > 0$. On the other hand we can commute with $\not{\mathcal{L}}_{\mathbf{T}}$ and derive an estimate of the form $\not{\mathcal{L}}_{\mathbf{T}}\widehat{X} = O\big(\epsilon_0 r^{-2}u^{-1-\delta_{dec}}\big)$ by making use of the fact that, according to **Ref 2**, $\not{\mathcal{L}}_{\mathbf{T}}A = O\big(\epsilon_0 r^{-3-\delta'}u^{-1-\delta_{dec}}\big)$.
3. We encounter a similar issue with estimates for B. To start with we can not use its natural transport equation

$$\nabla_4 B + 2\overline{\mathrm{tr}X}B = \frac{1}{2}\overline{\mathcal{D}}\cdot A + \frac{1}{2}(\overline{2Z + \underline{H}})\cdot A.$$

Indeed, this transport equation is seriously overshooting in r, i.e. its integration would require decay in r for A which is well beyond what is consistent with our estimates. We look instead at another Bianchi equation,

$$\mathcal{D}\widehat{\otimes}B + (Z+4H)\widehat{\otimes}B = \nabla_3 A + \left(\frac{1}{2}\mathrm{tr}\underline{X} - 4\underline{\omega}\right)A + 3\overline{P}\widehat{X}.$$

4. We need first to commute with $\mathcal{L}\!\!\!/_{\mathbf{T}}$ to take advantage of the previously derived information for $\mathcal{L}\!\!\!/_{\mathbf{T}}\widehat{X}$. This provides an estimate for $\mathcal{D}\widehat{\otimes}\mathcal{L}\!\!\!/_{\mathbf{T}}B$ from which we would have to recover $\mathcal{L}\!\!\!/_{\mathbf{T}}B$.
5. The problem however is that $\mathcal{D}\widehat{\otimes}$ is not an operator on $S(t,u)$. It is for this reason that we have to appeal to Proposition 6.29 and Corollary 6.30 to derive instead an estimate for $\mathcal{D}'\widehat{\otimes}\mathcal{L}\!\!\!/_{\mathbf{T}}B$.
6. In the process however we generate another $\mathcal{L}\!\!\!/_{\mathbf{T}}$ derivative for B, i.e. we first need an estimates for $\mathcal{L}\!\!\!/_{\mathbf{T}}^2B$. Fortunately this can be derived[141] by commuting the overshooting equation $\nabla_4B+2\overline{\mathrm{tr}X}B=\frac{1}{2}\overline{\mathcal{D}}\cdot A+\dots$ twice with $\mathcal{L}\!\!\!/_{\mathbf{T}}$ and thus derive a transport equation for $\mathcal{L}\!\!\!/_{\mathbf{T}}^2$ which is no longer overshooting, i.e. the RHS enjoys enough decay in r for the corresponding transport equation to be integrated. Thus we can first estimate $\mathcal{L}\!\!\!/_{\mathbf{T}}^2B$ from which we derive also an estimate for $\mathcal{D}'\widehat{\otimes}\mathcal{L}\!\!\!/_{\mathbf{T}}B$. Note that this step requires the full force of the assumptions **Ref 2** for A.
7. It remains to derive estimates for $\mathcal{L}\!\!\!/_{\mathbf{T}}B$ from the one of $\mathcal{D}'\widehat{\otimes}\mathcal{L}\!\!\!/_{\mathbf{T}}B$. In view of Corollary 6.39, we need first to estimate $\big(\overline{\mathcal{D}'}\cdot\mathcal{L}\!\!\!/_{\mathbf{T}}B\big)_{\ell=1}$. Starting with the equation for $\nabla_4\mathcal{L}\!\!\!/_{\mathbf{T}}B+2\overline{\mathrm{tr}X}\mathcal{L}\!\!\!/_{\mathbf{T}}B=\frac{1}{2}\overline{\mathcal{D}}\cdot\mathcal{L}\!\!\!/_{\mathbf{T}}A+\dots$ we first derive a transport equation[142] for the quantity $\frac{1}{\Sigma}\overline{q}\,\overline{\mathcal{D}}\cdot(\overline{q}^4\mathcal{L}\!\!\!/_{\mathbf{T}}B)J^{(p)}$ which can therefore be estimated, using our information on Σ_*. Using the transformation formulas between the prime and unprimed frames we then estimate $\big(\overline{\mathcal{D}'}\cdot\mathcal{L}\!\!\!/_{\mathbf{T}}B\big)_{\ell=1}$ and thus, by Corollary 6.39, the derivatives $(\not{\mathfrak{d}}')^k\mathcal{L}\!\!\!/_{\mathbf{T}}B$. Transforming back to the outgoing PG frame of ${}^{(ext)}\mathcal{M}$, we derive the desired estimates for $\mathcal{L}\!\!\!/_{\mathbf{T}}B$.
8. The estimates for $\mathcal{L}\!\!\!/_{\mathbf{T}}\check{Z}$ and $\mathcal{L}\!\!\!/_{\mathbf{T}}\check{P}$ are then similar to the ones for $\mathcal{L}\!\!\!/_{\mathbf{T}}\widehat{X}$. Our strategy therefore is to estimate first $\mathcal{L}\!\!\!/_{\mathbf{T}}\widehat{X}$, $\mathcal{L}\!\!\!/_{\mathbf{T}}^2B$, $\mathcal{L}\!\!\!/_{\mathbf{T}}B$, $\mathcal{L}\!\!\!/_{\mathbf{T}}\check{Z}$ and $\mathcal{L}\!\!\!/_{\mathbf{T}}\check{P}$, and then derive estimates for the primary quantities using elliptic theory on the spheres $S=S(r,u)$.
9. The estimates for the quantities $\widehat{X},B,\check{Z},\check{P}$, though more subtle, follow a similar pattern. The main difference is that they $\underbrace{\text{require}}$ the $\underbrace{\text{transport}}$ equations for the renormalized quantities $[H]_{ren}$, $[\mathcal{D}\cos\theta]_{ren}$, $[\widecheck{M}]_{ren}$, as well as $\int_{S(u,r)}\frac{rJ^{(0)}}{\Sigma}[\overline{\mathcal{D}}\cdot]_{ren}r^4[B]_{ren}$, derived in Proposition 6.41 and Proposition 6.43.
10. We first derive conditional decay estimates, see Proposition 6.53, in which the term $(\overline{\mathcal{D}}\cdot B)_{\ell=1}$ appear on the right hand side. The estimate

[141] Note that the corresponding transport equation for $\mathcal{L}\!\!\!/_{\mathbf{T}}B$ is still overshooting in r, but the one for $\mathcal{L}\!\!\!/_{\mathbf{T}}^2B$ is not.

[142] Note that this is simpler to derive than the renormalized version of the quantity $\frac{1}{\Sigma}\overline{q}\,\overline{\mathcal{D}}\cdot(\overline{q}^4B)J^{(p)}$ in Proposition 6.43.

are then made unconditional in Proposition 6.54. It is important to note that the estimates for $\widehat{X}, B, \check{Z}, \check{P}$ in Proposition 6.54 not only improve the corresponding assumptions in **Ref 1** by replacing ϵ with ϵ_0; they are also better in powers of r, that is they gain $r^{-\delta'}$ for $\delta' > 0$. These improvements are needed later to derive the correct estimates for the remaining quantities in Γ_b.

11. All remaining estimates, i.e. the $O(r^{-2}u^{-1/2-\delta_{dec}})$ decay estimates for Γ_g, and the decay estimates for Γ_b, are derived in Propositions 6.56 and Proposition 6.58.

6.4.4. Estimates for $\widetilde{\mathrm{tr} X}$

Proposition 6.50. *We have on* ${}^{(ext)}\mathcal{M}$

$$\max_{k\leq k_*} \sup_{{}^{(ext)}\mathcal{M}} r^2 u^{1+\delta_{dec}} |\mathfrak{d}^k \widetilde{trX}| \lesssim \epsilon_0.$$

Proof. We apply Proposition 6.47 to the equation

$$\nabla_4 \widetilde{\mathrm{tr} X} + \frac{2}{q}\widetilde{\mathrm{tr} X} = \Gamma_g \cdot \Gamma_g$$

and derive, for all $r_0 \leq r \leq r_*$,

$$r^2 \|\widetilde{\mathrm{tr} X}\|_{\infty,k}(u,r) \lesssim r_*^2 \|\widetilde{\mathrm{tr} X}\|_{\infty,k}(u,r_*) + \int_r^{r_*} \lambda^2 \|\Gamma_g \cdot \Gamma_g\|_{\infty,k}(u,\lambda) d\lambda.$$

Thus, in view of the estimates Proposition 6.48 for $\widetilde{\mathrm{tr} X}$ on Σ_*, bootstrap assumptions **Ref 1** for Γ_g and interpolation Remark 6.5, we derive

$$\sup_{r_0\leq r\leq r_*} r^2 \|\widetilde{\mathrm{tr} X}\|_{\infty,k}(u,r) \lesssim \epsilon_0 u^{-1-\delta_{dec}}, \qquad k \leq k_*, \qquad 1 \leq u \leq u_*,$$

which concludes the proof of the proposition. □

6.4.5. Estimates for renormalized quantities in $\widetilde{{}^{(ext)}\mathcal{M}}$ We extend the estimates for the renormalized quantities $[H]_{ren}$, $[\mathcal{D}\cos\theta]_{ren}$ and $[\widetilde{M}]_{ren}$ from Σ_*, as recorded in Proposition 6.48, to all of ${}^{(ext)}\mathcal{M}$. We do this with the help of the transport equations derived in Proposition 6.41. We also make use of the estimates for A in **Ref 2**, as well as the above estimate for $\widetilde{\mathrm{tr} X}$.

Lemma 6.51. *We have on* ${}^{(ext)}\mathcal{M}$

$$\max_{k\leq k_*} \sup_{{}^{(ext)}\mathcal{M}} r u^{1+\delta_{dec}} \left|\mathfrak{d}^k([H]_{ren})\right| \lesssim \epsilon_0,$$

$$\max_{k\le k_*}\sup_{{}^{(ext)}\mathcal{M}} ru^{1+\delta_{dec}}\left|\mathfrak{d}^k\left([\widecheck{\mathcal{D}\cos\theta}]_{ren}\right)\right| \lesssim \epsilon_0,$$

and

$$\max_{k\le k_*-1}\sup_{{}^{(ext)}\mathcal{M}} r^3u^{1+\delta_{dec}}\left|\mathfrak{d}^k\left([\widecheck{M}]_{ren}\right)\right| \lesssim \epsilon_0.$$

Proof. According to Proposition 6.41, we have

$$\begin{aligned}
\nabla_4\left(\overline{q}[\widecheck{H}]_{ren}\right) &= O(r^{-1})\widecheck{\mathrm{tr}X} + O(1)\,\mathfrak{d}\!\!/^{\le 1}A + r\Gamma_b\cdot\Gamma_g,\\
\nabla_4\left(q[\widecheck{\mathcal{D}\cos\theta}]_{ren}\right) &= O(1)\widecheck{\mathrm{tr}X} + O(r)A + r\Gamma_b\cdot\Gamma_g,\\
\nabla_4\left(\overline{q}q^2[\widecheck{M}]_{ren}\right) &= O(1)\,\mathfrak{d}\!\!/^{\le 1}\widecheck{\mathrm{tr}X} + O(r)\,\mathfrak{d}\!\!/^{\le 2}A + r^2\,\mathfrak{d}\!\!/^{\le 1}(\Gamma_g\cdot\Gamma_g) + r^3\Gamma_b\cdot A.
\end{aligned}$$

Applying Proposition 6.47 to the first identity, and using the estimate on Σ_* for $[\widecheck{H}]_{ren}$, we derive, for all $k\le k_*$, $r_0\le r\le r_*$ and $1\le u\le u_*$,

$$\begin{aligned}
r\big\|[\widecheck{H}]_{ren}\big\|_{\infty,k}(u,r) &\lesssim r_*\big\|[\widecheck{H}]_{ren}\big\|_{\infty,k}(u,r_*) + \int_r^{r_*}\|F\|_{\infty,k}(u,\lambda)d\lambda\\
&\lesssim \epsilon_0u^{-1-\delta_{dec}} + \int_r^{r_*}\|F\|_{\infty,k}(u,\lambda)d\lambda
\end{aligned}$$

where $F := O(r^{-1})\widecheck{\mathrm{tr}X} + O(1)\,\mathfrak{d}\!\!/^{\le 1}A + r\Gamma_b\cdot\Gamma_g$. Using the available estimates for $\widecheck{\mathrm{tr}X}$ and A, we obtain, for all $k\le k_*$, $r_0\le r\le r_*$ and $1\le u\le u_*$,

$$\int_r^{r_*}\|F\|_{\infty,k}(u,\lambda)d\lambda \lesssim \epsilon_0r^{-1}u^{-1-\delta_{dec}}.$$

Hence, we infer, for all $k\le k_*$, $r_0\le r\le r_*$ and $1\le u\le u_*$,

$$r\big\|[\widecheck{H}]_{ren}\big\|_{\infty,k}(u,r) \lesssim \epsilon_0u^{-1-\delta_{dec}}$$

as stated. The two other estimates are derived in the same manner. □

6.4.6. Estimates for some $\mathcal{L}\!\!\!/_{\mathbf{T}}$ derivatives in ${}^{(ext)}\mathcal{M}$

Proposition 6.52. *We have on*[143] ${}^{(ext)}\mathcal{M}$*, for* r_0 *sufficiently large,*

$$\max_{k\le k_*-1}\sup_{{}^{(ext)}\mathcal{M}} r^2u^{1+\delta_{dec}}|\mathfrak{d}^k\mathcal{L}\!\!\!/_{\mathbf{T}}\widehat{X}| \lesssim \epsilon_0,$$

[143] Recall that $r\ge r_0$ on ${}^{(ext)}\mathcal{M}$ for a sufficiently large r_0, and that the small constant $\delta'>0$, appearing in the estimate for $\mathcal{L}\!\!\!/_{\mathbf{T}}B$, is given by $\delta' = \frac{1}{2}\big(\delta_{extra}-\delta_{dec}\big)$.

$$\max_{k\le k_*-2}\sup_{^{(ext)}\mathcal{M}} r^4u^{1+\delta_{dec}}|\mathfrak{d}^k\mathcal{L}\mkern-9mu/_{\mathbf{T}}^2 B| \lesssim \epsilon_0,$$
$$\max_{k\le k_*-3}\sup_{^{(ext)}\mathcal{M}} r^{3+\delta'}u^{1+\delta_{dec}}|\mathfrak{d}^k\mathcal{L}\mkern-9mu/_{\mathbf{T}} B| \lesssim \epsilon_0,$$
$$\max_{k\le k_*-3}\sup_{^{(ext)}\mathcal{M}} r^2u^{1+\delta_{dec}}|\mathfrak{d}^k\mathcal{L}\mkern-9mu/_{\mathbf{T}} \check{Z}| \lesssim \epsilon_0,$$
$$\max_{k\le k_*-4}\sup_{^{(ext)}\mathcal{M}} r^3u^{1+\delta_{dec}}|\mathfrak{d}^k\mathcal{L}\mkern-9mu/_{\mathbf{T}} \check{P}| \lesssim \epsilon_0.$$

Proof. The proof contains several steps.

Step 1. First, we estimate $\mathcal{L}_{\mathbf{T}}\widehat{X}$. Starting with

$$\nabla_4\widehat{X}+\Re(\mathrm{tr}X)\widehat{X} = -A,$$

and commuting with $\mathcal{L}\mkern-9mu/_{\mathbf{T}}$ we derive, in view of Lemma 6.23,

$$\nabla_4\mathcal{L}\mkern-9mu/_{\mathbf{T}}\widehat{X}+\Re(\mathrm{tr}X)\mathcal{L}\mkern-9mu/_{\mathbf{T}}\widehat{X} = -\mathcal{L}\mkern-9mu/_{\mathbf{T}}A+\mathfrak{d}^{\le 1}(\Gamma_g)\cdot\mathfrak{d}^{\le 1}(\Gamma_g)$$

and hence

$$\nabla_4\mathcal{L}\mkern-9mu/_{\mathbf{T}}\widehat{X}+\Re\left(\frac{2}{q}\right)\mathcal{L}\mkern-9mu/_{\mathbf{T}}\widehat{X} = -\mathcal{L}\mkern-9mu/_{\mathbf{T}}A+\mathfrak{d}^{\le 1}(\Gamma_g)\cdot\mathfrak{d}^{\le 1}(\Gamma_g).$$

Hence, applying Proposition 6.47, we have, for all $r_0\le r\le r_*$ and $1\le u\le u_*$,

$$r^2\|\mathcal{L}\mkern-9mu/_{\mathbf{T}}\widehat{X}\|_{\infty,k}(u,r) \lesssim r_*^2\|\mathcal{L}\mkern-9mu/_{\mathbf{T}}\widehat{X}\|_{\infty,k}(u,r_*)+\int_r^{r_*}\lambda^2\|F\|_{\infty,k}(u,\lambda)d\lambda$$

where

$$F = -\mathcal{L}\mkern-9mu/_{\mathbf{T}}A+\mathfrak{d}^{\le 1}(\Gamma_g)\cdot\mathfrak{d}^{\le 1}(\Gamma_g).$$

According to the improved estimates for A of Theorem M1, we have, recalling that the small constant $\delta'>0$ is given by $\delta'=1/2(\delta_{extra}-\delta_{dec})$,

$$\max_{k\le k_*}\sup_{^{(ext)}\mathcal{M}} r^{2+\delta'}u^{1+\delta_{dec}}|\mathfrak{d}^kA|+\max_{k\le k_*-1}\sup_{^{(ext)}\mathcal{M}} r^{3+\delta'}u^{1+\delta_{dec}}|\mathfrak{d}^k\nabla_3A| \lesssim \epsilon_0.$$

Since $\mathcal{L}\mkern-9mu/_{\mathbf{T}}A=\nabla_3A+r^{-1}\mathfrak{d}^{\le 1}A$, and together with the estimates for Γ_g and the form of F, we infer

$$\max_{k\le k_*-1}\sup_{^{(ext)}\mathcal{M}} r^{3+\delta'}u^{1+\delta_{dec}}|\mathfrak{d}^kF| \lesssim \epsilon_0.$$

Taking into account the control on Σ_*, we deduce, for all $k\leq k_*-1$, $r_0\leq r\leq r_*$ and $1\leq u\leq u_*$,

$$r^2\|\mathcal{L}\mkern-10mu /_{\mathbf{T}}\widehat{X}\|_{\infty,k}(u,r) \lesssim \epsilon_0 u^{-1-\delta_{dec}}$$

as stated.

Step 2. Next, we estimate $\mathcal{L}\mkern-10mu /_{\mathbf{T}}^2 B$. Recall that we have

$$\nabla_4 B+\frac{4}{q}B = \frac{1}{2}\overline{\mathcal{D}}\cdot A+\frac{aq}{2|q|^2}\overline{\mathfrak{J}}\cdot A+\Gamma_g\cdot(A,B).$$

We commute with $\mathcal{L}\mkern-10mu /_{\mathbf{T}}$ and obtain, using Lemma 6.23,

$$\nabla_4\mathcal{L}\mkern-10mu /_{\mathbf{T}}B+\frac{4}{q}\mathcal{L}\mkern-10mu /_{\mathbf{T}}B = \frac{1}{2}\overline{\mathcal{D}}\cdot\mathcal{L}\mkern-10mu /_{\mathbf{T}}A+\frac{aq}{2|q|^2}\overline{\mathfrak{J}}\cdot\mathcal{L}\mkern-10mu /_{\mathbf{T}}A+\mathfrak{d}^{\leq 1}(\Gamma_g)\mathfrak{d}^{\leq 1}(A,B).$$

Commuting with $\mathcal{L}\mkern-10mu /_{\mathbf{T}}$ again, we derive

$$\nabla_4\mathcal{L}\mkern-10mu /_{\mathbf{T}}^2B+\frac{4}{q}\mathcal{L}\mkern-10mu /_{\mathbf{T}}^2B = \frac{1}{2}\overline{\mathcal{D}}\cdot\mathcal{L}\mkern-10mu /_{\mathbf{T}}^2A+\frac{aq}{2|q|^2}\overline{\mathfrak{J}}\cdot\mathcal{L}\mkern-10mu /_{\mathbf{T}}^2A+\mathfrak{d}^{\leq 2}(\Gamma_g)\mathfrak{d}^{\leq 2}(A,B).$$

Hence, applying Proposition 6.47, we infer, for all $r_0\leq r\leq r_*$ and $1\leq u\leq u_*$,

$$r^4\|\mathcal{L}\mkern-10mu /_{\mathbf{T}}^2B\|_{\infty,k}(u,r) \lesssim r_*^4\|\mathcal{L}\mkern-10mu /_{\mathbf{T}}^2B\|_{\infty,k}(u,r_*)+\int_r^{r_*}\lambda^4\|F\|_{\infty,k}(u,\lambda)d\lambda$$

where

$$F = \frac{1}{2}\overline{\mathcal{D}}\cdot\mathcal{L}\mkern-10mu /_{\mathbf{T}}^2A+O(r^{-2})\mathcal{L}\mkern-10mu /_{\mathbf{T}}^2A+\mathfrak{d}^{\leq 2}(\Gamma_g)\mathfrak{d}^{\leq 2}(A,B).$$

Making use of the improved estimate for A, $\nabla_3 A$ and $\nabla_3^2 A$ in **Ref 2**, we have, using also the relation between $\mathcal{L}\mkern-10mu /_{\mathbf{T}}^2$ and ∇_3^2 in (6.30),

$$\max_{k\leq k_*-1}\sup_{^{(ext)}\mathcal{M}} r^{4+\delta'}u^{1+\delta_{dec}}|\mathfrak{d}^k\mathcal{L}\mkern-10mu /_{\mathbf{T}}^2A| \lesssim \epsilon_0.$$

Therefore, together with the definition of F, and the control for B and Γ_g provided by **Ref 1**, we infer, for $k\leq k_*-1$, and for all $r_0\leq r\leq r_*$ and $1\leq u\leq u_*$,

$$\int_r^{r_*}\lambda^4\|F\|_{\infty,k}(u,\lambda)d\lambda \lesssim \epsilon_0 u^{-1-\delta_{dec}}.$$

Using also the estimates for $\nabla\!\!\!\!/\,\mathcal{L}_{\mathbf{T}}^2 B$ on Σ_*, we obtain, for all $k \leq k_* - 2$, $r_0 \leq r \leq r_*$ and $1 \leq u \leq u_*$,

$$r^4 \|\mathcal{L}\!\!\!/_{\mathbf{T}}^2 B\|_{\infty,k}(u,r) \quad \lesssim \quad \epsilon_0 u^{-1-\delta_{dec}}$$

as stated.

Step 3. Next, we estimate $\mathcal{D}\widehat{\otimes}\mathcal{L}\!\!\!/_{\mathbf{T}}B$. We start with the equation

$$\mathcal{D}\widehat{\otimes}B + (Z+4H)\widehat{\otimes}B \quad = \quad \nabla_3 A + \left(\frac{1}{2}\mathrm{tr}\underline{X} - 4\underline{\omega}\right)A + 3\overline{P}\widehat{X}.$$

Commuting with $\mathcal{L}\!\!\!/_{\mathbf{T}}$, and using Lemma 6.23 for the commutator, we infer

$$\begin{aligned}\mathcal{D}\widehat{\otimes}\mathcal{L}\!\!\!/_{\mathbf{T}}B + (Z+4H)\widehat{\otimes}\mathcal{L}\!\!\!/_{\mathbf{T}}B \quad &= \quad \mathcal{L}\!\!\!/_{\mathbf{T}}\nabla_3 A + \left(\frac{1}{2}\mathrm{tr}\underline{X} - 4\underline{\omega}\right)\mathcal{L}\!\!\!/_{\mathbf{T}}A + 3\overline{P}\mathcal{L}\!\!\!/_{\mathbf{T}}\widehat{X}\\ &\quad +r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1}(\Gamma_g) + \mathfrak{d}^{\leq 1}(\Gamma_b)A.\end{aligned}$$

Using the relation between $\mathcal{L}\!\!\!/_{\mathbf{T}}$ and ∇_3 in (6.30), as well as the fact that $Z = O(r^{-2}) + \Gamma_g$ and $H = O(r^{-2}) + \Gamma_b$, we deduce

$$\begin{aligned}\mathcal{D}\widehat{\otimes}\mathcal{L}\!\!\!/_{\mathbf{T}}B \quad &= \quad O(r^{-2})\mathcal{L}\!\!\!/_{\mathbf{T}}B + \nabla_3^2 A + O(r^{-1})\mathfrak{d}^{\leq 1}\nabla_3 A + O(r^{-2})\mathfrak{d}^{\leq 1}A\\ &\quad +O(r^{-3})\mathcal{L}\!\!\!/_{\mathbf{T}}\widehat{X} + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1}(\Gamma_g) + \mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1}(A,B).\end{aligned}$$

In view of Corollary 6.30 we have

$$\begin{aligned}\mathcal{D}' \quad &= \quad \Big(1+O(r^{-2})\Big)\mathcal{D} + O(r^{-1})\mathcal{L}\!\!\!/_{\mathbf{T}} + O(r^{-3}) + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1}\\ &= \quad \mathcal{D} + O(r^{-3})\,\mathfrak{d}\!\!\!/^{\leq 1} + O(r^{-1})\mathcal{L}\!\!\!/_{\mathbf{T}} + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1},\end{aligned}$$

and thus

$$\mathcal{D}'\widehat{\otimes}\mathcal{L}\!\!\!/_{\mathbf{T}}B \quad = \quad \mathcal{D}\widehat{\otimes}\mathcal{L}\!\!\!/_{\mathbf{T}}B + O(r^{-3})\,\mathfrak{d}\!\!\!/^{\leq 1}\mathcal{L}\!\!\!/_{\mathbf{T}}B + O(r^{-1})\mathcal{L}\!\!\!/_{\mathbf{T}}^2 B + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1}\mathcal{L}\!\!\!/_{\mathbf{T}}B.$$

Hence,

$$\begin{aligned}\mathcal{D}'\widehat{\otimes}\mathcal{L}\!\!\!/_{\mathbf{T}}B \quad &= \quad O(r^{-1})\mathcal{L}\!\!\!/_{\mathbf{T}}^2 B + O(r^{-2})\,\mathfrak{d}\!\!\!/^{\leq 1}\mathcal{L}\!\!\!/_{\mathbf{T}}B + \nabla_3^2 A + O(r^{-1})\mathfrak{d}^{\leq 1}\nabla_3 A\\ &\quad +O(r^{-2})\mathfrak{d}^{\leq 1}A + O(r^{-3})\mathcal{L}\!\!\!/_{\mathbf{T}}\widehat{X} + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1}(\Gamma_g)\\ &\quad +\mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1}(A,B).\end{aligned}$$

Using again Corollary 6.30 we can express $\mathfrak{d}\!\!\!/\mathcal{L}\!\!\!/_{\mathbf{T}}B$ in terms of $\mathfrak{d}\!\!\!/'\mathcal{L}\!\!\!/_{\mathbf{T}}B$ and derive

$$\mathcal{D}'\widehat{\otimes}\mathcal{L}\!\!\!/_{\mathbf{T}}B \quad = \quad O(r^{-1})\mathcal{L}\!\!\!/_{\mathbf{T}}^2 B + O(r^{-2})(\mathfrak{d}\!\!\!/')^{\leq 1}\mathcal{L}\!\!\!/_{\mathbf{T}}B + \nabla_3^2 A + O(r^{-1})\mathfrak{d}^{\leq 1}\nabla_3 A$$

$$+O(r^{-2})\mathfrak{d}^{\leq 1}A + O(r^{-3})\mathcal{L}\mkern-10mu /_{\mathbf{T}}\widehat{X} + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1}(\Gamma_g)$$
$$+\mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1}(A, B).$$

Differentiating this identity with respect to $(\not{\mathfrak{d}}')^k$, using the estimates for $\nabla_3^2 A$, $\nabla_3 A$ and A in **Ref 2**, the improved estimates for $\mathcal{L}\mkern-10mu /_{\mathbf{T}}\widehat{X}$ in Step 1, the improved estimate for $\mathcal{L}\mkern-10mu /_{\mathbf{T}}^2 B$ in Step 2, and the bootstrap assumptions for Γ_b, Γ_g and B, we deduce on any sphere $S = S(u,r)$ of ${}^{(ext)}\mathcal{M}$, for $k \leq k_* - 2$,

$$\|(\not{\mathfrak{d}}')^k \mathcal{D}'\widehat{\otimes}\mathcal{L}\mkern-10mu /_{\mathbf{T}}B\|_{L^2(S)} \lesssim O(r^{-2})\|(\not{\mathfrak{d}}')^{\leq k+1}\mathcal{L}\mkern-10mu /_{\mathbf{T}}B\|_{L^2(S)} + \epsilon_0 r^{-3-\delta'}u^{-1-\delta_{dec}}. \tag{6.46}$$

Step 4. Next, we derive a first estimate for $\mathcal{L}\mkern-10mu /_{\mathbf{T}}B$. To this end, we apply the first elliptic estimate of Corollary 6.39 to derive

$$\|(\not{\mathfrak{d}}')^{k+1}\mathcal{L}\mkern-10mu /_{\mathbf{T}}B\|_{L^2(S)} \lesssim r\|(\not{\mathfrak{d}}')^{\leq k}\mathcal{D}'\widehat{\otimes}\mathcal{L}\mkern-10mu /_{\mathbf{T}}B\|_{L^2(S)} + r^2\left|(\overline{\mathcal{D}'}\cdot\mathcal{L}\mkern-10mu /_{\mathbf{T}}B)_{\ell=1}\right|.$$

In view of (6.46), we deduce on any sphere $S = S(u,r)$ of ${}^{(ext)}\mathcal{M}$, for $k \leq k_* - 2$,

$$\|(\not{\mathfrak{d}}')^{k+1}\mathcal{L}\mkern-10mu /_{\mathbf{T}}B\|_{L^2(S)} \lesssim O(r^{-1})\|(\not{\mathfrak{d}}')^{\leq k+1}\mathcal{L}\mkern-10mu /_{\mathbf{T}}B\|_{L^2(S)} + r^2\left|(\overline{\mathcal{D}'}\cdot\mathcal{L}\mkern-10mu /_{\mathbf{T}}B)_{\ell=1}\right| + \epsilon_0 r^{-2-\delta'}u^{-1-\delta_{dec}}.$$

Thus, recalling that $r \geq r_0$ on ${}^{(ext)}\mathcal{M}$, and provided that r_0 is sufficiently large, we can absorb the first term on the RHS and deduce, for $k \leq k_* - 2$,

$$\|(\not{\mathfrak{d}}')^{k+1}\mathcal{L}\mkern-10mu /_{\mathbf{T}}B\|_{L^2(S)} \lesssim r^2\left|(\overline{\mathcal{D}'}\cdot\mathcal{L}\mkern-10mu /_{\mathbf{T}}B)_{\ell=1}\right| + \epsilon_0 r^{-2-\delta'}u^{-1-\delta_{dec}}. \tag{6.47}$$

Using again

$$\mathcal{D} = \mathcal{D}' + O(r^{-3})\not{\mathfrak{d}}^{\leq 1} + O(r^{-1})\mathcal{L}\mkern-10mu /_{\mathbf{T}} + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1},$$

and the above improved estimate for $\mathcal{L}\mkern-10mu /_{\mathbf{T}}^2 B$, we further deduce, for $k \leq k_* - 2$,

$$\|\not{\mathfrak{d}}^{k+1}\mathcal{L}\mkern-10mu /_{\mathbf{T}}B\|_{L^2(S)} \lesssim r^2\left|(\overline{\mathcal{D}}\cdot\mathcal{L}\mkern-10mu /_{\mathbf{T}}B)_{\ell=1}\right| + \epsilon_0 r^{-2-\delta'}u^{-1-\delta_{dec}}.$$

By Sobolev, we finally obtain on any sphere $S = S(u,r)$ of ${}^{(ext)}\mathcal{M}$, for $k \leq k_* - 3$,

$$\|\not{\mathfrak{d}}^{k}\mathcal{L}\mkern-10mu /_{\mathbf{T}}B\|_{L^\infty(S)} \lesssim r\left|(\overline{\mathcal{D}}\cdot\mathcal{L}\mkern-10mu /_{\mathbf{T}}B)_{\ell=1}\right| + \epsilon_0 r^{-3-\delta'}u^{-1-\delta_{dec}}. \tag{6.48}$$

Step 5. Next, we estimate $(\overline{\mathcal{D}}\cdot \not{\mathcal{L}}_{\mathbf{T}}B)_{\ell=1}$. Starting from

$$\nabla_4 B+\frac{4}{\overline{q}}B = \frac{1}{2}\overline{\mathcal{D}}\cdot A+\frac{aq}{2|q|^2}\overline{\mathfrak{J}}\cdot A+\Gamma_g\cdot(B,A),$$

we commute with $\not{\mathcal{L}}_{\mathbf{T}}$ and obtain, using Lemma 6.23, the following more precise transport equation for $\nabla_4\not{\mathcal{L}}_{\mathbf{T}}B$ compared the one used in Step 2

$$\begin{aligned}\nabla_4\not{\mathcal{L}}_{\mathbf{T}}B+\frac{4}{\overline{q}}\not{\mathcal{L}}_{\mathbf{T}}B &= \frac{1}{2}\overline{\mathcal{D}}\cdot\not{\mathcal{L}}_{\mathbf{T}}A+\frac{aq}{2|q|^2}\overline{\mathfrak{J}}\cdot\not{\mathcal{L}}_{\mathbf{T}}A\\ &\quad +(r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b),\not{\mathcal{L}}_{\mathbf{T}}\Gamma_g)\mathfrak{d}^{\leq 1}(A,B)+\Gamma_g\not{\mathcal{L}}_{\mathbf{T}}(A,B).\end{aligned}$$

Since $\nabla_3\Gamma_g=r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b$, and in view of the link between $\not{\mathcal{L}}_{\mathbf{T}}$ and ∇_3, we infer

$$\begin{aligned}&\nabla_4\not{\mathcal{L}}_{\mathbf{T}}B+\frac{4}{\overline{q}}\not{\mathcal{L}}_{\mathbf{T}}B\\ =\;&\frac{1}{2}\overline{\mathcal{D}}\cdot\not{\mathcal{L}}_{\mathbf{T}}A+\frac{aq}{2|q|^2}\overline{\mathfrak{J}}\cdot\not{\mathcal{L}}_{\mathbf{T}}A+r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1}(A,B)+\Gamma_g\nabla_3(A,B)\\ =\;&\frac{1}{2}\overline{\mathcal{D}}\cdot\not{\mathcal{L}}_{\mathbf{T}}A+O(r^{-2})\not{\mathcal{L}}_{\mathbf{T}}A+r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1}(A,B)+\Gamma_g\nabla_3(A,B)\\ =\;&\frac{1}{2}\overline{\mathcal{D}}\cdot\not{\mathcal{L}}_{\mathbf{T}}A+O(r^{-2})\nabla_3A+O(r^{-3})\mathfrak{d}^{\leq 1}A+r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1}(A,B)\\ &+\Gamma_g\nabla_3(A,B)\end{aligned}$$

and hence

$$\begin{aligned}\nabla_4(\overline{q}^4\not{\mathcal{L}}_{\mathbf{T}}B) &= \frac{\overline{q}^4}{2}\overline{\mathcal{D}}\cdot\not{\mathcal{L}}_{\mathbf{T}}A+O(r^2)\nabla_3A+O(r)\mathfrak{d}^{\leq 1}A\\ &\quad +r^3\mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1}(A,B)+r^4\Gamma_g\nabla_3(A,B).\end{aligned}$$

We commute with $\overline{q}\,\overline{\mathcal{D}}\cdot$ relying on the following commutator estimate (see Corollary 6.14)

$$[\nabla_4,\overline{q}\,\overline{\mathcal{D}}\cdot] = O(r^{-2})+\Gamma_g\cdot\not{\mathfrak{d}}^{\leq 1}.$$

Using also $\mathcal{D}(q)=O(r^{-1})+\Gamma_b$, $q=r+O(1)$, and the link between $\not{\mathcal{L}}_{\mathbf{T}}$ and ∇_3, we deduce

$$\begin{aligned}\nabla_4(\overline{q}\,\overline{\mathcal{D}}\cdot(\overline{q}^4\not{\mathcal{L}}_{\mathbf{T}}B)) &= O(r^2)\not{\mathcal{L}}_{\mathbf{T}}B+\frac{r^5}{2}\overline{\mathcal{D}}\cdot\overline{\mathcal{D}}\cdot\not{\mathcal{L}}_{\mathbf{T}}A+O(r^2)\not{\mathfrak{d}}^{\leq 2}\nabla_3A\\ &\quad +O(r)\mathfrak{d}^{\leq 3}A+r^3\not{\mathfrak{d}}^{\leq 1}(\mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1}(A,B))\end{aligned}$$

$$+r^4\mathfrak{d}^{\leq 1}(\Gamma_g\nabla_3(A,B)).$$

Since $e_4(J^{(p)})=0$ on ${}^{(ext)}\mathcal{M}$, and using $J^{(p)}=O(1)$, we derive, for $p=0,+,-$,

$$\begin{aligned}
&\nabla_4(\overline{q}\,\overline{\mathcal{D}}\cdot(\overline{q}^4\not{\mathcal{L}}_{\mathbf{T}}B)J^{(p)})\\
= \;& O(r^2)\not{\mathcal{L}}_{\mathbf{T}}B+\frac{r^5}{2}\overline{\mathcal{D}}\cdot\overline{\mathcal{D}}\cdot\not{\mathcal{L}}_{\mathbf{T}}AJ^{(p)}+O(r^2)\not{\mathfrak{d}}^{\leq 2}\nabla_3A+O(r)\mathfrak{d}^{\leq 3}A\\
&+r^3\not{\mathfrak{d}}^{\leq 1}(\mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1}(A,B))+r^4\mathfrak{d}^{\leq 1}(\Gamma_g\nabla_3(A,B)).
\end{aligned}$$

Next we integrate on S with the help of Corollary 6.32 according to which, for a scalar function h on $S=S(u,r)$,

$$e_4\left(\int_{S(u,r)}h\right) = \int_{S(u,r)}\frac{e_4(\Sigma h)}{\Sigma}+O\left(\frac{\epsilon}{u^{1+\delta_{dec}}}\right)h.$$

Applying this to

$$h:=\frac{1}{\Sigma}\overline{q}\,\overline{\mathcal{D}}\cdot(\overline{q}^4\not{\mathcal{L}}_{\mathbf{T}}B)J^{(p)}, \tag{6.49}$$

we deduce, since $\Sigma=r^2+O(1)$,

$$\begin{aligned}
&e_4\left(\int_{S(u,r)}h\right)\\
= \;& \int_S\frac{1}{\Sigma}\nabla_4(\overline{q}\,\overline{\mathcal{D}}\cdot(\overline{q}^4\not{\mathcal{L}}_{\mathbf{T}}B)J^{(p)})+O\left(\frac{\epsilon}{u^{1+\delta_{dec}}}\right)\frac{1}{\Sigma}\overline{q}\,\overline{\mathcal{D}}\cdot(\overline{q}^4\not{\mathcal{L}}_{\mathbf{T}}B)J^{(p)}\\
= \;& O(r^2)\not{\mathcal{L}}_{\mathbf{T}}B+\frac{r^3}{2}\left(\int_S\overline{\mathcal{D}}\cdot\overline{\mathcal{D}}\cdot\not{\mathcal{L}}_{\mathbf{T}}AJ^{(p)}\right)+O(r^2)\not{\mathfrak{d}}^{\leq 2}\nabla_3A+O(r)\mathfrak{d}^{\leq 3}A\\
&+r^3\not{\mathfrak{d}}^{\leq 1}(\mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1}(A,B))+r^4\mathfrak{d}^{\leq 1}(\Gamma_g\nabla_3(A,B))\\
&+O\left(\frac{r^2\epsilon}{u^{1+\delta_{dec}}}\right)\not{\mathfrak{d}}^{\leq 1}\not{\mathcal{L}}_{\mathbf{T}}B.
\end{aligned}$$

Now, using again $\mathcal{D}=\mathcal{D}'+O(r^{-3})\not{\mathfrak{d}}^{\leq 1}+O(r^{-1})\not{\mathcal{L}}_{\mathbf{T}}+r^{-1}\Gamma_b\mathfrak{d}^{\leq 1}$, we have

$$\begin{aligned}
\overline{\mathcal{D}}\cdot\overline{\mathcal{D}}\cdot\not{\mathcal{L}}_{\mathbf{T}}A = \;& \overline{\mathcal{D}'}\cdot\overline{\mathcal{D}'}\cdot\not{\mathcal{L}}_{\mathbf{T}}A+O(r^{-2})\mathfrak{d}^{\leq 1}\nabla_3^2A+O(r^{-3})\mathfrak{d}^{\leq 2}\nabla_3A\\
&+O(r^{-4})\mathfrak{d}^{\leq 3}A+r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\mathfrak{d}^{\leq 1}\nabla_3A)+r^{-2}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot A)
\end{aligned}$$

and hence

$$e_4\left(\int_{S(u,r)} h\right) = O(r^2)\not{\mathcal{L}}_{\mathbf{T}}B + \frac{r^3}{2}\left(\int_S \overline{\mathcal{D}'}\cdot\overline{\mathcal{D}'}\cdot\not{\mathcal{L}}_{\mathbf{T}}AJ^{(p)}\right) + O(r^3)\mathfrak{d}^{\leq 1}\nabla_3^2 A$$
$$+O(r^2)\mathfrak{d}^{\leq 2}\nabla_3 A + O(r)\mathfrak{d}^{\leq 3}A + r^3\mathfrak{d}^{\leq 3}(\Gamma_b(A,B))$$
$$+r^4\mathfrak{d}^{\leq 1}(\Gamma_g\mathfrak{d}^{\leq 1}\nabla_3(A,B)) + O\left(\frac{r^2\epsilon}{u^{1+\delta_{dec}}}\right)\not{\mathfrak{d}}^{\leq 1}\not{\mathcal{L}}_{\mathbf{T}}B.$$

Integrating by parts and making use of

$$\left|\mathcal{D}'\widehat{\otimes}\mathcal{D}'J^{(p)}\right| \lesssim \frac{\epsilon}{r^3u^{\frac{1}{2}+\delta_{dec}}} + \frac{1}{r^4},$$

see Proposition 6.35, we deduce

$$\int_S \overline{\mathcal{D}'}\cdot\overline{\mathcal{D}'}\cdot\not{\mathcal{L}}_{\mathbf{T}}AJ^{(p)} = \int_S \not{\mathcal{L}}_{\mathbf{T}}A\,\overline{\mathcal{D}'\widehat{\otimes}\mathcal{D}'J^{(p)}}$$
$$= O\left(\frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}}}\right)\not{\mathcal{L}}_{\mathbf{T}}A + O(r^{-2})\not{\mathcal{L}}_{\mathbf{T}}A.$$

Therefore

$$e_4\left(\int_{S(u,r)} h\right) = O(r^2)\not{\mathcal{L}}_{\mathbf{T}}B + O(r^3)\mathfrak{d}^{\leq 1}\nabla_3^2 A + O(r^2)\mathfrak{d}^{\leq 2}\nabla_3 A + O(r)\mathfrak{d}^{\leq 3}A$$
$$+r^3\mathfrak{d}^{\leq 3}(\Gamma_b(A,B)) + r^4\mathfrak{d}^{\leq 1}(\Gamma_g\mathfrak{d}^{\leq 1}\nabla_3(A,B))$$
$$+O\left(\frac{r^2\epsilon}{u^{1+\delta_{dec}}}\right)\not{\mathfrak{d}}^{\leq 1}\not{\mathcal{L}}_{\mathbf{T}}B + O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\not{\mathcal{L}}_{\mathbf{T}}A.$$

Together with the improved control on A, $\nabla_3 A$, $\nabla_3^2 A$ in **Ref 2**, as well as the bootstrap assumptions in **Ref 1**, we obtain

$$e_4\left(\int_{S(u,r)} h\right) = O(r^2)\not{\mathcal{L}}_{\mathbf{T}}B + O\left(\frac{\epsilon_0}{r^{1+\delta'}u^{1+\delta_{dec}}}\right).$$

Now, recall that h is given by

$$h = \frac{1}{\Sigma}\overline{q}\,\overline{\mathcal{D}}\cdot(\overline{q}^4\not{\mathcal{L}}_{\mathbf{T}}B)J^{(p)},$$

so that, together with $\Sigma = r^2 + O(1)$ and $q = r + O(1)$, we have

$$\int_S h = r^5\left(\overline{\mathcal{D}}\cdot\not{\mathcal{L}}_{\mathbf{T}}B\right)_{\ell=1} + O(r^3)\,\not{\mathfrak{d}}^{\leq 1}\not{\mathcal{L}}_{\mathbf{T}}B. \tag{6.50}$$

Together with (6.48), we infer

$$r^2 \not{\mathcal{L}}_{\mathbf{T}} B = O(r^{-2}) \int_S h + O(\epsilon_0 r^{-1-\delta'} u^{-1-\delta_{dec}}).$$

Plugging the second identity in the above, we obtain

$$e_4 \left(\int_{S(u,r)} h \right) = O(r^{-2}) \int_S h + O\left(\frac{\epsilon_0}{r^{1+\delta'} u^{1+\delta_{dec}}} \right).$$

Using Gronwall lemma, we deduce

$$\left| \int_{S(u,r)} h \right| \lesssim \left| \int_{S(r_*,u)} h \right| + \frac{\epsilon_0}{u^{1+\delta_{dec}}}.$$

Together with (6.50), and the control of $\left(\overline{\mathcal{D}} \cdot \not{\mathcal{L}}_{\mathbf{T}} B\right)_{\ell=1}$ and $\not{\mathcal{L}}_{\mathbf{T}} B$ on Σ_*, this yields, on any sphere $S = S(u,r)$ of ${}^{(ext)}\mathcal{M}$,

$$\left| (\overline{\mathcal{D}} \cdot \not{\mathcal{L}}_{\mathbf{T}} B)_{\ell=1} \right| \lesssim \epsilon_0 r^{-5} u^{-1-\delta_{dec}} + r^{-2} \| \not{\mathfrak{d}}^{\leq 1} \not{\mathcal{L}}_{\mathbf{T}} B \|_{L^\infty(S)}. \tag{6.51}$$

Step 6. We are now in position to conclude the estimate for $\not{\mathcal{L}}_{\mathbf{T}} B$. By combining (6.51) with the estimate (6.48) derived in Step 4, we have, on any sphere $S = S(u,r)$ of ${}^{(ext)}\mathcal{M}$, for $k \leq k_* - 3$,

$$\begin{aligned} \| \not{\mathfrak{d}}^k \not{\mathcal{L}}_{\mathbf{T}} B \|_{L^\infty(S)} &\lesssim r \left| (\overline{\mathcal{D}} \cdot \not{\mathcal{L}}_{\mathbf{T}} B)_{\ell=1} \right| + \epsilon_0 r^{-3-\delta'} u^{-1-\delta_{dec}} \\ &\lesssim r^{-1} \| \not{\mathfrak{d}}^{\leq 1} \not{\mathcal{L}}_{\mathbf{T}} B \|_{L^\infty(S)} + \epsilon_0 r^{-3-\delta'} u^{-1-\delta_{dec}}. \end{aligned}$$

Since $r \geq r_0$ on ${}^{(ext)}\mathcal{M}$, for r_0 sufficiently large, we may absorb the first term on the RHS. We deduce, for every sphere $S = S(u,r)$ in ${}^{(ext)}\mathcal{M}$,

$$\| \not{\mathfrak{d}}^{\leq k_*-3} \not{\mathcal{L}}_{\mathbf{T}} B \|_{L^\infty(S)} \lesssim \epsilon_0 r^{-3-\delta'} u^{-1-\delta_{dec}}.$$

Together with the estimates for $\not{\mathcal{L}}_{\mathbf{T}}^2 B$ of Step 2, we infer

$$\| (\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathfrak{d}})^{\leq k_*-3} \not{\mathcal{L}}_{\mathbf{T}} B \|_{L^\infty(S)} \lesssim \epsilon_0 r^{-3-\delta'} u^{-1-\delta_{dec}}.$$

Finally, recalling the transport equation for $\nabla_4 \not{\mathcal{L}}_{\mathbf{T}} B$ which yields

$$\nabla_4 \not{\mathcal{L}}_{\mathbf{T}} B = -\frac{4}{q} \not{\mathcal{L}}_{\mathbf{T}} B + O(r^{-1}) \mathfrak{d}^{\leq 1} \nabla_3 A + O(r^{-2}) \mathfrak{d}^{\leq 2} A$$

$$+r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1}(A,B)+\Gamma_g\nabla_3(A,B),$$

and together with the control of A and $\nabla_3 A$ provided by **Ref 2**, and the bootstrap assumptions in **Ref 1**, we obtain

$$\|(r\nabla_4, \not{\mathcal{L}}_{\mathbf{T}}, \not{\mathfrak{d}})^{\leq k_*-3}\not{\mathcal{L}}_{\mathbf{T}}B\|_{L^\infty(S)} \lesssim \epsilon_0 r^{-3-\delta'}u^{-1-\delta_{dec}}.$$

Together with the fact that e_3 is spanned by $\mathbf{T}$, e_4 and (e_1,e_2), we infer, for all $k\leq k_*-3$, $r_0\leq r\leq r_*$ and $1\leq u\leq u_*$,

$$r^{3+\delta'}\|\not{\mathcal{L}}_{\mathbf{T}}B\|_{\infty,k}(u,r) \lesssim \epsilon_0 u^{-1-\delta_{dec}} \tag{6.52}$$

as stated.

Step 7. Next, we estimate $\not{\mathcal{L}}_{\mathbf{T}}\check{P}$ with the help of the equation

$$\nabla_4\check{P}+\frac{3}{q}\check{P} = \frac{1}{2}\mathcal{D}\cdot\overline{B}-\frac{a\overline{q}}{2|q|^2}\mathfrak{J}\cdot\overline{B}+O(r^{-3})\widetilde{\mathrm{tr}X}+\Gamma_b\cdot A+r^{-1}\Gamma_g\cdot\Gamma_g.$$

We commute with $\not{\mathcal{L}}_{\mathbf{T}}$ using, see Lemma 6.23,

$$[\nabla_4,\not{\mathcal{L}}_{\mathbf{T}}] = r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1}, \qquad [\not{\mathcal{L}}_{\mathbf{T}},\nabla]=r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b)\mathfrak{d}^{\leq 1},$$

and use also the fact that $\nabla_3(\Gamma_g)=r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b$ and $\mathbf{T}(q)\in r\Gamma_b$. We obtain

$$\begin{aligned}\nabla_4\not{\mathcal{L}}_{\mathbf{T}}\check{P}+\frac{3}{q}\not{\mathcal{L}}_{\mathbf{T}}\check{P} &= \frac{1}{2}\mathcal{D}\cdot\not{\mathcal{L}}_{\mathbf{T}}\overline{B}-\frac{a\overline{q}}{2|q|^2}\mathfrak{J}\cdot\not{\mathcal{L}}_{\mathbf{T}}\overline{B}+O(r^{-3})\mathfrak{d}^{\leq 1}\widetilde{\mathrm{tr}X}\\ &\quad +\mathfrak{d}^{\leq 1}(\Gamma_b\cdot(A,B))+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b)\cdot\mathfrak{d}^{\leq 1}(\Gamma_g).\end{aligned}$$

We now make use of Proposition 6.47 and deduce, for all $r_0\leq r\leq r_*$ and $1\leq u\leq u_*$,

$$r^3\|\not{\mathcal{L}}_{\mathbf{T}}\check{P}\|_{\infty,k}(u,r) \lesssim r_*^3\|\not{\mathcal{L}}_{\mathbf{T}}\check{P}\|_{\infty,k}(u,r_*)+\int_r^{r_*}\lambda^3\|F\|_{\infty,k}(u,\lambda)d\lambda$$

with

$$\begin{aligned}F &= \frac{1}{2}\mathcal{D}\cdot\not{\mathcal{L}}_{\mathbf{T}}\overline{B}-\frac{a\overline{q}}{2|q|^2}\mathfrak{J}\cdot\not{\mathcal{L}}_{\mathbf{T}}\overline{B}+O(r^{-3})\mathfrak{d}^{\leq 1}\widetilde{\mathrm{tr}X}+\mathfrak{d}^{\leq 1}(\Gamma_b\cdot(A,B))\\ &\quad +r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b)\cdot\mathfrak{d}^{\leq 1}(\Gamma_g).\end{aligned}$$

Hence, using the estimates for $\not{\mathcal{L}}_{\mathbf{T}}B$ derived in Step 6, the estimate for $\widetilde{\mathrm{tr}X}$ of Proposition 6.50, the bootstrap assumptions in **Ref 1**, and the estimate for $\mathcal{L}\!\!\!/_{\mathbf{T}}\check{P}$ on Σ_*, we deduce, for all $k \le k_* - 4$, $r_0 \le r \le r_*$ and $1 \le u \le u_*$,

$$\|\mathcal{L}\!\!\!/_{\mathbf{T}}\check{P}\|_{\infty,k}(u,r) \lesssim \epsilon_0 r^{-3} u^{-1-\delta_{dec}} \tag{6.53}$$

as stated.

Step 8. Finally, we estimate $\mathcal{L}\!\!\!/_{\mathbf{T}}\check{Z}$. Recall

$$\nabla_4\check{Z} + \frac{2}{q}\check{Z} = -\frac{aq}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X} - B + O(r^{-2})\widetilde{\mathrm{tr}X} + \Gamma_g\cdot\Gamma_g.$$

We commute with $\mathcal{L}\!\!\!/_{\mathbf{T}}$. Using again $[\nabla_4, \mathcal{L}\!\!\!/_{\mathbf{T}}] = r^{-1}\mathfrak{d}^{\le 1}(\Gamma_b)\mathfrak{d}^{\le 1}$, and

$$\begin{aligned}
\mathbf{T}(q) &= \mathbf{T}(r) + ia\mathbf{T}(\cos\theta) \in r\Gamma_b,\\
\mathcal{L}\!\!\!/_{\mathbf{T}}\mathfrak{J} &= \widetilde{\nabla_3\mathfrak{J}} + O(r^{-1})\widetilde{\nabla\mathfrak{J}} + r^{-1}\mathfrak{d}(\Gamma_b)\mathfrak{J} \in r^{-1}\mathfrak{d}^{\le 1}\Gamma_b,
\end{aligned}$$

we infer

$$\begin{aligned}
&\nabla_4\mathcal{L}\!\!\!/_{\mathbf{T}}\check{Z} + \frac{2}{q}\mathcal{L}\!\!\!/_{\mathbf{T}}\check{Z}\\
&= -\frac{aq}{|q|^2}\overline{\mathfrak{J}}\cdot\mathcal{L}\!\!\!/_{\mathbf{T}}\widehat{X} - \mathcal{L}\!\!\!/_{\mathbf{T}}B + O(r^{-2})\mathfrak{d}^{\le 1}\widetilde{\mathrm{tr}X} + \mathfrak{d}^{\le 1}(\Gamma_g)\cdot\mathfrak{d}^{\le 1}(\Gamma_g)\\
&= O(r^{-2})\mathcal{L}\!\!\!/_{\mathbf{T}}\widehat{X} - \mathcal{L}\!\!\!/_{\mathbf{T}}B + O(r^{-2})\mathfrak{d}^{\le 1}\widetilde{\mathrm{tr}X} + \mathfrak{d}^{\le 1}(\Gamma_g)\cdot\mathfrak{d}^{\le 1}(\Gamma_g).
\end{aligned}$$

Proceeding as before with the help of Proposition 6.47, using the estimates for $\mathcal{L}\!\!\!/_{\mathbf{T}}\widehat{X}$ derived in Step 1, the estimates for $\mathcal{L}\!\!\!/_{\mathbf{T}}B$ derived in Step 6, the estimate for $\widetilde{\mathrm{tr}X}$ of Proposition 6.50, the bootstrap assumptions in **Ref 1**, and the estimate for $\mathcal{L}\!\!\!/_{\mathbf{T}}\check{Z}$ on Σ_*, we deduce, for all $k \le k_* - 3$, $r_0 \le r \le r_*$ and $1 \le u \le u_*$,

$$\|\mathcal{L}\!\!\!/_{\mathbf{T}}\check{Z}\|_{\infty,k}(u,r) \lesssim \epsilon_0 r^{-2} u^{-1-\delta_{dec}} \tag{6.54}$$

as stated. This concludes the proof of Proposition 6.52. □

6.5. Improved decay estimates for B, $\check{P}$, $\widehat{X}$, $\check{Z}$, $\check{H}$, and $\widetilde{\mathcal{D}\cos\theta}$

6.5.1. Conditional control of B, $\check{P}$, $\widehat{X}$, $\check{Z}$, $\check{H}$, and $\widetilde{\mathcal{D}\cos\theta}$ The goal of this section is to prove the following proposition.

Proposition 6.53. *We have on ${}^{(ext)}\mathcal{M}$, with $\delta' = \frac{1}{2}(\delta_{extra} - \delta_{dec}) > 0$, for all $k \le k_* - 5$,*

$$r\big(|\mathfrak{d}^{\le k}B| + |\mathfrak{d}^{\le k}\check{P}|\big) + |\mathfrak{d}^{\le k}\widehat{X}| + |\mathfrak{d}^{\le k}\check{Z}|$$

$$(6.55) \qquad \lesssim \quad r^2|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + \epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}$$

and

$$(6.56) \qquad |\mathfrak{d}^{\leq k}\check{H}| + |\mathfrak{d}^{\leq k}\widecheck{\mathcal{D}\cos\theta}| \quad \lesssim \quad r^2|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + \epsilon_0 r^{-1}u^{-1-\delta_{dec}}.$$

In addition we prove the following preliminary estimate[144] *for* B

$$(6.57) \qquad |\mathfrak{d}^{\leq k_*-2}B| \quad \lesssim \quad r|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + \epsilon r^{-3-\delta'}u^{-1/2-\delta_{dec}}.$$

Proof. We will repeatedly make use of the notation introduced in Definition 6.4, i.e. $U\in r^{-p}\mathrm{Good}_k$ if U satisfies $|\mathfrak{d}^{\leq k}U|\lesssim \epsilon_0 r^{-p}u^{-1-\delta_{dec}}$.

Step 1. In view of **Ref 2** and Proposition 6.50, we have on ${}^{(ext)}\mathcal{M}$

$$(6.58) \quad A\in r^{-2-\delta'}\mathrm{Good}_{k_*}, \quad \nabla_3 A\in r^{-3-\delta'}\mathrm{Good}_{k_*-1}, \quad \widecheck{\mathrm{tr}X}\in r^{-2}\mathrm{Good}_{k_*-1}.$$

Also, in view of Lemma 6.51, we have on ${}^{(ext)}\mathcal{M}$

$$(6.59) \qquad [\check{H}]_{ren},\ [\widecheck{\mathcal{D}\cos\theta}]_{ren}\in r^{-1}\mathrm{Good}_{k_*}, \quad [\widecheck{M}]_{ren}\in r^{-3}\mathrm{Good}_{k_*-1},$$

where we recall

$$\begin{aligned}
\overline{q}[\check{H}]_{ren} &= \overline{q}\check{H} - q\check{Z} + \frac{1}{3}\left(-\overline{q}^2+|q|^2\right)B + \frac{a}{2}(q-\overline{q})\overline{\mathfrak{J}}\cdot\widehat{X},\\
q[\widecheck{\mathcal{D}\cos\theta}]_{ren} &= q\widecheck{\mathcal{D}\cos\theta} + \frac{i}{2}|q|^2\overline{\mathfrak{J}}\cdot\widehat{X},\\
\overline{q}q^2[\widecheck{M}]_{ren} &= \overline{q}\,\overline{\mathcal{D}}\cdot\left(q^2\check{Z} + \left(-\frac{a}{2}q^2 - \frac{a}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\right) + 2\overline{q}^3\,\overline{\check{P}} - 2aq^2\overline{\mathfrak{J}}\cdot\check{Z}\\
&\quad + \left(-\frac{1}{3}q^2\overline{q}^2 - \frac{1}{3}q\overline{q}^3 + \frac{2}{3}\overline{q}^4\right)\overline{\mathcal{D}}\cdot B\\
&\quad + a\left(q^2\overline{q} + \frac{2}{3}q\overline{q}^2 - \frac{13}{6}\overline{q}^3\right)\overline{\mathfrak{J}}\cdot B + a^2(q^2+|q|^2)\overline{\mathfrak{J}}\cdot\widehat{X}\cdot\overline{\mathfrak{J}}.
\end{aligned}$$

Since $q=r+O(1)$, $\overline{q}=r+O(1)$, $-\overline{q}^2+|q|^2=O(r)$, and

$$-\frac{1}{3}q^2\overline{q}^2 - \frac{1}{3}q\overline{q}^3 + \frac{2}{3}\overline{q}^4 = O(r^3),$$

[144] A stronger estimate, i.e. with ϵ replaced by ϵ_0, will be proved later.

we infer

$$
\begin{aligned}
\check{H} &= \check{Z} + O(r^{-1})\check{Z} + O(1)B + O(r^{-2})\widehat{X} + r^{-1}\mathrm{Good}_{k_*},\\
\widetilde{\mathcal{D}\cos\theta} &= -\frac{i}{2}r\,\overline{\mathfrak{J}}\cdot\widehat{X} + O(r^{-1})\widehat{X} + r^{-1}\mathrm{Good}_{k_*},\\
r\overline{\mathcal{D}}\cdot\check{Z} &= -2r\,\overline{\check{P}} + O(r^{-1})\check{Z} + O(r^{-1})\,\mathfrak{d}^{\leq 1}\widehat{X} + O(1)\check{P}\\
&\quad + O(1)\,\mathfrak{d}^{\leq 1}B + r^{-2}\mathrm{Good}_{k_*-1}.
\end{aligned}
\tag{6.60}
$$

Step 2. Recall the linearized Codazzi for $\widehat{X}$

$$
\begin{aligned}
\frac{1}{2}\overline{\mathcal{D}}\cdot\widehat{X} &= \frac{1}{q}\check{Z} - B + O(r^{-2})\widehat{X} + O(r^{-2})\check{H} + O(r^{-1})\,\mathfrak{d}^{\leq 1}\widetilde{\mathrm{tr}X}\\
&\quad + O(r^{-2})\widetilde{\mathcal{D}(\cos\theta)} + \Gamma_b\cdot\Gamma_g.
\end{aligned}
$$

In view of the assumptions **Ref 1** and the estimate for $\widetilde{\mathrm{tr}X}$ in (6.58), we obtain

$$
\begin{aligned}
\overline{\mathcal{D}}\cdot\widehat{X} &= \frac{2}{r}\check{Z} - 2B + O(r^{-2})\check{Z} + O(r^{-2})\widehat{X} + O(r^{-2})\check{H} + O(r^{-2})\widetilde{\mathcal{D}(\cos\theta)}\\
&\quad + r^{-3}\mathrm{Good}_{k_*-1}.
\end{aligned}
$$

Eliminating $\check{H}$ and $\widetilde{\mathcal{D}(\cos\theta)}$ on the RHS with the help of the two first equations of (6.60), we infer

$$
\overline{\mathcal{D}}\cdot\widehat{X} = \frac{2}{r}\check{Z} - 2B + O(r^{-2})\check{Z} + O(r^{-2})\widehat{X} + O(r^{-2})B + r^{-3}\mathrm{Good}_{k_*-1}.
\tag{6.61}
$$

Step 3. Starting with the Bianchi identity

$$
\nabla_3 A - \frac{1}{2}\mathcal{D}\widehat{\otimes}B = -\frac{1}{2}\mathrm{tr}\underline{X}A + 4\underline{\omega}A + \frac{1}{2}(Z+4H)\widehat{\otimes}B - 3\overline{P}\widehat{X},
$$

we have

$$
\mathcal{D}\widehat{\otimes}B = 2\nabla_3 A + O(r^{-1})A + O(r^{-2})B + O(r^{-3})\widehat{X} + r^{-1}\Gamma_g\cdot\Gamma_g + \Gamma_b\cdot(A,B)
$$

and hence, using the estimates for $\nabla_3 A$ and A in (6.58), as well as **Ref 1** for the nonlinear terms, we deduce

$$
\mathcal{D}\widehat{\otimes}B = O(r^{-2})B + O(r^{-3})\widehat{X} + r^{-3-\delta'}\mathrm{Good}_{k_*-1}.
\tag{6.62}
$$

Step 4. Starting with the linearized Bianchi identity

$$\nabla_3 B - \mathcal{D}\overline{\check{P}} = \frac{2}{r}B + O(r^{-2})B + O(r^{-2})\check{P} + O(r^{-3})\check{H} + O(r^{-4})\mathcal{D}\widecheck{(\cos\theta)} + r^{-1}\Gamma_b\cdot\Gamma_g,$$

we deduce

$$\mathcal{D}\overline{\check{P}} = \nabla_3 B - \frac{2}{r}B + O(r^{-2})B + O(r^{-2})\check{P} + O(r^{-3})\check{H} + O(r^{-4})\mathcal{D}\widecheck{(\cos\theta)} + r^{-1}\Gamma_b\cdot\Gamma_g. \tag{6.63}$$

Using the identity

$$\not{\mathcal{L}}_{\mathbf{T}} = \frac{1}{2}\nabla_3 + \frac{1}{2}\frac{\Delta}{|q|^2}\nabla_4 + O(r^{-2})\,\not{\mathfrak{d}}^{\leq 1} + \Gamma_b,$$

we write

$$\nabla_3 B = 2\not{\mathcal{L}}_{\mathbf{T}}B - \frac{\Delta}{|q|^2}\nabla_4 B + O(r^{-2})\,\not{\mathfrak{d}}^{\leq 1}B + r^{-1}\Gamma_b\cdot\Gamma_g.$$

Also, using the linearized Bianchi identity

$$\nabla_4 B + \frac{4}{q}B = \frac{1}{2}\overline{\mathcal{D}}\cdot A + \frac{aq}{2|q|^2}\overline{\mathfrak{J}}\cdot A + \Gamma_g\cdot(B,A),$$

we have

$$\nabla_4 B = -\frac{4}{r}B + O(r^{-2})B + O(r^{-1})\,\not{\mathfrak{d}}^{\leq 1}A + r^{-1}\Gamma_g\cdot\Gamma_g.$$

Hence

$$\nabla_3 B = \frac{4}{r}B + 2\not{\mathcal{L}}_{\mathbf{T}}B + O(r^{-2})\,\not{\mathfrak{d}}^{\leq 1}B + O(r^{-1})\,\not{\mathfrak{d}}^{\leq 1}A + r^{-1}\Gamma_b\cdot\Gamma_g.$$

Back to (6.63) we substitute $\nabla_3 B$ to deduce

$$\mathcal{D}\overline{\check{P}} = \frac{2}{r}B + 2\not{\mathcal{L}}_{\mathbf{T}}B + O(r^{-2})\,\not{\mathfrak{d}}^{\leq 1}B + O(r^{-2})\check{P} + O(r^{-3})\check{H} + O(r^{-4})\mathcal{D}\widecheck{(\cos\theta)} + O(r^{-1})\,\not{\mathfrak{d}}^{\leq 1}A + r^{-1}\Gamma_b\cdot\Gamma_g.$$

In view of (6.58) for A and the control of $\not{\mathcal{L}}_{\mathbf{T}}B$ in Proposition 6.52, we deduce

$$\mathcal{D}\overline{\check{P}} = \frac{2}{r}B + O(r^{-2})\,\not{\mathfrak{d}}^{\leq 1}B + O(r^{-2})\check{P} + O(r^{-3})\check{H} + O(r^{-4})\mathcal{D}\widecheck{(\cos\theta)}$$

$$+r^{-3-\delta'}\text{Good}_{k_*-3}.$$

Using the first two equations of (6.60) to eliminate $\check{H}$ and $\widecheck{\mathcal{D}(\cos\theta)}$

$$\mathcal{D}\overline{\check{P}} = \frac{2}{r}B + O(r^{-2})\,\not{\mathfrak{d}}^{\leq 1}B + O(r^{-2})\check{P} + O(r^{-3})\check{Z} + O(r^{-4})\widehat{X} + r^{-3-\delta'}\text{Good}_{k_*-3}. \tag{6.64}$$

Step 5. We start with equation (6.62) of Step 3.

$$\mathcal{D}\widehat{\otimes}B \quad = \quad O(r^{-2})B + O(r^{-3})\widehat{X} + r^{-3-\delta'}\text{Good}_{k_*-1}.$$

We appeal again to Corollary 6.30 to write

$$\nabla' \quad = \quad \Big(1 + O(r^{-2})\Big)\nabla + O(r^{-1})\not{\mathcal{L}}_{\mathbf{T}} + O(r^{-3}) + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1}. \tag{6.65}$$

Together with the control of $\not{\mathcal{L}}_{\mathbf{T}}B$ provided by Proposition 6.52, we infer

$$\mathcal{D}'\widehat{\otimes}B \quad = \quad O(r^{-2})(\not{\mathfrak{d}}')^{\leq 1}B + O(r^{-3})\widehat{X} + r^{-3-\delta'}\text{Good}_{k_*-3}.$$

We appeal to the first elliptic estimate of Corollary 6.39 to derive for $k \leq k_* - 3$,

$$\begin{aligned}\|(\not{\mathfrak{d}}')^{\leq k+1}B\|_{L^2(S(u,r))} \quad \lesssim \quad & r^2|(\overline{\mathcal{D}'}\cdot B)_{\ell=1}| + r^{-1}\|(\not{\mathfrak{d}}')^{\leq k+1}B\|_{L^2(S(u,r))} \\ & + r^{-2}\|(\not{\mathfrak{d}}')^{\leq k}\widehat{X}\|_{L^2(S(u,r))} + \epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}.\end{aligned}$$

Using again (6.65) and the control of $\not{\mathcal{L}}_{\mathbf{T}}B$ as above we deduce

$$\begin{aligned}\|\not{\mathfrak{d}}^{\leq k+1}B\|_{L^2(S(u,r))} \quad \lesssim \quad & r^2|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + r^{-1}\|\not{\mathfrak{d}}^{\leq k+1}B\|_{L^2(S(u,r))} \\ & + r^{-2}\|\not{\mathfrak{d}}^{\leq k}\widehat{X}\|_{L^2(S(u,r))} + \epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}.\end{aligned}$$

Since $r \geq r_0$ on ${}^{(ext)}\mathcal{M}$, we infer, for r_0 large enough, for all $k \leq k_* - 3$,

$$\begin{aligned}\|\not{\mathfrak{d}}^{\leq k+1}B\|_{L^2(S(u,r))} \lesssim\; & r^2|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| \\ & + r^{-2}\|\not{\mathfrak{d}}^{\leq k}\widehat{X}\|_{L^2(S(u,r))} + \epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}.\end{aligned} \tag{6.66}$$

Step 6. We start with the last equation of (6.60)

$$\begin{aligned}r\overline{\mathcal{D}}\cdot\check{Z} \quad = \quad & -2r\,\overline{\check{P}} + O(r^{-1})\check{Z} + O(r^{-1})\,\not{\mathfrak{d}}^{\leq 1}\widehat{X} + O(1)\check{P} + O(1)\,\not{\mathfrak{d}}^{\leq 1}B \\ & + r^{-2}\text{Good}_{k_*-1}.\end{aligned}$$

Differentiating w.r.t. $r\mathcal{D}$, we infer

$$
\begin{aligned}
r^2\mathcal{D}\overline{\mathcal{D}}\cdot\check{Z} &= -2r^2\mathcal{D}\overline{\check{P}} + O(r^{-1})\,\mathfrak{d}^{\leq 1}\check{Z} + O(r^{-1})\,\mathfrak{d}^{\leq 2}\widehat{X} + O(1)\,\mathfrak{d}^{\leq 1}\check{P}\\
&\quad +O(1)\,\mathfrak{d}^{\leq 2}B + r^{-2}\text{Good}_{k_*-2}.
\end{aligned}
$$

On the other hand, recall (6.64),

$$
\begin{aligned}
\mathcal{D}\overline{\check{P}} &= \frac{2}{r}B + O(r^{-2})\,\mathfrak{d}^{\leq 1}B + O(r^{-2})\check{P} + O(r^{-3})\check{Z}\\
&\quad +O(r^{-4})\widehat{X} + r^{-3-\delta'}\text{Good}_{k_*-3}.
\end{aligned}
$$

Plugging in the previous identity, we infer

$$
\begin{aligned}
r^2\mathcal{D}\overline{\mathcal{D}}\cdot\check{Z} &= O(r)\,\mathfrak{d}^{\leq 2}B + O(r^{-1})\,\mathfrak{d}^{\leq 1}\check{Z} + O(r^{-1})\,\mathfrak{d}^{\leq 2}\widehat{X} + O(1)\,\mathfrak{d}^{\leq 1}\check{P}\\
&\quad +r^{-1-\delta'}\text{Good}_{k_*-3}.
\end{aligned}
$$

Using again formula (6.65) to pass to the prime frame, as well as the control of $\mathcal{L}_{\mathbf{T}}\check{Z}$ provided by Proposition 6.52, we deduce

$$
\begin{aligned}
r^2\mathcal{D}'\overline{\mathcal{D}'}\cdot\check{Z} &= O(r)\,\mathfrak{d}^{\leq 2}B + O(r^{-1})(\mathfrak{d}')^{\leq 2}\check{Z} + O(r^{-1})\,\mathfrak{d}^{\leq 2}\widehat{X} + O(1)\,\mathfrak{d}^{\leq 1}\check{P}\\
&\quad +r^{-1-\delta'}\text{Good}_{k_*-3}.
\end{aligned}
$$

Using the second elliptic estimate of Corollary 6.39, we deduce for all $k\leq k_*-3$,

$$
\begin{aligned}
\|(\mathfrak{d}')^{\leq k+2}\check{Z}\|_{L^2(S(u,r))} &\lesssim r\|\,\mathfrak{d}^{\leq k+2}B\|_{L^2(S(u,r))} + r^{-1}\|(\mathfrak{d}')^{\leq k+2}\check{Z}\|_{L^2(S(u,r))}\\
&\quad +r^{-1}\|\,\mathfrak{d}^{\leq k+2}\widehat{X}\|_{L^2(S(u,r))} + \|\,\mathfrak{d}^{\leq k+1}\check{P}\|_{L^2(S(u,r))}\\
&\quad +\epsilon_0 r^{-\delta'}u^{-1-\delta_{dec}}.
\end{aligned}
$$

Using again formula (6.65), to pass back to the un-prime frame, as well as the control of $\mathcal{L}_{\mathbf{T}}\check{Z}$ provided by Proposition 6.52, we deduce for all $k\leq k_*-3$,

$$
\begin{aligned}
\|\,\mathfrak{d}^{\leq k+2}\check{Z}\|_{L^2(S(u,r))} &\lesssim r\|\,\mathfrak{d}^{\leq k+2}B\|_{L^2(S(u,r))} + r^{-1}\|\,\mathfrak{d}^{\leq k+2}\check{Z}\|_{L^2(S(u,r))}\\
&\quad +r^{-1}\|\,\mathfrak{d}^{\leq k+2}\widehat{X}\|_{L^2(S(u,r))} + \|\,\mathfrak{d}^{\leq k+1}\check{P}\|_{L^2(S(u,r))}\\
&\quad +\epsilon_0 r^{-\delta'}u^{-1-\delta_{dec}}.
\end{aligned}
$$

Since $r\geq r_0$ on ${}^{(ext)}\mathcal{M}$, we infer, for r_0 large enough, and for $k\leq k_*-3$,

$$
\begin{aligned}
\|\,\mathfrak{d}^{\leq k+2}\check{Z}\|_{L^2(S(u,r))} \lesssim{}& r\|\,\mathfrak{d}^{\leq k+2}B\|_{L^2(S(u,r))} + r^{-1}\|\,\mathfrak{d}^{\leq k+2}\widehat{X}\|_{L^2(S(u,r))}\\
&+\|\,\mathfrak{d}^{\leq k+1}\check{P}\|_{L^2(S(u,r))} + \epsilon_0 r^{-\delta'}u^{-1-\delta_{dec}}.
\end{aligned}
\tag{6.67}
$$

Step 7. So far we have established for $k\leq k_*-3$, see (6.66) and (6.67),

$$\begin{aligned}\|\not{\mathfrak{d}}^{\leq k+1}B\|_{L^2(S(u,r))} &\lesssim r^2|(\overline{\mathcal{D}}\cdot B)_{\ell=1}|+r^{-2}\|\not{\mathfrak{d}}^{\leq k}\widehat{X}\|_{L^2(S(u,r))}\\ &\quad+\epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}},\\ \|\not{\mathfrak{d}}^{\leq k+2}\check{Z}\|_{L^2(S(u,r))} &\lesssim r\|\not{\mathfrak{d}}^{\leq k+2}B\|_{L^2(S(u,r))}+r^{-1}\|\not{\mathfrak{d}}^{\leq k+2}\widehat{X}\|_{L^2(S(u,r))}\\ &\quad+\|\not{\mathfrak{d}}^{\leq k+1}\check{P}\|_{L^2(S(u,r))}+\epsilon_0 r^{-\delta'}u^{-1-\delta_{dec}}.\end{aligned}$$

Combining them we derive, for $k\leq k_*-4$,

$$\begin{aligned}\|\not{\mathfrak{d}}^{\leq k+2}\check{Z}\|_{L^2(S(u,r))} &\lesssim r^3|(\overline{\mathcal{D}}\cdot B)_{\ell=1}|+r^{-1}\|\not{\mathfrak{d}}^{\leq k+2}\widehat{X}\|_{L^2(S(u,r))}\\ &\quad+\|\not{\mathfrak{d}}^{\leq k+1}\check{P}\|_{L^2(S(u,r))}+\epsilon_0 r^{-\delta'}u^{-1-\delta_{dec}}.\end{aligned}$$

Therefore, for any $S=S(u,r)$ in ${}^{(ext)}\mathcal{M}$

(6.68)
$$\begin{aligned}r\|\not{\mathfrak{d}}^{\leq k_*-2}B\|_{L^2(S)}+\|\not{\mathfrak{d}}^{\leq k_*-2}\check{Z}\|_{L^2(S)} &\lesssim r^3|(\overline{\mathcal{D}}\cdot B)_{\ell=1}|+r^{-1}\|\not{\mathfrak{d}}^{\leq k_*-1}\widehat{X}\|_{L^2(S)}\\ &\quad+\|\not{\mathfrak{d}}^{\leq k_*-2}\check{P}\|_{L^2(S)}+\epsilon_0 r^{-\delta'}u^{-1-\delta_{dec}}.\end{aligned}$$

It thus remains to eliminate the terms in $\check{P},\widehat{X}$ on the RHS.

Step 8. Using the last equation of (6.60), i.e.

$$\begin{aligned}r\overline{\mathcal{D}}\cdot\check{Z} &= -2r\,\overline{\check{P}}+O(r^{-1})\check{Z}+O(r^{-1})\not{\mathfrak{d}}^{\leq 1}\widehat{X}+O(1)\check{P}+O(1)\not{\mathfrak{d}}^{\leq 1}B\\ &\quad+r^{-2}\mathrm{Good}_{k_*-1},\end{aligned}$$

and $r\geq r_0$ sufficiently large, we have for all $k\leq k_*-1$

$$\begin{aligned}\|\not{\mathfrak{d}}^{\leq k-1}\check{P}\|_{L^2(S(u,r))} &\lesssim r^{-1}\|\not{\mathfrak{d}}^{\leq k}\check{Z}\|_{L^2(S(u,r))}+r^{-2}\|\not{\mathfrak{d}}^{\leq k}\widehat{X}\|_{L^2(S(u,r))}\\ &\quad+r^{-1}\|\not{\mathfrak{d}}^{\leq k}B\|_{L^2(S(u,r))}+\epsilon_0 r^{-2}u^{-1-\delta_{dec}}.\end{aligned}$$

Also, recall (6.64)

$$\begin{aligned}\mathcal{D}\overline{\check{P}} &= \frac{2}{r}B+O(r^{-2})\not{\mathfrak{d}}^{\leq 1}B+O(r^{-2})\check{P}+O(r^{-3})\check{Z}+O(r^{-4})\widehat{X}\\ &\quad+r^{-3-\delta'}\mathrm{Good}_{k_*-3}.\end{aligned}$$

Together with the previous estimate, we infer for all $k\leq k_*-3$,

$$\begin{aligned}\|\not{\mathfrak{d}}^{\leq k}\check{P}\|_{L^2(S(u,r))} &\lesssim r^{-1}\|\not{\mathfrak{d}}^{\leq k}\check{Z}\|_{L^2(S(u,r))}+\|\not{\mathfrak{d}}^{\leq k}B\|_{L^2(S(u,r))}\\ &\quad+r^{-2}\|\not{\mathfrak{d}}^{\leq k}\widehat{X}\|_{L^2(S(u,r))}+\epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}.\end{aligned}$$

Thus, for all $S = S(u, r) \subset {}^{(ext)}\mathcal{M}$,

$$\begin{aligned} \| \mathfrak{d}^{\leq k_*-3} \check{P} \|_{L^2(S)} \lesssim{}& r^{-1} \| \mathfrak{d}^{\leq k_*-3} \check{Z} \|_{L^2(S)} + \| \mathfrak{d}^{\leq k_*-3} B \|_{L^2(S)} \\ &+ r^{-2} \| \mathfrak{d}^{\leq k_*-3} \widehat{X} \|_{L^2(S)} + \epsilon_0 r^{-1-\delta'} u^{-1-\delta_{dec}}. \end{aligned} \tag{6.69}$$

Step 9. So far we have established, see (6.68) and (6.69),

$$\begin{aligned} r \| \mathfrak{d}^{\leq k_*-3} B \|_{L^2(S)} + \| \mathfrak{d}^{\leq k_*-3} \check{Z} \|_{L^2(S)} \lesssim{}& r^3 |(\overline{\mathcal{D}} \cdot B)_{\ell=1}| + r^{-1} \| \mathfrak{d}^{\leq k_*-2} \widehat{X} \|_{L^2(S)} \\ &+ \| \mathfrak{d}^{\leq k_*-3} \check{P} \|_{L^2(S)} + \epsilon_0 r^{-\delta'} u^{-1-\delta_{dec}} \end{aligned}$$

and

$$\begin{aligned} \| \mathfrak{d}^{\leq k_*-3} \check{P} \|_{L^2(S)} \lesssim{}& r^{-1} \| \mathfrak{d}^{\leq k_*-3} \check{Z} \|_{L^2(S)} + \| \mathfrak{d}^{\leq k_*-3} B \|_{L^2(S)} \\ &+ r^{-2} \| \mathfrak{d}^{\leq k_*-3} \widehat{X} \|_{L^2(S)} + \epsilon_0 r^{-1-\delta'} u^{-1-\delta_{dec}}. \end{aligned}$$

Combining we deduce

$$\begin{aligned} & r \| \mathfrak{d}^{\leq k_*-3} B \|_{L^2(S)} + \| \mathfrak{d}^{\leq k_*-3} \check{Z} \|_{L^2(S)} \\ \lesssim{}& r^3 |(\overline{\mathcal{D}} \cdot B)_{\ell=1}| + r^{-1} \| \mathfrak{d}^{\leq k_*-2} \widehat{X} \|_{L^2(S)} + \| \mathfrak{d}^{\leq k_*-3} \check{P} \|_{L^2(S)} \\ & + \epsilon_0 r^{-\delta'} u^{-1-\delta_{dec}} \\ \lesssim{}& r^3 |(\overline{\mathcal{D}} \cdot B)_{\ell=1}| + r^{-1} \| \mathfrak{d}^{\leq k_*-2} \widehat{X} \|_{L^2(S)} + \epsilon_0 r^{-\delta'} u^{-1-\delta_{dec}} \\ & + r^{-1} \| \mathfrak{d}^{\leq k_*-3} \check{Z} \|_{L^2(S)} + \| \mathfrak{d}^{\leq k_*-3} B \|_{L^2(S)} + r^{-2} \| \mathfrak{d}^{\leq k_*-3} \widehat{X} \|_{L^2(S)} \\ & + \epsilon_0 r^{-1} u^{-1-\delta_{dec}}. \end{aligned}$$

Hence, for $r \geq r_0$ sufficiently large, we derive

$$\begin{aligned} & r \| \mathfrak{d}^{\leq k_*-3} B \|_{L^2(S)} + \| \mathfrak{d}^{\leq k_*-3} \check{Z} \|_{L^2(S)} \\ \lesssim{}& r^3 |(\overline{\mathcal{D}} \cdot B)_{\ell=1}| + r^{-1} \| \mathfrak{d}^{\leq k_*-2} \widehat{X} \|_{L^2(S)} + \epsilon_0 r^{-\delta'} u^{-1-\delta_{dec}}. \end{aligned}$$

Back to (6.69), we deduce

$$\| \mathfrak{d}^{\leq k_*-3} \check{P} \|_{L^2(S))} \lesssim r^2 |(\overline{\mathcal{D}} \cdot B)_{\ell=1}| + r^{-2} \| \mathfrak{d}^{\leq k_*-2} \widehat{X} \|_{L^2(S))} + \epsilon_0 r^{-1-\delta'} u^{-1-\delta_{dec}}.$$

Combining the two estimates, we infer

$$\begin{aligned} & r \Big(\| \mathfrak{d}^{\leq k_*-3} B \|_{L^2(S)} + | \mathfrak{d}^{\leq k_*-3} \check{P} \|_{L^2(S)} \Big) + \| \mathfrak{d}^{\leq k_*-3} \check{Z} \|_{L^2(S)} \\ \lesssim{}& r^3 |(\overline{\mathcal{D}} \cdot B)_{\ell=1}| + r^{-1} \| \mathfrak{d}^{\leq k_*-2} \widehat{X} \|_{L^2(S)} + \epsilon_0 r^{-\delta'} u^{-1-\delta_{dec}}. \end{aligned} \tag{6.70}$$

It thus remains to estimate $\widehat{X}$.

Step 10. Recall (6.61)

$$\overline{\mathcal{D}}\cdot\widehat{X} = \frac{2}{r}\check{Z} - 2B + O(r^{-2})\check{Z} + O(r^{-2})\widehat{X} + O(r^{-2})B + r^{-3}\mathrm{Good}_{k_*-1}.$$

Using formula (6.65) to pass to the prime frame, as well as the control of $\not{\mathcal{L}}_{\mathbf{T}}\widehat{X}$ provided by Proposition 6.52, we deduce

$$r\overline{\mathcal{D}'}\cdot\widehat{X} = 2\check{Z} - 2rB + O(r^{-1})\check{Z} + O(r^{-1})\widehat{X} + O(r^{-1})B + r^{-2}\mathrm{Good}_{k_*-2}.$$

Using the third elliptic estimate of Corollary 6.39, we infer

$$\begin{aligned}\|(\not{\mathfrak{d}})'^{\leq k_*-2}\widehat{X}\|_{L^2(S(u,r))} &\lesssim \|(\not{\mathfrak{d}})^{\leq k_*-3}\check{Z}\|_{L^2(S(u,r))} + r\|(\not{\mathfrak{d}})^{\leq k_*-3}B\|_{L^2(S(u,r))}\\ &\quad + r^{-1}\|(\not{\mathfrak{d}})^{\leq k_*-3}\widehat{X}\|_{L^2(S(u,r))} + \epsilon_0 r^{-1}u^{-1-\delta_{dec}}.\end{aligned}$$

Passing back to the un-primed frame, and using again the control of $\not{\mathcal{L}}_{\mathbf{T}}\widehat{X}$ provided by Proposition 6.52, we deduce

$$\begin{aligned}\|(\not{\mathfrak{d}})^{\leq k_*-2}\widehat{X}\|_{L^2(S(u,r))} &\lesssim \|(\not{\mathfrak{d}})^{\leq k_*-3}\check{Z}\|_{L^2(S(u,r))} + r\|(\not{\mathfrak{d}})^{\leq k_*-3}B\|_{L^2(S(u,r))}\\ &\quad + r^{-1}\|(\not{\mathfrak{d}})^{\leq k_*-3}\widehat{X}\|_{L^2(S(u,r))} + \epsilon_0 r^{-1}u^{-1-\delta_{dec}}.\end{aligned}$$

Since $r \geq r_0$ on ${}^{(ext)}\mathcal{M}$, we infer, for r_0 large enough,

$$\begin{aligned}\|\not{\mathfrak{d}}^{\leq k_*-2}\widehat{X}\|_{L^2(S(u,r))} &\lesssim \|\not{\mathfrak{d}}^{\leq k_*-3}\check{Z}\|_{L^2(S(u,r))} + r\|\not{\mathfrak{d}}^{\leq k_*-3}B\|_{L^2(S(u,r))}\\ &\quad + \epsilon_0 r^{-1}u^{-1-\delta_{dec}}.\end{aligned}\tag{6.71}$$

Step 11. We combine (6.70) with (6.71). Thus for any $S = S(u,r) \subset {}^{(ext)}\mathcal{M}$, we have

$$\begin{aligned}& r\Big(\|\not{\mathfrak{d}}^{\leq k_*-3}B\|_{L^2(S)} + |\not{\mathfrak{d}}^{\leq k_*-3}P\|_{L^2(S)}\Big) + \|\not{\mathfrak{d}}^{\leq k_*-3}\check{Z}\|_{L^2(S)}\\ \lesssim\;& r^3|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + r^{-1}\|\not{\mathfrak{d}}^{\leq k_*-2}\widehat{X}\|_{L^2(S)} + \epsilon_0 r^{-\delta'}u^{-1-\delta_{dec}}\\ \lesssim\;& r^3|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + r^{-1}\|\not{\mathfrak{d}}^{\leq k_*-3}\check{Z}\|_{L^2(S)} + \|\not{\mathfrak{d}}^{\leq k_*-3}B\|_{L^2(S)}\\ & + \epsilon_0 r^{-\delta'}u^{-1-\delta_{dec}}.\end{aligned}$$

Thus, absorbing the terms in B and $\check{Z}$ on the RHS, we obtain

$$r\Big(\|\not{\mathfrak{d}}^{\leq k_*-3}B\|_{L^2(S)} + |\not{\mathfrak{d}}^{\leq k_*-3}P\|_{L^2(S)}\Big) + \|\not{\mathfrak{d}}^{\leq k_*-3}\check{Z}\|_{L^2(S)}$$

$$\lesssim \quad r^3|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + \epsilon_0 r^{-\delta'}u^{-1-\delta_{dec}},$$

and

$$\|\,\mathfrak{d}^{\leq k_*-2}\widehat{X}\|_{L^2(S)} \quad \lesssim \quad r^3|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + \epsilon_0 r^{-\delta'}u^{-1-\delta_{dec}}.$$

We infer

$$\begin{aligned} & r\Big(\|\,\mathfrak{d}^{\leq k_*-3}B\|_{L^2(S)} + |\,\mathfrak{d}^{\leq k_*-3}P\|_{L^2(S)}\Big) + \|\,\mathfrak{d}^{\leq k_*-3}\check{Z}\|_{L^2(S)} \\ & +\|\,\mathfrak{d}^{\leq k_*-2}\widehat{X}\|_{L^2(S(u,r))} \\ \lesssim \quad & r^3|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + \epsilon_0 r^{-\delta'}u^{-1-\delta_{dec}}. \end{aligned}$$

By Sobolev, we deduce

$$\begin{aligned} & r\big(|\,\mathfrak{d}^{\leq k_*-5}B| + |\,\mathfrak{d}^{\leq k_*-5}\check{P}|\big) + |\,\mathfrak{d}^{\leq k_*-5}\widehat{X}| + |\,\mathfrak{d}^{\leq k_*-5}\check{Z}| \\ \lesssim \quad & r^2|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + \epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}. \end{aligned}$$

It remains to estimate the derivatives in ∇_4, ∇_3. In view of the equations

$$\begin{aligned} \nabla_4\widehat{X} &= -A + O(r^{-1})\widehat{X} + \Gamma_g\cdot\Gamma_g, \\ \nabla_4\check{Z} &= -B + O(r^{-1})\check{Z} + O(r^{-2})\widehat{X} + O(r^{-2})\widetilde{\operatorname{tr}X} + \Gamma_g\cdot\Gamma_g, \\ \nabla_4 B &= O(r^{-1})\,\mathfrak{d}^{\leq 1}A + O(r^{-1})B + \Gamma_g\cdot(B,A), \\ \nabla_4\left(\check{P}\right) &= O(r^{-1})\,\mathfrak{d}^{\leq 1}B + O(r^{-1})\check{P} + O(r^{-3})\widetilde{\operatorname{tr}X} + r^{-1}\Gamma_g\cdot\Gamma_g + \Gamma_b\cdot A, \end{aligned}$$

and the control of A and $\widetilde{\operatorname{tr}X}$ in (6.58), we infer from the previous estimate

$$\begin{aligned} & r\big(|(r\nabla_4,\,\mathfrak{d})^{\leq k_*-5}B| + |(r\nabla_4,\,\mathfrak{d})^{\leq k_*-5}\check{P}|\big) + |(r\nabla_4,\,\mathfrak{d})^{\leq k_*-5}\widehat{X}| \\ & +|(r\nabla_4,\,\mathfrak{d})^{\leq k_*-5}\check{Z}| \\ \lesssim \quad & r^2|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + \epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}. \end{aligned}$$

Together with the control of $\mathcal{L}_{\mathbf{T}}B$, $\mathcal{L}_{\mathbf{T}}\check{P}$, $\mathcal{L}_{\mathbf{T}}\widehat{X}$ and $\mathcal{L}_{\mathbf{T}}\check{Z}$ provided by Proposition 6.52, we deduce

$$\begin{aligned} & r\big(|(r\nabla_4,\,\mathfrak{d},\mathcal{L}_{\mathbf{T}})^{\leq k_*-5}B| + |(r\nabla_4,\,\mathfrak{d},\mathcal{L}_{\mathbf{T}})^{\leq k_*-5}\check{P}|\big) \\ & +|(r\nabla_4,\,\mathfrak{d},\mathcal{L}_{\mathbf{T}})^{\leq k_*-5}\widehat{X}| + |(r\nabla_4,\,\mathfrak{d},\mathcal{L}_{\mathbf{T}})^{\leq k_*-5}\check{Z}| \\ \lesssim \quad & r^2|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + \epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}. \end{aligned}$$

Finally, since $\nabla_3 = 2\mathcal{L}\mkern-9mu/_{\bf T} - \Big(1+O(r^{-2})\Big)\nabla_4 + O(r^{-1})\nabla + O(r^{-3}) + \Gamma_b$, see Corollary 6.25, we infer that

$$\begin{aligned} & r\big(|\mathfrak{d}^{\leq k_*-5}B| + |\mathfrak{d}^{\leq k_*-5}\check{P}|\big) + |\mathfrak{d}^{\leq k_*-5}\widehat{X}| + |\mathfrak{d}^{\leq k_*-5}\check{Z}| \\ \lesssim\ & r^2|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + \epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}} \end{aligned}$$

which is precisely the estimate (6.55) of Proposition 6.53.

Step 12. We are now ready to prove the estimate (6.56). Indeed, combining the first two equations of (6.60) with the estimate (6.55) proved in Step 11, we obtain

$$|\mathfrak{d}^{\leq k_*-5}\check{H}| + |\mathfrak{d}^{\leq k_*-5}\widetilde{\mathcal{D}\cos\theta}| \ \lesssim\ r^2|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + \epsilon_0 r^{-1}u^{-1-\delta_{dec}}$$

which is the estimate (6.56) of Proposition 6.53.

Step 13. It remains to prove the auxiliary estimate (6.57), i.e.

$$|\mathfrak{d}^{\leq k_*-2}B| \ \lesssim\ r|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + \epsilon r^{-3-\delta'}u^{-1/2-\delta_{dec}}.$$

To do that, we start with the equation

$$\mathcal{D}\widehat{\otimes}B + (Z+4H)\widehat{\otimes}B \ =\ \nabla_3 A + \left(\frac{1}{2}\mathrm{tr}\underline{X} - 4\underline{\omega}\right)A + 3\overline{P}\widehat{X}$$

from which we deduce, making use of **Ref 1–2**, for $k\leq k_*-1$,

$$\begin{aligned} \big\|\mathfrak{d}^k\mathcal{D}\widehat{\otimes}B\big\|_{L^2(S)} &\lesssim r^{-2}\|\mathfrak{d}^{\leq k}B\|_{L^2(S)} + \epsilon_0 r^{-3-\delta'}u^{-\frac{1}{2}-\delta_{dec}} + r^{-3}\big\|\mathfrak{d}^{\leq k}\widehat{X}\big\|_{L^2(S)} \\ &\lesssim r^{-2}\|\mathfrak{d}^{\leq k}B\|_{L^2(S)} + \epsilon_0 r^{-3-\delta'}u^{-\frac{1}{2}-\delta_{dec}} + \epsilon r^{-4}u^{-\frac{1}{2}-\delta_{dec}} \end{aligned}$$

and hence, for $k\leq k_*-1$,

$$\big\|\mathfrak{d}^k\mathcal{D}\widehat{\otimes}B\big\|_{L^2(S)} \ \lesssim\ r^{-2}\|\mathfrak{d}^{\leq k}B\|_{L^2(S)} + \epsilon r^{-3-\delta'}u^{-\frac{1}{2}-\delta_{dec}}.$$

Using once more (6.65) to pass the prime frame, and the estimates for $\mathcal{L}\mkern-9mu/_{\bf T}B$ provided by Proposition 6.52, we deduce, for $k\leq k_*-1$,

$$\big\|(\mathfrak{d}')^k\mathcal{D}'\widehat{\otimes}B\big\|_{L^2(S)} \ \lesssim\ r^{-2}\|\mathfrak{d}^{\leq k}B\|_{L^2(S)} + \epsilon r^{-3-\delta'}u^{-\frac{1}{2}-\delta_{dec}}.$$

Applying the first estimate of Corollary 6.39, we deduce, for all $k\leq k_*-1$,

$$\|(\not{\mathfrak{d}}')^{\leq k+1}B\|_{L^2(S)} \ \lesssim\ r\|(\not{\mathfrak{d}}')^{\leq k}\mathcal{D}'\widehat{\otimes}B\|_{L^2(S)} + r^2\Big|(\overline{\mathcal{D}'}\cdot B)_{\ell=1}\Big|$$

$$\lesssim r^{-1}\|\mathfrak{d}^{\leq k}B\|_{L^2(S)} + r^2\Big|(\overline{\mathcal{D}'}\cdot B)_{\ell=1}\Big| + \epsilon r^{-2-\delta'}u^{-\frac{1}{2}-\delta_{dec}}.$$

Passing back to the un-prime frame and absorbing the B term on the RHS, we obtain, for all $k \leq k_* - 1$,

$$\|\not{\mathfrak{d}}^{\leq k+1}B\|_{L^2(S)} \lesssim r^2\Big|(\overline{\mathcal{D}}\cdot B)_{\ell=1}\Big| + \epsilon r^{-2-\delta'}u^{-\frac{1}{2}-\delta_{dec}}.$$

Thus, by Sobolev,

$$\|\not{\mathfrak{d}}^{\leq k_*-2}B\|_{L^\infty(S)} \lesssim r\Big|(\overline{\mathcal{D}}\cdot B)_{\ell=1}\Big| + \epsilon r^{-3-\delta'}u^{-\frac{1}{2}-\delta_{dec}}.$$

The estimates for ∇_4, ∇_3 derivatives can then be recovered as in Step 11. Thus, we finally obtain

$$|\mathfrak{d}^{\leq k_*-2}B| \lesssim r|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + \epsilon r^{-3-\delta'}u^{-\frac{1}{2}-\delta_{dec}}$$

as stated in (6.57). This concludes the proof of Proposition 6.53. ☐

6.5.2. $O(u^{-1-\delta_{dec}})$ type decay estimates for B, $\check{P}$, $\widehat{X}$, $\check{Z}$, $\check{H}$, $\widecheck{\mathcal{D}\cos\theta}$

The goal of this section is to prove the following proposition.

Proposition 6.54. *We have on* $^{(ext)}\mathcal{M}$

(6.72)
$$\begin{aligned} r\big(|\mathfrak{d}^{\leq k_*-5}B| + |\mathfrak{d}^{\leq k_*-5}\check{P}|\big) + |\mathfrak{d}^{\leq k_*-5}\widehat{X}| + |\mathfrak{d}^{\leq k_*-5}\check{Z}| &\lesssim \epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}},\\ |\mathfrak{d}^{\leq k_*-5}\check{H}| + |\mathfrak{d}^{\leq k_*-5}\widecheck{\mathcal{D}\cos\theta}| &\lesssim \epsilon_0 r^{-1}u^{-1-\delta_{dec}},\\ |(\overline{\mathcal{D}}\cdot B)_{\ell=1}| &\leq \epsilon_0 r^{-4-\delta'}u^{-1-\delta_{dec}}. \end{aligned}$$

Also, we have on $^{(ext)}\mathcal{M}$

$$\begin{aligned} & r|\mathfrak{d}^{\leq k_*-6}\nabla_3 B| + r|\mathfrak{d}^{\leq k_*-6}\nabla_3\check{P}| \\ & + |\mathfrak{d}^{\leq k_*-6}\nabla_3\widehat{X}| + |\mathfrak{d}^{\leq k_*-6}\nabla_3\check{Z}| \lesssim \epsilon_0 r^{-2}u^{-1-\delta_{dec}}. \end{aligned} \tag{6.73}$$

Remark 6.55. *The above $r^{-\delta'}$ gain for B, $\check{P}$, $\widehat{X}$ and $\check{Z}$ in (6.72) will be crucial to derive $O(u^{-1-\delta_{dec}})$ type decay estimates without log-loss in r in particular for $\widecheck{\mathrm{tr}\underline{X}}$, $\underline{B}$, $\widecheck{\mathcal{D}u}$, $\widecheck{\mathbf{D}\varphi}$, $\mathcal{D}\widehat{\otimes}\mathfrak{J}$ and $\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}$.*

Proof. Recall from Proposition 6.53 that we have on $^{(ext)}\mathcal{M}$

(6.74)
$$r\big(|\mathfrak{d}^{\leq k_*-5}B|+|\mathfrak{d}^{\leq k_*-5}\check{P}|\big)+|\mathfrak{d}^{\leq k_*-5}\widehat{X}|+|\mathfrak{d}^{\leq k_*-5}\check{Z}| \lesssim r^2|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + \epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}},$$
$$|\mathfrak{d}^{\leq k_*-5}\check{H}|+|\mathfrak{d}^{\leq k_*-5}\widetilde{\mathcal{D}\cos\theta}| \lesssim r^2|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + \epsilon_0 r^{-1}u^{-1-\delta_{dec}}.$$

Thus, to conclude the control of B, $\check{P}$, $\widehat{X}$, $\check{Z}$, $\check{H}$ and $\widetilde{\mathcal{D}\cos\theta}$ in (6.72), we need to control $(\overline{\mathcal{D}}\cdot B)_{\ell=1}$, which is the focus of Steps 1 and 2 below. Then, (6.73) is proved in Step 3.

Step 1. We derive transport equations in e_4 for the following scalars $h^{(p)}$, for $p=0,+,-$,

$$h^{(p)} = \left(\int_{S(u,r)}\frac{rJ^{(p)}}{\Sigma}[\overline{\mathcal{D}}\cdot]_{ren}\left(r^4[B]_{ren}\right)\right)$$
$$= \int_{S(u,r)}\frac{rJ^{(p)}}{\Sigma}\left(\overline{\mathcal{D}}\cdot -\frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_4-\frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_3\right)\left\{r^4\left(B-\frac{3a}{2}\overline{\check{P}}\mathfrak{J}-\frac{a}{4}\overline{\mathfrak{J}}\cdot A\right)\right\}.$$

To this end, we rely on the crucial identities (6.38) and (6.39) of Proposition 6.43, which we rewrite below in the following form

(6.75)
$$\begin{aligned}
e_4(h^{(0)}) &= O(1)\mathfrak{d}^{\leq 1}\widehat{X}+O(r)\mathfrak{d}^{\leq 2}B+O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 B+O(r)\mathfrak{d}^{\leq 2}\check{P}\\
&+O(1)\mathfrak{d}^{\leq 1}\widetilde{\mathrm{tr}X}+O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 A+O(r)\mathfrak{d}^{\leq 2}A+r^4\mathfrak{d}^{\leq 1}(\Gamma_g\cdot(B,A))\\
&+r^4\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\nabla_3 A)+r^2\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_g)\\
&+O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\Big(A,B,r^{-1}\check{P}\Big),
\end{aligned}$$

and

(6.76)
$$\begin{aligned}
e_4(h^{(\pm)})\mp\frac{a}{r^2}h^{(\mp)} &= O(1)\mathfrak{d}^{\leq 1}\widehat{X}+O(r)\mathfrak{d}^{\leq 2}B+O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 B+O(r)\mathfrak{d}^{\leq 2}\check{P}\\
&+O(1)\mathfrak{d}^{\leq 1}\widetilde{\mathrm{tr}X}+O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 A+O(r)\mathfrak{d}^{\leq 2}A\\
&+r^4\mathfrak{d}^{\leq 1}(\Gamma_g\cdot(B,A))+r^4\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\nabla_3 A)+r^2\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_g)\\
&+O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\Big(A,B,r^{-1}\check{P}\Big).
\end{aligned}$$

In view of the definition of $(\overline{\mathcal{D}}\cdot B)_{\ell=1}$ and $h^{(p)}$, and since $\Sigma = r^2+O(1)$, we have

$$(h^{(0)}, h^{(+)}, h^{(-)}) = r^5(\overline{\mathcal{D}}\cdot B)_{\ell=1} + O(r^3)\mathfrak{d}^{\leq 1}(B, \check{P}, A). \tag{6.77}$$

Together with the first equation in (6.74) we deduce

$$\begin{aligned} & r\big(|\mathfrak{d}^{\leq k_*-5}B| + |\mathfrak{d}^{\leq k_*-5}\check{P}|\big) + |\mathfrak{d}^{\leq k_*-5}\widehat{X}| + |\mathfrak{d}^{\leq k_*-5}\check{Z}| \\ \lesssim\ & r^{-3}|(h^{(0)}, h^{(+)}, h^{(-)})| + O(1)\mathfrak{d}^{\leq 1}(B, \check{P}, A) + \epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}. \end{aligned}$$

Since $r\geq r_0$ on ${}^{(ext)}\mathcal{M}$, we infer, for r_0 large enough,

$$\begin{aligned} & r\big(|\mathfrak{d}^{\leq k_*-5}B| + |\mathfrak{d}^{\leq k_*-5}\check{P}|\big) + |\mathfrak{d}^{\leq k_*-5}\widehat{X}| + |\mathfrak{d}^{\leq k_*-5}\check{Z}| \\ & \lesssim r^{-3}|(h^{(0)}, h^{(+)}, h^{(-)})| + O(1)\mathfrak{d}^{\leq 1}A + \epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}. \end{aligned} \tag{6.78}$$

Similarly, from the second equation in (6.74)

$$\begin{aligned} |\mathfrak{d}^{\leq k_*-5}\check{H}| + |\mathfrak{d}^{\leq k_*-5}\widetilde{\mathcal{D}\cos\theta}| \lesssim r^{-3}|(h^{(0)}, h^{(+)}, h^{(-)})| + O(1)\mathfrak{d}^{\leq 1}A \\ + \epsilon_0 r^{-1}u^{-1-\delta_{dec}}. \end{aligned} \tag{6.79}$$

Together with (6.78) and (6.79), we deduce[145] from (6.75) and (6.76)

$$\begin{aligned} e_4\left(h^{(0)}\right) =\ & O(r^{-3})|(h^{(0)}, h^{(+)}, h^{(-)})| + O(1)\mathfrak{d}^{\leq 1}\widetilde{\mathrm{tr}X} \\ & + O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 A + O(r)\mathfrak{d}^{\leq 2}A + r^4\mathfrak{d}^{\leq 1}(\Gamma_g\cdot(B, A)) \\ & + r^4\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\nabla_3 A) + r^2\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_g) \\ & + O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\Big(A, B, r^{-1}\check{P}\Big) + O\big(\epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}\big), \\ e_4\left(h^{(+)}\right) =\ & \frac{a}{r^2}h^{(-)} + O(r^{-3})|(h^{(0)}, h^{(+)}, h^{(-)})| + O(1)\mathfrak{d}^{\leq 1}\widetilde{\mathrm{tr}X} \\ & + O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 A + O(r)\mathfrak{d}^{\leq 2}A + r^4\mathfrak{d}^{\leq 1}(\Gamma_g\cdot(B, A)) \\ & + r^4\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\nabla_3 A) + r^2\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_g) \\ & + O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\Big(A, B, r^{-1}\check{P}\Big) + O\big(\epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}\big), \end{aligned}$$

[145] To bound the terms $O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 B$ on the RHS, we also use the following consequence of Bianchi

$$\nabla_3 B = O(r^{-1})\,\not{\mathfrak{d}}^{\leq 1}\check{P} + O(r^{-1})B + O(r^{-3})\check{H} + O(r^{-4})\widetilde{\mathcal{D}\cos\theta} + r^{-1}\Gamma_b\cdot\Gamma_g.$$

and

$$\begin{aligned}
e_4\left(h^{(-)}\right) &= -\frac{a}{r^2}h^{(+)} + O(r^{-3})|(h^{(0)}, h^{(+)}, h^{(-)})| + O(1)\mathfrak{d}^{\leq 1}\widecheck{\mathrm{tr}X} \\
&\quad +O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 A + O(r)\mathfrak{d}^{\leq 2}A + r^4\mathfrak{d}^{\leq 1}(\Gamma_g\cdot(B, A)) \\
&\quad +r^4\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\nabla_3 A) + r^2\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_g) \\
&\quad +O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}\left(A, B, r^{-1}\check{P}\right) + O\big(\epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}\big).
\end{aligned}$$

Using the control of A in **Ref 2**, the bootstrap assumptions **Ref 1**, and the control of $\widecheck{\mathrm{tr}X}$ derived in Proposition 6.50, we infer

$$\begin{aligned}
e_4\left(h^{(0)}\right) &= O(r^{-3})|(h^{(0)}, h^{(+)}, h^{(-)})| + O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}B \\
&\quad +O\big(\epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}\big), \\
e_4\left(h^{(+)}\right) &= \frac{a}{r^2}h^{(-)} + O(r^{-3})|(h^{(0)}, h^{(+)}, h^{(-)})| + O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}B \\
&\quad +O\big(\epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}\big),
\end{aligned}$$

and

$$\begin{aligned}
e_4\left(h^{(-)}\right) &= -\frac{a}{r^2}h^{(+)} + O(r^{-3})|(h^{(0)}, h^{(+)}, h^{(-)})| + O\left(\frac{r^2\epsilon}{u^{\frac{1}{2}+\delta_{dec}}}\right)\mathfrak{d}^{\leq 1}B \\
&\quad +O\big(\epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}\big).
\end{aligned}$$

In order to estimate the term involving $\mathfrak{d}^{\leq 1}B$ on the RHS, we rely on the estimate (6.57) in Proposition 6.53, i.e.

$$|\mathfrak{d}^{\leq k_*-2}B| \lesssim r|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| + \epsilon r^{-3-\delta'}u^{-1/2-\delta_{dec}}.$$

Together with (6.77), we deduce

$$\begin{aligned}
e_4\left(h^{(p)}\right) =& O(r^{-2})|(h^{(0)}, h^{(+)}, h^{(-)})| \\
&+ O\big(\epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}\big), \quad p = 0, +, -.
\end{aligned} \tag{6.80}$$

Step 2. Using the estimate

$$\sup_{\Sigma_*} r^5 u^{1+\delta_{dec}}\Big|([\mathcal{D}]_{ren}\cdot[B]_{ren})_{\ell=1}\Big| \lesssim \epsilon_0$$

of Proposition 6.48, and recalling that

$$h^{(p)} = \int_{S(u,r)} \frac{rJ^{(p)}}{\Sigma}[\overline{\mathcal{D}}\cdot]_{ren}\left(r^4[B]_{ren}\right),$$

we easily deduce on Σ_*, using also $\Sigma = r^2 + O(1)$ and the definition of $\ell = 1$ modes,

$$|(h^{(0)}, h^{(+)}, h^{(-)})| \lesssim \frac{\epsilon_0}{u^{1+\delta_{dec}}}.$$

Thus, integrating the transport equations in e_4 of (6.80) from the last slice Σ_*, we obtain on ${}^{(ext)}\mathcal{M}$

$$|(h^{(0)}, h^{(+)}, h^{(-)})| \lesssim \frac{\epsilon_0}{u^{1+\delta_{dec}}}.$$

Plugging these in (6.78) and (6.79), we deduce

$$\begin{aligned} r\big(|\mathfrak{d}^{\leq k_*-5}B| + |\mathfrak{d}^{\leq k_*-5}\check{P}|\big) + |\mathfrak{d}^{\leq k_*-5}\widehat{X}| + |\mathfrak{d}^{\leq k_*-5}\check{Z}| \lesssim & O(1)\mathfrak{d}^{\leq 1}A \\ & + O\big(\epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}\big), \\ |\mathfrak{d}^{\leq k_*-5}\check{H}| + |\mathfrak{d}^{\leq k_*-5}\widetilde{\mathcal{D}\cos\theta}| \lesssim & O(1)\mathfrak{d}^{\leq 1}A \\ & + O\big(\epsilon_0 r^{-1}u^{-1-\delta_{dec}}\big). \end{aligned}$$

Also, from (6.77),

$$|(\overline{\mathcal{D}}\cdot B)_{\ell=1}| \lesssim r^{-2}|\mathfrak{d}^{\leq 1}(B, \check{P}, A)| + \frac{\epsilon_0}{r^5 u^{1+\delta_{dec}}}.$$

Thanks to the control of A in **Ref 2**, we thus have obtained on ${}^{(ext)}\mathcal{M}$

$$\begin{aligned} r\big(|\mathfrak{d}^{\leq k_*-5}B| + |\mathfrak{d}^{\leq k_*-5}\check{P}|\big) + |\mathfrak{d}^{\leq k_*-5}\widehat{X}| + |\mathfrak{d}^{\leq k_*-5}\check{Z}| &\lesssim \epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}, \\ |\mathfrak{d}^{\leq k_*-5}\check{H}| + |\mathfrak{d}^{\leq k_*-5}\widetilde{\mathcal{D}\cos\theta}| &\lesssim \epsilon_0 r^{-1}u^{-1-\delta_{dec}}, \\ |(\overline{\mathcal{D}}\cdot B)_{\ell=1}| &\lesssim \epsilon_0 r^{-4-\delta'}u^{-1-\delta_{dec}}, \end{aligned}$$

as stated in (6.72).

Step 3. It remains to prove the estimate (6.73) for $\nabla_3 B$, $\nabla_3\check{P}$, $\nabla_3\widehat{X}$ and $\nabla_3\check{Z}$. This follows easily by relying on the formula, see Corollary 6.25,

$$\nabla_3 = 2\not{\mathcal{L}}_{\mathbf{T}} + O(r^{-1})\mathfrak{d}^{\leq 1} + \Gamma_b,$$

the control of $\not{\mathcal{L}}_{\mathbf{T}}(B, \check{P}, \widehat{X}, \check{Z})$ provided by Proposition 6.52, and the estimates (6.72) derived in Step 2 above. This concludes the proof of Proposition 6.54. ◻

6.5.3. $O(u^{-\frac{1}{2}-\delta_{dec}})$ type decay estimates for B, $\check{P}$, $\widehat{X}$ and $\check{Z}$ The goal of this section is to prove the following proposition.

Proposition 6.56. *We have on* ${}^{(ext)}\mathcal{M}$

$$
\begin{aligned}
r|\mathfrak{d}^{\leq k_*-6}\check{P}| + |\mathfrak{d}^{\leq k_*-6}\widehat{X}| + |\mathfrak{d}^{\leq k_*-6}\check{Z}| &\lesssim \epsilon_0 r^{-2}u^{-1/2-\delta_{dec}},\\
|\mathfrak{d}^{\leq k_*-6}B| &\lesssim \epsilon_0 \min\left\{r^{-\frac{7}{2}-\delta_{dec}},\, r^{-3-\delta'}u^{-1/2-\delta_{dec}}\right\}.
\end{aligned}
$$

Proof. As mentioned in Section 6.4.3, the proof of these estimates is much easier than the ones derived so far. We sketch the main steps below.

Step 1. To get the estimate for $\widehat{X}$ we apply Proposition 6.47 to the equation

$$
\nabla_4\widehat{X} + \frac{2r}{|q|^2}\widehat{X} = -A + \Gamma_g\cdot\Gamma_g.
$$

Thus, using the assumptions **Ref 1–2**, and the estimates for $\widehat{X}$ on Σ_* in Proposition 6.48, we derive, for all $r_0 \leq r \leq r_*$ and $1 \leq u \leq u_*$,

$$
r^2\big|\mathfrak{d}^{\leq k_*}\widehat{X}(r,u)\big| \lesssim r_*^2\big|\mathfrak{d}^{\leq k_*}\widehat{X}(r_*,u)\big| + \epsilon_0 r^{-\delta'}u^{-1/2-\delta_{dec}} \lesssim \epsilon_0 u^{-1/2-\delta_{dec}}
$$

as stated.

Step 2. To get the estimate for B we proceed exactly as in the proof for the auxiliary estimate (6.57) of Proposition 6.53. More precisely we start with

$$
\mathcal{D}\widehat{\otimes}B + (Z+4H)\widehat{\otimes}B = \nabla_3 A + \left(\frac{1}{2}\mathrm{tr}\underline{X} - 4\underline{\omega}\right)A + 3\overline{P}\widehat{X}
$$

from which we deduce, making use of **Ref 1–2**, and the above estimate for $\widehat{X}$, for $k \leq k_*$,

$$
\begin{aligned}
\|\not{\mathfrak{d}}^k\mathcal{D}\widehat{\otimes}B\|_{L^2(S)} &\lesssim r^{-2}\|\not{\mathfrak{d}}^{\leq k}B\|_{L^2(S)} + \epsilon_0\min\left\{r^{-\frac{7}{2}-\delta_{dec}},\, r^{-3-\delta'}u^{-1/2-\delta_{dec}}\right\}\\
&\quad + r^{-3}\|\not{\mathfrak{d}}^{\leq k}\widehat{X}\|_{L^2(S)}\\
&\lesssim r^{-2}\|\not{\mathfrak{d}}^{\leq k}B\|_{L^2(S)} + \epsilon_0\min\left\{r^{-\frac{7}{2}-\delta_{dec}},\, r^{-3-\delta'}u^{-1/2-\delta_{dec}}\right\}\\
&\quad + \epsilon_0 r^{-4}u^{-1/2-\delta_{dec}},
\end{aligned}
$$

which we write in the form

$$
\|\not{\mathfrak{d}}^k\mathcal{D}\widehat{\otimes}B\|_{L^2(S)} \lesssim r^{-2}\|\not{\mathfrak{d}}^{\leq k}B\|_{L^2(S)} + \epsilon_0\min\left\{r^{-\frac{7}{2}-\delta_{dec}},\, r^{-3-\delta'}u^{-1/2-\delta_{dec}}\right\}.
$$

We then proceed exactly as for the proof of (6.57) and deduce, for all $r_0 \leq r \leq r_*$ and $1 \leq u \leq u_*$,

$$|\mathfrak{d}^{\leq k_*-4} B| \lesssim \epsilon_0 \min\{r^{-\frac{7}{2}-\delta_{dec}}, r^{-3-\delta'} u^{-1/2-\delta_{dec}}\}$$

as stated.

Step 3. To get the estimate for $\check{Z}$, we apply Proposition 6.47 to the equation

$$\nabla_4 \check{Z} + \frac{2}{q}\check{Z} = -B + O(r^{-2})\widehat{X} + O(r^{-2})\widetilde{\mathrm{tr} X} + \Gamma_g \cdot \Gamma_g.$$

Using the estimate already derived for $\widehat{X}, B, \widetilde{\mathrm{tr} X}$, assumption **Ref 1**, and the estimate on the last slice for $\check{Z}$, we easily derive, for all $r_0 \leq r \leq r_*$ and $1 \leq u \leq u_*$,

$$r^2|\mathfrak{d}^{\leq k_*-4}\check{Z}(u,r)| \lesssim r_*^2|\mathfrak{d}^{\leq k_*-4}\check{Z}(u,r_*)| + \epsilon_0 r^{-\delta'} u^{-1/2-\delta_{dec}} \lesssim \epsilon_0 u^{-1/2-\delta_{dec}}$$

as stated.

Step 4. To get the estimate for $\check{P}$, we apply Proposition 6.47 to the equation

$$\nabla_4\left(\check{P}\right) + \frac{3}{q}\check{P} = O(r^{-1})\,\mathfrak{d}\!\!/^{\leq 1} B + O(r^{-3})\widetilde{\mathrm{tr} X} + r^{-1}\Gamma_g \cdot \Gamma_g + \Gamma_b \cdot A.$$

Using the estimates already derived for B and $\widetilde{\mathrm{tr} X}$, assumption **Ref 1**, and the estimate for $\check{P}$ on the last slice, we easily deduce, for all $r_0 \leq r \leq r_*$ and $1 \leq u \leq u_*$,

$$r^3|\mathfrak{d}^{\leq k_*-5}\check{P}(u,r)| \lesssim r_*^3|\mathfrak{d}^{\leq k_*-5}\check{P}(u,r_*)| + \epsilon_0 r^{-\delta'} u^{-1/2-\delta_{dec}} \lesssim \epsilon_0 u^{-1/2-\delta_{dec}}$$

as stated. This concludes the proof of Proposition 6.56. ☐

6.6. End of the proof of Proposition 6.49

Remark 6.57. *We summarize the estimates proved so far:*

1. *According to Proposition 6.50, we have*

$$|\mathfrak{d}^{\leq k_*}\widetilde{trX}| \lesssim \epsilon_0 r^{-2} u^{-1-\delta_{dec}}.$$

2. *According to Proposition 6.54, we have*

$$r\big(|\mathfrak{d}^{\leq k_*-5}B|+|\mathfrak{d}^{\leq k_*-5}\check{P}|\big)+|\mathfrak{d}^{\leq k_*-5}\widehat{X}|+|\mathfrak{d}^{\leq k_*-5}\check{Z}| \lesssim \epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}},$$
$$|\mathfrak{d}^{\leq k_*-5}\check{H}|+|\mathfrak{d}^{\leq k_*-5}\widetilde{\mathcal{D}\cos\theta}| \lesssim \epsilon_0 r^{-1}u^{-1-\delta_{dec}},$$

and

$$r|\mathfrak{d}^{\leq k_*-6}\nabla_3 B|+r|\mathfrak{d}^{\leq k_*-6}\nabla_3\check{P}|+|\mathfrak{d}^{\leq k_*-6}\nabla_3\widehat{X}|+|\mathfrak{d}^{\leq k_*-6}\nabla_3\check{Z}| \lesssim \epsilon_0 r^{-2}u^{-1-\delta_{dec}}.$$

3. *According to Proposition 6.56, we have*

$$\begin{aligned} r|\mathfrak{d}^{\leq k_*-6}\check{P}|+|\mathfrak{d}^{\leq k_*-6}\widehat{X}| & \\ +|\mathfrak{d}^{\leq k_*-6}\check{Z}| &\lesssim \epsilon_0 r^{-2}u^{-1/2-\delta_{dec}},\\ |\mathfrak{d}^{\leq k_*-6}B| &\lesssim \epsilon_0 \min\left\{r^{-\frac{7}{2}-\delta_{dec}},\, r^{-3-\delta'}u^{-1/2-\delta_{dec}}\right\}. \end{aligned}$$

This provides the desired Γ_g estimates of Theorem M4 for the quantities

$$\widetilde{trX},\quad \widehat{X},\quad \check{Z},\quad rB,\quad r\check{P},$$

as well as the desired Γ_b estimates for the quantities

$$\check{H},\quad \widetilde{\mathcal{D}\cos\theta}.$$

We now recover the remaining components of the outgoing PG structure of ${}^{(ext)}\mathcal{M}$. They are stated in the following proposition.

Proposition 6.58. *The following estimates hold true on ${}^{(ext)}\mathcal{M}$*

$$\begin{aligned} |\mathfrak{d}^{\leq k_*-7}\widetilde{tr\underline{X}}|+|\mathfrak{d}^{\leq k_*-6}\widehat{\underline{X}}|+|\mathfrak{d}^{\leq k_*-7}\underline{\Xi}| & \\ +|\mathfrak{d}^{\leq k_*-6}\check{\underline{\omega}}|+|\mathfrak{d}^{\leq k_*-7}\underline{A}| &\lesssim \epsilon_0 r^{-1}u^{-1-\delta_{dec}},\\ |\mathfrak{d}^{\leq k_*-6}\underline{B}|+|\mathfrak{d}^{\leq k_*-8}\nabla_3\widetilde{tr\underline{X}}| &\lesssim \epsilon_0 r^{-2}u^{-1-\delta_{dec}},\\ |\mathfrak{d}^{\leq k_*-7}\widetilde{tr\underline{X}}| &\lesssim \epsilon_0 r^{-2}u^{-1/2-\delta_{dec}}, \end{aligned} \tag{6.81}$$

$$\begin{aligned} |\mathfrak{d}^{\leq k_*-6}\widetilde{e_3(r)}|+|\mathfrak{d}^{\leq k_*-6}\widetilde{e_3(u)}| &\lesssim \epsilon_0 u^{-1-\delta_{dec}},\\ |\mathfrak{d}^{\leq k_*-6}e_3(\cos\theta)|+|\mathfrak{d}^{\leq k_*-6}\widetilde{\mathcal{D}u}| &\lesssim \epsilon_0 r^{-1}u^{-1-\delta_{dec}}, \end{aligned} \tag{6.82}$$

$$|\mathfrak{d}^{\leq k_*-6}\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}|+|\mathfrak{d}^{\leq k_*-6}\mathcal{D}\widehat{\otimes}\mathfrak{J}|+|\mathfrak{d}^{\leq k_*-6}\widetilde{\nabla_3\mathfrak{J}}| \lesssim \epsilon_0 r^{-2}u^{-1-\delta_{dec}}, \tag{6.83}$$

$$|\mathfrak{d}^{\leq k_*-6}\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm}|+|\mathfrak{d}^{\leq k_*-6}\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm|+|\mathfrak{d}^{\leq k_*-6}\widetilde{\nabla_3\mathfrak{J}_\pm}| \lesssim \epsilon_0 r^{-2}u^{-1-\delta_{dec}}, \tag{6.84}$$

and

$$(6.85)\qquad |\mathfrak{d}^{\leq k_*-6}\widetilde{\mathcal{D}(J^{(\pm)})}| + |\mathfrak{d}^{\leq k_*-6}\widetilde{e_3(J^{(\pm)})}| \lesssim \epsilon_0 r^{-1}u^{-1-\delta_{dec}}.$$

Proof. We proceed in steps as follows.

Step 1. We first derive the estimates for $\mathcal{D}\widehat{\otimes}\mathfrak{J}$, $\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}$, $\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm$ and $\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm}$ in (6.83), (6.84), with the help of their transport equations, see Lemma 6.17, and their estimates on the last slice Σ_*, see Proposition 6.48. Recall from Lemma 6.17 that we have

$$\begin{aligned}
\nabla_4(\mathcal{D}\widehat{\otimes}\mathfrak{J}) + \frac{2}{q}\mathcal{D}\widehat{\otimes}\mathfrak{J} &= O(r^{-1})B + O(r^{-2})\widetilde{\mathrm{tr}X} + O(r^{-2})\widehat{X}\\
&\quad + O(r^{-2})\check{Z} + O(r^{-3})\widetilde{\mathcal{D}(\cos\theta)} + r^{-1}\Gamma_b\cdot\Gamma_g,\\
\nabla_4(\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}) + \Re\left(\frac{2}{q}\right)\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}} &= O(r^{-1})B + O(r^{-2})\widetilde{\mathrm{tr}X} + O(r^{-2})\widehat{X}\\
&\quad + O(r^{-2})\check{Z} + O(r^{-3})\widetilde{\mathcal{D}(\cos\theta)} + r^{-1}\Gamma_b\cdot\Gamma_g,\\
\nabla_4(\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm) + \frac{2}{q}\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm &= O(r^{-1})B + O(r^{-2})\widetilde{\mathrm{tr}X} + O(r^{-2})\widehat{X}\\
&\quad + O(r^{-2})\check{Z} + O(r^{-3})\widetilde{\mathcal{D}(\cos\theta)} + r^{-1}\Gamma_b\cdot\Gamma_g,\\
\nabla_4(\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm}) + \Re\left(\frac{2}{q}\right)\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm} &= O(r^{-1})B + O(r^{-2})\widetilde{\mathrm{tr}X} + O(r^{-2})\widehat{X}\\
&\quad + O(r^{-2})\check{Z} + O(r^{-3})\widetilde{\mathcal{D}(\cos\theta)} + r^{-1}\Gamma_b\cdot\Gamma_g.
\end{aligned}$$

Let F denote the schematic right hand side of these transport equations, i.e.

$$\begin{aligned}
F &= O(r^{-1})B + O(r^{-2})\widetilde{\mathrm{tr}X} + O(r^{-2})\widehat{X} + O(r^{-2})\check{Z} + O(r^{-3})\widetilde{\mathcal{D}(\cos\theta)}\\
&\quad + r^{-1}\Gamma_b\cdot\Gamma_g.
\end{aligned}$$

In view of the estimates already derived for B, $\widetilde{\mathrm{tr}X}, \check{Z}, \widetilde{\mathcal{D}(\cos\theta)}$, see Remark 6.57, and using **Ref 1** for the nonlinear terms, we have on ${}^{(ext)}\mathcal{M}$

$$\|\mathfrak{d}^{\leq k_*-6}F\|_\infty(u,r) \lesssim \epsilon_0 r^{-3-\delta'}u^{-1-\delta_{dec}} + \epsilon_0 r^{-4}u^{-1-\delta_{dec}}.$$

Thus, applying Proposition 6.47 and making use of the estimates on the last slice Σ_*, we deduce

$$r^2\left|\mathfrak{d}^{\leq k_*-6}(\mathcal{D}\widehat{\otimes}\mathfrak{J})\right| + r^2\left|\mathfrak{d}^{\leq k_*-6}(\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}})\right| \lesssim \epsilon_0 u^{-1-\delta_{dec}},$$

$$r^2\Big|\mathfrak{d}^{\leq k_*-6}(\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm)\Big| + r^2\Big|\mathfrak{d}^{\leq k_*-6}(\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm})\Big| \lesssim \epsilon_0 u^{-1-\delta_{dec}},$$

as stated in (6.83), (6.84).

Step 2. Next, we estimate $\widetilde{\mathrm{tr}\underline{X}}$ with the help of the following equation, see Lemma 6.15,

$$\nabla_4\widetilde{\mathrm{tr}\underline{X}} + \frac{1}{q}\widetilde{\mathrm{tr}\underline{X}} = F,$$

where

$$F = -\mathcal{D}\cdot\check{\overline{Z}} + 2\check{\overline{P}} + O(r^{-2})\check{Z} + O(r^{-1})\widetilde{\mathrm{tr}X} + O(r^{-1})\widetilde{\mathcal{D}\cdot\overline{\mathfrak{J}}}$$
$$+O(r^{-3})\mathcal{D}\widetilde{(\cos\theta)} + \Gamma_b\cdot\Gamma_g.$$

In view of the estimates already derived for $\check{Z}$, $\widetilde{\mathrm{tr}X}$, $\mathcal{D}\widetilde{(\cos\theta)}$, $\check{P}$, see Remark 6.57, and the estimate for $\widetilde{\mathcal{D}\cdot\overline{\mathfrak{J}}}$ obtained in Step 1 above, we have on ${}^{(ext)}\mathcal{M}$

$$\big|\mathfrak{d}^{\leq k_*-7}F\big| \lesssim \epsilon_0\min\{r^{-3}u^{-1/2-\delta_{dec}}, r^{-2-\delta'}u^{-1-\delta_{dec}}\}.$$

Thus, applying Proposition 6.47, we infer on ${}^{(ext)}\mathcal{M}$

$$r\|\mathfrak{d}^{\leq k_*-7}\widetilde{\mathrm{tr}\underline{X}}\|_\infty(u,r) \lesssim r_*\|\mathfrak{d}^{\leq k_*-7}\widetilde{\mathrm{tr}\underline{X}}\|_\infty(u,r_*)$$
$$+\epsilon_0\min\left\{r^{-1}u^{-1/2-\delta_{dec}}, r^{-\delta'}u^{-1-\delta_{dec}}\right\}.$$

Hence, making use of the estimates for $\widetilde{\mathrm{tr}\underline{X}}$ on Σ_*, we derive

$$|\mathfrak{d}^{\leq k_*-7}\widetilde{\mathrm{tr}\underline{X}}| \lesssim \epsilon_0\min\left\{r^{-2}u^{-1/2-\delta_{dec}}, r^{-1}u^{-1-\delta_{dec}}\right\}$$

as stated in (6.81).

Step 3. Next, we derive the desired estimate for $\widehat{\underline{X}}$ using the following equation

$$\nabla_4\widehat{\underline{X}} + \frac{1}{q}\widehat{\underline{X}} = -\frac{1}{2}\mathcal{D}\widehat{\otimes}\check{Z} + O(r^{-2})\check{Z} + O(r^{-1})\widehat{X} + O(r^{-1})\mathcal{D}\widehat{\otimes}\mathfrak{J}$$
$$+O(r^{-3})\mathcal{D}\widetilde{(\cos\theta)} + \Gamma_b\cdot\Gamma_g.$$

Denoting the right hand side by F and using the estimates already derived for $\check{Z}$, $\widetilde{\mathrm{tr}X}$, $\mathcal{D}\widetilde{(\cos\theta)}$, $\check{P}$, see Remark 6.57, and the estimate for $\mathcal{D}\widehat{\otimes}\mathfrak{J}$ obtained in Step 1 above, we have on ${}^{(ext)}\mathcal{M}$

$$\big|\mathfrak{d}^{\leq k_*-6}F\big| \quad \lesssim \quad \epsilon_0 r^{-2-\delta'}u^{-1-\delta_{dec}}.$$

Thus, applying Proposition 6.47, we infer on ${}^{(ext)}\mathcal{M}$

$$r\big\|\mathfrak{d}^{\leq k_*-6}\widehat{\underline{X}}\big\|_\infty(u,r) \quad \lesssim \quad r_*\big\|\mathfrak{d}^{\leq k_*-6}\widehat{\underline{X}}\big\|_\infty(u,r_*)+\epsilon_0 u^{-1-\delta_{dec}}.$$

Hence, using the estimates on the last slice Σ_* for $\widehat{\underline{X}}$, we obtain

$$\big|\mathfrak{d}^{\leq k_*-6}\widehat{\underline{X}}\big| \quad \lesssim \quad \epsilon_0 r^{-1}u^{-1-\delta_{dec}}$$

as stated in (6.81).

Step 4. Next, we estimate $\underline{\check{\omega}}$ using the following equation, see Lemma 6.15,

$$\nabla_4(\underline{\check{\omega}}) \quad = \quad \Re(\check{P})+O(r^{-2})\check{Z}+O(r^{-2})\check{H}+\Gamma_b\cdot\Gamma_g.$$

Denoting the right hand side by F and using the estimates already derived for $\check{Z}$, $\check{P}$, $\check{H}$, see Remark 6.57, we have on ${}^{(ext)}\mathcal{M}$

$$\big|\mathfrak{d}^{\leq k_*-6}F\big| \quad \lesssim \quad \epsilon_0 r^{-2-\delta'}u^{-1-\delta_{dec}}.$$

Hence, applying Proposition 6.47, and making use of the estimates on the last slice Σ_* for $\underline{\check{\omega}}$, we deduce

$$\big|\mathfrak{d}^{\leq k_*-6}\underline{\check{\omega}}\big|(r,u) \quad \lesssim \epsilon_0 r_*^{-1}u^{-1-\delta_{dec}}+\epsilon_0 r^{-1-\delta'}u^{-1-\delta_{dec}}, \tag{6.86}$$

which implies the estimate for $\underline{\check{\omega}}$ in (6.81).

Step 5. Next, we estimate $\underline{\Xi}$ using the following equation, see Lemma 6.15,

$$\begin{aligned}\nabla_4\underline{\Xi}+\frac{1}{q}\underline{\Xi} \quad &= \quad O(r^{-1})\,\not{\mathfrak{d}}^{\leq 1}(\underline{\check{\omega}})+O(r^{-2})\check{Z}+O(r^{-2})\check{H}+O(r^{-2})\widetilde{\mathrm{tr}\underline{X}}\\ &\quad +O(r^{-3})\widetilde{\mathcal{D}(\cos\theta)}+\Gamma_b\cdot\Big(\underline{\check{\omega}},\Gamma_g\Big).\end{aligned}$$

Denoting the right hand side by F and using the estimates already derived for $\check{Z}$, $\check{H}$, $\widetilde{\mathcal{D}(\cos\theta)}$, see Remark 6.57, as well as the estimate (6.86) for $\underline{\check{\omega}}$, and the estimate for $\widetilde{\mathrm{tr}\underline{X}}$ of Step 2 above, we derive on ${}^{(ext)}\mathcal{M}$

$$\big|\mathfrak{d}^{\leq k_*-7}F\big| \quad \lesssim \quad \epsilon_0 r^{-1}r_*^{-1}u^{-1-\delta_{dec}}+\epsilon_0 r^{-2-\delta'}u^{-1-\delta_{dec}}.$$

Applying Proposition 6.47, and making use of the estimate for $\underline{\Xi}$ on the last slice, we deduce

$$\begin{aligned} r\big|\mathfrak{d}^{\leq k_*-7}\underline{\Xi}\big|(r,u) &\lesssim r_*\big|\mathfrak{d}^{\leq k_*-7}\underline{\Xi}\big|(r_*,u) + \frac{(r_*-r)\epsilon_0}{r_* u^{1+\delta_{dec}}} + \frac{\epsilon_0}{u^{1+\delta_{dec}}} \\ &\lesssim \epsilon_0 u^{-1-\delta_{dec}}. \end{aligned}$$

Consequently, we obtain

$$\big|\mathfrak{d}^{\leq k_*-7}\underline{\Xi}\big|(r,u) \lesssim \epsilon_0 r^{-1} u^{-1-\delta_{dec}} \tag{6.87}$$

as stated.

Step 6. To estimate $\underline{B}$, we make use of the following equation

$$\begin{aligned} \nabla_4\underline{B} + \frac{2}{q}\underline{B} &= -\mathcal{D}\left(\check{P}\right) + O(r^{-2})\check{P} + O(r^{-3})\check{Z} + O(r^{-4})\widetilde{\mathcal{D}(\cos\theta)} \\ &\quad + r^{-1}\Gamma_b\cdot\Gamma_g. \end{aligned}$$

Denoting the right hand side by F and making use of the estimates already derived for $\check{P}$, $\check{Z}$, $\widetilde{\mathcal{D}(\cos\theta)}$, see Remark 6.57, we have on $^{(ext)}\mathcal{M}$

$$\big|\mathfrak{d}^{\leq k_*-6}F\big| \lesssim \epsilon_0 r^{-3-\delta'} u^{-1-\delta_{dec}}.$$

Thus, integrating with the help of Proposition 6.47, and making use of the estimate for $\underline{B}$ on the last slice, we deduce

$$\big|\mathfrak{d}^{\leq k_*-6}\underline{B}\big| \lesssim \epsilon_0 r^{-2} u^{-1-\delta_{dec}}$$

as stated in (6.81).

Step 7. To estimate $\underline{A}$, we make use of the following equation, see Proposition 6.9,

$$\nabla_4\underline{A} + \frac{1}{2}\mathcal{D}\widehat{\otimes}\underline{B} = -\frac{1}{2}\overline{\mathrm{tr}X}\,\underline{A} + \frac{5}{2}Z\widehat{\otimes}\underline{B} - 3P\underline{\widehat{X}},$$

which yields

$$\nabla_4\underline{A} + \frac{1}{q}\underline{A} = O(r^{-1})\,\not{\mathfrak{d}}^{\leq 1}\underline{B} + O(r^{-3})\underline{\widehat{X}} + \Gamma_g\cdot\Gamma_b.$$

Denoting the right hand side by F and making use of the estimates already derived for $\underline{B}$ and $\widehat{\underline{X}}$, we have on ${}^{(ext)}\mathcal{M}$

$$\left|\mathfrak{d}^{\leq k_*-7}F\right| \lesssim \epsilon_0 r^{-3}u^{-1-\delta_{dec}}.$$

Thus, integrating with the help of Proposition 6.47, and making use of the estimate for $\underline{A}$ on the last slice, we deduce

$$\left|\mathfrak{d}^{\leq k_*-7}\underline{A}\right| \lesssim \epsilon_0 r^{-1}u^{-1-\delta_{dec}}$$

as stated in (6.81).

Step 8. Next, we estimate $\widetilde{e_3(r)}$ using the following equation, see Lemma 6.16,

$$e_4\left(\widetilde{e_3(r)}\right) = -2\underline{\check{\omega}}.$$

Integrating, and using the estimate (6.86) for $\underline{\check{\omega}}$ derived above, we deduce

$$\begin{aligned}\left|\mathfrak{d}^{\leq k_*-6}\left(\widetilde{e_3(r)}\right)(r,u)\right| &\lesssim \left|\mathfrak{d}^{\leq k_*-6}\left(\widetilde{e_3(r)}\right)\right|(r_*,u)+\frac{(r_*-r)\epsilon_0}{r_*u^{1+\delta_{dec}}}+\frac{\epsilon_0}{u^{1+\delta_{dec}}}\\ &\lesssim \frac{\epsilon_0}{u^{1+\delta_{dec}}}.\end{aligned}$$

Hence

$$\left|\mathfrak{d}^{\leq k_*-6}\left(\widetilde{e_3(r)}\right)\right| \lesssim \epsilon_0 u^{-1-\delta_{dec}}$$

as stated in (6.82).

Step 9. Next, we derive estimates for $\widetilde{\mathcal{D}\cos\theta}$, $\widetilde{e_3(u)}$, and $e_3(\cos\theta)$ by relying on the following equations, see Lemma 6.16,

$$\begin{aligned}\nabla_4\widetilde{\mathcal{D}u}+\frac{1}{q}\widetilde{\mathcal{D}u} &= O(r^{-1})\widetilde{\mathrm{tr}X}+O(r^{-1})\widehat{X}+\Gamma_b\cdot\Gamma_g,\\ e_4\left(\widetilde{e_3(u)}\right) &= O(r^{-1})\check{H}+O(r^{-1})\check{Z}+O(r^{-2})\widetilde{\mathcal{D}u}+\Gamma_b\cdot\Gamma_b,\\ e_4(e_3(\cos\theta)) &= O(r^{-1})\check{H}+O(r^{-1})\check{Z}+O(r^{-2})\widetilde{\mathcal{D}\cos\theta}+\Gamma_b\cdot\Gamma_b.\end{aligned}$$

Note that the right side F_1 of the first equation verifies

$$\left|\mathfrak{d}^{\leq k_*-6}F_1\right| \lesssim \epsilon_0 r^{-2-\delta'}u^{-1-\delta_{dec}}.$$

Proceeding exactly as before, we infer

$$\begin{aligned} r\|\mathfrak{d}^{\leq k_*-6}\widecheck{\mathcal{D}u}\|_{L^\infty}(u,r) &\lesssim r_*\|\mathfrak{d}^{\leq k_*-6}\widecheck{\mathcal{D}u}\|_{L^\infty}(u,r_*)+\epsilon_0 r^{-\delta'}u^{-1-\delta_{dec}}\\ &\lesssim \epsilon_0 u^{-1-\delta_{dec}}. \end{aligned}$$

Thus,

$$\left|\mathfrak{d}^{\leq k_*-6}\widecheck{\mathcal{D}u}\right| \lesssim \epsilon_0 r^{-1}u^{-1-\delta_{dec}}$$

as stated in (6.82).

Also, the right hand side F_2 of the second equation verifies

$$\left|\mathfrak{d}^{\leq k_*-6}F_2\right| \lesssim \epsilon_0 r^{-2}u^{-1-\delta_{dec}}.$$

Hence, by integration, in the same manner, we obtain

$$\|\mathfrak{d}^{\leq k_*-6}\widecheck{e_3(u)}\|_{L^\infty}(u,r) \lesssim \|\mathfrak{d}^{\leq k_*-6}\widecheck{e_3(u)}\|_{L^\infty}(u,r_*)+\epsilon_0 r^{-1}u^{-1-\delta_{dec}}.$$

Thus, according to the estimate on the last slice for $\widecheck{e_3(u)}$, see Proposition 6.48, we deduce

$$\left|\mathfrak{d}^{\leq k_*-6}\widecheck{e_3(u)}\right| \lesssim \epsilon_0 u^{-1-\delta_{dec}}$$

as stated in (6.82). The estimate (6.82) for $e_3(\cos\theta)$ follows exactly in the same manner. This ends the proof of the estimates (6.82).

Step 10. Next, we estimate $\nabla_3\widecheck{\mathrm{tr}\underline{X}}$. In view of Proposition 2.8, we have

$$\nabla_3\mathrm{tr}\underline{X} = -\frac{1}{2}(\mathrm{tr}\underline{X})^2-2\underline{\omega}\mathrm{tr}\underline{X}+O(r^{-1})\,\mathfrak{d}\!\!/^{\leq 1}\underline{\Xi}+\Gamma_b\cdot\Gamma_b$$

which implies

$$\begin{aligned} \nabla_3\widecheck{\mathrm{tr}\underline{X}} &= O(r^{-1})\,\mathfrak{d}\!\!/^{\leq 1}\underline{\Xi}+O(r^{-1})\widecheck{\mathrm{tr}\underline{X}}+O(r^{-1})\widecheck{\underline{\omega}}+O(r^{-2})\widecheck{e_3(r)}\\ &\quad +O(r^{-2})e_3(\cos\theta)+\Gamma_b\cdot\Gamma_b. \end{aligned}$$

We deduce from the above estimates for $\underline{\Xi}$, $\widecheck{\mathrm{tr}\underline{X}}$, $\widecheck{\underline{\omega}}$, $\widecheck{e_3(r)}$ and $e_3(\cos\theta)$ that

$$\left|\mathfrak{d}^{\leq k_*-8}\nabla_3\widecheck{\mathrm{tr}\underline{X}}\right| \lesssim \epsilon_0 r^{-2}u^{-1-\delta_{dec}}.$$

This ends the proof of the estimates (6.81).

Step 11. Next, we estimate $\widetilde{\nabla_3\mathfrak{J}}$ and $\widetilde{\nabla_3\mathfrak{J}_\pm}$. According to Lemma 6.17, we have

$$\begin{aligned}
\nabla_4(\widetilde{\nabla_3\mathfrak{J}})+\frac{1}{q}\widetilde{\nabla_3\mathfrak{J}} &= O(r^{-3})\widetilde{e_3(r)}+O(r^{-3})e_3(\cos\theta)+O(r^{-2})\underline{\check{\omega}}\\
&\quad +O(r^{-2})\check{H}+O(r^{-2})\check{Z}+O(r^{-2})\widecheck{\nabla\mathfrak{J}}+O(r^{-1})\check{P}\\
&\quad +r^{-1}\Gamma_b\cdot\Gamma_g,\\
\nabla_4(\widetilde{\nabla_3\mathfrak{J}_\pm})+\frac{1}{q}\widetilde{\nabla_3\mathfrak{J}_\pm} &= O(r^{-3})\widetilde{e_3(r)}+O(r^{-3})e_3(\cos\theta)+O(r^{-2})\underline{\check{\omega}}\\
&\quad +O(r^{-2})\check{H}+O(r^{-2})\check{Z}+O(r^{-2})\widecheck{\nabla\mathfrak{J}}+O(r^{-1})\check{P}\\
&\quad +r^{-1}\Gamma_b\cdot\Gamma_b.
\end{aligned}$$

Denoting the right hand sides by F, we easily check, using the previously derived estimates,

$$\left|\mathfrak{d}^{\leq k_*-6}F\right| \lesssim \epsilon_0 r^{-3}u^{-1-\delta_{dec}}.$$

Proceeding exactly as before, we infer

$$\begin{aligned}
r\|\mathfrak{d}^{\leq k_*-6}\widetilde{\nabla_3(\mathfrak{J},\mathfrak{J}_\pm)}\|_{L^\infty}(u,r) &\lesssim r_*\|\mathfrak{d}^{\leq k_*-6}\widetilde{\nabla_3\mathfrak{J}}\|_{L^\infty}(u,r_*)+\epsilon_0 r^{-1}u^{-1-\delta_{dec}}\\
&\lesssim \epsilon_0 r^{-1}u^{-1-\delta_{dec}}.
\end{aligned}$$

Hence

$$\left|\mathfrak{d}^{\leq k_*-6}\widetilde{\nabla_3\mathfrak{J}}\right| \lesssim \epsilon_0 r^{-2}u^{-1-\delta_{dec}}, \qquad \left|\mathfrak{d}^{\leq k_*-6}\widetilde{\nabla_3\mathfrak{J}_\pm}\right|\lesssim \epsilon_0 r^{-2}u^{-1-\delta_{dec}},$$

as stated. This ends the proof of the estimates (6.83) and (6.84).

Step 12. It only remains to prove the estimates (6.85) for $J^{(\pm)}$. We make use of Lemma 6.18 according to which

$$\begin{aligned}
\nabla_4(\widetilde{\mathcal{D}(J^{(\pm)})})+\frac{1}{q}\widetilde{\mathcal{D}(J^{(\pm)})} &= O(r^{-1})\widetilde{\mathrm{tr}X}+O(r^{-1})\widehat{X}+\Gamma_b\cdot\Gamma_g,\\
\nabla_4(\widetilde{\nabla_3 J^{(\pm)}}) &= O(r^{-2})\widetilde{\mathcal{D}(J^{(\pm)})}+O(r^{-1})\check{Z}+O(r^{-1})\check{H}+\Gamma_b\cdot\Gamma_b.
\end{aligned}$$

From the first equation we easily derive, using the estimates for $\widetilde{\mathrm{tr}X}$ and $\widehat{X}$ derived before,

$$\begin{aligned}
r\left\|\mathfrak{d}^{\leq k_*-6}\widetilde{\mathcal{D}J^{(\pm)}}\right\|_{L^\infty}(u,r) &\lesssim r_*\left\|\mathfrak{d}^{\leq k_*-6}\widetilde{\mathcal{D}J^{(\pm)}}\right\|_{L^\infty}(u,r_*)+\epsilon_0 r^{-\delta'}u^{-1-\delta_{dec}}\\
&\lesssim \epsilon_0 u^{-1-\delta_{dec}}.
\end{aligned}$$

Thus, we obtain

$$\left|\mathfrak{d}^{\leq k_*-6}\widetilde{\mathcal{D}J^{(\pm)}}\right| \lesssim \epsilon_0 r^{-1}u^{-1-\delta_{dec}}$$

as stated. Finally integrating the equation for $\nabla_4\big(\widetilde{\nabla_3 J^{(\pm)}}\big)$ we derive

$$\left|\mathfrak{d}^{\leq k_*-6}\widetilde{\nabla_3 J^{(\pm)}}\right| \lesssim \epsilon_0 r^{-1}u^{-1-\delta_{dec}}.$$

This ends the proof of (6.85) and concludes the proof of Proposition 6.58. ☐

The estimates in Remark 6.57, together with the ones of Proposition 6.58, conclude the proof of Proposition 6.49. Also, recalling from (6.1) that $k_* = k_{small} + 60$ in this chapter, this concludes the proof of Theorem M4 as stated in Section 3.7.1.

7. DECAY ESTIMATES ON ${}^{(int)}\mathcal{M}$ AND ${}^{(top)}\mathcal{M}$ (THEOREM M5)

The goal of this chapter is to prove Theorem M5, i.e. to derive decay estimates on ${}^{(int)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$.

7.1. Linearized equations for ingoing PG structures

Ingoing PG structures have been introduced in Section 2.7. In particular, recall that such structures verify the following identities, see Section 2.7.1,

$$\begin{aligned}&\underline{\xi}=0, \qquad \underline{\omega}=0, \qquad \eta=\zeta,\\ &e_3(r)=-1, \qquad \nabla(r)=0, \qquad e_3(\underline{u})=e_3(\theta)=e_3(\varphi)=0,\\ &\nabla_3\mathfrak{J}=\frac{1}{\overline{q}}\mathfrak{J}, \qquad \nabla_3\mathfrak{J}_\pm=\frac{1}{\overline{q}}\mathfrak{J}_\pm, \qquad e_3(J^{(p)})=0,\ p=0,+,-.\end{aligned}$$

In this section we provide the linearized equations for ingoing PG structures that will be used to derive decay estimates for the ingoing PG structures of ${}^{(int)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$. Recall that the definition of the linearized quantities for ingoing PG structures can be found in Definition 2.72.

Remark 7.1. *The equations for ingoing PG structures stated in this section can be easily deduced from their analog for outgoing PG structures by performing the following substitutions*

$$\begin{aligned}&u\to\underline{u}, \quad r\to r, \quad \theta\to\theta, \quad \varphi\to\varphi, \quad e_4\to e_3, \quad e_3\to e_4, \quad e_a\to e_a,\\ &\alpha\to\underline{\alpha}, \quad \beta\to-\underline{\beta}, \quad \rho\to\rho, \quad {}^*\rho\to-\,{}^*\rho, \quad \underline{\beta}\to-\beta, \quad \underline{\alpha}\to\alpha,\\ &\xi\to\underline{\xi}, \quad \omega\to\underline{\omega}, \quad \chi\to\underline{\chi}, \quad \eta\to\underline{\eta}, \quad \underline{\eta}\to\eta, \quad \zeta\to-\zeta, \quad \underline{\chi}\to\chi,\\ &\underline{\omega}\to\omega, \quad \underline{\xi}\to\xi, \quad J^{(p)}\to J^{(p)}, \quad \mathfrak{J}\to\mathfrak{J}, \quad \mathfrak{J}_\pm\to\mathfrak{J}_\pm.\end{aligned}$$

In view of Remark 7.1, the following lemma can be easily obtained from its analog for the outgoing case in Lemma 6.15.

Lemma 7.2. *The linearized null structure equations in the e_3 direction are*

$$\begin{aligned}\nabla_3(\widetilde{tr\underline{X}})-\frac{2}{\overline{q}}\widetilde{tr\underline{X}} &= \Gamma_b\cdot\Gamma_b,\\ \nabla_3\widehat{\underline{X}}-\frac{2r}{|q|^2}\widehat{\underline{X}} &= -\underline{A}+\Gamma_b\cdot\Gamma_b,\\ \nabla_3\check{Z}-\frac{2}{\overline{q}}\check{Z} &= -\underline{B}+O(r^{-2})\widehat{\underline{X}}+O(r^{-2})\widetilde{tr\underline{X}}+\Gamma_b\cdot\Gamma_g,\end{aligned}$$

$$
\begin{aligned}
\nabla_3 \widecheck{\underline{H}} - \frac{1}{q}\widecheck{\underline{H}} &= \underline{B} + O(r^{-1})\check{Z} + O(r^{-2})\widehat{\underline{X}} + O(r^{-2})\widecheck{tr\underline{X}} + \Gamma_b \cdot \Gamma_g,\\
\nabla_3 \widecheck{trX} - \frac{1}{q}\widecheck{trX} &= \mathcal{D}\cdot\overline{\check{Z}} + 2\check{P} + O(r^{-2})\check{Z} + O(r^{-1})\widecheck{tr\underline{X}}\\
&\quad + O(r^{-1})\widecheck{\mathcal{D}\cdot\overline{\mathfrak{J}}} + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} + \Gamma_b\cdot\Gamma_g,\\
\nabla_3 \widehat{X} - \frac{1}{q}\widehat{X} &= \frac{1}{2}\mathcal{D}\widehat{\otimes}\check{Z} + O(r^{-2})\check{Z} + O(r^{-1})\widehat{\underline{X}} + O(r^{-1})\mathcal{D}\widehat{\otimes}\mathfrak{J}\\
&\quad + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} + \Gamma_b\cdot\Gamma_g,\\
\nabla_3(\check{\omega}) &= \Re(\check{P}) + O(r^{-2})\check{Z} + O(r^{-2})\widecheck{\underline{H}} + \Gamma_b\cdot\Gamma_g,\\
\nabla_3\Xi - \frac{1}{q}\Xi &= O(r^{-1})\,\mathfrak{d}^{\leq 1}(\check{\omega}) + O(r^{-2})\check{Z} + O(r^{-2})\widecheck{\underline{H}} + O(r^{-2})\widecheck{trX}\\
&\quad + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} + \Gamma_b\cdot\Big(\check{\omega}, \Gamma_g\Big).
\end{aligned}
$$

The linearized Bianchi equations for $B, P, \underline{B}$ are

$$
\begin{aligned}
\nabla_3\underline{B} - \frac{4}{q}\underline{B} &= \frac{1}{2}\overline{\mathcal{D}}\cdot\underline{A} + O(r^{-2})\underline{A} + \Gamma_b\cdot(\underline{B}, \underline{A}),\\
\nabla_3\left(\check{P}\right) + \frac{1}{2}\overline{\mathcal{D}}\cdot\underline{B} &= \frac{3}{q}\check{P} + O(r^{-2})\underline{B} + O(r^{-3})\widecheck{tr\underline{X}} + r^{-1}\Gamma_b\cdot\Gamma_g + \Gamma_g\cdot\underline{A},\\
\nabla_3 B - \mathcal{D}\overline{\check{P}} &= \frac{2}{q}B + O(r^{-2})\check{P} + O(r^{-3})\check{Z} + O(r^{-4})\widecheck{\mathcal{D}(\cos\theta)}\\
&\quad + r^{-1}\Gamma_b\cdot\Gamma_g.
\end{aligned}
$$

In view of Remark 7.1, the following lemma can be easily obtained from its analog for the outgoing case in Lemma 6.16.

Lemma 7.3. *We have*

$$
\begin{aligned}
e_3\left(\widecheck{e_4(r)}\right) &= -2\check{\omega},\\
\nabla_3\widecheck{\mathcal{D}\underline{u}} - \frac{1}{q}\widecheck{\mathcal{D}\underline{u}} &= O(r^{-1})\widecheck{tr\underline{X}} + O(r^{-1})\widehat{\underline{X}} + \Gamma_b\cdot\Gamma_b,\\
e_3\left(\widecheck{e_4(\underline{u})}\right) &= O(r^{-1})\widecheck{\underline{H}} + O(r^{-1})\check{Z} + O(r^{-2})\widecheck{\mathcal{D}\underline{u}} + \Gamma_g\cdot\Gamma_g,\\
\nabla_3\widecheck{\mathcal{D}\cos\theta} - \frac{1}{q}\widecheck{\mathcal{D}\cos\theta} &= \frac{i}{2}\mathfrak{J}\cdot\widehat{\underline{X}} + O(r^{-1})\widecheck{tr\underline{X}} + \Gamma_b\cdot\Gamma_b,\\
e_3(e_4(\cos\theta)) &= O(r^{-1})\widecheck{\underline{H}} + O(r^{-1})\check{Z} + O(r^{-2})\widecheck{\mathcal{D}\cos\theta} + \Gamma_g\cdot\Gamma_g.
\end{aligned}
$$

In view of Remark 7.1, the following lemma can be easily obtained from its analog for the outgoing case in Lemma 6.17.

Lemma 7.4. *The following equations hold for the tensors $\mathfrak{J}, \mathfrak{J}_\pm$.*

1. *We have*

$$
\begin{aligned}
\nabla_3(\mathcal{D}\widehat{\otimes}\mathfrak{J}) - \frac{2}{\overline{q}}\mathcal{D}\widehat{\otimes}\mathfrak{J} &= O(r^{-1})\underline{B} + O(r^{-2})\widecheck{tr\underline{X}} + O(r^{-2})\widehat{\underline{X}} \\
&\quad +O(r^{-2})\check{Z} + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} \\
&\quad +r^{-1}\Gamma_b\cdot\Gamma_b, \\
\nabla_3(\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}) - \Re\left(\frac{2}{\overline{q}}\right)\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}} &= O(r^{-1})\underline{B} + O(r^{-2})\widecheck{tr\underline{X}} + O(r^{-2})\widehat{\underline{X}} \\
&\quad +O(r^{-2})\check{Z} + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} \\
&\quad +r^{-1}\Gamma_b\cdot\Gamma_b, \\
\nabla_3(\widecheck{\nabla_4\mathfrak{J}}) - \frac{1}{\overline{q}}\widecheck{\nabla_4\mathfrak{J}} &= O(r^{-3})\widecheck{e_4(r)} + O(r^{-3})e_4(\cos\theta) \\
&\quad +O(r^{-2})\check{\omega} + O(r^{-2})\widecheck{\underline{H}} + O(r^{-2})\check{Z} \\
&\quad +O(r^{-2})\widecheck{\nabla\mathfrak{J}} + O(r^{-1})\check{P} + r^{-1}\Gamma_b\cdot\Gamma_g.
\end{aligned}
$$

2. *We also have*

$$
\begin{aligned}
\nabla_3(\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm) - \frac{2}{\overline{q}}\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm &= O(r^{-1})\underline{B} + O(r^{-2})\widecheck{tr\underline{X}} + O(r^{-2})\widehat{\underline{X}} \\
&\quad +O(r^{-2})\check{Z} + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} \\
&\quad +r^{-1}\Gamma_b\cdot\Gamma_b, \\
\nabla_3(\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm}) - \Re\left(\frac{2}{\overline{q}}\right)\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm} &= O(r^{-1})\underline{B} + O(r^{-2})\widecheck{tr\underline{X}} + O(r^{-2})\widehat{\underline{X}} \\
&\quad +O(r^{-2})\check{Z} + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} \\
&\quad +r^{-1}\Gamma_b\cdot\Gamma_b, \\
\nabla_3(\widecheck{\nabla_4\mathfrak{J}_\pm}) - \frac{1}{\overline{q}}\widecheck{\nabla_4\mathfrak{J}_\pm} &= O(r^{-3})\widecheck{e_4(r)} + O(r^{-3})e_4(\cos\theta) \\
&\quad +O(r^{-2})\check{\omega} + O(r^{-2})\widecheck{\underline{H}} + O(r^{-2})\check{Z} \\
&\quad +O(r^{-2})\widecheck{\nabla\mathfrak{J}} + O(r^{-1})\check{P} \\
&\quad +r^{-1}\Gamma_b\cdot\Gamma_g.
\end{aligned}
$$

In view of Remark 7.1, the following lemma can be easily obtained from its analog for the outgoing case in Lemma 6.18.

Lemma 7.5. *The following equations hold true*[146].

[146]Similar equations hold for $J^{(0)} = \cos\theta$, see Lemma 7.3.

(7.1)
$$\nabla_3(\widetilde{\mathcal{D}(J^{(\pm)})}) - \frac{1}{q}\widetilde{\mathcal{D}(J^{(\pm)})} = O(r^{-1})\widecheck{tr\underline{X}} + O(r^{-1})\widehat{\underline{X}} + \Gamma_b\cdot\Gamma_g,$$
$$\nabla_3(\widetilde{\nabla_4 J^{(\pm)}}) = O(r^{-2})\widetilde{\mathcal{D}(J^{(\pm)})} + O(r^{-2})\widecheck{Z} + O(r^{-2})\widecheck{\underline{H}} + \Gamma_b\cdot\Gamma_g.$$

7.2. Decay estimates for the PG structure of ${}^{(int)}\mathcal{M}$ on $\mathcal{T}$

To simplify the notations, in this section, we denote

- by (e_4, e_3, e_1, e_2) the outgoing PG frame of ${}^{(ext)}\mathcal{M}$, with all quantities associated to the outgoing PG structure of ${}^{(ext)}\mathcal{M}$ being unprimed,
- by (e_4', e_3', e_1', e_2') the ingoing PG frame of ${}^{(int)}\mathcal{M}$, with all quantities associated to the ingoing PG structure of ${}^{(int)}\mathcal{M}$ being primed.

Recall that ${}^{(ext)}\mathcal{M} \cap {}^{(int)}\mathcal{M} = \mathcal{T} = \{r = r_0\}$. In view of the above notations, and the initialization of the ingoing PG structure of ${}^{(int)}\mathcal{M}$ from the outgoing PG structure of ${}^{(ext)}\mathcal{M}$ on $\mathcal{T}$, see Section 3.2.5, we have

$$\underline{u} = u, \quad r' = r, \quad J'^{(p)} = J^{(p)},\ p = 0,+,-, \quad \mathfrak{J}' = \mathfrak{J}, \quad \mathfrak{J}'_\pm = \mathfrak{J}_\pm \text{ on } \mathcal{T}, \tag{7.2}$$

and

$$e_4' = \lambda e_4, \qquad e_3' = \lambda^{-1}e_3, \qquad e_a' = e_a,\ a = 1,2, \quad \text{on} \quad \mathcal{T}, \tag{7.3}$$

where λ is given by

$$\lambda = \frac{\Delta}{|q|^2}. \tag{7.4}$$

In order to derive decay estimates for the ingoing PG structure of ${}^{(int)}\mathcal{M}$ on $\mathcal{T}$, we will rely on the following lemma.

Lemma 7.6. *We have on* $\mathcal{T}$

$$A' = \lambda^2 A, \quad B' = \lambda B, \quad \check{P}' = \check{P}, \quad \underline{B}' = \lambda^{-1}\underline{B}, \quad \underline{A}' = \lambda^{-2}\underline{A},$$
$$\underline{\Xi}' = 0, \quad \underline{\omega}' = 0, \quad H' = Z',$$
$$\widecheck{trX}' = \lambda\widecheck{trX}, \quad \widehat{X}' = \lambda\widehat{X}, \quad \widecheck{tr\underline{X}}' = \lambda^{-1}\widecheck{tr\underline{X}}, \quad \widehat{\underline{X}}' = \lambda^{-1}\widehat{\underline{X}},$$
$$e_3'(r') = -1, \quad \nabla'(r') = 0, \quad e_3'(\underline{u}) = 0, \quad e_3'(J'^{(p)}) = 0,\ p = 0,+,-,$$
$$\nabla_3'\mathfrak{J}' = \frac{1}{\overline{q}'}\mathfrak{J}', \quad \nabla_3'\mathfrak{J}'_\pm = \frac{1}{\overline{q}'}\mathfrak{J}'_\pm,$$
$$\widetilde{\nabla'(\underline{u})} = \widetilde{\nabla(u)}, \quad \widetilde{\nabla'(J'^{(p)})} = \widetilde{\nabla(J^{(p)})},\ p = 0,+,-,$$

$$\widecheck{\overline{\mathcal{D}}'\cdot\mathfrak{J}'} = \widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}, \quad \widecheck{\mathcal{D}'\widehat{\otimes}\mathfrak{J}'} = \widecheck{\mathcal{D}\widehat{\otimes}\mathfrak{J}}, \quad \widecheck{\overline{\mathcal{D}}'\cdot\mathfrak{J}'_\pm} = \widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm}, \quad \widecheck{\mathcal{D}'\widehat{\otimes}\mathfrak{J}'_\pm} = \widecheck{\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm},$$

$$\begin{aligned}
\check{Z}' &= \check{Z} + \frac{1}{q}\widecheck{\mathcal{D}(q)} + \frac{1}{\overline{q}}\widecheck{\mathcal{D}(\overline{q})},\\
\widecheck{\underline{H}}' &= -\check{Z} - \frac{1}{e_3(r)}\underline{\Xi},\\
\Xi' &= \frac{\lambda^2}{e_3(r)}\left(\check{Z} + \frac{1}{q}\widecheck{\mathcal{D}(q)} + \frac{1}{\overline{q}}\widecheck{\mathcal{D}(\overline{q})} - \check{H}\right),\\
\check{\omega}' &= \frac{\lambda}{e_3(r)}\underline{\check{\omega}} + \frac{1}{2e_3(r)}\partial_r\left(\frac{\Delta}{|q|^2}\right)\widecheck{e_3(r)} - \frac{4a^2\lambda\cos\theta}{e_3(r)|q|^2}e_3(\cos\theta),
\end{aligned}$$

and

$$\begin{aligned}
\widecheck{e_4'(r')} &= -\frac{\lambda}{e_3(r)}\widecheck{e_3(r)},\\
\widecheck{e_4'(\underline{u})} &= -\frac{\lambda}{e_3(r)}\widecheck{e_3(u)} - \frac{2(r^2+a^2)}{e_3(r)|q|^2}\widecheck{e_3(r)},\\
e_4'(J'^{(0)}) &= -\frac{\lambda}{e_3(r)}e_3(J^{(0)}),\\
\widecheck{e_4'(J'^{(\pm)})} &= -\frac{\lambda}{e_3(r)}\widecheck{e_3(J^{(\pm)})} \pm \frac{2a}{|q|^2e_3(r)}J^{(\mp)}\widecheck{e_3(r)},\\
\widecheck{\nabla_{e_4'}\mathfrak{J}'} &= -\frac{\lambda}{e_3(r)}\widecheck{\nabla_3\mathfrak{J}},\\
\widecheck{\nabla_{e_4'}\mathfrak{J}'_\pm} &= -\frac{\lambda}{e_3(r)}\widecheck{\nabla_{e_3}\mathfrak{J}_\pm} \pm \frac{2a}{|q|e_3(r)}\widecheck{e_3(r)}\mathfrak{J}_\mp,
\end{aligned}$$

where the definition of the linearized quantities for the outgoing PG structure of ${}^{(ext)}\mathcal{M}$ can be found in Definition 2.66, while definition of the linearized quantities for the ingoing PG structure of ${}^{(int)}\mathcal{M}$ can be found in Definition 2.72.

Proof. The identities for $\underline{\Xi}'$, $\underline{\omega}'$, $H' - Z'$, $e_3'(r')$, $e_3'(\underline{u})$, $e_3'(J'^{(p)})$, $\nabla_3'\mathfrak{J}'$ and $\nabla_3'\mathfrak{J}'_\pm$ come from the ingoing PG structure assumption on ${}^{(int)}\mathcal{M}$. Also, the identities for A', B', $\underline{B}'$, $\underline{A}'$, $\widehat{X}'$ and $\widehat{\underline{X}}'$ follow immediately from the change of frame formulas of Proposition 2.12 with coefficients $(f=0, \underline{f}=0, \lambda)$ and the fact that (e_1, e_2) are tangent to $\mathcal{T}$. Also, the identities for $\check{P}'$, $\widecheck{\mathrm{tr}X}'$ and $\widecheck{\mathrm{tr}\underline{X}}'$, follow immediately from the change of frame formulas of Proposition 2.12 with coefficients $(f=0, \underline{f}=0, \lambda)$, the explicit choice for λ, the fact that $q' = q$ on $\mathcal{T}$, and the fact that (e_1, e_2) are tangent to $\mathcal{T}$. Also, the identities for $\nabla'(r')$,

$\widetilde{\nabla'(\underline{u})}$, $\widetilde{\nabla'(J'^{(p)})}$, $\widetilde{\overline{\mathcal{D}}'\cdot\mathfrak{J}'}$, $\mathcal{D}'\widehat{\otimes}\mathfrak{J}'$, $\widetilde{\overline{\mathcal{D}}'\cdot\mathfrak{J}'_\pm}$, and $\mathcal{D}'\widehat{\otimes}\mathfrak{J}'_\pm$, follow immediately from the fact that we have, on $\mathcal{T}$, $\nabla'=\nabla$, $r'=r$, $\underline{u}=u$, $J'^{(p)}=J^{(p)}$, $\mathfrak{J}'=\mathfrak{J}$, $\mathfrak{J}'_\pm=\mathfrak{J}_\pm$, together with the fact that ∇ is tangent to $\mathcal{T}$.

It remains to derive the identities for $\check{Z}'$, Ξ', ω', $\widetilde{\underline{H}}'$, $\widetilde{e'_4(r)}$, $\widetilde{e'_4(\underline{u})}$, $e'_4(J'^{(0)})$, $\widetilde{e'_4(J'^{(\pm)})}$, $\widetilde{\nabla'_4\mathfrak{J}'}$ and $\widetilde{\nabla'_4\mathfrak{J}'_\pm}$. We start with Z'. In view of the change of frame formulas of Proposition 2.12 with coefficients $(f=0,\underline{f}=0,\lambda)$, and the fact that (e_1,e_2) are tangent to $\mathcal{T}$, we have

$$Z' = Z-\mathcal{D}'(\log\lambda).$$

Since $\mathcal{D}'=\mathcal{D}$, and using the explicit form of λ, as well as $\nabla(r)=0$, we infer

$$Z' = Z-\mathcal{D}\left(\log\left(\frac{\Delta}{|q|^2}\right)\right)=Z+\frac{1}{|q|^2}\mathcal{D}(|q|^2)=Z+\frac{1}{q}\mathcal{D}(q)+\frac{1}{\overline{q}}\mathcal{D}(\overline{q})$$

which yields, together with the fact that $\mathfrak{J}'=\mathfrak{J}$ and $q'=q$ on $\mathcal{T}$, in view of the linearization of the various quantities, and taking the different linearization for Z' (ingoing PG structure) and Z (outgoing PG structure) into account,

$$\check{Z}' = \check{Z}+\frac{1}{q}\widetilde{\mathcal{D}(q)}+\frac{1}{\overline{q}}\widetilde{\mathcal{D}(\overline{q})}$$

as desired.

Next, since $e'_4=\lambda e_4$, $e'_3=\lambda^{-1}e_3$, and $e'_a=e_a$, $a=1,2$, on $\mathcal{T}$, and since $e_3-e_3(r)e_4$ is tangent to $\mathcal{T}$, we have

$$\begin{aligned}
\mathbf{g}(\mathbf{D}_{e_3-e_3(r)e_4}e'_3,e'_a) &= \mathbf{g}(\mathbf{D}_{e_3-e_3(r)e_4}(\lambda^{-1}e_3),e_a),\\
\mathbf{g}(\mathbf{D}_{e_3-e_3(r)e_4}e'_4,e'_a) &= \mathbf{g}(\mathbf{D}_{e_3-e_3(r)e_4}(\lambda e_4),e_a),\\
\mathbf{g}(\mathbf{D}_{e_3-e_3(r)e_4}e'_4,e'_3) &= \mathbf{g}(\mathbf{D}_{e_3-e_3(r)e_4}(\lambda e_4),\lambda^{-1}e_3).
\end{aligned}$$

Hence, using again that $e'_4=\lambda e_4$ and $e'_3=\lambda^{-1}e_3$ on $\mathcal{T}$, we deduce

$$\begin{aligned}
\mathbf{g}(\mathbf{D}_{\lambda e'_3-e_3(r)\lambda^{-1}e'_4}e'_3,e'_a) &= \lambda^{-1}\mathbf{g}(\mathbf{D}_{e_3-e_3(r)e_4}e_3,e_a),\\
\mathbf{g}(\mathbf{D}_{\lambda e'_3-e_3(r)\lambda^{-1}e'_4}e'_4,e'_a) &= \lambda\mathbf{g}(\mathbf{D}_{e_3-e_3(r)e_4}e_4,e_a),\\
\mathbf{g}(\mathbf{D}_{\lambda e'_3-e_3(r)\lambda^{-1}e'_4}e'_4,e'_3) &= -2(e_3-e_3(r)e_4)\log(\lambda)+\mathbf{g}(\mathbf{D}_{e_3-e_3(r)e_4}e_4,e_3),
\end{aligned}$$

and hence

$$\begin{aligned}
\lambda\underline{\xi}'-\lambda^{-1}e_3(r)\underline{\eta}' &= \lambda^{-1}(\underline{\xi}-e_3(r)\underline{\eta}),\\
\lambda\eta'-e_3(r)\lambda^{-1}\xi' &= \lambda(\eta-e_3(r)\xi),
\end{aligned}$$

$$-\lambda\underline{\omega}' - e_3(r)\lambda^{-1}\omega' \quad = \quad -\underline{\omega} - e_3(r)\omega - 2(e_3 - e_3(r)e_4)\log(\lambda).$$

Since $\underline{\xi}' = 0$, $\xi = 0$, $\underline{\omega}' = 0$, $\omega = 0$, $\eta' = \zeta'$ and $\underline{\eta} = -\zeta$, we infer

$$\begin{aligned}\underline{\eta}' &= -\zeta - \frac{1}{e_3(r)}\underline{\xi},\\ \xi' &= \frac{\lambda^2}{e_3(r)}(\zeta' - \eta),\\ \omega' &= \frac{\lambda}{e_3(r)}\underline{\omega} + \frac{2\lambda}{e_3(r)}(e_3 - e_3(r)e_4)\log(\lambda).\end{aligned}$$

In view of the lineraizations for ingoing and outgoing PG structures, the fact that $\mathfrak{J}' = \mathfrak{J}$ and $q' = q$ on $\mathcal{T}$, and the above identity for $\check{Z}'$, we obtain

$$\begin{aligned}\widecheck{\underline{H}}' &= -\check{Z} - \frac{1}{e_3(r)}\underline{\Xi},\\ \Xi' &= \frac{\lambda^2}{e_3(r)}\left(\check{Z} + \frac{1}{q}\widecheck{\mathcal{D}(q)} + \frac{1}{\overline{q}}\widecheck{\mathcal{D}(\overline{q})} - \check{H}\right),\\ \check{\omega}' &= \frac{\lambda}{e_3(r)}\underline{\check{\omega}} + \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right)\left(1 + \frac{\lambda}{e_3(r)}\right) + \frac{2\lambda}{e_3(r)}(e_3 - e_3(r)e_4)\log(\lambda).\end{aligned}$$

These are the desired identities for $\widecheck{\underline{H}}'$ and Ξ'. For $\check{\omega}'$, we note that, in view of the formula for λ, we have $e_3(r) = -\lambda + \widecheck{e_3(r)}$. Also, $(e_3 - e_3(r)e_4)(\Delta) = 0$ and $e_4(\theta) = 0$. Hence

$$\begin{aligned}&\frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right)\left(1 + \frac{\lambda}{e_3(r)}\right) + \frac{2\lambda}{e_3(r)}(e_3 - e_3(r)e_4)\log(\lambda)\\ &= \frac{1}{2e_3(r)}\partial_r\left(\frac{\Delta}{|q|^2}\right)\widecheck{e_3(r)} - \frac{4a^2\lambda\cos\theta}{e_3(r)|q|^2}e_3(\cos\theta)\end{aligned}$$

which yields

$$\check{\omega}' \quad = \quad \frac{\lambda}{e_3(r)}\underline{\check{\omega}} + \frac{1}{2e_3(r)}\partial_r\left(\frac{\Delta}{|q|^2}\right)\widecheck{e_3(r)} - \frac{4a^2\lambda\cos\theta}{e_3(r)|q|^2}e_3(\cos\theta)$$

as desired.

It remains to derive the identities for $\widecheck{e_4'(r)}$, $\widecheck{e_4'(\underline{u})}$, $e_4'(J'^{(0)})$, $e_4'(\widecheck{J'^{(\pm)}})$, $\widecheck{\nabla_4'\mathfrak{J}'}$ and $\widecheck{\nabla_4'\mathfrak{J}_\pm'}$. Since we have $r' = r$, $\underline{u} = u$, $J'^{(p)} = J^{(p)}$, $\mathfrak{J}' = \mathfrak{J}$ and $\mathfrak{J}'_\pm = \mathfrak{J}$ on $\mathcal{T}$, and since $e_3 - e_3(r)e_4$ is tangent to $\mathcal{T}$, we have

$$(e_3 - e_3(r)e_4)r' \quad = \quad (e_3 - e_3(r)e_4)r,$$

$$\begin{aligned}
(e_3 - e_3(r)e_4)\underline{u} &= (e_3 - e_3(r)e_4)u,\\
(e_3 - e_3(r)e_4)J'^{(p)} &= (e_3 - e_3(r)e_4)J^{(p)}, \quad p = 0, +, -,\\
\nabla_{e_3 - e_3(r)e_4}\mathfrak{J}' &= \nabla_{e_3 - e_3(r)e_4}\mathfrak{J},\\
\nabla_{e_3 - e_3(r)e_4}\mathfrak{J}'_{\pm} &= \nabla_{e_3 - e_3(r)e_4}\mathfrak{J}_{\pm}.
\end{aligned}$$

Using the fact that $e_4' = \lambda e_4$ and $e_3' = \lambda^{-1}e_3$ on $\mathcal{T}$, and since

$$\begin{aligned}
& e_3'(r') = -1, \qquad e_3'(\underline{u}) = 0, \qquad e_3'(J'^{(p)}) = 0, \; p = 0, +, -,\\
& \nabla_3'\mathfrak{J}' = \frac{1}{\overline{q}'}\mathfrak{J}', \qquad \nabla_3'\mathfrak{J}'_{\pm} = \frac{1}{\overline{q}'}\mathfrak{J}'_{\pm},\\
& e_4(r) = 1, \qquad e_4(u) = 0, \qquad e_4(J^{(p)}) = 0, \; p = 0, +, -,\\
& \nabla_4\mathfrak{J} = -\frac{1}{q}\mathfrak{J}, \qquad \nabla_4\mathfrak{J}_{\pm} = -\frac{1}{q}\mathfrak{J}_{\pm},
\end{aligned}$$

we infer

$$\begin{aligned}
e_4'(r') &= -\frac{\lambda^2}{e_3(r)},\\
e_4'(\underline{u}) &= -\frac{\lambda}{e_3(r)}e_3(u),\\
e_4'(J'^{(p)}) &= -\frac{\lambda}{e_3(r)}e_3(J^{(p)}), \quad p = 0, +, -,\\
\nabla_{e_4'}\mathfrak{J}' &= -\frac{\lambda}{e_3(r)}\left(\nabla_{e_3}\mathfrak{J} + \frac{e_3(r)}{q}\mathfrak{J} - \frac{\lambda}{\overline{q}'}\mathfrak{J}'\right),\\
\nabla_{e_4'}\mathfrak{J}'_{\pm} &= -\frac{\lambda}{e_3(r)}\left(\nabla_{e_3}\mathfrak{J}_{\pm} + \frac{e_3(r)}{q}\mathfrak{J}_{\pm} - \frac{\lambda}{\overline{q}'}\mathfrak{J}'_{\pm}\right).
\end{aligned}$$

Since $r' = r$, $q' = q$, $J'^{(p)} = J^{(p)}$, $\mathfrak{J}' = \mathfrak{J}$ and $\mathfrak{J}'_{\pm} = \mathfrak{J}_{\pm}$ on $\mathcal{T}$, and since $e_3(r) = -\lambda + \widecheck{e_3(r)}$, we deduce, in view of the linerizations for ingoing and outgoing PG structures,

$$\begin{aligned}
\widecheck{e_4'(r')} &= -\frac{\lambda}{e_3(r)}\widecheck{e_3(r)},\\
\widecheck{e_4'(\underline{u})} &= -\frac{\lambda}{e_3(r)}\widecheck{e_3(u)} - \frac{2(r^2+a^2)}{e_3(r)|q|^2}\widecheck{e_3(r)},\\
e_4'(J'^{(0)}) &= -\frac{\lambda}{e_3(r)}e_3(J^{(0)}),\\
\widecheck{e_4'(J'^{(\pm)})} &= -\frac{\lambda}{e_3(r)}e_3\widecheck{(J^{(\pm)})} \pm \frac{2a}{|q|^2 e_3(r)}J^{(\mp)}\widecheck{e_3(r)},
\end{aligned}$$

$$\widetilde{\nabla_{e'_4}\mathfrak{J}'} = -\frac{\lambda}{e_3(r)}\widetilde{\nabla_3\mathfrak{J}},$$
$$\widetilde{\nabla_{e'_4}\mathfrak{J}'_\pm} = -\frac{\lambda}{e_3(r)}\widetilde{\nabla_{e_3}\mathfrak{J}_\pm} \pm \frac{2a}{|q|e_3(r)}\widetilde{e_3(r)}\mathfrak{J}_\mp,$$

as desired. This concludes the proof of the lemma. □

We are now ready to derive decay estimates for the ingoing PG structure of ${}^{(int)}\mathcal{M}$ on $\mathcal{T}$.

Lemma 7.7. *The following decay estimates hold on $\mathcal{T}$ for the ingoing PG structure of ${}^{(int)}\mathcal{M}$*

$$\sup_{\mathcal{T}} r'\underline{u}^{1+\delta_{dec}}|\mathfrak{d}^{\leq k_{small}+40}(\Gamma'_g, \Gamma'_b)| \lesssim \epsilon_0. \tag{7.5}$$

Proof. In view of the control of outgoing PG structure of ${}^{(ext)}\mathcal{M}$ established in Theorem M4 and the fact that $\mathcal{T} \subset {}^{(ext)}\mathcal{M}$, we have

$$\sup_{\mathcal{T}} r u^{1+\delta_{dec}}|\mathfrak{d}^{\leq k_{small}+40}(\Gamma_g, \Gamma_b)| \lesssim \epsilon_0.$$

Note that the tangential derivatives to $\mathcal{T} = \{r = r_0\}$ are generated by ∇ and $\nabla_3 - e_3(r)\nabla_4$. We introduce the following notation for r-weighted tangential derivatives to $\mathcal{T}$

$$\widetilde{\mathfrak{d}} := (\nabla_3 - e_3(r)\nabla_4, r\nabla).$$

From the above estimate for (Γ_g, Γ_b), together the identities of Lemma 7.6 on $\mathcal{T}$ and the fact that $r' = r$ and $\underline{u} = u$ on $\mathcal{T}$, we infer

$$\sup_{\mathcal{T}} r'\underline{u}^{1+\delta_{dec}}|\widetilde{\mathfrak{d}}^{\leq k_{small}+40}(\Gamma'_g, \Gamma'_b)| \lesssim \epsilon_0.$$

Finally, since $\mathfrak{d}$ is generated by e'_3 and $\widetilde{\mathfrak{d}}$, the previous estimate and the control of e'_3 derivatives provided by the null structure equations and Bianchi identities of the ingoing PG structure of ${}^{(int)}\mathcal{M}$ immediately imply

$$\sup_{\mathcal{T}} r'\underline{u}^{1+\delta_{dec}}|\mathfrak{d}^{\leq k_{small}+40}(\Gamma'_g, \Gamma'_b)| \lesssim \epsilon_0$$

as stated. □

7.3. Decay estimates for $\underline{A}$ in ${}^{(int)}\mathcal{M}$

In this section, we initiate the proof of the part of Theorem M5 concerning ${}^{(int)}\mathcal{M}$, i.e. to derive decay estimates for the ingoing PG structure of ${}^{(int)}\mathcal{M}$, by first controlling $\underline{A}$ in ${}^{(int)}\mathcal{M}$. This is done in the following proposition.

Proposition 7.8. *Relative to to the ingoing PG structure of ${}^{(int)}\mathcal{M}$, $\underline{A}$ verifies the following estimate in ${}^{(int)}\mathcal{M}$*

$$\sup_{{}^{(int)}\mathcal{M}} \underline{u}^{1+\delta_{dec}}|\mathfrak{d}^{\leq k_{small}+33}\underline{A}| \lesssim \epsilon_0. \tag{7.6}$$

Proof. The proof proceeds in the following steps.

Step 1. *Construction of a frame on part of ${}^{(int)}\mathcal{M}$ for which $\Xi = 0$.*

Recall from Lemma 7.7 that we control the ingoing PG frame of ${}^{(int)}\mathcal{M}$, and in particular $\underline{A}$, on $\mathcal{T}$. We would like to propagate this control for $\underline{A}$ from $\mathcal{T}$ to part of ${}^{(int)}\mathcal{M}$ by making use of the Teukolsky-Starobinsky (TS) formula, derived in Proposition 5.4.1 of [28], which relates $\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X\right)^4\underline{A}$ to $\mathfrak{d}^{\leq 4}A$ up to quadratic and higher order terms, where the notation ${}^{(c)}\nabla_4$ has been introduced in section 2.2.9 of [28] and is given, for a horizontal tensor U, by

$$ {}^{(c)}\nabla_4 U := \nabla_4 U + 2s\omega U, \qquad s = \mathrm{signature}(U), \tag{7.7}$$

i.e. the coefficient in front of ω depends on the signature of U. However, Proposition 5.4.1 of [28] holds true in a frame for which $\Xi = 0$. We have thus to construct, first, a new frame (e_3', e_4', e_1', e_2'), which coincides with that of ${}^{(int)}\mathcal{M}$ on $\mathcal{T}$, and which satisfies $\Xi' = 0$. This is the goal of this first step.

In order to construct the above mentioned frame, we look for a frame transformation of the form (2.6), see section 2.2.1, in the particular case where the transition coefficients $(f, \underline{f}, \lambda)$ satisfy $\underline{f} = 0$, $\lambda = 1$, i.e.

$$\begin{aligned} e_4' &= e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3, \\ e_a' &= e_a + \frac{1}{2} f_a e_3, \quad a = 1, 2, \\ e_3' &= e_3. \end{aligned}$$

We also define u' such that

$$e_4'(u') = 0$$

and initialize both f and u' on $\mathcal{T}$ such that

$$f|_{\mathcal{T}} = 0, \qquad (u'-u)|_{\mathcal{T}} = 0.$$

Moreover, we introduce the following subregion of ${}^{(int)}\mathcal{M}$

$${}^{(int)}\mathcal{M}_1 := {}^{(int)}\mathcal{M}(r \geq r_+(1+\delta_{red})) \cap \{u' \leq u_*\},$$

where $\delta_{red} > 0$ is a sufficiently small constant.

According to Proposition 2.12, and given that the transformation satisfies $\lambda = 1$ and $\underline{f} = 0$, ξ transforms as follows

$$\begin{aligned}
\xi' &= \xi + \frac{1}{2}\nabla'_4 f + \frac{1}{4}(\mathrm{tr}\,\chi f - {}^{(a)}\mathrm{tr}\chi\ {}^*f) + \omega f + \mathrm{Err}(\xi, \xi'),\\
\mathrm{Err}(\xi, \xi') &= \frac{1}{2} f \cdot \widehat{\chi} + \frac{1}{4}|f|^2\eta + \frac{1}{2}(f\cdot\zeta)\, f - \frac{1}{4}|f|^2\underline{\eta}\\
&\quad + \left(\frac{1}{2}(f\cdot\xi')\,\underline{f} + \frac{1}{2}(f\cdot\underline{f})\,\xi'\right) + O(f^3)\Gamma + O(f^2)\Gamma_b.
\end{aligned}$$

To enforce $\xi' = 0$ is equivalent to require that f satisfies the following transport equation

$$\nabla'_4 f + \frac{1}{2}(\mathrm{tr}\,\chi f - {}^{(a)}\mathrm{tr}\chi\ {}^*f) + 2\omega f \quad = \quad \Gamma_g + f\cdot\Gamma_g + O(f^2)\Gamma + O(f^2)\Gamma_b.$$

Introducing the anti-selfdual 1-from F given by

$$F := f + i\ {}^*f,$$

this transport equation is equivalent to

$$\nabla'_4 F + \frac{1}{2}\mathrm{tr}X F + 2\omega F = \Gamma_g + f\cdot\Gamma_g + O(f^2)\Gamma + O(f^2)\Gamma_b, \qquad F|_{\mathcal{T}} = 0,$$

which we rewrite as

$$\nabla'_4 F + \left(\frac{1}{q}\frac{\Delta}{|q|^2} - \partial_r\left(\frac{\Delta}{|q|^2}\right)\right) F = h, \qquad F|_{\mathcal{T}} = 0, \tag{7.8}$$

where the initialization of F on $\mathcal{T}$ comes from the one of f, and where h has the following schematic form

$$h \quad = \quad \Gamma_g + f\cdot\Gamma_g + O(f^2)\Gamma + O(f^2)\Gamma_b. \tag{7.9}$$

Step 2. *Control of the change of frame.*

Next, we estimate f by relying on the transport equation (7.8). To this end, we assume the following local bootstrap assumption in ${}^{(int)}\mathcal{M}_1$

$$|\mathfrak{d}^{k_{small}+40}f| \leq \frac{\sqrt{\epsilon}}{\underline{u}^{1+\frac{\delta_{dec}}{2}}} \quad \text{on} \quad r_1\leq r\leq r_0, \tag{7.10}$$

where

$$r_+(1+\delta_{red})\leq r_1<r_0.$$

Since $f=0$ on $\mathcal{T}=\{r=r_0\}$, (7.10) holds for r_1 close enough to r_0, and our goal is to prove that we may in fact choose $r_1=r_+(1+\delta_{red})$ and replace $\sqrt{\epsilon}$ with ϵ in (7.10).

Interpolating between the bootstrap assumptions on energy and on decay, we have on ${}^{(int)}\mathcal{M}$, for all $k\leq k_{small}+40$,

$$|\mathfrak{d}^k(\Gamma_b,\Gamma_g)| \lesssim \epsilon\underline{u}^{-1-\frac{\delta_{dec}}{2}}, \tag{7.11}$$

see Lemma 5.15 for the corresponding statement on Σ_*. In view of (7.9) and (7.11), we infer that the RHS h of the transport equation (7.8) satisfies on ${}^{(int)}\mathcal{M}_1$, for all $k\leq k_{small}+40$,

$$|\mathfrak{d}^k h| \lesssim \frac{\epsilon}{\underline{u}^{1+\frac{\delta_{dec}}{2}}}(1+|\mathfrak{d}^{\leq k}f|)+|\mathfrak{d}^{\leq k}f|^2. \tag{7.12}$$

Next, we rewrite (7.8) as

$$\begin{aligned}
&\nabla'_4\left(q\left(\frac{\Delta}{|q|^2}\right)^{-1}F\right)=q\left(\frac{\Delta}{|q|^2}\right)^{-1}h+\left(\frac{1}{q}\frac{\Delta}{|q|^2}-\partial_r\left(\frac{\Delta}{|q|^2}\right)\right)\left(\frac{\Delta}{|q|^2}\right)^{-1}\Gamma_g,\\
&F\big|_{\mathcal{T}}=0.
\end{aligned}$$

Integrating this transport equation from $\mathcal{T}$, and using (7.10), (7.12), and the fact that $r_1\geq r_+(1+\delta_{red})$ so that $\frac{\Delta}{|q|^2}\gtrsim\delta_{red}>0$, we deduce[147]

$$|f| \leq \frac{\epsilon}{\underline{u}^{1+\frac{\delta_{dec}}{2}}} \quad \text{on} \quad r_1\leq r\leq r_0.$$

[147]We also use the fact that u' is constant along the integral curves of e'_4 since $e'_4(u')=0$, and that $\underline{u}\sim u'$ along such integral curves as shown below in (7.15).

Similarly, commuting (7.8) with $\not{\mathcal{L}}_{\mathbf{T}}$ and ∇_3, proceeding as above, and using (7.8) to recover ∇_4 derivatives, we obtain, for all $k \leq k_{small} + 40$,

$$|\mathfrak{d}^k f| \quad \leq \quad \frac{\epsilon}{\underline{u}^{1+\frac{\delta_{dec}}{2}}} \quad \text{on} \quad r_1 \leq r \leq r_0.$$

This is an improvement of the local bootstrap assumption (7.10) and we may thus choose $r_1 = r_+(1 + \delta_{red})$ and replace $\sqrt{\epsilon}$ with ϵ in (7.10), i.e. we have obtained

$$|\mathfrak{d}^{k_{small}+40} f| \quad \leq \quad \frac{\epsilon}{\underline{u}^{1+\frac{\delta_{dec}}{2}}} \quad \text{on} \quad {}^{(int)}\mathcal{M}_1. \tag{7.13}$$

Remark 7.9. *In view of (7.13), the second frame (e'_3, e'_4, e'_1, e'_2) is defined everywhere on ${}^{(int)}\mathcal{M}_1$. Also, by the choice of the transport equation for f in Step 1, it satisfies*

$$\Xi' = 0 \quad on \quad {}^{(int)}\mathcal{M}_1$$

as desired. Finally, note from the transformation formula for $\underline{\alpha}$ in Proposition 2.12 and the fact that $\underline{f} = 0$ and $\lambda = 1$ that the following identity holds

$$\underline{A}' = \underline{A} \quad on \quad {}^{(int)}\mathcal{M}_1.$$

We conclude this step by deriving two consequences on the estimate (7.13). First, as an immediate consequence of the control of (Γ_g, Γ_b) in ${}^{(int)}\mathcal{M}$ provided by (7.11), the control of f provided by (7.13), and the change of frame formulas of Proposition 2.12, we have on ${}^{(int)}\mathcal{M}_1$, for all $k \leq k_{small} + 39$,

$$|\mathfrak{d}^k(\Gamma'_b, \Gamma'_g)| \quad \lesssim \quad \epsilon \underline{u}^{-1-\frac{\delta_{dec}}{2}}. \tag{7.14}$$

Second, note that we have in ${}^{(int)}\mathcal{M}_1$

$$\begin{aligned} e'_4(\underline{u}) &= e_4(\underline{u}) + f \cdot \nabla(\underline{u}) + \frac{1}{4}|f|^2 e_3(\underline{u}) \\ &= \frac{2(r^2+a^2)}{|q|^2} + \Gamma_g + O(f) \end{aligned}$$

which together with (7.13) and (7.11) implies

$$|e'_4(\underline{u} - u')| \lesssim 1 \quad \text{on} \quad {}^{(int)}\mathcal{M}_1,$$

where we also used the fact that $e'_4(u')=0$. Since $u'=u=\underline{u}$ on $\mathcal{T}$, since $r_+(1+\delta_{red})\leq r\leq r_0$ on ${}^{(int)}\mathcal{M}_1$, and since $e'_4(r)\gtrsim\delta_{red}$ on ${}^{(int)}\mathcal{M}_1$, we infer

$$|u'-\underline{u}|\leq c_{\delta_{red}} \tag{7.15}$$

for a constant $c_{\delta_{red}}>0$ depending on δ_{red}. In particular, in view of the definition of ${}^{(int)}\mathcal{M}_1$, we deduce

$${}^{(int)}\mathcal{M}(r\geq r_+(1+\delta_{red}))\cap\{\underline{u}\leq u_*-c_{red}\}\subset {}^{(int)}\mathcal{M}_1. \tag{7.16}$$

Step 3. *Estimates for $\underline{A}$ in ${}^{(int)}\mathcal{M}_1$.*

We use the Teukolsky-Starobinsky identity for $\underline{A}'$ in ${}^{(int)}\mathcal{M}'_1$ and the fact that we have complete control for $\underline{A}'=\underline{A}$ on $\mathcal{T}$ to derive estimates for $\underline{A}'$ in ${}^{(int)}\mathcal{M}'_1$. The precise TS formula we need is as follows.

Proposition 7.10. *Relative to a frame*[148] *in ${}^{(int)}\mathcal{M}$ for which $\Xi=0$, the complex tensors A, $\underline{A}\in\mathfrak{s}_2(\mathbb{C})$ satisfy the following relation*[149] *in the region $r\leq r_0$*

$$\left({}^{(c)}\nabla_4+2trX\right)^4\underline{A} \;=\; r^{-4}\mathfrak{d}^{\leq 4}A+\mathfrak{d}^{\leq 3}(\Gamma_b\cdot\Gamma_g). \tag{7.17}$$

Proof. See Proposition 5.4.1 of [28]. □

We apply (7.17) in the new prime frame of Step 1 for which $\Xi'=0$, in the subregion ${}^{(int)}\mathcal{M}_1$ of ${}^{(int)}\mathcal{M}$. We thus obtain in that frame

$$\left({}^{(c)}\nabla_4+2\mathrm{tr}X\right)^4\underline{A}' \;=\; h \tag{7.18}$$

where the RHS h is given schematically by

$$h \;=\; r^{-4}\mathfrak{d}^{\leq 4}A'+\mathfrak{d}^{\leq 3}(\Gamma'_b\cdot\Gamma'_g).$$

In view of (7.14), we have on ${}^{(int)}\mathcal{M}_1$, for $k\leq k_{small}+36$,

$$|\mathfrak{d}^k h| \;\lesssim\; |\mathfrak{d}^{\leq 4}A'|+\frac{\epsilon^2}{\underline{u}^{2+\delta_{dec}}}.$$

[148] Both A, $\underline{A}$ here are defined relative to the frame for which $\Xi=0$ in ${}^{(int)}\mathcal{M}$. Below, we will thus apply the identity as one relating $\underline{A}'$ and A'.

[149] See (7.7) for the definition of notation ${}^{(c)}\nabla_4$ which has been introduced in section 2.2.9 of [28].

Using the transformation formula for $\underline{\alpha}$ in Proposition 2.12, the fact that $\underline{f} = 0$ and $\lambda = 1$, the control for f provided by (7.13), and the control for (Γ_b, Γ_g) in (7.11), we infer on ${}^{(int)}\mathcal{M}_1$, for $k \leq k_{small} + 36$,

$$|\mathfrak{d}^k h| \lesssim |\mathfrak{d}^{\leq 4} A| + \frac{\epsilon^2}{\underline{u}^{2+\delta_{dec}}}.$$

Together with the control for A derived in Theorem M1, we deduce on ${}^{(int)}\mathcal{M}_1$, for $k \leq k_{small} + 36$,

$$|\mathfrak{d}^k h| \lesssim \frac{\epsilon_0}{\underline{u}^{1+\delta_{dec}}}. \tag{7.19}$$

Next, we integrate the fourth order transport equation (7.18) in e_4' from $\mathcal{T}$ using:

- the control of the RHS h provided by (7.19),
- the control of $\underline{A}$ on $\mathcal{T}$ provided by Lemma 7.7, and the fact that $\underline{A}' = \underline{A}$,

and obtain on ${}^{(int)}\mathcal{M}_1$

$$|\underline{A}'| \lesssim \frac{\epsilon_0}{\underline{u}^{1+\delta_{dec}}}.$$

Differentiating (7.18) w.r.t. $\not{\mathcal{L}}_{\mathbf{T}}$ and ∇_3, using (7.18) to recover e_4 derivatives, and proceeding as above, we deduce on ${}^{(int)}\mathcal{M}_1$, for $k \leq k_{small} + 36$,

$$|\mathfrak{d}^k \underline{A}'| \lesssim \frac{\epsilon_0}{\underline{u}^{1+\delta_{dec}}}.$$

Since $\underline{A}' = \underline{A}$ on ${}^{(int)}\mathcal{M}_1$ in view of Remark 7.9, we infer

$$|\mathfrak{d}^{\leq k_{small}+36} \underline{A}| \lesssim \epsilon_0 \underline{u}^{-1-\delta_{dec}} \quad \text{on} \quad {}^{(int)}\mathcal{M}_1.$$

In view of (7.16), this implies

$$|\mathfrak{d}^{\leq k_{small}+36} \underline{A}| \lesssim \epsilon_0 \underline{u}^{-1-\delta_{dec}} \quad \text{on} \quad {}^{(int)}\mathcal{M}(r \geq r_+(1+\delta_{red})) \cap \{\underline{u} \leq u_* - c_{red}\}. \tag{7.20}$$

Step 4. *Extension of (7.20) to* ${}^{(int)}\mathcal{M} \cap \{\underline{u} \leq u_* - c_{red}\}$.

In view of (7.20), we still need to control $\underline{A}$ in the region

$${}^{(int)}\mathcal{M}(r \leq r_+(1+\delta_{red})) \cap \{\underline{u} \leq u_* - c_{red}\}.$$

This is done by making use of the general red shift estimates of Proposition 9.4.2 in [28] applied to the Teukolsky equation for $\underline{A}$. More precisely, in view of Lemma 5.3.3. in [28], the Teukolsky equation for $\underline{A}$ can be written in the following form

$$\begin{aligned}\dot{\square}_2\underline{A} &= -4\omega\nabla_3\underline{A} + O(r^{-1})\nabla_4\underline{A} + O(ar^{-2})\nabla\underline{A} + O(r^{-2})\underline{A} + N,\\ N &= r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b) + \Gamma_b\cdot\Gamma_b\cdot\Gamma_g.\end{aligned} \tag{7.21}$$

We introduce the following notations

$$\psi := e^{c_0\tau}\underline{A}, \qquad \tau_{|\,^{(int)}\mathcal{M}} := \underline{u} + \frac{m^2}{r}, \qquad c_0 = \frac{r_+ - m}{r_+^2 + a^2} > 0, \tag{7.22}$$

and note from section D.3 that τ can be extended to $\mathcal{M}$ such that its level sets are spacelike. In view of (7.21), and using the fact that $\omega = -\frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right) + \widecheck{\omega}$,

$$e_4(\tau) = e_4(\underline{u}) - \frac{m^2}{r^2}e_4(r) = \frac{2(r^2+a^2)}{|q|^2} - \frac{m^2}{r^2}\frac{\Delta}{|q|^2} + \widecheck{e_4(\underline{u})} - \frac{m^2}{r^2}\widecheck{e_4(r)},$$

as well as the definition of ψ, we infer in the red shift region $|\frac{r}{r_+} - 1| \leq 2\delta_{red}$

$$\begin{aligned}\dot{\square}_2\psi - V\psi &= \left(C_+ + O\left(\frac{r}{r_+} - 1\right)\right)\nabla_3\psi + O(1)\nabla\psi + O(1)\nabla_4\psi + O(1)\psi\\ &\quad + e^{c_0\tau}N,\end{aligned} \tag{7.23}$$

where $V = \frac{4\Delta}{(r^2+a^2)|q|^2}$, where N has the same structure as in (7.21), and where C_+ is a function of $\cos\theta$ given by

$$C_+ := 2\partial_r\left(\frac{\Delta}{|q|^2}\right)_{|r=r_+} - c_0\left(\frac{2(r^2+a^2)}{|q|^2} - \frac{m^2}{r^2}\frac{\Delta}{|q|^2}\right)_{|r=r_+}$$

which together with the choice of the constant c_0 implies

$$C_+ = \frac{4(r_+ - m) - 2c_0(r_+^2 + a^2)}{r_+^2 + a^2(\cos\theta)^2} = \frac{2(r_+ - m)}{r_+^2 + a^2(\cos\theta)^2} > 0$$

so that C_+ is a positive function.

We then appeal to the following version of the red shift estimates.

Proposition 7.11 (Redshift estimates)**.** *Let ψ a solution of a wave equation which, in the red shift region $|\frac{r}{r_+} - 1| \leq 2\delta_{red}$, takes the form (7.23) where C_+*

is a positive function. Then, for $|a| < m$, there exists a small enough constant $\delta_{red} > 0$ such that $\delta_{red} = \delta_{red}(m - |a|)$ with $\delta_{red} \geq \delta_{\mathcal{H}}$, such that the following estimate holds true in $\mathcal{M}(\tau_1, \tau_2)$, for all $s \leq k_{small} + 40$,

$$
\begin{aligned}
E^s_{r\leq r_+(1+\delta_{red})}[\psi](\tau_2) \quad \lesssim \quad & E^s_{r\leq r_+(1+2\delta_{red})}[\psi](\tau_1) \\
& +\delta_{red}^{-1} Mor^s_{r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[\psi](\tau_1, \tau_2) \\
& + \int_{\mathcal{M}(\tau_1,\tau_2)\cap\{r\leq r_+(1+2\delta_{red})\}} e^{2c_0\tau}|\mathfrak{d}^{\leq s} N|^2,
\end{aligned}
\tag{7.24}
$$

where

$$
\begin{aligned}
E[\psi](\tau) =& \int_{\Sigma(\tau)} \Big(|\nabla_4\psi|^2 + r^{-2}|\nabla_3\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big), \\
E^s[\psi] =& E[\mathfrak{d}^{\leq s}\psi],
\end{aligned}
\tag{7.25}
$$

and[150]

$$
\begin{aligned}
& Mor^s_{r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[\psi](\tau_1, \tau_2) \\
\lesssim & \int_{\tau_1}^{\tau_2} E^s_{r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[\psi](\tau) d\tau.
\end{aligned}
\tag{7.26}
$$

Proof. See Proposition 9.4.2 of [28]. □

Since $\underline{u} \leq \tau$ in ${}^{(int)}\mathcal{M}$ in view of (7.22), we have

$$
\begin{aligned}
& {}^{(int)}\mathcal{M}(r \geq r_+(1+\delta_{red})) \cap \{\tau \leq u_* - c_{red}\} \\
\subset \quad & {}^{(int)}\mathcal{M}(r \geq r_+(1+\delta_{red})) \cap \{\underline{u} \leq u_* - c_{red}\}
\end{aligned}
$$

which together with (7.20) implies[151]

$$
E^{k_{small}+35}_{r_+(1+\delta_{red})\leq r\leq r_0}[\underline{A}](\tau) \lesssim \epsilon_0 \tau^{-1-\delta_{dec}} \quad \text{for} \quad \tau \leq u_* - c_{red}. \tag{7.27}
$$

Since $\psi = e^{c_0\tau}\underline{A}$, we have in view of Proposition 7.11 and (7.27), using also the bootstrap assumptions to control N, for $1 \leq \tau \leq u_* - c_{red}$,

$$
E^{k_{small}+35}_{r\leq r_+(1+\delta_{red})}[\underline{A}](\tau) \quad \lesssim \quad e^{-2c_0\tau} E^{k_{small}+35}_{r\leq r_+(1+2\delta_{red})}[\psi](1)
$$

[150] The estimate (7.26) follows immediately from the definition (7.25) of the energy norm and from the definition of the Morawetz norm $\mathrm{Mor}^s[\psi]$ given in section 6.1.5 of [28].

[151] Note that the choice of τ in section D.3 is such that $\tau \sim \underline{u}$ in ${}^{(int)}\mathcal{M}$.

$$
\begin{aligned}
& + e^{-2c_0\tau}\int_1^\tau e^{2c_0\tau'}\epsilon_0^2\tau'^{-2-2\delta_{dec}}d\tau',\\
\lesssim\ & \epsilon_0^2\tau^{-2-2\delta_{dec}}
\end{aligned}
$$

where we used the fact that $c_0 > 0$. Together with (7.27), we deduce

$$
E^{k_{small}+35}_{r\leq r_0}[\underline{A}](\tau) \lesssim \epsilon_0^2\tau^{-2-2\delta_{dec}} \quad \text{for} \quad 1\leq\tau\leq u_* - c_{red}. \tag{7.28}
$$

Using Sobolev and the fact that $\tau\sim\underline{u}$ on ${}^{(int)}\mathcal{M}$, we deduce, for all $k\leq k_{small}+33$,

$$
|\mathfrak{d}^k\underline{A}| \lesssim \epsilon_0\underline{u}^{-1-\delta_{dec}} \quad \text{on} \quad {}^{(int)}\mathcal{M}(\tau\leq u_*-c_{red}). \tag{7.29}
$$

Step 5. In view of (7.29), we still need to control $\underline{A}$ on ${}^{(int)}\mathcal{M}(\tau\geq u_*-c_{red})$. To this end, we denote with primes quantities with respect to the global frame of Proposition 3.33, and we use:

- the control of the change of frame from the frame of ${}^{(int)}\mathcal{M}$ and the frame of ${}^{(ext)}\mathcal{M}$ to the one of Proposition 3.33 provided by property (f) of that proposition and (3.63),
- the change of frame formula for $\underline{A}$ provided by Proposition 2.12,
- the control of $\underline{A}$ in ${}^{(ext)}\mathcal{M}$ provided by Theorem M4[152],
- the control of $\underline{A}$ in ${}^{(int)}\mathcal{M}$ provided by (7.28),

to deduce

$$
E^{k_{small}+35}[\underline{A}'](\tau) \lesssim \epsilon_0^2\tau^{-2-2\delta_{dec}} \quad \text{for} \quad 1\leq\tau\leq u_*-c_{red}. \tag{7.30}
$$

Next, notice that $\mathcal{M}(\tau\geq u_*-c_{red})$ is in fact a local existence type region. Starting from (7.30), we may thus rely on standard local energy decay estimates applied to the Teukolsky equation for $\underline{A}'$. This can be done exactly as in the end of section 12.4.3 of [28], see Steps 3–5. We leave the details to the reader. We derive

$$
E^{k_{small}+35}[\underline{A}'](\tau) \lesssim \epsilon_0^2\tau^{-2-2\delta_{dec}} \quad \text{for} \quad \tau\geq u_*-c_{red},
$$

which together with (7.28) implies

$$
E^{k_{small}+35}[\underline{A}'](\tau) \lesssim \epsilon_0^2\tau^{-2-2\delta_{dec}} \quad \text{for} \quad \tau\geq 1. \tag{7.31}
$$

[152] Note that the choice of τ in section D.3 is such that $\tau\sim u$ in ${}^{(ext)}\mathcal{M}$.

Using again the control of the change of frame from the frame of ${}^{(int)}\mathcal{M}$ to the one of Proposition 3.33 provided by property (f) of that proposition, and the change of frame formula for $\underline{A}$ provided by Proposition 2.12, we deduce from (7.31)

$$E^{k_{small}+35}_{r\leq r_0}[\underline{A}](\tau) \lesssim \epsilon_0^2 \tau^{-2-2\delta_{dec}} \quad \text{for} \quad \tau\geq 1.$$

Using Sobolev and the fact that $\tau\sim \underline{u}$ on ${}^{(int)}\mathcal{M}$, we deduce, for all $k\leq k_{small}+33$,

$$|\mathfrak{d}^k\underline{A}| \lesssim \epsilon_0 \underline{u}^{-1-\delta_{dec}} \quad \text{on} \quad {}^{(int)}\mathcal{M}.$$

as stated. This ends the proof of Proposition 7.8. □

7.4. Decay estimates in ${}^{(int)}\mathcal{M}$

In this section, all quantities appearing correspond to the ingoing PG structure of ${}^{(int)}\mathcal{M}$. We are now ready to prove the part of Theorem M5 concerning ${}^{(int)}\mathcal{M}$, i.e. to derive decay estimates for the ingoing PG structure of ${}^{(int)}\mathcal{M}$. To this end, recall first that $\underline{A}$ has already been estimated in Proposition 7.8, see (7.6).

Relying on the estimates of the ingoing PG structure of ${}^{(int)}\mathcal{M}$ on $\mathcal{T}$ derived in Lemma 7.7, we propagate these estimates to ${}^{(int)}\mathcal{M}$ thanks to the linearized transport equations in the e_3 direction of Section 7.1 for ingoing PG structures. Recalling that $\underline{A}$ has already been estimated, see (7.6), the other quantities are recovered in the following order:

1. We recover $\widecheck{\mathrm{tr}\underline{X}}$, with a control of $k_{small}+40$ derivatives, from

$$\nabla_3(\widecheck{\mathrm{tr}\underline{X}}) - \frac{2}{q}\widecheck{\mathrm{tr}\underline{X}} = \Gamma_b\cdot\Gamma_b.$$

2. We recover $\widehat{\underline{X}}$, with a control of $k_{small}+33$ derivatives, from

$$\nabla_3\widehat{\underline{X}} - \frac{2r}{|q|^2}\widehat{\underline{X}} = -\underline{A} + \Gamma_b\cdot\Gamma_b.$$

3. We recover $\underline{B}$, with a control of $k_{small}+32$ derivatives, from

$$\nabla_3\underline{B} - \frac{4}{q}\underline{B} = \frac{1}{2}\overline{\mathcal{D}}\cdot\underline{A} + O(r^{-2})\underline{A} + \Gamma_b\cdot(\underline{B},\underline{A}).$$

4. We recover $\check{Z}$, with a control of $k_{small}+32$ derivatives, from

$$\nabla_3\check{Z} - \frac{2}{\overline{q}}\check{Z} = -\underline{B} + O(r^{-2})\widehat{\underline{X}} + O(r^{-2})\widecheck{\mathrm{tr}\underline{X}} + \Gamma_b\cdot\Gamma_g.$$

5. We recover $\widecheck{\underline{H}}$, with a control of $k_{small}+32$ derivatives, from

$$\nabla_3\widecheck{\underline{H}} - \frac{1}{q}\widecheck{\underline{H}} = \underline{B} + O(r^{-1})\check{Z} + O(r^{-2})\widehat{\underline{X}} + O(r^{-2})\widecheck{\mathrm{tr}\underline{X}} + \Gamma_b\cdot\Gamma_g.$$

6. We recover $\widecheck{\mathcal{D}\cos\theta}$, with a control of $k_{small}+33$ derivatives, from

$$\nabla_3\widecheck{\mathcal{D}\cos\theta} - \frac{1}{\overline{q}}\widecheck{\mathcal{D}\cos\theta} = \frac{i}{2}\overline{\mathfrak{J}}\cdot\widehat{\underline{X}} + O(r^{-1})\widecheck{\mathrm{tr}\underline{X}} + \Gamma_b\cdot\Gamma_g.$$

7. We recover $\mathcal{D}\widehat{\otimes}\mathfrak{J}$, with a control of $k_{small}+32$ derivatives, from

$$\begin{aligned}\nabla_3(\mathcal{D}\widehat{\otimes}\mathfrak{J}) - \frac{2}{\overline{q}}\mathcal{D}\widehat{\otimes}\mathfrak{J} &= O(r^{-1})\underline{B} + O(r^{-2})\widecheck{\mathrm{tr}\underline{X}} + O(r^{-2})\widehat{\underline{X}} \\ &\quad + O(r^{-2})\check{Z} + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)}.\end{aligned}$$

8. We recover $\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}$, with a control of $k_{small}+32$ derivatives, from

$$\begin{aligned}\nabla_3(\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}) - \Re\left(\frac{2}{\overline{q}}\right)\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}} &= O(r^{-1})\underline{B} + O(r^{-2})\widecheck{\mathrm{tr}\underline{X}} + O(r^{-2})\widehat{\underline{X}} \\ &\quad + O(r^{-2})\check{Z} + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)}.\end{aligned}$$

9. We recover $e_4(\cos\theta)$, with a control of $k_{small}+32$ derivatives, from

$$e_3(e_4(\cos\theta)) = O(r^{-1})\widecheck{\underline{H}} + O(r^{-1})\check{Z} + O(r^{-2})\widecheck{\mathcal{D}\cos\theta} + \Gamma_b\cdot\Gamma_b.$$

10. We recover $\check{P}$, with a control of $k_{small}+31$ derivatives, from

$$\begin{aligned}\nabla_3\left(\check{P}\right) - \frac{3}{q}\check{P} &= -\frac{1}{2}\overline{\mathcal{D}}\cdot\underline{B} + O(r^{-2})\underline{B} + O(r^{-3})\widecheck{\mathrm{tr}\underline{X}} + r^{-1}\Gamma_b\cdot\Gamma_g \\ &\quad + \Gamma_g\cdot\underline{A}.\end{aligned}$$

11. We recover $\widecheck{\mathrm{tr}X}$, with a control of $k_{small}+31$ derivatives, from

$$\begin{aligned}\nabla_3\widecheck{\mathrm{tr}X} - \frac{1}{\overline{q}}\widecheck{\mathrm{tr}X} &= \mathcal{D}\cdot\overline{\check{Z}} + 2\overline{\check{P}} + O(r^{-2})\check{Z} + O(r^{-1})\widecheck{\mathrm{tr}\underline{X}} \\ &\quad + O(r^{-1})\widecheck{\mathcal{D}\cdot\overline{\mathfrak{J}}} + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} + \Gamma_b\cdot\Gamma_g.\end{aligned}$$

12. We recover $\widehat{X}$, with a control of $k_{small}+31$ derivatives, from

$$\nabla_3\widehat{X}-\frac{1}{q}\widehat{X} = \frac{1}{2}\mathcal{D}\widehat{\otimes}\check{Z}+O(r^{-2})\check{Z}+O(r^{-1})\underline{\widehat{X}}+O(r^{-1})\mathcal{D}\widehat{\otimes}\mathfrak{J} +O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)}+\Gamma_b\cdot\Gamma_g.$$

13. We recover $\check{\omega}$, with a control of $k_{small}+31$ derivatives, from

$$\nabla_3(\check{\omega}) = \Re(\check{P})+O(r^{-2})\check{Z}+O(r^{-2})\widecheck{\underline{H}}+\Gamma_b\cdot\Gamma_g.$$

14. We recover $\widecheck{e_4(r)}$, with a control of $k_{small}+31$ derivatives, from

$$e_3\left(\widecheck{e_4(r)}\right) = -2\check{\omega}.$$

15. We recover $\widecheck{\nabla_4\mathfrak{J}}$, with a control of $k_{small}+31$ derivatives, from

$$\nabla_3(\widecheck{\nabla_4\mathfrak{J}})-\frac{1}{q}\widecheck{\nabla_4\mathfrak{J}} = O(r^{-3})\widecheck{e_4(r)}+O(r^{-3})e_4(\cos\theta)+O(r^{-2})\check{\omega} +O(r^{-2})\widecheck{\underline{H}}+O(r^{-2})\check{Z}+O(r^{-2})\widecheck{\nabla\mathfrak{J}} +O(r^{-1})\check{P}.$$

16. We recover B, with a control of $k_{small}+30$ derivatives, from

$$\nabla_3 B-\frac{2}{q}B = -\mathcal{D}\overline{\check{P}}+O(r^{-2})\check{P}+O(r^{-3})\check{Z}+O(r^{-4})\widecheck{\mathcal{D}(\cos\theta)} +r^{-1}\Gamma_b\cdot\Gamma_g.$$

17. We recover Ξ, with a control of $k_{small}+30$ derivatives, from

$$\nabla_3\Xi-\frac{1}{q}\Xi = O(r^{-1})\,\not{\mathfrak{d}}^{\leq 1}(\check{\omega})+O(r^{-2})\check{Z}+O(r^{-2})\widecheck{\underline{H}}+O(r^{-2})\widecheck{\mathrm{tr}X} +O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)}+\Gamma_b\cdot\Big(\check{\omega},\Gamma_g\Big).$$

18. We recover A, with a control of $k_{small}+29$ derivatives, from

$$\nabla_3 A-\frac{1}{q}A = \frac{1}{2}\mathcal{D}\widehat{\otimes}B+O(r^{-2})B+O(r^{-3})\widehat{X}.$$

19. We recover $\widecheck{\mathcal{D}\underline{u}}$, with a control of $k_{small}+33$ derivatives, from

$$\nabla_3\widecheck{\mathcal{D}\underline{u}}-\frac{1}{q}\widecheck{\mathcal{D}\underline{u}} = O(r^{-1})\widecheck{\mathrm{tr}\underline{X}}+O(r^{-1})\widehat{\underline{X}}+\Gamma_b\cdot\Gamma_g.$$

20. We recover $\widetilde{e_4(\underline{u})}$, with a control of $k_{small}+32$ derivatives, from

$$e_3\left(\widetilde{e_4(\underline{u})}\right) = O(r^{-1})\widecheck{\underline{H}}+O(r^{-1})\widecheck{Z}+O(r^{-2})\widecheck{\mathcal{D}\underline{u}}+\Gamma_b\cdot\Gamma_b.$$

21. We recover $\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm$, with a control of $k_{small}+32$ derivatives, from

$$\begin{aligned}\nabla_3(\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm)-\frac{2}{\overline{q}}\mathcal{D}\widehat{\otimes}\mathfrak{J}_\pm &= O(r^{-1})\underline{B}+O(r^{-2})\widecheck{\mathrm{tr}\underline{X}}+O(r^{-2})\widehat{\underline{X}}\\ &+O(r^{-2})\widecheck{Z}+O(r^{-3})\widetilde{\mathcal{D}(\cos\theta)}.\end{aligned}$$

22. We recover $\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm}$, with a control of $k_{small}+32$ derivatives, from

$$\begin{aligned}\nabla_3(\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm})-\Re\left(\frac{2}{\overline{q}}\right)\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm} &= O(r^{-1})\underline{B}+O(r^{-2})\widecheck{\mathrm{tr}\underline{X}}+O(r^{-2})\widehat{\underline{X}}\\ &+O(r^{-2})\widecheck{Z}+O(r^{-3})\widetilde{\mathcal{D}(\cos\theta)}.\end{aligned}$$

23. We recover $\widetilde{\nabla_4\mathfrak{J}_\pm}$, with a control of $k_{small}+31$ derivatives, from

$$\begin{aligned}\nabla_3(\widetilde{\nabla_4\mathfrak{J}_\pm})-\frac{1}{\overline{q}}\widetilde{\nabla_4\mathfrak{J}_\pm} &= O(r^{-3})\widetilde{e_4(r)}+O(r^{-3})e_4(\cos\theta)+O(r^{-2})\widecheck{\omega}\\ &+O(r^{-2})\widecheck{\underline{H}}+O(r^{-2})\widecheck{Z}+O(r^{-2})\widecheck{\nabla\mathfrak{J}}\\ &+O(r^{-1})\widecheck{P}.\end{aligned}$$

24. We recover $\widetilde{\mathcal{D}(J^{(\pm)})}$, with a control of $k_{small}+33$ derivatives, from

$$\nabla_3(\widetilde{\mathcal{D}(J^{(\pm)})})-\frac{1}{\overline{q}}\widetilde{\mathcal{D}(J^{(\pm)})} = O(r^{-1})\widecheck{\mathrm{tr}\underline{X}}+O(r^{-1})\widehat{\underline{X}}+\Gamma_b\cdot\Gamma_g.$$

25. We recover $\widetilde{\nabla_4 J^{(\pm)}}$, with a control of $k_{small}+32$ derivatives, from

$$\nabla_3(\widetilde{\nabla_4 J^{(\pm)}}) = O(r^{-2})\widetilde{\mathcal{D}(J^{(\pm)})}+O(r^{-2})\widecheck{Z}+O(r^{-2})\widecheck{\underline{H}}+\Gamma_b\cdot\Gamma_b.$$

As the estimates are significantly simpler to derive[153] and in the same spirit as the corresponding ones in Theorem M4, we leave the details to the reader. This concludes the proof of Theorem M5 for the part of ${}^{(int)}\mathcal{M}$.

[153]Note that r is bounded on ${}^{(int)}\mathcal{M}$ and that all quantities behave the same in ${}^{(int)}\mathcal{M}$.

7.5. Decay estimates for the PG structure of ${}^{(top)}\mathcal{M}$ on $\{u = u_*\}$

To simplify the notations, in this section, we denote

- by (e_4, e_3, e_1, e_2) the outgoing PG frame of ${}^{(ext)}\mathcal{M}$, with all quantities associated to the outgoing PG structure of ${}^{(ext)}\mathcal{M}$ being unprimed,
- by (e'_4, e'_3, e'_1, e'_2) the ingoing PG frame of ${}^{(top)}\mathcal{M}$, with all quantities associated to the ingoing PG structure of ${}^{(top)}\mathcal{M}$ being primed.

Remark 7.12. *Note that in ${}^{(top)}\mathcal{M}$, we do not need to define φ', $J'^{(\pm)}$ and $\mathfrak{J}'_\pm$. In particular, recall from Remark 3.14 that the quantities Γ'_g, Γ'_b in ${}^{(top)}\mathcal{M}$ correspond to the ones in Definition 2.73 where all linearized quantities based on $J'^{(\pm)}$ and $\mathfrak{J}'_\pm$ have been removed.*

Recall that ${}^{(ext)}\mathcal{M} \cap {}^{(top)}\mathcal{M} = \{u = u_*\}$. In view of the above notations, and the initialization of the ingoing PG structure of ${}^{(top)}\mathcal{M}$ from the outgoing PG structure of ${}^{(ext)}\mathcal{M}$ on $\{u = u_*\}$, see Section 3.2.5, we have

$$r' = r, \quad J'^{(0)} = J^{(0)}, \ p = 0, +, -, \quad \mathfrak{J}' = \mathfrak{J}, \tag{7.32}$$

$$\underline{u} = u + 2\int_{r_0}^{r} \frac{\tilde{r}^2 + a^2}{\tilde{r}^2 - 2m\tilde{r} + a^2} d\tilde{r}, \tag{7.33}$$

and

$$\begin{aligned} e'_4 &= \lambda\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right), \\ e'_a &= e_a + \frac{1}{2} f_a e_3, \\ e'_3 &= \lambda^{-1} e_3, \end{aligned} \tag{7.34}$$

where

$$\lambda = \frac{\Delta}{|q|^2}, \qquad f = h\widetilde{e_3(r)}\nabla(u), \tag{7.35}$$

with the scalar function h given by

$$h \;=\; -\frac{2}{\lambda e_3(u)}. \tag{7.36}$$

Note in particular that $h = 1 + O(r^{-2})$, which together with the fact that $\widetilde{e_3(r)} \in r\Gamma_b$ and $\nabla(u) = O(r^{-1})$ implies

$$f \in \Gamma_b. \tag{7.37}$$

In order to derive decay estimates for the ingoing PG structure of ${}^{(top)}\mathcal{M}$ on $\{u=u_*\}$, we will rely on the following lemma.

Lemma 7.13. *We have on* $\{u=u_*\}$

$$\begin{aligned}
&A'=\lambda^2A+\Gamma_b\cdot B+r^{-3}\Gamma_b\cdot\Gamma_b, \quad B'=\lambda B+r^{-3}\Gamma_b,\\
&\check{P}'=\check{P}+r^{-1}\Gamma_b\cdot\Gamma_b, \quad \underline{B}'=\lambda^{-1}\underline{B}+r^{-3}\Gamma_b, \quad \underline{A}'=\lambda^{-2}\underline{A},\\
&\underline{\Xi}'=0, \quad \underline{\omega}'=0, \quad H'=Z',\\
&\Xi'=r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b, \quad \omega'=-\frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right)+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b, \quad \widecheck{\underline{H}}'=-\check{Z}+r^{-1}\Gamma_b,\\
&e_3'(r')=-1, \quad \nabla'(r')=0, \quad e_3'(\underline{u})=0, \quad e_3'(J'^{(0)})=0, \quad \nabla_3'\mathfrak{J}'=\frac{1}{\overline{q'}}\mathfrak{J}',\\
&\widecheck{e_4'(r')}=\Gamma_b\cdot\Gamma_b, \quad \widecheck{e_4'(\underline{u})}=r^{-1}\Gamma_b, \quad e_4'(J'^{(0)})=r^{-1}\Gamma_b, \quad \widecheck{\nabla_{e_4'}\mathfrak{J}'}=r^{-2}\Gamma_b,
\end{aligned}$$

$$\begin{aligned}
\widecheck{\nabla'(\underline{u})}&=\widecheck{\nabla(u)}+\Gamma_b, \qquad \widecheck{\nabla'(J'^{(0)})}=\widecheck{\nabla(J^{(0)})}+r^{-1}\Gamma_b,\\
\mathcal{D}'\widehat{\otimes}\mathfrak{J}'&=\mathcal{D}\widehat{\otimes}\mathfrak{J}+r^{-2}\Gamma_b, \qquad \overline{\mathcal{D}}'\cdot\mathfrak{J}'=\overline{\mathcal{D}}\cdot\mathfrak{J}+r^{-2}\Gamma_b,
\end{aligned}$$

$$\begin{aligned}
\widecheck{trX}'&=\lambda\widecheck{trX}+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b, \qquad \widehat{X}'=\lambda\widehat{X}+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b,\\
\widecheck{tr\underline{X}}'&=\lambda^{-1}\widecheck{tr\underline{X}}+r^{-1}\Gamma_b, \qquad \widehat{\underline{X}}'=\lambda^{-1}\widehat{\underline{X}}+r^{-1}\Gamma_b,
\end{aligned}$$

and

$$\check{Z}' \;=\; \check{Z}+\frac{1}{q}\widecheck{\mathcal{D}(q)}+\frac{1}{\overline{q}}\widecheck{\mathcal{D}(\overline{q})}+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b,$$

where the definition of the linearized quantities for the outgoing PG structure of ${}^{(ext)}\mathcal{M}$ *can be found in Definition 2.66, while definition of the linearized quantities for the ingoing PG structure of* ${}^{(top)}\mathcal{M}$ *can be found in Definition 2.72.*

Proof. The identities for $\underline{\Xi}'$, $\underline{\omega}'$, $H'-Z'$, $e_3'(r')$, $e_3'(\underline{u})$, $e_3'(J'^{(0)})$ and $\nabla_3'\mathfrak{J}'$ come from the ingoing PG structure assumption on ${}^{(top)}\mathcal{M}$. Also, the identities for A', B', $\underline{B}'$ and $\underline{A}'$ follow immediately from the change of frame formulas of Proposition 2.12 with coefficients $(f,\underline{f}=0,\lambda)$ and the fact that $f\in\Gamma_b$, and the identity for $\check{P}'$ follows using additionally the fact that $q'=q$ on $\{u=u_*\}$.

Next, note that we have

$$\begin{aligned}
\lambda^{-1}e_4' &= e_4+f^be_b+\frac{1}{4}|f|^2e_3\\
&= e_4+f^b\left(e_b'-\frac{1}{2}f_b\lambda e_3'\right)+\frac{1}{4}|f|^2\lambda e_3'
\end{aligned}$$

and hence

$$e_4 = \lambda^{-1}e_4' - f^b e_b' + \frac{1}{4}|f|^2\lambda e_3'.$$

In particular, since $f \in \Gamma_b$, we infer

$$e_4 = \lambda^{-1}e_4' + r^{-1}\Gamma_b \cdot \mathfrak{d}.$$

Since e_4 is tangent to $\{u = u_*\}$, this implies

$$\begin{aligned}
\mathbf{g}(\mathbf{D}_{\lambda^{-1}e_4'}e_4', e_a') &= \mathbf{g}\left(\mathbf{D}_{e_4}\left(\lambda\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right)\right), e_a + \frac{1}{2}f_a e_3\right)\\
&\quad + r^{-1}\Gamma_b,\\
\mathbf{g}(\mathbf{D}_{\lambda^{-1}e_4'}e_4', e_3') &= \mathbf{g}\left(\mathbf{D}_{e_4}\left(\lambda\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right)\right), \lambda^{-1}e_3\right) + r^{-1}\Gamma_b,\\
\mathbf{g}(\mathbf{D}_{\lambda^{-1}e_4'}e_3', e_a') &= \mathbf{g}\left(\mathbf{D}_{e_4}\left(\lambda^{-1}e_3\right), e_a + \frac{1}{2}f_a e_3\right) + r^{-1}\Gamma_b,
\end{aligned}$$

which yields

$$\begin{aligned}
\lambda^{-2}\Xi' &= \Xi + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b,\\
\lambda^{-1}\omega' &= \omega - \frac{1}{2}\lambda^{-1}e_4(\lambda) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b,\\
\underline{H}' &= \underline{H} + r^{-1}\Gamma_b.
\end{aligned}$$

Together with the explicit choice for λ, the fact that $\Xi = \omega = 0$ and $\underline{H} = -Z$, and the fact that $q' = q$ and $\mathfrak{J}' = \mathfrak{J}$ on $\{u = u_*\}$, this immediately yields the identities for Ξ', $\check{\omega}'$ and $\widecheck{\underline{H}}'$. Also, the identities for $\widecheck{e_4'(r')}$, $\widecheck{e_4'(\underline{u})}$, $e_4'(J'^{(0)})$ and $\widecheck{\nabla_{e_4'}\mathfrak{J}'}$ follow immediately from the fact that we have, on $\{u = u_*\}$, $e_4 = \lambda^{-1}e_4' + r^{-1}\Gamma_b \cdot \mathfrak{d}$, $r' = r$, $J'^{(0)} = J^{(0)}$, $\mathfrak{J}' = \mathfrak{J}$, and

$$\underline{u} = u + 2\int_{r_0}^{r} \frac{\tilde{r}^2 + a^2}{\tilde{r}^2 - 2m\tilde{r} + a^2}d\tilde{r},$$

together with the fact that e_4 is tangent to $\{u = u_*\}$, and the fact that $e_4(r) = 1$, $e_4(u) = 0$, $e_4(J^{(0)}) = 0$ and $\nabla_4\mathfrak{J} = -\frac{1}{q}\mathfrak{J}$.

It remains to derive the identities for $\nabla'(r')$, $\check{Z}'$, $\widecheck{\mathrm{tr}X}'$, $\widehat{X}'$, $\widecheck{\mathrm{tr}\underline{X}}'$, $\widehat{\underline{X}}'$, $\widecheck{\nabla'(\underline{u})}$, $\widecheck{\nabla'(J'^{(0)})}$, $\widecheck{\overline{\mathcal{D}}'} \cdot \mathfrak{J}'$ and $\mathcal{D}'\widehat{\otimes}\mathfrak{J}'$. Since, $\nabla - \frac{1}{e_3(u)}\nabla(u)\nabla_3$ is tangent to $\{u = u_*\}$ and since

$$\nabla - \frac{1}{e_3(u)}\nabla(u)\nabla_3 = \nabla' - \frac{1}{2}f\lambda\nabla_{e_3'} - \frac{1}{e_3(u)}\nabla(u)\nabla_{\lambda e_3'},$$

we have on $\{u=u_*\}$

$$\left(\nabla'-\frac{1}{2}f\lambda\nabla_{e_3'}-\frac{1}{e_3(u)}\nabla(u)\nabla_{\lambda e_3'}\right)r' \;=\; \left(\nabla-\frac{1}{e_3(u)}\nabla(u)\nabla_3\right)r.$$

Since $\nabla(r)=0$ and $e_3'(r')=-1$, we infer, using also $e_3(r)=-\lambda+\widecheck{e_3(r)}$,

$$\nabla'(r')+\frac{\lambda}{2}f \;=\; -\frac{\widecheck{e_3(r)}}{e_3(u)}\nabla(u).$$

Now, recall that we have chosen $f=h\widecheck{e_3(r)}\nabla(u)$. We infer

$$\nabla'(r') \;=\; -\frac{\lambda}{2}\left(h+\frac{2}{\lambda e_3(u)}\right)\widecheck{e_3(r)}\nabla(u).$$

Since $h=-\frac{2}{\lambda e_3(u)}$, we infer

$$\nabla'(r') \;=\; 0$$

as desired.

Next, using $f\in\Gamma_b$, note that we have

$$\begin{aligned}\nabla-\frac{1}{e_3(u)}\nabla(u)\nabla_3 &= \nabla'-\frac{1}{2}f\lambda\nabla_{e_3'}-\frac{1}{e_3(u)}\nabla(u)\nabla_{\lambda e_3'}\\ &= \nabla'-\frac{\lambda}{e_3(u)}\nabla(u)\nabla_{e_3'}+\Gamma_b\cdot\nabla_{e_3'}.\end{aligned}$$

Together with the fact that that we have, on $\{u=u_*\}$, $J'^{(0)}=J^{(0)}$, $\mathfrak{J}'=\mathfrak{J}$, and

$$\underline{u}=u+2\int_{r_0}^{r}\frac{\tilde{r}^2+a^2}{\tilde{r}^2-2m\tilde{r}+a^2}d\tilde{r},$$

and together with the fact that $\nabla-\frac{1}{e_3(u)}\nabla(u)\nabla_3$ is tangent to $\{u=u_*\}$, we infer

$$\begin{aligned}\left(\nabla'-\frac{\lambda}{e_3(u)}\nabla(u)\nabla_{e_3'}+\Gamma_b\cdot\nabla_{e_3'}\right)\underline{u} \;&=\; \left(\nabla-\frac{1}{e_3(u)}\nabla(u)\nabla_3\right)\\ &\quad\left(u+2\int_{r_0}^{r}\frac{\tilde{r}^2+a^2}{\tilde{r}^2-2m\tilde{r}+a^2}d\tilde{r}\right),\end{aligned}$$

$$\begin{aligned}
\left(\nabla' - \frac{\lambda}{e_3(u)}\nabla(u)\nabla_{e_3'} + \Gamma_b\cdot\nabla_{e_3'}\right) J'^{(0)} &= \left(\nabla - \frac{1}{e_3(u)}\nabla(u)\nabla_3\right) J^{(0)},\\
\left(\nabla' - \frac{\lambda}{e_3(u)}\nabla(u)\nabla_{e_3'} + \Gamma_b\cdot\nabla_{e_3'}\right)\mathfrak{J}' &= \left(\nabla - \frac{1}{e_3(u)}\nabla(u)\nabla_3\right)\mathfrak{J} + r^{-2}\Gamma_b.
\end{aligned}$$

Using the fact that $q' = q$ on $\{u = u_*\}$, and since

$$e_3'(r') = -1, \quad e_3'(\underline{u}) = 0, \quad e_3'(J'^{(0)}) = 0, \quad \nabla_3'\mathfrak{J}' = \frac{1}{\overline{q}'}\mathfrak{J}', \quad \nabla(r) = 0,$$

we infer

$$\begin{aligned}
\nabla'(\underline{u}) &= \nabla(u) - \frac{1}{e_3(u)}\nabla(u)\left(e_3(u) + 2e_3(r)\frac{r^2+a^2}{\Delta}\right)\\
&= \nabla(u) + \Gamma_b,\\
\nabla'(J'^{(0)}) &= \nabla(J^{(0)}) + r^{-1}\Gamma_b,\\
\nabla'\mathfrak{J}' - \frac{\lambda}{e_3(u)\overline{q}}\nabla(u)\mathfrak{J}' &= \left(\nabla - \frac{1}{e_3(u)}\nabla(u)\nabla_3\right)\mathfrak{J} + r^{-2}\Gamma_b,
\end{aligned}$$

and hence, since $q' = q$ and $\mathfrak{J}' = \mathfrak{J}$ on $\{u = u_*\}$, we obtain

$$\begin{aligned}
&\widetilde{\nabla'(\underline{u})} = \widetilde{\nabla(u)} + \Gamma_b, \qquad \nabla'\widetilde{(J'^{(0)})} = \nabla\widetilde{(J^{(0)})} + r^{-1}\Gamma_b,\\
&\mathcal{D}'\widehat{\otimes}\mathfrak{J}' = \mathcal{D}\widehat{\otimes}\mathfrak{J} + r^{-2}\Gamma_b, \qquad \overline{\mathcal{D}}'\cdot\mathfrak{J}' = \overline{\mathcal{D}}\cdot\mathfrak{J} + r^{-2}\Gamma_b,
\end{aligned}$$

as stated.

It remains to derive the identities for $\check{Z}'$, $\widetilde{\mathrm{tr}X}'$, $\widehat{X}'$, $\widetilde{\mathrm{tr}\underline{X}}'$ and $\widehat{\underline{X}}'$. In view of

$$\begin{aligned}
e_a &= e_a' - \frac{1}{2}f_a\lambda e_3',\\
e_3 &= \lambda e_3',
\end{aligned}$$

we have on $\{u = u_*\}$, since $f \in \Gamma_b$,

$$e_a' - \frac{\lambda}{e_3(u)}e_a(u)e_3' = e_a - \frac{1}{e_3(u)}e_a(u)e_3 + r^{-1}\Gamma_b\mathfrak{d}.$$

Since $e_a - \frac{1}{e_3(u)}e_a(u)e_3$ is tangent to $\{u = u_*\}$, and since

$$
\begin{aligned}
e_4' &= \lambda\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right),\\
e_a' &= e_a + \frac{1}{2} f_a e_3,\\
e_3' &= \lambda^{-1} e_3,
\end{aligned}
\tag{7.38}
$$

we infer on $\{u = u_*\}$

$$
\begin{aligned}
& \mathbf{g}\left(\mathbf{D}_{e_a' - \frac{\lambda}{e_3(u)}e_a(u)e_3'} e_4', e_3'\right)\\
=& \mathbf{g}\left(\mathbf{D}_{e_a - \frac{1}{e_3(u)}e_a(u)e_3}\left(\lambda\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right)\right), \lambda^{-1} e_3\right) + r^{-1}\Gamma_b,\\
& \mathbf{g}\left(\mathbf{D}_{e_a' - \frac{\lambda}{e_3(u)}e_a(u)e_3'} e_4', e_b'\right)\\
=& \mathbf{g}\left(\mathbf{D}_{e_a - \frac{1}{e_3(u)}e_a(u)e_3}\left(\lambda\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right)\right), e_b + \frac{1}{2} f_b e_3\right) + r^{-1}\Gamma_b,\\
& \mathbf{g}\left(\mathbf{D}_{e_a' - \frac{\lambda}{e_3(u)}e_a(u)e_3'} e_3', e_b'\right)\\
=& \mathbf{g}\left(\mathbf{D}_{e_a - \frac{1}{e_3(u)}e_a(u)e_3}\left(\lambda^{-1} e_3\right), e_b + \frac{1}{2} f_b e_3\right) + r^{-1}\Gamma_b,
\end{aligned}
$$

and hence, using again $f \in \Gamma_b$,

$$
\begin{aligned}
\mathbf{g}\left(\mathbf{D}_{e_a' - \frac{\lambda}{e_3(u)}e_a(u)e_3'} e_4', e_3'\right) &= \lambda^{-1}\mathbf{g}\left(\mathbf{D}_{e_a - \frac{1}{e_3(u)}e_a(u)e_3}(\lambda e_4), e_3\right) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b,\\
\mathbf{g}\left(\mathbf{D}_{e_a' - \frac{\lambda}{e_3(u)}e_a(u)e_3'} e_4', e_b'\right) &= \lambda\mathbf{g}\left(\mathbf{D}_{e_a - \frac{1}{e_3(u)}e_a(u)e_3} e_4, e_b\right) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b,\\
\mathbf{g}\left(\mathbf{D}_{e_a' - \frac{\lambda}{e_3(u)}e_a(u)e_3'} e_3', e_b'\right) &= \lambda^{-1}\mathbf{g}\left(\mathbf{D}_{e_a - \frac{1}{e_3(u)}e_a(u)e_3} e_3, e_b\right) + r^{-1}\Gamma_b.
\end{aligned}
$$

We infer on $\{u = u_*\}$

$$
\begin{aligned}
2\zeta_a' + \frac{4\lambda}{e_3(u)}e_a(u)\underline{\omega}' &= -2\left(e_a - \frac{1}{e_3(u)}e_a(u)e_3\right)\log\lambda + 2\zeta_a\\
&\quad + \frac{4}{e_3(u)}e_a(u)\underline{\omega} + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b,\\
\chi_{ab}' - \frac{2\lambda}{e_3(u)}e_a(u)\eta_b' &= \lambda\chi_{ab} - \frac{2\lambda}{e_3(u)}e_a(u)\eta_b + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b,
\end{aligned}
$$

$$\underline{\chi}'_{ab} - \frac{2\lambda}{e_3(u)}e_a(u)\underline{\xi}'_b = \lambda^{-1}\underline{\chi}_{ab} - \frac{2\lambda^{-1}}{e_3(u)}e_a(u)\underline{\xi}_b + r^{-1}\Gamma_b.$$

Since $\underline{\xi}' = 0$, $\underline{\omega}' = 0$ and $\eta' = \zeta'$, and since $\underline{\xi} \in \Gamma_b$ and $\underline{\check{\omega}} \in \Gamma_b$, we deduce on $\{u = u_*\}$

$$\begin{aligned}
2\zeta'_a &= -2\left(e_a - \frac{1}{e_3(u)}e_a(u)e_3\right)\log\lambda + 2\zeta_a \\
&\quad + \frac{2}{e_3(u)}e_a(u)\partial_r\left(\frac{\Delta}{|q|^2}\right) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b, \\
\chi'_{ab} - \frac{2\lambda}{e_3(u)}e_a(u)\zeta'_b &= \lambda\chi_{ab} - \frac{2\lambda}{e_3(u)}e_a(u)\eta_b + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b, \\
\underline{\chi}'_{ab} &= \lambda^{-1}\underline{\chi}_{ab} + r^{-1}\Gamma_b.
\end{aligned}$$

In particular, since $q' = q$ on $\{u = u_*\}$, we have

$$\widetilde{\mathrm{tr}\underline{X}}' = \lambda^{-1}\widetilde{\mathrm{tr}\underline{X}} + r^{-1}\Gamma_b, \qquad \widehat{\underline{X}}' = \lambda^{-1}\widehat{\underline{X}} + r^{-1}\Gamma_b,$$

as stated. Also, since we have

$$-2\left(-\frac{1}{e_3(u)}e_a(u)e_3\right)\log\lambda + \frac{2}{e_3(u)}e_a(u)\partial_r\left(\frac{\Delta}{|q|^2}\right) = r^{-1}\Gamma_b,$$

we infer from the first identity

$$Z' = Z - \mathcal{D}(\log\lambda) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b.$$

Using the explicit for of λ, as well as $\nabla(r) = 0$, we infer on $\{u = u_*\}$

$$Z' = Z - \mathcal{D}\left(\log\left(\frac{\Delta}{|q|^2}\right)\right) + r^{-1}\Gamma_b = Z + \frac{1}{q}\mathcal{D}(q) + \frac{1}{\overline{q}}\mathcal{D}(\overline{q}) + r^{-1}\Gamma_b,$$

which yields, together with the fact that $\mathfrak{J}' = \mathfrak{J}$ and $q' = q$ on $\{u = u_*\}$, in view of the linearization of the various quantities, and taking the different linearization for Z' (ingoing PG structure) and Z (outgoing PG structure) into account,

$$\check{Z}' = \check{Z} + \frac{1}{q}\widetilde{\mathcal{D}(q)} + \frac{1}{\overline{q}}\widetilde{\mathcal{D}(\overline{q})} + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b$$

as desired.

Finally, coming back to

$$\chi'_{ab} - \frac{2\lambda}{e_3(u)} e_a(u)\zeta'_b = \lambda\chi_{ab} - \frac{2\lambda}{e_3(u)} e_a(u)\eta_b + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b$$

and since we have in view of the above on $\{u = u_*\}$

$$\begin{aligned} Z' - H &= Z - \mathcal{D}(\log\lambda) - H + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b = Z + \frac{1}{q}\mathcal{D}(q) + \frac{1}{\overline{q}}\mathcal{D}(\overline{q}) - H \\ &\quad + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b \\ &= \check{Z} + \frac{1}{q}\widetilde{\mathcal{D}(q)} + \frac{1}{\overline{q}}\widetilde{\mathcal{D}(\overline{q})} - \check{H} + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b = \mathfrak{d}^{\leq 1}\Gamma_b, \end{aligned}$$

we deduce

$$\widetilde{\mathrm{tr} X}' = \lambda\widetilde{\mathrm{tr} X} + r^{-1}\Gamma_b, \qquad \widehat{X}' = \lambda\widehat{X} + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b,$$

as desired. This concludes the proof of the lemma. □

We are now ready to derive decay estimates for the ingoing PG structure of ${}^{(top)}\mathcal{M}$ on $\{u = u_*\}$.

Lemma 7.14. *The following decay estimates hold on $\{u = u_*\}$ for the ingoing PG structure of ${}^{(top)}\mathcal{M}$*

$$\begin{aligned} &\sup_{\{u=u_*\}}\Big(ru^{1+\delta_{dec}} + r^2 u^{\frac{1}{2}+\delta_{dec}}\Big)|\mathfrak{d}^{\leq k_{small}+39}\Gamma'_g| \\ &\qquad + \sup_{\{u=u_*\}} ru^{1+\delta_{dec}}|\mathfrak{d}^{\leq k_{small}+39}\Gamma'_b| \lesssim \epsilon_0. \end{aligned} \tag{7.39}$$

Proof. In view of the control of outgoing PG structure of ${}^{(ext)}\mathcal{M}$ established in Theorem M4 and the fact that $\{u = u_*\} \subset {}^{(ext)}\mathcal{M}$, we have

$$\begin{aligned} &\sup_{\{u=u_*\}}\Big(ru^{1+\delta_{dec}} + r^2 u^{\frac{1}{2}+\delta_{dec}}\Big)|\mathfrak{d}^{\leq k_{small}+40}\Gamma_g| \\ &\qquad + \sup_{\{u=u_*\}} ru^{1+\delta_{dec}}|\mathfrak{d}^{\leq k_{small}+40}\Gamma_b| \lesssim \epsilon_0. \end{aligned}$$

Note that the tangential derivatives to $\{u = u_*\}$ are generated by e_4 and $\nabla - \frac{1}{e_3(u)}\nabla(u)\nabla_3$. We introduce the following notation for r-weighted tangential derivatives to $\{u = u_*\}$

$$\widetilde{\mathfrak{d}} := \left(r\left(\nabla - \frac{1}{e_3(u)}\nabla(u)\nabla_3\right), r\nabla_4\right).$$

From the above estimate for (Γ_g, Γ_b), together the identities of the previous lemma on $\{u = u_*\}$, we infer

$$\sup_{\{u=u_*\}}\Big(ru^{1+\delta_{dec}} + r^2u^{\frac{1}{2}+\delta_{dec}}\Big)|\widetilde{\mathfrak{d}}^{\leq k_{small}+39}\Gamma'_g| + \sup_{\{u=u_*\}} ru^{1+\delta_{dec}}|\widetilde{\mathfrak{d}}^{\leq k_{small}+39}\Gamma'_b| \lesssim \epsilon_0.$$

Finally, since $\mathfrak{d}$ is generated by e'_3 and $\widetilde{\mathfrak{d}}$, the previous estimate and the control of e'_3 derivatives provided by the null structure equations and Bianchi identities of the ingoing PG structure of ${}^{(top)}\mathcal{M}$ immediately imply

$$\sup_{\{u=u_*\}}\Big(ru^{1+\delta_{dec}} + r^2u^{\frac{1}{2}+\delta_{dec}}\Big)|\mathfrak{d}^{\leq k_{small}+39}\Gamma'_g| + \sup_{\{u=u_*\}} ru^{1+\delta_{dec}}|\mathfrak{d}^{\leq k_{small}+39}\Gamma'_b| \lesssim \epsilon_0$$

as stated. □

7.6. Decay estimates for $\underline{A}$ in ${}^{(top)}\mathcal{M}$

In this section, we initiate the proof of the part of Theorem M5 concerning ${}^{(top)}\mathcal{M}$, i.e. to derive decay estimates for the ingoing PG structure of ${}^{(top)}\mathcal{M}$, by first controlling $\underline{A}$ in ${}^{(top)}\mathcal{M}$. Recall that ${}^{(top)}\mathcal{M}$ is endowed with a PG structure defined in Sections 2.7 and 3.2.2. In particular $\underline{\Xi} = 0$, $\underline{\omega} = 0$.

The goal of this section is to prove the following proposition.

Proposition 7.15. *$\underline{A}$, corresponding to the ingoing PG structure of ${}^{(top)}\mathcal{M}$, verifies the following estimate in ${}^{(top)}\mathcal{M}$*

$$\sup_{{}^{(top)}\mathcal{M}(r\leq r_0)} \underline{u}^{1+\delta_{dec}}|\mathfrak{d}^{\leq k_{small}+31}\underline{A}| + \sup_{{}^{(top)}\mathcal{M}(r\geq r_0)} ru^{1+\delta_{dec}}|\mathfrak{d}^{\leq k_{small}+31}\underline{A}| \lesssim \epsilon_0. \tag{7.40}$$

Proof. The proof proceeds in the following steps.

Step 1. We denote with primes quantities with respect to the global null frame of Proposition 3.33. Recall from (7.31) that we have obtained the following estimate for $\underline{A}'$ on $\mathcal{M}$

$$E^{k_{small}+35}[\underline{A}'](\tau) \lesssim \epsilon_0^2\tau^{-2-2\delta_{dec}} \quad \text{for} \quad \tau \geq 1,$$

see (7.25) for the definition of the energy.

We then use the following lemma.

Lemma 7.16. *We have the estimate*

$$r^{-1}\int_{S(\tau,r)}|\mathfrak{d}^s\psi|^2 \lesssim E^s[\psi](\tau).$$

Proof. See section 5.4.4 in [39]. □

In view of the above lemma, we infer on $\mathcal{M}$, for $k\leq k_{small}+35$,

$$r^{-1}\int_{S(\tau,r)}|\mathfrak{d}^k\underline{A}'|^2\lesssim\epsilon_0^2\tau^{-2-2\delta_{dec}}.$$

We now restrict the estimate to ${}^{(top)}\mathcal{M}$ and deduce using Sobolev, for $k\leq k_{small}+33$,

$$|\mathfrak{d}^k\underline{A}'|\lesssim\epsilon_0 r^{-\frac{1}{2}}\tau^{-1-\delta_{dec}}.$$

Using the control of the change of frame from the frame of ${}^{(top)}\mathcal{M}$ to the one of Proposition 3.33 provided by property (f) of that proposition, and the change of frame formula for $\underline{A}$ provided by Proposition 2.12, we deduce, for $k\leq k_{small}+33$,

$$|\mathfrak{d}^k\underline{A}|\lesssim\epsilon_0 r^{-\frac{1}{2}}\tau^{-1-\delta_{dec}}. \tag{7.41}$$

Step 2. Since $\tau\sim\underline{u}$ for $r\leq r_0$ and $\tau\sim u$ for $r\geq r_0$, the estimate (7.41) has a loss of $r^{\frac{1}{2}}$ compared to the desired estimate for $\underline{A}$ in ${}^{(top)}\mathcal{M}(r\geq r_0)$. In order to improve it, we first derive estimates for $(\underline{\widehat{X}},\underline{B})$ with a loss of $r^{\frac{1}{2}}$ using the linearized null structure equations of Lemma 7.2

$$\begin{aligned}\nabla_3\underline{\widehat{X}}-\frac{2r}{|q|^2}\underline{\widehat{X}} &= -\underline{A}+\Gamma_b\cdot\Gamma_b,\\ \nabla_3\underline{B}-\frac{4}{q}\underline{B} &= \frac{1}{2}\overline{\mathcal{D}}\cdot\underline{A}+O(r^{-2})\underline{A}+\Gamma_b\cdot(\underline{B},\underline{A}).\end{aligned}$$

Integrating these transport equations from $\{u=u_*\}$ and using:

- the control of $\underline{B}$ and $\underline{\widehat{X}}$ on $\{u=u_*\}$ provided by Lemma 7.14,
- the control of $\underline{A}$ in ${}^{(top)}\mathcal{M}$ provided by (7.41),
- the bootstrap assumptions on decay and energy,

we easily infer[154] on ${}^{(top)}\mathcal{M}(r \geq r_0)$, for $k \leq k_{small} + 32$,

$$
\begin{aligned}
|\mathfrak{d}^s \widehat{\underline{X}}| &\lesssim \epsilon_0 r^{-\frac{1}{2}} \tau^{-1-\delta_{dec}}, \\
|\mathfrak{d}^s \underline{B}| &\lesssim \epsilon_0 r^{-\frac{3}{2}} \tau^{-1-\delta_{dec}}.
\end{aligned} \tag{7.42}
$$

Step 3. Notice that (7.41) yields the desired estimate for $\underline{A}$ in ${}^{(top)}\mathcal{M}(r \leq r_0)$. It thus remains to improve (7.41) in ${}^{(top)}\mathcal{M}(r \geq r_0)$ by $r^{-\frac{1}{2}}$. To this end, we rely on the following Bianchi identity of Proposition 2.9

$$
\nabla_4 \underline{A} + \frac{1}{2}\mathcal{D}\widehat{\otimes}\underline{B} \;\; = \;\; -\frac{1}{2}\mathrm{tr}X\underline{A} + 4\omega\underline{A} + \frac{1}{2}(Z - 4\,\underline{H})\widehat{\otimes}\underline{B} - 3P\widehat{\underline{X}}
$$

which we rewrite in ${}^{(top)}\mathcal{M}(r \geq r_0)$ under the following form

$$
\nabla_4(q\underline{A}) \;\; = \;\; O(r^{-1})\underline{A} + O(1)\,\mathfrak{d}\!\!/^{\leq 1}\underline{B} + O(r^{-2})\widehat{\underline{X}} + r\Gamma_g \cdot \Gamma_b.
$$

Using the bootstrap assumptions on decay and energy, the control of $\underline{A}$ in ${}^{(top)}\mathcal{M}$ provided by (7.41), and the control of $(\widehat{\underline{X}}, \underline{B})$ in ${}^{(top)}\mathcal{M}$ provided by (7.42), we deduce in ${}^{(top)}\mathcal{M}(r \geq r_0)$

$$
\nabla_4(q\underline{A}) = h, \qquad |\mathfrak{d}^k h| \lesssim \epsilon_0 r^{-\frac{3}{2}} \tau^{-1-\delta_{dec}} \quad \text{for} \quad k \leq k_{small} + 31.
$$

Integrating this transport equation forward from ${}^{(top)}\mathcal{M} \cap \{r = r_0\}$ where $\underline{A}$ is under control in view of (7.41), we infer on ${}^{(top)}\mathcal{M}(r \geq r_0)$, for $k \leq k_{small} + 31$,

$$
|\mathfrak{d}^k \underline{A}| \lesssim \epsilon_0 r^{-1} \tau^{-1-\delta_{dec}}.
$$

Since $\tau \sim \underline{u}$ for $r \leq r_0$ and $\tau \sim u$ for $r \geq r_0$, we deduce from (7.41) and the above estimate that $\underline{A}$ verifies the following estimate in ${}^{(top)}\mathcal{M}$

$$
\sup_{{}^{(top)}\mathcal{M}(r \leq r_0)} \underline{u}^{1+\delta_{dec}} |\mathfrak{d}^{\leq k_{small}+31} \underline{A}| + \sup_{{}^{(top)}\mathcal{M}(r \geq r_0)} r u^{1+\delta_{dec}} |\mathfrak{d}^{\leq k_{small}+31} \underline{A}| \;\; \lesssim \;\; \epsilon_0
$$

as stated. This concludes the proof of Proposition 7.15. □

[154] Along a level set of $\underline{u}$ in ${}^{(top)}\mathcal{M}$, denoting $r_+(\underline{u})$ the maximal value of ${}^{(top)}r$, i.e. the one on $\{u = u_*\}$, and $r_-(\underline{u})$ the minimal value of ${}^{(top)}r$, i.e. the one on ${}^{(top)}\Sigma$, we have $0 < r_+(\underline{u}) - r_-(\underline{u}) \lesssim 1$, see (3.7). In particular, the integration along e_3 is always finite in ${}^{(top)}\mathcal{M}$ and hence easy.

7.7. Decay estimates in ${}^{(top)}\mathcal{M}$

We are now ready to prove the part of Theorem M5 concerning ${}^{(top)}\mathcal{M}$, i.e. to derive decay estimates for the ingoing PG structure of ${}^{(top)}\mathcal{M}$. To this end, recall first that $\underline{A}$ has already been estimated on ${}^{(top)}\mathcal{M}$ in Proposition 7.15, see (7.40). Relying on the estimates of the ingoing PG structure of ${}^{(top)}\mathcal{M}$ on $\{u = u_*\}$ derived in Lemma 7.14, we propagate these estimates to ${}^{(top)}\mathcal{M}$ thanks to the linearized transport equations in the e_3 direction of Section 7.1 for ingoing PG structures. Recalling that $\underline{A}$ has already been estimated in (7.40), the other quantities are recovered following the same scheme[155] as the one for ${}^{(int)}\mathcal{M}$ outlined in Section 7.4.

As the estimates are significantly simpler to derive[156] and in the same spirit as the corresponding ones in Theorem M4, we leave the details to the reader. This concludes the proof of Theorem M5 for ${}^{(top)}\mathcal{M}$. Together with the proof of Theorem M5 for ${}^{(int)}\mathcal{M}$ in Section 7.4, this concludes the proof of Theorem M5.

[155] Note that the steps 21–25 in Section 7.4 are not needed in the case of ${}^{(top)}\mathcal{M}$ in view of Remark 7.12.

[156] Along a level set of $\underline{u}$ in ${}^{(top)}\mathcal{M}$, denoting $r_+(\underline{u})$ the maximal value of ${}^{(top)}r$, i.e. the one on $\{u = u_*\}$, and $r_-(\underline{u})$ the minimal value of ${}^{(top)}r$, i.e. the one on ${}^{(top)}\Sigma$, we have $0 < r_+(\underline{u}) - r_-(\underline{u}) \lesssim 1$, see (3.7). In particular, the integration along e_3 is always finite in ${}^{(top)}\mathcal{M}$.

8. INITIALIZATION AND EXTENSION (THEOREMS M0, M6 AND M7)

The goal of this chapter is to prove Theorems M0, M6, and M7. To this end, we first review our GCM procedure in Section 8.1, and construct an auxiliary outgoing geodesic foliation in the part ${}^{(ext)}\mathcal{L}_0$ of the initial data layer in Section 8.2. Theorems M0, M6, and M7 are then proved respectively in Sections 8.3, 8.4 and 8.5.

8.1. GCM procedure

In this section, we review the main results on the existence of GCM spheres in [40] [41] and on the existence of GCM hypersurfaces in [50]. These results will be used repeatedly in the proof of Theorems M0, M6 and M7.

8.1.1. Background spacetime Our GCM results hold true in a vacuum spacetime region, denoted $\mathcal{R}$, foliated by two functions (u,s) such that:

1. On $\mathcal{R}$, (u,s) is a geodesic foliation of lapse ς, i.e.
   - u is an optical function, $L=-\mathbf{g}^{\alpha\beta}\partial_\beta u\partial_\alpha u$, and $L(\varsigma)=1$,
   - $e_4=\varsigma L$ and $L(s)=1$,
   - e_3 is the correspond null companion to e_4, perpendicular to the surfaces $S(u,s)$ induced by the level surfaces of (u,s), and such that $\mathbf{g}(e_3,e_4)=-2$.

   In particular, it follows from the above that

$$ {}^{(a)}\mathrm{tr}\chi={}^{(a)}\mathrm{tr}\underline{\chi}=0, \qquad \omega=\xi=0, \qquad \underline{\eta}=-\zeta, \qquad \varsigma=\frac{2}{e_3(u)}. $$

2. We define the following renormalized quantities

$$ \widetilde{\mathrm{tr}\,\chi}:=\mathrm{tr}\,\chi-\frac{2}{r}, \qquad \widetilde{\mathrm{tr}\,\underline{\chi}}:=\mathrm{tr}\,\underline{\chi}+\frac{2\Upsilon}{r}, \qquad \check{\underline{\omega}}:=\underline{\omega}-\frac{m}{r^2}, $$
$$ \check{K}:=K-\frac{1}{r^2}, \qquad \check{\rho}:=\rho+\frac{2m}{r^3}, \qquad \check{\mu}:=\mu-\frac{2m}{r^3}, $$
$$ \check{\underline{\Omega}}:=\underline{\Omega}+\Upsilon, \qquad \check{\varsigma}:=\varsigma-1, $$

   where

$$ \underline{\Omega}:=e_3(s), \qquad \Upsilon:=1-\frac{2m}{r}, $$

   and group them in the sets Γ_g and Γ_b defined as follows

$$
\begin{aligned}
\Gamma_g :=& \Big\{ \widetilde{\mathrm{tr}\,\chi},\ \widehat{\chi},\ \zeta,\ \widetilde{\mathrm{tr}\,\underline{\chi}},\ r\check{\mu},\ r\check{\rho},\ r\,{}^*\rho,\ r\beta,\ r\alpha,\ r\check{K}, \\
& r^{-1}\big(e_4(r)-1\big),\ r^{-1}e_4(m)\Big\}, \\
\Gamma_b :=& \Big\{ \eta,\ \underline{\widehat{\chi}},\ \underline{\check{\omega}},\ \underline{\xi},\ r\underline{\beta},\ \underline{\alpha},\ r^{-1}\check{\underline{\Omega}},\ r^{-1}\check{\varsigma}, \\
& r^{-1}\big(e_3(r)+\Upsilon\big),\ r^{-1}e_3(m)\Big\}.
\end{aligned}
\tag{8.1}
$$

3. Let $(\overset{\circ}{u}, \overset{\circ}{s})$ two real numbers. Let $\overset{\circ}{S} := S(\overset{\circ}{u}, \overset{\circ}{s})$, $\overset{\circ}{r}$ the area radius of $\overset{\circ}{S}$, and $\overset{\circ}{m}$ the Hawking mass of $\overset{\circ}{S}$.
4. $\mathcal{R}$ is covered by two coordinates charts $\mathcal{R} = \mathcal{R}_N \cup \mathcal{R}_S$ such that:
   (a) The North coordinate chart $\mathcal{R}_N$ is given by (u, s, y^1_N, y^2_N) with $(y^1_N)^2 + (y^2_N)^2 < 2$.
   (b) The South coordinate chart $\mathcal{R}_S$ is given by (u, s, y^1_S, y^2_S) with $(y^1_S)^2 + (y^2_S)^2 < 2$.
   (c) The two coordinate charts intersect in the open equatorial region $\mathcal{R}_{Eq} := \mathcal{R}_N \cap \mathcal{R}_S$ in which both coordinate systems are defined.
   (d) In $\mathcal{R}_{Eq}$, the transition functions between the two coordinate systems are given by the smooth functions φ_{SN} and $\varphi_{NS} = \varphi_{SN}^{-1}$.
5. The metric coefficients for the two coordinate systems are given by

$$
\begin{aligned}
\mathbf{g} &= -2\varsigma du ds + \varsigma^2\underline{\Omega}du^2 + g^N_{ab}\big(dy^a_N - \varsigma\underline{B}^a_N du\big)\big(dy^b_N - \varsigma\underline{B}^b_N du\big), \\
\mathbf{g} &= -2\varsigma du ds + \varsigma^2\underline{\Omega}du^2 + g^S_{ab}\big(dy^a_S - \varsigma\underline{B}^a_S du\big)\big(dy^b_S - \varsigma\underline{B}^b_S du\big),
\end{aligned}
$$

   where

$$
\underline{\Omega} = e_3(s), \qquad \underline{B}^a_N = \frac{1}{2}e_3(y^a_N), \qquad \underline{B}^a_S = \frac{1}{2}e_3(y^a_S).
$$

6. We restrict the region $\mathcal{R}$ such that, for $\overset{\circ}{\epsilon}$ sufficiently small, $\overset{\circ}{r}$ the area radius of $S(\overset{\circ}{u}, \overset{\circ}{s})$ sufficiently large, i.e $\overset{\circ}{\epsilon} \ll m_0 \ll \overset{\circ}{r}$,

$$
\mathcal{R} := \Big\{ |u - \overset{\circ}{u}| \leq \overset{\circ}{\epsilon}, \quad |s - \overset{\circ}{s}| \leq \overset{\circ}{\epsilon} \Big\}. \tag{8.2}
$$

7. We assume that on $\mathcal{R}$ the following assumptions are verified, for an integer s_{max} sufficiently large

A1. For $k \leq s_{max}$

$$\|\Gamma_g\|_{k,\infty} \leq \mathring{\epsilon} r^{-2}, \qquad \|\Gamma_b\|_{k,\infty} \leq \mathring{\epsilon} r^{-1}. \tag{8.3}$$

A2. The Hawking mass $m = m(u,s)$ of $S(u,s)$ verifies

$$\sup_{\mathcal{R}} \left| \frac{m}{m_0} - 1 \right| \leq \mathring{\epsilon}. \tag{8.4}$$

A3. In the region of their respective validity[157] we have

$$\underline{B}^a_N, \ \underline{B}^a_S \in r^{-1}\Gamma_b, \tag{8.5}$$

and

$$r^{-2}\check{g}^N_{ab}, \ r^{-2}\check{g}^S_{ab} \in r\Gamma_g, \tag{8.6}$$

where

$$\begin{aligned} \check{g}^N{}_{ab} &= g^N_{ab} - \frac{4r^2}{1 + (y^1_N)^2 + (y^2_N)^2)}\delta_{ab}, \\ \check{g}^S{}_{ab} &= g^S_{ab} - \frac{4r^2}{(1 + (y^1_S)^2 + (y^2_S)^2)}\delta_{ab}. \end{aligned}$$

A4. We assume the existence of a smooth family of scalar functions $J^{(p)} : \mathcal{R} \to \mathbb{R}$, for $p = 0, +, -$, verifying the following properties

(a) On the sphere $\mathring{S}$ of the background foliation, there holds

$$\begin{aligned} \left((\mathring{r})^2 \mathring{\Delta} + 2\right) J^{(p)} &= O(\mathring{\epsilon}), \qquad p = 0, +, -, \\ \frac{1}{|\mathring{S}|} \int_{\mathring{S}} J^{(p)} J^{(q)} &= \frac{1}{3}\delta_{pq} + O(\mathring{\epsilon}), \qquad p, q = 0, +, -, \\ \frac{1}{|\mathring{S}|} \int_{\mathring{S}} J^{(p)} &= O(\mathring{\epsilon}), \qquad p = 0, +, -. \end{aligned} \tag{8.7}$$

(b) We extend $J^{(p)}$ from $\mathring{S}$ to $\mathcal{R}$ by $\partial_s J^{(p)} = \partial_u J^{(p)} = 0$, i.e.

$$J^{(p)}(u, s, y^1, y^2) = J^{(p)}(\mathring{u}, \mathring{s}, y^1, y^2). \tag{8.8}$$

[157] That is the quantities on the left verify the same estimates as those for Γ_b, respectively Γ_g.

8.1.2. Deformations of surfaces We review the results in [40] on deformations of surfaces that will be useful in this chapter.

Definition 8.1. *We say that* $\mathbf{S}$ *is a deformation of* $\overset{\circ}{S}$ *if there exist smooth scalar functions* U, S *defined on* $\overset{\circ}{S}$ *and a map a map* $\Psi : \overset{\circ}{S} \to \mathbf{S}$ *verifying, on either coordinate chart* (y^1, y^2) *of* $\overset{\circ}{S}$,

$$\Psi(\overset{\circ}{u}, \overset{\circ}{s}, y^1, y^2) = \Big(\overset{\circ}{u} + U(y^1, y^2),\, \overset{\circ}{s} + S(y^1, y^2), y^1, y^2\Big). \tag{8.9}$$

Definition 8.2. *Given a deformation* $\Psi : \overset{\circ}{S} \to \mathbf{S}$ *we say that a new frame* (e_3', e_4', e_1', e_2') *on* $\mathbf{S}$*, obtained from the standard frame* (e_3, e_4, e_1, e_2) *via the general frame transformation* (2.6)*, is* $\mathbf{S}$*-adapted if the horizontal vectorfields* e_1', e_2' *are tangent to* $\mathbf{S}$.

The following result combines Lemma 5.8, Corollary 5.9 and Corollary 5.17 in [40].

Proposition 8.3. *Let* $\overset{\circ}{S} \subset \mathcal{R}$*. Let* $\Psi : \overset{\circ}{S} \to \mathbf{S}$ *be a deformation generated by the functions* (U, S) *as in Definition 8.1. Assume the bound*

$$\|(U, S)\|_{L^\infty(\overset{\circ}{S})} + r\|\overset{\circ}{\nabla}(U, S)\|_{L^\infty(\overset{\circ}{S})} + r^2\|\overset{\circ}{\nabla}{}^2 (U, S)\|_{L^\infty(\overset{\circ}{S})} \lesssim \overset{\circ}{\delta}. \tag{8.10}$$

Then

1. *We have*

$$\sum_{a,b=1}^{2} |g^{\mathbf{S},\#}_{ab} - \overset{\circ}{g}_{ab}| \lesssim r\overset{\circ}{\delta}. \tag{8.11}$$

2. *We have*

$$\frac{r^{\mathbf{S}}}{\overset{\circ}{r}} = 1 + O(r^{-1}\overset{\circ}{\delta}) \tag{8.12}$$

where $r^{\mathbf{S}}$ *is the area radius of* $\mathbf{S}$ *and* $\overset{\circ}{r}$ *that of* $\overset{\circ}{S}$.

3. *Let* $m = m(u, s)$ *the Hawking mass of the surfaces* $S(u, s)$ *and* m^S *the Hawking mass of* S*. We have*

$$\sup_{\mathbf{S}} |m - m^{\mathbf{S}}| \lesssim \overset{\circ}{\delta}.$$

4. *For an arbitrary scalar function f on $\mathcal{R}$,*

$$\left|\int_{\mathbf{S}} f - \int_{\overset{\circ}{S}} f\right| \lesssim \overset{\circ}{\delta}\,\overset{\circ}{r}\left(\sup_{\mathcal{R}}|f| + \overset{\circ}{r}\sup_{\mathcal{R}}\left(|\partial_u f| + |\partial_s f|\right)\right).$$

The following results combine Lemma 7.3 with Corollary 7.7 in [40].

Lemma 8.4. *Let $\Psi : \overset{\circ}{S} \to \mathbf{S}$ be a deformation defined by (U, S), as in Definition 8.1 with $(f, \underline{f})$ the transition function of the frame transformation from the frame of $\mathcal{R}$ to that adapted to $\mathbf{S}$, as in Definition 8.2. There exists a small enough constant δ_1 such that for given $f, \underline{f}$ on $\mathcal{R}$ satisfying*

$$\|f\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} + (r^{\mathbf{S}})^{-1}\|\underline{f}\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} \leq \delta_1,$$

the following holds

1. *We have*

$$(\overset{\circ}{r})^{-1}\|U\|_{\mathfrak{h}_{s_{max}+1}(\overset{\circ}{S})} + (\overset{\circ}{r})^{-2}\|S\|_{\mathfrak{h}_{s_{max}+1}(\overset{\circ}{S})} \lesssim \delta_1.$$

   In particular, we have

$$\sup_{\mathbf{S}}|u - \overset{\circ}{u}| \lesssim \delta_1, \qquad \sup_{\mathbf{S}}|s - \overset{\circ}{s}| \lesssim \overset{\circ}{r}\delta_1.$$

2. *We have,*

$$\sum_{a,b=1}^{2}|g^{\mathbf{S},\#}_{ab} - \overset{\circ}{g}_{ab}| \lesssim r^2\delta_1.$$

3. *We have*

$$\left|\frac{r^{\mathbf{S}}}{\overset{\circ}{r}} - 1\right| + \sup_{\mathbf{S}}\left|\frac{r^{\mathbf{S}}}{r} - 1\right| \lesssim \delta_1.$$

4. *The following estimate holds true for an arbitrary scalar function h on $\mathcal{R}$,*

$$\left|h^{\#} - h\right| \lesssim \delta_1 \sup_{\mathcal{R}}|\mathfrak{d}h|.$$

5. *The following estimate holds true for an arbitrary scalar function h on $\mathcal{R}$,*

$$\left|\int_{\mathbf{S}} h - \int_{\overset{\circ}{S}} h\right| \lesssim \delta_1(\overset{\circ}{r})^2\left(\sup_{\mathcal{R}}\left(|f| + |\partial_u f|\right) + \overset{\circ}{r}\sup_{\mathcal{R}}|\partial_s f|\right).$$

6. *If $V\in\mathfrak{h}_s(\mathbf{S})$ and $V^\#$ is its pull-back by Ψ, we have for all $0\le s\le s_{max}$,*

$$\|V\|_{\mathfrak{h}_s(\mathbf{S})}=\|V^\#\|_{\mathfrak{h}_s(\overset{\circ}{S},\, g^{\mathbf{S},\#})}=\|V^\#\|_{\mathfrak{h}_s(\overset{\circ}{S},\overset{\circ}{g})}(1+O(\delta_1)). \tag{8.13}$$

7. *For any tensor h on $\mathcal{R}$*

$$\|h\|_{\mathfrak{h}_s(\mathbf{S})}\lesssim r\sup_{\mathcal{R}}\Big(|\not{\mathfrak{d}}^{\le s}h|+\delta_1|\mathfrak{d}^{\le s}h|\Big),\qquad 0\le s\le s_{max}. \tag{8.14}$$

8. *We have*

$$\sum_{a,b,c=1,2}\Big\|(\Gamma^{\mathbf{S},\#})^c_{ab}-(\overset{\circ}{\Gamma})^c_{ab}\Big\|_{\mathfrak{h}_{s_{max}-1}(\overset{\circ}{S})}\lesssim r^2\delta_1. \tag{8.15}$$

9. *We also have, for $\overset{\circ}{m}$ the Hawking mass of $\overset{\circ}{S}$,*

$$|m^{\mathbf{S}}-\overset{\circ}{m}|\lesssim\delta_1+(\overset{\circ}{\epsilon})^2.$$

8.1.3. Existence of intrinsic GCM spheres We review in this section the results of [41] useful for this chapter. We start with the following definition of canonical $\ell=1$ modes on a deformed sphere $\mathbf{S}$.

Definition 8.5. *Given a deformation map $\Psi:\overset{\circ}{S}\to\mathbf{S}$ and a fixed effective uniformization map $(\overset{\circ}{\Phi},\overset{\circ}{\phi})$ for $\overset{\circ}{S}$ we let (Φ,ϕ) be the unique effective uniformization map of $\mathbf{S}$ calibrated with $(\overset{\circ}{\Phi},\overset{\circ}{\phi})$ relative to the map Ψ, in the sense of Definition 5.6. With this choice, we define the canonical $\ell=1$ modes of $\mathbf{S}$ by the formula*

$$J^{(p,\mathbf{S})} = J^{(p,\mathbb{S}^2)}\circ\Phi^{-1} \tag{8.16}$$

with $J^{(p,\mathbb{S}^2)}$ denoting the $\ell=1$ spherical harmonics of $\mathbb{S}^2$.

Consider as before a vacuum spacetime region $\mathcal{R}$ verifying the assumptions **A1**–**A4**. In addition we make the following stronger assumptions on **A1** and **A4**.

A1-Strong. For $k\le s_{max}$, and for a small enough constant $\delta_1>0$, with $\delta_1\ge\overset{\circ}{\epsilon}$,

$$\Big\|(\Gamma_g,\Gamma_b)\Big\|_{k,\infty}\lesssim\delta_1 r^{-2},\qquad \Big\|\nabla_3\Gamma_g\Big\|_{k,\infty}\lesssim\delta_1 r^{-3}. \tag{8.17}$$

A4-Strong. We assume the existence of a smooth family of scalar functions $J^{(p)}:\mathcal{R}\to\mathbb{R}$, for $p=0,+,-$, verifying the following properties

1. On the sphere $\overset{\circ}{S}$ of the background foliation, there holds the following stronger version of (8.7)

$$
\begin{aligned}
\Big((\overset{\circ}{r})^2\overset{\circ}{\Delta}+2\Big)J^{(p)} &= O(\overset{\circ}{\epsilon}r^{-1}), \qquad p=0,+,-,\\
\frac{1}{|S|}\int_S J^{(p)}J^{(q)} &= \frac{1}{3}\delta_{pq}+O(\overset{\circ}{\epsilon}r^{-1}), \qquad p,q=0,+,-,\\
\frac{1}{|S|}\int_S J^{(p)} &= O(\overset{\circ}{\epsilon}r^{-1}), \qquad p=0,+,-.
\end{aligned}
\tag{8.18}
$$

2. On $\overset{\circ}{S}$ we have

$$
\max_{p=0,-,+}\|J^{(p)}-J^{(p,\overset{\circ}{S})}\|_{\mathfrak{h}_{s_{max}+1}(\overset{\circ}{S})}\lesssim\overset{\circ}{\delta},
\tag{8.19}
$$

   where $J^{(p,\overset{\circ}{S})}$ denotes the canonical basis of $\ell=1$ modes on $\overset{\circ}{S}$ corresponding to the effective uniformization map $(\overset{\circ}{\Phi},\overset{\circ}{\phi})$ for $\overset{\circ}{S}$ appearing in Definition 8.5.
3. We extend $J^{(p)}$ from $\overset{\circ}{S}$ to $\mathcal{R}$ as in (8.8), i.e. by $\partial_s J^{(p)}=\partial_u J^{(p)}=0$.

We state below the results of Corollary 7.2. in [41].

Corollary 8.6. *Let $J^{(p)}$ satisfying* **A4-Strong**, *and let $J^{(p,\overset{\circ}{S})}$ denotes the canonical basis of $\ell=1$ modes on $\overset{\circ}{S}$ corresponding to the effective uniformization map $(\overset{\circ}{\Phi},\overset{\circ}{\phi})$ for $\overset{\circ}{S}$ appearing in Definition 8.5. Let a deformation $\Psi:\overset{\circ}{S}\to\mathbf{S}$ with the corresponding deformation parameters (U,S) satisfying*

$$
\|(U,S)\|_{\mathfrak{h}_{s_{max}+1}(\overset{\circ}{S})}\lesssim r\overset{\circ}{\delta}, \qquad s_{max}\geq 2.
\tag{8.20}
$$

Let $J^{(p,\mathbf{S})}$ be the corresponding canonical basis of $\ell=1$ modes of **S** *calibrated according to Definition 8.5. Then, the following estimate holds true*

$$
\max_{p=0,-,+}\sup_{\mathbf{S}}|J^{(p)}-J^{(p,\mathbf{S})}|\lesssim r^{-1}\overset{\circ}{\delta}.
\tag{8.21}
$$

We state below Theorem 7.3. in [41], which is the main result in that paper, concerning the construction of intrinsic GCM spheres. Recall, see Definition 2.61, that the $\ell=1$ modes of a scalar function f are defined to be the triplet[158]

[158]Note that this definition differs from the one in [41] by a factor of r^{-2}.

$$(f)_{\ell=1} \;=\; \left(\frac{1}{|S|}\int_S fJ^{(0)}, \frac{1}{|S|}\int_S fJ^{(+)}, \frac{1}{|S|}\int_S fJ^{(-)}\right).$$

Theorem 7.3. in [41] holds for spacetime regions $\mathcal{R}$ verifying, in addition to **A1–A4**,

$$\begin{aligned}\kappa &= \frac{2}{r} + \dot{\kappa},\\ \underline{\kappa} &= -\frac{2\Upsilon}{r} + \underline{C}_0 + \sum_p \underline{C}^{(p)} J^{(p)} + \dot{\underline{\kappa}},\\ \mu &= \frac{2m}{r^3} + M_0 + \sum_p M^{(p)} J^{(p)} + \dot{\mu},\end{aligned} \tag{8.22}$$

where the scalar functions $\underline{C}_0 = \underline{C}_0(u,s)$, $\underline{C}^{(p)} = \underline{C}^{(p)}(u,s)$, $M_0 = M_0(u,s)$ and $M^{(p)} = M^{(p)}(u,s)$, defined on the spacetime region $\mathcal{R}$, depend only on the coordinates (u,s), and where $\dot{\kappa}$, $\dot{\underline{\kappa}}$ and $\dot{\mu}$ satisfy the following estimates

$$\sup_{\mathcal{R}} |\mathfrak{d}^{\leq s_{max}}(\dot{\kappa}, \dot{\underline{\kappa}})| \lesssim r^{-2}\overset{\circ}{\delta}, \qquad \sup_{\mathcal{R}} |\mathfrak{d}^{\leq s_{max}}\dot{\mu}| \lesssim r^{-3}\overset{\circ}{\delta}. \tag{8.23}$$

Theorem 8.7 (Existence of intrinsic GCM spheres)**.** *Assume that the spacetime region $\mathcal{R}$ verifies the assumptions* **A1-Strong, A2, A3**, **A4-Strong**, *as well as* (8.22) (8.23)*. We further assume that, relative to the $\ell = 1$ modes of the background foliation,*

$$(div\,\beta)_{\ell=1} = O(\overset{\circ}{\delta} r^{-5}), \quad (\widecheck{tr\chi})_{\ell=1} = O(\overset{\circ}{\delta} r^{-3}), \quad (\widecheck{tr\underline{\chi}})_{\ell=1} = O(\overset{\circ}{\delta} r^{-3}). \tag{8.24}$$

Then, there exist unique constants $M^{(\mathbf{S},p)}$, $p \in \{-,0,+\}$, such that

$$\begin{aligned}\kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}},\\ \underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}}\Upsilon^{\mathbf{S}},\\ \mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + \sum_p M^{(\mathbf{S},p)} J^{(p,\mathbf{S})},\end{aligned} \tag{8.25}$$

and

$$\int_{\mathbf{S}} div^{\mathbf{S}} \beta^{\mathbf{S}} J^{(p,\mathbf{S})} = 0, \tag{8.26}$$

where $J^{(p,\mathbf{S})}$ *is a canonical* $\ell = 1$ *basis for* $\mathbf{S}$ *calibrated, relative by* Ψ*, with the canonical* $\ell = 1$ *basis of* $\overset{\circ}{S}$*. Moreover the deformation verifies the properties:*

1. *The volume radius* $r^{\mathbf{S}}$ *verifies*

$$\left|\frac{r^{\mathbf{S}}}{\overset{\circ}{r}} - 1\right| \lesssim r^{-1}\overset{\circ}{\delta}.$$

2. *The parameter functions* U, S *of the deformation verify*

$$\|(U, S)\|_{\mathfrak{h}_{s_{max}+1}(\overset{\circ}{S})} \lesssim r\overset{\circ}{\delta}.$$

3. *The Hawking mass* $m^{\mathbf{S}}$ *of* $\mathbf{S}$ *verifies the estimate*

$$|m^{\mathbf{S}} - \overset{\circ}{m}| \lesssim \overset{\circ}{\delta}.$$

4. *The well defined*[159] *Ricci and curvature coefficients of* $\mathbf{S}$ *verify,*

$$\|\Gamma_g^{\mathbf{S}}\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} \lesssim \overset{\circ}{\epsilon} r^{-1}, \qquad \|\Gamma_b^{\mathbf{S}}\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} \lesssim \overset{\circ}{\epsilon}.$$

The following corollary is Corollary 7.7 in [41].

Corollary 8.8. *Under the same assumptions as in Theorem 8.7 we have, in addition to* (8.25) *and* (8.26)*,*

- *either, for any choice of a canonical* $\ell = 1$ *basis of* $\mathbf{S}$*,*

$$(curl^{\mathbf{S}}\beta^{\mathbf{S}})_{\ell=1} = 0,$$

- *or there exists a canonical basis of* $\ell = 1$ *modes of* $\mathbf{S}$ *such that*

$$\int_{\mathbf{S}} curl^{\mathbf{S}}\beta^{\mathbf{S}}\, J^{(\pm,\mathbf{S})} = 0, \qquad \int_{\mathbf{S}} curl^{\mathbf{S}}\beta^{\mathbf{S}}\, J^{(0,\mathbf{S})} \neq 0. \tag{8.27}$$

We then define the angular parameter $a^{\mathbf{S}}$ *on* $\mathbf{S}$ *by the formula*

$$a^{\mathbf{S}} := \frac{(r^{\mathbf{S}})^3}{8\pi m^{\mathbf{S}}} \int_{\mathbf{S}} curl^{\mathbf{S}}\beta^{\mathbf{S}} J^{(0,\mathbf{S})}. \tag{8.28}$$

---

[159]Note that while the Ricci coefficients $\kappa^{\mathbf{S}}, \underline{\kappa}^{\mathbf{S}}, \widehat{\chi}^{\mathbf{S}}, \underline{\widehat{\chi}}^{\mathbf{S}}, \zeta^{\mathbf{S}}$ as well as all curvature components and mass aspect function $\mu^{\mathbf{S}}$ are well defined on $\mathbf{S}$, this in not the case of $\eta^{\mathbf{S}}, \underline{\eta}^{\mathbf{S}}, \xi^{\mathbf{S}}, \underline{\xi}^{\mathbf{S}}, \omega^{\mathbf{S}}, \underline{\omega}^{\mathbf{S}}$ which require the derivatives of the frame in the $e_3^{\mathbf{S}}$ and $e_4^{\mathbf{S}}$ directions.

With this definition, we have $a^{\mathbf{S}} = 0$ in the first case, while $a^{\mathbf{S}} \neq 0$ in the second case.

Remark 8.9. *In the case $a^{\mathbf{S}} \neq 0$, with $a^{\mathbf{S}}$ given by (8.28), let a canonical basis $J^{(p,\mathbf{S})}$ satisfying the condition (8.27). Then, $J^{(0,\mathbf{S})}$ is unique, see Remark 7.8 in [41]. Thus, on an intrinsic GCM sphere $\mathbf{S}$,*

- *$a^{\mathbf{S}}$ given by (8.28) is a well defined notion of angular momentum,*
- *the condition (8.27) is a canonical way to define a notion of axis[160] when $a^{\mathbf{S}} \neq 0$.*

Finally we state below the results of Proposition 8.1 in [41].

Proposition 8.10. *Let a fixed spacetime region $\mathcal{R}$ verifying assumptions* **A1–A4** *and (8.22) (8.23), as well as, for any background sphere S of $\mathcal{R}$,*

$$|(\text{div}\,\beta)_{\ell=1}| \lesssim r^{-4}\overset{\circ}{\delta}, \qquad |(\check{\underline{\kappa}})_{\ell=1}| \lesssim r^{-2}\overset{\circ}{\delta}. \tag{8.29}$$

Assume that $\mathbf{S}$ is a deformed sphere in $\mathcal{R}$ which verifies the GCM conditions

$$\begin{aligned}
\kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}},\\
\underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}}\Upsilon^{\mathbf{S}} + \underline{C}_0^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S},p)}\widetilde{J}^{(p)},\\
\mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + M_0^{\mathbf{S}} + \sum_p M^{(\mathbf{S},p)}\widetilde{J}^{(p)},
\end{aligned} \tag{8.30}$$

for some basis[161] of $\ell = 1$ modes $\widetilde{J}^{(p)}$ on $\mathbf{S}$, such that for a small enough constant $\delta_1 > 0$,

- *The transition coefficients $(f, \underline{f}, \lambda)$ from the background frame of $\mathcal{R}$ to that of $\mathbf{S}$ verifies, for some $4 \leq s \leq s_{max}$, the bound*

$$\|f\|_{\mathfrak{h}_s(\mathbf{S})} + (r^{\mathbf{S}})^{-1}\|(\underline{f}, \overset{\circ}{\lambda})\|_{\mathfrak{h}_s(\mathbf{S})} \leq \delta_1, \tag{8.31}$$

- *The difference between the basis of $\ell = 1$ modes $\widetilde{J}^{(p)}$ on $\mathbf{S}$ and the basis of $\ell = 1$ modes of the background foliation $J^{(p)}$ verifies*

$$r^{-1}\|\widetilde{J}^{(p)} - J^{(p)}\|_{\mathfrak{h}_s(\mathbf{S})} \leq \delta_1. \tag{8.32}$$

[160] Note that in Kerr, the axis corresponds to $J^{(0,\mathbf{S})} = \pm 1$.

[161] $\widetilde{J}^{(p)}$ is not assumed to be a canonical basis of $\ell = 1$ modes on $\mathbf{S}$.

Assume in addition that we have, with respect to the basis of $\ell = 1$ modes $\widetilde{J}^{(p)}$ on $\mathbf{S}$,

$$|(div^{\mathbf{S}}\beta^{\mathbf{S}})_{\ell=1}| \lesssim r^{-4}\overset{\circ}{\delta}, \qquad |(\underline{\check{\kappa}}^{\mathbf{S}})_{\ell=1}| \lesssim r^{-2}\overset{\circ}{\delta}. \tag{8.33}$$

Then $(f, \underline{f}, \lambda = 1 + \overset{\circ}{\lambda})$ *verify the estimates*

$$\|(f, \underline{f}, \check{\overset{\circ}{\lambda}}^{\mathbf{S}})\|_{\mathfrak{h}_{s+1}(\mathbf{S})} \lesssim r\overset{\circ}{\delta} + r(\overset{\circ}{\epsilon})^2 + r\delta_1\left(\frac{1}{\overset{\circ}{r}} + \overset{\circ}{\epsilon} + \delta_1\right)$$

and

$$r|\overline{\overset{\circ}{\lambda}}^{\mathbf{S}}| \lesssim r\overset{\circ}{\delta} + r(\overset{\circ}{\epsilon})^2 + r\delta_1\left(\frac{1}{\overset{\circ}{r}} + \overset{\circ}{\epsilon} + \delta_1\right) + \sup_{\mathbf{S}}|r - r^{\mathbf{S}}|.$$

8.1.4. Existence of GCM hypersurfaces In this section, we review the results on the construction of GCM hypersurfaces (GCMH) from [50]. We state below Theorem 4.1 in [50] on the construction of GCM hypersurfaces.

Theorem 8.11 (Construction of GCM hypersurfaces)**.** *Assume that the spacetime region* $\mathcal{R}$ *verifies the assumptions* **A1**–**A4**. *We further assume that, relative to the* $\ell = 1$ *modes of the background foliation,*

$$\sup_{\mathcal{R}} r\left|\widehat{\mathfrak{d}}^{\le s_{max}} e_3(J^{(p)})\right| \lesssim \overset{\circ}{\delta}, \tag{8.34}$$

where

$$\widehat{\mathfrak{d}} := (e_3 - (e_3(u) + e_3(s))e_4, \, \not{\mathfrak{d}})$$

denotes the weighted derivatives tangential to the level hypersurfaces of $u+s$. *In addition, we assume on* $\mathcal{R}$

$$\sup_{\mathcal{R}} |\widehat{\mathfrak{d}}^{\le s_{max}}(\dot{\kappa}, \underline{\dot{\kappa}})| \lesssim r^{-2}\overset{\circ}{\delta}, \qquad \sup_{\mathcal{R}} |\widehat{\mathfrak{d}}^{\le s_{max}}\dot{\mu}| \lesssim r^{-3}\overset{\circ}{\delta}, \tag{8.35}$$

where $\dot{\kappa}$, $\underline{\dot{\kappa}}$ *and* $\dot{\mu}$ *are given by* (8.22), *and*

$$|(div\eta)_{\ell=1}| \lesssim \overset{\circ}{\delta}, \quad |(div\underline{\xi})_{\ell=1}| \lesssim \overset{\circ}{\delta}, \tag{8.36}$$

$$|r - s| + |e_3(r) - e_3(s)| \lesssim \overset{\circ}{\delta}, \tag{8.37}$$

as well as the existence of a constant $m^{(0)}$ *such that we have on* $\mathcal{R}$

$$\left|\overline{e_3(u)+e_3(s)} - 1 - \frac{2m^{(0)}}{r}\right| \lesssim \overset{\circ}{\delta}, \tag{8.38}$$

where $\overline{e_3(u)+e_3(s)}$ *denotes the average of* $e_3(u)+e_3(s)$ *on the spheres of the background foliation.*

Let $\mathbf{S}_0$ *be a fixed sphere included in the region* $\mathcal{R}$, *let a pair of triplets* $\Lambda_0, \underline{\Lambda}_0 \in \mathbb{R}^3$ *such that*

$$|\Lambda_0| + |\underline{\Lambda}_0| \lesssim r^{-2}\overset{\circ}{\delta}, \tag{8.39}$$

and let $J^{(p)}[\mathbf{S}_0]$ *a basis of* $\ell = 1$ *modes on* $\mathbf{S}_0$, *such that we have on* $\mathbf{S}_0$

$$\begin{aligned}
\kappa^{\mathbf{S}_0} &= \frac{2}{r^{\mathbf{S}_0}},\\
\underline{\kappa}^{\mathbf{S}_0} &= -\frac{2\Upsilon^{\mathbf{S}_0}}{r^{\mathbf{S}_0}} + \underline{C}_0^{\mathbf{S}_0} + \sum_p \underline{C}^{(\mathbf{S}_0,p)} J^{(p)}[\mathbf{S}_0],\\
\mu^{\mathbf{S}_0} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}_0})^3} + M_0^{\mathbf{S}_0} + \sum_p M^{(p,\mathbf{S}_0)} J^{(p)}[\mathbf{S}_0],
\end{aligned} \tag{8.40}$$

as well as

$$(div\, f_0)_{\ell=1} = \Lambda_0, \qquad (div\, \underline{f}_0)_{\ell=1} = \underline{\Lambda}_0, \tag{8.41}$$

with $(f_0, \underline{f}_0)$ *corresponding to the coefficients from the background frame to the frame adapted to* $\mathbf{S}_0$, *and the* $\ell = 1$ *modes being taken w.r.t. the basis* $J^{(p)}[\mathbf{S}_0]$, *and where*

$$\|J^{(p)}[\mathbf{S}_0] - J^{(p)}\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S}_0)} \lesssim r\overset{\circ}{\delta}.$$

Then, there exists a unique, local, smooth, space like hypersurface Σ_0 *passing through* $\mathbf{S}_0$, *a scalar function* $u^{\mathbf{S}}$ *defined on* Σ_0, *whose level sets a topological spheres denoted by* $\mathbf{S}$, *a smooth collection of constants* $\Lambda^{\mathbf{S}}, \underline{\Lambda}^{\mathbf{S}}$ *and a triplet of functions* $J^{(p)}[\mathbf{S}]$ *defined on* Σ_0 *verifying*

$$\Lambda^{\mathbf{S}_0} = \Lambda_0, \qquad \underline{\Lambda}^{\mathbf{S}_0} = \underline{\Lambda}_0, \qquad J^{(p)}[\mathbf{S}]\Big|_{\mathbf{S}_0} = J^{(p)}[\mathbf{S}_0],$$

such that the following conditions are verified:

1. *The following GCM conditions hold on* Σ_0

$$\kappa^{\mathbf{S}} = \frac{2}{r^{\mathbf{S}}},$$
$$\underline{\kappa}^{\mathbf{S}} = -\frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} + \underline{C}_0^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S},p)} J^{(p)}[\mathbf{S}], \tag{8.42}$$
$$\mu^{\mathbf{S}} = \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + M_0^{\mathbf{S}} + \sum_p M^{(\mathbf{S},p)} J^{(p)}[\mathbf{S}].$$

2. *There exists a constant* c_0 *such that*

$$u^{\mathbf{S}} + r^{\mathbf{S}} = c_0 \quad along \quad \Sigma_0. \tag{8.43}$$

3. *Let* $\nu^{\mathbf{S}}$ *the unique vectorfield tangent to the hypersurfaces* Σ_0*, normal to* $\mathbf{S}$*, and normalized by* $\mathbf{g}(\nu^{\mathbf{S}}, e_4^{\mathbf{S}}) = -2$*, and let* $b^{\mathbf{S}}$ *be the unique scalar function on* Σ_0 *such that* $\nu^{\mathbf{S}}$ *is given by*

$$\nu^{\mathbf{S}} = e_3^{\mathbf{S}} + b^{\mathbf{S}} e_4^{\mathbf{S}}. \tag{8.44}$$

Then, the following normalization condition holds true

$$\overline{b^{\mathbf{S}}} = -1 - \frac{2m^{(0)}}{r^{\mathbf{S}}}, \tag{8.45}$$

where $\overline{b^{\mathbf{S}}}$ *denotes the average of* $b^{\mathbf{S}}$ *on the spheres* $\mathbf{S}$ *foliating* Σ_0.

4. *The triplet of functions* $J^{(p)}[\mathbf{S}]$ *verifies on* Σ_0

$$\nu^{\mathbf{S}}(J^{(p)}[\mathbf{S}]) = 0, \quad p = 0, +, -. \tag{8.46}$$

5. *The following transversality conditions are assumed on* Σ_0

$$\xi^{\mathbf{S}} = 0, \;\; \omega^{\mathbf{S}} = 0, \;\; \underline{\eta}^{\mathbf{S}} + \zeta^{\mathbf{S}} = 0, \;\; e_4^{\mathbf{S}}(r^{\mathbf{S}}) = 1, \;\; e_4^{\mathbf{S}}(u^{\mathbf{S}}) = 0. \tag{8.47}$$

6. *In view of* (8.47)*, the Ricci coefficients* $\eta^{\mathbf{S}}$ *and* $\underline{\xi}^{\mathbf{S}}$ *are well defined on* Σ_0*. They verify on* Σ_0

$$(div^{\mathbf{S}}\eta^{\mathbf{S}})_{\ell=1} = 0, \qquad (div^{\mathbf{S}}\underline{\xi}^{\mathbf{S}})_{\ell=1} = 0, \tag{8.48}$$

where the $\ell = 1$ *modes are taken w.r.t.* $J^{(p)}[\mathbf{S}]$.

7. *The transition coefficients* $(f, \underline{f}, \lambda)$ *from the background foliation to that of* Σ_0 *verify*

$$\|(f, \underline{f}, \lambda - 1)\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S})} + \|\mathfrak{d}(f, \underline{f}, \lambda - 1)\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} \lesssim \overset{\circ}{\delta}. \tag{8.49}$$

We state below Corollary 4.2 in [50].

Corollary 8.12. *Assume that the spacetime region* $\mathcal{R}$ *verifies the assumptions* **A1**–**A4** *and* (8.34)–(8.38)*, as well as*

$$|\widehat{\mathfrak{d}}^{\leq s_{max}}(div\,\eta)_{\ell=1}| \lesssim \overset{\circ}{\delta}, \qquad |\widehat{\mathfrak{d}}^{\leq s_{max}}(div\,\underline{\xi})_{\ell=1}| \lesssim \overset{\circ}{\delta}, \tag{8.50}$$

and

$$|\widehat{\mathfrak{d}}^{\leq s_{max}}(e_3(r) - e_3(s))| \lesssim \overset{\circ}{\delta}, \tag{8.51}$$

where $\widehat{\mathfrak{d}} = (e_3 - (e_3(u) + e_3(s))e_4,\ \not{\mathfrak{d}})$*. Also, assume given a GCM hypersurface* $\Sigma_0 \subset \mathcal{R}$ *foliated by hypersurfaces* $\mathbf{S}$ *such that*

$$\begin{aligned} \kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}, \\ \underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}}\Upsilon^{\mathbf{S}} + \underline{C}_0^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S},p)} J^{(p)}[\mathbf{S}], \\ \mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + M_0^{\mathbf{S}} + \sum_p M^{(\mathbf{S},p)} J^{(p)}[\mathbf{S}], \end{aligned} \tag{8.52}$$

and

$$(div^{\mathbf{S}}\eta^{\mathbf{S}})_{\ell=1} = 0, \qquad (div^{\mathbf{S}}\underline{\xi}^{\mathbf{S}})_{\ell=1} = 0, \tag{8.53}$$

where the triplet of functions $J^{(p)}[\mathbf{S}]$ *verifies on* Σ_0

$$\nu^{\mathbf{S}}(J^{(p)}[\mathbf{S}]) = 0, \quad p = 0, +, -. \tag{8.54}$$

1. *If we assume in addition that for a given sphere* $\mathbf{S}_0$ *on* Σ_0 *the transition coefficients* $(f, \underline{f}, \lambda)$ *from the background foliation to* $\mathbf{S}_0$ *verify*

$$\|(f, \underline{f}, \lambda - 1)\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S}_0)} \lesssim \overset{\circ}{\delta}, \tag{8.55}$$

then

$$\|\mathfrak{d}^{\leq s_{max}+1}(f, \underline{f}, \lambda - 1)\|_{L^2(\mathbf{S}_0)} \lesssim \overset{\circ}{\delta}. \tag{8.56}$$

2. *If we assume in addition that for a given sphere $\mathbf{S}_0$ on Σ_0 the transition coefficients $(f, \underline{f}, \lambda)$ from the background foliation to $\mathbf{S}_0$ verify*

$$\|f\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S}_0)} + (r^{\mathbf{S}_0})^{-1}\|(\underline{f}, \lambda - 1)\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S}_0)} \lesssim \overset{\circ}{\delta}, \tag{8.57}$$

then

$$\|\mathfrak{d}^{\leq s_{max}+1} f\|_{L^2(\mathbf{S}_0)} + (r^{\mathbf{S}_0})^{-1}\|\mathfrak{d}^{\leq s_{max}+1}(\underline{f}, \lambda - 1)\|_{L^2(\mathbf{S}_0)} + \|\mathfrak{d}^{\leq s_{max}} \nabla^{\mathbf{S}}_{\nu^{\mathbf{S}}}(\underline{f}, \lambda - 1)\|_{L^2(\mathbf{S}_0)} \lesssim \overset{\circ}{\delta}. \tag{8.58}$$

We conclude this section with the following simple consequence of Lemma 4.19 in [50].

Lemma 8.13. *Assume given a hypersurface $\Sigma_0 \subset \mathcal{R}$ foliated by hypersurfaces $\mathbf{S}$ such that*

$$\sup_{\mathbf{S} \subset \Sigma_0} \|\mathfrak{d}^{\leq s_{max}}(f, \underline{f}, \lambda - 1)\|_{L^2(\mathbf{S})} \lesssim \overset{\circ}{\delta},$$

where $\nu^{\mathbf{S}} = e_3^{\mathbf{S}} + b^{\mathbf{S}} e_4^{\mathbf{S}}$ is tangent to Σ_0, and where $(f, \underline{f}, \lambda)$ denote the coefficients of the change of frame from the background frame to the frame adapted to the spheres $\mathbf{S}$. Then, we have, for any scalar function h on $\mathcal{R}$ and for any $1 \leq j \leq s_{max}$,

$$\begin{aligned} &\|(\nu^{\mathbf{S}})^j h\|_{\mathfrak{h}_{s_{max}-j}(\mathbf{S})} \\ \lesssim\ & r \sup_{\mathcal{R}} \left(|\widehat{\mathfrak{d}}^{\leq s_{max}} h| + \overset{\circ}{\delta}|\mathfrak{d}^{\leq s_{max}} h|\right) \\ &+ \left(\sum_{l=0}^{j-1} \|\nabla^l_{\nu^{\mathbf{S}}}(b^{\mathbf{S}} + e_3(u) + e_3(r))\|_{\mathfrak{h}_{s_{max}-l}(\mathbf{S})}\right) \sup_{\mathcal{R}} |\mathfrak{d}^{\leq s_{max}} h|. \end{aligned}$$

8.2. An auxiliary geodesic foliation in $^{(ext)}\mathcal{L}_0$

Recall from Section 3.1 that the initial data layer $\mathcal{L}_0$ is given by $\mathcal{L}_0 = {}^{(int)}\mathcal{L}_0 \cup {}^{(ext)}\mathcal{L}_0$, with ${}^{(int)}\mathcal{L}_0$ and ${}^{(ext)}\mathcal{L}_0$ covered by PG structures. The goal of this section is to construct and control an auxiliary outgoing geodesic foliation in ${}^{(ext)}\mathcal{L}_0$ that will be used in the proof of Theorem M0 and Theorem M6. To this end:

- We recall basic properties of the outgoing PG structure of ${}^{(ext)}\mathcal{L}_0$ in Section 8.2.1.

- We construct an auxiliary outgoing geodesic foliation in ${}^{(ext)}\mathcal{L}_0$ in Section 8.2.2.
- We use the transformation formulas to compare the Ricci coefficients and curvature components of the auxiliary outgoing geodesic foliation to the Ricci coefficients and curvature components of the outgoing PG structure of ${}^{(ext)}\mathcal{L}_0$ in Section 8.2.3.
- Finally, we control the auxiliary outgoing geodesic foliation of ${}^{(ext)}\mathcal{L}_0$ in Section 8.2.4, see Proposition 8.20.

8.2.1. Preliminaries In Section 8.2, we concentrate to the region ${}^{(ext)}\mathcal{L}_0$. To ease notations, throughout Section 8.2, we denote:

- (a, m) instead of (a_0, m_0),
- by (u, r, θ, φ) the PG coordinates of ${}^{(ext)}\mathcal{L}_0$,
- by $E = (e_1, e_2, e_3, e_4)$ the outgoing PG frame of ${}^{(ext)}\mathcal{L}_0$,
- by χ, $\underline{\chi}$, ζ, η, $\underline{\eta}$, ξ, $\underline{\xi}$, ω, $\underline{\omega}$, and α, $\underline{\alpha}$, β, $\underline{\beta}$, ρ, ${}^*\rho$ the Ricci and curvature coefficients of the outgoing PG structure of ${}^{(ext)}\mathcal{L}_0$.

Recall that the PG structure of ${}^{(ext)}\mathcal{L}_0$ verifies the following identities

$$e_4(r) = 1, \qquad e_4(u) = e_4(\theta) = e_4(\varphi) = 0, \qquad e_1(r) = e_2(r) = 0,$$

as well as

$$\xi = 0, \qquad \omega = 0, \qquad \underline{\eta} = -\zeta.$$

Moreover, ${}^{(ext)}\mathcal{L}_0$ is also endowed with a complex 1-form $\mathfrak{J}$ verifying

$$\nabla_{e_4}(r\mathfrak{J}) = 0, \qquad {}^*\mathfrak{J} = -i\mathfrak{J}, \qquad \left|\Re(\mathfrak{J})\right|^2 = \frac{(\sin\theta)^2}{|q|^2}.$$

Also, we define the linearized quantities for the outgoing PG structure of ${}^{(ext)}\mathcal{L}_0$ as in Definition 2.66, and the corresponding quantities Γ_g, Γ_b as in Definition 2.67. Since u is bounded in ${}^{(ext)}\mathcal{L}_0$, we simply write $\Gamma_g = r^{-1}\Gamma_b$. Our initial data control, see (3.52), implies in particular the bounds

$$ {}^{(ext)}\mathfrak{I}_{k_{large}+10} \;\leq\; \epsilon_0, \qquad {}^{(ext)}\mathfrak{I}_3 \leq \epsilon_0^2, \tag{8.59}$$

where, see Section 3.3.6,

$$ {}^{(ext)}\mathfrak{I}_k \;=\; \sup_{{}^{(ext)}\mathcal{L}_0}\left\{r|\mathfrak{d}^{\leq k}\Gamma_b| + r^{\frac{7}{2}+\frac{\delta_B}{2}}\left(|\mathfrak{d}^{\leq k}A| + |\mathfrak{d}^{\leq k}B|\right)\right\}.$$

We make the additional assumption

$$\widecheck{e_3(u)} \in \Gamma_b. \tag{8.60}$$

Remark 8.14. *To justify the above additional assumption in* ${}^{(ext)}\mathcal{L}_0$*, we recall the following equation, see Lemma 6.16,*

$$\begin{aligned} e_4\left(\widecheck{e_3(u)}\right) &= O(r^{-1})\check{H} + O(r^{-1})\check{Z} + O(r^{-2})\widecheck{\mathcal{D}u} \\ &\quad +\Gamma_b \cdot \Gamma_b \in r^{-1}\Gamma_b. \end{aligned} \tag{8.61}$$

Thus, the assumption (8.60) follows from integrating backwards the transport equation $e_4\left(\widecheck{e_3(u)}\right) \in r^{-1}\Gamma_b$*, and from using fact that* $\widecheck{e_3(u)} \to 0$ *as* $r \to \infty$*.*

Finally, note that we have, in view of Definitions 2.66 and 2.67,

$$\begin{aligned} & e_3(u) = 2 + \frac{2a^2\sin^2\theta}{r^2} + \Gamma_b + O(r^{-4}), \qquad \nabla u = a\Re(\mathfrak{J}) + \Gamma_b, \\ & e_3(r) = -\Upsilon + r\Gamma_b + O(r^{-2}), \qquad \nabla(\cos\theta) = \nabla(J^{(0)}) = -\Im(\mathfrak{J}) + \Gamma_b, \\ & e_3(\cos\theta) = \Gamma_b, \end{aligned}$$

where $\Upsilon = 1 - \frac{2m}{r}$.

8.2.2. Construction and asymptotic of the geodesic foliation We look for an optical function $\widetilde{u}$ such that $\widetilde{u} \sim u$ as $r \to \infty$. Its existence is provided by the following lemma.

Lemma 8.15. *There exists a unique optical function* $\widetilde{u}$ *defined in* ${}^{(ext)}\mathcal{L}_0$ *and verifying*

$$\widetilde{u} = u - \frac{a^2(\sin\theta)^2}{2r} + h, \qquad h = \Gamma_b + O(r^{-3}). \tag{8.62}$$

Proof. Let

$$\widetilde{u}_0 := u - \frac{a^2(\sin\theta)^2}{2r}.$$

We calculate

$$\begin{aligned} \mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u &= -e_3(u)e_4(u) + |\nabla u|^2 = |\nabla u|^2 \\ &= |a\Re(\mathfrak{J}) + \Gamma_b|^2 = \frac{a^2\sin^2\theta}{|q|^2} + r^{-1}\Gamma_b \end{aligned}$$

$$= \frac{a^2\sin^2\theta}{r^2} + O(r^{-4}) + r^{-1}\Gamma_b.$$

Hence, since $e_4(u)=0$, and in view of the definition of $\widetilde{u}_0$, we infer

$$\begin{aligned}
&\mathbf{g}^{\alpha\beta}\partial_\alpha\widetilde{u}_0\partial_\beta\widetilde{u}_0\\
= \ & \mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u - 2\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta\left(\frac{a^2(\sin\theta)^2}{2r}\right)\\
&+\mathbf{g}^{\alpha\beta}\partial_\alpha\left(\frac{a^2(\sin\theta)^2}{2r}\right)\partial_\beta\left(\frac{a^2(\sin\theta)^2}{2r}\right)\\
= \ & \frac{a^2\sin^2\theta}{r^2} + O(r^{-4}) + r^{-1}\Gamma_b + e_3(u)e_4\left(\frac{a^2(\sin\theta)^2}{2r}\right)\\
&-2\nabla(u)\cdot\nabla\left(\frac{a^2(\sin\theta)^2}{2r}\right) - e_4\left(\frac{a^2(\sin\theta)^2}{2r}\right)e_3\left(\frac{a^2(\sin\theta)^2}{2r}\right)\\
&+\left|\nabla\left(\frac{a^2(\sin\theta)^2}{2r}\right)\right|^2.
\end{aligned}$$

Note that

$$\begin{aligned}
&e_3(u)e_4\left(\frac{a^2(\sin\theta)^2}{2r}\right) - 2\nabla(u)\cdot\nabla\left(\frac{a^2(\sin\theta)^2}{2r}\right)\\
&-e_4\left(\frac{a^2(\sin\theta)^2}{2r}\right)e_3\left(\frac{a^2(\sin\theta)^2}{2r}\right) + \left|\nabla\left(\frac{a^2(\sin\theta)^2}{2r}\right)\right|^2\\
= \ & -\Big(2+\Gamma_b+O(r^{-2})\Big)\frac{a^2(\sin\theta)^2}{2r^2}\\
&+4\Big(a\Re(\mathfrak{J})+\Gamma_b\Big)\cdot\frac{a^2\cos\theta}{2r}\Big(-\Im(\mathfrak{J})+\Gamma_b\Big) + O(r^{-4}) + r^{-2}\Gamma_b\\
= \ & -\frac{a^2(\sin\theta)^2}{r^2} + O(r^{-4}) + r^{-2}\Gamma_b
\end{aligned}$$

where we used the fact that $\Re(\mathfrak{J})\cdot\Im(\mathfrak{J})=0$ since $\Im(\mathfrak{J}) = {}^*\Re(\mathfrak{J})$. Hence, we infer

$$\mathbf{g}^{\alpha\beta}\partial_\alpha\widetilde{u}_0\partial_\beta\widetilde{u}_0 = O(r^{-4}) + r^{-1}\Gamma_b.$$

Thus, since $\widetilde{u}=\widetilde{u}_0+h$, we deduce

$$\mathbf{g}^{\alpha\beta}\partial_\alpha\widetilde{u}\partial_\beta\widetilde{u} = \mathbf{g}^{\alpha\beta}\partial_\alpha(\widetilde{u}_0+h)\partial_\beta(\widetilde{u}_0+h) = 2\mathbf{g}^{\alpha\beta}\partial_\alpha\widetilde{u}_0\partial_\beta h + \mathbf{g}^{\alpha\beta}\partial_\alpha h\partial_\beta h$$

$$
\begin{aligned}
& +O(r^{-4})+r^{-1}\Gamma_b \\
= & -e_3(\widetilde{u}_0)e_4(h)-e_4(\widetilde{u}_0)e_3(h)+2\nabla(\widetilde{u}_0)\cdot\nabla(h)+\mathbf{g}^{\alpha\beta}\partial_\alpha h\partial_\beta h \\
& +O(r^{-4})+r^{-1}\Gamma_b.
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
e_3(\widetilde{u}_0) &= e_3(u)+\frac{a^2\sin^2\theta}{2r^2}e_3(r)+r^{-1}\Gamma_b = 2+O(r^{-2})+\Gamma_b,\\
e_4(\widetilde{u}_0) &= e_4(u)+\frac{a^2\sin^2\theta}{2r^2} = \frac{a^2\sin^2\theta}{2r^2},\\
\nabla(\widetilde{u}_0) &= \nabla(u)+O(r^{-2})+r^{-1}\Gamma_b = a\Re(\mathfrak{J})+O(r^{-2})+\Gamma_b,
\end{aligned}
$$

and hence

$$
\begin{aligned}
\mathbf{g}^{\alpha\beta}\partial_\alpha\widetilde{u}\partial_\beta\widetilde{u} &= -\Big(2+O(r^{-2})+\Gamma_b\Big)e_4(h)+O(r^{-2})e_3(h)\\
&\quad +2\Big(a\Re(\mathfrak{J})+O(r^{-2})+\Gamma_b\Big)\cdot\nabla(h)+\mathbf{g}^{\alpha\beta}\partial_\alpha h\partial_\beta h+O(r^{-4})\\
&\quad +r^{-1}\Gamma_b.
\end{aligned}
$$

Since $\widetilde{u}$ is an optical function, we are thus led to solve the following equation for h

$$
\begin{aligned}
\Big(1+O(r^{-2})+\Gamma_b\Big)e_4(h)+O(r^{-2})e_3(h)&\\
-\Big(a\Re(\mathfrak{J})+O(r^{-2})+\Gamma_b\Big)\cdot\nabla(h)+\mathbf{g}^{\alpha\beta}\partial_\alpha h\partial_\beta h &= O(r^{-4})+r^{-1}\Gamma_b,
\end{aligned}
\tag{8.63}
$$

with the initialization

$$
\lim_{r\to+\infty} h = 0. \tag{8.64}
$$

(8.63) is a nonlinear transport equation for h. If $h=\Gamma_b+O(r^{-3})$, h satisfies in particular $|\mathfrak{d}^{\leq 1}h|\lesssim\epsilon_0 r^{-1}+r^{-3}$, and the uniqueness follows immediately from backward integration of (8.63) and the initialization (8.64). Also, commuting (8.63), using (8.59) to estimate the RHS of (8.63), integrating backward, and using the initialization (8.64), we easily obtain the following a priori bounds for h

$$
\big|\mathfrak{d}^{\leq k_{large}+10}h\big|\lesssim\epsilon_0 r^{-1}+r^{-3}, \qquad \big|\mathfrak{d}^{\leq 3}h\big|\lesssim\epsilon_0^2 r^{-1}+r^{-3}.
$$

These bounds yield $h=\Gamma_b+O(r^{-3})$, and can be used to prove the existence of h. Thus, there exists a unique solution h of (8.63) defined on ${}^{(ext)}\mathcal{L}_0$, such that $h \to 0$ as $r \to \infty$ and $h = \Gamma_b + O(r^{-3})$. This concludes the proof of the lemma. $\square$

Definition 8.16. *Let $\widetilde{u}$ the outgoing optical function of Lemma 8.15. Then, we define the following:*

- *Let $\widetilde{e}_4 := -\mathbf{g}^{\alpha\beta}\partial_\alpha \widetilde{u} \partial_\beta$ the null outgoing geodesic vectorfield associated to $\widetilde{u}$.*
- *Let $\widetilde{s}$ be the associated affine parameter, i.e. $\widetilde{e}_4(\widetilde{s}) = 1$, with $\widetilde{s}$ normalized such that $\widetilde{s} = r$ as $r \to \infty$.*
- *We define the region ${}^{(ext)}\widetilde{\mathcal{L}_0} \subset {}^{(ext)}\mathcal{L}_0$ to be the region*[162]

$$(8.65) \qquad {}^{(ext)}\widetilde{\mathcal{L}_0} := \left\{ 0 \le \widetilde{u} \le 2, \quad \widetilde{s} \ge \frac{\delta_*}{2}\epsilon_0^{-1} \right\},$$

  with $\delta_* > 0$ the small constant introduced in Section 3.4.1. See Figure 7 below where ${}^{(ext)}\widetilde{\mathcal{L}_0}$ is sketched in red inside the initial data layer $\mathcal{L}_0$.
- *We denote by $\widetilde{r}$ the area radius of the spheres $S(\widetilde{u}, \widetilde{s})$.*
- *The foliation induced by $(\widetilde{u}, \widetilde{s})$ is called the outgoing geodesic foliation of ${}^{(ext)}\mathcal{L}_0$ normalized at infinity. We denote by $\widetilde{E} = (\widetilde{e}_4, \widetilde{e}_3, \widetilde{e}_1, \widetilde{e}_2)$ the corresponding null frame.*
- *We denote by $(f, \underline{f}, \lambda)$ the transition coefficients of the frame transformation which takes the outgoing PG frame of ${}^{(ext)}\mathcal{L}_0$, denoted E, into $\widetilde{E}$.*

Proposition 8.17. *The following holds true in ${}^{(ext)}\widetilde{\mathcal{L}_0}$:*

1. *The transition functions $(f, \underline{f}, \lambda)$ are given by the formulas*

$$(8.66) \qquad \begin{aligned} \lambda &= 1 + \frac{3a^2(\sin\theta)^2}{4r^2} + O(r^{-3}) + r^{-1}\mathfrak{d}^{\le 1}\Gamma_b, \\ f &= -a\Re(\mathfrak{J}) + \frac{a^2\cos\theta}{r}\Im(\mathfrak{J}) + O(r^{-3}) + \mathfrak{d}^{\le 1}\Gamma_b, \\ \underline{f} &= -a\Upsilon\Re(\mathfrak{J}) - \frac{a^2\cos\theta}{r}\Im(\mathfrak{J}) + O(r^{-3}) + \mathfrak{d}^{\le 2}\Gamma_b. \end{aligned}$$

   Using complex notations, $F = f + i\,{}^*f$ and $\underline{F} = \underline{f} + i\,{}^*\underline{f}$ satisfy

$$F \quad = \quad -a\left(1 + i\frac{a\cos\theta}{r}\right)\mathfrak{J} + O(r^{-3}) + \mathfrak{d}^{\le 1}\Gamma_b = -\frac{aq}{r}\mathfrak{J} + O(r^{-3})$$

[162] Recall that $0 \le u \le 3$ and $r \ge r_0$ in ${}^{(ext)}\mathcal{L}_0$, so that we have indeed ${}^{(ext)}\widetilde{\mathcal{L}_0} \subset {}^{(ext)}\mathcal{L}_0$ in view of the asymptotic of $\widetilde{u}$ provided by Lemma 8.15 and the asymptotic for $\widetilde{s}$ provided by Proposition 8.17.

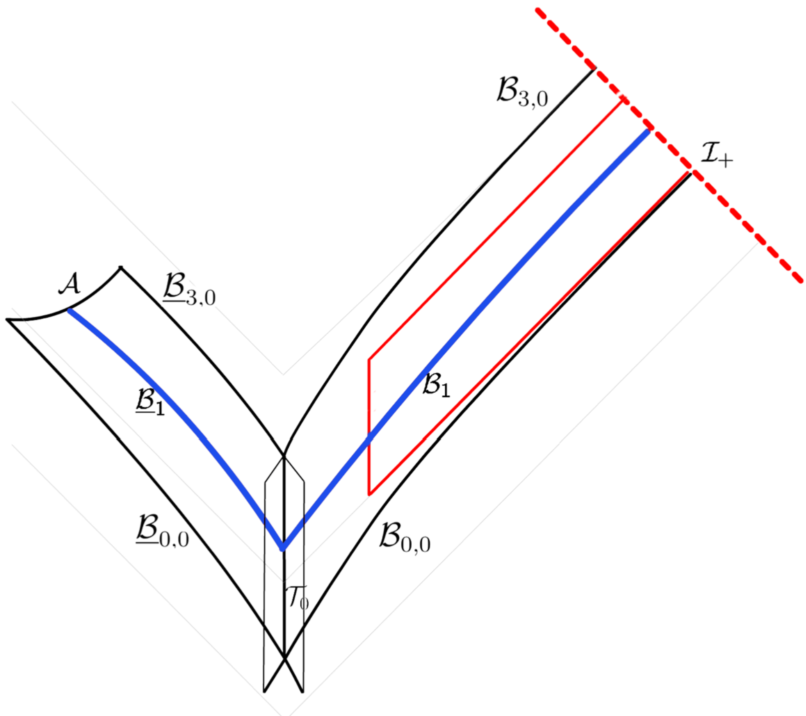

Figure 7: The initial data layer $\mathcal{L}_0$ and the region ${}^{(ext)}\widetilde{\mathcal{L}_0}$ (in red).

$$\begin{aligned} & +\mathfrak{d}^{\leq 1}\Gamma_b, \\ \underline{F} \quad = \quad & -a\left(\Upsilon - i\frac{a\cos\theta}{r}\right)\mathfrak{J} + O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_b = -\frac{a\Upsilon\overline{q}}{r}\mathfrak{J} + O(r^{-3}) \\ & +\mathfrak{d}^{\leq 2}\Gamma_b. \end{aligned}$$

2. *The function $\widetilde{s}$ behaves as follows*

$$\widetilde{s} = r + \frac{a^2\sin^2\theta}{2r} + O(r^{-2}) + \mathfrak{d}^{\leq 1}\Gamma_b. \tag{8.67}$$

Proof. The proof proceeds in several steps.

Step 1. We start with the control of f. To this end, we rely on the formula

$$\widetilde{e}_4 \quad = \quad \lambda\left(e_4 + f^a e_a + \frac{1}{4}|f|^2 e_3\right).$$

On the other hand, in view of the definition of $\widetilde{e}_4$, we have

$$\widetilde{e}_4 = -\mathbf{g}^{\alpha\beta}\partial_\alpha(\widetilde{u})\partial_\beta = \frac{1}{2}e_3(\widetilde{u})e_4 + \frac{1}{2}e_4(\widetilde{u})e_3 - e_a(\widetilde{u})e_a.$$

Hence

$$\lambda = \frac{1}{2}e_3(\widetilde{u}), \qquad f = -\frac{2}{\widetilde{e}_3(\widetilde{u})}\nabla(\widetilde{u}).$$

We calculate, using the following identities $e_3(r) = -\Upsilon + O(r^{-2}) + r\Gamma_b$, $e_3(u) = 2 + \frac{2a^2\sin^2\theta}{r^2} + O(r^{-4}) + \Gamma_b$, $e_3(\cos\theta) = \Gamma_b$, $\widetilde{u} = u - \frac{a^2(\sin\theta)^2}{2r} + h$ and $h = O(r^{-3}) + \Gamma_b$,

$$\begin{aligned} e_3(\widetilde{u}) &= e_3\left(u - \frac{a^2(\sin\theta)^2}{2r} + h\right) = 2 + \frac{2a^2\sin^2\theta}{r^2} + O(r^{-4}) + \Gamma_b \\ &\quad + \frac{a^2(\sin\theta)^2}{2r^2}e_3(r) + \frac{a^2}{r}\cos\theta e_3(\cos\theta) + e_3(h) \\ &= 2 + \frac{3a^2\sin^2\theta}{2r^2} + O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_b. \end{aligned}$$

Also, since $\nabla u = a\Re(\mathfrak{J}) + \Gamma_b$ and $\nabla(\cos\theta) = -\Im(\mathfrak{J}) + \Gamma_b$, we have

$$\begin{aligned} \nabla(\widetilde{u}) &= \nabla\left(u - \frac{a^2(\sin\theta)^2}{2r} + h\right) = a\Re(\mathfrak{J}) + \Gamma_b + \frac{a^2\cos\theta}{r}\nabla(\cos\theta) + \nabla(h) \\ &= a\Re(\mathfrak{J}) - \frac{a^2\cos\theta}{r}\Im(\mathfrak{J}) + O(r^{-4}) + \mathfrak{d}^{\leq 1}\Gamma_b. \end{aligned}$$

We deduce,

$$\begin{aligned} \lambda &= \frac{1}{2}e_3(\widetilde{u}) = 1 + \frac{3a^2\sin^2\theta}{4r^2} + O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_b, \\ f &= -\frac{2}{\widetilde{e}_3(\widetilde{u})}\nabla(\widetilde{u}) = -a\Re(\mathfrak{J}) + \frac{a^2\cos\theta}{r}\Im(\mathfrak{J}) + O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_b, \end{aligned}$$

which is the desired estimate for f, but not for λ.

Step 2. Next, we derive the desired estimate for λ. To this end, we need to improve the estimate for λ of Step 1. This improvement will be needed to get the correct asymptotic for $\widetilde{s}$.

Lemma 8.18. *We have*

$$\lambda = 1 + \frac{3a^2\sin^2\theta}{4r^2} + O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b.$$

Proof. Recall the transport equation for $e_4(\log\lambda)$ in Corollary 2.13, which, in view of that fact that $\omega = 0$ and $\underline{\eta} = -\zeta$, takes the following form

$$\begin{aligned}\lambda^{-1}\nabla_{\widetilde{e}_4}(\log\lambda) &= 2f\cdot\zeta + E_2(f,\Gamma),\\ E_2(f,\Gamma) &= -\frac{1}{2}|f|^2\underline{\omega} - \frac{1}{4}\mathrm{tr}\,\underline{\chi}|f|^2 + O(f^3\Gamma + f^2\widehat{\underline{\chi}}).\end{aligned}$$

Using in particular the form of f in Step 1, note that

$$E_2(f,\Gamma) = -\frac{1}{4}\mathrm{tr}\,\underline{\chi}|f|^2 + O(r^{-4}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b.$$

We deduce

$$\begin{aligned}\widetilde{e}_4(\log\lambda) &= 2f\cdot\zeta - \frac{1}{4}\mathrm{tr}\,\underline{\chi}|f|^2 + O(r^{-4}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b\\ &= \Re(F\cdot\overline{Z}) - \frac{1}{4}\mathrm{tr}\,\underline{\chi}|f|^2 + O(r^{-4}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b.\end{aligned}$$

Next, recall that

$$Z = \frac{a\overline{q}}{|q|^2}\mathfrak{J} + \Gamma_g, \qquad \mathrm{tr}\,\underline{\chi} = -\frac{2\Upsilon}{r} + O(r^{-3}) + \Gamma_g.$$

Thus, together with the form of f in Step 1, we infer

$$\begin{aligned}F\cdot\overline{Z} &= \left(-\frac{aq}{r}\mathfrak{J} + O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_b\right)\cdot\left(\frac{aq}{|q|^2}\overline{\mathfrak{J}} + \Gamma_g\right)\\ &= -\frac{aq}{r}\mathfrak{J}\cdot\left(\frac{aq}{|q|^2}\overline{\mathfrak{J}}\right) + O(r^{-4}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b = -\frac{2a^2\sin^2\theta}{r^3} + O(r^{-4})\\ &\quad + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b,\end{aligned}$$

$$\begin{aligned}|f|^2 &= \left|-a\Re(\mathfrak{J}) + \frac{a^2\cos\theta}{r}\Im(\mathfrak{J}) + O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_b\right|^2\\ &= \frac{a^2\sin^2\theta}{r^2} + O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{4}\mathrm{tr}\,\underline{\chi}|f|^2 &= \left(-\frac{\Upsilon}{2r} + O(r^{-3}) + \Gamma_g\right)\left(\frac{a^2\sin^2\theta}{r^2} + O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b\right)\\ &= -\frac{a^2\sin^2\theta}{2r^3} + O(r^{-4}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b.\end{aligned}$$

We deduce

$$\begin{aligned}\widetilde{e}_4(\log\lambda) &= \Re(F\cdot\overline{Z}) - \frac{1}{4}\mathrm{tr}\,\underline{\chi}|f|^2 + O(r^{-4}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b\\ &= -\frac{2a^2\sin^2\theta}{r^3} + \frac{a^2\sin^2\theta}{2r^3} + O(r^{-4}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b\\ &= -\frac{3}{2}\frac{a^2\sin^2\theta}{r^3} + O(r^{-4}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b.\end{aligned}$$

Integrating backwards from $r\to+\infty$, and noticing that $\log(\lambda)=0$ at $r=+\infty$ in view of the control for λ derived in Step 1, we obtain

$$\log\lambda = \frac{3}{4}\frac{a^2\sin^2\theta}{r^2} + O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b$$

i.e.

$$\lambda = 1 + \frac{3}{4}\frac{a^2\sin^2\theta}{r^2} + O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b$$

as stated. This concludes the proof of Lemma 8.18. □

Step 3. We look for $\widetilde{s}$ in the form

(8.68) $$\widetilde{s} = r + \widetilde{s}_0, \qquad \lim_{r\to+\infty}\widetilde{s}_0 = 0.$$

To have $\widetilde{e}_4(\widetilde{s})=1$ we thus need to solve the following transport equation for $\widetilde{s}_0$

(8.69) $$\widetilde{e}_4(\widetilde{s}_0) = 1 - \widetilde{e}_4(r).$$

Since $e_a(r)=0$ and $\left|\Re(\mathfrak{J})\right|^2 = \frac{(\sin\theta)^2}{|q|^2} = \frac{(\sin\theta)^2}{r^2} + O(r^{-4})$, we have

$$\begin{aligned}\widetilde{e}_4(r) &= \lambda\left(e_4 + f^a e_a + \frac{1}{4}|f|^2 e_3\right)r = \lambda + \frac{1}{4}\lambda|f|^2 e_3(r)\\ &= \lambda + \frac{1}{4}\lambda|f|^2\Big(-\Upsilon + r\Gamma_b + O(r^{-2})\Big) = \left(\lambda - \frac{1}{4}|f|^2\right) + O(r^{-3})\\ &\quad + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b.\end{aligned}$$

Using the improved asymptotic for λ and the asymptotic of f derived above

$$\lambda - \frac{1}{4}|f|^2 = 1 + \frac{3}{4}\frac{a^2\sin^2\theta}{r^2} + O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b$$

$$-\frac{1}{4}\left|-a\Re(\mathfrak{J})+\frac{a^2\cos\theta}{r}\Im(\mathfrak{J})+O(r^{-3})+\mathfrak{d}^{\leq 1}\Gamma_b\right|^2$$
$$= 1+\frac{3}{4}\frac{a^2\sin^2\theta}{r^2}-\frac{1}{4}\frac{a^2\sin^2\theta}{r^2}+O(r^{-3})+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b$$
$$= 1+\frac{a^2\sin^2\theta}{r^2}+O(r^{-3})+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b.$$

Hence

$$\widetilde{e}_4(r) = \left(\lambda-\frac{1}{4}|f|^2\right)+O(r^{-3})+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b = 1+\frac{a^2\sin^2\theta}{r^2}+O(r^{-3}) +r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b$$

and we deduce

$$\widetilde{e}_4(\widetilde{s}_0) = 1-\widetilde{e}_4(r) = -\frac{a^2\sin^2\theta}{r^2}+O(r^{-3})+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b.$$

Integrating backwards from $r\to+\infty$, and since $\widetilde{s}_0=0$ at $r=+\infty$, we infer

$$\widetilde{s}_0 = \frac{a^2\sin^2\theta}{r}+O(r^{-2})+\mathfrak{d}^{\leq 1}\Gamma_b.$$

Therefore

$$\widetilde{s} = r+\widetilde{s}_0 = r+\frac{a^2\sin^2\theta}{r}+O(r^{-2})+\mathfrak{d}^{\leq 1}\Gamma_b$$

as stated.

Step 4. In this last step, we control $\underline{f}$. To this end, we recall the formula

$$\widetilde{e}_a = \left(\delta_a^b+\frac{1}{2}\underline{f}_a f^b\right)e_b+\frac{1}{2}\underline{f}_a e_4+\left(\frac{1}{2}f_a+\frac{1}{8}|f|^2\underline{f}_a\right)e_3.$$

Since $e_a(r)=0$, $\widetilde{s}=r+\frac{a^2\sin^2\theta}{2r}+O(r^{-2})+\mathfrak{d}^{\leq 1}\Gamma_b$ and $f=O(r^{-1})$, we derive

$$\nabla(\widetilde{s}) = \nabla\left(\frac{a^2\sin^2\theta}{2r}\right)+O(r^{-3})+r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b$$
$$= -\frac{a^2}{r}\cos\theta\nabla(\cos\theta)+O(r^{-3})+r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b$$
$$= -\frac{a^2}{r}\cos\theta\big(-\Im(\mathfrak{J})+\Gamma_b\big)+O(r^{-3})+r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b$$

$$
\begin{aligned}
&= \frac{a^2}{r}\cos\theta\Im(\mathfrak{J}) + O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b,\\
e_3(\widetilde{s}) &= \left(1-\frac{a^2\sin^2\theta}{2r^2}\right)\Big(-\Upsilon + r\Gamma_b\Big) + O(r^{-3}) + \mathfrak{d}^{\leq 2}\Gamma_b\\
&= -\Upsilon + \frac{a^2\sin^2\theta}{r^2} + O(r^{-3}) + r\mathfrak{d}^{\leq 2}\Gamma_b,\\
e_4(\widetilde{s}) &= 1-\frac{a^2\sin^2\theta}{2r^2} + O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b.
\end{aligned}
$$

Therefore, since $\widetilde{e}_a(\widetilde{s}) = 0$,

$$
\begin{aligned}
0 &= \nabla(\widetilde{s}) + \frac{1}{2}(f\cdot\nabla(\widetilde{s}))\underline{f} + \frac{1}{2}\underline{f}e_4(\widetilde{s}) + \left(\frac{1}{2}f + \frac{1}{8}|f|^2\underline{f}\right)e_3(\widetilde{s})\\
&= \nabla(\widetilde{s}) + \frac{1}{2}\underline{f}e_4(\widetilde{s}) + \frac{1}{2}fe_3(\widetilde{s}) + O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b\\
&= \frac{a^2}{r}\cos\theta\Im(\mathfrak{J}) + O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b\\
&\quad +\frac{1}{2}\underline{f}\left(1-\frac{a^2\sin^2\theta}{2r^2} + O(r^{-3}) + \mathfrak{d}^{\leq 2}\Gamma_b\right)\\
&\quad +\frac{1}{2}f\left(-\Upsilon + \frac{a^2\sin^2\theta}{r^2} + O(r^{-3}) + r\mathfrak{d}^{\leq 2}\Gamma_b\right) + O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b.
\end{aligned}
$$

We deduce,

$$
0 = \frac{a^2}{r}\cos\theta\Im(\mathfrak{J}) + \frac{1}{2}\underline{f} - \frac{1}{2}\Upsilon f + O(r^{-3}) + \mathfrak{d}^{\leq 2}\Gamma_b
$$

i.e.

$$
\begin{aligned}
\underline{f} &= \Upsilon f - \frac{2a^2\cos\theta}{r}\Im(\mathfrak{J}) + O(r^{-3}) + \mathfrak{d}^{\leq 2}\Gamma_b\\
&= \Upsilon\left(-a\Re(\mathfrak{J}) + \frac{a^2\cos\theta}{r}\Im(\mathfrak{J}) + O(r^{-3}) + \mathfrak{d}^{\leq 2}\Gamma_b\right) - \frac{2a^2\cos\theta}{r}\Im(\mathfrak{J})\\
&\quad +O(r^{-3}) + \mathfrak{d}^{\leq 2}\Gamma_b\\
&= -a\Upsilon\Re(\mathfrak{J}) - \frac{a^2\cos\theta}{r}\Im(\mathfrak{J}) + O(r^{-3}) + \mathfrak{d}^{\leq 2}\Gamma_b
\end{aligned}
$$

as stated. This ends the proof of Proposition 8.17. □

8.2.3. Ricci and curvature coefficients in the geodesic frame of ${}^{(ext)}\mathcal{L}_0$

Lemma 8.19. *The Ricci and curvature coefficients relative to the outgoing geodesic foliation of ${}^{(ext)}\mathcal{L}_0$ verify following:*

1. *The curvature coefficients satisfy*

$$\begin{aligned}
\widetilde{\alpha} &= \alpha + O(r^{-5}) + r^{-3}\mathfrak{d}^{\leq 2}\Gamma_b,\\
\widetilde{\beta} &= \beta + \frac{3am}{r^3}\Re(\mathfrak{J}) + O(r^{-5}) + r^{-3}\mathfrak{d}^{\leq 2}\Gamma_b,\\
\widetilde{\rho} &= -\frac{2m}{r^3} + O(r^{-5}) + r^{-3}\mathfrak{d}^{\leq 2}\Gamma_b,\\
{}^*\widetilde{\rho} &= \frac{6am\cos\theta}{r^4} + O(r^{-5}) + r^{-3}\mathfrak{d}^{\leq 2}\Gamma_b,\\
\widetilde{\underline{\beta}} &= \underline{\beta} + O(r^{-5}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b,\\
\widetilde{\underline{\alpha}} &= \underline{\alpha} + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b + O(r^{-5}).
\end{aligned} \tag{8.70}$$

2. *The Ricci coefficients satisfy*[163]

$$\begin{aligned}
\widetilde{tr\chi} &= \frac{2}{r} + O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b,\\
\widetilde{\widehat{\chi}} &= O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b,\\
\widetilde{tr\underline{\chi}} &= -\frac{2\Upsilon}{r} + O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b,\\
\widetilde{\widehat{\underline{\chi}}} &= O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b,\\
\widetilde{\zeta} &= O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b,\\
\widetilde{\underline{\xi}} &= O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b,\\
\widetilde{\underline{\omega}} &= \frac{m}{r^2} + O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b.
\end{aligned} \tag{8.71}$$

3. *Let $\widetilde{r}$ be the area radius of $S(\widetilde{u},\widetilde{s})$, i.e. $4\pi(\widetilde{r})^2 = |S(\widetilde{u},\widetilde{s})|$. Then, we have*

$$\widetilde{r} \;=\; r + \frac{a^2(\sin\theta)^2}{2r} + O(r^{-2}) + r\mathfrak{d}^{\leq 2}\Gamma_b. \tag{8.72}$$

[163] Recall that $\widetilde{\xi} = 0$, $\widetilde{\omega} = 0$ and $\widetilde{\eta} = \widetilde{\zeta} = -\widetilde{\underline{\eta}}$ as the foliation is outgoing geodesic.

4. *Let*

$$\widetilde{\widetilde{tr\chi}} := \widetilde{tr\chi} - \frac{2}{\widetilde{r}}, \qquad \widetilde{\widetilde{tr\underline{\chi}}} := \widetilde{tr\underline{\chi}} + \frac{2(1-\frac{2m}{\widetilde{r}})}{\widetilde{r}}.$$

Then

$$\widetilde{\widetilde{tr\chi}} = O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b, \quad \widetilde{\widetilde{tr\underline{\chi}}} = O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b. \tag{8.73}$$

5. *Let the basis of $\ell = 1$ modes $\widetilde{J}^{(p)}$ of $S(\widetilde{u}, \widetilde{s})$ be given by $\nabla_{\widetilde{e}_4}\widetilde{J}^{(p)} = 0$ and $\widetilde{J}^{(p)} = J^{(p)}$ as $r \to \infty$. Then, we have*

$$\widetilde{J}^{(p)} - J^{(p)} = O(r^{-1}) + \mathfrak{d}^{\leq 1}\Gamma_b, \qquad p = 0, +, -. \tag{8.74}$$

6. *We have*

$$\begin{aligned} \widetilde{div}\,\widetilde{\beta} &= O(r^{-6}) + r^{-3}\mathfrak{d}^{\leq 3}\Gamma_b, \\ \widetilde{curl}\,\widetilde{\beta} &= \frac{6a_0m_0}{\widetilde{r}^5}\widetilde{J}^{(0)} + O(r^{-6}) + r^{-3}\mathfrak{d}^{\leq 3}\Gamma_b. \end{aligned} \tag{8.75}$$

Proof. The proof proceeds in several steps.

Step 1. Recall from Proposition 8.17 that the transition coefficients $(f, \underline{f}, \lambda)$ from the frame E to the frame $\widetilde{E}$ are given by the formulas

$$\begin{aligned} \lambda &= 1 + \frac{3a^2(\sin\theta)^2}{4r^2} + O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b, \\ f &= -a\Re(\mathfrak{J}) + \frac{a^2\cos\theta}{r}\Im(\mathfrak{J}) + O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_b, \\ \underline{f} &= -a\Upsilon\Re(\mathfrak{J}) - \frac{a^2\cos\theta}{r}\Im(\mathfrak{J}) + O(r^{-3}) + \mathfrak{d}^{\leq 2}\Gamma_b. \end{aligned}$$

To derive the formulas for the curvature and Ricci coefficients, it suffices to consider the simplified formulas

$$\begin{aligned} \lambda &= 1 + O(r^{-2}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b, \\ f &= -a\Re(\mathfrak{J}) + O(r^{-2}) + \mathfrak{d}^{\leq 1}\Gamma_b, \\ \underline{f} &= -a\Upsilon\Re(\mathfrak{J}) + O(r^{-2}) + \mathfrak{d}^{\leq 2}\Gamma_b. \end{aligned}$$

Together with the transformation formulas of Proposition 2.12, this easily yields the formulas for the curvature and Ricci coefficients. Note that, modulo the terms in Γ_b, the formulas are exactly the same as those in Lemma 2.52 regarding the Ricci and curvature coefficients for the integrable frame.

Step 2. Next, we derive more precise transformation formulas for $\widetilde{\operatorname{tr}\chi}, \widetilde{\operatorname{tr}\underline{\chi}}$. In view of the transformation formulas of Proposition 2.12, and the fact that ${}^{(a)}\operatorname{tr}\chi$, $\underline{\omega} = O(r^{-2}) + \Gamma_b$ and $f, \underline{f} = O(r^{-1})$, we have

$$\begin{aligned}
\lambda^{-1}\widetilde{\operatorname{tr}\chi} &= \operatorname{tr}\chi + \widetilde{\operatorname{div}} f + f\cdot\eta + f\cdot\zeta + \operatorname{Err}(\operatorname{tr}\chi, \widetilde{\operatorname{tr}\chi}),\\
\operatorname{Err}(\operatorname{tr}\chi, \widetilde{\operatorname{tr}\chi}) &= -\frac{1}{4}|f|^2\operatorname{tr}\underline{\chi} + O(r^{-4}) + r^{-2}\mathfrak{d}^{\leq 2}\Gamma_b,\\
\lambda\widetilde{\operatorname{tr}\underline{\chi}} &= \operatorname{tr}\underline{\chi} + \widetilde{\operatorname{div}}\,\underline{f} + \underline{f}\cdot\underline{\eta} - \underline{f}\cdot\zeta + \operatorname{Err}(\operatorname{tr}\underline{\chi}, \widetilde{\operatorname{tr}\underline{\chi}}),\\
\operatorname{Err}(\operatorname{tr}\underline{\chi}, \widetilde{\operatorname{tr}\underline{\chi}}) &= \frac{1}{2}(f\cdot\underline{f})\operatorname{tr}\underline{\chi} - \frac{1}{4}|\underline{f}|^2\operatorname{tr}\chi + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b,
\end{aligned}$$

where $(f, \underline{f}, \lambda)$ are the transition coefficients from the frame E to the frame $\widetilde{E}$. In view of

$$\begin{aligned}
f &= -a\Re(\mathfrak{J}) + O(r^{-2}) + \mathfrak{d}^{\leq 1}\Gamma_b,\\
\underline{f} &= -a\Re(\mathfrak{J}) + O(r^{-2}) + \mathfrak{d}^{\leq 2}\Gamma_b,\\
\operatorname{tr}\chi &= \frac{2}{r} - \frac{2a^2(\cos\theta)^2}{r^3} + O(r^{-5}) + \Gamma_g,\\
\operatorname{tr}\underline{\chi} &= -\frac{2\Upsilon}{r} - \frac{2a^2}{r^3} + \frac{4a^2(\cos\theta)^2}{r^3} + O(r^{-4}) + \Gamma_g,\\
\zeta &= \Re\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right) + \Gamma_g = \frac{a}{r}\Re(\mathfrak{J}) + O(r^{-3}) + \Gamma_g,\\
\eta &= \Re\left(\frac{aq}{|q|^2}\mathfrak{J}\right) + \Gamma_b = \frac{a}{r}\Re(\mathfrak{J}) + O(r^{-3}) + \Gamma_b,
\end{aligned}$$

and $\underline{\eta} = -\zeta$, we deduce

$$\begin{aligned}
\lambda^{-1}\widetilde{\operatorname{tr}\chi} =& \frac{2}{r} - \frac{2a^2(\cos\theta)^2}{r^3} - \frac{3a^2(\sin\theta)^2}{2r^3} + \widetilde{\operatorname{div}} f + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b,\\
\lambda\widetilde{\operatorname{tr}\underline{\chi}} =& -\frac{2\Upsilon}{r} - \frac{2a^2}{r^3} + \frac{4a^2(\cos\theta)^2}{r^3} + \frac{a^2(\sin\theta)^2}{2r^3} + \widetilde{\operatorname{div}}\,\underline{f} + O(r^{-4})\\
&+ r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b.
\end{aligned}$$

Next, we compute $\widetilde{\operatorname{div}} f$ and $\widetilde{\operatorname{div}}\,\underline{f}$. Arguing similarly to Section 6.2.2, see in particular 6.30, and using $f, \underline{f} = O(r^{-1})$ and $\lambda = 1 + O(r^{-2})$, we obtain

$$\widetilde{\operatorname{div}} f \quad = \quad \operatorname{div} f + \frac{1}{2}f\cdot(\nabla_3 f) + \frac{1}{2}\underline{f}\cdot(\nabla_4 f) + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b,$$

$$\widetilde{\operatorname{div}}\,\underline{f} \quad = \quad \operatorname{div}\underline{f} + \frac{1}{2}f\cdot(\nabla_3\underline{f}) + \frac{1}{2}\underline{f}\cdot(\nabla_4\underline{f}) + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b.$$

Using again

$$f = -a\Re(\mathfrak{J}) + \frac{a^2\cos\theta}{r}\Im(\mathfrak{J}) + O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_b,$$
$$\underline{f} = -a\Upsilon\Re(\mathfrak{J}) - \frac{a^2\cos\theta}{r}\Im(\mathfrak{J}) + O(r^{-3}) + \mathfrak{d}^{\leq 2}\Gamma_b,$$

we infer

$$\begin{aligned}
\widetilde{\operatorname{div}}\,f &= \operatorname{div}\left(-a\Re(\mathfrak{J}) + \frac{a^2\cos\theta}{r}\Im(\mathfrak{J})\right) + \frac{a^2}{2}\Re(\mathfrak{J})\cdot\nabla_3\Re(\mathfrak{J})\\
&\quad + \frac{a^2}{2}\Re(\mathfrak{J})\cdot\nabla_4\Re(\mathfrak{J}) + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b,\\
\widetilde{\operatorname{div}}\,\underline{f} &= \operatorname{div}\left(-a\Upsilon\Re(\mathfrak{J}) - \frac{a^2\cos\theta}{r}\Im(\mathfrak{J})\right) + \frac{a^2}{2}\Re(\mathfrak{J})\cdot\nabla_3\Re(\mathfrak{J})\\
&\quad + \frac{a^2}{2}\Re(\mathfrak{J})\cdot\nabla_4\Re(\mathfrak{J}) + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b.
\end{aligned}$$

Note that

$$\begin{aligned}
2\Re(\mathfrak{J})\cdot\nabla_3\Re(\mathfrak{J}) + 2\Re(\mathfrak{J})\cdot\nabla_4\Re(\mathfrak{J}) &= \nabla_{e_3+e_4}(|\Re(\mathfrak{J})|^2)\\
&= O(r^{-3})\Big(e_3(r) + e_4(r)\Big)\\
&\quad + O(r^{-3})e_3(\cos\theta)\\
&= O(r^{-4}) + r^{-2}\Gamma_b
\end{aligned}$$

where we used the identities $e_4(\theta) = 0$, $e_3(\cos\theta) \in \Gamma_b$, $e_4(r) = 1$ and $e_3(r) = -1 + O(r^{-1}) + r\Gamma_b$. This yields

$$\begin{aligned}
\widetilde{\operatorname{div}}\,f &= \operatorname{div}\left(-a\Re(\mathfrak{J}) + \frac{a^2\cos\theta}{r}\Im(\mathfrak{J})\right) + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b,\\
\widetilde{\operatorname{div}}\,\underline{f} &= \operatorname{div}\left(-a\Upsilon\Re(\mathfrak{J}) - \frac{a^2\cos\theta}{r}\Im(\mathfrak{J})\right) + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b.
\end{aligned}$$

Next, we use

$$\overline{\mathcal{D}}\cdot\mathfrak{J} = \frac{4i(r^2+a^2)\cos\theta}{|q|^4} + r^{-1}\Gamma_b$$

which yields

$$\mathrm{div}\,(\Re(\mathfrak{J})) = r^{-1}\Gamma_b, \qquad \mathrm{div}\,(\Im(\mathfrak{J})) = \frac{2\cos\theta}{r^2} + O(r^{-4}) + r^{-1}\Gamma_b,$$

and hence

$$\begin{aligned}
\widetilde{\mathrm{div}}\, f &= \frac{2a^2(\cos\theta)^2}{r^3} + \frac{a^2}{r}\nabla(\cos\theta)\cdot\Im(\mathfrak{J}) + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b,\\
\widetilde{\mathrm{div}}\, \underline{f} &= -\frac{2a^2(\cos\theta)^2}{r^3} - \frac{a^2}{r}\nabla(\cos\theta)\cdot\Im(\mathfrak{J}) + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b.
\end{aligned}$$

Since $\nabla(\cos\theta) = -\Im(\mathfrak{J}) + \Gamma_b$ and $|\Im(\mathfrak{J})|^2 = \frac{a^2(\sin\theta)^2}{|q|^2}$, we obtain

$$\begin{aligned}
\widetilde{\mathrm{div}}\, f &= \frac{2a^2(\cos\theta)^2}{r^3} - \frac{a^2(\sin\theta)^2}{|q|^2} + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b,\\
\widetilde{\mathrm{div}}\, \underline{f} &= -\frac{2a^2(\cos\theta)^2}{r^3} + \frac{a^2(\sin\theta)^2}{|q|^2} + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b.
\end{aligned}$$

Plugging in

$$\begin{aligned}
\lambda^{-1}\widetilde{\mathrm{tr}\,\chi} &= \frac{2}{r} - \frac{2a^2(\cos\theta)^2}{r^3} - \frac{3a^2(\sin\theta)^2}{2r^3} + \widetilde{\mathrm{div}}\, f + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b,\\
\lambda\widetilde{\mathrm{tr}\,\underline{\chi}} &= -\frac{2\Upsilon}{r} - \frac{2a^2}{r^3} + \frac{4a^2(\cos\theta)^2}{r^3} + \frac{a^2(\sin\theta)^2}{2r^3} + \widetilde{\mathrm{div}}\,\underline{f} + O(r^{-4})\\
&\quad + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b,
\end{aligned}$$

we infer

$$\begin{aligned}
\lambda^{-1}\widetilde{\mathrm{tr}\,\chi} &= \frac{2}{r} - \frac{5a^2(\sin\theta)^2}{2r^3} + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b,\\
\lambda\widetilde{\mathrm{tr}\,\underline{\chi}} &= -\frac{2\Upsilon}{r} - \frac{2a^2}{r^3} + \frac{2a^2(\cos\theta)^2}{r^3} + \frac{3a^2(\sin\theta)^2}{2r^3} + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b.
\end{aligned}$$

As

$$\lambda = 1 + \frac{3a^2(\sin\theta)^2}{4r^2} + O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b,$$

we deduce

$$\widetilde{\mathrm{tr}\,\chi} = \frac{2}{r} - \frac{a^2(\sin\theta)^2}{r^3} + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b,$$

$$\widetilde{\mathrm{tr}\,\underline{\chi}} = -\frac{2\Upsilon}{r} + \frac{a^2(\sin\theta)^2}{r^3} + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b,$$

which is the precise form that will be used in Step 4 and Step 5.

Step 3. Next, we derive a first, non-sharp, asymptotic for $r - \widetilde{r}$. Given coordinates[164] (x^1, x^2) on the spheres $S(u, r)$ with $e_4(x^A) = 0$, we consider the coordinates $(\widetilde{x}^1, \widetilde{x}^2)$ on $S(\widetilde{u}, \widetilde{s})$ given by

$$\widetilde{x}^1 = x^1, \qquad \widetilde{x}^2 = x^2.$$

We introduce the vectorfields

$$\widetilde{X}_A = \partial_{x^A} + \frac{1}{2}\mathbf{g}(\widetilde{e}_4, \partial_{x^A})\widetilde{e}_3 + \frac{1}{2}\mathbf{g}(\widetilde{e}_3, \partial_{x^A})\widetilde{e}_4,$$

which are tangent to the spheres $S(\widetilde{u}, \widetilde{s})$. We have

$$\widetilde{X}_A(\widetilde{x}^B) = \delta_A^B + \frac{1}{2}\mathbf{g}(\widetilde{e}_4, \partial_{x^A})\widetilde{e}_3(\widetilde{x}^B) + \frac{1}{2}\mathbf{g}(\widetilde{e}_3, \partial_{x^A})\widetilde{e}_4(\widetilde{x}^B).$$

Since $\widetilde{x}^A = x^A$, $e_4(x^A) = 0$, $e_3(x^A) = O(r^{-2}) + \Gamma_b$, $e_B(x^A) = O(r^{-1})$, as well as[165] $\mathbf{g}(e_3, \partial_{x^a}) = O(1)$ and $\mathbf{g}(e_4, \partial_{x^a}) = O(1)$, and since

$$\begin{aligned}
\widetilde{e}_4 &= \lambda\left(e_4 + f^c e_c + \frac{1}{4}|f|^2 e_3\right) = \lambda\Big(e_4 + f^c e_c + O(r^{-2})e_3\Big),\\
\widetilde{e}_3 &= \lambda^{-1}\left(\left(1 + \frac{1}{2}f\cdot\underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right)e_3 + \left(\underline{f}^b + \frac{1}{4}|\underline{f}|^2 f^b\right)e_b + \frac{1}{4}|\underline{f}|^2 e_4\right)\\
&= \lambda^{-1}\Big(\Big(1 + O(r^{-2})\Big)e_3 + \Big(\underline{f}^b + O(r^{-3})\Big)e_b + O(r^{-2})e_4\Big),
\end{aligned}$$

we infer

$$\begin{aligned}
\widetilde{X}_A(\widetilde{x}^B) &= \delta_A^B + \frac{1}{2}\mathbf{g}(\widetilde{e}_4, \partial_{x^A})\widetilde{e}_3(\widetilde{x}^B) + \frac{1}{2}\mathbf{g}(\widetilde{e}_3, \partial_{x^A})\widetilde{e}_4(\widetilde{x}^B)\\
&= \delta_A^B + \frac{1}{2}\mathbf{g}\Big(e_4 + f^c e_c + O(r^{-2})e_3, \partial_{x^A}\Big)\\
&\quad\times\Big(\Big(1 + O(r^{-2})\Big)e_3 + \Big(\underline{f}^b + O(r^{-3})\Big)e_b\Big)(x^B)
\end{aligned}$$

[164]In practice, as in Sections 2.4.4 and 2.4.7, we cover the spheres $S(u, r)$ with the coordinates systems $(x^1, x^2) = (\theta, \varphi)$ and $(x^1, x^2) = (J^{(+)}, J^{(-)})$.

[165]Note that $\partial_{x^A} = Y_A^B e_B + z_A^3 e_3 + z_A^4 e_4$, with $Y_A^B = O(r)$ and $z_A^3 e_3(u) = -Y_A^B e_B(u)$, $z_A^4 = -z_A^3 e_3(r)$, so that we have z_A^3, $z_A^4 = O(1)$, and hence $\mathbf{g}(e_3, \partial_{x^A}) = -2z_A^4 = O(1)$ and $\mathbf{g}(e_4, \partial_{x^A}) = -2z_A^3 = O(1)$.

$$
\begin{aligned}
& +\frac{1}{2}\mathbf{g}\Big(\Big(1+O(r^{-2})\Big)e_3+\Big(\underline{f}^b+O(r^{-3})\Big)e_b+O(r^{-2})e_4,\partial_{x^A}\Big)\\
& \times\Big(f^c e_c+O(r^{-2})e_3\Big)(x^B)\\
= \;& \delta_A^B+O(r^{-2})+\Gamma_b.
\end{aligned}
$$

Also,

$$
\begin{aligned}
& \mathbf{g}(\widetilde{X}_A,\widetilde{X}_B)\\
= \;& \mathbf{g}\Bigg(\partial_{x^A}+\frac{1}{2}\mathbf{g}(\widetilde{e}_4,\partial_{x^A})\widetilde{e}_3+\frac{1}{2}\mathbf{g}(\widetilde{e}_3,\partial_{x^A})\widetilde{e}_4,\\
& \partial_{x^B}+\frac{1}{2}\mathbf{g}(\widetilde{e}_4,\partial_{x^B})\widetilde{e}_3+\frac{1}{2}\mathbf{g}(\widetilde{e}_3,\partial_{x^B})\widetilde{e}_4\Bigg)\\
= \;& \mathbf{g}(\partial_{x^A},\partial_{x^B})+\frac{1}{2}\mathbf{g}(\widetilde{e}_4,\partial_{x^A})\mathbf{g}(\widetilde{e}_3,\partial_{x^B})+\frac{1}{2}\mathbf{g}(\widetilde{e}_4,\partial_{x^B})\mathbf{g}(\widetilde{e}_3,\partial_{x^A})\\
= \;& g_{AB}+\frac{1}{2}\mathbf{g}\Big(e_4+f^c e_c+O(r^{-2})e_3,\partial_{x^A}\Big)\\
& \times\mathbf{g}\Big(\Big(1+O(r^{-2})\Big)e_3+\Big(\underline{f}^b+O(r^{-3})\Big)e_b+O(r^{-2})e_4,\partial_{x^B}\Big)\\
& +\frac{1}{2}\mathbf{g}\Big(e_4+f^c e_c+O(r^{-2})e_3,\partial_{x^B}\Big)\\
& \times\mathbf{g}\Big(\Big(1+O(r^{-2})\Big)e_3+\Big(\underline{f}^b+O(r^{-3})\Big)e_b+O(r^{-2})e_4,\partial_{x^A}\Big)\\
= \;& g_{AB}+O(1),
\end{aligned}
$$

where g_{AB} denotes the induced metric on $S(u,r)$. Denoting the induced metric on $S(\widetilde{u},\widetilde{s})$ by $\widetilde{g}_{AB}$, we easily infer from the above computation of $\widetilde{X}_A(\widetilde{x}^B)$ and $\mathbf{g}(\widetilde{X}_A,\widetilde{X}_B)$, and from the fact that the vectorfields $\widetilde{X}_1$ and $\widetilde{X}_2$ are tangent to the spheres $S(\widetilde{u},\widetilde{s})$, that

$$
\widetilde{g}_{AB} \quad = \quad g_{AB}+O(1)+r^2\Gamma_b.
$$

In particular, we infer

$$
\begin{aligned}
|S(\widetilde{u},\widetilde{s})| \;&=\; \int\sqrt{|\widetilde{g}_{AB}|}d\widetilde{x}^A d\widetilde{x}^B\\
&=\; \int\sqrt{|g_{AB}|}dx^A dx^B+O(1)+r^2\Gamma_b\\
&=\; |S(u,r)|+O(1)+r^2\Gamma_b.
\end{aligned}
$$

Together with the bound $|S(u,r)| = 4\pi r^2 + O(1)$ which is a non-sharp consequence of Lemma A.2, we deduce

$$|S(\widetilde{u},\widetilde{s})| \;\; = \;\; 4\pi r^2 + O(1) + r^2\Gamma_b,$$

and hence, since $|S(\widetilde{u},\widetilde{s})| = 4\pi(\widetilde{r})^2$, we infer the following non-sharp bound for $\widetilde{r} - r$

$$\widetilde{r} \;\; = \;\; r + O(r^{-1}) + r\Gamma_b.$$

Step 4. We now improve the bound for $\widetilde{r} - r$ of Step 3. Recall that, as $(\widetilde{u},\widetilde{s})$ is an outgoing geodesic foliation, we have

$$\widetilde{e}_4\left(\widetilde{\operatorname{tr}\chi} - \frac{2}{\widetilde{r}}\right) \;\; = \;\; -\frac{1}{2}(\widetilde{\operatorname{tr}\chi})^2 - |\widehat{\widetilde{\chi}}|^2 + \frac{2}{(\widetilde{r})^2}\frac{\widetilde{r}}{2}\overline{\widetilde{\operatorname{tr}\chi}}^{S(\widetilde{u},\widetilde{s})}.$$

In view of the above decomposition of $\widehat{\widetilde{\chi}}$, we infer

$$\widetilde{e}_4\left(\widetilde{\operatorname{tr}\chi} - \frac{2}{\widetilde{r}}\right) \;\; = \;\; -\frac{1}{2}(\widetilde{\operatorname{tr}\chi})^2 + \frac{1}{\widetilde{r}}\overline{\widetilde{\operatorname{tr}\chi}}^{S(\widetilde{u},\widetilde{s})} + O(r^{-6}) + r^{-4}\mathfrak{d}^{\leq 2}\Gamma_b.$$

We infer

$$\overline{\widetilde{e}_4\left(\widetilde{\operatorname{tr}\chi} - \frac{2}{\widetilde{r}}\right)}^{S(\widetilde{u},\widetilde{s})} \;\; = \;\; -\frac{1}{2}\overline{(\widetilde{\operatorname{tr}\chi})^2}^{S(\widetilde{u},\widetilde{s})} + \frac{1}{\widetilde{r}}\overline{\widetilde{\operatorname{tr}\chi}}^{S(\widetilde{u},\widetilde{s})} + O(r^{-6}) + r^{-4}\mathfrak{d}^{\leq 2}\Gamma_b.$$

Since, for any scalar function h, we have

$$\begin{aligned}
\widetilde{e}_4\left(\overline{h}^{S(\widetilde{u},\widetilde{s})}\right) \;\; &= \;\; \frac{1}{|S(\widetilde{u},\widetilde{s})|}\int_{S(\widetilde{u},\widetilde{s})}\left(\widetilde{e}_4(h) + \widetilde{\operatorname{tr}\chi}h\right) - \frac{2\widetilde{e}_4(\widetilde{r})}{\widetilde{r}}\overline{h}^{S(\widetilde{u},\widetilde{s})} \\
&= \;\; \overline{e_4(h)}^{S(\widetilde{u},\widetilde{s})} + \overline{\widetilde{\operatorname{tr}\chi}h}^{S(\widetilde{u},\widetilde{s})} - \overline{\widetilde{\operatorname{tr}\chi}}^{S(\widetilde{u},\widetilde{s})}\overline{h}^{S(\widetilde{u},\widetilde{s})},
\end{aligned}$$

we infer

$$\begin{aligned}
\widetilde{e}_4\left(\overline{\widetilde{\operatorname{tr}\chi}}^{S(\widetilde{u},\widetilde{s})} - \frac{2}{\widetilde{r}}\right) \;\; &= \;\; -\frac{1}{2}\overline{(\widetilde{\operatorname{tr}\chi})^2}^{S(\widetilde{u},\widetilde{s})} + \frac{1}{\widetilde{r}}\overline{\widetilde{\operatorname{tr}\chi}}^{S(\widetilde{u},\widetilde{s})} + O(r^{-6}) + r^{-4}\mathfrak{d}^{\leq 2}\Gamma_b \\
&\quad + \overline{\widetilde{\operatorname{tr}\chi}\left(\widetilde{\operatorname{tr}\chi} - \frac{2}{\widetilde{r}}\right)}^{S(\widetilde{u},\widetilde{s})} - \overline{\widetilde{\operatorname{tr}\chi}}^{S(\widetilde{u},\widetilde{s})}\overline{\left(\widetilde{\operatorname{tr}\chi} - \frac{2}{\widetilde{r}}\right)}^{S(\widetilde{u},\widetilde{s})} \\
&= \;\; -\frac{1}{\widetilde{r}}\left(\overline{\widetilde{\operatorname{tr}\chi}}^{S(\widetilde{u},\widetilde{s})} - \frac{2}{\widetilde{r}}\right) + \frac{1}{2}\overline{\left(\widetilde{\operatorname{tr}\chi} - \frac{2}{\widetilde{r}}\right)^2}^{S(\widetilde{u},\widetilde{s})}
\end{aligned}$$

$$-\left(\overline{\widetilde{\mathrm{tr}\,\chi}}^{S(\widetilde{u},\widetilde{s})}-\frac{2}{\widetilde{r}}\right)^2+O(r^{-6})+r^{-4}\mathfrak{d}^{\leq 2}\Gamma_b$$

and hence

$$\begin{aligned}\widetilde{e}_4\left(\overline{\widetilde{r}\widetilde{\mathrm{tr}\,\chi}}^{S(\widetilde{u},\widetilde{s})}-2\right) &= \frac{1}{2\widetilde{r}}\overline{\left(\widetilde{r}\widetilde{\mathrm{tr}\,\chi}-2\right)^2}^{S(\widetilde{u},\widetilde{s})}-\frac{1}{2\widetilde{r}}\left(\overline{\widetilde{r}\widetilde{\mathrm{tr}\,\chi}-2}^{S(\widetilde{u},\widetilde{s})}\right)^2\\ &\quad +O(r^{-5})+r^{-3}\mathfrak{d}^{\leq 2}\Gamma_b.\end{aligned}$$

Now, we have, in view of the decomposition of Step 3 for $\widetilde{r}-r$, and in view of the decomposition of Step 1 for $\widetilde{\mathrm{tr}\,\chi}$,

$$\widetilde{r}\widetilde{\mathrm{tr}\,\chi}-2 = O(r^{-1})(\widetilde{r}-r)+r\widetilde{\mathrm{tr}\,\chi}-2=O(r^{-2})+\mathfrak{d}^{\leq 2}\Gamma_b$$

and hence

$$\widetilde{e}_4\left(\overline{\widetilde{r}\widetilde{\mathrm{tr}\,\chi}}^{S(\widetilde{u},\widetilde{s})}-2\right)=O(r^{-5})+r^{-2}\mathfrak{d}^{\leq 2}\Gamma_b, \qquad \lim_{r\to+\infty}\left(\overline{\widetilde{r}\widetilde{\mathrm{tr}\,\chi}}^{S(\widetilde{u},\widetilde{s})}-2\right)=0.$$

Integrating backwards from infinity, we infer

$$\overline{\widetilde{r}\widetilde{\mathrm{tr}\,\chi}}^{S(\widetilde{u},\widetilde{s})}-2 = O(r^{-4})+r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b.$$

Plugging the precise asymptotic for $\widetilde{\mathrm{tr}\,\chi}$ of Step 2, we infer

$$\overline{\widetilde{r}\left(\frac{2}{r}-\frac{a^2(\sin\theta)^2}{r^3}\right)}^{S(\widetilde{u},\widetilde{s})}-2 = O(r^{-3})+\mathfrak{d}^{\leq 2}\Gamma_b$$

which we rewrite

$$\overline{\left(\frac{\widetilde{r}}{r+\frac{a^2(\sin\theta)^2}{2r}}\right)}^{S(\widetilde{u},\widetilde{s})}-1 = O(r^{-3})+\mathfrak{d}^{\leq 2}\Gamma_b.$$

On the other hand, we have

$$\begin{aligned}&\widetilde{\nabla}\left(r+\frac{a^2(\sin\theta)^2}{2r}\right)\\ &= \left(\left(\delta_a^b+\frac{1}{2}\underline{f}_a f^b\right)e_b+\frac{1}{2}\underline{f}_a e_4+\left(\frac{1}{2}f_a+\frac{1}{8}|f|^2\underline{f}_a\right)e_3\right)\left(r+\frac{a^2(\sin\theta)^2}{2r}\right)\end{aligned}$$

$$= \frac{1}{2}\underline{f}_a + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right)e_3(r) + e_a\left(\frac{a^2(\sin\theta)^2}{2r}\right) + O(r^{-3}).$$

In view of the form of f, $\underline{f}$, and the control of $e_3(r)$, we infer

$$\begin{aligned}\widetilde{\nabla}\left(r + \frac{a^2(\sin\theta)^2}{2r}\right) &= \frac{1}{2}\left(-a\Upsilon\Re(\mathfrak{J}) - \frac{a^2\cos\theta}{r}\Im(\mathfrak{J})\right)\\ &\quad + \frac{1}{2}\left(-a\Re(\mathfrak{J}) + \frac{a^2\cos\theta}{r}\Im(\mathfrak{J})\right)(-\Upsilon)\\ &\quad - \frac{a^2\cos\theta}{r}\nabla(\cos\theta) + O(r^{-3}) + \mathfrak{d}^{\leq 2}\Gamma_b\\ &= -\frac{a^2\cos\theta}{r}\Big(\nabla(\cos\theta) + \Im(\mathfrak{J})\Big) + O(r^{-3}) + \mathfrak{d}^{\leq 2}\Gamma_b.\end{aligned}$$

Since $\nabla(\cos\theta) = -\Im(\mathfrak{J}) + \Gamma_b$, we obtain

$$\widetilde{\nabla}\left(r + \frac{a^2(\sin\theta)^2}{2r}\right) = O(r^{-3}) + \mathfrak{d}^{\leq 2}\Gamma_b.$$

Together with

$$\overline{\left(\frac{\widetilde{r}}{r + \frac{a^2(\sin\theta)^2}{2r}}\right)}^{S(\widetilde{u},\widetilde{s})} - 1 = O(r^{-3}) + \mathfrak{d}^{\leq 2}\Gamma_b,$$

we infer

$$\widetilde{r} = r + \frac{a^2(\sin\theta)^2}{2r} + O(r^{-2}) + r\mathfrak{d}^{\leq 2}\Gamma_b$$

which is the stated control of $\widetilde{r}$.

Step 5. Next, recall the precise asymptotic for $\widetilde{\mathrm{tr}\,\chi}$ and $\widetilde{\mathrm{tr}\,\underline{\chi}}$ derived in Step 2

$$\begin{aligned}\widetilde{\mathrm{tr}\,\chi} &= \frac{2}{r} - \frac{a^2(\sin\theta)^2}{r^3} + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b,\\ \widetilde{\mathrm{tr}\,\underline{\chi}} &= -\frac{2\Upsilon}{r} + \frac{a^2(\sin\theta)^2}{r^3} + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b.\end{aligned}$$

In view of the control for $\widetilde{r}-r$ of Step 4, we deduce

$$\widetilde{\operatorname{tr}\chi} = \frac{2}{\widetilde{r}} + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b,$$
$$\widetilde{\operatorname{tr}\underline{\chi}} = -\frac{2\widetilde{\Upsilon}}{\widetilde{r}} + O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b.$$

In view of the definition of $\widetilde{\widetilde{\operatorname{tr}\chi}}$ and $\widetilde{\widetilde{\operatorname{tr}\underline{\chi}}}$, we infer

$$\widetilde{\widetilde{\operatorname{tr}\chi}} = O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b,$$
$$\widetilde{\widetilde{\operatorname{tr}\underline{\chi}}} = O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b,$$

as stated.

Step 6. Next, we control $\widetilde{J}^{(p)} - J^{(p)}$ for $p = 0, +, -$. Since $\widetilde{e}_4(\widetilde{J}^{(p)}) = 0$, we have

$$\begin{aligned} \widetilde{e}_4(\widetilde{J}^{(p)} - J^{(p)}) &= -\widetilde{e}_4(J^{(0)}) = -\lambda\left(e_4 + f\cdot\nabla + \frac{1}{4}|f|^2 e_3\right)J^{(p)} \\ &= -\lambda\left(f\cdot\nabla(J^{(p)}) + \frac{1}{4}|f|^2 e_3(J^{(p)})\right). \end{aligned}$$

Using the control of λ and f, we infer

$$\widetilde{e}_4(\widetilde{J}^{(p)} - J^{(p)}) = O(r^{-2}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b.$$

Integrating backwards from $r = +\infty$ where $\widetilde{J}^{(p)} = J^{(p)}$, we infer

$$\widetilde{J}^{(p)} - J^{(p)} = O(r^{-1}) + \mathfrak{d}^{\leq 1}\Gamma_b, \qquad p = 0, +, -,$$

as desired.

Step 7. Finally, we derive precise asymptotic for $\widetilde{\operatorname{div}}\,\widetilde{\beta}$ and $\widetilde{\operatorname{curl}}\,\widetilde{\beta}$. Recall that we have obtained in Step 1

$$\widetilde{\beta} = \beta + \frac{3am}{r^3}\Re(\mathfrak{J}) + O(r^{-5}) + r^{-3}\mathfrak{d}^{\leq 2}\Gamma_b.$$

Since $\beta \in r^{-2}\Gamma_b$, we infer

$$\widetilde{\beta} = \frac{3am}{r^3}\Re(\mathfrak{J}) + O(r^{-5}) + r^{-2}\mathfrak{d}^{\leq 2}\Gamma_b$$

and hence

$$\begin{aligned}\widetilde{\operatorname{div}}\,\widetilde{\beta} &= \widetilde{\operatorname{div}}\left(\frac{3am}{r^3}\Re(\mathfrak{J})\right) + O(r^{-6}) + r^{-3}\mathfrak{d}^{\leq 3}\Gamma_b,\\ \widetilde{\operatorname{curl}}\,\widetilde{\beta} &= \widetilde{\operatorname{curl}}\left(\frac{3am}{r^3}\Re(\mathfrak{J})\right) + O(r^{-6}) + r^{-3}\mathfrak{d}^{\leq 3}\Gamma_b.\end{aligned}$$

One easily infers

$$\begin{aligned}\widetilde{\operatorname{div}}\,\widetilde{\beta} &= \frac{3am}{r^3}\operatorname{div}\,(\Re(\mathfrak{J})) + O(r^{-6}) + r^{-3}\mathfrak{d}^{\leq 3}\Gamma_b,\\ \widetilde{\operatorname{curl}}\,\widetilde{\beta} &= \frac{3am}{r^3}\operatorname{curl}\,(\Re(\mathfrak{J})) + O(r^{-6}) + r^{-3}\mathfrak{d}^{\leq 3}\Gamma_b.\end{aligned}$$

Next, we use

$$\overline{\mathcal{D}}\cdot\mathfrak{J} = \frac{4i(r^2+a^2)\cos\theta}{|q|^4} + r^{-1}\Gamma_b$$

which yields

$$\operatorname{div}\,(\Re(\mathfrak{J})) = r^{-1}\Gamma_b, \qquad \operatorname{curl}\,(\Re(\mathfrak{J})) = \frac{2\cos\theta}{r^2} + O(r^{-4}) + r^{-1}\Gamma_b.$$

We infer, together with the control for $\widetilde{r} - r$,

$$\begin{aligned}\widetilde{\operatorname{div}}\,\widetilde{\beta} &= O(r^{-6}) + r^{-3}\mathfrak{d}^{\leq 3}\Gamma_b,\\ \widetilde{\operatorname{curl}}\,\widetilde{\beta} &= \frac{6am\cos\theta}{\widetilde{r}^5} + O(r^{-6}) + r^{-3}\mathfrak{d}^{\leq 3}\Gamma_b.\end{aligned}$$

This is the desired control for $\operatorname{div}\widetilde{\beta}$. For $\widetilde{\operatorname{curl}}\,\widetilde{\beta}$, we use in addition the following control derived in Step 6

$$\widetilde{J}^{(0)} - J^{(0)} = O(r^{-1}) + \mathfrak{d}^{\leq 1}\Gamma_b.$$

Plugging in the above, and since $J^{(0)} = \cos\theta$, we infer

$$\widetilde{\operatorname{curl}}\,\widetilde{\beta} = \frac{6am\widetilde{J}^{(0)}}{\widetilde{r}^5} + O(r^{-6}) + r^{-3}\mathfrak{d}^{\leq 3}\Gamma_b$$

as desired. This ends the proof of Lemma 8.19. □

8.2.4. Control of the geodesic foliation of $^{(ext)}\mathcal{L}_0$ We recall in what follows that the frame $\widetilde{E}$, the scalar functions $(\widetilde{u},\widetilde{s})$ and the scalars $\widetilde{J}^{(p)}$, $p=0,+,-$, correspond to the outgoing geodesic foliation of $^{(ext)}\widetilde{\mathcal{L}_0}$, while E, r, u, $J^{(p)}$ and $\mathfrak{J}$ correspond to the outgoing PG structure of $^{(ext)}\mathcal{L}_0$. We also recall that $(f,\underline{f},\lambda)$ denote the transition coefficients from the frame E to $\widetilde{E}$. In addition, we define

$$f_0 := r\Re(\mathfrak{J}). \tag{8.76}$$

We define the following norms on $^{(ext)}\widetilde{\mathcal{L}_0}$.

$$^{(ext)}\widetilde{\mathfrak{I}}_k \quad := \quad ^{(ext)}\widetilde{\mathfrak{I}}'_k + {}^{(ext)}\widetilde{\mathfrak{I}}''_{k+1}$$

where, for $k=0$,

$$\begin{aligned}
^{(ext)}\widetilde{\mathfrak{I}}'_0 \quad := \quad & \sup_{^{(ext)}\widetilde{\mathcal{L}_0}} \Bigg[r^{\frac{7}{2}+\delta_B}\left(|\widetilde{\alpha}| + \left|\widetilde{\beta} - \frac{3a_0m_0}{\widetilde{r}^4}f_0\right|\right) + r^3\,\left|\widetilde{\rho} + \frac{2m_0}{\widetilde{r}^3}\right| + r^2|\widetilde{\underline{\beta}}| \\
& \qquad\qquad + r|\widetilde{\underline{\alpha}}|\Bigg] \\
& + \sup_{^{(ext)}\widetilde{\mathcal{L}_0}} r^{\frac{9}{2}+\delta_B}\left(|\widetilde{\mathrm{div}\,\beta}| + \left|\widetilde{\mathrm{curl}}\,\widetilde{\beta} - \frac{6a_0m_0}{\widetilde{r}^5}\widetilde{J}^{(0)}\right|\right) \\
& + \sup_{^{(ext)}\widetilde{\mathcal{L}_0}} r^2\left(|\widetilde{\widehat{\chi}}| + \left|\widetilde{\mathrm{tr}\,\chi} - \frac{2}{\widetilde{r}}\right| + |\widetilde{\zeta}| + \left|\widetilde{\mathrm{tr}\,\underline{\chi}} + \frac{2\left(1-\frac{2m_0}{\widetilde{r}}\right)}{\widetilde{r}}\right|\right) \\
& + \sup_{^{(ext)}\widetilde{\mathcal{L}_0}} r\Bigg(|\widetilde{\underline{\widehat{\chi}}}| + \left|\widetilde{\underline{\omega}} - \frac{m_0}{\widetilde{r}^2}\right| + |\widetilde{\underline{\xi}}| + \left|\widetilde{\nabla}\widetilde{J}^{(0)} + \frac{1}{\widetilde{r}}\,{}^*f_0\right| \\
& \qquad\qquad + \left|\widetilde{\nabla}f_0 - \frac{\widetilde{J}^{(0)}}{\widetilde{r}}\in\right|\Bigg) \\
& + \sup_{^{(ext)}\widetilde{\mathcal{L}_0}}\left(\left|\widetilde{e}_3(\widetilde{r}) + 1 - \frac{2m_0}{\widetilde{r}}\right| + |\widetilde{e}_3(\widetilde{u}) - 2| + |\widetilde{r} - r|\right), \\
^{(ext)}\widetilde{\mathfrak{I}}''_0 \quad := \quad & \sup_{^{(ext)}\widetilde{\mathcal{L}_0}} r\left(\left|f + \frac{a_0}{r}f_0\right| + \left|\underline{f} + \frac{a_0\left(1-\frac{2m_0}{r}\right)}{r}f_0\right| + |\log(\lambda)|\right) \\
& + \sup_{^{(ext)}\widetilde{\mathcal{L}_0}}\max_{p=0,+,-}\left|\widetilde{J}^{(p)} - J^{(p)}\right| + \sup_{^{(ext)}\widetilde{\mathcal{L}_0}} r\left|\mathfrak{J} - \frac{1}{|q|}(f_0 + {}^*f_0)\right|.
\end{aligned}$$

The higher derivative norms $^{(ext)}\widetilde{\mathfrak{I}}'$, $^{(ext)}\widetilde{\mathfrak{I}}''$ are then defined by replacing each component with $\mathfrak{d}^{\leq k}$ of it.

The following proposition provides the control of the norm ${}^{(ext)}\widetilde{\mathfrak{I}}_k$ under the assumptions (8.59), i.e.

$$ {}^{(ext)}\mathfrak{I}_{k_{large}+10} \le \epsilon_0, \qquad {}^{(ext)}\mathfrak{I}_3 \le \epsilon_0^2. $$

Proposition 8.20. *The following holds true:*

1. *Under the assumption* ${}^{(ext)}\mathfrak{I}_{k_{large}+10} \le \epsilon_0$, *we have*

$$ {}^{(ext)}\widetilde{\mathfrak{I}}_{k_{large}+7} \lesssim \epsilon_0. \tag{8.77} $$

2. *If in addition the assumption* ${}^{(ext)}\mathfrak{I}_3 \le \epsilon_0^2$ *also holds true, then*

$$ \begin{aligned} &\sup_{{}^{(ext)}\widetilde{\mathcal{L}}_0 \cap \{\widetilde{r}\sim\epsilon_0^{-1}\}} r^5 \left(\left|\widetilde{div}\,\widetilde{\beta}\right| + \left|\widetilde{curl}\,\widetilde{\beta} - \frac{6a_0 m_0}{\widetilde{r}^5}\widetilde{J}^{(0)}\right| \right) \\ &+ \sup_{{}^{(ext)}\widetilde{\mathcal{L}}_0 \cap \{\widetilde{r}\sim\epsilon_0^{-1}\}} r^3 \left(\left|\widetilde{\widetilde{tr\chi}}\right| + \left|\widetilde{\widetilde{tr\underline{\chi}}}\right| \right) \lesssim \epsilon_0. \end{aligned} \tag{8.78} $$

Remark 8.21. *According to (3.52), we have the stronger bound* $\mathfrak{I}_{k_{large}+10} \le \epsilon_0^2$. *As it turns out, we only need the weaker bounds* $\mathfrak{I}_{k_{large}+10} \le \epsilon_0$ *and* ${}^{(ext)}\mathfrak{I}_3 \le \epsilon_0^2$ *as emphasized in Remark 3.24. In particular,* ${}^{(ext)}\mathfrak{I}_3 \le \epsilon_0^2$ *is only used to ensure (8.78) which will be used in the proof of Theorem M0 and Theorem M6.*

Proof. We start with the first estimate. In view of the definition of ${}^{(ext)}\widetilde{\mathfrak{I}}_k$, Proposition 8.17 and Lemma 8.19 imply

$$ \begin{aligned} {}^{(ext)}\widetilde{\mathfrak{I}}_k \lesssim\; & \sup_{{}^{(ext)}\widetilde{\mathcal{L}}_0} r \left(\left|\mathfrak{d}^{\le k}\left(\nabla J^{(0)} + \frac{1}{r}\,{}^*f_0\right)\right| + \left|\mathfrak{d}^{\le k}\left(\nabla f_0 - \frac{J^{(0)}}{r}\in\right)\right| \right) \\ & + \sup_{{}^{(ext)}\widetilde{\mathcal{L}}_0} \left(\left|\mathfrak{d}^{\le k+1}\left(f_0 - r\Re(\mathfrak{J})\right)\right| + r^{-1} + r|\mathfrak{d}^{\le k+3}\Gamma_b| \right). \end{aligned} $$

Together with $f_0 = r\Re(\mathfrak{J})$, $r|\mathfrak{d}^{\le k+3}\Gamma_b| \lesssim \mathfrak{I}_{k+3}$, and ${}^{(ext)}\mathfrak{I}_{k_{large}+10} \le \epsilon_0$, we infer, for $k \le k_{large}+7$,

$$ \begin{aligned} {}^{(ext)}\widetilde{\mathfrak{I}}_k \lesssim\; & \sup_{{}^{(ext)}\widetilde{\mathcal{L}}_0} r \left(\left|\mathfrak{d}^{\le k}\left(\nabla J^{(0)} + {}^*\Re(\mathfrak{J})\right)\right| + \left|\mathfrak{d}^{\le k}\left(r\nabla\Re(\mathfrak{J}) - \frac{J^{(0)}}{r}\in\right)\right| \right) \\ & +\epsilon_0. \end{aligned} $$

where we also used the fact that $r \gtrsim \epsilon_0^{-1}$ on ${}^{(ext)}\widetilde{\mathcal{L}}_0$, since by definition $\widetilde{s} \gtrsim \epsilon_0^{-1}$ on ${}^{(ext)}\widetilde{\mathcal{L}}_0$, and since $\widetilde{s} \sim \widetilde{r} \sim r$ in view of Proposition 8.17 and Lemma 8.19.

Since we have, in view of Definitions 2.66 and 2.67,

$$\nabla J^{(0)} = - \,{}^*\Re(\mathfrak{J}) + \Gamma_b, \qquad \nabla \Re(\mathfrak{J}) = \frac{J^{(0)}}{r^2} \in +O(r^{-4}) + r^{-1}\Gamma_b,$$

we deduce

$${}^{(ext)}\widetilde{\mathfrak{J}}_{k_{large}+7} \lesssim \epsilon_0$$

which is the first stated estimate.

For the second estimate, we rely on the following identities in Lemma 8.19

$$\widetilde{\widetilde{\mathrm{tr}\,\chi}} = O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b, \qquad \widetilde{\widetilde{\mathrm{tr}\,\underline{\chi}}} = O(r^{-4}) + r^{-1}\mathfrak{d}^{\leq 3}\Gamma_b,$$
$$\widetilde{\mathrm{div}}\,\widetilde{\beta} = O(r^{-6}) + r^{-3}\mathfrak{d}^{\leq 3}\Gamma_b, \quad \widetilde{\mathrm{curl}}\,\widetilde{\beta} = \frac{6a_0m_0}{\widetilde{r}^5}\widetilde{J}^{(0)} + O(r^{-6}) + r^{-3}\mathfrak{d}^{\leq 3}\Gamma_b,$$

which yields

$$r^5\left(\left|\widetilde{\mathrm{div}}\,\widetilde{\beta}\right| + \left|\widetilde{\mathrm{curl}}\,\widetilde{\beta} - \frac{6a_0m_0}{\widetilde{r}^5}\widetilde{J}^{(0)}\right|\right) + r^3\left(\left|\widetilde{\widetilde{\mathrm{tr}\,\chi}}\right| + \left|\widetilde{\widetilde{\mathrm{tr}\,\underline{\chi}}}\right|\right) \lesssim r^{-1} + r^2|\mathfrak{d}^{\leq 3}\Gamma_b|.$$

We infer

$$\sup_{{}^{(ext)}\widetilde{\mathcal{L}}_0\cap\{\widetilde{r}\sim\epsilon_0^{-1}\}} r^5\left(\left|\widetilde{\mathrm{div}}\,\widetilde{\beta}\right| + \left|\widetilde{\mathrm{curl}}\,\widetilde{\beta} - \frac{6a_0m_0}{\widetilde{r}^5}\widetilde{J}^{(0)}\right|\right) + \sup_{{}^{(ext)}\widetilde{\mathcal{L}}_0\cap\{\widetilde{r}\sim\epsilon_0^{-1}\}} r^3\left(\left|\widetilde{\widetilde{\mathrm{tr}\,\chi}}\right| + \left|\widetilde{\widetilde{\mathrm{tr}\,\underline{\chi}}}\right|\right) \lesssim \epsilon_0 + \epsilon_0^{-1}\,{}^{(ext)}\mathfrak{J}_3$$

and the second stated estimate follows from the assumption ${}^{(ext)}\mathfrak{J}_3 \leq \epsilon_0^2$. This concludes the proof of the proposition. □

8.3. Proof of Theorem M0

We recall the reader, see also Remark 3.41, that Theorem M0 comes first in the sequence of steps, Theorems M0–M8, and therefore its proof can only rely on the assumption (3.52) of our main theorem[166]

$$\mathfrak{J}_{k_{large}+10} \leq \epsilon_0, \qquad {}^{(ext)}\mathfrak{J}_3 \leq \epsilon_0^2,$$

[166] In fact, under (3.52), we have the stronger bound $\mathfrak{J}_{k_{large}+10} \leq \epsilon_0^2$. As it turns out, we only need the weaker bounds $\mathfrak{J}_{k_{large}+10} \leq \epsilon_0$ and ${}^{(ext)}\mathfrak{J}_3 \leq \epsilon_0^2$ as emphasized in Remark 8.21.

as well as the fact that the space $\mathcal{M}$ which we consider is a GCM admissible spacetime verifying our bootstrap assumptions $\mathbf{BA}_\epsilon$ made in Section 3.5.

Notation. *Since we will be using various frames in the proof it is important to recall the main definitions. The PG structure of* $^{(ext)}\mathcal{M}$ *is denoted by the usual symbols* $\{E, r, u, \mathfrak{J}, J\}$ *where* $E = \{e_1, e_2, e_3, e_4\}$*. The PG quantities of the initial layer* $\mathcal{L}_0$ *are denoted with* 0 *indices, i.e.* $\{E_0, r_0, u_0, \mathfrak{J}, J_0\}$*. The quantities related to the outgoing geodesic frame of* $^{(ext)}\widetilde{\mathcal{L}_0}$ *are denoted by tildes, i.e.* $\{\widetilde{E}, \widetilde{s}, \widetilde{r}, \widetilde{u}, \widetilde{J}\}$*. We shall also make use of a second outgoing geodesic foliation in* $^{(ext)}\mathcal{L}_0$ *defined starting with the sphere* S_1*, of the PG structure of* $^{(ext)}\mathcal{M}$ *on* Σ_**, whose related quantities will be denoted by primes. At various stages of the proof, when only two foliations are needed, we will redefine notations accordingly.*

The proof of Theorem M0 proceeds in 24 steps which we summarize below for convenience:

1. In Steps 1–7, we propagate from S_* along Σ_* the $\ell = 1$ modes of $\operatorname{div}\beta$, $\operatorname{curl}\beta$, $\check{\rho}$ and $\check{\underline{\kappa}}$ to arrive at the estimate (8.79) on S_1. This sequence of steps makes use of the GCM assumptions on S_* and the results of Section 5.4.1.
2. In Steps 8–16, we derive the control of $m - m_0$, see (8.111) in Step 13, and $a - a_0$, see (8.118) in Step 15 and (8.125) in Step 16. We also provide estimates for the transition coefficients between the auxiliary outgoing geodesic frame of $^{(ext)}\widetilde{\mathcal{L}_0}$, introduced in Section 8.2, and the frame of Σ_* induced on the sphere $S_1 = \Sigma_* \cap \{u = 1\}$. In particular, we show that the sphere S_1 of Σ_* is contained in $^{(ext)}\widetilde{\mathcal{L}_0} \subset {}^{(ext)}\mathcal{L}_0$.
3. In Steps 17–19, we control the change of frame coefficients from the outgoing PG frame of $^{(ext)}\mathcal{L}_0$ to the outgoing PG frame of $^{(ext)}\mathcal{M}$ on the sphere S_1 in (8.135).
4. In steps 20–22, we propagate the control on the change of frame coefficients from the outgoing PG frame of $^{(ext)}\mathcal{L}_0$ to the outgoing PG frame of $^{(ext)}\mathcal{M}$ from S_1 to $\{u = 1\}$ in (8.143).
5. In step 23, we control on the change of frame coefficients from the ingoing PG frame of $^{(int)}\mathcal{L}_0$ to the ingoing PG frame of $^{(int)}\mathcal{M}$ on $\{\underline{u} = 1\}$ in (8.145).
6. Finally, we conclude the proof of Theorem M0 in step 24 by using the control of the change of frame coefficients, the control of the initial data layer, and the change of frame formulas to infer the control of the curvature components of $^{(ext)}\mathcal{M}$ on $\{u = 1\}$ and of $^{(int)}\mathcal{M}$ on $\{\underline{u} = 1\}$.

8.3.1. Steps 1–7 We state the main result of Step 1–7 in the following

Proposition 8.22. *The following estimates*[167] *are true on* Σ_*

$$\sup_{\Sigma_*}\Bigg(r^5|(div\,\beta)_{\ell=1}|+r^5|(curl\,\beta)_{\ell=1,\pm}| \tag{8.79}$$
$$+r^5\left|(curl\,\beta)_{\ell=1,0}-\frac{2am}{r^5}\right|\Bigg)\lesssim\epsilon_0,$$
$$\sup_{\Sigma_*}\Big(r^3|(\check{\rho})_{\ell=1}|+r^2|(\check{\underline{\kappa}})_{\ell=1}|\Big)\lesssim\epsilon_0,$$

where the quantities and the definition of the $\ell=1$ modes correspond to the frame of Σ_*.

Proof. The proof, which uses the GCM conditions on S_* and the results of Section 5.4.1 is outlined in Steps 1–7 below. □

Step 1. We start by recalling Lemma 5.37 on the control of the $\ell=1$ basis on Σ_*.

Lemma 8.23. *The functions $J^{(p)}$ verify the following properties*

1. *We have on* Σ_*

$$\int_S J^{(p)}=O\left(\epsilon r u^{-\delta_{dec}}\right), \tag{8.80}$$
$$\int_S J^{(p)}J^{(q)}=\frac{4\pi}{3}r^2\delta_{pq}+O\left(\epsilon r u^{-\delta_{dec}}\right).$$

2. *We have on* Σ_*

$$\nabla_\nu\Big[(r^2\Delta+2)J^{(p)}\Big]=O(\not{\mathfrak{d}}^{\leq 1}\Gamma_b). \tag{8.81}$$

3. *For any $k\leq k_{small}$, we have on* Σ_*

$$\left|\mathfrak{d}_*^k\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right|\lesssim\epsilon r^{-3}u^{-\frac{\delta_{dec}}{2}}.$$

4. *We have for any $k\leq k_{small}$ on* Σ_*

$$\left|\mathfrak{d}_*^k\,\not{d}_2^*\not{d}_1^*J^{(p)}\right|\lesssim\epsilon r^{-3}u^{-\frac{\delta_{dec}}{2}},$$

[167] Recall the definition of a and m in Section 3.2.4.

where by $\not{d}_1^* J^{(p)}$*, we mean either* $\not{d}_1^*(J^{(p)}, 0)$ *or* $\not{d}_1^*(0, J^{(p)})$*.*

Also, we recall Lemma 5.47 on the control of $(\check{\rho} - \frac{1}{2}\widehat{\chi} \cdot \underline{\widehat{\chi}})_{\ell=1}$ on S_*.

Lemma 8.24. *The following holds on* S_*

$$\left|\left(\check{\rho} - \frac{1}{2}\widehat{\chi} \cdot \underline{\widehat{\chi}}\right)_{\ell=1}\right| \lesssim \epsilon_0 r^{-3} u^{-2-2\delta_{dec}}. \tag{8.82}$$

Step 2. On Σ_* we assume the following local bootstrap assumptions

$$\begin{aligned}
&\sup_{\Sigma_*}\Bigg(r^5|(\mathrm{div}\,\beta)_{\ell=1}| + r^5|(\mathrm{curl}\,\beta)_{\ell=1,\pm}| \\
&\qquad + r^5\left|(\mathrm{curl}\,\beta)_{\ell=1,0} - \frac{2am}{r^5}\right|\Bigg) \leq \epsilon, \\
&\sup_{\Sigma_*}\left(r^3 u^{1+\delta_{dec}}|(\check{\rho})_{\ell=1}| + r^2 u^{1+\delta_{dec}}|(\check{\kappa})_{\ell=1}|\right) \leq \epsilon,
\end{aligned} \tag{8.83}$$

which will be improved in Steps 2–7.

We start with the control of $(\mathrm{div}\,\zeta)_{\ell=1}$. Recall the following consequence of the Codazzi equation for $\widehat{\chi}$

$$\not{d}_2\widehat{\chi} = \frac{1}{r}\zeta - \beta + \Gamma_g \cdot \Gamma_g.$$

Differentiating w.r.t. div , we infer

$$\mathrm{div}\,\not{d}_2\widehat{\chi} = \frac{1}{r}\mathrm{div}\,\zeta - \mathrm{div}\,\beta + r^{-1}\,\not{\mathfrak{d}}^{\leq 1}(\Gamma_g \cdot \Gamma_g).$$

Projecting on the $\ell = 1$ modes, this yields

$$(\mathrm{div}\,\not{d}_2\widehat{\chi})_{\ell=1} = \frac{1}{r}(\mathrm{div}\,\zeta)_{\ell=1} - (\mathrm{div}\,\beta)_{\ell=1} + r^{-1}\,\not{\mathfrak{d}}^{\leq 1}(\Gamma_g \cdot \Gamma_g).$$

Next, we estimate $(\not{d}_1\,\not{d}_2\widehat{\chi})_{\ell=1}$. We have

$$(\mathrm{div}\,\not{d}_2\widehat{\chi})_{\ell=1,p} = \frac{1}{|S|}\int_S \mathrm{div}\,\not{d}_2\widehat{\chi} J^{(p)} = \frac{1}{|S|}\int_S \widehat{\chi} \cdot \not{d}_2^*\nabla J^{(p)}$$

and hence

$$|(\mathrm{div}\,\not{d}_2\widehat{\chi})_{\ell=1}| \lesssim |\not{d}_2^*\not{d}_1^* J^{(p)}||\Gamma_g|.$$

We deduce

$$(\mathrm{div}\,\zeta)_{\ell=1} \quad = \quad r(\mathrm{div}\,\beta)_{\ell=1} + r|\not{d}_2^*\not{d}_1^* J^{(p)}\Gamma_g| + \not{\mathfrak{d}}^{\leq 1}(\Gamma_g\cdot\Gamma_g).$$

Together with the local bootstrap assumption (8.83) for $(\mathrm{div}\,\beta)_{\ell=1}$, the control of Lemma 8.23 for $\not{d}_2^*\not{d}_1^* J^{(p)}$, and the bootstrap assumptions for Γ_g, we infer on Σ_*

$$|(\mathrm{div}\,\zeta)_{\ell=1}| \quad \lesssim \quad \frac{\epsilon}{r^4}. \tag{8.84}$$

Step 3. Next, we control of $(\mathrm{div}\,\underline{\beta})_{\ell=1}$. Recall the following consequence of the Codazzi equation for $\widehat{\underline{\chi}}$

$$\mathrm{div}\,\widehat{\underline{\chi}} \quad = \quad \frac{1}{2}\nabla\underline{\check{\kappa}} + \frac{\Upsilon}{r}\zeta + \underline{\beta} + \Gamma_b\cdot\Gamma_g.$$

Differentiating w.r.t. div , we infer

$$\mathrm{div}\,\not{d}_2\widehat{\underline{\chi}} \quad = \quad \frac{1}{2}\Delta\underline{\check{\kappa}} + \frac{\Upsilon}{r}\mathrm{div}\,\zeta + \mathrm{div}\,\underline{\beta} + r^{-1}\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_g).$$

Projecting on the $\ell = 1$ modes, this yields

$$(\mathrm{div}\,\not{d}_2\widehat{\underline{\chi}})_{\ell=1} \quad = \quad \frac{1}{2}(\Delta\underline{\check{\kappa}})_{\ell=1} + \frac{\Upsilon}{r}(\mathrm{div}\,\zeta)_{\ell=1} + (\mathrm{div}\,\underline{\beta})_{\ell=1} + r^{-1}\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_g).$$

As in Step 2, we have

$$|(\mathrm{div}\,\not{d}_2\widehat{\underline{\chi}})_{\ell=1}| \quad \lesssim \quad |\not{d}_2^*\not{d}_1^* J^{(p)}||\Gamma_b|.$$

Also, we have

$$(\Delta\underline{\check{\kappa}})_{\ell=1,p} = \frac{1}{|S|}\int_S \Delta\underline{\check{\kappa}} J^{(p)} = -\frac{2}{r^2}\frac{1}{|S|}\int_S \underline{\check{\kappa}} J^{(p)} + \frac{1}{|S|}\int_S \underline{\check{\kappa}}\left(\Delta + \frac{2}{r^2}\right) J^{(p)}$$

and hence

$$|(\Delta\underline{\check{\kappa}})_{\ell=1}| \quad \lesssim \quad r^{-2}|(\underline{\check{\kappa}})_{\ell=1}| + \left|\left(\Delta + \frac{2}{r^2}\right) J^{(p)}\right| |\Gamma_g|.$$

We deduce

$$|(\mathrm{div}\,\underline{\beta})_{\ell=1}| \quad \lesssim \quad r^{-2}|(\underline{\check{\kappa}})_{\ell=1}| + r^{-1}|(\mathrm{div}\,\zeta)_{\ell=1}| + \left|\left(\Delta + \frac{2}{r^2}\right) J^{(p)}\right| |\Gamma_g|$$

$$+|\not{d}_2^\star \not{d}_1^\star J^{(p)}||\Gamma_b| + r^{-1}|\not{\mathfrak{d}}^{\leq 1}(\Gamma_b \cdot \Gamma_g)|.$$

Together with the local bootstrap assumption (8.83) for $(\underline{\check{\kappa}})_{\ell=1}$, the control for $(\operatorname{div}\zeta)_{\ell=1}$ of Step 2, the control of Lemma 8.23 for $\not{d}_2^\star \not{d}_1^\star J^{(p)}$ and $(\Delta + \frac{2}{r^2})J^{(p)}$, and the bootstrap assumptions for Γ_b and Γ_g, we infer on Σ_*

$$|(\operatorname{div}\underline{\beta})_{\ell=1}| \lesssim \frac{\epsilon}{r^4 u^{1+\delta_{dec}}} + \frac{\epsilon}{r^5}.$$

Using the dominance condition (3.50) on r on Σ_*, we infer on Σ_*

$$|(\operatorname{div}\underline{\beta})_{\ell=1}| \lesssim \frac{\epsilon}{r^4 u^{1+\delta_{dec}}}. \tag{8.85}$$

Step 4. Recall from Corollary 5.41 the following transport along Σ_*, for $p = 0, +, -$,

$$\begin{aligned} & \nu\left(\int_S \left(\Delta\underline{\check{\kappa}} + \frac{2\Upsilon}{r}\operatorname{div}\zeta\right) J^{(p)}\right) \\ = \; & O(r^{-3})\int_S \underline{\check{\kappa}} J^{(p)} + O(r^{-2})\int_S \operatorname{div}\zeta J^{(p)} + O(r^{-1})\int_S \operatorname{div}\underline{\beta} J^{(p)} \\ & +O(r^{-2})\int_S \check{\rho} J^{(p)} + O(r^{-1})\int_S \operatorname{div}\beta J^{(p)} + r\left|\left(\Delta + \frac{2}{r^2}\right) J^{(p)}\right| \not{\mathfrak{d}}^{\leq 1}\Gamma_b \\ & + \not{\mathfrak{d}}^{\leq 2}(\Gamma_b \cdot \Gamma_b), \end{aligned}$$

and

$$\begin{aligned} \nu\left(\int_S \left(\check{\rho} - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right) J^{(p)}\right) = \; & -\int_S \operatorname{div}\underline{\beta} J^{(p)} - (1 + O(r^{-1}))\int_S \operatorname{div}\beta J^{(p)} \\ & +O(r^{-1})\int_S \left(\check{\rho} - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right) J^{(p)} \\ & +O(r^{-3})\int_S \underline{\check{\kappa}} J^{(p)} + O(r^{-2})\int_S \operatorname{div}\zeta J^{(p)} \\ & +r\left|\left(\Delta + \frac{2}{r^2}\right) J^{(p)}\right| \Gamma_b + r\,\not{\mathfrak{d}}^{\leq 1}(\Gamma_b \cdot \Gamma_b), \end{aligned}$$

where by $\not{d}_1^\star J^{(p)}$, we mean $\not{d}_1^\star(J^{(p)}, 0)$ or $\not{d}_1^\star(0, J^{(p)})$. Together with the local bootstrap assumption (8.83) for $(\operatorname{div}\beta)_{\ell=1}$, $(\underline{\check{\kappa}})_{\ell=1}$, and $(\check{\rho})_{\ell=1}$, the control for $(\operatorname{div}\zeta)_{\ell=1}$ of Step 2, the control for $(\operatorname{div}\underline{\beta})_{\ell=1}$ of Step 3, the control of Lemma 8.23 for $(\Delta + \frac{2}{r^2})J^{(p)}$, and the bootstrap assumptions for Γ_b and Γ_g, we infer on Σ_*, for $p = 0, +, -$,

$$\left|\nu\left(\int_S \left(\Delta\underline{\check{\kappa}} + \frac{2\Upsilon}{r}\operatorname{div}\zeta\right) J^{(p)}\right)\right| \lesssim \frac{\epsilon^2}{r^2 u^{2+2\delta_{dec}}} + \frac{\epsilon}{r^3 u^{1+\delta_{dec}}} + \frac{\epsilon}{r^4},$$

$$\left|\nu\left(\int_S\left(\check{\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)J^{(p)}\right)\right| \lesssim \frac{\epsilon^2}{ru^{2+2\delta_{dec}}}+\frac{\epsilon}{r^2u^{1+\delta_{dec}}}+\frac{\epsilon}{r^3}.$$

Using the dominance condition (3.50) on r on Σ_*, we infer on Σ_*, for $p = 0,+,-$,

$$\left|\nu\left(\int_S\left(\Delta\check{\underline{\kappa}}+\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right)J^{(p)}\right)\right| \lesssim \frac{\epsilon_0}{r^2u^{2+2\delta_{dec}}},$$
$$\left|\nu\left(\int_S\left(\check{\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)J^{(p)}\right)\right| \lesssim \frac{\epsilon_0}{ru^{2+2\delta_{dec}}}.$$

Integrating from S_*, and using the fact that $\nu(u) = 2 + O(\epsilon)$, we deduce on Σ_*, for $p = 0,+,-$,

$$\left|\int_S\left(\Delta\check{\underline{\kappa}}+\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right)J^{(p)}\right| \lesssim \left|\int_{S_*}\left(\Delta\check{\underline{\kappa}}+\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right)J^{(p)}\right|+\frac{\epsilon_0}{r^2u^{1+\delta_{dec}}},$$
$$\left|\int_S\left(\check{\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)J^{(p)}\right| \lesssim \left|\int_{S_*}\left(\check{\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)J^{(p)}\right|+\frac{\epsilon_0}{ru^{1+\delta_{dec}}}.$$

Now, since $\check{\underline{\kappa}} = 0$ on S_* in view of our GCM conditions, and in view of the control $(\check{\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}})_{\ell=1}$ on S_* provided by Lemma 8.24, and the control for $(\mathrm{div}\,\zeta)_{\ell=1}$ of Step 2, we have

$$\left|\int_{S_*}\left(\Delta\check{\underline{\kappa}}+\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right)J^{(p)}\right| \lesssim r|(\mathrm{div}\,\zeta)_{\ell=1}|\lesssim\frac{\epsilon}{r^3},$$
$$\left|\int_{S_*}\left(\check{\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)J^{(p)}\right| \lesssim \frac{\epsilon_0}{ru^{1+\delta_{dec}}},$$

and hence, we obtain on Σ_*, for $p = 0,+,-$,

$$\left|\int_S\left(\Delta\check{\underline{\kappa}}+\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right)J^{(p)}\right| \lesssim \frac{\epsilon_0}{r^2u^{1+\delta_{dec}}}+\frac{\epsilon}{r^3}\lesssim\frac{\epsilon_0}{r^2u^{1+\delta_{dec}}}$$

where we used in the last inequality the dominance condition (3.50) on r on Σ_*, and

$$\left|\int_S\left(\check{\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)J^{(p)}\right| \lesssim \frac{\epsilon_0}{ru^{1+\delta_{dec}}}.$$

We infer on Σ_*, for $p = 0,+,-$,

$$\left|\int_S\check{\underline{\kappa}}J^{(p)}\right| \lesssim r^2\left|\int_S\left(\Delta+\frac{2}{r^2}\right)\check{\underline{\kappa}}J^{(p)}\right|+r\left|\int_S\mathrm{div}\,\zeta J^{(p)}\right|+\frac{\epsilon_0}{u^{1+\delta_{dec}}},$$

$$\left|\int_S \check{\rho} J^{(p)}\right| \lesssim r^2\Gamma_b\cdot\Gamma_g + \frac{\epsilon_0}{ru^{1+\delta_{dec}}}.$$

Together with the control for $(\operatorname{div}\zeta)_{\ell=1}$ of Step 2, the control of Lemma 8.23 for $(\Delta+\frac{2}{r^2})J^{(p)}$, and the bootstrap assumptions for Γ_b and Γ_g, we infer on Σ_*,

$$r|(\check{\rho})_{\ell=1}| + |(\check{\underline{\kappa}})_{\ell=1}| \lesssim \frac{\epsilon_0}{r^2u^{1+\delta_{dec}}} + \frac{\epsilon}{r^3}.$$

Using the dominance condition (3.50) on r on Σ_*, we deduce

$$\sup_{\Sigma_*}\Big(r^3u^{1+\delta_{dec}}|(\check{\rho})_{\ell=1}| + r^2u^{1+\delta_{dec}}|(\check{\underline{\kappa}})_{\ell=1}|\Big) \lesssim \epsilon_0. \tag{8.86}$$

Step 5. Recall from Corollary 5.41 the following transport equation along Σ_*, for $p=0,+,-$,

$$\begin{aligned}\nu\left(\int_S \operatorname{div}\beta J^{(p)}\right) &= O(r^{-1})\int_S \operatorname{div}\beta J^{(p)} + O(r^{-2})\int_S\check{\rho}J^{(p)}\\ &\quad + r\left(\left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right| + \left|{\not{d}}_2^*{\not{d}}_1^*J^{(p)}\right|\right)\Gamma_g + \not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_g).\end{aligned}$$

Together with the local bootstrap assumption (8.83) for $(\operatorname{div}\beta)_{\ell=1}$, the control for $(\check{\rho})_{\ell=1}$ of Step 4, the control of Lemma 8.23 for $(\Delta+\frac{2}{r^2})J^{(p)}$ and ${\not{d}}_2^*{\not{d}}_1^*J^{(p)}$, and the bootstrap assumptions for Γ_b and Γ_g, we infer on Σ_*, for $p=0,+,-$,

$$\left|\nu\left(\int_S \operatorname{div}\beta J^{(p)}\right)\right| \lesssim \frac{\epsilon_0+\epsilon^2}{r^3u^{1+\delta_{dec}}} + \frac{\epsilon}{r^4},$$

and using the dominance condition (3.50) on r on Σ_*, we deduce

$$\left|\nu\left(\int_S \operatorname{div}\beta J^{(p)}\right)\right| \lesssim \frac{\epsilon_0}{r^3u^{1+\delta_{dec}}}.$$

Integrating from S_*, using the fact that $\nu(u)=2+O(\epsilon)$, and the GCM condition $(\operatorname{div}\beta)_{\ell=1}=0$ on S_*, we deduce on Σ_*, for $p=0,+,-$,

$$\sup_{\Sigma_*} r^5|(\operatorname{div}\beta)_{\ell=1}| \lesssim \epsilon_0. \tag{8.87}$$

Step 6. Next, we control $({}^*\rho)_{\ell=1}$. To this end, we first control $(\operatorname{curl}\zeta)_{\ell=1}$. Recall the following consequence of the Codazzi equation for $\widehat{\chi}$

$${\not{d}}_2\widehat{\chi} = \frac{1}{r}\zeta - \beta + \Gamma_g\cdot\Gamma_g.$$

Differentiating w.r.t. curl, we infer

$$\mathrm{curl}\, \not{d}_2\widehat{\chi} = \frac{1}{r}\mathrm{curl}\,\zeta - \mathrm{curl}\,\beta + r^{-1}\not{\mathfrak{d}}^{\leq 1}(\Gamma_g\cdot\Gamma_g).$$

Proceeding as for the control of $(\mathrm{div}\,\zeta)_{\ell=1}$ in Step 2, we infer

$$(\mathrm{curl}\,\zeta)_{\ell=1} = r(\mathrm{curl}\,\beta)_{\ell=1} + r|\not{d}_2^*\not{d}_1^*J^{(p)}|\Gamma_g + \not{\mathfrak{d}}^{\leq 1}(\Gamma_g\cdot\Gamma_g).$$

Together with the local bootstrap assumption (8.83) for $(\mathrm{curl}\,\beta)_{\ell=1}$, the control of Lemma 8.23 for $\not{d}_2^*\not{d}_1^*J^{(p)}$, and the bootstrap assumptions for Γ_g, we infer on Σ_*

$$|(\mathrm{curl}\,\zeta)_{\ell=1,\pm}| + \left|(\mathrm{curl}\,\zeta)_{\ell=1,0} - \frac{2am}{r^4}\right| \lesssim \frac{\epsilon}{r^4}.$$

Now, note that we have from the null structure equations

$${}^*\rho = \mathrm{curl}\,\zeta + \Gamma_b\cdot\Gamma_g.$$

Together with the above control of $(\mathrm{curl}\,\zeta)_{\ell=1}$ and the bootstrap assumptions for Γ_g, we infer on Σ_*

$$|({}^*\rho)_{\ell=1,\pm}| + \left|({}^*\rho)_{\ell=1,0} - \frac{2am}{r^4}\right| \lesssim \frac{\epsilon}{r^4} + \frac{\epsilon^2}{r^3u^{1+\delta_{dec}}}$$

and using the dominance condition (3.50) on r on Σ_*, we deduce

$$|({}^*\rho)_{\ell=1,\pm}| + \left|({}^*\rho)_{\ell=1,0} - \frac{2am}{r^4}\right| \lesssim \frac{\epsilon_0}{r^3u^{1+\delta_{dec}}}. \tag{8.88}$$

Step 7. Recall from Corollary 5.41 the following transport along Σ_*, for $p=0,+,-$,

$$\begin{aligned}
&\nu\left(\int_S \mathrm{curl}\,\beta J^{(p)}\right)\\
&= \frac{4}{r}(1+O(r^{-1}))\int_S \mathrm{curl}\,\beta J^{(p)} + \frac{2}{r^2}(1+O(r^{-1}))\int_S {}^*\rho J^{(p)}\\
&\quad + r\left(\left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right| + \left|\not{d}_2^*\not{d}_1^*J^{(p)}\right|\right)\Gamma_g + \not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

In the case $p=\pm$, we have $(\mathrm{curl}\,\beta)_{\ell=1,\pm}=0$ on S_*, so using Step 6 to control $({}^*\rho)_{\ell=1,\pm}$, and arguing exactly as for the control of $(\mathrm{div}\,\beta)_{\ell=1}$ in Step 5, we obtain the corresponding estimate, i.e.

$$\sup_{\Sigma_*} r^5|(\mathrm{curl}\,\beta)_{\ell=1,\pm}| \lesssim \epsilon_0. \tag{8.89}$$

Next, we focus on the case $p=0$. We rewrite the above transport equation in this particular case

$$\begin{aligned}
&\nu\left(\int_S \mathrm{curl}\,\beta J^{(0)}\right)\\
=\ & \frac{4}{r}(1+O(r^{-1}))\int_S \mathrm{curl}\,\beta J^{(0)} + \frac{2}{r^2}(1+O(r^{-1}))\int_S {}^*\rho J^{(0)}\\
&+r\left(\left|\left(\Delta+\frac{2}{r^2}\right)J^{(0)}\right| + \left|\not{d}_2^*\not{d}_1^*J^{(0)}\right|\right)\Gamma_g + \not{\mathfrak{d}}^{\le 1}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

Since $\nu(r) = -2 + r\Gamma_b$, we have

$$\begin{aligned}
\nu\left(r^3\int_S \mathrm{curl}\,\beta J^{(0)}\right) &= r^3\nu\left(\int_S \mathrm{curl}\,\beta J^{(0)}\right) + 3r^2\nu(r)\int_S \mathrm{curl}\,\beta J^{(0)}\\
&= r^3\nu\left(\int_S \mathrm{curl}\,\beta J^{(0)}\right) - 6r^2\int_S \mathrm{curl}\,\beta J^{(0)}\\
&\quad +r^5\Gamma_b(\mathrm{curl}\,\beta)_{\ell=1,0},
\end{aligned}$$

and hence

$$\begin{aligned}
&\nu\left(r^3\int_S \mathrm{curl}\,\beta J^{(0)} - 8\pi am\right)\\
=\ & -\frac{2}{r}r^3(1+O(r^{-1}))\int_S \mathrm{curl}\,\beta J^{(0)} + 2r(1+O(r^{-1}))\int_S {}^*\rho J^{(0)}\\
&+r^5\Gamma_b(\mathrm{curl}\,\beta)_{\ell=1,0} + r^4\left(\left|\left(\Delta+\frac{2}{r^2}\right)J^{(0)}\right| + \left|\not{d}_2^*\not{d}_1^*J^{(0)}\right|\right)\Gamma_g\\
&+r^3\,\not{\mathfrak{d}}^{\le 1}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

In view of the local bootstrap assumption (8.83) for $(\mathrm{curl}\,\beta)_{\ell=1,0}$, the control of $({}^*\rho)_{\ell=1}$ of Step 6, the control of Lemma 8.23 for $(\Delta+\frac{2}{r^2})J^{(p)}$ and $\not{d}_2^*\not{d}_1^*J^{(p)}$, and the bootstrap assumptions for Γ_g, we infer on Σ_*

$$\left|\nu\left(r^3\int_S \mathrm{curl}\,\beta J^{(0)} - 8\pi am\right)\right| \lesssim \frac{1}{r} + \frac{\epsilon_0+\epsilon^2}{u^{1+\delta_{dec}}}.$$

Using the dominance condition (3.50) on r on Σ_*, we obtain

$$\left|\nu\left(r^3\int_S \mathrm{curl}\,\beta J^{(0)} - 8\pi am\right)\right| \lesssim \frac{\epsilon_0}{u^{1+\delta_{dec}}}.$$

Integrating from S_* where there holds $(\mathrm{curl}\,\beta)_{\ell=1,0} = \frac{2am}{r^5}$ on S_*, we deduce on Σ_*

$$\left|(\mathrm{curl}\,\beta)_{\ell=1,0} - \frac{2am}{r^5}\right| \lesssim \frac{\epsilon_0}{r^5}. \tag{8.90}$$

Thus, in view of the control for $(\check{\underline{\kappa}})_{\ell=1}$ and $(\check{\rho})_{\ell=1}$ of Step 4, the control for $(\mathrm{div}\,\beta)_{\ell=1}$ of Step 5, and the control for $(\mathrm{curl}\,\beta)_{\ell=1}$ of this step, we have finally obtained on Σ_* the desired estimate (8.79), i.e.

$$\begin{aligned}
&\sup_{\Sigma_*}\left(r^5|(\mathrm{div}\,\beta)_{\ell=1}| + r^5|(\mathrm{curl}\,\beta)_{\ell=1,\pm}| + r^5\left|(\mathrm{curl}\,\beta)_{\ell=1,0} - \frac{2am}{r^5}\right|\right) \lesssim \epsilon_0,\\
&\sup_{\Sigma_*}\left(r^3u^{1+\delta_{dec}}|(\check{\rho})_{\ell=1}| + r^2u^{1+\delta_{dec}}|(\check{\kappa})_{\ell=1}|\right) \lesssim \epsilon_0,
\end{aligned}$$

thus improving the local bootstrap assumption (8.83), and concluding the proof of Proposition 8.22.

8.3.2. Steps 8–16 As a consequence of (8.79) we have, in particular on $S_1 = \Sigma_* \cap \{u=1\}$, the estimate

$$\sup_{S_1}\left(r^2\,|(\check{\underline{\kappa}})_{\ell=1}| + r^5\,|(\mathrm{div}\,\beta)_{\ell=1}|\right) \lesssim \epsilon_0. \tag{8.91}$$

On S_1, we also have the GCM conditions

$$\begin{aligned}
\kappa &= \frac{2}{r}, \qquad \underline{\kappa} = -\frac{2\Upsilon}{r} + \underline{C}_0 + \sum_p \underline{C}_p J^{(p)},\\
\mu &= \frac{2m}{r^3} + M_0 + \sum_p M_p J^{(p)}.
\end{aligned} \tag{8.92}$$

We introduce the following auxiliary construction.

Definition 8.25 (The outgoing cone $\mathcal{C}_1'$)**.** *Starting with the sphere S_1 we define the outgoing geodesic null cone $\mathcal{C}_1'$ emanating from S_1 in the direction of e_4. We denote:*

- *by e_4' the geodesic extension of e_4, and by s' the corresponding affine parameter, i.e. $e_4'(s') = 1$, normalized such that $s' = r$ on S_1,*
- *by S' the spheres of constant s' along $\mathcal{C}_1'$ and by r' the corresponding area radius,*
- *by $J'^{(p)}$ the basis of $\ell = 1$ modes verifying $e_4'(J'^{(p)}) = 0$ with $J'^{(p)} = J^{(p)}$ on S_1, for $p = 0, +, -$.*

We restrict $\mathcal{C}_1'$ to the region $\{\delta_*\epsilon_0^{-1} \leq r' \leq r(S_1)\}$. With this restriction, we will show below, see (8.98), that $\mathcal{C}_1' \subset {}^{(ext)}\widetilde{\mathcal{L}_0}$. Finally, we denote by $(f, \underline{f}, \lambda)$ the transition coefficients from the outgoing geodesic null frame of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ of Section 8.2 to the outgoing geodesic null frame of $\mathcal{C}_1'$ initialized on $S_1' = S_1$.

Local Bootstrap Assumptions:

1. Along $\mathcal{C}_1'$, we assume

$$\sup_{S'\subset\mathcal{C}_1'}\Big(\|f\|_{\mathfrak{h}_4(S')} + (r')^{-1}\|(\underline{f}, \log\lambda)\|_{\mathfrak{h}_4(S')}\Big) \leq \epsilon. \tag{8.93}$$

2. On $S_1' = S_1$, we assume

$$\|f\|_{\mathfrak{h}_{k_{large}}(S_1')} + r^{-1}\|(\underline{f}, \log(\lambda))\|_{\mathfrak{h}_{k_{large}}(S_1')} \leq \epsilon. \tag{8.94}$$

3. In the case $a_0 \neq 0$, we make the following assumption[168], on S_1', on the difference between the basis of $\ell = 1$ modes $J^{(p)}$ of Σ_*, and the basis of $\ell = 1$ modes $\widetilde{J}^{(p)}$ of ${}^{(ext)}\widetilde{\mathcal{L}_0}$

$$\max_{p=0,+,-}\Big\|\mathfrak{d}_*^{\leq k_{large}}(J^{(p)} - \widetilde{J}^{(p)})\Big\|_{L^\infty(S_1')} \leq \epsilon. \tag{8.95}$$

Remark 8.26. (8.94) *will be improved in Step 13,* (8.93) *will be improve in Step 14, and* (8.95) *will be improved in Step 17.*

Remark 8.27. *The discrepancy between the top number of derivatives in* (8.94) *and* (8.95) *is due to the fact that the former depends only on the total number of derivatives allowed in ${}^{(ext)}\widetilde{\mathcal{L}_0}$ while the latter reflects the total number of derivatives allowed by the global bootstrap assumptions on ${}^{(ext)}\mathcal{M}$.*

Remark 8.28. *The anomalous behavior for $\underline{f}$ and λ in* (8.93) (8.94)*, i.e. the fact that they display a r loss compared to f, does not affect the desired estimates for the curvature components, see* (8.147)*. This is due to the fact that, in the change of frame formulas for the curvature components, λ and $\underline{f}$ are multiplied by terms that decay faster in r. We refer to Remark 8.31 for a heuristic explanation of this anomalous behavior.*

[168] Recall that $\mathfrak{d}_*$ refers to the properly normalized tangential derivatives along Σ_*, i.e. $\mathfrak{d}_* = (\not{\mathfrak{d}}, \nabla_\nu)$.

Remark 8.29. *Note that the control of* $J^{(\pm)}_{{}^{(ext)}\widetilde{\mathcal{L}_0}}$ *provided by Proposition 8.20 is invariant under the change*[169]

$$J^{(\pm)}_{{}^{(ext)}\widetilde{\mathcal{L}_0}} \longrightarrow \cos(\varphi_0) J^{(\pm)}_{{}^{(ext)}\widetilde{\mathcal{L}_0}} \pm \sin(\varphi_0) J^{(\mp)}_{{}^{(ext)}\widetilde{\mathcal{L}_0}}, \qquad \varphi_0 \in [0, 2\pi),$$

which corresponds to the invariance $\varphi_{{}^{(ext)}\widetilde{\mathcal{L}_0}} \to \varphi_{{}^{(ext)}\widetilde{\mathcal{L}_0}} - \varphi_0$, *and hence to the fact that Kerr is axially symmetric. We may thus use this freedom to choose the particular pair* $J^{(\pm)}_{{}^{(ext)}\widetilde{\mathcal{L}_0}}$ *such that*

$$\int_{S_1} J^{(+)}_{{}^{(ext)}\widetilde{\mathcal{L}_0}} J^{(-)} = 0. \tag{8.96}$$

This normalization will be needed to improve the bootstrap assumptions (8.95).

Assumption (8.93) allows to apply Lemma 7.3 in [40] (recalled here in Lemma 8.4) with $\delta_1 = \epsilon$ on each sphere[170] $S' = S(s')$ of $\mathcal{C}_1'$. This allows us to deduce the following comparison estimate

$$\sup_{\mathcal{C}_1'} \Big(\widetilde{r}^{-1}|r' - \widetilde{r}| + |1 - \widetilde{u}| + \widetilde{r}^{-1}|s' - \widetilde{s}| \Big) \lesssim \epsilon. \tag{8.97}$$

In particular, since $r' \geq \delta_* \epsilon_0^{-1}$ on $\mathcal{C}_1'$, we infer

$$\sup_{\mathcal{C}_1'} |\widetilde{u} - 1| \lesssim \epsilon, \qquad \inf_{\mathcal{C}_1'} \widetilde{s} \geq \frac{\delta_*}{2} \epsilon_0^{-1}.$$

Since ${}^{(ext)}\widetilde{\mathcal{L}_0} = \{0 \leq \widetilde{u} \leq 2, \, \widetilde{s} \geq \frac{\delta_*}{2}\epsilon_0^{-1}\}$, we deduce

$$\mathcal{C}_1' \subset {}^{(ext)}\widetilde{\mathcal{L}_0}. \tag{8.98}$$

[169] This holds true provided one also changes the quantities $J^{(\pm)}_{{}^{(ext)}\mathcal{L}_0}$ and $\mathfrak{J}_{\pm, {}^{(ext)}\mathcal{L}_0}$, associated to the part ${}^{(ext)}\mathcal{L}_0$ of the initial data layer, according to

$$J^{(\pm)}_{{}^{(ext)}\mathcal{L}_0} \to \cos(\varphi_0) J^{(\pm)}_{{}^{(ext)}\mathcal{L}_0} \pm \sin(\varphi_0) J^{(\mp)}_{{}^{(ext)}\mathcal{L}_0},$$
$$\mathfrak{J}_{\pm, {}^{(ext)}\mathcal{L}_0} \to \cos(\varphi_0) \mathfrak{J}_{\pm, {}^{(ext)}\mathcal{L}_0} \pm \sin(\varphi_0) \mathfrak{J}_{\mp, {}^{(ext)}\mathcal{L}_0}.$$

Note that these transformations leave invariant the linearized quantities in Definition 2.66, and hence (Γ_b, Γ_g) in Definition 2.67. This corresponds to the fact that Kerr is axially symmetric.

[170] Note that each sphere S' of $\mathcal{C}_1'$ intersects a unique sphere $\widetilde{S}$ of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ at its south pole. Hence, we may consider S' as a deformation of $\widetilde{S}$.

We note also that $r, \widetilde{r}, r'$ are all comparable along $\mathcal{C}'_1$.

Also, since the sphere $S'_1 = S_1$ satisfies the following:

- S'_1 is a sphere of ${}^{(ext)}\mathcal{M}$ in ${}^{(ext)}\widetilde{\mathcal{L}_0}$,
- S'_1 is a sphere of the GCM hypersurface Σ_*,
- the estimate (8.91) holds on S'_1,
- the estimate (8.94) holds on S'_1,
- the estimate (8.95) holds on S'_1,

we can invoke Proposition 8.1 in [41] (restated here in Proposition 8.10) with the choice $\mathring{\epsilon} = \mathring{\delta} = \epsilon_0$, $\delta_1 = \epsilon$, $s_{max} = k_{large}$, and with the background foliation being the outgoing geodesic foliation of the part ${}^{(ext)}\widetilde{\mathcal{L}_0}$ of the initial data layer. We obtain

$$r^{-1}\|(f, \underline{f}, \lambda - \overline{\lambda}^{S'_1})\|_{\mathfrak{h}_{k_{large}+1}(S'_1)} \lesssim \epsilon_0, \tag{8.99}$$

and, with $\overline{\lambda}^{S'_1}$ denoting the average of λ on S'_1,

$$|\overline{\lambda}^{S'_1} - 1| \lesssim \epsilon_0 + r^{-1}\sup_{S'_1}|r - \widetilde{r}|\,. \tag{8.100}$$

Remark 8.30. *In order to improve the bootstrap assumption (8.94) we need in particular to improve the estimate for f in (8.99) by r^{-1}. Obtaining this improvement is the focus*[171] *of Step 8 to 13.*

Remark 8.31. *In view of (8.97), while $|1 - \widetilde{u}| \lesssim \epsilon$ on S_1, we only have $|s' - \widetilde{s}| \lesssim r\epsilon$ on S_1. This, as well as the anomalous behavior of $\underline{f}$ mentioned in Remark 8.28, shows that the sphere S_1 is a large deformation, along the outgoing direction, of spheres of the initial data layer ${}^{(ext)}\widetilde{\mathcal{L}_0}$. This reflects the fact that S_1 (and Σ_*) captures the center of mass frame of the limiting Kerr solution, while the initial data layer foliation captures the center of mass frame of the initial Kerr solution. The behavior of $s' - \widetilde{s}$, as well as the one of $\underline{f}$, is consistent with the presence of a large Lorentz boost between these two center of mass frames.*

Local Notation for Steps 8–18. *In Steps 8–18 below, $(f, \underline{f}, \lambda)$ denote the transition coefficients from the frame $\widetilde{E} = (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4)$ of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ to the prime one of $\mathcal{C}'_1$. Also, we will only make use of the prime frame along $\mathcal{C}'_1$ and the geodesic frame of ${}^{(ext)}\widetilde{\mathcal{L}_0}$. Since we are not making reference here to*

[171]In fact, all intermediate estimates in Step 8 to 13 are only needed to derive this improvement on f.

the PG structure of $^{(ext)}\mathcal{L}_0$ *and the one of* $^{(ext)}\mathcal{M}$*, we drop the tilde on the quantities associated to* $^{(ext)}\widetilde{\mathcal{L}_0}$.

Step 8. We derive estimates for β' and $\not{d}_1'^*(-\rho', {}^*\rho')$ on S_1' with the help of transformation formulas of Proposition 2.12. To start with, we make use of

$$\beta' = \lambda\left(\beta + \frac{3}{2}(f\rho + {}^*f\ {}^*\rho) + \frac{1}{2}\alpha\cdot\underline{f} + \text{ l.o.t.}\right),$$

together with the estimate (8.99) for f to estimate the linear terms ρf and ${}^*\rho\ {}^*f$, the estimate (8.94) for $(f, \underline{f}, \lambda)$ to estimate the other terms, and the control provided by Proposition 8.20 for the part $^{(ext)}\widetilde{\mathcal{L}_0}$ of the initial data layer to control[172] the $\mathfrak{h}_j(S_1')$ norm of the Ricci coefficients and curvature components of the initial data foliation of $^{(ext)}\widetilde{\mathcal{L}_0}$ in terms of their sup norm, we have

$$\sup_{k\le k_{large}} r^2\|\not{\mathfrak{d}}'^k\beta'\|_{L^2(S_1')} \lesssim \epsilon_0. \tag{8.101}$$

Also, we have

$$\begin{aligned}
\rho' &= \rho + \underline{f}\cdot\beta - f\cdot\underline{\beta} + \frac{3}{2}\rho(f\cdot\underline{f}) - \frac{3}{2}\ {}^*\rho(f\wedge\underline{f}) + \text{ l.o.t.},\\
{}^*\rho' &= {}^*\rho - \underline{f}\cdot{}^*\beta - f\cdot{}^*\underline{\beta} + \frac{3}{2}\ {}^*\rho(f\cdot\underline{f}) + \frac{3}{2}\rho(f\wedge\underline{f}) + \text{ l.o.t.}
\end{aligned}$$

Differentiating the two equations w.r.t. e_a', and using the decomposition of e_a', we infer

$$\begin{aligned}
e_a'(\rho') &= \left(\left(\delta_a^b + \frac{1}{2}\underline{f}_a f^b\right)e_b + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right)e_3\right)\rho\\
&\quad + e_a'\left(\underline{f}\cdot\beta - f\cdot\underline{\beta} + \frac{3}{2}\rho(f\cdot\underline{f}) - \frac{3}{2}\ {}^*\rho(f\wedge\underline{f}) + \text{ l.o.t.}\right),\\
e_a'({}^*\rho') &= \left(\left(\delta_a^b + \frac{1}{2}\underline{f}_a f^b\right)e_b + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right)e_3\right){}^*\rho\\
&\quad + e_a'\left(-\underline{f}\cdot{}^*\beta - f\cdot{}^*\underline{\beta} + \frac{3}{2}\ {}^*\rho(f\cdot\underline{f}) + \frac{3}{2}\rho(f\wedge\underline{f}) + \text{ l.o.t.}\right),
\end{aligned}$$

and hence

$$e_a'(\rho') = e_a(\rho) + \frac{1}{2}\underline{f}_a e_4(\rho) + \frac{1}{2}f_a e_3(\rho)$$

[172]In order to control $\mathfrak{h}_j(S_1')$ norms by sup norms, we use, here and in the remainder of the proof, Lemma 7.3 in [40] restated here in Lemma 8.4.

$$
\begin{aligned}
& +e'_a\left(\underline{f}\cdot\beta - f\cdot\underline{\beta} + \frac{3}{2}\rho(f\cdot\underline{f}) - \frac{3}{2}\,{}^*\rho(f\wedge\underline{f})\right) + \text{ l.o.t.},\\
e'_a({}^*\rho') \;\; = \;\; & e_a(\rho) + \frac{1}{2}\underline{f}_a e_4(\rho) + \frac{1}{2}f_a e_3({}^*\rho)\\
& +e'_a\left(-\underline{f}\cdot{}^*\beta - f\cdot{}^*\underline{\beta} + \frac{3}{2}\,{}^*\rho(f\cdot\underline{f}) + \frac{3}{2}\rho(f\wedge\underline{f})\right) + \text{ l.o.t.},
\end{aligned}
$$

and hence

$$
\begin{aligned}
\nabla'(\rho') = & \nabla(\rho) + \frac{1}{2}\underline{f}e_4(\rho) + \frac{1}{2}fe_3(\rho)\\
& + \nabla'\left(\underline{f}\cdot\beta - f\cdot\underline{\beta} + \frac{3}{2}\rho(f\cdot\underline{f}) - \frac{3}{2}\,{}^*\rho(f\wedge\underline{f})\right) + \text{ l.o.t.},\\
{}^*\nabla'({}^*\rho') = & {}^*\nabla(\rho) + \frac{1}{2}\,{}^*\underline{f}e_4(\rho) + \frac{1}{2}\,{}^*fe_3({}^*\rho)\\
& + {}^*\nabla'\left(-\underline{f}\cdot{}^*\beta - f\cdot{}^*\underline{\beta} + \frac{3}{2}\,{}^*\rho(f\cdot\underline{f}) + \frac{3}{2}\rho(f\wedge\underline{f})\right) + \text{ l.o.t.}
\end{aligned}
$$

Together with the estimate (8.99) for f and $\underline{f}$ to estimate the linear terms $\underline{f}e_4(\rho)$, ${}^*\underline{f}e_4(\rho)$, $fe_3(\rho)$ and ${}^*fe_3(\rho)$, and the bootstrap estimate (8.94) for $(f, \underline{f}, \lambda)$ to estimate the other terms, we deduce[173],

$$
\max_{k\leq k_{large}-1} r^3\|\mathfrak{\not{d}}'^k \,\not{d}_1'^*(-\rho',\ {}^*\rho')\|_{L^2(S'_1)} \;\; \lesssim \;\; \epsilon_0. \tag{8.102}
$$

Step 9. Recall the definition of the mass aspect function μ'

$$
\mu' \;\; = \;\; -\text{div}\,'\zeta' - \rho' + \frac{1}{2}\widehat{\chi}'\cdot\widehat{\underline{\chi}}',
$$

and the the null structure equation

$$
\text{curl}\,'\zeta' \;\; = \;\; {}^*\rho' - \frac{1}{2}\widehat{\chi}'\wedge\widehat{\underline{\chi}}'.
$$

Together with the GCM conditions for μ on Σ_*, this yields on S'_1, recalling that $S'_1\subset\Sigma_*$,

$$
\not{d}_1'\zeta' \;\; = \;\; (-\mu', 0) + (-\rho',\ {}^*\rho') + \frac{1}{2}\left(\widehat{\chi}'\cdot\widehat{\underline{\chi}}', -\widehat{\chi}'\wedge\widehat{\underline{\chi}}'\right)
$$

---

[173] We also make use of a standard elliptic estimate on S_1 and the r dominance condition $r_*\sim\epsilon_0^{-1}$, see (3.50), on Σ_*.

$$= \quad -\left(M_0' + \sum_p M_p' J'^{(p)}, 0\right) + (-\rho', \ {}^*\rho') + \frac{1}{2}\left(\widehat{\chi}' \cdot \underline{\widehat{\chi}}', -\widehat{\chi}' \wedge \underline{\widehat{\chi}}'\right)$$

and hence, since M_0' and M_p' are constant on S_1', we infer

$$\begin{aligned} \not{d}_2^* \not{d}_1^* \not{d}_1 \zeta' &= -\sum_p M_p' \not{d}_2^* \not{d}_1^* (J'^{(p)}, 0) + \not{d}_2^* \not{d}_1^* (-\rho', \ {}^*\rho') \\ & \quad + \frac{1}{2} \not{d}_2^* \not{d}_1^* \left(\widehat{\chi}' \cdot \underline{\widehat{\chi}}', -\widehat{\chi}' \wedge \underline{\widehat{\chi}}'\right). \end{aligned}$$

In view of the identity $\not{d}_1^{*\prime} \not{d}_1{}' = \not{d}_2{}' \not{d}_2^{*\prime} + 2K'$, we infer

$$\begin{aligned} (\not{d}_2^{*\prime} \not{d}_2{}' + 2K') \not{d}_2^{*\prime} \zeta' &= \not{d}_2^* \not{d}_1^* (-\rho', \ {}^*\rho') - \sum_p M_p' \not{d}_2^* \not{d}_1^* (J'^{(p)}, 0) \\ & \quad + \frac{1}{2} \not{d}_2^* \not{d}_1^* \left(\widehat{\chi}' \cdot \underline{\widehat{\chi}}', -\widehat{\chi}' \wedge \underline{\widehat{\chi}}'\right) + \nabla'(K') \widehat{\otimes} \zeta'. \end{aligned}$$

Using the estimate for $(\rho', \ {}^*\rho')$ of Step 8, the fact that $M_p' \in r^{-1}\Gamma_g$ in view of Corollary 5.39, the control of $\not{d}_2^* \not{d}_1^* (J'^{(p)}, 0)$ provided by Lemma 8.23, and an elliptic estimate for $\not{d}_2^{*\prime} \not{d}_2{}' + 2K'$, we obtain

$$\max_{k \leq k_{large}-1} r^2 \| \not{\mathfrak{d}}'^k \not{d}_2^{*\prime} \zeta' \|_{L^2(S_1')} \quad \lesssim \quad \epsilon_0. \tag{8.103}$$

Note that the quadratic terms involving $\widehat{\chi}' \cdot \underline{\widehat{\chi}}'$, $\widehat{\chi}' \wedge \underline{\widehat{\chi}}'$ and $\nabla'(K') \widehat{\otimes} \zeta'$ are estimated using the transformation formulas[174], the estimates (8.94) for $(f, \underline{f}, \lambda)$, and the control provided by Proposition 8.20 for the curvature components and the Ricci coefficients of the part ${}^{(ext)}\widetilde{\mathcal{L}}_0$ of the initial data layer.

Step 10. Recall Codazzi for $\widehat{\chi}'$

$$\not{d}_2{}' \widehat{\chi}' + \zeta' \cdot \widehat{\chi}' \quad = \quad \frac{1}{2} \nabla' \kappa' + \frac{1}{2} \kappa' \zeta' - \beta'.$$

We differentiate w.r.t. $\not{d}_2^{*\prime}$ and use the GCM condition $\kappa' = 2/r'$ which holds on Σ_* and $S_1' \subset \Sigma_*$ to deduce

$$\not{d}_2^{*\prime} \not{d}_2{}' \widehat{\chi}' \quad = \quad -\not{d}_2^{*\prime} \beta' + \frac{1}{r'} \not{d}_2^{*\prime} \zeta' - \not{d}_2^{*\prime} (\zeta' \cdot \widehat{\chi}').$$

[174] In fact, in view of the Gauss equation $K' = -\rho' - \frac{1}{4}\kappa' \underline{\kappa}' + \frac{1}{2} \widehat{\chi}' \underline{\widehat{\chi}}'$, the GCM conditions for κ' and $\underline{\kappa}'$, and the control of ρ' in Step 8, we only need the transformation formulas for $\widehat{\chi}'$ and $\underline{\widehat{\chi}}'$. These formulas involve at most one angular derivative of f and $\underline{f}$, and no transversal derivative.

Together with the estimate of Step 8 for β', the estimate of Step 9 for ${\not{d}}_2^{*\prime}\zeta'$, dealing with the quadratic terms as above, and using an elliptic estimate, we infer,

$$\max_{k\leq k_{large}} r\|\not{\mathfrak{d}}'^k\widehat{\chi}'\|_{L^2(S_1')} \lesssim \epsilon_0.$$

Next, recall from Proposition 2.12 the transformation formula

$$\begin{aligned}\widehat{\chi}' &= \lambda\Big(\widehat{\chi}+\nabla'\widehat{\otimes}f+f\widehat{\otimes}\eta+f\widehat{\otimes}\zeta+\frac{1}{4}\underline{f}\widehat{\otimes}f\kappa-\underline{\omega}f\widehat{\otimes}f\\ &+\frac{1}{4}(f\widehat{\otimes}\underline{f})\lambda^{-1}\kappa'+\frac{1}{2}\underline{f}\widehat{\otimes}(f\cdot\lambda^{-1}\widehat{\chi}')+\text{ l.o.t.}\Big)\end{aligned}$$

where we have used the fact that ${}^{(a)}\mathrm{tr}\chi' = {}^{(a)}\mathrm{tr}\chi = 0$ and ${}^{(a)}\mathrm{tr}\underline{\chi} = {}^{(a)}\mathrm{tr}\underline{\chi}' = 0$ since both frame are outgoing geodesic and hence integrable, and also that $\omega = 0$ and $\xi = 0$. Together with the above estimate for $\widehat{\chi}'$, the estimate (8.94) for $(f, \underline{f}, \lambda)$, and the estimates (3.52) for the Ricci coefficients of ${}^{(ext)}\widetilde{\mathcal{L}_0} \subset {}^{(ext)}\mathcal{L}_0$, we infer

$$\max_{k\leq k_{large}} r\|\not{\mathfrak{d}}'^k{\not{d}}_2^{*\prime}f\|_{L^2(S_1')} \lesssim \epsilon_0+\epsilon^2\lesssim\epsilon_0.$$

Together with the Hodge elliptic estimates of Lemma 5.28, chapter 5, we infer

$$\max_{k\leq k_{large}+1} \|\not{\mathfrak{d}}'^k f\|_{L^2(S_1')} \lesssim \epsilon_0+r^2|({\not{d}}_1' f)_{\ell=1}|. \tag{8.104}$$

Step 11. Next, recall from Proposition 2.12 the transformation formula

$$\begin{aligned}\lambda^{-1}\,{}^{(a)}\mathrm{tr}\chi' &= {}^{(a)}\mathrm{tr}\chi+\mathrm{curl}'f+f\wedge\eta+f\wedge\zeta+\underline{f}\wedge\xi\\ &+\frac{1}{4}\left(\underline{f}\wedge f\mathrm{tr}\,\chi+(f\cdot\underline{f})\,{}^{(a)}\mathrm{tr}\chi\right)\\ &+\omega f\wedge\underline{f}-\frac{1}{4}|f|^2\,{}^{(a)}\mathrm{tr}\underline{\chi}-\frac{1}{4}(f\cdot\underline{f})\lambda^{-1}\,{}^{(a)}\mathrm{tr}\chi'\\ &+\frac{1}{4}\lambda^{-1}(f\wedge\underline{f})\mathrm{tr}\,\chi'+\text{ l.o.t.}\end{aligned}$$

Since ${}^{(a)}\mathrm{tr}\chi' = {}^{(a)}\mathrm{tr}\chi = 0$ and ${}^{(a)}\mathrm{tr}\underline{\chi} = {}^{(a)}\mathrm{tr}\underline{\chi}' = 0$ as both frame are outgoing geodesic and hence integrable, and also since $\omega = 0$ and $\xi = 0$, we infer

$$\mathrm{curl}'f = -f\wedge\eta-f\wedge\zeta-\frac{1}{4}\underline{f}\wedge f\kappa-\frac{1}{4}\lambda^{-1}(f\wedge\underline{f})\kappa'+\text{ l.o.t.}$$

Together with the estimate (8.94) for $(f, \underline{f}, \lambda)$, and our estimates for the Ricci coefficients of ${}^{(ext)}\widetilde{\mathcal{L}_0}$, we deduce

$$\max_{k\leq k_{large}+7} r\|\mathfrak{d}'^k \mathrm{curl}' f\|_{L^2(S_1')} \lesssim \epsilon_0.$$

Together with the estimate for f of Step 10, and since $\not{d}_1' = (\mathrm{div}', \mathrm{curl}')$, we infer

$$\max_{k\leq k_{large}+8} \|\mathfrak{d}'^k f\|_{L^2(S_1')} \lesssim \epsilon_0 + r^2|(\mathrm{div}' f)_{\ell=1}|. \tag{8.105}$$

Step 12. In view of Step 11, it remains to control $(\mathrm{div}' f)_{\ell=1}$. We begin by making the following local bootstrap assumptions

$$\sup_{\mathcal{C}_1'}\left(r^2|\mathfrak{d}'^{\leq 5}(\check{\kappa}', \widehat{\chi}', \zeta')| + r^2|\mathfrak{d}'^{\leq 4}\zeta'| + r^3|\mathfrak{d}'^{\leq 4}\beta'| + \left|\frac{r'}{r} - 1\right|\right) \leq \epsilon, \tag{8.106}$$

where $\check{\kappa}' = \kappa' - \frac{2}{r'}$, and where we recall that $\mathcal{C}_1'$ denotes the portion of the past directed outgoing null cone initialized on the sphere S_1' and restricted to $r' \geq \delta_* \epsilon_0^{-1}$. Recall also that $\mathcal{C}_1' \subset {}^{(ext)}\widetilde{\mathcal{L}_0}$ and that r denotes the area radius for the outgoing geodesic foliation of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ while r' denotes the area radius of the spheres $S' \subset \mathcal{C}_1'$. The local bootstrap assumptions (8.106) will be improved in Step 14. The goal of this step is to prove the following.

Lemma 8.32. *The following estimate holds true*

$$\sup_{\mathcal{C}_1'} r^2 |(div'(f))_{\ell=1}| \lesssim \epsilon_0. \tag{8.107}$$

Proof. We proceed in four steps.

Step 12a. We start by deriving an estimate for $\nabla' \log\lambda$ and $\underline{f}$. In view of Corollary 2.13, since ${}^{(a)}\mathrm{tr}\chi = 0$ and $\xi = \omega = 0$, we have

$$\lambda^{-1} e_4'(\log\lambda) = 2f\cdot\zeta + E_2(f, \Gamma)$$

and hence

$$\nabla_4'(r'\nabla' \log\lambda) = r'\nabla'\left(2\lambda f\cdot\zeta + \lambda E_2(f,\Gamma)\right) + [\nabla_4', r'\nabla'](\log\lambda)$$

$$
\begin{aligned}
&= r'\nabla'\left(2\lambda f\cdot\zeta+\lambda E_2(f,\Gamma)\right)\\
&\quad +r'\left(-\frac{1}{2}(\check{\kappa}'-\overline{\check{\kappa}'})\nabla'-\widehat{\chi}'\cdot\nabla'\right)\log\lambda.
\end{aligned}
$$

Together with the bootstrap assumptions (8.106) for $\check{\kappa}'$ and $\widehat{\chi}'$, the bootstrap assumption (8.93) for (f,λ) along $\mathcal{C}_1'$, and the estimates (3.52) for the Ricci coefficients of the part ${}^{(ext)}\widetilde{\mathcal{L}_0}$ of the initial data layer, we infer along $\mathcal{C}_1'$

$$
|\nabla_4'(r'\nabla'\log\lambda)| \lesssim \frac{\epsilon_0+\epsilon^2}{r'^2}\lesssim\frac{\epsilon_0}{r'^2}.
$$

Integrating from S_1' where $\nabla'\log\lambda$ verifies (8.99), we deduce along $\mathcal{C}_1'$

$$
r'|\nabla'\lambda| \lesssim \epsilon_0.
$$

Also, using the transformation formula for ζ', we derive

$$
|\underline{f}| \lesssim r\Big(|\nabla'\lambda|+|\zeta'|+|\widehat{\chi}'|+|\check{\kappa}'|+|\Gamma_g|\Big)+|\not{\mathfrak{d}}'^{\leq 1}f|+\text{ l.o.t.},
$$

and hence, together with the above estimate for $\nabla'\lambda$, the bootstrap assumptions (8.106) for ζ', $\check{\kappa}'$ and $\widehat{\chi}'$, the bootstrap assumption (8.93) for f, and the properties of the background foliation of ${}^{(ext)}\widetilde{\mathcal{L}_0}$, we infer along $\mathcal{C}_1'$, recalling that $r'\gtrsim\epsilon_0^{-1}$ along $\mathcal{C}_1'$,

$$
|\underline{f}| \lesssim \epsilon_0+\frac{\epsilon}{r}\lesssim\epsilon_0
$$

and hence, we have obtained along $\mathcal{C}_1'$

$$
r'|\nabla'\lambda|+|\underline{f}| \lesssim \epsilon_0. \tag{8.108}
$$

Step 12b. We derive next an estimate for $(\mathrm{div}'\beta')_{\ell=1}$ along $\mathcal{C}_1'$.

We start with the following identity for the outgoing geodesic foliation initialized on S_1'

$$
\begin{aligned}
e_4'\left(r'^3\int_{S'}\mathrm{div}'\beta' J'^{(p)}\right) &= r'^3\int_{S'}\Big(e_4'\mathrm{div}'\beta'+\kappa'\mathrm{div}'\beta'\Big)J'^{(p)}\\
&\quad +3e_4'(r')r'^2\int_{S'}\mathrm{div}'\beta' J'^{(p)}\\
&= r'^3\int_{S'}\Big(\mathrm{div}'\nabla_4'\beta'+[\nabla_4',\mathrm{div}']\beta'+\kappa'\mathrm{div}'\beta'\Big)J'^{(p)}\\
&\quad +\frac{3}{2}\overline{\kappa'}r'^3\int_{S'}\mathrm{div}'\beta' J'^{(p)}.
\end{aligned}
$$

Using the Bianchi identity for $\nabla'_4\beta'$ and the structure of the commutator, we infer

$$\begin{aligned}
& e'_4\left(r'^3\int_{S'}\mathrm{div}'\beta' J'^{(p)}\right)\\
= \;& r'^3\int_{S'}\Big(\mathrm{div}'\mathrm{div}'\alpha' - 2\nabla'\kappa'\cdot\beta' + \mathrm{div}'(\alpha'\cdot\zeta') - \frac{1}{2}\kappa'\zeta'\cdot\beta'\\
& +|\beta'|^2 - \widehat{\chi}'\cdot\nabla'\beta' + \zeta'\cdot\widehat{\chi}'\cdot\beta'\Big)J'^{(p)} - \frac{3}{2}(\check{\kappa}' - \overline{\check{\kappa}'})r'^3\int_{S'}\mathrm{div}'\beta' J'^{(p)},
\end{aligned}$$

which yields, after integration by parts,

$$\begin{aligned}
& e'_4\left(r'^3\int_{S'}\mathrm{div}'\beta' J'^{(p)}\right)\\
= \;& r'^3\int_{S'}\alpha' \, {\not{d}}_2^*{\not{d}}_1^*(J'^{(p)},0) + r'^3\int_{S'}\Big(- 2\nabla'\kappa'\cdot\beta' + \mathrm{div}'(\alpha'\cdot\zeta')\\
& -\frac{1}{2}\kappa'\zeta'\cdot\beta' + |\beta'|^2 - \widehat{\chi}'\cdot\nabla'\beta' + \zeta'\cdot\widehat{\chi}'\cdot\beta'\Big)J'^{(p)}\\
& -\frac{3}{2}(\check{\kappa}' - \overline{\check{\kappa}'})r'^3\int_{S'}\mathrm{div}'\beta' J'^{(p)}.
\end{aligned}$$

Together with the bootstrap assumptions (8.106) and the control of the quantity ${\not{d}}_2^*{\not{d}}_1^*(J'^{(p)},0)$ provided by Lemma 8.23, we obtain along $\mathcal{C}'_1$,

$$\left|e'_4\left(r'^3\int_{S'}\mathrm{div}'\beta' J'^{(p)}\right)\right| \;\lesssim\; \frac{\epsilon^2}{r'^{\frac{3}{2}}} \lesssim \frac{\epsilon_0}{r'^{\frac{3}{2}}}.$$

Transporting along $\mathcal{C}'_1$ from S'_1, and using the control of $(\mathrm{div}'\beta')_{\ell=1}$ in (8.91) on S'_1, we infer

$$\sup_{\mathcal{C}'_1} r'^5\,|(\mathrm{div}'\beta')_{\ell=1}| \;\lesssim\; \epsilon_0.$$

In particular, consider the sphere $S'(\delta_*\epsilon_0^{-1}) = \mathcal{C}'_1\cap\{r' = \delta_*\epsilon_0^{-1}\}$. Then

$$r'^5\,|(\mathrm{div}'\beta')_{\ell=1}| \;\lesssim\; \epsilon_0 \quad \text{on} \quad S'(\delta_*\epsilon_0^{-1}). \tag{8.109}$$

Step 12c. We next use estimate (8.109) to derive an estimate for $(\mathrm{div}'(f))_{\ell=1}$ on $S'(\delta_*\epsilon_0^{-1})$, see (8.110).

To this end, we invoke again the transformation formula

$$\beta' \;=\; \lambda\left(\beta + \frac{3}{2}(f\rho + {}^*f\;{}^*\rho) + \frac{1}{2}\alpha\cdot\underline{f} + \text{ l.o.t.}\right)$$

from which we derive

$$
\begin{aligned}
\mathrm{div}'\beta' &= \lambda\left(\mathrm{div}'\beta + \frac{3}{2}\mathrm{div}'(f\rho + {}^*f\ {}^*\rho) + \frac{1}{2}\mathrm{div}'(\alpha\cdot\underline{f}) + \text{ l.o.t.}\right)\\
&\quad +\nabla'\lambda\cdot\left(\beta + \frac{3}{2}(f\rho + {}^*f\ {}^*\rho) + \frac{1}{2}\alpha\cdot\underline{f} + \text{ l.o.t.}\right)\\
&= \mathrm{div}\,\beta + \frac{1}{2}f\cdot\nabla_3\beta + \frac{1}{2}\underline{f}\cdot\nabla_4\beta - \frac{3m}{r'^3}\mathrm{div}'(f)\\
&\quad +3m\left(\frac{1}{r^3} - \frac{1}{r'^3}\right)\mathrm{div}'(f)\\
&\quad +\frac{3}{2}\left(\mathrm{div}'(f)\left(\rho + \frac{2m}{r^3}\right) + \mathrm{curl}'(f)\ {}^*\rho\right)\\
&\quad +\frac{3}{2}\Bigg(f\cdot\nabla\rho + \frac{1}{2}f\cdot(f\nabla_3 + \underline{f}\nabla_4)\rho + {}^*f\cdot\nabla\ {}^*\rho\\
&\quad +\frac{1}{2}\ {}^*f\cdot(f\nabla_3 + \underline{f}\nabla_4)\ {}^*\rho\Bigg)\\
&\quad +\frac{1}{2}\mathrm{div}\,(\alpha\cdot\underline{f}) + (\lambda - 1)\left(\mathrm{div}\,\beta + \frac{3}{2}(\mathrm{div}'(f)\rho + \mathrm{curl}'(f)\ {}^*\rho)\right)\\
&\quad +\nabla'\lambda\cdot\left(\beta + \frac{3}{2}(f\rho + {}^*f\ {}^*\rho)\right) + \text{ l.o.t.}
\end{aligned}
$$

Together with the bootstrap assumptions (8.93) for $(f, \underline{f}, \lambda)$, the bootstrap assumption (8.106) for $r - r'$, and the control provided by Proposition 8.20 for ${}^{(ext)}\widetilde{\mathcal{L}_0}$, we infer

$$
\begin{aligned}
& r^5\left|\mathrm{div}'\beta' + \frac{3m}{r'^3}\mathrm{div}'(f) - \mathrm{div}\,\beta - \frac{1}{2}f\cdot\nabla_3\beta - \frac{1}{2}\underline{f}\cdot\nabla_4\beta - \nabla'\lambda\cdot\beta\right|\\
\lesssim\ & r^{\frac{1}{2}}\epsilon\epsilon_0 + \epsilon^2\\
\lesssim\ & r^{\frac{1}{2}}\epsilon_0^{\frac{5}{3}} + \epsilon_0
\end{aligned}
$$

where we used the fact that $\epsilon = \epsilon_0^{\frac{2}{3}}$. Also, using the Bianchi identities[175] for $\nabla_4\beta$ and $\nabla_3\beta$, the control of $(\underline{f}, \lambda)$ provided by (8.108), we obtain along $\mathcal{C}_1'$

[175]Concerning the term $\underline{f}\cdot\nabla_4\beta$, note that Bianchi identities imply $\nabla_4\beta = -2\mathrm{tr}\,\chi\beta + \text{ l.o.t.}$, thus, since $\beta = \frac{3a_0m_0}{r^4}f_0 + O(\epsilon_0 r^{-7/2})$, we obtain $\underline{f}\cdot\nabla_4\beta = -\frac{12a_0m_0}{r^5}f_0\cdot\underline{f} + O(r^{-\frac{9}{2}}\epsilon_0)\underline{f} = O(r^{-5} + r^{-\frac{9}{2}}\epsilon_0)\underline{f}$ so that $\underline{f}\cdot\nabla_4\beta = O(r^{-5}\epsilon_0 + r^{-\frac{9}{2}}\epsilon_0^2)$ thanks to the estimate (8.108) for $\underline{f}$.

$$r^5 \left| \mathrm{div}\,'\beta' + \frac{3m}{r'^3}\mathrm{div}\,'(f) - \mathrm{div}\,\beta \right| \lesssim r^{\frac{1}{2}}\epsilon_0^{\frac{5}{3}} + \epsilon_0 + r^{-1}.$$

Thus, on the sphere $S'(\delta_*\epsilon_0^{-1})$, where $r' = \delta_*\epsilon_0^{-1}$, we infer, see also Remark 8.33 below,

$$\sup_{S'(\delta_*\epsilon_0^{-1})} r^5 \left| \mathrm{div}\,'\beta' + \frac{3m}{r'^3}\mathrm{div}\,'(f) \right| \lesssim \epsilon_0.$$

Remark 8.33. *In the estimate above, we used in particular the following estimate of Proposition 8.20 for the control of* $div\,\beta$ *which relies on the stronger bound on the initial data layer norm* ${}^{(ext)}\mathfrak{I}_3 \le \epsilon_0^2$, *i.e.*

$$\sup_{{}^{(ext)}\widetilde{\mathcal{L}_0}\cap\{r\sim\epsilon_0^{-1}\}} \Big(r^5 |div\,\beta| \Big) \lesssim \epsilon_0.$$

This yields on $S'(\delta_*\epsilon_0^{-1})$

$$mr^2 |(\mathrm{div}\,'(f))_{\ell=1}| \lesssim \epsilon_0 + r^5 |(\mathrm{div}\,'\beta')_{\ell=1}|.$$

Since $|m - m_0| \le \epsilon$ and[176] $|r - r'| \le \epsilon r$, together with the estimate (8.109) for $r^5|(\mathrm{div}\,'\beta')_{\ell=1}|$ on $S'(\delta_*\epsilon_0^{-1})$, we infer on $S'(\delta_*\epsilon_0^{-1})$

$$m_0 r^2 |(\mathrm{div}\,'(f))_{\ell=1}| \lesssim \epsilon_0$$

and hence

$$r^2 |(\mathrm{div}\,'(f))_{\ell=1}| \lesssim \epsilon_0 \quad \text{on} \quad S'(\delta_*\epsilon_0^{-1}). \tag{8.110}$$

Step 12d. We will next propagate forward the information provided by estimate (8.110).

We use the following identity for the outgoing geodesic foliation initialized on S'_1

$$\begin{aligned} e'_4 \left(\int_{S'} \mathrm{div}\,'(f) J'^{(p)} \right) &= \int_{S'} \Big(e'_4 \mathrm{div}\,'(f) + \kappa' \mathrm{div}\,'(f) \Big) J'^{(p)} \\ &= \int_{S'} \Big(\mathrm{div}\,'\nabla'_4(f) + [\nabla'_4, \mathrm{div}\,']f + \kappa' \mathrm{div}\,'(f) \Big) J'^{(p)}. \end{aligned}$$

Also recall, see Corollary 2.13, that f satisfies the transport equation[177]

[176] By the local bootstrap assumption (8.106).

[177] Recall that ${}^{(a)}\mathrm{tr}\chi = 0$ and $\omega = \xi = 0$ for the outgoing geodesic foliation of ${}^{(ext)}\widetilde{\mathcal{L}_0}$.

along $\mathcal{C}'_1$

$$\nabla_{\lambda^{-1}e'_4} f + \frac{1}{2}\kappa f \quad = \quad -f\cdot\widehat{\chi} + E_1(f,\Gamma).$$

Plugging in the above, and using a commutator formula for $[\nabla'_4, \mathrm{div}\,']$, we infer

$$\begin{aligned}
& e'_4\left(\int_{S'} \mathrm{div}\,'(f) J'^{(p)}\right)\\
= & \int_{S'}\left[\mathrm{div}\,'\left(\lambda\left(-\frac{1}{2}\kappa f - f\cdot\widehat{\chi} + E_1(f,\Gamma)\right)\right)\right.\\
+ & \left.\left(-\frac{1}{2}\kappa'\mathrm{div}\,' - \frac{1}{2}\kappa'\zeta'\cdot + {}^*\beta'\cdot\ {}^* - \widehat{\chi}'\cdot\nabla' + \zeta'\cdot\widehat{\chi}'\cdot\right) f + \kappa'\mathrm{div}\,'(f)\right] J'^{(p)}\\
= & \int_{S'}\left[-\frac{1}{2}\nabla'\kappa'\cdot f - \frac{1}{2}\mathrm{div}\,'((\lambda\kappa-\kappa')f) + \mathrm{div}\,'(\lambda(-f\cdot\widehat{\chi} + E_1(f,\Gamma)))\right.\\
& \left. + \left(-\frac{1}{2}\kappa'\zeta'\cdot + {}^*\beta'\cdot\ {}^* - \widehat{\chi}'\cdot\nabla' + \zeta'\cdot\widehat{\chi}'\cdot\right) f\right] J'^{(p)}.
\end{aligned}$$

In view of the bootstrap assumption (8.93) for f and λ, and the bootstrap assumptions (8.106), we deduce

$$\left|e'_4\left(\int_{S'} \mathrm{div}\,'(f) J'^{(p)}\right)\right| \lesssim \frac{\epsilon_0+\epsilon^2}{r^2} + \epsilon\left|\mathfrak{d}'^{\le 1}\left(\kappa - \lambda^{-1}\kappa'\right)\right| + r\left|\mathfrak{d}'^{\le 1}\left(E_1(f,\Gamma)\right)\right|.$$

In view of the form of the error term E_1 in Corollary 2.13, using the transformation formula for κ', together with the bootstrap assumption (8.93) for f, and the bootstrap assumptions (8.106), we obtain

$$\left|e'_4\left(\int_{S'} \mathrm{div}\,'(f) J'^{(p)}\right)\right| \quad \lesssim \quad \frac{\epsilon_0+\epsilon^2}{r^2} \lesssim \frac{\epsilon_0}{r^2}.$$

Integrating forward from $r = \epsilon_0^{-1}$, and using estimate (8.110) for $(\mathrm{div}\,'(f))_{\ell=1}$ on $S'(\delta_*\epsilon_0^{-1})$, we obtain

$$\sup_{\mathcal{C}'_1} r^2\left|(\mathrm{div}\,'(f))_{\ell=1}\right| \quad \lesssim \quad \epsilon_0.$$

This ends the proof of Lemma 8.32. □

Step 13. Combining the estimate (8.105) for $\not{\mathfrak{d}}'^k f$ of Step 11 and the estimate for $(\text{div}'(f))_{\ell=1}$ of Step 12, we obtain on $S_1 = S_1'$,

$$\max_{k\leq k_{large}+1} \|\not{\mathfrak{d}}'^k f\|_{L^2(S_1')} \lesssim \epsilon_0.$$

Together with (8.99), we infer

$$\|f\|_{\mathfrak{h}_{k_{large}+1}(S_1')} + r^{-1}\|(\underline{f}, \lambda - \overline{\lambda}^{S_1'})\|_{\mathfrak{h}_{k_{large}+1}(S_1')} \lesssim \epsilon_0.$$

In particular, the above estimate for $(f, \underline{f})$ allows to reapply Lemma 7.3 in [40] (restated here in Lemma 8.4), with $\delta_1 = \epsilon_0$ which yields

$$\sup_{S_1'} \left|\frac{r'}{r} - 1\right| \lesssim \epsilon_0.$$

Together with (8.100), we infer

$$\|f\|_{\mathfrak{h}_{k_{large}+1}(S_1')} + r^{-1}\|(\underline{f}, \log\lambda)\|_{\mathfrak{h}_{k_{large}+1}(S_1')} \lesssim \epsilon_0.$$

This improves the iteration assumption (8.94).

We next appeal to Corollary 4.2 in [50] (restated here in Corollary 8.12) with $\overset{\circ}{\delta} = \epsilon_0$, with background foliation given by ${}^{(ext)}\widetilde{\mathcal{L}_0}$ and $s_{max} = k_{large}$ which allows us to make use of the above estimate for $(f, \underline{f}, \lambda)$ on $S_1' \subset \Sigma_*$ to derive

$$\sup_{k\leq k_{large}+1} \Big(\|\mathfrak{d}^k f\|_{L^2(S_1')} + r^{-1}\|\mathfrak{d}^k(\underline{f}, \log\lambda)\|_{L^2(S_1')} + \|\mathfrak{d}^{\leq k-1}\nabla_\nu'(\underline{f}, \log\lambda)\|_{L^2(S_1')}\Big)$$
$$\lesssim \epsilon_0.$$

The above control of $(f, \underline{f})$, together with Lemma 7.3 in [40] (restated here in Lemma 8.4) for $\delta_1 = \epsilon_0$, and Corollary 7.7 in [40] (restated here in Lemma 8.4) with $\overset{\circ}{\epsilon} = \epsilon_0$, implies

$$\sup_{S_1'} \left(\left|\frac{m_H'}{m_0} - 1\right| + \left|\frac{r'}{r} - 1\right|\right) \lesssim \epsilon_0,$$

where m_H' denotes the Hawking mass of S'.

We appeal next to the argument used to derive estimate $m_H' - m$ in Proposition 5.52 which leads to

$$\sup_{\Sigma_*} u^{1+2\delta_{dec}}|m_H' - m| \lesssim \epsilon_0.$$

We have thus obtained on S'_1

$$\begin{aligned}
&\sup_{k\le k_{large}+1}\Big(\|\mathfrak{d}^k f\|_{L^2(S'_1)}+r^{-1}\|\mathfrak{d}^k(\underline{f},\log\lambda)\|_{L^2(S'_1)}\\
&\qquad +\|\mathfrak{d}^{\le k-1}\nabla'_\nu(\underline{f},\log\lambda)\|_{L^2(S'_1)}\Big)\lesssim\epsilon_0,\\
&\sup_{S'_1}\left(\left|\frac{m}{m_0}-1\right|+\left|\frac{r'}{r}-1\right|\right)\lesssim\epsilon_0.
\end{aligned} \tag{8.111}$$

In view of (8.111) and the transformation formulas from the frame[178] of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ to that of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ we deduce, in addition to $\check{\kappa}'=0$, $\xi'=0$, $\omega'=0$ and $\underline{\eta}'=\zeta'$,

$$\begin{aligned}
&r\|\mathfrak{d}^{\le k_{large}}\widehat{\chi}'\|_{L^2(S'_1)}+r\|\mathfrak{d}^{\le k_{large}-1}\zeta'\|_{L^2(S'_1)}\\
&+\|\mathfrak{d}^{\le k_{large}}(\eta',\widecheck{\mathrm{tr}\underline{\chi}}',\widehat{\underline{\chi}}',\check{\underline{\omega}}',\underline{\xi}')\|_{L^2(S'_1)}\ \lesssim\ \epsilon_0.
\end{aligned} \tag{8.112}$$

Remark 8.34. *The estimate for $\widecheck{\mathrm{tr}\underline{\chi}}'$ in (8.112) displays a loss of r^{-1} with respect to the expected behavior. It is due to the anomalous behavior for $\underline{f}$ and λ in (8.111). A priori, the same loss should also occur for ζ' which would then create problems in Step 22. To avoid this issue, we estimate first β' using the transformation formulas and the control of $(f,\underline{f},\lambda)$ and $r'-r$ in (8.111) to obtain*

$$r^2\|\mathfrak{d}^{\le k_{large}}\beta'\|_{L^2(S'_1)}\ \lesssim\ \epsilon_0.$$

We can then use the Codazzi equation for $\widehat{\chi}'$ to estimate ζ', Indeed together with the estimate for $\widehat{\chi}'$ in (8.112) and the fact that $\check{\kappa}'=0$, we obtain the claimed estimate for ζ' in (8.112), where the loss of one derivative is due to the term $\mathrm{div}'\widehat{\chi}'$ in Codazzi.

Step 14. In this step, we improve the bootstrap assumption (8.93) for $(f,\underline{f},\lambda)$ and the bootstrap assumptions (8.106) on $(\check{\kappa}',\widehat{\chi}',\zeta')$, β' and $r'-r$. In view of Corollary 2.13, and since ${}^{(a)}\mathrm{tr}\chi=0$ and $\xi=\omega=0$, we have

$$\begin{aligned}
\nabla_{\lambda^{-1}e'_4}f+\frac{1}{2}\kappa f &= -f\cdot\widehat{\chi}+E_1(f,\Gamma),\\
\lambda^{-1}e'_4(\log\lambda) &= 2f\cdot\zeta+E_2(f,\Gamma).
\end{aligned}$$

Since

[178] Recall that $S'_1=S_1$ and the primed frame on S'_1 coincides with that induced by Σ_*.

$$\lambda^{-1}e_4' = e_4 + f^a e_a + \frac{1}{4}|f|^2 e_3, \qquad e_4(r) = \frac{r}{2}\overline{\kappa}, \qquad e_4(e_3(r)) = -2\underline{\omega},$$

we infer

$$\nabla_{\lambda^{-1}e_4'}(rf) = -\frac{r}{2}(\check{\kappa} - \overline{\check{\kappa}})f - rf\cdot\widehat{\chi} + rE_1(f,\Gamma) + \frac{1}{4}|f|^2 e_3(r)f.$$

Then, we proceed as follows for the estimates of $(f, \underline{f}, \lambda)$, $(\check{\kappa}', \widehat{\chi}', \zeta')$, β' and $r' - r$:

1. Integrating the above transport equations for f and λ from S_1' where (8.111) holds, we obtain on $\mathcal{C}_1'$

$$r'|\mathfrak{d}'^{\leq 5} f| + |\mathfrak{d}'^{\leq 5}\log\lambda| \lesssim \epsilon_0. \tag{8.113}$$

2. We estimate $(\check{\kappa}', \widehat{\chi}', \zeta')$ and β' as follows:
   (a) one first controls 5 derivatives of α' relying on the corresponding change of frame formula in Proposition 2.12, the control of the foliation of ${}^{(ext)}\widetilde{\mathcal{L}_0}$, and using in particular the fact that the change of frame formula for α' only involves (f, λ) but not $\underline{f}$,
   (b) one then controls 5 derivatives $\widehat{\chi}'$ using the null structure equation for $\nabla_4'\widehat{\chi}'$, the above control for 5 derivatives of α', and integrating from S_1' where $\widehat{\chi}'$ is under control from (8.112),
   (c) one then also controls 5 derivatives of $\check{\kappa}'$ by integrating Raychadhuri from S_1' where $\check{\kappa}' = 0$ in view of the GCM condition on Σ_*,
   (d) then, using the transformation formula for β', the control of the foliation of ${}^{(ext)}\widetilde{\mathcal{L}_0}$, and the above control of f and λ, we control 4 derivatives β',
   (e) then, we control 4 derivatives of ζ' from the codazzi for $\widehat{\chi}'$, thanks to the control of 4 derivatives of β', and of 5 derivatives of $\widehat{\chi}'$ and $\check{\kappa}'$.

   The above steps thus lead to the following control on $\mathcal{C}_1'$

$$r'^2|\mathfrak{d}'^{\leq 5}(\widehat{\chi}', \check{\kappa}')| + r'^2|\mathfrak{d}'^{\leq 4}\zeta'| + r'^3|\mathfrak{d}'^{\leq 4}\beta'| + r'^{\frac{7}{2}}|\mathfrak{d}'^{\leq 5}\alpha'| \lesssim \epsilon_0. \tag{8.114}$$

3. Using the transformation formulas for ζ' and the control of 5 derivatives of f and λ, and of 4 derivatives of ζ', we obtain the control of 4 derivatives of $\underline{f}$ on $\mathcal{C}_1'$:

$$|\mathfrak{d}'^{\leq 4}\underline{f}| \lesssim \epsilon_0. \tag{8.115}$$

4. Finally, the above control of $(f, \underline{f})$, together with Lemma 7.3 in [40] (recalled here in Lemma 8.4) for $\delta_1 = \epsilon_0$ implies the following control of $r' - r$ on $\mathcal{C}'_1$:

$$|r' - r| \lesssim \epsilon_0 r. \tag{8.116}$$

In view of the above, we have improved the bootstrap assumption (8.93) for $(f, \underline{f}, \lambda)$ and the bootstrap assumptions (8.106) on $(\check{\kappa}', \widehat{\chi}', \zeta')$, β' and $r' - r$.

Proceeding as above for β', and using in addition the above estimate for $r' - r$, we get the following improved estimates for β'

$$|\mathfrak{d}'^{\leq 4}\beta'| \lesssim \frac{\epsilon_0}{r'^{\frac{7}{2}}} + \frac{1}{r'^4}. \tag{8.117}$$

Step 15. In Steps 15–16, we estimate[179] $a - a_0$. Proceeding as in Step 12b we obtain

$$\begin{aligned}
& e'_4\left(r'^3 \int_{S'} \mathrm{curl}'\beta' J'^{(p)}\right) \\
= \; & r'^3 \int_{S'} \alpha' \, \not{d}_2^* \not{d}_1^*(0, J'^{(p)}) + r'^3 \int_{S'} \Big(-2\, {}^*\nabla'\kappa' \cdot \beta' + \mathrm{curl}'(\alpha' \cdot \zeta') \\
& -\frac{1}{2}\kappa'\zeta' \cdot {}^*\beta' - {}^*\widehat{\chi}' \cdot \nabla'\beta' + \zeta' \cdot {}^*\widehat{\chi}' \cdot \beta'\Big) J'^{(p)} \\
& -\frac{3}{2}(\check{\kappa}' - \overline{\check{\kappa}'}) r'^3 \int_{S'} \mathrm{curl}'\beta' J'^{(p)}.
\end{aligned}$$

Together with the control of $(\check{\kappa}', \widehat{\chi}', \zeta', \beta', \alpha')$ from Step 14 and the control of $\not{d}_2^* \not{d}_1^*(0, J'^{(p)})$ provided by Lemma 8.23, we obtain along $\mathcal{C}'_1$

$$\left| e'_4\left(r'^3 \int_{S'} \mathrm{curl}'\beta' J'^{(p)} - 8\pi m a \delta_{p0}\right)\right| \lesssim \frac{\epsilon_0}{r'^{\frac{3}{2}}}.$$

Transporting along $\mathcal{C}'_1$ from S'_1, using the control of $(\mathrm{curl}'\beta')_{\ell=1}$ in (8.79) on Σ_*, and hence on S'_1, we infer

$$\sup_{\mathcal{C}'_1} r'^5 \left(|(\mathrm{curl}'\beta')_{\ell=1,\pm}| + \left| (\mathrm{curl}'\beta')_{\ell=1,0} - \frac{2am}{r'^5} \right| \right) \lesssim \epsilon_0.$$

In particular, consider the sphere $S'(\delta_*\epsilon_0^{-1}) = \mathcal{C}'_1 \cap \{r' = \delta_*\epsilon_0^{-1}\}$. Then

$$r'^5 \left(|(\mathrm{curl}'\beta')_{\ell=1,\pm}| + \left| (\mathrm{curl}'\beta')_{\ell=1,0} - \frac{2am}{r'^5} \right| \right) \lesssim \epsilon_0 \quad \text{on} \quad S'(\delta_*\epsilon_0^{-1}).$$

[179] Recall that a is defined in Section 3.2.4.

Also, using the change of frame formula for β' in Proposition 2.12, the control for $(f, \underline{f}, \lambda)$ of Step 14, and the control of the the curvature components of ${}^{(ext)}\widetilde{\mathcal{L}}_0$, we have

$$|\mathrm{curl}'\beta' - \mathrm{curl}\,\beta| \lesssim \frac{\epsilon_0}{r'^5} + \frac{\epsilon_0^2}{r'^{\frac{9}{2}}}.$$

Together with the above, we obtain on $S'(\delta_*\epsilon_0^{-1})$

$$r'^3\left(\left|\int J'^{(+)}\mathrm{curl}\,\beta\right| + \left|\int J'^{(-)}\mathrm{curl}\,\beta\right| + \left|\int_{S'(\delta_*\epsilon_0^{-1})} J'^{(0)}\mathrm{curl}\,\beta - \frac{8\pi m a}{r'^3}\right|\right) \lesssim \epsilon_0.$$

Using the estimates for $m - m_0$ of Step 13, we deduce

$$\begin{aligned} r'^3\Bigg(\left|\int_{S'(\delta_*\epsilon_0^{-1})} J'^{(+)}\mathrm{curl}\,\beta\right| + \left|\int_{S'(\delta_*\epsilon_0^{-1})} J'^{(-)}\mathrm{curl}\,\beta\right| \\ + \left|\int_{S'(\delta_*\epsilon_0^{-1})} J'^{(0)}\mathrm{curl}\,\beta - \frac{8\pi m_0 a}{r'^3}\right|\Bigg) \lesssim \epsilon_0. \end{aligned}$$

Also, making use of the estimate for $\mathrm{curl}\,\beta$ in Proposition 8.20

$$\mathrm{curl}\,\beta = \frac{6a_0 m_0}{r^5} J^{(0)} + O(r^{-5}\epsilon_0) \quad \text{on } S'(\delta_*\epsilon_0^{-1}).$$

Using the estimates for $r' - r$ of Step 14, this yields on $S'(\delta_*\epsilon_0^{-1})$

$$\mathrm{curl}\,\beta = \frac{6a_0 m_0}{r'^5} J^{(0)} + O(r'^{-5}\epsilon_0).$$

Plugging in the above, and dividing by m_0, we deduce

$$\begin{aligned} r'^{-2}\Bigg(\left|\int_{S'(\delta_*\epsilon_0^{-1})} J'^{(+)} a_0 J^{(0)}\right| + \left|\int_{S'(\delta_*\epsilon_0^{-1})} J'^{(-)} a_0 J^{(0)}\right| \\ + \left|\int_{S'(\delta_*\epsilon_0^{-1})} J'^{(0)} a_0 J^{(0)} - \frac{4\pi a}{3} r'^2\right|\Bigg) \lesssim \epsilon_0. \end{aligned}$$

Now, recall that we have either $a_0 = 0$ or $|a_0| \gg \epsilon_0$. In particular, we have

$$|a| \lesssim \epsilon_0 \quad \text{if} \quad a_0 = 0. \tag{8.118}$$

In the other case, we have, since $|a_0| \gg \epsilon_0$,

$$r'^{-2}\left(\left|\int_{S'(\delta_*\epsilon_0^{-1})} J'^{(+)}J^{(0)}\right| + \left|\int_{S'(\delta_*\epsilon_0^{-1})} J'^{(-)}J^{(0)}\right| \right.$$
$$\left. + \left|\int_{S'(\delta_*\epsilon_0^{-1})} J'^{(0)}J^{(0)} - \frac{4\pi a}{3a_0}r'^2\right|\right) \lesssim \epsilon_0. \tag{8.119}$$

Step 16. In this step, we consider the case $a_0 \neq 0$.

Step 16a. We introduce the following scalar function

$$\widetilde{J} := J^{(0)} - (1+c_0)J'^{(0)}, \qquad c_0 := \frac{\int_{S'(\delta_*\epsilon_0^{-1})} J'^{(0)}J^{(0)}}{\int_{S'(\delta_*\epsilon_0^{-1})} (J'^{(0)})^2} - 1. \tag{8.120}$$

We summarize the results of this step in the lemma below.

Lemma 8.35. *The following estimates hold true*

$$\|\widetilde{J}\|_{L^\infty(S'(\delta_*\epsilon_0^{-1}))} \lesssim \epsilon_0, \qquad |c_0| \lesssim \epsilon_0. \tag{8.121}$$

Proof. In view of the bootstrap assumption (8.95) for $J'^{(p)} - J^{(p)}$, we have

$$|c_0| \lesssim \epsilon. \tag{8.122}$$

In view of the definition of $\widetilde{J}$ and c_0, we have

$$\int_{S'(\delta_*\epsilon_0^{-1})} J'^{(0)}\widetilde{J} = 0,$$

i.e. $(\widetilde{J})_{\ell=1,0} = 0$ on $S'(\delta_*\epsilon_0^{-1})$. Using an elliptic estimate for $\Delta' + \frac{2}{r'^2}$, see Section 5.1.11, we deduce

$$r^{-1}\|\widetilde{J}\|_{\mathfrak{h}_2(S'(\delta_*\epsilon_0^{-1}))} \lesssim r\left\|\left(\Delta' + \frac{2}{r'^2}\right)\widetilde{J}\right\|_{L^2(S'(\delta_*\epsilon_0^{-1}))} + |(\widetilde{J})_{\ell=1,\pm}|.$$

Since $J'^{(p)}$ was extended from S'_1 by $e'_4(J'^{(p)}) = 0$, we have

$$e'_4\left(r'^{-2}\int_{S'} J'^{(p)}J'^{(q)} - \frac{4\pi}{3}\delta_{pq}\right) = r'^{-2}\int_{S'}(\check{\kappa}' - \overline{\check{\kappa}'})J'^{(p)}J'^{(q)} = O\left(\frac{\epsilon_0}{r'^2}\right)$$

where we have used the estimate for $\check{\kappa}'$ of Step 14, see (8.114). Integrating from $S'_1 \subset \Sigma_*$ where we have[180] (8.80), we infer, for any $S' \subset \mathcal{C}'_1$,

$$\left|\int_{S'} J'^{(p)} J'^{(q)} - \frac{4\pi}{3} r'^2 \delta_{pq}\right| \lesssim \epsilon r'. \tag{8.123}$$

Hence, since $r' \geq \delta_* \epsilon_0^{-1}$ on $\mathcal{C}'_1$, we deduce

$$\left|\int_{S'} J'^{(p)} J'^{(q)} - \frac{4\pi}{3} r'^2 \delta_{pq}\right| \lesssim r'^2 \epsilon_0, \qquad S' \subset \mathcal{C}'_1.$$

We deduce, in particular, $|(\widetilde{J})_{\ell=1,\pm}| \lesssim \epsilon_0$ and hence

$$r^{-1} \|\widetilde{J}\|_{\mathfrak{h}_2(S'(\delta_* \epsilon_0^{-1}))} \lesssim r \left\| \left(\Delta' + \frac{2}{r'^2}\right) \widetilde{J} \right\|_{L^2(S'(\delta_* \epsilon_0^{-1}))} + \epsilon_0. \tag{8.124}$$

Also, since we have extended $J'^{(p)}$ from S'_1 by $e'_4(J'^{(p)}) = 0$,

$$\begin{aligned} e'_4 \left[r'^2 \left(\Delta' + \frac{2}{r'^2}\right) J'^{(p)} \right] &= [e'_4, r'^2 \Delta'] J'^{(p)} = \not{\mathfrak{d}}'^{\leq 1} (\check{\kappa}' - \overline{\check{\kappa}'}, \widehat{\chi}', \zeta', r' \beta') \not{\mathfrak{d}}'^{\leq 2} J'^{(p)} \\ &= O(\epsilon_0 r'^{-2}) \end{aligned}$$

where we have used the estimate for $\check{\kappa}'$, $\widehat{\chi}'$, ζ' and β' of Step 14. Integrating from $S'_1 \subset \Sigma_*$, and using the control on Σ_* (and hence on S'_1) provided by Lemma 8.23, we infer along $\mathcal{C}'_1$

$$\left| \left(\Delta' + \frac{2}{r'^2}\right) J'^{(p)} \right| \lesssim \frac{\epsilon}{r'^3}.$$

In particular

$$\left| \left(\Delta' + \frac{2}{r'^2}\right) J'^{(p)} \right| \lesssim \frac{\epsilon_0}{r'^2} \qquad \text{on } S'(\delta_* \epsilon_0^{-1}).$$

In view of the definition of $\widetilde{J}$ we infer from (8.124)

$$r^{-1} \|\widetilde{J}\|_{\mathfrak{h}_2(S'(\delta_* \epsilon_0^{-1}))} \lesssim r \left\| \left(\Delta' + \frac{2}{r'^2}\right) J^{(0)} \right\|_{L^2(S'(\delta_* \epsilon_0^{-1}))} + \epsilon_0.$$

[180] Note the change of notation, the unprimed quantities in (8.80) are primed here.

Together with the control of $r'-r$ of Step 14, this yields

$$r^{-1}\|\widetilde{J}\|_{\mathfrak{h}_2(S'(\delta_*\epsilon_0^{-1}))} \lesssim r^{-1}\left\|\left(r^2\Delta'+2\right)J^{(0)}\right\|_{L^2(S'(\delta_*\epsilon_0^{-1}))}+\epsilon_0.$$

Using the change of frame formula for Δ', and the control of f and $\underline{f}$ in Step 14, we infer

$$r^{-1}\|\widetilde{J}\|_{\mathfrak{h}_2(S'(\delta_*\epsilon_0^{-1}))} \lesssim r^{-1}\left\|\left(r^2\Delta+2\right)J^{(0)}\right\|_{L^2(S'(\delta_*\epsilon_0^{-1}))}+\epsilon_0.$$

Finally, in view of the control we have for the geodesic foliation of ${}^{(ext)}\widetilde{\mathcal{L}_0}$, using also Sobolev, we deduce

$$\|\widetilde{J}\|_{L^\infty(S'(\delta_*\epsilon_0^{-1}))} \lesssim \epsilon_0$$

as stated in Lemma 8.35.

Next, we estimate c_0. Using our control for ${}^{(ext)}\widetilde{\mathcal{L}_0}$, see Proposition 8.20,

$$\left|\int_S (J^{(0)})^2-\frac{4\pi}{3}r^2\right| \lesssim \epsilon_0 r.$$

Using Lemma 7.3 in [40] (restated here in Lemma 8.4) on the comparison between integrals on S and on S', and using also the control of $r'-r$ of Step 14, we obtain

$$\left|\int_{S'(\delta_*\epsilon_0^{-1})}(J^{(0)})^2-\frac{4\pi}{3}r'^2\right| \lesssim r'^2\epsilon_0.$$

Also, in view of (8.123),

$$\left|\int_{S'(\delta_*\epsilon_0^{-1})}(J'^{(0)})^2-\frac{4\pi}{3}r'^2\right| \lesssim r'^2\epsilon_0,$$

and hence

$$\left|\int_{S'(\delta_*\epsilon_0^{-1})}(J^{(0)})^2-\int_{S'(\delta_*\epsilon_0^{-1})}(J'^{(0)})^2\right| \lesssim r'^2\epsilon_0.$$

On the other hand, in view of the above control for $\widetilde{J}$ and its definition, we have

$$\int_{S'(\delta_*\epsilon_0^{-1})}(J^{(0)})^2 = \int_{S'(\delta_*\epsilon_0^{-1})}\left((1+c_0)J'^{(0)}+\widetilde{J}\right)^2$$

$$= (1+c_0)^2 \int_{S'(\delta_* \epsilon_0^{-1})} (J'^{(0)})^2 + 2(1+c_0) \int_{S'(\delta_* \epsilon_0^{-1})} J'^{(0)} \widetilde{J} + \int_{S'(\delta_* \epsilon_0^{-1})} (\widetilde{J})^2.$$

Together with our weak control for c_0 in (8.122) and the above control for $\widetilde{J}$, we deduce

$$2c_0 \int_{S'(\delta_* \epsilon_0^{-1})} (J'^{(0)})^2 = \int_{S'(\delta_* \epsilon_0^{-1})} (J^{(0)})^2 - \int_{S'(\delta_* \epsilon_0^{-1})} (J'^{(0)})^2 + r'^2 O(\epsilon_0 + \epsilon^2).$$

We obtain

$$|c_0| r'^2 \lesssim \left| \int_{S'(\delta_* \epsilon_0^{-1})} (J^{(0)})^2 - \int_{S'(\delta_* \epsilon_0^{-1})} (J'^{(0)})^2 \right| + r'^2 \epsilon_0$$

and hence, in view of the above, we infer

$$|c_0| \lesssim \epsilon_0.$$

This ends the proof of Lemma 8.35. ◻

Step 16b. The goal of this step is to prove the following lemma.

Lemma 8.36. *The following estimates hold true.*

$$|a - a_0| + \max_{p=0,+,-} \sup_{S'_1} \left| J^{(p)} - J'^{(p)} \right| \lesssim \epsilon_0 \quad \textit{if} \quad a_0 \neq 0. \tag{8.125}$$

Proof. In view of the definition of $\widetilde{J}$, and the estimate we have derived for it and c_0 in Lemma 8.35, we have

$$\sup_{S'(\delta_* \epsilon_0^{-1})} |J^{(0)} - J'^{(0)}| \lesssim \sup_{S'(\delta_* \epsilon_0^{-1})} (|\widetilde{J}| + |c_0|) \lesssim \epsilon_0. \tag{8.126}$$

Now, recalling from (8.119) that we have

$$r'^{-2} \left| \int_{S'(\delta_* \epsilon_0^{-1})} J'^{(0)} J^{(0)} - \frac{4\pi a}{3a_0} r'^2 \right| \lesssim \epsilon_0,$$

we infer

$$r'^{-2} \left| \int_{S'(\delta_* \epsilon_0^{-1})} (J'^{(0)})^2 - \frac{4\pi a}{3a_0} r'^2 \right| \lesssim \epsilon_0.$$

Together with the above control of $\int_{S'(\delta_*\epsilon_0^{-1})}(J'^{(0)})^2$, we deduce

$$r'^{-2}\left|\frac{4\pi}{3}r'^2-\frac{4\pi a}{3a_0}r'^2\right|\lesssim\epsilon_0$$

and hence

$$|a-a_0| \quad \lesssim \quad \epsilon_0$$

as stated in (8.125).

It remains to prove the estimate for $|J^{(\pm)}-J'^{(\pm)}|$. To achieve this, we introduce the following scalar functions

$$\begin{aligned}
\widetilde{J}^{+} &:= J^{(+)}-\Big((1+c_{++})J'^{(+)}+c_{+-}J'^{(-)}\Big),\\
\widetilde{J}^{-} &:= J^{(-)}-\Big(c_{-+}J'^{(+)}+(1+c_{--})J'^{(-)}\Big),
\end{aligned}$$

where the constants c_{++} and c_{+-} are the solutions of the following 2 by 2 system

$$\begin{aligned}
& c_{++}\int_{S'(\delta_*\epsilon_0^{-1})}(J'^{(+)})^2+c_{+-}\int_{S'(\delta_*\epsilon_0^{-1})}J'^{(+)}J'^{(-)}\\
= & \int_{S'(\delta_*\epsilon_0^{-1})}(J^{(+)}-J'^{(+)})J'^{(+)},\\
& c_{++}\int_{S'(\delta_*\epsilon_0^{-1})}J'^{(+)}J'^{(-)}+c_{+-}\int_{S'(\delta_*\epsilon_0^{-1})}(J'^{(-)})^2\\
= & \int_{S'(\delta_*\epsilon_0^{-1})}(J^{(+)}-J'^{(+)})J'^{(-)},
\end{aligned}$$

and where the constants c_{-+} and c_{--} are the solutions of the following 2 by 2 system

$$\begin{aligned}
& c_{--}\int_{S'(\delta_*\epsilon_0^{-1})}(J'^{(-)})^2+c_{-+}\int_{S'(\delta_*\epsilon_0^{-1})}J'^{(-)}J'^{(+)}\\
= & \int_{S'(\delta_*\epsilon_0^{-1})}(J^{(-)}-J'^{(-)})J'^{(-)},\\
& c_{--}\int_{S'(\delta_*\epsilon_0^{-1})}J'^{(-)}J'^{(+)}+c_{-+}\int_{S'(\delta_*\epsilon_0^{-1})}(J'^{(+)})^2\\
= & \int_{S'(\delta_*\epsilon_0^{-1})}(J^{(-)}-J'^{(-)})J'^{(+)}.
\end{aligned}$$

In view of the bootstrap assumption (8.95) for $J'^{(p)} - J^{(p)}$, we have

$$|c_{++}| + |c_{+-}| + |c_{-+}| + |c_{--}| \lesssim \epsilon. \tag{8.127}$$

Also, in view of the definition of $\widetilde{J}^{\pm}$ and c_{++}, c_{+-}, c_{-+} and c_{--}, we have

$$\int_{S'(\delta_*\epsilon_0^{-1})} J'^{(\pm)}\widetilde{J}^{+} = 0, \qquad \int_{S'(\delta_*\epsilon_0^{-1})} J'^{(\pm)}\widetilde{J}^{-} = 0,$$

i.e. $(\widetilde{J}^{+})_{\ell=1,\pm} = 0$ and $(\widetilde{J}^{-})_{\ell=1,\pm} = 0$ on $S'(\delta_*\epsilon_0^{-1})$. This yields, together with a Hodge elliptic estimate,

$$r^{-1}\|\widetilde{J}^{\pm}\|_{\mathfrak{h}_2(S'(\delta_*\epsilon_0^{-1}))} \lesssim r\left\|\left(\Delta' + \frac{2}{r'^2}\right)\widetilde{J}^{\pm}\right\|_{L^2(S'(\delta_*\epsilon_0^{-1}))} + |(\widetilde{J}^{\pm})_{\ell=1,0}|.$$

Arguing as above for $\widetilde{J}$, we have

$$\begin{aligned} & r\left\|\left(\Delta' + \frac{2}{r'^2}\right)\widetilde{J}^{\pm}\right\|_{L^2(S'(\delta_*\epsilon_0^{-1}))} \\ \lesssim & \sup_{S'(\delta_*\epsilon_0^{-1})}\left|\left(\Delta' + \frac{2}{r'^2}\right)J^{(\pm)}\right| + \sup_{S'(\delta_*\epsilon_0^{-1})}\left|\left(\Delta' + \frac{2}{r'^2}\right)J'^{(\pm)}\right| \\ \lesssim & \epsilon_0, \end{aligned}$$

and hence

$$r^{-1}\|\widetilde{J}^{\pm}\|_{\mathfrak{h}_2(S'(\delta_*\epsilon_0^{-1}))} \lesssim |(\widetilde{J}^{\pm})_{\ell=1,0}| + \epsilon_0.$$

Also, in view of the definition of $\widetilde{J}^{\pm}$, we have

$$\begin{aligned} |(\widetilde{J}^{\pm})_{\ell=1,0}| \lesssim & r^{-2}\left|\int_{S'(\delta_*\epsilon_0^{-1})}\widetilde{J}^{\pm}J'^{(0)}\right| \\ \lesssim & \sup_{S'(\delta_*\epsilon_0^{-1})}|J'^{(0)} - J^{(0)}| + r^{-2}\left|\int_{S'(\delta_*\epsilon_0^{-1})}J'^{(\pm)}J'^{(0)}\right| \\ & +r^{-2}\left|\int_{S'(\delta_*\epsilon_0^{-1})}J^{(\pm)}J^{(0)}\right|. \end{aligned}$$

In view of the above, we infer

$$|(\widetilde{J}^{\pm})_{\ell=1,0}| \lesssim \epsilon_0 + r^{-2}\left|\int_{S'(\delta_*\epsilon_0^{-1})}J^{(\pm)}J^{(0)}\right|.$$

Also, denoting S_0 the sphere of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ sharing the same south pole with $S'(\delta_*\epsilon_0^{-1})$, we have in view of the control of ${}^{(ext)}\widetilde{\mathcal{L}_0}$

$$\begin{aligned}|(\widetilde{J}^{\pm})_{\ell=1,0}| &\lesssim \epsilon_0 + r^{-2}\left|\int_{S'(\delta_*\epsilon_0^{-1})} J^{(\pm)}J^{(0)} - \int_{S_0} J^{(\pm)}J^{(0)}\right| \\ &\quad + r^{-2}\left|\int_{S_0} J^{(\pm)}J^{(0)}\right| \\ &\lesssim \epsilon_0 + r^{-2}\left|\int_{S'(\delta_*\epsilon_0^{-1})} J^{(\pm)}J^{(0)} - \int_{S_0} J^{(\pm)}J^{(0)}\right|.\end{aligned}$$

Finally, in view of the control of $(f, \underline{f})$ in Step 14, we may apply Lemma 7.3 in [40] (restated here in Lemma 8.4) with $\delta_1 = \epsilon_0$ which yields, together with $e_4(J^{(p)}) = 0$,

$$r^{-2}\left|\int_{S'(\delta_*\epsilon_0^{-1})} J^{(\pm)}J^{(0)} - \int_{S_0} J^{(\pm)}J^{(0)}\right| \lesssim \epsilon_0 \max_{p=0,+,-} \sup_{{}^{(ext)}\widetilde{\mathcal{L}_0}} |\mathfrak{d}^{\leq 1}J^{(p)}| \lesssim \epsilon_0$$

so that

$$|(\widetilde{J}^{\pm})_{\ell=1,0}| \lesssim \epsilon_0$$

and hence

$$r^{-1}\|\widetilde{J}^{\pm}\|_{\mathfrak{h}_2(S'(\delta_*\epsilon_0^{-1}))} \lesssim \epsilon_0.$$

Together with Sobolev, we deduce

$$\sup_{S'(\delta_*\epsilon_0^{-1})} |\widetilde{J}^{\pm}| \lesssim \epsilon_0.$$

Next, using again Lemma 7.3 in [40] (restated here in Lemma 8.4) with $\delta_1 = \epsilon_0$, we have

$$\begin{aligned}& r^{-2}\left|\int_{S'(\delta_*\epsilon_0^{-1})} (J^{(\pm)})^2 - \int_{S_0} (J^{(\pm)})^2\right| \\ & + r^{-2}\left|\int_{S'(\delta_*\epsilon_0^{-1})} J^{(+)}J^{(-)} - \int_{S_0} J^{(+)}J^{(-)}\right| \lesssim \epsilon_0.\end{aligned}$$

Together with the control of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ and the control of $r' - r$ of Step 14, we deduce

$$\int_{S'_{\delta_*\epsilon_0^{-1}}} (J^{(\pm)})^2 = \frac{4\pi}{3} r'^2 \Big(1 + O(\epsilon_0)\Big), \qquad \int_{S'_{\delta_*\epsilon_0^{-1}}} J^{(+)} J^{(-)} = O(r'^2 \epsilon_0).$$

Together with the above control of $\widetilde{J}^{\pm}$, we infer

$$\begin{aligned}
\frac{4\pi}{3} r'^2 \Big(1 + O(\epsilon_0)\Big) &= \int_{S'_{\delta_*\epsilon_0^{-1}}} (J^{(+)})^2 \\
&= \int_{S'_{\delta_*\epsilon_0^{-1}}} \Big(\widetilde{J}^{+} + \Big((1 + c_{++}) J'^{(+)} + c_{+-} J'^{(-)}\Big)\Big)^2 \\
&= \frac{4\pi}{3} r'^2 \Big((1 + c_{++})^2 + (c_{+-})^2 + O(\epsilon_0)\Big), \\
O(r'^2 \epsilon_0) &= \int_{S'_{\delta_*\epsilon_0^{-1}}} J^{(+)} J^{(-)} \\
&= \int_{S'_{\delta_*\epsilon_0^{-1}}} \Big(\widetilde{J}^{+} + \Big((1 + c_{++}) J'^{(+)} + c_{+-} J'^{(-)}\Big)\Big) \\
&\quad \times \Big(\widetilde{J}^{-} + \Big(c_{-+} J'^{(+)} + (1 + c_{--}) J'^{(-)}\Big)\Big) \\
&= \frac{4\pi}{3} r'^2 \Big((1 + c_{++}) c_{-+} + + c_{+-}(1 + c_{--}) + O(\epsilon_0)\Big),
\end{aligned}$$

and

$$\begin{aligned}
\frac{4\pi}{3} r'^2 \Big(1 + O(\epsilon_0)\Big) &= \int_{S'_{\delta_*\epsilon_0^{-1}}} (J^{(-)})^2 \\
&= \int_{S'_{\delta_*\epsilon_0^{-1}}} \Big(\widetilde{J}^{-} + \Big(c_{-+} J'^{(+)} + (1 + c_{--}) J'^{(-)}\Big)\Big)^2 \\
&= \frac{4\pi}{3} r'^2 \Big((c_{-+})^2 + (1 + c_{--})^2 + O(\epsilon_0)\Big),
\end{aligned}$$

which yields

$$\begin{aligned}
&(1 + c_{++})^2 + (c_{+-})^2 = 1 + O(\epsilon_0), \qquad (c_{-+})^2 + (1 + c_{--})^2 = 1 + O(\epsilon_0), \\
&(1 + c_{++}) c_{-+} + c_{+-}(1 + c_{--}) = O(\epsilon_0).
\end{aligned}$$

Together with the above control of c_{++}, c_{+-}, c_{--} and c_{--}, we deduce

$$\begin{aligned}
&1 + c_{++} = \sqrt{1 - (c_{+-})^2} + O(\epsilon_0), \qquad 1 + c_{--} = \sqrt{1 - (c_{-+})^2} + O(\epsilon_0), \\
&c_{-+} = c_{+-} + O(\epsilon_0).
\end{aligned}$$

In particular, there exists a real number φ_0 such that

$$c_{+-} = \sin(\varphi_0), \qquad c_{-+} = -\sin(\varphi_0) + O(\epsilon_0), \qquad |\varphi_0| \lesssim \epsilon, \\ c_{++} = \cos(\varphi_0) - 1 + O(\epsilon_0), \qquad c_{--} = \cos(\varphi_0) - 1 + O(\epsilon_0). \tag{8.128}$$

Together with the above definition, and the above control, of $\widetilde{J}^{\pm}$, we infer

$$\sup_{S'_{\delta_*\epsilon_0^{-1}}} \left|J^{(\pm)} - \left(\cos(\varphi_0)J'^{(\pm)} \pm \sin(\varphi_0)J'^{(\mp)}\right)\right| \lesssim \epsilon_0. \tag{8.129}$$

Also, we have

$$e'_4\left(J^{(0)} - J'^{(0)}\right) = e'_4\left(J^{(0)}\right) = \lambda\left(e_4 + f\cdot\nabla' + \frac{1}{4}|f|^2 e_3\right)J^{(0)} = O(r^{-1}f)$$

and

$$\begin{aligned} & e'_4\left(J^{(\pm)} - \left(\cos(\varphi_0)J'^{(\pm)} \pm \sin(\varphi_0)J'^{(\mp)}\right)\right) \\ = \;& e'_4\left(J^{(\pm)}\right) = \lambda\left(e_4 + f\cdot\nabla' + \frac{1}{4}|f|^2 e_3\right)J^{(\pm)} = O(r^{-1}f). \end{aligned}$$

Together with the control of f in Step 14, we obtain on $\mathcal{C}'_1$

$$\left|e'_4\left(J^{(0)} - J'^{(0)}\right)\right| + \left|e'_4\left(J^{(\pm)} - \left(\cos(\varphi_0)J'^{(\pm)} \pm \sin(\varphi_0)J'^{(\mp)}\right)\right)\right| \lesssim \frac{\epsilon_0}{r'^2}$$

and hence, integrating forward from $S'_{\delta_*\epsilon_0^{-1}}$, and using the above estimate for $J^{(0)} - J'^{(0)}$ and for $J^{(\pm)} - (\cos(\varphi_0)J'^{(\pm)} \pm \sin(\varphi_0)J'^{(\mp)})$ on $S'_{\delta_*\epsilon_0^{-1}}$, we deduce on $\mathcal{C}'_1$

$$\left|J^{(0)} - J'^{(0)}\right| + \left|J^{(\pm)} - (\cos(\varphi_0)J'^{(\pm)} \pm \sin(\varphi_0)J'^{(\mp)})\right| \lesssim \epsilon_0.$$

This estimate holds thus in particular on S'_1. Finally, recalling (8.96), we have

$$\begin{aligned} 0 &= \int_{S'_1} J^{(+)}J'^{(-)} = \int_{S'_1}\left(\cos(\varphi_0)J'^{(+)} + \sin(\varphi_0)J'^{(-)} + O(\epsilon_0)\right)J'^{(-)} \\ &= \frac{4\pi}{3}r'^2\left(\sin(\varphi_0) + O(\epsilon_0)\right) \end{aligned}$$

which implies $\sin(\varphi_0) = O(\epsilon_0)$. We have thus obtain

$$\max_{p=0,+,-}\sup_{S'_1}\left|J^{(p)} - J'^{(p)}\right| \lesssim \epsilon_0 \quad \text{if} \quad a_0 \neq 0,$$

which together with the previous estimate for $a - a_0$ establishes (8.125). This ends the proof of Lemma 8.36. $\square$

8.3.3. Steps 17–24

Step 17. In the case $a_0 \neq 0$, we derive higher derivative estimates for $f'_0 - f_0$ and $J^{(p)} - J'^{(p)}$ on Σ_*.

In what follows we recall that ϕ is the effective uniformization factor on S_* with (θ, φ) the corresponding coordinates and $J^{(p)}$ the corresponding $\ell = 1$ balanced modes, see (5.19). We also recall the definition of the $1-$ forms f_-, f_0, f_+, see Definition 5.56. Following our conventions above[181] we denote these by $J'^{(p)}$ and f'_-, f'_0, f'_+.

Lemma 8.37. *The effective uniformization factor ϕ of S_* verifies*

$$\|\phi\|_{\mathfrak{h}_{k+2}(S_*)} \lesssim r' \left\|\Gamma'_g\right\|_{\mathfrak{h}_k(S_*)}, \qquad 0 \leq k \leq k_{large}. \tag{8.130}$$

Proof. According to Theorem 5.2, we have

$$\|\phi\|_{\mathfrak{h}_{k+2}(S_*)} \lesssim r'^2 \left\|K' - \frac{1}{r'^2}\right\|_{\mathfrak{h}_k(S_*)}, \qquad 0 \leq k \leq k_{large}.$$

Also, in view of the linearized Gauss equation of Proposition 5.18, we have $\check{K}' \in r^{-1}\Gamma'_g$ and hence

$$\|\phi\|_{\mathfrak{h}_{k+2}(S_*)} \lesssim r' \left\|\Gamma'_g\right\|_{\mathfrak{h}_k(S_*)}$$

which concludes the proof of the lemma. $\square$

We now proceed as follows:

1. Using Lemmas 5.61 and 5.62, and simple elliptic estimates, we deduce

$$\begin{aligned}
&\|\phi\|_{\mathfrak{h}_{k+2}(S_*)} \lesssim r'\|\Gamma'_g\|_{\mathfrak{h}_k(S_*)},\\
&\max_{p=0,+,-} \|r'\widetilde{\nabla' J'^{(p)}}\|_{\mathfrak{h}_{k+2}(S_*)} \lesssim \|\phi\|_{\mathfrak{h}_{k+2}(S_*)},\\
&\|r'\nabla f'_0 - J'^{(0)} \in \|_{\mathfrak{h}_{k+1}(S_*)} + \|r'\nabla f'_\pm + J'^{(\pm)}\delta\|_{\mathfrak{h}_{k+1}(S_*)} \lesssim \|\phi\|_{\mathfrak{h}_{k+2}(S_*)}.
\end{aligned}$$

   Together with the bootstrap assumption for Γ'_g and Γ'_b, we infer, for $k \leq k_{large}$,

$$\|\phi\|_{\mathfrak{h}_{k+2}(S_*)} \lesssim \epsilon,$$

[181] Quantities related to the PG frame are denoted by primes.

$$\max_{p=0,+,-}\|r'\widetilde{\nabla' J'^{(p)}}\|_{\mathfrak{h}_{k+2}(S_*)}\lesssim\epsilon,$$
$$\|r'\nabla f_0' - J'^{(0)}\in\|_{\mathfrak{h}_{k+1}(S_*)}+\|r'\nabla f_\pm' + J'^{(\pm)}\delta\|_{\mathfrak{h}_{k+1}(S_*)}\lesssim\epsilon.$$

By Sobolev, we deduce

$$\begin{aligned}
&\max_{p=0,+,-} r'\|\not{\mathfrak{d}}'^{\leq k_{large}} r'\widetilde{\nabla' J'^{(0)}}\|_{L^\infty(S_*)}\\
&+r'\|\not{\mathfrak{d}}'^{\leq k_{large}-1}(r'\nabla f_0' - J'^{(0)}\in)\|_{L^\infty(S_*)}\\
&+r'\|\not{\mathfrak{d}}'^{\leq k_{large}-1}(r'\nabla f_\pm' + J'^{(\pm)}\delta)\|_{L^\infty(S_*)} \quad\lesssim\quad \epsilon.
\end{aligned}$$

2. On Σ_* we have, using the transport equations along Σ_* of Lemma 5.66

$$\begin{aligned}
\nabla_\nu\Big[r'\nabla f_0' - J'^{(0)}\in\Big] &= \Gamma_b'\cdot\not{\mathfrak{d}}^{\leq 1}f_0',\\
\nabla_\nu\Big[r'\nabla f_\pm' + J'^{(\pm)}\delta\Big] &= \Gamma_b'\cdot\not{\mathfrak{d}}^{\leq 1}f_\pm',\\
\nabla_\nu\Big[r'\widetilde{\nabla' J'^{(p)}}\Big] &= \Gamma_b'\cdot\not{\mathfrak{d}}^{\leq 1}J'^{(p)}, \quad p=0,+,-.
\end{aligned}$$

We integrate from S_* and obtain, using the bootstrap assumption for Γ_b',

$$\begin{aligned}
&\|\not{\mathfrak{d}}'^{\leq k_{large}-1}(r'\nabla f_0' - J'^{(0)}\in)\|_{L^\infty(\Sigma_*)}\\
\lesssim\;&\|\not{\mathfrak{d}}'^{\leq k_{large}-1}(r'\nabla f_0' - J'^{(0)}\in)\|_{\mathfrak{h}_k(S_*)}+\frac{u_*}{r},\\
&\|\not{\mathfrak{d}}'^{\leq k_{large}-1}(r'\nabla f_\pm' + J'^{(\pm)}\delta)\|_{L^\infty(\Sigma_*)}\\
\lesssim\;&\|\not{\mathfrak{d}}'^{\leq k_{large}-1}(r'\nabla f_\pm' + J'^{(\pm)}\delta)\|_{\mathfrak{h}_k(S_*)}+\frac{u_*}{r},\\
&\|\not{\mathfrak{d}}'^{\leq k_{large}}(r'\widetilde{\nabla' J'^{(p)}})\|_{L^\infty(\Sigma_*)}\\
\lesssim\;&\|\not{\mathfrak{d}}'^{\leq k_{large}}(r'\widetilde{\nabla' J'^{(p)}})\|_{\mathfrak{h}_k(S_*)}+\frac{u_*}{r}, \quad p=0,+,-.
\end{aligned}$$

Together the above control on S_* and the dominance condition for r on Σ_*, we deduce

$$\begin{aligned}
\|\not{\mathfrak{d}}'^{\leq k_{large}-1}(r'\nabla f_0' - J'^{(0)}\in)\|_{L^\infty(\Sigma_*)} &\lesssim \epsilon_0,\\
\|\not{\mathfrak{d}}'^{\leq k_{large}-1}(r'\nabla f_\pm' + J'^{(\pm)}\delta)\|_{L^\infty(\Sigma_*)} &\lesssim \epsilon_0,\\
\|\not{\mathfrak{d}}'^{\leq k_{large}}(r'\widetilde{\nabla' J'^{(p)}})\|_{L^\infty(\Sigma_*)} &\lesssim \epsilon_0, \quad p=0,+,-.
\end{aligned}$$

In view of the above, using Definition 5.57 of $\widetilde{\nabla' J'^{(p)}}$, we infer on S'_1

$$\begin{aligned}
(8.131)\qquad & \left\|\mathfrak{d}'^{\leq k_{large}-1}\left(\nabla' f'_0 - \frac{J'^{(0)}}{r'}\in\right)\right\|_{L^\infty(S'_1)}\\
&+\left\|\mathfrak{d}'^{\leq k_{large}-1}\left(\nabla' f'_\pm + \frac{J'^{(\pm)}}{r'}\delta\right)\right\|_{L^\infty(S'_1)}\\
&+\left\|\mathfrak{d}'^{\leq k_{large}}\left(\nabla' J'^{(0)} + \frac{1}{r'}\,{}^*f'_0\right)\right\|_{L^\infty(S'_1)}\\
&+\left\|\mathfrak{d}'^{\leq k_{large}}\left(\nabla' J'^{(\pm)} - \frac{1}{r'}f'_\pm\right)\right\|_{L^\infty(S'_1)} \lesssim \frac{\epsilon_0}{r'}.
\end{aligned}$$

On the other hand, from the control of ${}^{(ext)}\widetilde{\mathcal{L}_0}$, the change of frame formula for ∇', the control of the change of frame $(f,\underline{f},\lambda)$ from ${}^{(ext)}\widetilde{\mathcal{L}_0}$ to Σ_* of Step 13, and the control of $r'-r$ on S'_1 provided by (8.111), we have

$$\begin{aligned}
&\left\|\mathfrak{d}'^{\leq k_{large}-1}\left(\nabla' f_0 - \frac{J^{(0)}}{r'}\in\right)\right\|_{L^\infty(S'_1)} + \left\|\mathfrak{d}'^{\leq k_{large}-1}\left(\nabla' f_\pm + \frac{J^{(\pm)}}{r'}\delta\right)\right\|_{L^\infty(S'_1)}\\
&+\left\|\mathfrak{d}'^{\leq k_{large}}\left(\nabla' J^{(0)} + \frac{1}{r'}\,{}^*f_0\right)\right\|_{L^\infty(S'_1)} + \left\|\mathfrak{d}'^{\leq k_{large}}\left(\nabla' J^{(\pm)} - \frac{1}{r'}f_\pm\right)\right\|_{L^\infty(S'_1)}\\
&\lesssim \frac{\epsilon_0}{r},
\end{aligned}$$

and hence

$$\begin{aligned}
&\left\|\mathfrak{d}'^{\leq k_{large}-1}\left(\nabla'(f'_0 - f_0) - \frac{J'^{(0)}-J^{(0)}}{r'}\in\right)\right\|_{L^\infty(S'_1)}\\
&+\left\|\mathfrak{d}'^{\leq k_{large}-1}\left(\nabla'(f'_\pm - f_\pm) + \frac{J'^{(\pm)}-J^{(\pm)}}{r'}\delta\right)\right\|_{L^\infty(S'_1)}\\
&+\left\|\mathfrak{d}'^{\leq k_{large}}\left(\nabla'(J'^{(0)}-J^{(0)}) + \frac{1}{r'}\,{}^*(f'_0 - f_0)\right)\right\|_{L\infty(S'_1)}\\
&+\left\|\mathfrak{d}'^{\leq k_{large}}\left(\nabla'(J'^{(\pm)}-J^{(\pm)}) - \frac{1}{r'}\,{}^*(f'_\pm - f_\pm)\right)\right\|_{L\infty(S'_1)} \lesssim \epsilon_0.
\end{aligned}$$

We deduce

$$\left\|\mathfrak{d}'^{\leq k_{large}}(f'_0 - f_0)\right\|_{L^\infty(S'_1)} + \left\|\mathfrak{d}'^{\leq k_{large}+1}(J'^{(0)} - J^{(0)})\right\|_{L^\infty(S'_1)}$$

$$\lesssim \epsilon_0 + \|J'^{(0)} - J^{(0)}\|_{L^\infty(S'_1)}$$

and

$$\begin{aligned} & \left\| \not{\mathfrak{d}}'^{\leq k_{large}}(f'_\pm - f_\pm)\right\|_{L^\infty(S'_1)} + \left\| \not{\mathfrak{d}}'^{\leq k_{large}+1}(J'^{(\pm)} - J^{(\pm)})\right\|_{L^\infty(S'_1)} \\ \lesssim \; & \epsilon_0 + \|J'^{(\pm)} - J^{(\pm)}\|_{L^\infty(S'_1)}. \end{aligned}$$

Together with (8.125), we obtain in the case $a_0 \neq 0$

$$\max_{p=0,+,-} \left(\left\| \not{\mathfrak{d}}'^{\leq k_{large}}(f'_p - f_p)\right\|_{L^\infty(S'_1)} + \left\| \not{\mathfrak{d}}'^{\leq k_{large}+1}(J'^{(p)} - J^{(p)})\right\|_{L^\infty(S'_1)} \right) \lesssim \epsilon_0.$$

In view of the fact that $\nabla_\nu f'_p = 0$ for $p = 0, +, -$ on Σ_*, and in view of the control of ${}^{(ext)}\widetilde{\mathcal{L}_0}$, and hence of f_p for $p = 0, +, -$, we deduce for $a_0 \neq 0$

$$\begin{aligned} \max_{p=0,+,-} & \Bigg(\left\| \mathfrak{d}_*^{\leq k_{large}}(f'_p - f_p)\right\|_{L^\infty(S'_1)} \\ & + \left\| \mathfrak{d}_*^{\leq k_{large}+1}(J'^{(p)} - J^{(p)})\right\|_{L^\infty(S'_1)} \Bigg) \lesssim \epsilon_0. \end{aligned} \tag{8.132}$$

In particular, this improves the bootstrap assumption (8.95) on $J'^{(p)} - J^{(p)}$ for $p = 0, +, -$.

Step 18. Next, we control $\nu(r')$ and b_* on S'_1. First, recall from (8.112) that we have, for $k \leq k_{large} + 7$ for the frame of Σ_*,

$$r' |\mathfrak{d}_*^k(\eta', \underline{\xi}', \underline{\breve{\omega}}')| \lesssim \epsilon_0.$$

Together with Lemma 5.12, and since $b_* = -y' - z'$ in view of Lemma 5.12, we infer, for $k \leq k_{large} + 7$,

$$r' |\nabla' \mathfrak{d}_*^k(e_3(r)', e_3(u)', b_*)| \lesssim \epsilon_0. \tag{8.133}$$

Then, proceeding as in Step 7 in the proof of Proposition 5.42 for the averages, we obtain on S'_1, for $k \leq k_{large} + 6$,

$$\left| \mathfrak{d}_*^k \left(\nu(r') + 2, b_* + 1 + \frac{2m}{r'} \right) \right| \lesssim \epsilon_0. \tag{8.134}$$

Step 19. We consider the following change of frame coefficients:

- $(f, \underline{f}, \lambda)$ are the change of frame coefficients from the outgoing geodesic frame of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ to the frame of Σ_*. They satisfy, according to (8.111),

$$\sup_{k\leq k_{large}+1}\Big(\|\mathfrak{d}^k f\|_{L^2(S_1')}+r^{-1}\|\mathfrak{d}^k(\underline{f},\log\lambda)\|_{L^2(S_1')}\\ +\|\mathfrak{d}^{\leq k-1}\nabla'_{\nu}(\underline{f},\log\lambda)\|_{L^2(S_1')}\Big)\lesssim\epsilon_0.$$

- $(f', \underline{f}', \lambda')$ are the change of frame coefficients from the outgoing PG frame of ${}^{(ext)}\mathcal{L}_0$ to the outgoing geodesic frame of ${}^{(ext)}\widetilde{\mathcal{L}_0}$. $(f', \underline{f}', \lambda')$ satisfies in view of Proposition 8.20

$$\sup_{{}^{(ext)}\widetilde{\mathcal{L}_0}} r\left|\mathfrak{d}^{\leq k_{large}+8}\left(f'+\frac{a_0}{r}f_0, \underline{f}'+\frac{a_0\Upsilon}{r}f_0, \log\lambda'\right)\right| \lesssim \epsilon_0.$$

- $(f'', \underline{f}'', \lambda'')$ are the change of frame coefficients from the frame of Σ_* to the outgoing PG frame of ${}^{(ext)}\mathcal{M}$. $(f'', \underline{f}'', \lambda'')$ satisfies by the initialization of the PG structure on Σ_*, see Section 3.2.5,

$$\lambda''=1, \qquad f''=\frac{a}{r'}f_0', \qquad \underline{f}''=-\frac{(\nu(r')-b_*)}{1-\frac{1}{4}b_*|f''|^2}f''.$$

We now consider the change of frame coefficients $(f''', \underline{f}''', \lambda''')$ from the outgoing PG frame of ${}^{(ext)}\mathcal{L}_0$ to the outgoing PG frame of ${}^{(ext)}\mathcal{M}$. In view of:

- the above estimates for $(f, \underline{f}, \lambda)$ and $(f', \underline{f}', \lambda')$,
- the above formula for $(f'', \underline{f}'', \lambda'')$,
- the control for $r-r'$ and $m-m_0$ given by (8.111), and the control for $\nu(r')$ and b_* in Step 18,
- the control of a in (8.118) in the case $a_0=0$,
- the control for $a-a_0$ in (8.125) and the control for $f_0'-f_0$ in Step 17 in the case $a_0\neq 0$,

we infer the following estimates

$$\sup_{S_1'} r|\mathfrak{d}^{\leq k_{large}}f'''|+\sup_{S_1'}|\mathfrak{d}^{\leq k_{large}}(\underline{f}''', \log\lambda''')| \lesssim \epsilon_0+\frac{1}{r}.$$

Together with the dominance condition for r on Σ_*, we infer

$$\sup_{S_1'} r|\mathfrak{d}^{\leq k_{large}}f'''|+\sup_{S_1'}|\mathfrak{d}^{\leq k_{large}}(\underline{f}''', \log\lambda''')| \lesssim \epsilon_0. \tag{8.135}$$

Step 20. In Steps 20–22, (e_1, e_2, e_3, e_4) denotes the outgoing PG frame of ${}^{(ext)}\mathcal{L}_0$, and (e_1', e_2', e_3', e_4') denotes the outgoing PG frame of ${}^{(ext)}\mathcal{M}$. Also, from now on, $(f, \underline{f}, \lambda)$ denotes[182] the change of frame coefficients from the outgoing PG frame of ${}^{(ext)}\mathcal{L}_0$ to the outgoing PG frame of ${}^{(ext)}\mathcal{M}$. In view of Step 19, we have on S_1'

$$\sup_{S_1'} r|\mathfrak{d}^{\leq k_{large}} f| + \sup_{S_1'} |\mathfrak{d}^{\leq k_{large}}(\underline{f}, \log\lambda)| \quad \lesssim \quad \epsilon_0.$$

Let

$$F = f + i\,{}^*f.$$

Since

$$\Xi' = 0, \qquad \omega' = 0, \qquad \Xi = 0, \qquad \omega = 0,$$

we have, in view of Corollary 2.14,

$$\begin{aligned} \nabla_{\lambda^{-1}e_4'}(\overline{q}F) &= E_4(f, \Gamma), \\ \lambda^{-1}\nabla_4'(\log\lambda) &= f\cdot(\zeta - \underline{\eta}) + E_2(f, \Gamma). \end{aligned}$$

We integrate the above transport equations for F and λ in the order they appear from S_1'. In view of the control for (f, λ) on S_1' derived in Step 19, and in view of the assumptions on the initial data layer norm, we infer

$$\sup_{\{u'=1\}} \Big(r|\mathfrak{d}^{\leq k_{large}} f| + |\mathfrak{d}^{\leq k_{large}} \log(\lambda)|\Big) \quad \lesssim \quad \epsilon_0, \tag{8.136}$$

where u' denotes from now on the scalar function of the PG structure of ${}^{(ext)}\mathcal{M}$. In particular, we have by construction $S_1' = \Sigma_* \cap \{u' = 1\}$, $e_4'(u') = 0$ and $\{u' = 1\} \subset {}^{(ext)}\mathcal{L}_0$.

Step 21. In this step, we estimate $r' - r$, as well as A', $\widetilde{\mathrm{tr}X}'$ and $\widehat{X}'$. Moreover, in the case $a_0 \neq 0$, we also estimate $J'^{(0)} - J^{(0)}$ and $\mathfrak{J}' - \mathfrak{J}$. First, since $J'^{(0)}$ is propagated from Σ_* by $e_4'(J'^{(0)}) = 0$, and using the change of frame formula between the PG frame of ${}^{(ext)}\mathcal{L}_0$ and the PG frame of ${}^{(ext)}\mathcal{M}$, we infer

$$e_4'(J'^{(0)} - J^{(0)}) \quad = \quad -\lambda\left(f\cdot\nabla + \frac{1}{4}|f|^2 e_3\right) J^{(0)}.$$

[182] Denoted earlier by triple prime.

Together with the control of f and λ of Step 20, and the control of ${}^{(ext)}\mathcal{L}_0$, we infer

$$\sup_{\{u'=1\}} r^2|\mathfrak{d}^{\leq k_{large}}(e_4'(J'^{(0)} - J^{(0)}))| \lesssim \epsilon_0.$$

Integrating from S_1', where $J'^{(0)} - J^{(0)}$ is under control in view of Step 17 in the case $a_0 \neq 0$, we infer

$$\sup_{\{u'=1\}} |\mathfrak{d}^{\leq k_{large}}(J'^{(0)} - J^{(0)})| \lesssim \epsilon_0 \quad \text{if} \quad a_0 \neq 0. \tag{8.137}$$

Next, we control $\widetilde{\mathrm{tr}X}'$. To this end, we also need to control A' and $\widehat{X}'$. First, note that the change of frame formula for A', the control of the foliation of ${}^{(ext)}\mathcal{L}_0$, the control of f and λ of Step 20, and the fact that the transformation formula for A' does not depend on $\underline{f}$ implies

$$\sup_{\{u'=1\}} r^{\frac{7}{2}+\delta_B}|\mathfrak{d}^{\leq k_{large}}A'| \lesssim \epsilon_0. \tag{8.138}$$

Then, using the control of the Ricci coefficients of the frame of Σ_* obtained in (8.112), the control of a in (8.118) in the case $a_0 = 0$, the control for $a - a_0$ in (8.125) and the control for $f_0' - f_0$ in (8.131) in the case $a_0 \neq 0$, the control of $\nu(r')$ and b_* of Step 18, and the change of frame formula between the frame of Σ_* and the outgoing PG frame of ${}^{(ext)}\mathcal{M}$ on Σ_*, we infer[183]

$$\sup_{S_1'} r^2\Big(|\mathfrak{d}^{\leq k_{large}-1}\widetilde{\mathrm{tr}X}'| + |\mathfrak{d}^{\leq k_{large}-1}\widehat{X}'|\Big) \lesssim \epsilon_0.$$

Then, propagating Raychadhuri for $\widetilde{\mathrm{tr}X}'$ and the null structure equation for $\nabla_4'\widehat{X}'$ from S_1' where $\widetilde{\mathrm{tr}X}'$ and $\widehat{X}'$ are under control in view of the above estimate, we infer, using the above control of A',

$$\sup_{\{u'=1\}} r^2\Big(|\mathfrak{d}^{\leq k_{large}-1}\widetilde{\mathrm{tr}X}'| + |\mathfrak{d}^{\leq k_{large}-1}\widehat{X}'|\Big) \lesssim \epsilon_0. \tag{8.139}$$

Next, we notice

$$\mathrm{tr}X' - \lambda\mathrm{tr}X = \frac{2}{q'} - \frac{2}{q} + (\lambda - 1)\mathrm{tr}X + \widetilde{\mathrm{tr}X}' - \widetilde{\mathrm{tr}X}$$

[183]Note that while the control of the Ricci coefficients $\widetilde{\mathrm{tr}\underline{\chi}}$ of Σ_* in (8.112) displays a loss of r^{-1}, this allows nevertheless to obtain the correct power of r of $\widetilde{\mathrm{tr}X}'$ and $\widehat{X}'$ on S_1'.

so that

$$q' - q = \frac{qq'}{2}\Big(-\lambda\Big(\lambda^{-1}\mathrm{tr}X' - \mathrm{tr}X\Big) + (\lambda-1)\mathrm{tr}X + \widetilde{\mathrm{tr}X}' - \widetilde{\mathrm{tr}X}\Big).$$

Together with the control of Step 20 for f and λ, the above control $\widetilde{\mathrm{tr}X}'$, the above control for $J'^{(0)} - J^{(0)}$ if $a_0 \neq 0$, the control of a in (8.118) in the case $a_0 = 0$, the control for $a - a_0$ in (8.125) in the case $a_0 \neq 0$, the control of the foliation ${}^{(ext)}\mathcal{L}_0$, and the fact that $q = r + ia_0 J^{(0)}$ and $q' = r' + iaJ'^{(0)}$, we infer

$$\sup_{\{u'=1\}}\left|\mathfrak{d}^{\leq k_{large}-1}\left(\frac{r'}{r} - 1 + \frac{qq'}{2r}\Big(\lambda^{-1}\mathrm{tr}X' - \mathrm{tr}X\Big)\right)\right| \lesssim \epsilon_0.$$

Moreover, from the change of frame formulas for $\mathrm{tr}\,\chi$ and ${}^{(a)}\mathrm{tr}\chi$ we have, schematically,

$$\lambda^{-1}\mathrm{tr}X' - \mathrm{tr}X = r'^{-1}\mathfrak{d}'f + \Gamma\cdot f + f\cdot\underline{f}\cdot\Gamma + f\cdot\underline{f}\cdot\mathrm{tr}X' + \text{ l.o.t.}$$

Together with the control of Step 20 for f, the above control for $\widetilde{\mathrm{tr}X}'$ and the control of the foliation ${}^{(ext)}\mathcal{L}_0$, we deduce

$$\sup_{\{u'=1\}}\left|\mathfrak{d}^{\leq k_{large}-1}\left(\frac{r'}{r} - 1\right)\right| \lesssim \epsilon_0 + \epsilon_0\sup_{\{u'=1\}}\left|\mathfrak{d}^{\leq k_{large}-1}\underline{f}\right|. \tag{8.140}$$

Next, recall the definition of $\mathfrak{J}'$ on Σ_*

$$\mathfrak{J}' = \frac{1}{|q'|}(f_0' + i\,{}^*f_0') = \frac{1}{\sqrt{r'^2 + a^2(J'^{(0)})^2}}(f_0' + i\,{}^*f_0').$$

Recall that we control $\mathfrak{J}' - \mathfrak{J}$ only in the case $a_0 \neq 0$. Together with the control of $r - r'$ and $m - m_0$ given by (8.111), the control of $a - a_0$ in Step 16 for $a_0 \neq 0$, the control for $f_0' - f_0$ and $J'^{(0)} - J^{(0)}$ in Step 17 for $a_0 \neq 0$, and the control of ${}^{(ext)}\mathcal{L}_0$, we infer

$$r'\|\mathfrak{d}_*^{\leq k_{large}}(\mathfrak{J}' - \mathfrak{J})\|_{L^\infty(S_1')} \lesssim \epsilon_0 + \frac{1}{r}.$$

Together with the dominance condition for r on Σ_*, we infer

$$r'\|\mathfrak{d}_*^{\leq k_{large}}(\mathfrak{J}' - \mathfrak{J})\|_{L^\infty(S_1')} \lesssim \epsilon_0.$$

Together with the identity

$$\begin{aligned} q'\mathfrak{J}' - q\mathfrak{J} &= q'(\mathfrak{J}' - \mathfrak{J}) + (q' - q)\mathfrak{J} \\ &= q'(\mathfrak{J}' - \mathfrak{J}) + \Big(r' - r + i(aJ'^{(0)} - a_0 J^{(0)})\Big)\mathfrak{J}, \end{aligned}$$

the control of $r' - r$ given by (8.111), the control of $J'^{(0)} - J^{(0)}$ in Step 17 and the control of $a - a_0$ of Step 16, we obtain

$$\|\mathfrak{d}_*^{\leq k_{large}}(q'\mathfrak{J}' - q\mathfrak{J})\|_{L^\infty(S_1')} \lesssim \epsilon_0.$$

Also, recall that $\mathfrak{J}'$ and $\mathfrak{J}$ satisfy in $\{u' = 1\}$

$$\nabla_4'\mathfrak{J}' = -\frac{1}{q'}\mathfrak{J}', \qquad \nabla_4\mathfrak{J} = -\frac{1}{q}\mathfrak{J},$$

and hence

$$\nabla_4'(q'\mathfrak{J}') = 0, \qquad \nabla_4(q\mathfrak{J}) = 0.$$

We infer

$$\nabla_{\lambda^{-1}4'}(q'\mathfrak{J}') = 0, \qquad \nabla_{\lambda^{-1}4'}(q\mathfrak{J}) = \left(f\cdot\nabla + \frac{1}{4}|f|^2 e_3\right)(q\mathfrak{J}).$$

Together with the control of f and λ of Step 20, and the control of ${}^{(ext)}\mathcal{L}_0$, we obtain

$$\sup_{\{u'=1\}} r^2|\mathfrak{d}^{\leq k_{large}}(\nabla_4'(q'\mathfrak{J}' - q\mathfrak{J}))| \lesssim \epsilon_0.$$

Integrating from S_1' where $q'\mathfrak{J}' - q\mathfrak{J}$ is under control in view of the above, we infer

$$\sup_{\{u'=1\}} |\mathfrak{d}^{\leq k_{large}}(q'\mathfrak{J}' - q\mathfrak{J})| \lesssim \epsilon_0.$$

Using again the above identity for $q'\mathfrak{J}' - q\mathfrak{J}$, as well as the above control of $J'^{(0)} - J^{(0)}$ and $r' - r$, and the control of $a - a_0$ of Step 16, we deduce

$$\sup_{\{u'=1\}} r|\mathfrak{d}^{\leq k_{large}-1}(\mathfrak{J}' - \mathfrak{J})| \lesssim \epsilon_0 + \epsilon_0 \sup_{\{u'=1\}}\left|\mathfrak{d}^{\leq k_{large}-1}\underline{f}\right| \quad \text{if} \quad a_0 \neq 0.$$

Step 22. In this step, we control $\underline{f}$ on ${}^{(ext)}\mathcal{M}$. To this end, we first control B' and Z'. The change of frame formula for B', the control of the foliation of

${}^{(ext)}\mathcal{L}_0$, the control of f and λ of Step 20, and the fact that the terms involving $\underline{f}$ in the transformation formula for B' are at least quadratic, implies

$$\sup_{\{u'=1\}} r^{\frac{7}{2}+\delta_B}|\mathfrak{d}^{\leq k_{large}-1}B'| \lesssim \epsilon_0 + \epsilon_0 \sup_{\{u'=1\}}\left|\mathfrak{d}^{\leq k_{large}-1}\underline{f}\right|. \tag{8.141}$$

Then, propagating the null structure equation for $\nabla'_4\check{Z}'$ from S'_1 where it is under control in view of (8.112) for the frame of Σ_* and the change of frame formula, we infer, using the above control of B',

$$\sup_{\{u'=1\}} r^2|\mathfrak{d}^{\leq k_{large}-1}\check{Z}'| \lesssim \epsilon_0 + \epsilon_0 \sup_{\{u'=1\}}\left|\mathfrak{d}^{\leq k_{large}-1}\underline{f}\right|. \tag{8.142}$$

Next, the transformation formula for Z', together with the control of the foliation of ${}^{(ext)}\mathcal{L}_0$, the control of f and λ of Step 20, and the control of $\widetilde{\mathrm{tr}X}'$ and $\widehat{X}'$ of Step 21, yields

$$\sup_{\{u'=1\}} r\left|\mathfrak{d}^{\leq k_{large}-1}\left(Z'-Z-\frac{1}{4}\mathrm{tr}X(\underline{f}+i\,{}^*\underline{f})\right)\right| \lesssim \epsilon_0 + \epsilon_0 \sup_{\{u'=1\}}\left|\mathfrak{d}^{\leq k_{large}-1}\underline{f}\right|.$$

Since we have

$$Z'-Z = \frac{a\overline{q'}}{|q'|^2}\mathfrak{J}' - \frac{a_0\overline{q}}{|q|^2}\mathfrak{J} + \check{Z}' - \check{Z},$$

we deduce, together with the control of the foliation of ${}^{(ext)}\mathcal{L}_0$, the above control of $\check{Z}'$, the control of $r'-r$ of Step 21, the control of a in (8.118) in the case $a_0=0$, the control for $a-a_0$ in (8.125) in the case $a_0\neq 0$, and the control of $J'^{(0)}-J^{(0)}$ and $\mathfrak{J}'-\mathfrak{J}$ of Step 21 in the case $a_0\neq 0$,

$$\sup_{\{u'=1\}}\left|\mathfrak{d}^{\leq k_{large}-1}\underline{f}\right| \lesssim \epsilon_0 + \epsilon_0 \sup_{\{u'=1\}}\left|\mathfrak{d}^{\leq k_{large}-1}\underline{f}\right|$$

and hence

$$\sup_{\{u'=1\}}\left|\mathfrak{d}^{\leq k_{large}-1}\underline{f}\right| \lesssim \epsilon_0.$$

Together with the control of f and λ of Step 20, we have finally obtained

$$\sup_{\{u'=1\}}\Big(r|\mathfrak{d}^{\leq k_{large}}f| + |\mathfrak{d}^{\leq k_{large}}\log(\lambda)| + |\mathfrak{d}^{\leq k_{large}-1}\underline{f}|\Big) \lesssim \epsilon_0. \tag{8.143}$$

Also, together with the estimates of Step 21 for $r' - r$, we obtain

$$\sup_{\{u'=1\}}\left|\mathfrak{d}^{\leq k_{large}-1}\left(\frac{r'}{r}-1\right)\right| \lesssim \epsilon_0. \tag{8.144}$$

Step 23. Let $(f', \underline{f}', \lambda')$ denote the change of frame coefficients from the principal outgoing null frame of ${}^{(int)}\mathcal{L}_0$ to the principal outgoing null frame of ${}^{(int)}\mathcal{M}$. From

- the estimates of Step 22 on $\{u' = 1\}$,
- the fact that ${}^{(int)}\mathcal{M} \cap {}^{(ext)}\mathcal{M} = \{r' = r_0\}$,
- the fact that $\{u = 1\} \cap \{\underline{u}' = 1\}$ is included in ${}^{(ext)}\mathcal{L}_0 \cap {}^{(int)}\mathcal{L}_0$,
- the initialization of the frame of ${}^{(int)}\mathcal{M}$ as an explicit renormalization of the frame of ${}^{(ext)}\mathcal{M}$ on $\{r' = r_0\}$,
- the control in ${}^{(ext)}\mathcal{L}_0 \cap {}^{(int)}\mathcal{L}_0$ of the difference between the frame of ${}^{(int)}\mathcal{L}_0$ and an explicit renormalization of the frame of ${}^{(ext)}\mathcal{L}_0$,

we easily infer, using also $\underline{u}' = u'$ on $\{r = r_0\}$,

$$\sup_{\{r=r_0\}\cap\{\underline{u}'=1\}}\left(|\mathfrak{d}^{\leq k_{large}-1}(f', \underline{f}', \log\lambda')|\right) \lesssim \epsilon_0.$$

Next, we proceed as in Step 20, exchanging the role of e_3 and e_4, and we propagate along e_3 the above estimate to $\{\underline{u}' = 1\}$ for $\underline{f}'$ and λ'. We also propagate the control of Step 21 for $J'^{(0)} - J^{(0)}$ on $\{u' = 1\}$ in the case $a_0 \neq 0$, and hence on its boundary $\{r' = r_0\}$ to $\{\underline{u}' = 1\}$. Also one propagates the control of Step 22 for $r - r'$ on $\{u' = 1\}$, and hence on its boundary $\{r' = r_0\}$ to $\{\underline{u}' = 1\}$ using the transport equation[184]

$$\begin{aligned} e_3'(r' - r) &= 1 - \lambda'\left(e_3 + \underline{f}'^a e_a + \frac{1}{4}|\underline{f}'|^2 e_4\right) r \\ &= -(\lambda' - 1) + \underline{f}' \cdot \nabla(r) + \frac{1}{4}|\underline{f}'|^2 e_4(r). \end{aligned}$$

Finally, we propagate f similarly to Step 22. We finally obtain

$$\sup_{\{\underline{u}'=1\}}\left(|\mathfrak{d}^{\leq k_{large}-1}(\underline{f}', \log\lambda')| + |\mathfrak{d}^{\leq k_{large}-2}(r' - r, f')|\right) \lesssim \epsilon_0, \tag{8.145}$$

[184]Note that we could not have used this transport equation in ${}^{(ext)}\mathcal{L}_0$ in view of the lack of decay in r for $\lambda - 1$. This is why we avoided this transport equation in Step 21 and used instead the control of $\widetilde{\mathrm{tr} X}'$. On the other hand, r is bounded in ${}^{(int)}\mathcal{L}_0$ so that one can simply rely on the transport equation for $e_3'(r' - r)$ in ${}^{(int)}\mathcal{L}_0$.

and

$$\sup_{\{\underline{u}'=1\}}\left|\mathfrak{d}^{\leq k_{large}-2}\left(J'^{(0)}-J^{(0)}\right)\right| \lesssim \epsilon_0, \quad \text{if} \quad a_0\neq 0. \tag{8.146}$$

Step 24. Note that the desired estimate for $m-m_0$ has been obtained in Step 13. Also, note that the desired estimate for $a-a_0$ has been obtained in Step 15 in the case $a_0=0$, and in Step 16 in the case $a_0\neq 0$. To conclude the proof of Theorem M0, it remains to control $k_{large}-2$ derivatives, with suitable r-weights and $O(\epsilon_0)$ smallness constant, of A', B', $\check{P}'$, $\underline{B}'$ and $\underline{A}'$ in $\{u'=1\}\cup\{\underline{u}'=1\}$, i.e.

$$\begin{aligned}\max_{0\leq k\leq k_{large}-2}\Bigg\{\sup_{\{u'=1\}}\Bigg[& r^{\frac{7}{2}+\delta_B}\left(|\mathfrak{d}^{k\,(ext)}A'|+|\mathfrak{d}^{k\,(ext)}B'|\right)\\ &+r^3\left|\mathfrak{d}^k\left({}^{(ext)}P'+\frac{2m}{q'^3}\right)\right|+r^2|\mathfrak{d}^{k\,(ext)}\underline{B}'|+r|\mathfrak{d}^{k\,(ext)}\underline{A}'|\Bigg]\Bigg\}\\ &+\max_{0\leq k\leq k_{large}-2}\sup_{\underline{\mathcal{B}}_1}\Bigg[|\mathfrak{d}^{k\,(int)}A'|+|\mathfrak{d}^{k\,(int)}B'|\\ &+\left|\mathfrak{d}^k\left({}^{(int)}P'+\frac{2m}{q'^3}\right)\right|+|\mathfrak{d}^{k\,(int)}\underline{B}'|+|\mathfrak{d}^{k\,(int)}\underline{A}'|\Bigg] \lesssim \epsilon_0.\end{aligned} \tag{8.147}$$

This follows from:

- the control of $(f,\underline{f},\lambda)$ on $\{u'=1\}$ derived in Step 22,
- the control of $(f',\underline{f}',\lambda')$ on $\{\underline{u}'=1\}$ derived in Step 23,
- the fact that $(f,\underline{f},\lambda)$ denote the change of frame coefficients from the PG frame of ${}^{(ext)}\mathcal{L}_0$ to the PG frame of ${}^{(ext)}\mathcal{M}$, and the fact that $(f',\underline{f}',\lambda')$ denote the change of frame coefficients from the principal outgoing null frame of ${}^{(int)}\mathcal{L}_0$ to the principal outgoing null frame of ${}^{(int)}\mathcal{M}$,
- the change of frame formulas for the curvature components,
- in the particular case of the estimate for $\check{P}$, the fact that

$$P'-P = -\frac{2m}{q'^3}+\frac{2m_0}{q^3}+\check{P}'-\check{P},$$

  together with the control of $m-m_0$ derived in Step 13, the control of $r'-r$ in Steps 22 and 23, the control of a in Step 15 in the case $a_0=0$, the control of $a-a_0$ in Step 16 in the case $a_0\neq 0$, and the control of $J'^{(0)}-J^{(0)}$ in Step 21 and 23 in the case $a_0\neq 0$,

- the assumptions on the initial data layer norm.

The proof of Theorem M0 is now complete.

8.4. Proof of Theorem M6

The proof of Theorem M6 proceeds in 8 steps which we summarize below for convenience:

1. In Steps 1–3, we construct our last sphere S_*, and then our last slice Σ_* inside the part ${}^{(ext)}\widetilde{\mathcal{L}_0}$ of the initial data layer, by relying on the control of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ provided by Proposition 8.20, and the GCM constructions of [41] and [50] recalled in Section 8.1.
2. In Steps 4–8, we construct from Σ_* a GCM admissible spacetime $\mathcal{M}$ and we control the change of frame coefficients between the frames of the initial data layer $\mathcal{L}_0$, and the corresponding frames of $\mathcal{M}$. In view of the control of $\mathcal{L}_0$ and of the change of frames coefficients, we infer the desired control of $\mathcal{M}$ thanks to the change of frame formulas.

We now proceed with the proof of Theorem M6.

Step 1. Let $r_{(0)}$ such that

$$r_{(0)} := d_0\delta_*\epsilon_0^{-1}, \tag{8.148}$$

where the small constant δ_* appears in (3.50), and where the constant d_0 satisfies

$$\frac{1}{2} \leq d_0 \leq 2$$

and will be suitably chosen in Step 3. Also, let $\delta_0 > 0$ sufficiently small. Consider the unique sphere $\overset{\circ}{S}$ of the part ${}^{(ext)}\widetilde{\mathcal{L}_0}$ of the initial data layer on $\{\widetilde{u} = 1 + \delta_0\}$ with area radius $r_{(0)}$. Then, denoting $S(\widetilde{u}, \widetilde{s})$ the spheres of the outgoing geodesic foliation of ${}^{(ext)}\widetilde{\mathcal{L}_0}$, we have

$$\overset{\circ}{S} = S(\overset{\circ}{u}, \overset{\circ}{s}), \qquad \overset{\circ}{u} = 1 + \delta_0, \qquad |\overset{\circ}{s} - r_{(0)}| \lesssim \epsilon_0,$$

where the control of $\overset{\circ}{s} - r_{(0)}$ follows from the assumptions on the control of ${}^{(ext)}\widetilde{\mathcal{L}_0}$. Relying on the control of the initial data layer given by (3.52), i.e.

$$\mathfrak{I}_{k_{large}+10} \leq \epsilon_0, \qquad {}^{(ext)}\mathfrak{I}_3 \leq \epsilon_0^2,$$

we are in position to apply Theorem 7.3 of [41] (restated here as Theorem 8.7) and Corollary 7.7 of [41] (restated here as Corollary 8.8), with the choices

$$\overset{\circ}{\delta} = \overset{\circ}{\epsilon} = \epsilon_0, \quad s_{max} = k_{large} + 7,$$

to produce a unique GCM sphere S_*, which is a deformation of $\overset{\circ}{S}$, satisfying

$$\begin{aligned} &\check{\kappa}^{S_*} = 0, \qquad \check{\underline{\kappa}}^{S_*} = 0, \qquad \check{\mu}^{S_*} = \sum_p M_p^{S_*} J^{(p,S_*)}, \\ &(\text{div}^{S_*} \beta^{S_*})_{\ell=1} = 0, \\ &(\text{curl}^{S_*} \beta^{S_*})_{\ell=1,\pm} = 0, \qquad (\text{curl}^{S_*} \beta^{S_*})_{\ell=1,0} = \frac{2a^{S_*} m^{S_*}}{(r^{S_*})^5}, \end{aligned} \tag{8.149}$$

where

- $J^{(p,S_*)}$ denotes the canonical basis of $\ell = 1$ mode on S_* in the sense of Definition 3.10 of [41] (recalled here in Definition 5.3),
- the $\ell = 1$ modes in (8.149) are defined w.r.t. the basis of $\ell = 1$ modes $J^{(p,S_*)}$,
- m^{S_*} denotes the Hawking mass of S_*, r^{S_*} denotes the area radius of S_*, and the identity for $(\text{curl}^{S_*} \beta^{S_*})_{\ell=1,0}$ in (8.149) should be understood as providing the definition of a^{S_*}.

Remark 8.38. *In order to apply Theorem 7.3 of [41] (restated here as Theorem 8.7) and Corollary 7.7 of [41] (restated here as Corollary 8.8) to the above setting, one needs to check that foliation of* ${}^{(ext)}\widetilde{\mathcal{L}_0}$ *satisfies the assumptions of the theorem, and in particular, in the region* $r \sim r_{(0)}$ *of* ${}^{(ext)}\widetilde{\mathcal{L}_0}$*,*

$$r^5 \left(|(div\,\beta)_{\ell=1}| + |(curl\,\beta)_{\ell=1,\pm}| + \left| (curl\,\beta)_{\ell=1,0} - \frac{2a_0 m_0}{r^5} \right| \right) + r^3 |(\check{\kappa})_{\ell=1}| + r^3 |(\check{\underline{\kappa}})_{\ell=1}| \lesssim \overset{\circ}{\delta},$$

as well as

$$r^2 |\mathfrak{d}^k \Gamma_b| \lesssim \delta_*, \qquad k \le s_{max}.$$

Now, in view of the above choice for s_{max}, $\overset{\circ}{\delta}$, $\overset{\circ}{\epsilon}$ *and* $r_{(0)}$, *this follows from*

$$r|\mathfrak{d}^{\le k_{large}+7} \Gamma_b| \lesssim \epsilon_0$$

and

$$\sup_{{}^{(ext)}\widetilde{\mathcal{L}}_0\cap\{r\sim\epsilon_0^{-1}\}} r^5\left(|div\,\beta|+\left|curl\,\beta-\frac{6a_0m_0}{r^5}J^{(0)}\right|\right) \\ +\sup_{{}^{(ext)}\widetilde{\mathcal{L}}_0\cap\{r\sim\epsilon_0^{-1}\}} r^3\left(|\check{\kappa}|+|\check{\underline{\kappa}}|\right) \lesssim \epsilon_0$$

and hence, in view of Proposition 8.20, from

$$\mathfrak{I}_{k_{large}+10}\leq\epsilon_0, \qquad {}^{(ext)}\mathfrak{I}_3\leq\epsilon_0^2.$$

From now on, we denote for simplicity

$$m := m^{S_*}, \qquad a := a^{S_*}. \tag{8.150}$$

Step 2. Starting from S_* constructed in Step 1, and relying on the control provided by Proposition 8.20 for the foliation of ${}^{(ext)}\widetilde{\mathcal{L}}_0$, we may then apply Theorem 4.1 in [50] (restated here in Theorem 8.11), with $s_{max}=k_{large}+7$, which yields the existence of a smooth small piece of spacelike hypersurface Σ_* passing through the sphere S_*, together with a scalar function u defined on Σ_*, whose level surfaces are topological spheres denoted by S, so that

- The following GCM conditions are verified on Σ_*

$$\check{\kappa}=0, \qquad \check{\underline{\kappa}}=\underline{C}_0+\sum_p\underline{C}_pJ^{(p)}, \qquad \check{\mu}=M_0+\sum_p M_pJ^{(p)},$$
$$(\mathrm{div}\,\eta)_{\ell=1}=0, \qquad (\mathrm{div}\,\underline{\xi})_{\ell=1}=0.$$

- $\underline{C}_0$, $\underline{C}_p$, M_0 and M_p are constant on each leaf of the u-foliation of Σ_*.
- We have, for some constant c_{Σ_*},

$$u+r=c_{\Sigma_*}, \quad \text{along} \quad \Sigma_*,$$

  where r denotes the area radius of the spheres S of the u-foliation of Σ_*.
- The following normalization condition holds true

$$\overline{b_*}=-1-\frac{2m}{r},$$

  where $\overline{b_*}$ denotes the average of b_* on the spheres foliating Σ_*, and where b_* is such that we have

$$\nu=e_3+b_*e_4,$$

with ν the unique vectorfield tangent to the hypersurface Σ_*, normal to S, and normalized by $\mathbf{g}(\nu, e_4) = -2$.

- The basis of $\ell = 1$ modes $J^{(p)}$ is given by $J^{(p)} = J^{(p,S_*)}$ on S_*, and extended to Σ_* by $\nu(J^{(p)}) = 0$. Also, the $\ell = 1$ modes of $\text{div}\,\eta$ and $\text{div}\,\underline{\xi}$ above are computed with respect to this basis.

Furthermore, we have[185]

$$\max_{k\leq k_{large}+6}\sup_{\Sigma_*} r\Big(|\mathfrak{d}^k f| + |\mathfrak{d}^k \underline{f}| + |\mathfrak{d}^k \log(\lambda)|\Big) \lesssim \epsilon_0, \tag{8.151}$$

and

$$|m - m_0| + \sup_{\Sigma_*}|r - r_{(0)}| \lesssim \epsilon_0, \tag{8.152}$$

where $(f, \underline{f}, \lambda)$ are the transition function from the frame of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ to the frame of Σ_*.

Remark 8.39. *To fix u, we need to pick a specific constant c_{Σ_*} such that $u + r = c_{\Sigma_*}$ along Σ_*. We choose $c_{\Sigma_*} = 1 + r(S_1)$ where S_1 is the only sphere of Σ_* intersecting the curve of the south poles[186] of the outgoing null cone $\{\widetilde{u} = 1\}$ of ${}^{(ext)}\widetilde{\mathcal{L}_0}$.*

Step 3. From now on, u is calibrated according to[187] Remark 8.39, which also fixes the sphere $S_1 = \Sigma_* \cap \{u = 1\}$. We can then compare $\mathring{u} = 1 + \delta_0$ to $u(S_*)$ and obtain

$$|u(S_*) - 1 - \delta_0| \lesssim \epsilon_0\delta_0,$$

[185] We have in fact

$$\max_{k\leq k_{large}+8}\sup_{\Sigma_*}\Big(\|\mathfrak{d}^k f\|_{L^2(S)} + \|\mathfrak{d}^k \underline{f}\|_{L^2(S)} + \|\mathfrak{d}^k \log(\lambda)\|_{L^2(S)}\Big) \lesssim \epsilon_0,$$

and then use the Sobolev embedding on the 2-spheres S foliating Σ_* to deduce (8.151).

[186] Note that this curve is transversal to Σ_* and hence intersect Σ_* at exactly one point given by

$$\Sigma_* \cap (\{\widetilde{u} = 1\} \cap \{\widetilde{\theta} = \pi\}).$$

[187] Indeed, provided $\delta_0 > 0$ has been chosen sufficiently small, the spacelike hypersurface Σ_* of Step 2 intersects the curve of the south poles of the spheres foliating the outgoing cone $\{\widetilde{u} = 1\}$ of the part ${}^{(ext)}\widetilde{\mathcal{L}_0}$ of the initial data layer, which allows to calibrate u as in Remark 8.39.

so that

$$1 \leq u \leq u(S_*) \quad \text{ on } \Sigma_* \text{ where } 1 < u(S_*) < 1 + 2\delta_0.$$

Together with the estimate (8.152), and in view of the choice (8.148) for $r_{(0)}$, we have

$$\begin{aligned} r(S_*) &= r_{(0)} + O(\epsilon_0) = d_0\delta_*\epsilon_0^{-1} + O(\epsilon_0) \\ &= \delta_*\epsilon_0^{-1}(u(S_*))^{1+\delta_{dec}}\left(d_0 + O(\delta_0) + O\left(\delta_*^{-1}\epsilon_0^2\right)\right). \end{aligned}$$

Thus, we may choose the constant d_0 in the range $\frac{1}{2} \leq d_0 \leq 2$ such that

$$r(S_*) = \delta_*\epsilon_0^{-1}(u(S_*))^{1+\delta_{dec}}$$

so that the condition (3.50) for r is satisfied.

Step 4. In view of Step 1 to Step 3, Σ_* satisfies all the required properties for the future spacelike boundary of a GCM admissible spacetime, see Section 3.2.3. We now introduce

- the outgoing geodesic frame $(\widetilde{e}_4, \widetilde{e}_3, \widetilde{e}_1, \widetilde{e}_2)$ of ${}^{(ext)}\widetilde{\mathcal{L}_0}$,
- the outgoing PG frame $((e_0)_4, (e_0)_3, (e_0)_1, (e_0)_2)$ of ${}^{(ext)}\mathcal{L}_0$,
- the outgoing PG frame (e'_4, e'_3, e'_1, e'_2) initialized on Σ_* from the GCM frame (e_4, e_3, e_1, e_2) by the change of frame with coefficients $(f'', \underline{f}'', \lambda'')$ given by

$$\lambda'' = 1, \qquad f'' = \frac{a}{r}f_0, \qquad \underline{f}'' = -\frac{(\nu(r) - b_*)}{1 - \frac{1}{4}b_*|f''|^2}f'',$$

  where the 1-form f_0 is chosen on Σ_* by

$$(f_0)_1 = 0, \quad (f_0)_2 = \sin(\theta), \quad \text{on} \quad S_*, \qquad \nabla_\nu f_0 = 0 \quad \text{on} \quad \Sigma_*,$$

  with (e_1, e_2) specified on S_* by (2.51).

We have the following change of frame coefficients:

- $(f, \underline{f}, \lambda)$, introduced in Step 2, and corresponding to the change from the outgoing geodesic frame $(\widetilde{e}_4, \widetilde{e}_3, \widetilde{e}_1, \widetilde{e}_2)$ of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ to the GCM frame (e_4, e_3, e_1, e_2) of Σ_*,
- $(f', \underline{f}', \lambda')$, which we now introduce, corresponding to the change from the outgoing geodesic frame $(\widetilde{e}_4, \widetilde{e}_3, \widetilde{e}_1, \widetilde{e}_2)$ of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ to the outgoing PG frame $((e_0)_4, (e_0)_3, (e_0)_1, (e_0)_2)$ of ${}^{(ext)}\mathcal{L}_0$,

- $(f'', \underline{f}'', \lambda'')$, provided explicitly above, and corresponding to the change from the GCM frame (e_4, e_3, e_1, e_2) of Σ_* to the PG frame (e_4', e_3', e_1', e_2'),
- $(f''', \underline{f}''', \lambda''')$, which we now introduce, corresponding to the change from the outgoing PG frame $((e_0)_4, (e_0)_3, (e_0)_1, (e_0)_2)$ of the part ${}^{(ext)}\mathcal{L}_0$ of the initial data layer to the outgoing PG frame (e_4', e_3', e_1', e_2') initialized on Σ_*.

In this step, our goal is to control the change of frame coefficients $(f''', \underline{f}''', \lambda''')$. In view of the above, we have schematically

$$(f''', \underline{f}''', \lambda''') = (f'', \underline{f}'', \lambda'') \circ (f, \underline{f}, \lambda) \circ (f', \underline{f}', \lambda')^{-1}$$

where $(f', \underline{f}', \lambda')^{-1}$ denote the coefficients corresponding to the inverse transformation coefficients of the transformation with coefficients $(f', \underline{f}', \lambda')$. We infer

$$\begin{aligned}\sup_{\Sigma_*}\left|\mathfrak{d}_*^k(f''', \underline{f}''', \lambda'''-1)\right| \;\lesssim\; & \sup_{\Sigma_*}\left|\mathfrak{d}_*^k(f, \underline{f}, \lambda-1)\right| \\ & +\sup_{\Sigma_*}\left|\mathfrak{d}_*^k(f''-f', \underline{f}''-\underline{f}', \lambda''-\lambda')\right|.\end{aligned}$$

Together with (8.151), we infer, for $k\leq k_{large}+6$,

$$\sup_{\Sigma_*} r\left|\mathfrak{d}_*^k(f''', \underline{f}''', \lambda'''-1)\right| \;\lesssim\; \epsilon_0+\sup_{\Sigma_*} r\left|\mathfrak{d}_*^k(f''-f', \underline{f}''-\underline{f}', \lambda''-\lambda')\right|.$$

Together with the explicit formulas above for $(f'', \underline{f}'', \lambda'')$, we obtain, for $k\leq k_{large}+6$,

$$\begin{aligned}&\sup_{\Sigma_*} r\left|\mathfrak{d}_*^k(f''', \underline{f}''', \lambda'''-1)\right| \\ \lesssim\;& \epsilon_0+\sup_{\Sigma_*} r\left|\mathfrak{d}_*^k\left(f'-\frac{a}{r}f_0, \underline{f}'+\frac{(\nu(r)-b_*)}{1-\frac{1}{4}b_*\frac{a^2}{r^2}|f_0|^2}\frac{a}{r}f_0, \lambda'-1\right)\right|.\end{aligned}$$

We deduce, using also the control (8.152) of $m-m_0$, for $k\leq k_{large}+6$,

$$\begin{aligned}&\sup_{\Sigma_*} r\left|\mathfrak{d}_*^k(f''', \underline{f}''', \lambda'''-1)\right| \\ \lesssim\;& \epsilon_0+|a-a_0| \\ & +\sup_{\Sigma_*} r\left|\mathfrak{d}_*^k\left(f'-\frac{a_0}{{}^{(ext)}r_{\mathcal{L}_0}}{}^{(\mathcal{L}_0)}f_0, \underline{f}'-\frac{a_0\left(1-\frac{2m_0}{{}^{(ext)}r_{\mathcal{L}_0}}\right)}{{}^{(ext)}r_{\mathcal{L}_0}}{}^{(\mathcal{L}_0)}f_0, \lambda'-1\right)\right|\end{aligned}$$

$$+\sup_{\Sigma_*}\left(|\mathfrak{d}_*^k(f_0 - {}^{(\mathcal{L}_0)}f_0)| + r^{-1}|\mathfrak{d}_*^k(r - {}^{(ext)}r_{\mathcal{L}_0})| + \left|\mathfrak{d}_*^k\left(b_* + 1 + \frac{2m}{r}\right)\right|\right.$$
$$\left. + |\mathfrak{d}_*^k(\nu(r)+2)|\right).$$

Also, since the change of frame coefficients $(f', \underline{f}', \lambda')$ correspond to the change from the outgoing geodesic frame $(\widetilde{e}_4, \widetilde{e}_3, \widetilde{e}_1, \widetilde{e}_2)$ of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ to the outgoing PG frame $((e_0)_4, (e_0)_3, (e_0)_1, (e_0)_2)$ of ${}^{(ext)}\mathcal{L}_0$, we have by the control provided by Proposition 8.20, for $k \leq k_{large} + 7$,

$$\sup_{\Sigma_*} r\left|\mathfrak{d}_*^k\left(f' - \frac{a_0}{{}^{(ext)}r_{\mathcal{L}_0}}{}^{(\mathcal{L}_0)}f_0, \underline{f}' - \frac{a_0\left(1 - \frac{2m_0}{{}^{(ext)}r_{\mathcal{L}_0}}\right)}{{}^{(ext)}r_{\mathcal{L}_0}}{}^{(\mathcal{L}_0)}f_0, \lambda' - 1\right)\right| \lesssim \epsilon_0.$$

We deduce, for $k \leq k_{large} + 6$,

$$\begin{aligned}\sup_{\Sigma_*} r\left|\mathfrak{d}_*^k(f''', \underline{f}''', \lambda''' - 1)\right| \\ \lesssim \quad & \epsilon_0 + \sup_{\widetilde{\Sigma}_*}\left(|\mathfrak{d}_*^k(af_0 - a_0{}^{(\mathcal{L}_0)}f_0)| + r^{-1}|\mathfrak{d}_*^k(r - {}^{(ext)}r_{\mathcal{L}_0})|\right. \\ & \left. + \left|\mathfrak{d}_*^k\left(b_* + 1 + \frac{2m}{r}\right)\right| + |\mathfrak{d}_*^k(\nu(r)+2)|\right).\end{aligned} \tag{8.153}$$

Step 5. In this step, we focus on the control of the terms on the RHS of (8.153). To this end, we first estimate $r - {}^{(ext)}r_{\mathcal{L}_0}$ on Σ_*. In view of the control of the part ${}^{(ext)}\widetilde{\mathcal{L}_0}$ of the initial data layer, we have, for $k \leq k_{large} + 7$,

$$\sup_{{}^{(ext)}\widetilde{\mathcal{L}_0}} (\widetilde{r})^2|\widetilde{\check{\kappa}}| \lesssim \epsilon_0.$$

Together with the GCM condition $\check{\kappa} = 0$, we infer, for $k \leq k_{large} + 7$,

$$\sup_{\Sigma_*} r^2\left|\mathfrak{d}_*^k\left(\check{\kappa} - \widetilde{\check{\kappa}}\right)\right| \lesssim \epsilon_0.$$

Now, we have

$$\check{\kappa} - \widetilde{\check{\kappa}} = \kappa - \widetilde{\kappa} - \frac{2}{r} + \frac{2}{\widetilde{r}} = \kappa - \widetilde{\kappa} - \frac{2(\widetilde{r} - r)}{r\widetilde{r}}$$

so that

$$r-\widetilde{r} \;=\; \frac{r\widetilde{r}}{2}\left(\kappa-\widetilde{\kappa}-\left(\breve{\kappa}-\breve{\widetilde{\kappa}}\right)\right)$$

and hence, using the above estimate for $\breve{\kappa}-\breve{\widetilde{\kappa}}$, we have, for $k\leq k_{large}+7$,

$$\sup_{\Sigma_*}\left|\mathfrak{d}_*^k(r-\widetilde{r})\right| \;\lesssim\; \epsilon_0+\sup_{\Sigma_*}r^2\left|\mathfrak{d}_*^k(\kappa-\widetilde{\kappa})\right|.$$

Using the change of frame formula for κ, together with the control (8.151) for $(f,\underline{f},\lambda)$ and the part ${}^{(ext)}\widetilde{\mathcal{L}_0}$ of the initial data layer, we deduce, for $k\leq k_{large}+5$,

$$\sup_{\Sigma_*}\left|\mathfrak{d}_*^k(r-\widetilde{r})\right| \;\lesssim\; \epsilon_0. \tag{8.154}$$

Together with the control of $\widetilde{r}-{}^{(ext)}r_{\mathcal{L}_0}$ provided by Proposition 8.20, we infer, for $k\leq k_{large}+5$,

$$\sup_{\Sigma_*}\left|\mathfrak{d}_*^k\left(r-{}^{(ext)}r_{\mathcal{L}_0}\right)\right| \;\lesssim\; \epsilon_0. \tag{8.155}$$

Next, we control $b_*+1+\frac{2m}{r}$ and $\nu(r)+2$. First, note that we have

$$\nu(r-\widetilde{r}) \;=\; \nu(r)-e_3(\widetilde{r})-b_*e_4(\widetilde{r}).$$

Together with the control of $r-\widetilde{r}$ in (8.154), the fact that ν is tangent to Σ_*, the change of frame formulas, the control (8.151) for $(f,\underline{f},\lambda)$ and the control of the part ${}^{(ext)}\widetilde{\mathcal{L}_0}$ of the initial data layer, we deduce for $k\leq k_{large}+4$,

$$\sup_{\Sigma_*}\left|\mathfrak{d}_*^k\left(\nu(r)+1-\frac{2m_0}{\widetilde{r}}-b_*\right)\right| \;\lesssim\; \epsilon_0.$$

Together with the control of $r-\widetilde{r}$ in (8.154) and the control (8.152) of $m-m_0$, we infer, for $k\leq k_{large}+4$,

$$\sup_{\Sigma_*}\left|\mathfrak{d}_*^k\left(\nu(r)+1-\frac{2m}{r}-b_*\right)\right| \;\lesssim\; \epsilon_0. \tag{8.156}$$

Since $\nu(u+r)=0$ on Σ_*, we infer for $k\leq k_{large}+4$

$$\sup_{\Sigma_*}\left|\mathfrak{d}_*^k\left(b_*+\nu(u)-\left(1-\frac{2m}{r}\right)\right)\right| \;\lesssim\; \epsilon_0.$$

Now, as part of the construction of Σ_*, the following transversality conditions on Σ_* are assumed, see (8.47) in Theorem 8.11,

$$\xi=\omega=0, \qquad \underline{\eta}=-\zeta, \qquad e_4(r)=1, \qquad e_4(u)=0. \tag{8.157}$$

We infer

$$\nu(u)=e_3(u)+b_*e_4(u)=e_3(u)$$

and hence, for $k\leq k_{large}+4$,

$$\sup_{\Sigma_*}\left|\mathfrak{d}_*^k\left(b_*+e_3(u)-\left(1-\frac{2m}{r}\right)\right)\right| \;\lesssim\; \epsilon_0.$$

Also, using again the transversality conditions (8.157), we have

$$\nabla(e_3(u))=(\zeta-\eta)e_3(u).$$

We deduce, for $k\leq k_{large}+3$,

$$\sup_{\Sigma_*} r\left|\mathfrak{d}_*^k\left(\nabla(b_*)-(\zeta-\eta)\left(b_*-\left(1-\frac{2m}{r}\right)\right)\right)\right| \;\lesssim\; \epsilon_0.$$

The control for ζ and η inferred from the transformation formula, the control of $(f,\underline{f},\lambda)$ and the control of the initial data layer implies, for $k\leq k_{large}+2$,

$$\left\|\mathfrak{d}_*^k\nabla(b_*)\right\|_{\mathfrak{h}_1(S)} \;\lesssim\; \epsilon_0+r^{-1}\epsilon_0\left\|\mathfrak{d}_*^k(b_*)\right\|_{\mathfrak{h}_1(S)}.$$

Also, by our GCM condition on Σ_* for b_*, we have

$$\overline{b_*}=-1-\frac{2m}{r} \quad \text{on} \quad \Sigma_*,$$

and hence, since ν is tangent to Σ_*, we have

$$\nu^k\left(\overline{b_*}+1+\frac{2m}{r}\right)=0 \quad \text{on} \quad \Sigma_*.$$

Thus, introducing the scalar h on Σ_* given by

$$h \;:=\; b_*+1+\frac{2m}{r},$$

we have obtained so far on Σ_*, for $k\leq k_{large}+2$,

$$\left\|\mathfrak{d}_*^k\nabla(h)\right\|_{\mathfrak{h}_1(S)} \lesssim \epsilon_0+r^{-1}\epsilon_0\left\|\mathfrak{d}_*^k(h)\right\|_{\mathfrak{h}_1(S)}$$

and for any k

$$\nu^k\left(\overline{h}\right)=0 \quad \text{on} \quad \Sigma_*.$$

Together with Corollary 5.32, we deduce from the above identity, for any k,

$$\overline{\nu^k(h)} = \nu^{\leq k}(r\Gamma_b\cdot h).$$

The control of Γ_b inferred from the transformation formula, the control of $(f,\underline{f},\lambda)$ and the control of the initial data layer implies, for $k\leq k_{large}+2$

$$|\overline{\nu^k(h)}| \lesssim \epsilon_0|\overline{\nu^{\leq k}(h)}|+\epsilon_0\sum_{j=0}^k|\nu^jh-\overline{\nu^jh}|$$

and hence, by iteration and together with Poincaré and Sobolev, we infer, for $k\leq k_{large}+2$,

$$|\overline{\nu^k(h)}| \lesssim \epsilon_0\left\|\nabla\mathfrak{d}_*^{\leq k}h\right\|_{\mathfrak{h}_1(S)}.$$

In view of the above, and using again Poincaré inequality, we deduce for $k\leq k_{large}+2$,

$$\begin{aligned} r^{-1}\left\|\mathfrak{d}_*^kh\right\|_{\mathfrak{h}_2(S)} &\lesssim |\overline{\nu^k(h)}|+\left\|\mathfrak{d}_*^k\nabla h\right\|_{\mathfrak{h}_1(S)}\lesssim\left\|\mathfrak{d}_*^{\leq k}\nabla h\right\|_{\mathfrak{h}_1(S)}\\ &\lesssim r^{-1}\epsilon_0\left\|\mathfrak{d}_*^{\leq k}h\right\|_{\mathfrak{h}_1(S)}+\epsilon_0. \end{aligned}$$

For ϵ_0 small enough, we infer, for $k\leq k_{large}+2$,

$$r^{-1}\left\|\mathfrak{d}_*^kh\right\|_{\mathfrak{h}_2(S)} \lesssim \epsilon_0.$$

Using Sobolev, and recalling the definition of h, we infer, for $k\leq k_{large}+2$,

$$\sup_{\Sigma_*}\left|\mathfrak{d}_*^k\left(b_*+1+\frac{2m}{r}\right)\right| \lesssim \epsilon_0.$$

Together with (8.156), we have obtain, for $k \le k_{large}+2$,

$$\sup_{\Sigma_*}\left|\mathfrak{d}_*^k\left(b_*+1+\frac{2m}{r},\, \nu(r)+2\right)\right| \;\lesssim\; \epsilon_0. \tag{8.158}$$

In view of (8.153) and (8.155), this yields, for $k \le k_{large}+2$,

$$\sup_{\Sigma_*} r\left|\mathfrak{d}_*^k(f''', \underline{f}''', \lambda'''-1)\right| \;\lesssim\; \epsilon_0 + \sup_{\widetilde{\Sigma}_*}|\mathfrak{d}_*^k(af_0 - a_0{}^{(\mathcal{L}_0)}f_0)|. \tag{8.159}$$

Step 6. In this step, we focus on the control of the terms on the RHS of (8.159). To this end, we first control a. We have in view of Proposition 8.20

$$\sup_{{}^{(ext)}\widetilde{\mathcal{L}}_0(\widetilde{r}\sim\epsilon_0^{-1})}(\widetilde{r})^4\left|\widetilde{\mathrm{curl}}\,\widetilde{\beta} - \frac{6a_0m_0}{(\widetilde{r})^5}\widetilde{J}^{(0)}\right| \lesssim \epsilon_0.$$

Since, in view of the transformation formulas, the control of $(f, \underline{f}, \lambda)$ and the control provided by Proposition 8.20 for the foliation of ${}^{(ext)}\widetilde{\mathcal{L}_0}$, we have

$$\sup_{S_*}\left|\mathrm{curl}\,\beta - \widetilde{\mathrm{curl}}\,\widetilde{\beta}\right| \;\lesssim\; \frac{\epsilon_0}{r^5_{(0)}},$$

we infer

$$\max_{p=0,+,-}(\widetilde{r})^5\left|\frac{1}{|S_*|}\int_{S_*}\mathrm{curl}\,\beta J^{(p)} - \frac{1}{|S_*|}\int_{S_*}\frac{6a_0m_0}{(\widetilde{r})^5}\widetilde{J}^{(0)}J^{(p)}\right| \;\lesssim\; \epsilon_0.$$

Recalling from (8.149) that the following holds on the sphere S_* of Σ_*

$$\frac{1}{|S_*|}\int_{S_*}\mathrm{curl}\,\beta J^{(0)} = \frac{2am}{r^5}, \qquad \frac{1}{|S_*|}\int_{S_*}\mathrm{curl}\,\beta J^{(\pm)} = 0,$$

we infer, using also (8.154) to control $r-\widetilde{r}$ on S_*,

$$\left|am - 3a_0m_0\frac{1}{|S_*|}\int_{S_*}\widetilde{J}^{(0)}J^{(0)}\right| + |a_0|m_0\left|\frac{1}{|S_*|}\int_{S_*}\widetilde{J}^{(0)}J^{(\pm)}\right| \;\lesssim\; \epsilon_0.$$

Together with the control of $m-m_0$ in (8.152), and dividing by m_0, we obtain

$$\left|a - 3a_0\frac{1}{|S_*|}\int_{S_*}\widetilde{J}^{(0)}J^{(0)}\right| + |a_0|\left|\frac{1}{|S_*|}\int_{S_*}\widetilde{J}^{(0)}J^{(\pm)}\right| \;\lesssim\; \epsilon_0.$$

Next, in view of Corollary 7.2 of [41] (restated here as Corollary 8.6), there exists a canonical basis of $\ell = 1$ modes on S_* in the sense of Definition

3.10 of [41] (recalled here in Definition 5.3), which we denote by $J_0^{(p,S_*)}$, such that

$$\max_{p=0,+,-} \left\| J_0^{(p,S_*)} - \widetilde{J}^{(p)} \right\|_{\mathfrak{h}_{k_{large}+7}(S_*)} \lesssim \epsilon_0.$$

Also, recall that $J^{(p)} = J^{(p,S_*)}$ on S_*, where $J^{(p,S_*)}$ is in general another canonical basis of $\ell = 1$ modes on S_*. In view of Definition 5.3, note that the canonical basis of $\ell = 1$ modes on S_* are unique modulo isometries of $\mathbb{S}^2$, i.e. there exists $O \in O(3)$ such that

$$J^{(p,S_*)} = \sum_{q=0,+,-} O_{pq} J_0^{(q,S_*)}, \qquad p = 0, +, -. \tag{8.160}$$

Remark 8.40. *In general, we have $O \neq I$ in (8.160). In fact, the role of O corresponds in Step 1 to the application of Corollary 8.8 which ensures that the following holds on S_* w.r.t. the canonical basis of $\ell = 1$ modes $J^{(p,S_*)}$, see (8.149),*

$$(curl\, \beta)_{\ell=1,\pm} = 0.$$

This corresponds to fixing the axis of S_*. Note that this condition (and hence the axis of S_*) is preserved by multiplying the basis $J^{(p,S_*)}$ by $O = -I$ or by any O fixing $J^{(0,S_*)}$, so that we may assume in (8.160) that O satisfies

$$O_{00} \geq 0, \qquad O_{++} \geq 0, \qquad O_{+-} = 0, \qquad O_{--} \geq 0. \tag{8.161}$$

Since $J^{(p)} = J^{(p,S_*)}$ on S_*, we infer

$$\max_{p=0,+,-} \left\| J^{(p)} - \sum_{q=0,+,-} O_{pq} \widetilde{J}^{(q)} \right\|_{\mathfrak{h}_{k_{large}+7}(S_*)} \lesssim \epsilon_0, \tag{8.162}$$

where O satisfies (8.161). Plugging (8.162) in the above, we deduce

$$\left| a - 3a_0 \sum_{q=0,+,-} O_{0q} \frac{1}{|S_*|} \int_{S_*} \widetilde{J}^{(0)} \widetilde{J}^{(q)} \right| + |a_0| \left| \frac{1}{|S_*|} \sum_{q=0,+,-} O_{\pm q} \int_{S_*} \widetilde{J}^{(0)} \widetilde{J}^{(q)} \right| \lesssim \epsilon_0.$$

Now, recall that $\overset{\circ}{S} = S(\overset{\circ}{u}, \overset{\circ}{s})$ is the sphere of the foliation of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ which shares the same south pole a S_*. Relying on Corollary 5.9 in [40] (see also Proposition 8.3 here), we have, for $q = 0, +, -$,

$$\begin{aligned}&\left|\int_{S_*} \widetilde{J}^{(0)}\widetilde{J}^{(q)} - \int_{\overset{\circ}{S}} \widetilde{J}^{(0)}\widetilde{J}^{(q)}\right| \\ \lesssim\;& r\epsilon_0 \left(\sup_{\widetilde{\mathcal{R}}} |\mathfrak{d}^{\leq 1}(\widetilde{J}^{(0)}\widetilde{J}^{(q)})| + r\sup_{\widetilde{\mathcal{R}}}(|\nabla_3(\widetilde{J}^{(0)}\widetilde{J}^{(q)})| + |\nabla_4(\widetilde{J}^{(0)}\widetilde{J}^{(q)})|)\right).\end{aligned}$$

Together with the control of $\widetilde{J}^{(p)}$ in ${}^{(ext)}\widetilde{\mathcal{L}_0}$ provided by Proposition 8.20, we deduce

$$\left|\int_{S_*} \widetilde{J}^{(0)}\widetilde{J}^{(q)} - \int_{\overset{\circ}{S}} \widetilde{J}^{(0)}\widetilde{J}^{(q)}\right| \lesssim r\epsilon_0.$$

Using the properties of $\widetilde{J}^{(p)}$ on the sphere $\overset{\circ}{S}$ of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ and the control of $r - \widetilde{r}$ in (8.154), this yields

$$\left|\frac{1}{|S_*|}\int_{S_*} \widetilde{J}^{(0)}\widetilde{J}^{(q)} - \frac{1}{3}\delta_{0q}\right| \lesssim \epsilon_0,$$

and hence, plugging in the above, we obtain

$$|a - a_0 O_{00}| + |a_0|\,|O_{+0}| + |a_0|\,|O_{-0}| \lesssim \epsilon_0. \tag{8.163}$$

Now, recall that we have either $a_0 = 0$ or $|a_0| \gg \epsilon_0$. In particular, we have in view of the above estimate

$$|a| \lesssim \epsilon_0 \quad \text{if} \quad a_0 = 0. \tag{8.164}$$

In the other case, we have, since $|a_0| \gg \epsilon_0$,

$$\left|\frac{a}{a_0} - O_{00}\right| + |O_{+0}| + |O_{-0}| \lesssim \epsilon_0 \quad \text{if} \quad a_0 \neq 0. \tag{8.165}$$

This allows us to control, in the case $a_0 = 0$, the change of frame coefficients $(f''', \underline{f}''', \lambda''')$ introduced in Step 4 from the outgoing PG frame $((e_0)_4, (e_0)_3, (e_0)_1, (e_0)_2)$ of the part ${}^{(ext)}\mathcal{L}_0$ of the initial data layer to the outgoing PG frame (e'_4, e'_3, e'_1, e'_2) initialized on Σ_*. Indeed, (8.159) and (8.164) yield, for $k \leq k_{large} + 2$,

$$\sup_{\Sigma_*} r\left|\mathfrak{d}_*^k(f''', \underline{f}''', \lambda''' - 1)\right| \lesssim \epsilon_0 \quad \text{if} \quad a_0 = 0. \tag{8.166}$$

Step 7. Next, we focus on controlling the RHS of (8.159) in the case $a_0 \neq 0$. Since $O \in O(3)$, we have $\sum_p O_{p0}^2 = 1$, and recalling also that $O_{00} \geq 0$ in view of (8.161), we infer from (8.165)

$$|a - a_0| \lesssim \epsilon_0 \quad \text{if} \quad a_0 \neq 0 \tag{8.167}$$

and

$$|O_{00} - 1| + |O_{+0}| + |O_{-0}| \lesssim \epsilon_0.$$

Also, since $O \in O(3)$, we also have

$$0 = \sum_p O_{p+} O_{p0} = O_{0+} + O(\epsilon_0), \qquad 0 = \sum_p O_{p-} O_{p0} = O_{0-} + O(\epsilon_0),$$

and hence

$$|O_{0+}| + |O_{0-}| \lesssim \epsilon_0.$$

Together with the fact that $O_{+-} = 0$ and $O_{--} \geq 0$ in view of (8.161), and since $\sum_p O_{p-}^2 = 1$, we infer

$$|O_{--} - 1| \lesssim \epsilon_0.$$

Finally $O_{++} \geq 0$ in view of (8.161), since we have obtained above that $|O_{0+}| \lesssim \epsilon_0$, and since $\sum_p O_{p-}^2 = 1$ and $\sum_p O_{p+} O_{p-} = 0$, we infer

$$|O_{++} - 1| + |O_{-+}| \lesssim \epsilon_0.$$

We have thus obtained

$$|O - I| \lesssim \epsilon_0,$$

which together with (8.162) implies

$$\max_{p=0,+,-} r^{-1} \left\| J^{(p)} - \widetilde{J}^{(q)} \right\|_{\mathfrak{h}_{k_{large}+7}(S_*)} \lesssim \epsilon_0 \quad \text{if} \quad a_0 \neq 0.$$

Next, we control $J^{(p)} - \widetilde{J}^{(p)}$ for $p = 0, +, -$ on Σ_*. Recall that we have $\nu(J^{(p)}) = 0$ along Σ_*. We infer

$$\nu\left(J^{(p)} - \widetilde{J}^{(p)}\right) = -\nu\left(\widetilde{J}^{(p)}\right) = -e_3\left(\widetilde{J}^{(p)}\right) - b_* e_4\left(\widetilde{J}^{(p)}\right).$$

Using the change of frame formulas, and the control (8.151) of the change of frame coefficients $(f, \underline{f}, \lambda)$, and the control of $\widetilde{J}^{(p)}$, we easily obtain, for $k \leq k_{large} + 6$,

$$\sup_{\Sigma_*} r \left| \mathfrak{d}_*^k \nu \left(J^{(p)} - \widetilde{J}^{(p)} \right) \right| \lesssim \epsilon_0.$$

Integrating along Σ_* from S_*, and using the above control on S_* and Sobolev, as well as the fact that $r \sim \epsilon_0^{-1}$ on Σ_*, we infer, for $k \leq k_{large} + 5$,

$$\max_{p=0,+,-} \sup_{\Sigma_*} \left| \mathfrak{d}_*^k \left(J^{(p)} - \widetilde{J}^{(p)} \right) \right| \lesssim \epsilon_0 \quad \text{if} \quad a_0 \neq 0. \tag{8.168}$$

Next, we control $f_0 - {}^{(\mathcal{L}_0)}f_0$ on Σ_*. First, from the change of frame formulas, the control (8.151) for $(f, \underline{f}, \lambda)$, and the control of the part ${}^{(ext)}\widetilde{\mathcal{L}_0}$ of the initial data layer, we have, for $k \leq k_{large} + 7$,

$$\sup_{\Sigma_*} \left(r^2 \left| \mathfrak{d}_*^k \Gamma_g \right| + r \left| \mathfrak{d}_*^k \Gamma_b \right| \right) \lesssim \epsilon_0. \tag{8.169}$$

Proceeding as in Proposition 5.59, we infer, for $k \leq k_{large} + 7$,

$$\sup_{\Sigma_*} r^2 \left| \mathfrak{d}_*^k \left(\nabla J^{(0)} + \frac{1}{r} {}^*f_0 \right) \right| \lesssim \epsilon_0.$$

Also, in view of the control of the part ${}^{(ext)}\widetilde{\mathcal{L}_0}$ of the initial data layer, we have, for $k \leq k_{large} + 7$,

$$\sup_{{}^{(ext)}\widetilde{\mathcal{L}_0}} (\widetilde{r})^2 \left| \mathfrak{d}^k \left(\widetilde{\nabla} \widetilde{J}^{(0)} + \frac{1}{\widetilde{r}} {}^*({}^{(\mathcal{L}_0)}f_0) \right) \right| \lesssim \epsilon_0.$$

Together with the control (8.155) for $r - {}^{(ext)}r_{\mathcal{L}_0}$, the change of frame formulas, the control (8.151) for $(f, \underline{f}, \lambda)$, and the control of the part ${}^{(ext)}\widetilde{\mathcal{L}_0}$ of the initial data layer, we infer, for $k \leq k_{large} + 7$,

$$\sup_{\Sigma_*} r^2 \left| \mathfrak{d}^k \left(\nabla \widetilde{J}^{(0)} + \frac{1}{r} {}^*({}^{(\mathcal{L}_0)}f_0) \right) \right| \lesssim \epsilon_0.$$

Subtracting the two estimates, we infer, for $k \leq k_{large} + 7$,

$$\sup_{\Sigma_*} r^2 \left| \mathfrak{d}_*^k \left(\nabla (J^{(0)} - \widetilde{J}^{(0)}) + \frac{1}{r} {}^*(f_0 - {}^{(\mathcal{L}_0)}f_0) \right) \right| \lesssim \epsilon_0.$$

Together with the above control for $J^{(0)} - \widetilde{J}^{(0)}$, we deduce, for $k \leq k_{large} + 7$,

$$\sup_{\Sigma_*} \left| \mathfrak{d}_*^k \left(f_0 - {}^{(\mathcal{L}_0)} f_0 \right) \right| \lesssim \epsilon_0 \quad \text{if} \quad a_0 \neq 0. \tag{8.170}$$

We are now ready to control, in the case $a_0 \neq 0$, the change of frame coefficients $(f''', \underline{f}''', \lambda''')$ introduced in Step 4 and corresponding to the change from the outgoing PG frame $((e_0)_4, (e_0)_3, (e_0)_1, (e_0)_2)$ of ${}^{(ext)}\mathcal{L}_0$ to the outgoing PG frame (e_4', e_3', e_1', e_2') initialized on Σ_*. Indeed, (8.159), and the above control of $a - a_0$, $J^{(0)} - \widetilde{J}^{(0)}$ and $f_0 - {}^{(\mathcal{L}_0)} f_0$ in the case $a_0 \neq 0$ yields, for $k \leq k_{large} + 5$,

$$\sup_{\Sigma_*} r \left| \mathfrak{d}_*^k (f''', \underline{f}''', \lambda''' - 1) \right| \lesssim \epsilon_0 \quad \text{if} \quad a_0 \neq 0. \tag{8.171}$$

We conclude this step with the control of $\mathfrak{J} - \mathfrak{J}_{\mathcal{L}_0}$ on Σ_* in the case $a_0 \neq 0$. Recall that $\mathfrak{J}$ is given on Σ_* by

$$\mathfrak{J} = \frac{1}{|q|} (f_0 + i \, {}^* f_0) \quad \text{on} \quad \Sigma_*.$$

Together with the above estimates for $a - a_0$, $f_0 - {}^{(\mathcal{L}_0)} f_0$, $r - \widetilde{r}$ and $J^{(0)} - \widetilde{J}^{(0)}$ in the case $a_0 \neq 0$, we infer, for $k \leq k_{large} + 7$,

$$\sup_{\Sigma_*} r \left| \mathfrak{d}_*^k (\mathfrak{J} - \mathfrak{J}_{\mathcal{L}_0}) \right| \lesssim \epsilon_0 + \sup_{{}^{(ext)}\widetilde{\mathcal{L}_0}} \widetilde{r} \left| \mathfrak{d}^k \left(\mathfrak{J}_{\mathcal{L}_0} - \frac{1}{|q_{\mathcal{L}_0}|} \left({}^{(\mathcal{L}_0)} f_0 + i \, {}^*({}^{(\mathcal{L}_0)} f_0) \right) \right) \right|$$

which together with the control provided by Proposition 8.20 for the part ${}^{(ext)}\widetilde{\mathcal{L}_0}$ of the initial data layer implies, for $k \leq k_{large} + 7$,

$$\sup_{\Sigma_*} r \left| \mathfrak{d}_*^k (\mathfrak{J} - \mathfrak{J}_{\mathcal{L}_0}) \right| \lesssim \epsilon_0 \quad \text{if} \quad a_0 \neq 0. \tag{8.172}$$

Finally, we have obtained in this step, for $k \leq k_{large} + 2$,

$$\begin{aligned} |a - a_0| + \sup_{\Sigma_*} \Bigg(& r \left| \mathfrak{d}_*^k (f''', \underline{f}''', \lambda''' - 1) \right| \\ & + \left| \mathfrak{d}_*^k \left(J^{(0)} - J^{(0)}_{\mathcal{L}_0} \right) \right| + r \left| \mathfrak{d}_*^k (\mathfrak{J} - \mathfrak{J}_{\mathcal{L}_0}) \right| \Bigg) \lesssim \epsilon_0 \quad \text{if} \quad a_0 \neq 0, \end{aligned} \tag{8.173}$$

where we have also used the control for $\widetilde{J}^{(0)} - J^{(0)}_{\mathcal{L}_0}$ provided by Proposition 8.20.

Step 8. We now control the outgoing PG structure initialized on Σ_*, and covering the region we denote by ${}^{(ext)}\mathcal{M}$, which is included in the initial data layer. For convenience, we change our notation. From now on:

- (e_4, e_3, e_1, e_2) denotes the outgoing PG frame of the part ${}^{(ext)}\mathcal{L}_0$ of the initial data layer,
- (e'_4, e'_3, e'_1, e'_2) denotes the outgoing PG frame of ${}^{(ext)}\mathcal{M}$ initialized on Σ_*,
- $(f, \underline{f}, \lambda)$ denote the transition coefficients from the outgoing PG frame (e_4, e_3, e_1, e_2) to the outgoing PG frame (e'_4, e'_3, e'_1, e'_2),
- $(r, f_0, J^{(0)}, \mathfrak{J})$ and $(r', f'_0, {J^{(0)}}', \mathfrak{J}')$ correspond respectively to the outgoing PG structure of ${}^{(ext)}\mathcal{L}_0$ and to the outgoing PG structure of ${}^{(ext)}\mathcal{M}$.

In view of (8.152), (8.155), (8.164), (8.166), (8.173), using the above new notations, and noticing that the structure equations in the e'_4 direction for the outgoing PG structure initialized on Σ_* allow to recover the e'_4 derivatives (which are transversal to Σ_*), we have, for $k \leq k_{large} + 2$,

$$
\begin{aligned}
&|m - m_0| + |a - a_0| + \sup_{\Sigma_*}\Bigg(r\left|\mathfrak{d}_*^k(f, \underline{f}, \lambda - 1)\right| + \left|\mathfrak{d}_*^k\,(r' - r)\right| \\
&\qquad + \left|\mathfrak{d}_*^k\left(a{J'}^{(0)} - a_0 J^{(0)}\right)\right| + r\left|\mathfrak{d}_*^k\,(a\mathfrak{J}' - a_0\mathfrak{J})\right|\Bigg) \lesssim \epsilon_0.
\end{aligned}
\tag{8.174}
$$

We introduce the notations

$$
F := f + i\,{}^*f, \qquad \underline{F} := \underline{f} + i\,{}^*\underline{f}.
$$

Since (e_4, e_3, e_1, e_2) and (e'_4, e'_3, e'_1, e'_2) are outgoing PG frames, we have

$$
\Xi = 0, \qquad \omega = 0, \qquad \underline{H} + Z = 0, \qquad \Xi' = 0, \qquad \omega' = 0, \qquad \underline{H}' + Z' = 0.
$$

In view of Corollary 2.14, we have the following transport equations

$$
\begin{aligned}
\nabla_{\lambda^{-1}e'_4}(qF) &= E_4(f, \Gamma),\\
\lambda^{-1}\nabla_{e'_4}(\log\lambda) &= 2f\cdot\zeta + E_2(f, \Gamma),\\
\nabla_{\lambda^{-1}e'_4}\left[q\Big(\underline{F} - 2q\mathcal{D}'(\log\lambda) + e_3(r)F\Big)\right] &= -3q^2\mathcal{D}'\,(f\cdot\zeta)\\
&\quad + E_5(\nabla'^{\leq 1}f, \underline{f}, \nabla'^{\leq 1}\lambda, \mathbf{D}^{\leq 1}\Gamma),
\end{aligned}
$$

where E_2, E_4 and E_5 are given in Corollary 2.14. Integrating these transport equations from Σ_* in the order they appear, using the control in (8.174) for

$(f, \underline{f}, \lambda)$ on Σ_*, and together with the control of the part ${}^{(ext)}\mathcal{L}_0$ of the initial data layer, we obtain, for $k \leq k_{large}+2$,

$$\sup_{{}^{(ext)}\mathcal{M}} r\Big(|\mathfrak{d}^k(f, \log(\lambda))| + |\mathfrak{d}^{k-1}\underline{f}|\Big) \lesssim \epsilon_0. \tag{8.175}$$

Also, we have

$$e_4'(r' - \lambda^{-1}r) \;=\; 1 - \lambda^{-1}e_4(r) + \lambda^{-1}e_4'(\log(\lambda)).$$

Using the change of frame formula and the above transport equation for $\log(\lambda)$, we infer

$$\begin{aligned} e_4(r' - \lambda^{-1}r) &= 1 - \left(e_4 + f\cdot\nabla + \frac{1}{4}|f|^2 e_3\right) r + \frac{3}{2} f\cdot\zeta + E_2(f, \Gamma) \\ &= -\frac{1}{4}|f|^2 e_3(r) + \frac{3}{2} f\cdot\zeta + E_2(f, \Gamma). \end{aligned}$$

Integrating from Σ_* where $r' - r$ is under control in view of (8.174), and using the control (8.175) for f and λ as well as the control of ${}^{(ext)}\mathcal{L}_0$, we infer, for $k \leq k_{large}+2$,

$$\sup_{{}^{(ext)}\mathcal{M}} \Big|\mathfrak{d}^k(r' - r)\Big| \;\lesssim\; \epsilon_0. \tag{8.176}$$

Also, we have

$$\begin{aligned} e_4'(aJ'^{(0)} - a_0J^{(0)}) &= -e_4'(a_0J^{(0)}) = -a_0\lambda\left(e_4 + f\cdot\nabla + \frac{1}{4}|f|^2 e_3\right) J^{(0)} \\ &= -a_0\lambda\left(f\cdot\nabla + \frac{1}{4}|f|^2 e_3\right) J^{(0)} \end{aligned}$$

and

$$\begin{aligned} \nabla_4'(aq'\mathfrak{J}' - a_0q\mathfrak{J}) &= -\nabla_4'(a_0q\mathfrak{J}) = -a_0\lambda\left(\nabla_4 + f\cdot\nabla + \frac{1}{4}|f|^2\nabla_3\right)(q\mathfrak{J}) \\ &= -a_0\lambda\left(f\cdot\nabla + \frac{1}{4}|f|^2\nabla_3\right)(q\mathfrak{J}). \end{aligned}$$

Integrating from Σ_* where $J'^{(0)} - J^{(0)}$ and $\mathfrak{J}' - \mathfrak{J}$ are under control in view of (8.174), and using the control (8.175) for f and λ as well as the control of ${}^{(ext)}\mathcal{L}_0$, we infer, for $k \leq k_{large}+2$,

$$\sup_{{}^{(ext)}\mathcal{M}} \left(\Big|\mathfrak{d}^k(aJ'^{(0)} - a_0J^{(0)})\Big| + r\Big|\mathfrak{d}^k(a\mathfrak{J}' - a_0\mathfrak{J})\Big|\right) \;\lesssim\; \epsilon_0. \tag{8.177}$$

Then, using the outgoing PG structure of ${}^{(ext)}\mathcal{M}$, we initialize

- the ingoing PG structure of ${}^{(int)}\mathcal{M}$ on $\mathcal{T}=\{r'=r_0\}$,
- the ingoing PG structure of ${}^{(top)}\mathcal{M}$ on $\{u'=u_*\}$,

as in Section 3.2.5. Using the control of $(f, \underline{f}, \lambda)$, $r'-r$, $J'^{(0)}-J^{(0)}$ and $\mathfrak{J}'-\mathfrak{J}$ induced on $\{r'=r_0\}$ and $\{u'=u_*\}$ by (8.175), (8.176) and (8.177), and using the analog in the e_3' direction for ingoing PG structures of the above transport equation in the e_4' direction for outgoing PG structures, we obtain for ${}^{(int)}\mathcal{M}$ and $k\leq k_{large}+1$

$$\begin{aligned}\sup_{{}^{(int)}\mathcal{M}}\Big(&|\mathfrak{d}^k(\underline{f}, \log(\lambda))|+|\mathfrak{d}^{k-1}f|+|\mathfrak{d}^k(r'-r)|\\ &+|\mathfrak{d}^k(aJ'^{(0)}-a_0J^{(0)})|+|\mathfrak{d}^k(a\mathfrak{J}'-a_0\mathfrak{J})|\Big) \;\lesssim\; \epsilon_0,\end{aligned}\tag{8.178}$$

and a similar estimate for ${}^{(top)}\mathcal{M}$.

Let now

$$\mathcal{M} \;:=\; {}^{(ext)}\mathcal{M}\cup{}^{(int)}\mathcal{M}\cup{}^{(top)}\mathcal{M}.$$

Then, in view of (8.175)–(8.178), the control of $a-a_0$ and $m-m_0$ in (8.174), and using the transformation formulas of Proposition 2.12, and well as the definition of the linearized quantities based on a, m, r, $J^{(0)}=\cos\theta$ and $\mathfrak{J}$, we deduce

$$\mathfrak{N}^{(Sup)}_{k_{large}}+\mathfrak{N}^{(Dec)}_{k_{small}} \;\lesssim\; \epsilon_0$$

which concludes the proof of Theorem M6.

8.5. Proof of Theorem M7

The proof of Theorem M7 proceeds in 18 steps which we summarize below for convenience:

1. In Steps 1–5, we use local existence to extend the spacetime $\mathcal{M}$ a little bit, and then focus on the region in the future of Σ_* in which we derive additional estimates.
2. In Steps 6–7, we construct a new last sphere $\widetilde{S}_*$, and a new last slice $\widetilde{\Sigma}_*$ inside the region of the extended spacetime in the future of Σ_* by relying on the control derived in Step 1–5 and the GCM constructions of [41] and [50] recalled in Section 8.1.

3. In Steps 8–11, we show that the new last slice $\widetilde{\Sigma}_*$, which a priori exists only in a small neighborhood of the new last sphere $\widetilde{S}_*$, extends in fact all the way to the initial data layer.
4. In Steps 12–13, we complete the proof of the fact that the new last slice $\widetilde{\Sigma}_*$ satisfies all the required properties for being the future spacelike boundary of a GCM admissible spacetime.
5. In Steps 14–18, we construct from $\widetilde{\Sigma}_*$ a new GCM admissible spacetime $\widetilde{\mathcal{M}}$ and we control the change of frame coefficients between the frames of the extended spacetime, and the corresponding frames of $\widetilde{\mathcal{M}}$. In view of the control of the extended spacetime and of the change of frames coefficients, we infer the desired control of $\widetilde{\mathcal{M}}$ thanks to the change of frame formulas.

We now proceed with the proof of Theorem M7. For convenience, we introduce the following notation

$$k_* \quad := \quad k_{small} + 20. \tag{8.179}$$

Then, in view of the assumptions, we are given a GCM admissible spacetime $\mathcal{M} = \mathcal{M}(u_*) \in \aleph(u_*)$ verifying the following improved bounds

$$\mathfrak{N}^{(Dec)}_{k_*}(\mathcal{M}) \leq C\epsilon_0, \tag{8.180}$$

for a universal constant $C > 0$ provided by Theorems M1–M5.

8.5.1. Steps 1–5

Step 1. We extend $\mathcal{M}$ by a local existence argument, to a strictly larger spacetime $\mathcal{M}^{(extend)}$, with a naturally extended foliation and the following slightly increased bounds[188]

$$\mathfrak{N}^{(Dec)}_{k_*-3}(\mathcal{M}^{(extend)}) \leq 2C\epsilon_0,$$

but which may not verify our admissibility criteria.

Step 2. We then invoke Theorem 4.1 in [50] (restated here in Theorem 8.11) to extend Σ_* in $\mathcal{M}^{(extend)} \setminus \mathcal{M}$ as a smooth spacelike hypersurface $\Sigma_*^{(extend)}$, together with a scalar function $u^{(extend)}$, satisfying the same GCM conditions than Σ_*.

[188]The loss of three derivatives occurs due to the fact that local existence holds in L^2 based spaces while $\mathfrak{N}^{(Dec)}_k$ is based on L^∞.

Step 3. We consider the outgoing geodesic foliation $(u^{(extend)}, s^{(extend)})$ initialized on $\Sigma_*^{(extend)}$ to the future of $\Sigma_*^{(extend)}$ in $\mathcal{M}^{(extend)}$. Note in particular that we have from the definition of Σ_* and $\Sigma_*^{(extend)}$

$$u^{(extend)} + s^{(extend)} = c_{\Sigma_*} \quad \text{on} \quad \Sigma_*^{(extend)}.$$

We define the following spacetime region to the future of $\Sigma_*^{(extend)}$

$$\widetilde{\mathcal{R}} := \Big\{1 \leq u^{(extend)} \leq u_* + \delta_{ext}, \quad c_{\Sigma_*} \leq u^{(extend)} + s^{(extend)} \leq c_{\Sigma_*} + \Delta_{ext}\Big\},$$

where

$$\Delta_{ext} := \frac{d_0 r_*}{u_*}\delta_{ext}, \qquad r_* := r(S_*), \qquad S_* := \Sigma_* \cap \mathcal{C}_*,$$

with $\delta_{ext} > 0$ chosen sufficiently small so that $\widetilde{\mathcal{R}} \subset \mathcal{M}^{(extend)}$, and with d_0 a constant satisfying

$$\frac{1}{2} \leq d_0 \leq 1$$

which will be suitably chosen in Step 12 below. From now on, for convenience, we drop the index $(extend)$ and simply denote $u^{(extend)}$ and $s^{(extend)}$ by u and s.

Step 4. On $\Sigma_*^{(extend)}$, the following GCM conditions hold by construction

$$\begin{aligned}
&\check{\kappa} = 0, \qquad \underline{\check{\kappa}} = \underline{C}_0 + \sum_p \underline{C}_p J^{(p)}, \qquad \check{\mu} = M_0 + \sum_p M_p J^{(p)},\\
&(\operatorname{div}\eta)_{\ell=1} = 0, \qquad (\operatorname{div}\underline{\xi})_{\ell=1} = 0,
\end{aligned} \tag{8.181}$$

where the basis of $\ell = 1$ modes satisfies in particular

$$\nu(J^{(p)}) = 0, \quad p = 0, +, -, \quad \text{along} \quad \Sigma_*^{(extend)}, \tag{8.182}$$

and where the scalar functions $\underline{C}_0$, $\underline{C}_p$, M_0 and M_p are constant on the leaves of the u-foliation of $\Sigma_*^{(extend)}$, i.e. they are functions of u along $\Sigma_*^{(extend)}$. We propagate $J^{(p)}$, $\underline{C}_0$, $\underline{C}_p$, M_0 and M_p from $\Sigma_*^{(extend)}$ to the spacetime region $\widetilde{\mathcal{R}}$ along e_4 as follows

$$\begin{aligned}
&e_4(J^{(p)}) = 0, \qquad e_4(r^2\underline{C}_0) = 0, \qquad e_4(r^2\underline{C}_p) = 0,\\
&e_4(r^3 M_0) = 0, \qquad e_4(r^3 M_p) = 0 \quad \text{on} \quad \widetilde{\mathcal{R}},
\end{aligned} \tag{8.183}$$

so that we have[189] $\underline{C}_0 = \underline{C}_0(u,s)$, $\underline{C}_p = \underline{C}_p(u,s)$, $M_0 = M_0(u,s)$, and $M_p = M_p(u,s)$ in $\widetilde{\mathcal{R}}$. In view of (8.183), we have in particular

$$\begin{aligned} e_4\left(r^2\left(\underline{\check{\kappa}} - \left(\underline{C}_0 + \sum_p \underline{C}_p J^{(p)}\right)\right)\right) &= e_4(r^2\underline{\check{\kappa}}), \\ e_4\left(r^3\left(\check{\mu} - \left(M_0 + \sum_p M_p J^{(p)}\right)\right)\right) &= e_4(r^3\check{\mu}). \end{aligned}$$

Propagating from $\Sigma_*^{(extend)}$ where (8.181) holds, and using the bounds of Step 1 on $\mathcal{M}^{(extend)}$, and hence on $\widetilde{\mathcal{R}}$, for $\check{\kappa}$, $\underline{\check{\kappa}}$, $\check{\mu}$, η and $\underline{\xi}$, we obtain, for all $k \leq k_* - 4$,

$$\begin{aligned} \sup_{\widetilde{\mathcal{R}}}\Bigg(r^2\left|\widehat{\mathfrak{d}}^k(\check{\kappa})\right| + r^2\left|\widehat{\mathfrak{d}}^k\left(\underline{\check{\kappa}} - \left(\underline{C}_0 + \sum_p \underline{C}_p J^{(p)}\right)\right)\right| \\ + r^3\left|\widehat{\mathfrak{d}}^k\left(\check{\mu} - \left(M_0 + \sum_p M_p J^{(p)}\right)\right)\right|\Bigg) &\lesssim \frac{\epsilon_0}{r}\Delta_{ext} \end{aligned} \tag{8.184}$$

and

$$\sup_{\widetilde{\mathcal{R}}} r^2\Big(|\widehat{\mathfrak{d}}^k(\mathrm{div}\,\eta)_{\ell=1}| + |\widehat{\mathfrak{d}}^k(\mathrm{div}\,\underline{\xi})_{\ell=1}|\Big) \lesssim \frac{\epsilon_0}{r}\Delta_{ext}, \tag{8.185}$$

where[190] $\widehat{\mathfrak{d}} = (e_3 - (e_3(u) + e_3(s))e_4, \not{\mathfrak{d}})$ denotes weighted derivatives tangential to the level hypersurfaces of $u+s$.

Next, recall that $\nu = e_3 + b_* e_4$ denotes the unique tangent vectorfield to Σ_* which is orthogonal to the u-foliation and normalized by $\mathbf{g}(\nu, e_4) = -2$. In view of Corollary 5.55, we have on $\Sigma_*^{(extend)}$

$$\begin{aligned} |\nu((\mathrm{div}\,\beta)_{\ell=1})| + |\nu((\mathrm{curl}\,\beta)_{\ell=1,\pm})| \\ + \left|\nu\left((\mathrm{curl}\,\beta)_{\ell=1,0} - \frac{2am}{r^5}\right)\right| &\lesssim \frac{\epsilon_0}{r^5 u^{1+\delta_{dec}}}, \end{aligned}$$

---

[189] More precisely, we have $\underline{C}_0 = r^{-2}\widetilde{\underline{C}}_0$, $\underline{C}_p = r^{-2}\widetilde{\underline{C}}_p$, $M_0 = r^{-3}\widetilde{M}_0$, and $M_p = r^{-3}\widetilde{M}_p$,, with $\widetilde{\underline{C}}_0$, $\widetilde{\underline{C}}_p$, $\widetilde{M}_0$ and $\widetilde{M}_p$ given by the restriction of $r^2\underline{C}_0$, $r^2\underline{C}_p$, r^3M_0 and r^3M_p to $\Sigma_*^{(extend)}$ so that $\widetilde{\underline{C}}_0 = \widetilde{\underline{C}}_0(u)$, $\widetilde{\underline{C}}_p = \widetilde{\underline{C}}_p(u)$, $\widetilde{M}_0 = \widetilde{M}_0(u)$ and $\widetilde{M}_p = \widetilde{M}_p(u)$. Note also that $r = r(u,s)$.

[190] Since $s = r$ and $u + r$ is constant on $\Sigma_*^{(extend)}$, the restriction of $\widehat{\mathfrak{d}}$ to $\Sigma_*^{(extend)}$ corresponds to the weighted derivatives tangent to $\Sigma_*^{(extend)}$. In particular, note that the identities (8.181) on $\Sigma_*^{(extend)}$ are preserved under differentiation by $\widehat{\mathfrak{d}}$.

$$|\nu((\check{\underline{\kappa}})_{\ell=1})| \lesssim \frac{\epsilon_0}{r^3 u^{1+\delta_{dec}}} + \frac{\epsilon_0^2}{r^2 u^{2+2\delta_{dec}}}.$$

In particular, since $r(S_*) = \delta_* \epsilon_0^{-1}(u(S_*))^{1+\delta_{dec}}$ in view of (3.50) and $u(S_*) = u_*$, we infer $r \sim \delta_* \epsilon_0^{-1} u^{1+\delta_{dec}}$ on $\Sigma_*^{(extend)}(u_* \leq u \leq u_* + \delta_{ext})$ and hence

$$\begin{aligned}|\nu((\mathrm{div}\,\beta)_{\ell=1})| + |\nu((\mathrm{curl}\,\beta)_{\ell=1,\pm})| \\ + \left|\nu\left((\mathrm{curl}\,\beta)_{\ell=1,0} - \frac{2am}{r^5}\right)\right| &\lesssim \frac{\epsilon_0}{r^5 u^{1+\delta_{dec}}}, \\ |\nu((\check{\underline{\kappa}})_{\ell=1})| &\lesssim \frac{\epsilon_0}{r^3 u^{1+\delta_{dec}}}.\end{aligned}$$

We integrate from S_* where we have

$$(\mathrm{div}\,\beta)_{\ell=1} = 0, \quad (\mathrm{curl}\,\beta)_{\ell=1,\pm} = 0, \quad (\mathrm{curl}\,\beta)_{\ell=1,0} = \frac{2am}{r^5}, \quad (\check{\underline{\kappa}})_{\ell=1} = 0,$$

and obtain

$$\begin{aligned}\sup_{\Sigma_*^{(extend)}(u_* \leq u \leq u_* + \delta_{ext})} \Bigg[r^5 \Bigg(|(\mathrm{div}\,\beta)_{\ell=1}| + |(\mathrm{curl}\,\beta)_{\ell=1,\pm}| \\ + \left|(\mathrm{curl}\,\beta)_{\ell=1,0} - \frac{2am}{r^5}\right| \Bigg) + r^3 |(\check{\underline{\kappa}})_{\ell=1}| \Bigg] \lesssim \frac{\epsilon_0}{u_*}\delta_{ext}.\end{aligned}$$

We now integrate in the e_4 direction from $\Sigma_*^{(extend)}(u_* \leq u \leq u_* + \delta_{ext})$ where we have the above estimate as well as $\check{\kappa} = 0$. We obtain

$$\begin{aligned}\sup_{\widetilde{\mathcal{R}} \cap \{u \geq u_*\}} \Bigg[r^5 \Bigg(|(\mathrm{div}\,\beta)_{\ell=1}| + |(\mathrm{curl}\,\beta)_{\ell=1,\pm}| + \left|(\mathrm{curl}\,\beta)_{\ell=1,0} - \frac{2am}{r^5}\right| \Bigg) \\ + r^3 |(\check{\kappa})_{\ell=1}| + r^3 |(\check{\underline{\kappa}})_{\ell=1}| \Bigg] \\ \lesssim \frac{\epsilon_0}{u_*}\delta_{ext} + \frac{\epsilon_0}{r}\Delta_{ext} \\ \lesssim \frac{\epsilon_0}{r}\Delta_{ext}.\end{aligned} \tag{8.186}$$

Next, recall that $s = r$ on $\Sigma_*^{(extend)}$. Since ν is tangent to $\Sigma_*^{(extend)}$ with $\nu = e_3 + b e_4$, since $e_4(r) = \frac{r}{2}\overline{\kappa} = 1$ on $\Sigma_*^{(extend)}$, and since $e_4(s) = 1$ for an outgoing geodesic foliation, we have

$$s - r = 0, \qquad e_4(s) - e_4(r) = 0, \qquad e_3(s) - e_3(r) = 0 \quad \text{on} \quad \Sigma_*^{(extend)}.$$

Using the above identities on $\Sigma_*^{(extend)}$, integrating the following transport equation valid for an outgoing geodesic foliation

$$e_4(r-s) = \frac{r}{2}\left(\overline{\kappa} - \frac{2}{r}\right) = \frac{r}{2}\overline{\check{\kappa}},$$

from $\Sigma_*^{(extend)}$, and using the bounds of Step 1 on $\mathcal{M}^{(extend)}$, and hence on $\widetilde{\mathcal{R}}$, for $\check{\kappa}$, we infer, for all $k \leq k_* - 3$,

$$\begin{aligned}\sup_{\widetilde{\mathcal{R}}}\Big(|\widehat{\mathfrak{d}}^k(r-s)| + |\widehat{\mathfrak{d}}^{k-1}(e_3(r)-e_3(s))|& \\ +|\widehat{\mathfrak{d}}^{k-1}(e_4(r)-e_4(s))|\Big) \quad &\lesssim \quad \frac{\epsilon_0}{r}\Delta_{ext}.\end{aligned} \tag{8.187}$$

Also, one has, since $u+r$ is constant on $\Sigma_*^{(extend)}$ and $s=r$ on $\Sigma_*^{(extend)}$

$$0 = \nu(u+s) = e_3(u) + b_* e_4(u) + e_3(s) + b_* e_4(s) = e_3(u) + e_3(s) + b_*,$$

where we used $e_4(s) = 1$ and $e_4(u) = 0$ for an outgoing geodesic foliation, and hence

$$b_* \quad = \quad -e_3(u) - e_3(s) \text{ on } \Sigma_*.$$

Together with the GCM condition on b_*, we infer

$$\overline{e_3(u)+e_3(s)} \quad = \quad 1 + \frac{2m}{r} \text{ on } \Sigma_*,$$

where $\overline{e_3(u)+e_3(s)}$ denotes the average of $e_3(u)+e_3(s)$ on the spheres foliating Σ_*. As above, propagating forward in e_4, and using the bounds of Step 1 on $\mathcal{M}^{(extend)}$, and hence on $\widetilde{\mathcal{R}}$, for $\widecheck{e_3(u)}$ and $\widecheck{e_3(s)}$, we infer

$$\sup_{\widetilde{\mathcal{R}}}\left|\overline{e_3(u)+e_3(s)} - \left(1+\frac{2m}{r}\right)\right| \quad \lesssim \quad \frac{\epsilon_0}{r}\Delta_{ext}. \tag{8.188}$$

Also, arguing as we did above on $\Sigma_*^{(extend)}(u_* \leq u \leq u_* + \delta_{ext})$, we have $r \sim \delta_* \epsilon_0^{-1} u^{1+\delta_{dec}}$ on $\widetilde{\mathcal{R}} \cap \{u \geq u_*\}$ and hence, for all $k \leq k_* - 3$,

$$\sup_{\widetilde{\mathcal{R}}\cap\{u\geq u_*\}} r^2|\mathfrak{d}^k\Gamma_b| \quad \lesssim \quad \sup_{\widetilde{\mathcal{R}}\cap\{u\geq u_*\}}\left(\frac{r\epsilon_0}{u^{1+\delta_{dec}}}\right) \lesssim \delta_*, \tag{8.189}$$

which corresponds, for $\delta_* > 0$ small enough, to assumption **A1-strong** in [41], see (8.17).

Finally, we consider the control of the Hawking mass m_H of the sphere $S(u,s)$ of the (u,s) foliation in the region $\widetilde{\mathcal{R}}\cap\{u\geq u_*\}$, where we recall that m_H is given by the formula

$$\frac{2m_H}{r} = 1+\frac{1}{16\pi}\int_S\kappa\underline{\kappa}.$$

First, we have from Lemma 5.51 on $\Sigma_*^{(extend)}$

$$\nu(m_H) = r^2\mathfrak{d}\!\!/^{\leq 1}(\Gamma_b\cdot\Gamma_b)$$

and hence, since m is a constant, and using the bounds of Step 1 on $\mathcal{M}^{(extend)}$,

$$\sup_{\Sigma_*^{(extend)}} u^{2+2\delta_{dec}}|\nu(m_H-m)| \lesssim \epsilon_0^2.$$

Since $m_H=m$ on S_* by definition of m, we infer, propagating the above transport equation from S_*,

$$\sup_{\Sigma_*^{(extend)}(u_*\leq u\leq u_*+\delta_{ext})}|m_H-m| \lesssim \frac{\epsilon_0^2\delta_{ext}}{u_*^{2+2\delta_{dec}}}.$$

Next, recall from the proof of Lemma 5.51 the following computation

$$\begin{aligned}e_4(\kappa\underline{\kappa})+\kappa^2\underline{\kappa} &= 2\kappa\rho-2\kappa\mathrm{div}\,\zeta+\kappa\left(2|\zeta|^2-\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)-\underline{\kappa}|\widehat{\chi}|^2\\ &= 2\kappa\rho-2\kappa\mathrm{div}\,\zeta+r^{-1}\Gamma_b\cdot\Gamma_g.\end{aligned}$$

This yields, using a well-known identity for the e_4 derivative of the integral on S of a scalar function in an outgoing geodesic foliation,

$$\begin{aligned}e_4\left(\int_S\kappa\underline{\kappa}\right) &= \int_S\left(e_4(\kappa\underline{\kappa})+\kappa^2\underline{\kappa}\right)\\ &= \int_S\left(2\kappa\rho-2\kappa\mathrm{div}\,\zeta+r^{-1}\Gamma_b\cdot\Gamma_g\right)\end{aligned}$$

and hence, using integration by parts,

$$e_4\left(\int_S\kappa\underline{\kappa}\right) = \int_S\left(2\kappa\rho+r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}\Gamma_g\right).$$

Together with the definition of m_H, we infer

$$2e_4(m_H) = \frac{2m_H}{r}e_4(r)+\frac{r}{16\pi}e_4\left(\int_S\kappa\underline{\kappa}\right)$$

$$= \quad m_H\overline{\kappa} + \frac{r}{8\pi}\int_S \Big(\kappa\rho + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}\Gamma_g\Big).$$

Now, using in particular Gauss equation and Gauss-Bonnet formula, we have

$$\begin{aligned}\int_S \kappa\rho &= \overline{\kappa}\int_S \rho + \int_S(\kappa-\overline{\kappa})(\rho-\overline{\rho}) = \overline{\kappa}\int_S\rho + \int_S r^{-1}\Gamma_g\cdot\Gamma_g\\ &= \overline{\kappa}\int_S\left(-K - \frac{1}{4}\kappa\underline{\kappa} + \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right) + \int_S r^{-1}\Gamma_g\cdot\Gamma_g\\ &= \overline{\kappa}\left(-4\pi - \frac{1}{4}\int_S\kappa\underline{\kappa}\right) + \int_S r^{-1}\Gamma_g\cdot\Gamma_g.\end{aligned}$$

In view of the above, and using again the definition of m_H, we infer

$$e_4(m_H) \quad = \quad r^2\Gamma_b\cdot\mathfrak{d}^{\leq 1}\Gamma_g.$$

Hence, since m is a constant, and using the bounds of Step 1 on $\mathcal{M}^{(extend)}$, we have

$$\sup_{\widetilde{\mathcal{R}}\cap\{u\geq u_*\}} ru^{\frac{3}{2}+2\delta_{dec}}|e_4(m_H-m)| \quad \lesssim \quad \epsilon_0^2.$$

Integrating from $\Sigma_*^{(extend)}(u_*\leq u\leq u_*+\delta_{ext})$ where we control m_H-m in view of the above, we infer

$$\sup_{\widetilde{\mathcal{R}}\cap\{u\geq u_*\}}|m_H-m| \quad \lesssim \quad \frac{\epsilon_0^2}{ru_*^{\frac{3}{2}+2\delta_{dec}}}\Delta_{ext} + \frac{\epsilon_0^2\delta_{ext}}{u_*^{2+2\delta_{dec}}} \lesssim \frac{\epsilon_0^2}{ru_*}\Delta_{ext}. \tag{8.190}$$

Step 5. In this step, we control the basis of $\ell=1$ mode $J^{(p)}$ in the spacetime region $\widetilde{\mathcal{R}}$. Recall that $J^{(p)}$ is chosen on S_* to be a canonical basis of $\ell=1$ modes in the sense of Definition 3.10 of [41] (recalled here in Definition 5.3), i.e. on S_* there exist coordinates (θ,φ) such that:

1. The induced metric g on S_* takes the form

$$g = r^2e^{2\phi}\Big((d\theta)^2+\sin^2\theta(d\varphi)^2\Big). \tag{8.191}$$

2. The functions

$$J^{(0)} := \cos\theta, \quad J^{(-)} := \sin\theta\sin\varphi, \quad J^{(+)} := \sin\theta\cos\varphi, \tag{8.192}$$

verify the balanced conditions

$$(8.193) \qquad \int_{S_*} J^{(p)} = 0, \qquad p = 0, +, -.$$

Recall also that we extend (θ, φ) and $J^{(p)}$ to $\Sigma_*^{(extend)}$ by setting

$$(8.194) \qquad \nu(\theta) = 0, \quad \nu(\varphi) = 0, \quad \nu(\phi) = 0, \quad \nu(J^{(p)}) = 0, \; p = 0, +, -,$$

where we have also extended the conformal fact ϕ. We then extend (θ, φ), ϕ and $J^{(p)}$ to $\widetilde{\mathcal{R}}$ as follows

$$(8.195) \qquad e_4(\theta) = 0, \quad e_4(\varphi) = 0, \quad e_4(\phi) = 0, \quad e_4(J^{(p)}) = 0, \; p = 0, +, -.$$

In what follows, to avoid the singularities at $\theta = 0$ and $\theta = \pi$ of the (θ, φ) coordinates system on the spheres $S = S(u, s)$ of the outgoing geodesic $(u, s,)$-foliation of $\widetilde{\mathcal{R}}$, we use instead two regular coordinates charts based on (θ, φ).

Definition 8.41. *We define two coordinates charts on $S(u, s)$ as follows*

1. *The coordinates (x_N^1, x_N^2) are defined for $0 \leq \theta < \pi$ by*

$$x_N^1 := \frac{\sin\theta\cos\varphi}{1 + \cos\theta}, \qquad x_N^2 := \frac{\sin\theta\sin\varphi}{1 + \cos\theta}.$$

2. *The coordinates (x_S^1, x_S^2) are defined for $0 < \theta \leq \pi$ by*

$$x_S^1 := \frac{\sin\theta\cos\varphi}{1 - \cos\theta}, \qquad x_S^2 := \frac{\sin\theta\sin\varphi}{1 - \cos\theta}.$$

Lemma 8.42. *Let g denote the metric induced by $\mathbf{g}$ on $S(u, s)$. Then, on S_*, the metric g takes the following form in the (x_N^1, x_N^2) coordinates system and in the (x_S^1, x_S^2) coordinates system, for $(x^1, x^2) \in \mathbb{R}^2$,*

$$(8.196) \qquad g \;=\; \frac{4r^2 e^{2\phi}}{\left(1 + (x^1)^2 + (x^2)^2\right)^2}\Big[(dx^1)^2 + (dx^2)^2\Big].$$

Proof. This follows immediately from (8.191) and the definition of (x_N^1, x_N^2) and (x_S^1, x_S^2) in terms of (θ, φ). □

From now on, let (x^1, x^2) denote either (x_N^1, x_N^2) or (x_S^1, x_S^2). In view of the definition of (x^1, x^2), we have $\nu(x^1) = \nu(x^2) = 0$ on $\Sigma_*^{(extend)}$. Since $\nu = e_3 + b_* e_4$ and $\nu(u) = e_3(u)$, we infer $\partial_u = \frac{1}{e_3(u)}\nu$. We easily derive the following formula on $\Sigma_*^{(extend)}$ in the (x^1, x^2) coordinates system of S

$$\partial_u g_{ab} \;=\; 2\chi\left(\frac{\partial}{\partial_a}, \frac{\partial}{\partial_b}\right) + 2b_*\chi\left(\frac{\partial}{\partial_a}, \frac{\partial}{\partial_b}\right),$$

and hence

$$\partial_u g_{ab} \quad = \quad (\underline{\kappa} + b_* \kappa) g_{ab} + 2\underline{\widehat{\chi}}\left(\frac{\partial}{\partial_a}, \frac{\partial}{\partial_b}\right) + 2b_* \widehat{\chi}\left(\frac{\partial}{\partial_a}, \frac{\partial}{\partial_b}\right),$$

which we rewrite as follows, recalling in particular that $\check{\kappa} = 0$ on $\Sigma_*^{(extend)}$, and using $\nu(x^a) = 0$ and $\nu(\phi) = 0$,

$$\begin{aligned}
&\partial_u\left(r^{-2}g_{ab} - \frac{4e^{2\phi}}{1+(x^1)^2+(x^2)^2}\delta_{ab}\right)\\
= \quad &\left(\check{\underline{\kappa}} + \frac{2}{r}\left(1 - \frac{1}{e_3(u)}\right)\check{b}_* - \frac{2}{re_3(u)}\widecheck{e_3(u)}\right) r^{-2}g_{ab}\\
&+2r^{-2}\underline{\widehat{\chi}}\left(\frac{\partial}{\partial_a}, \frac{\partial}{\partial_b}\right) + 2r^{-2}b_*\widehat{\chi}\left(\frac{\partial}{\partial_a}, \frac{\partial}{\partial_b}\right).
\end{aligned}$$

Together with the control of Step 1 on $\mathcal{M}^{(extend)}$, and hence on $\Sigma_*^{(extend)}$, for $\check{\underline{\kappa}}$, $\check{b}_*$, $\widecheck{e_3(u)}$, $\underline{\widehat{\chi}}$ and $\widehat{\chi}$, we infer, for all $k \leq k_* - 3$,

$$\left|\not{\mathfrak{d}}^k \partial_u\left(r^{-2}g_{ab} - \frac{4e^{2\phi}}{1+(x^1)^2+(x^2)^2}\delta_{ab}\right)\right| \quad \lesssim \quad \frac{\epsilon_0}{ru^{1+\delta_{dec}}} r^{-2}|g|.$$

Integrating from S_*, where (8.196) holds, we infer on $\Sigma_*^{(extend)}(u_* \leq u \leq u_* + \delta_{ext})$, for all $k \leq k_* - 3$,

$$\left|\not{\mathfrak{d}}^k \left(r^{-2}g_{ab} - \frac{4e^{2\phi}}{1+(x^1)^2+(x^2)^2}\delta_{ab}\right)\right| \quad \lesssim \quad \frac{\epsilon_0}{ru_*^{1+\delta_{dec}}}\delta_{ext}.$$

Next, we estimate $r^{-2}g_{ab}$ in $\widetilde{\mathcal{R}} \cap \{u \geq u_*\}$. In view of the definition of (x^1, x^2), we have $e_4(x^1) = e_4(x^2) = 0$. Since we also have $e_4(u) = 0$ and $e_4(s) = 1$, we infer $e_4 = \partial_s$. We easily derive the following formula on $\widetilde{\mathcal{R}}$ in the (x^1, x^2) coordinates system of S

$$\partial_s g_{ab} \quad = \quad 2\chi\left(\frac{\partial}{\partial_a}, \frac{\partial}{\partial_b}\right),$$

and hence

$$\partial_s g_{ab} \quad = \quad \kappa g_{ab} + 2\widehat{\chi}\left(\frac{\partial}{\partial_a}, \frac{\partial}{\partial_b}\right),$$

which we rewrite as follows, using $e_4(x^a) = 0$ and $e_4(\phi) = 0$,

$$\partial_s\left(r^{-2}g_{ab}-\frac{4e^{2\phi}}{1+(x^1)^2+(x^2)^2}\delta_{ab}\right) = \left(\check{\kappa}-\overline{\check{\kappa}}\right)r^{-2}g_{ab}+2r^{-2}\widehat{\chi}\left(\frac{\partial}{\partial_a},\frac{\partial}{\partial_b}\right).$$

Together with the control of Step 1 on $\mathcal{M}^{(extend)}$, and hence on $\widetilde{\mathcal{R}}$, for $\check{\kappa}$ and $\widehat{\chi}$, we infer, for all $k\leq k_*-3$,

$$\left|\mathfrak{d}\!\!/^k\partial_s\left(r^{-2}g_{ab}-\frac{4e^{2\phi}}{1+(x^1)^2+(x^2)^2}\delta_{ab}\right)\right| \lesssim \frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta_{dec}}}r^{-2}|g|.$$

Integrating from $\Sigma_*^{(extend)}(u_*\leq u\leq u_*+\delta_{ext})$, and using the above control of $r^{-2}g_{ab}$ on $\Sigma_*^{(extend)}(u_*\leq u\leq u_*+\delta_{ext})$, we infer in $\widetilde{\mathcal{R}}\cap\{u\geq u_*\}$, for all $k\leq k_*-3$,

$$\begin{aligned}\left|\mathfrak{d}\!\!/^k\left(r^{-2}g_{ab}-\frac{4e^{2\phi}}{1+(x^1)^2+(x^2)^2}\delta_{ab}\right)\right| &\lesssim \frac{\epsilon_0}{ru_*^{1+\delta_{dec}}}\delta_{ext}+\frac{\epsilon_0}{r^2u_*^{\frac{1}{2}+\delta_{dec}}}\Delta_{ext}\\ &\lesssim \frac{\epsilon_0}{r^2}\Delta_{ext}.\end{aligned}\tag{8.197}$$

Next, we estimate ϕ in $\widetilde{\mathcal{R}}\cap\{u\geq u_*\}$. First, recall from Corollary 5.60 that we have obtained on S_*

$$\|\mathfrak{d}\!\!/^{\leq k_*}\phi\|_{L^\infty(S_*)} \lesssim \frac{\epsilon_0}{ru_*^{\frac{1}{2}+\delta_{dec}}}.$$

Since we have extended ϕ to $\Sigma_*^{(extend)}$ by $\nu(\phi)=0$ and then to $\widetilde{\mathcal{R}}$ by $e_4(\phi)=0$, we easily infer in $\widetilde{\mathcal{R}}\cap\{u\geq u_*\}$, for all $k\leq k_*-3$,

$$|\mathfrak{d}^k\phi| \lesssim \frac{\epsilon_0}{ru_*^{\frac{1}{2}+\delta_{dec}}}.\tag{8.198}$$

Next, we estimate $\int_S J^{(p)}$ for $p=0,+,-$ in $\widetilde{\mathcal{R}}\cap\{u\geq u_*\}$. Recall that $J^{(p)}$ is balanced on S_*, i.e.

$$\int_{S_*}J^{(p)}=0,\quad p=0,+,-.$$

Also, since $\nu(J^{(p)})=0$ on $\Sigma_*^{(extend)}$, we have, in view of Corollary 5.32,

$$\nu\left(r^{-2}\int_S J^{(p)}\right)=\Gamma_b,\quad\text{for}\quad S\subset\Sigma_*^{(extend)},\quad p=0,+,-,$$

and since $J^{(p)}$ is extended to $\widetilde{\mathcal{R}}$ by $e_4(J^{(p)})=0$, we have

$$e_4\left(r^{-2}\int_S J^{(p)}\right) = \Gamma_g, \quad \text{for} \quad S\subset\widetilde{\mathcal{R}}, \quad p=0,+,-.$$

We deduce in $\widetilde{\mathcal{R}}\cap\{u\geq u_*\}$

$$r^{-2}\left|\int_S J^{(p)}\right| \lesssim \frac{\epsilon_0}{ru_*^{1+\delta_{dec}}}\delta_{ext} + \frac{\epsilon_0}{r^2u_*^{\frac{1}{2}+\delta_{dec}}}\Delta_{ext} \lesssim \frac{\epsilon_0}{r^2}\Delta_{ext}$$

and hence

$$r^{-2}\left|\int_S J^{(p)}\right| \lesssim \frac{\epsilon_0}{r^2}\Delta_{ext} \text{ for } S\subset\widetilde{\mathcal{R}}\cap\{u\geq u_*\},\ p=0,+,-. \tag{8.199}$$

In view of (8.197), (8.198) and (8.199), we may apply Proposition 4.15 in [41] (restated here as Proposition 5.5) which yields on any sphere S of $\widetilde{\mathcal{R}}\cap\{u\geq u_*\}$ the existence of a canonical basis $J^{(p,S)}$ of $\ell=1$ modes such that

$$\max_{p=0,+,-} r^{-1}\|J^{(p)}-J^{(p,S)}\|_{\mathfrak{h}_{k_*-2}(S)} \lesssim \frac{\epsilon_0}{r^2}\Delta_{ext}, \text{ for } S\subset\widetilde{\mathcal{R}}\cap\{u\geq u_*\}. \tag{8.200}$$

This, together with (8.202) below, corresponds to assumption **A4-strong** which is stated at the beginning of Section 8.1.3.

Next, we estimate $e_3(J^{(p)})$ in the region $\widetilde{\mathcal{R}}$. Since $\nu(J^{(p)})=0$ on $\Sigma_*^{(extend)}$, since $\nu=e_3+b_*e_4$ and since $J^{(p)}$ is extended from $\Sigma_*^{(extend)}$ by $e_4(J^{(p)})=0$, we have

$$e_3(J^{(p)})=0 \quad \text{on} \quad \Sigma_*^{(extend)}.$$

Then, we compute, using $e_4(J^{(p)})=0$, $\omega=0$ and $\underline{\eta}=-\zeta$,

$$\begin{aligned} e_4(e_3(J^{(p)})) &= [e_4,e_3]J^{(p)} = \Big(2\omega e_3 - 2\underline{\omega}e_4 + 2(\underline{\eta}-\eta)\cdot\nabla\Big)J^{(p)} \\ &= -2(\eta+\zeta)\cdot\nabla J^{(p)}. \end{aligned}$$

Together with the control of Step 1 on $\mathcal{M}^{(extend)}$, and hence on $\widetilde{\mathcal{R}}$, for η and ζ, we infer on $\widetilde{\mathcal{R}}$, for all $k\leq k_*-3$,

$$|\mathfrak{d}^k e_4(e_3(J^{(p)}))| \lesssim \frac{\epsilon_0}{r^2u^{1+\delta_{dec}}}.$$

Since $e_3(J^{(p)})=0$ on $\Sigma_*^{(extend)}$, we infer, for all $k\leq k_*-3$,

$$|\widehat{\mathfrak{d}}^k e_3(J^{(p)})| \lesssim \frac{\epsilon_0}{r^2u^{1+\delta_{dec}}}\Delta_{ext} \quad \text{on} \quad \widetilde{\mathcal{R}}. \tag{8.201}$$

Finally, arguing as in Corollary 5.45, the following holds on $\Sigma_*^{(extend)}$

$$\begin{aligned}\Big(r^2\Delta+2\Big)J^{(p)} &= O(\epsilon_0 r^{-1}u^{-\frac{1}{2}-\delta_{dec}}), \qquad p=0,+,-,\\ \frac{1}{|S|}\int_S J^{(p)}J^{(q)} &= \frac{1}{3}\delta_{pq}+O(\epsilon_0 r^{-1}u^{-\frac{1}{2}-\delta_{dec}}), \qquad p,q=0,+,-,\\ \frac{1}{|S|}\int_S J^{(p)} &= O(\epsilon_0 r^{-1}u^{-\frac{1}{2}-\delta_{dec}}), \qquad p=0,+,-.\end{aligned}$$

Since $J^{(p)}$ is extended to $\widetilde{\mathcal{R}}$ by $e_4(J^{(p)})=0$, we propagate from $\Sigma_*^{(extend)}$ and easily obtain on $\widetilde{\mathcal{R}}$

$$\begin{aligned}\Big(r^2\Delta+2\Big)J^{(p)} &= O(\epsilon_0 r^{-1}u^{-\frac{1}{2}-\delta_{dec}}), \quad p=0,+,-,\\ \frac{1}{|S|}\int_S J^{(p)}J^{(q)} &= \frac{1}{3}\delta_{pq}+O(\epsilon_0 r^{-1}u^{-\frac{1}{2}-\delta_{dec}}), \quad p,q=0,+,-,\\ \frac{1}{|S|}\int_S J^{(p)} &= O(\epsilon_0 r^{-1}u^{-\frac{1}{2}-\delta_{dec}}), \quad p=0,+,-.\end{aligned} \tag{8.202}$$

8.5.2. Steps 6–13

Step 6. We fix the following sphere of the $(u^{(extend)}, s^{(extend)})$ foliation in $\widetilde{\mathcal{R}}\cap\{u\geq u_*\}$

$$\overset{\circ}{S} := S(\overset{\circ}{u},\overset{\circ}{s}), \qquad \overset{\circ}{u} := u_*+\frac{\delta_{ext}}{2}, \qquad \overset{\circ}{s} := r_*+\frac{3d_0 r_*}{4u_*}\delta_{ext}. \tag{8.203}$$

Define

$$\overset{\circ}{\delta} := \frac{\epsilon_0}{r_*}\Delta_{ext} = \frac{d_0\epsilon_0\delta_{ext}}{u_*}, \qquad \overset{\circ}{\epsilon} := \epsilon_0,$$

and the small spacetime neighborhood of $\overset{\circ}{S}$

$$\mathcal{R}(\overset{\circ}{\epsilon},\overset{\circ}{\delta}) := \Big\{|u-\overset{\circ}{u}|\leq\delta_{\mathcal{R}}, \quad |s-\overset{\circ}{s}|\leq\delta_{\mathcal{R}}\Big\}, \qquad \delta_{\mathcal{R}}=\overset{\circ}{\delta}(\overset{\circ}{\epsilon})^{-\frac{1}{2}}.$$

Note that $\mathcal{R}(\overset{\circ}{\epsilon},\overset{\circ}{\delta})\subset\widetilde{\mathcal{R}}$. In view of (8.184), (8.186), (8.189), (8.200) and (8.202), we are in position to apply Theorem 7.3 and Corollary 7.7 of [41] (restated here in Theorem 8.7 and Corollary 8.8), with $s_{max}=k_{small}+k_*-4$, which yields the existence of a unique sphere $\widetilde{S}_*$, which is a deformation of $\overset{\circ}{S}$, is included in $\mathcal{R}(\overset{\circ}{\epsilon},\overset{\circ}{\delta})$, and is such that the following GCM conditions hold on it

$$
\begin{aligned}
&\check{\widetilde{\kappa}} = 0, \qquad \check{\underline{\widetilde{\kappa}}} = 0, \qquad \check{\widetilde{\mu}} = \sum_p \widetilde{M}_p J^{(p,\widetilde{S}_*)},\\
&(\widetilde{\mathrm{div}}\,\widetilde{\beta})_{\ell=1} = 0, \qquad (\widetilde{\mathrm{curl}}\,\widetilde{\beta})_{\ell=1,\pm} = 0, \qquad (\widetilde{\mathrm{curl}}\,\widetilde{\beta})_{\ell=1,0} = \frac{2\widetilde{a}\widetilde{m}}{\widetilde{r}^5},
\end{aligned} \tag{8.204}
$$

where

- the tilde refer to the quantities and tangential operators on $\widetilde{S}_*$,
- $J^{(p,\widetilde{S}_*)}$ denotes the canonical basis of $\ell = 1$ mode on $\widetilde{S}_*$ in the sense of Definition 3.10 of [41] (recalled here in Definition 5.3),
- the $\ell = 1$ modes in (8.204) are defined w.r.t. the basis of $\ell = 1$ modes $J^{(p,\widetilde{S}_*)}$,
- $\widetilde{m}$ denotes the Hawking mass of $\widetilde{S}_*$, $\widetilde{r}$ denotes the area radius of $\widetilde{S}_*$, and the identity for $(\widetilde{\mathrm{curl}}\,\widetilde{\beta})_{\ell=1,0}$ in (8.204) should be understood as providing the definition of $\widetilde{a}$.

Step 7. Starting from $\widetilde{S}_*$ constructed in Step 6, and in view of (8.184), (8.185), (8.187), (8.188) and (8.201), we may apply Theorem 4.1 in [50] (restated here in Theorem 8.11), with $s_{max} = k_{small} + k_* - 4$, which yields the existence of a smooth small piece of spacelike hypersurface $\widetilde{\Sigma}_*$ starting from $\widetilde{S}_*$ towards the initial data layer, together with a scalar function $\widetilde{u}$ defined on $\widetilde{\Sigma}_*$, whose level surfaces are topological spheres denoted by $\widetilde{S}$, so that

- The following GCM conditions are verified on $\widetilde{\Sigma}_*$

$$
\begin{aligned}
&\check{\widetilde{\kappa}} = 0, \qquad \check{\underline{\widetilde{\kappa}}} = \underline{\widetilde{C}}_0 + \sum_p \underline{\widetilde{C}}_p \widetilde{J}^{(p)}, \qquad \check{\widetilde{\mu}} = \widetilde{M}_0 + \sum_p \widetilde{M}_p \widetilde{J}^{(p)},\\
&(\widetilde{\mathrm{div}}\,\widetilde{\eta})_{\ell=1} = 0, \qquad (\widetilde{\mathrm{div}}\,\underline{\widetilde{\xi}})_{\ell=1} = 0,
\end{aligned}
$$

  where the tilde refer to the quantities and tangential operators on $\widetilde{\Sigma}_*$.
- $\underline{\widetilde{C}}_0$, $\underline{\widetilde{C}}_p$, $\widetilde{M}_0$ and $\widetilde{M}_p$ are constant on each leaf of the $\widetilde{u}$-foliation of $\widetilde{\Sigma}_*$.
- We have, for some constant $c_{\widetilde{\Sigma}_*}$,

$$
\widetilde{u} + \widetilde{r} = c_{\widetilde{\Sigma}_*}, \quad \text{along} \quad \widetilde{\Sigma}_*,
$$

  where $\widetilde{r}$ denotes the area radius of the spheres $\widetilde{S}$ of the $\widetilde{u}$-foliation of $\widetilde{\Sigma}_*$.
- The following normalization condition holds true

$$
\overline{\widetilde{b}} = -1 - \frac{2\widetilde{m}}{\widetilde{r}},
$$

where $\overline{\widetilde{b}}$ denotes the average of $\widetilde{b}$ on the spheres foliating $\widetilde{\Sigma}_*$, and where $\widetilde{b}$ is such that we have

$$\widetilde{\nu} = \widetilde{e}_3 + \widetilde{b}\widetilde{e}_4,$$

with $\widetilde{\nu}$ the unique vectorfield tangent to the hypersurface $\widetilde{\Sigma}_*$, normal to $\widetilde{S}$, and normalized by $\mathbf{g}(\widetilde{\nu}, \widetilde{e}_4) = -2$.

- The basis of $\ell = 1$ modes $\widetilde{J}^{(p)}$ is given by $\widetilde{J}^{(p)} = J^{(p,\widetilde{S}_*)}$ on $\widetilde{S}_*$, and extended to $\widetilde{\Sigma}_*$ by $\widetilde{\nu}(\widetilde{J}^{(p)}) = 0$. Also, the $\ell = 1$ modes of $\widetilde{\mathrm{div}\,\eta}$ and $\widetilde{\mathrm{div}\,\underline{\xi}}$ above are computed with respect to this basis.
- The transition functions $(f, \underline{f}, \lambda)$ from the frame of $\mathcal{M}^{(extend)}$ to the frame of $\widetilde{\Sigma}_*$ satisfy on each sphere $\widetilde{S} \subset \widetilde{\Sigma}_*$

$$\|(f, \underline{f}, \log(\lambda))\|_{\mathfrak{h}_{k_*-3}(\widetilde{S})} \lesssim \overset{\circ}{\delta}.$$

Step 8. The spacelike GCM hypersurface $\widetilde{\Sigma}_*$ has been constructed in Step 7 in a small neighborhood of $\widetilde{S}_*$. We now focus on proving that it in fact extends all the way to the initial data layer. To this end, we denote by u_1 with

$$1 \leq u_1 < \overset{\circ}{u},$$

the minimal value of u such that

- We have

$$\widetilde{\Sigma}_* \cap \mathcal{C}_u \neq \emptyset \text{ for any } u_1 \leq u \leq \overset{\circ}{u}. \tag{8.205}$$

- There exists a large constant $D \geq 1$ such that we have for any sphere $\widetilde{S}$ of $\widetilde{\Sigma}_*(u \geq u_1)$

$$\|(f, \underline{f}, \log(\lambda))\|_{\mathfrak{h}_{k_*-3}(\widetilde{S})} \leq D u_* \overset{\circ}{\delta}. \tag{8.206}$$

- For the same large constant $D \geq 1$ as above, we have along $\widetilde{\Sigma}_*(u \geq u_1)$

$$|\psi(s)| \leq D u_* \overset{\circ}{\delta}, \tag{8.207}$$

where the function $\psi(s)$ is such that the curve

$$\Big(u = -s + c_{\widetilde{\Sigma}_*} + \psi(s),\ s,\ \theta = \pi, \varphi\Big) \text{ with } \psi(\overset{\circ}{s}) = 0, \tag{8.208}$$

coincides with the south poles of the sphere $\widetilde{S}$ of $\widetilde{\Sigma}_*$ and the constant $c_{\widetilde{\Sigma}_*}$ is fixed by the condition $\psi(\overset{\circ}{s}) = 0$.

The fact that $\psi(\overset{\circ}{s}) = 0$ together with the bounds of Step 7 implies that (8.205) (8.206) (8.207) hold for $u_1 < \overset{\circ}{u}$ with u_1 close enough to $\overset{\circ}{u}$. By a continuity argument based on reapplying Theorem 4.1 in [50] (restated here in Theorem 8.11), it suffices to show that we may improve the bounds (8.206) (8.207) independently of the value of u_1.

Step 9. We now focus on improving the bounds (8.206) (8.207). We first prove that $\widetilde{\Sigma}_*(u \geq u_1)$ is included in $\widetilde{\mathcal{R}}$. Indeed, (8.206) (8.207) imply, using also the dominant condition on r in $\widetilde{\mathcal{R}}$,

$$\begin{aligned}
\sup_{\widetilde{\Sigma}_*(u\geq u_1)} |u+s-c_{\widetilde{\Sigma}_*}| &\lesssim \sup_{\widetilde{\Sigma}_*(u\geq u_1)} \Big(|\psi| + r|f| + r|\underline{f}|\Big)\\
&\lesssim Du_*\overset{\circ}{\delta}\\
&\lesssim \frac{Du_*}{r}\epsilon_0\Delta_{ext}\\
&\lesssim \epsilon_0 D\epsilon_0\Delta_{ext}\\
&\lesssim \epsilon_0\Delta_{ext}.
\end{aligned}$$

On the other hand, by construction, $\psi(\overset{\circ}{s}) = 0$ and the south pole of $\overset{\circ}{S}$ and $\widetilde{S}_*$ coincide, so that we have

$$\begin{aligned}
c_{\widetilde{\Sigma}_*} &= \overset{\circ}{u} + \overset{\circ}{s} = u_* + r_* + \frac{\delta_{ext}}{2} + \frac{3d_0 r_*}{4u_*}\delta_{ext}\\
&= c_{\Sigma_*} + \frac{3}{4}\left(1 + \frac{2u_*}{3d_0 r_*}\right)\Delta_{ext}
\end{aligned}$$

and hence, using also the dominant condition on r in $\widetilde{\mathcal{R}}$,

$$\begin{aligned}
\sup_{\widetilde{\Sigma}_*(u\geq u_1)} \left|u+s-c_{\Sigma_*} - \frac{3}{4}\Delta_{ext}\right| &\lesssim \left(\frac{u_*}{2d_0 r_*} + \epsilon_0\right)\Delta_{ext}\\
&\lesssim \epsilon_0\Delta_{ext}.
\end{aligned}$$

In view of the definition of $\widetilde{\mathcal{R}}$, we infer

$$\widetilde{\Sigma}_*(u \geq u_1) \subset \widetilde{\mathcal{R}} \tag{8.209}$$

as claimed.

Step 10. Since $\widetilde{\Sigma}_*(u \geq u_1) \subset \widetilde{\mathcal{R}}$, the bounds (8.188), (8.184) and (8.185) apply, and hence we have

$$\sup_{\widetilde{\mathcal{R}}} \left| \overline{e_3(u) + e_3(s)} - \left(1 + \frac{2m}{r}\right) \right| \lesssim \frac{\epsilon_0}{r} \Delta_{ext} \lesssim \mathring{\delta},$$

and for all $k \leq k_* - 4$

$$\begin{aligned} \sup_{\widetilde{\mathcal{R}}} \Bigg(r^2 \left| \widehat{\mathfrak{d}}^k \left(\check{\kappa}\right) \right| + r^2 \left| \widehat{\mathfrak{d}}^k \left(\underline{\check{\kappa}} - \left(\underline{C}_0 + \sum_p \underline{C}_p J^{(p)} \right) \right) \right| \\ + r^2 \left| \widehat{\mathfrak{d}}^k \left(\check{\mu} - \left(M_0 + \sum_p M_p J^{(p)} \right) \right) \right| \Bigg) \lesssim \frac{\epsilon_0}{r} \Delta_{ext} \lesssim \mathring{\delta}, \end{aligned}$$

as well as

$$\sup_{\widetilde{\mathcal{R}}} r^2 \Big(|\widehat{\mathfrak{d}}^k (\mathrm{div}\, \eta)_{\ell=1}| + |\widehat{\mathfrak{d}}^k (\mathrm{div}\, \underline{\xi})_{\ell=1}| \Big) \lesssim \frac{\epsilon_0}{r} \Delta_{ext} \lesssim \mathring{\delta}.$$

Together with the a priori estimates in the proof of Theorem 4.1 in [50], this yields

$$\begin{aligned} |\psi'(s)| &\lesssim \left| 1 + \frac{2\widetilde{m}}{\widetilde{r}} - \overline{e_3(u) + e_3(s)} \right| + |\lambda - 1| \\ &\lesssim \left| \frac{\widetilde{m}}{\widetilde{r}} - \frac{m}{r} \right| + |\lambda - 1| + \mathring{\delta}. \end{aligned}$$

Now, we need to estimate $\widetilde{r} - r$ and $\widetilde{m} - m$. We claim

$$|\widetilde{r} - r| + |\widetilde{m} - m| \lesssim D u_* \mathring{\delta}. \tag{8.210}$$

Indeed, in view of (8.206), and using Lemma 5.8 in [40] (restated here in Proposition 8.3), we have

$$|\widetilde{r} - r| \lesssim \sup_{\widetilde{S}} r(|f| + |\underline{f}|) \lesssim D u_* \mathring{\delta},$$

which is the stated estimate for $\widetilde{r} - r$ in (8.210). Next, recall that $\mathring{S} = S(\mathring{u}, \mathring{s})$ is the sphere of the foliation of $\widetilde{\mathcal{R}} \cap \{u \geq u_*\}$ which shares the same south pole a $\widetilde{S}_*$. We denote $\mathring{m}$ the Hawking mass of $\mathring{S}$ and recall that $\widetilde{m}$ denotes the Hawking mass of $\widetilde{S}_*$. Then, in view of (8.206), and using Corollary 5.17 in [40] (restated here in Proposition 8.3), we have

$$\left|\widetilde{m}-\mathring{m}\right| \lesssim \sup_{\widetilde{S}} r(|f|+|\underline{f}|)\lesssim Du_*\mathring{\delta}.$$

Also, since $\mathring{S}\subset\widetilde{\mathcal{R}}\cap\{u\geq u_*\}$, and since $\mathring{m}$ denotes the Hawking mass of $\mathring{S}$, we have in view of (8.190)

$$|\mathring{m}-m| \lesssim \mathring{\delta}.$$

We deduce

$$|\widetilde{m}-m| \lesssim Du_*\mathring{\delta}$$

which concludes the proof of (8.210).

We infer from (8.210) and the estimate immediately above for $|\psi'(s)|$, using also the dominant condition on r in $\widetilde{\mathcal{R}}$,

$$\begin{aligned}|\psi'(s)| &\lesssim \frac{Du_*}{r}\mathring{\delta}+\mathring{\delta}\\ &\lesssim (1+\epsilon_0 D)\,\mathring{\delta}\\ &\lesssim \mathring{\delta}.\end{aligned}$$

Integrating from $\mathring{s}$ where $\psi(\mathring{s})=0$, we infer

$$\begin{aligned}|\psi(s)| &\lesssim |s-\mathring{s}|\mathring{\delta}\\ &\lesssim u_*\mathring{\delta}\end{aligned}$$

which improves (8.207) for $D\geq 1$ large enough.

Similarly, we obtain, using the a-priori estimates for GCM spheres in [40],

$$\begin{aligned}&\|(f,\underline{f},\log(\lambda))\|_{\mathfrak{h}_{k_*-3}(\widetilde{S})}\\ \lesssim\ & \max_{k\leq k_*-4}\sup_{\widetilde{\mathcal{R}}}\left(r^2\left|\mathfrak{d}^k\left(\check{\kappa}\right)\right|+r^2\left|\mathfrak{d}^k\left(\underline{\check{\kappa}}-\left(\underline{C}_0+\sum_p\underline{C}_pJ^{(p)}\right)\right)\right|\right.\\ &\left.+r^2\left|\mathfrak{d}^k\left(\check{\mu}-\left(M_0+\sum_p M_pJ^{(p)}\right)\right)\right|\right)+r\left(\left|(\widetilde{\operatorname{div}} f)_{\ell=1}\right|+\left|(\widetilde{\operatorname{div}}\,\underline{f})_{\ell=1}\right|\right),\end{aligned}$$

and hence

$$\|(f,\underline{f},\log(\lambda))\|_{\mathfrak{h}_{k_*-3}(\widetilde{S})} \lesssim r^2\left(\left|(\widetilde{\operatorname{div}} f)_{\ell=1}\right|+\left|(\widetilde{\operatorname{div}}\,\underline{f})_{\ell=1}\right|\right)+\mathring{\delta},$$

where the $\ell = 1$ modes are taken w.r.t. the basis $\widetilde{J}^{(p)}$. Also, using the a priori estimates in the proof of Theorem 4.1 in [50], we have

$$\left|\tilde{\nu}\left((\widetilde{\mathrm{div}}\, f)_{\ell=1}\right)\right| + \left|\tilde{\nu}\left((\widetilde{\mathrm{div}}\, \underline{f})_{\ell=1}\right)\right| \lesssim r^{-2}\overset{\circ}{\delta} + \frac{1}{r}\left(\left|(\widetilde{\mathrm{div}}\, f)_{\ell=1}\right| + \left|(\widetilde{\mathrm{div}}\, \underline{f})_{\ell=1}\right|\right).$$

In view of (8.206), we infer

$$\left|\tilde{\nu}\left((\widetilde{\mathrm{div}}\, f)_{\ell=1}\right)\right| + \left|\tilde{\nu}\left((\widetilde{\mathrm{div}}\, \underline{f})_{\ell=1}\right)\right| \lesssim r^{-2}\overset{\circ}{\delta} + r^{-3}Du_*\overset{\circ}{\delta}$$

and integrating from $\widetilde{S}_*$, we infer, using also the dominant condition for r in $\widetilde{\mathcal{R}}$,

$$\begin{aligned} r^2\left(\left|(\widetilde{\mathrm{div}}\, f)_{\ell=1}\right| + \left|(\widetilde{\mathrm{div}}\, \underline{f})_{\ell=1}\right|\right) &\lesssim u_*\overset{\circ}{\delta} + \frac{D(u_*)^2}{r}\overset{\circ}{\delta} \\ &\lesssim (1+\epsilon_0 D)\, u_*\overset{\circ}{\delta} \\ &\lesssim u_*\overset{\circ}{\delta}. \end{aligned}$$

This yields

$$\|(f, \underline{f}, \log(\lambda))\|_{\mathfrak{h}_{k_*-3}(\widetilde{S})} \lesssim u_*\overset{\circ}{\delta}$$

which improves (8.206) for $D \geq 1$ large enough. We thus conclude that $u_1 = 1$, i.e. $\widetilde{\Sigma}_*$ extends all the way to the initial data layer, $\widetilde{\Sigma}_* \subset \widetilde{\mathcal{R}}$, and we have the bounds

$$\|(f, \underline{f}, \log(\lambda))\|_{\mathfrak{h}_{k_*-3}(\widetilde{S})} \lesssim u_*\overset{\circ}{\delta}, \qquad |\psi(s)| \lesssim u_*\overset{\circ}{\delta}.$$

In view of the definition of $\overset{\circ}{\delta}$, we infer in particular for any sphere $\widetilde{S}$ of $\widetilde{\Sigma}_*$

$$\|(f, \underline{f}, \log(\lambda))\|_{\mathfrak{h}_{k_*-3}(\widetilde{S})} \lesssim \epsilon_0\delta_{ext}, \qquad |\psi(s)| \lesssim \epsilon_0\delta_{ext}. \tag{8.211}$$

Step 11. As $\widetilde{\Sigma}_*$ extends all the way to the initial data layer, this allows us to calibrate $\tilde{u}$ along $\widetilde{\Sigma}_*$ by fixing the value $\widetilde{u} = 1$ as in Remark 8.39:

$$\widetilde{S}_1 = \widetilde{\Sigma}_* \cap \{\tilde{u} = 1\} \text{ is such that } \widetilde{S}_1 \cap \{\widetilde{u}_{\mathcal{L}_0} = 1\} \cap \{\widetilde{\theta}_{\mathcal{L}_0} = \pi\} \neq \emptyset,$$

i.e. $\widetilde{S}_1$ is the unique sphere of $\widetilde{\Sigma}_*$ intersecting the curve of the south poles of $\{\widetilde{u}_{\mathcal{L}_0} = 1\}$ in the part ${}^{(ext)}\widetilde{\mathcal{L}_0}$ of the initial data layer constructed in Section 8.2.

Now that $\widetilde{u}$ is calibrated, we define

$$\tilde{u}_* := \tilde{u}(\widetilde{S}_*). \tag{8.212}$$

For the proof of Theorem M7, we need in particular to show that $\tilde{u}_* > u_*$. First, note that, since $\tilde{u} + \tilde{r}$ is constant along $\widetilde{\Sigma}_*$, we have

$$\widetilde{\Sigma}_* = \left\{\widetilde{u} + \widetilde{r} = 1 + \widetilde{r}(\widetilde{S}_1)\right\}. \tag{8.213}$$

Since $\widetilde{S}_* \subset \widetilde{\Sigma}_*$, and in view of (8.213), (8.203), (8.208), we infer,

$$\begin{aligned}\left|\widetilde{u}(\widetilde{S}_*) - \left(u_* + \frac{\delta_{ext}}{2}\right)\right| &= \left|\widetilde{u}(\widetilde{S}_*) - u(\mathring{S})\right| \\ &= \left|1 + \widetilde{r}(\widetilde{S}_1) - \widetilde{r}(\widetilde{S}_*) - \left(-s(\mathring{S}) + c_{\widetilde{\Sigma}_*}\right)\right|.\end{aligned}$$

Together with (8.210) and (8.187), this yields

$$\left|\widetilde{u}(\widetilde{S}_*) - \left(u_* + \frac{\delta_{ext}}{2}\right)\right| \lesssim \left|1 + \widetilde{r}(\widetilde{S}_1) - c_{\widetilde{\Sigma}_*}\right| + \epsilon_0\delta_{ext}.$$

Since $c_{\widetilde{\Sigma}_*}$ in (8.208) is a constant, we have in particular

$$c_{\widetilde{\Sigma}_*} = u(\widetilde{S}_1) + r(\widetilde{S}_1) - \psi(s(\widetilde{S}_1))$$

and thus

$$\begin{aligned}\left|\widetilde{u}(\widetilde{S}_*) - \left(u_* + \frac{\delta_{ext}}{2}\right)\right| &\lesssim \left|1 + \widetilde{r}(\widetilde{S}_1) - u(\widetilde{S}_1) - r(\widetilde{S}_1) + \psi(s(\widetilde{S}_1))\right| + \epsilon_0\delta_{ext} \\ &\lesssim \left|1 - u(\widetilde{S}_1)\right| + \left|\widetilde{r}(\widetilde{S}_1) - r(\widetilde{S}_1)\right| + \left|\psi(s(\widetilde{S}_1))\right| \\ &\quad + \epsilon_0\delta_{ext}.\end{aligned}$$

In view of (8.211) and (8.210), we infer

$$\left|\widetilde{u}(\widetilde{S}_*) - \left(u_* + \frac{\delta_{ext}}{2}\right)\right| \lesssim \left|1 - u(\widetilde{S}_1)\right| + \epsilon_0\delta_{ext}.$$

Also, since:

- we have by the calibration of u

$$u = 1 \text{ on } S_1 \cap \{\widetilde{u}_{\mathcal{L}_0} = 1\} \cap \{\widetilde{\theta}_{\mathcal{L}_0} = \pi\},$$

i.e. S_1 is the unique sphere of Σ_* intersecting the curve of the south poles of $\{\widetilde{u}_{\mathcal{L}_0}=1\}$ in the part ${}^{(ext)}\widetilde{\mathcal{L}_0}$ of the initial data layer,

- we have, for the change of frame coefficients $(f_0, \underline{f}_0, \lambda_0)$ between the outgoing geodesic frame of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ and the outgoing geodesic foliation (u,s) initialized on Σ_*, the following control

$$|f_0| \lesssim \frac{\epsilon_0}{r}, \qquad |\lambda_0 - 1| + |\underline{f}_0| \lesssim \epsilon_0,$$

see Step 14 in the proof of Theorem M0, and hence

$$\widetilde{e}_4^{\mathcal{L}_0}(u-1) = \lambda_0\left(e_4 + f_0\cdot\nabla + \frac{1}{4}|f_0|^2 e_3\right)u = \frac{\lambda_0}{4}|f_0|^2 e_3(u) = O\left(\frac{\epsilon_0}{r^2}\right),$$

we infer

$$\sup_{\widetilde{\mathcal{R}}\cap\{\widetilde{u}_{\mathcal{L}_0}=1\}\cap\{\widetilde{\theta}_{\mathcal{L}_0}=\pi\}} |u-1| \lesssim \Delta_{ext}\frac{\epsilon_0}{r^2} \lesssim \epsilon_0\delta_{ext}.$$

This yields $|1-u(\widetilde{S}_1)| \lesssim \epsilon_0\delta_{ext}$ and hence

$$\left|\widetilde{u}(\widetilde{S}_*) - \left(u_* + \frac{\delta_{ext}}{2}\right)\right| \lesssim \epsilon_0\delta_{ext}. \tag{8.214}$$

In particular, we deduce, for ϵ_0 small enough,

$$\widetilde{u}(\widetilde{S}_*) > u_* \tag{8.215}$$

as desired.

Step 12. We would like to check that the condition (3.50) for r on $\widetilde{\Sigma}_*$ holds, i.e. we need to prove that there exists a choice of constant d_0 satisfying $\frac{1}{2} \leq d_0 \leq 1$ such that

$$\widetilde{r}(\widetilde{S}_*) = \delta_*\epsilon_0^{-1}(\widetilde{u}(\widetilde{S}_*))^{1+\delta_{dec}}.$$

To this end, note that we have in view of (8.210), (8.187) and (8.214)

$$\begin{aligned} & \widetilde{r}(\widetilde{S}_*) - \delta_*\epsilon_0^{-1}(\widetilde{u}(\widetilde{S}_*))^{1+\delta_{dec}} \\ = \; & s(\overset{\circ}{S}) + O(\epsilon_0\delta_{ext}) - \delta_*\epsilon_0^{-1}\left(u_* + \frac{\delta_{ext}}{2} + O(\epsilon_0\delta_{ext})\right)^{1+\delta_{dec}} \\ = \; & s(\overset{\circ}{S}) - \delta_*\epsilon_0^{-1}(u_*)^{1+\delta_{dec}} - \frac{1+\delta_{dec}}{2}\delta_*\epsilon_0^{-1}(u_*)^{\delta_{dec}}\delta_{ext} \end{aligned}$$

$$+\delta_*\epsilon_0^{-1}(u_*)^{\delta_{dec}}\delta_{ext}O\left(\frac{\delta_{ext}}{u_*}+\epsilon_0\right)+O\left(\epsilon_0\delta_{ext}\right).$$

Together with (8.203), we infer

$$\begin{aligned}
&\widetilde{r}(\widetilde{S}_*)-\delta_*\epsilon_0^{-1}(\widetilde{u}(\widetilde{S}_*))^{1+\delta_{dec}}\\
=\ & r_*+\frac{3d_0r_*}{4u_*}\delta_{ext}-\delta_*\epsilon_0^{-1}(u_*)^{1+\delta_{dec}}-\frac{1+\delta_{dec}}{2}\delta_*\epsilon_0^{-1}(u_*)^{\delta_{dec}}\delta_{ext}\\
&+\delta_*\epsilon_0^{-1}(u_*)^{\delta_{dec}}\delta_{ext}O\left(\frac{\delta_{ext}}{u_*}+\epsilon_0\right)+O\left(\epsilon_0\delta_{ext}\right)\\
=\ & r_*-\delta_*\epsilon_0^{-1}(u_*)^{1+\delta_{dec}}+\left(\frac{3d_0r_*}{4}-\frac{1+\delta_{dec}}{2}\delta_*\epsilon_0^{-1}(u_*)^{1+\delta_{dec}}\right)\frac{\delta_{ext}}{u_*}\\
&+\delta_*\epsilon_0^{-1}(u_*)^{\delta_{dec}}\delta_{ext}O\left(\frac{\delta_{ext}}{u_*}+\epsilon_0\right)+O\left(\epsilon_0\delta_{ext}\right).
\end{aligned}$$

Since we have by the condition (3.50) of r on Σ_*

$$r_*=\delta_*\epsilon_0^{-1}u_*^{1+\delta_{dec}},$$

we deduce

$$\begin{aligned}
&\widetilde{r}(\widetilde{S}_*)-\delta_*\epsilon_0^{-1}(\widetilde{u}(\widetilde{S}_*))^{1+\delta_{dec}}\\
=\ & \left(\frac{3d_0}{4}-\frac{1+\delta_{dec}}{2}\right)\frac{r_*\delta_{ext}}{u_*}+\delta_*\epsilon_0^{-1}(u_*)^{\delta_{dec}}\delta_{ext}O\left(\frac{\delta_{ext}}{u_*}+\epsilon_0\right)+O\left(\epsilon_0\delta_{ext}\right)\\
=\ & \frac{3r_*\delta_{ext}}{4u_*}\left(d_0-\frac{2+2\delta_{dec}}{3}+O\left(\epsilon_0+\frac{\delta_{ext}}{u_*}\right)\right).
\end{aligned}$$

Thus, we may choose the constant d_0 such that $\frac{1}{2}\leq d_0\leq 1$ and

$$\widetilde{r}(\widetilde{S}_*)\ =\ \delta_*\epsilon_0^{-1}(\widetilde{u}(\widetilde{S}_*))^{1+\delta_{dec}}$$

as desired.

Step 13. We summarize the properties of $\widetilde{\Sigma}_*$ obtained so far:

- $\widetilde{\Sigma}_*$ is a spacelike hypersurface included in the spacetime region $\widetilde{\mathcal{R}}$.
- The scalar function $\widetilde{u}$ is defined on $\widetilde{\Sigma}_*$ and its level sets are topological 2-spheres denoted by $\widetilde{S}$.
- The following GCM conditions holds on $\widetilde{\Sigma}_*$

$$\begin{aligned}
&\widetilde{\check{\kappa}}=0, \qquad \widetilde{\underline{\check{\kappa}}}=\widetilde{\underline{C}}_0+\sum_p\widetilde{\underline{C}}_p\widetilde{J}^{(p)}, \qquad \widetilde{\check{\mu}}=\widetilde{M}_0+\sum_p\widetilde{M}_p\widetilde{J}^{(p)},\\
&(\widetilde{\text{div}}\,\widetilde{\eta})_{\ell=1}=0, \qquad (\widetilde{\text{div}}\,\widetilde{\underline{\xi}})_{\ell=1}=0.
\end{aligned}$$

- In addition, the following GCM conditions holds on the sphere $\widetilde{S}_*$ of $\widetilde{\Sigma}_*$

$$\underline{\check{\widetilde{\kappa}}} = 0, \quad (\widetilde{\mathrm{div}}\,\widetilde{\beta})_{\ell=1} = 0, \quad (\widetilde{\mathrm{curl}}\,\widetilde{\beta})_{\ell=1,\pm} = 0, \quad (\widetilde{\mathrm{curl}}\,\widetilde{\beta})_{\ell=1,0} = \frac{2\widetilde{a}\widetilde{m}}{\widetilde{r}^5}.$$

- We have, for some constant $c_{\widetilde{\Sigma}_*}$,

$$\widetilde{u} + \widetilde{r} = c_{\widetilde{\Sigma}_*}, \quad \text{along} \quad \widetilde{\Sigma}_*.$$

- The following normalization condition holds true

$$\overline{\widetilde{b}_*} = -1 - \frac{2\widetilde{m}}{\widetilde{r}},$$

  where $\overline{\widetilde{b}_*}$ denotes the average of $\widetilde{b}_*$ on the spheres foliating $\widetilde{\Sigma}_*$, and where $\widetilde{b}_*$ is such that we have

$$\widetilde{\nu} = \widetilde{e}_3 + \widetilde{b}_* \widetilde{e}_4,$$

  with $\widetilde{\nu}$ the unique vectorfield tangent to the hypersurface $\widetilde{\Sigma}_*$, normal to $\widetilde{S}$, and normalized by $g(\widetilde{\nu}, \widetilde{e}_4) = -2$.
- The condition (3.50) for r on $\widetilde{S}_*$ holds, i.e. we have

$$\widetilde{r}(\widetilde{S}_*) \quad = \quad \delta_* \epsilon_0^{-1} (\widetilde{u}(\widetilde{S}_*))^{1+\delta_{dec}}.$$

- $\tilde{u}$ is calibrated along $\widetilde{\Sigma}_*$ by fixing the value $\widetilde{u} = 1$:

  $\widetilde{S}_1 = \widetilde{\Sigma}_* \cap \{\tilde{u} = 1\}$ is such that $\widetilde{S}_1 \cap \{u'_{\mathcal{L}_0} = 1\} \cap \{{}^{(ext)}\theta'_{\mathcal{L}_0} = \pi\} \neq \emptyset$,

  i.e. $\widetilde{S}_1$ is the unique sphere of $\widetilde{\Sigma}_*$ intersecting the curve of the south poles of $\{u'_{\mathcal{L}_0} = 1\}$ in the part ${}^{(ext)}\widetilde{\mathcal{L}_0}$ of the initial data layer.

Thus, $\widetilde{\Sigma}_*$ satisfies all the required properties for the future spacelike boundary of a GCM admissible spacetime, see Section 3.2.3. Furthermore, we have on $\widetilde{\Sigma}_*$

$$\widetilde{u}(\widetilde{S}_*) > u_*. \tag{8.216}$$

8.5.3. Steps 14–18

Step 14. We introduce:

- the outgoing PG frame (e'_4, e'_3, e'_1, e'_2) of ${}^{(ext)}\mathcal{M}$ extended to the spacetime $\mathcal{M}^{(extend)}$,

- the outgoing PG frame $(\widetilde{e}'_4, \widetilde{e}'_3, \widetilde{e}'_1, \widetilde{e}'_2)$ initialized on $\widetilde{\Sigma}_*$ from the GCM frame $(\widetilde{e}_4, \widetilde{e}_3, \widetilde{e}_1, \widetilde{e}_2)$ by the change of frame with coefficients $(f'', \underline{f}'', \lambda'')$ given by

$$\lambda'' = 1, \qquad f'' = \frac{\widetilde{a}}{\widetilde{r}}\widetilde{f}_0, \qquad \underline{f}'' = -\frac{(\widetilde{\nu}(\widetilde{r}) - \widetilde{b}_*)}{1 - \frac{1}{4}\widetilde{b}_*|f''|^2} f'',$$

  where the 1-form $\widetilde{f}_0$ is chosen on $\widetilde{\Sigma}_*$ by

$$(\widetilde{f}_0)_1 = 0, \quad (\widetilde{f}_0)_2 = \sin(\widetilde{\theta}), \quad \text{on} \quad \widetilde{S}_*, \qquad \widetilde{\nabla}_{\widetilde{\nu}}\widetilde{f}_0 = 0 \quad \text{on} \quad \widetilde{\Sigma}_*,$$

  with (e_1, e_2) specified on $\widetilde{S}_*$ by (2.51).

We have the following change of frame coefficients:

- $(f, \underline{f}, \lambda)$, introduced in Step 7, and corresponding to the change from the outgoing geodesic frame (e_4, e_3, e_1, e_2) of $\mathcal{M}^{(extend)}$ to the GCM frame $(\widetilde{e}_4, \widetilde{e}_3, \widetilde{e}_1, \widetilde{e}_2)$ of $\widetilde{\Sigma}_*$,
- $(f', \underline{f}', \lambda')$, which we now introduce, corresponding to the change from the outgoing geodesic frame (e_4, e_3, e_1, e_2) of $\mathcal{M}^{(extend)}$ to the outgoing PG frame (e'_4, e'_3, e'_1, e'_2) of ${}^{(ext)}\mathcal{M}$ extended to the spacetime $\mathcal{M}^{(extend)}$,
- $(f'', \underline{f}'', \lambda'')$, provided explicitly above, and corresponding to the change from the GCM frame $(\widetilde{e}_4, \widetilde{e}_3, \widetilde{e}_1, \widetilde{e}_2)$ of $\widetilde{\Sigma}_*$ to the outgoing PG frame $(\widetilde{e}'_4, \widetilde{e}'_3, \widetilde{e}'_1, \widetilde{e}'_2)$ initialized on $\widetilde{\Sigma}_*$,
- $(f''', \underline{f}''', \lambda''')$, which we now introduce, corresponding to the change from the outgoing PG frame (e'_4, e'_3, e'_1, e'_2) of ${}^{(ext)}\mathcal{M}$ extended to the spacetime $\mathcal{M}^{(extend)}$ and the outgoing PG frame $(\widetilde{e}'_4, \widetilde{e}'_3, \widetilde{e}'_1, \widetilde{e}'_2)$ initialized on $\widetilde{\Sigma}_*$.

In this step, our goal is to control the change of frame coefficients $(f''', \underline{f}''', \lambda''')$. In view of the above, we have schematically

$$(f''', \underline{f}''', \lambda''') = (f'', \underline{f}'', \lambda'') \circ (f, \underline{f}, \lambda) \circ (f', \underline{f}', \lambda')^{-1}$$

where $(f', \underline{f}', \lambda')^{-1}$ denote the coefficients corresponding to the inverse transformation coefficients of the transformation with coefficients $(f', \underline{f}', \lambda')$. We infer

$$\begin{aligned}\sup_{\widetilde{\Sigma}_*}\left|\widetilde{\mathfrak{d}}_*^k(f''', \underline{f}''', \lambda''' - 1)\right| \;\lesssim\;& \sup_{\widetilde{\Sigma}_*}\left|\widetilde{\mathfrak{d}}_*^k(f, \underline{f}, \lambda - 1)\right| \\ &+ \sup_{\widetilde{\Sigma}_*}\left|\widetilde{\mathfrak{d}}_*^k(f'' - f', \underline{f}'' - \underline{f}', \lambda'' - \lambda')\right|.\end{aligned}$$

Now, recall that $(f, \underline{f}, \lambda)$ satisfy in view of (8.211) and Corollary 4.2 in [50] (restated here in Corollary 8.12)

$$\sup_{\widetilde{S}\subset\widetilde{\Sigma}_*}\|\mathfrak{d}^{\leq k_*-3}(f, \underline{f}, \log(\lambda))\|_{L^2(\widetilde{S})} \lesssim \epsilon_0\delta_{ext}.$$

Together with the Sobolev embedding on the spheres $\widetilde{S}$, we find

$$\sup_{\widetilde{\Sigma}_*}\widetilde{r}\,|\mathfrak{d}^{\leq k_*-5}(f, \underline{f}, \log(\lambda))| \lesssim \epsilon_0\delta_{ext}. \tag{8.217}$$

We infer, for $\leq k_*-5$,

$$\sup_{\widetilde{\Sigma}_*} r\left|\widetilde{\mathfrak{d}}_*^k(f''', \underline{f}''', \lambda'''-1)\right| \quad\lesssim\quad \epsilon_0\delta_{ext} + \sup_{\widetilde{\Sigma}_*} r\left|\widetilde{\mathfrak{d}}_*^k(f''-f', \underline{f}''-\underline{f}', \lambda''-\lambda')\right|.$$

Together with the explicit formulas above for $(f'', \underline{f}'', \lambda'')$, we obtain, for $\leq k_*-5$,

$$\begin{aligned}
&\sup_{\widetilde{\Sigma}_*} r\left|\widetilde{\mathfrak{d}}_*^k(f''', \underline{f}''', \lambda'''-1)\right| \\
\lesssim\quad & \epsilon_0\delta_{ext} + \sup_{\widetilde{\Sigma}_*} r\left|\widetilde{\mathfrak{d}}_*^k\left(f' - \frac{\widetilde{a}}{\widetilde{r}}\widetilde{f}_0, \underline{f}' + \frac{(\widetilde{\nu}(\widetilde{r})-\widetilde{b}_*)}{1-\frac{1}{4}\widetilde{b}_*\frac{(\widetilde{a})^2}{(\widetilde{r})^2}|\widetilde{f}_0|^2}\frac{\widetilde{a}}{\widetilde{r}}\widetilde{f}_0, \lambda'-1\right)\right|.
\end{aligned}$$

We deduce, for $\leq k_*-5$,

$$\begin{aligned}
&\sup_{\widetilde{\Sigma}_*} r\left|\widetilde{\mathfrak{d}}_*^k(f''', \underline{f}''', \lambda'''-1)\right| \\
\lesssim\quad & \epsilon_0\delta_{ext} \\
&+\sup_{\widetilde{\Sigma}_*} r\left|\widetilde{\mathfrak{d}}_*^k\left(f' - \frac{a}{r}f_0, \underline{f}' + \frac{e_3(s)}{1+\frac{1}{4}(e_3(u)+e_3(s))\frac{a^2}{r^2}|f_0|^2}\frac{a}{r}f_0, \lambda'-1\right)\right| \\
&+\sup_{\widetilde{\Sigma}_*}\Big(|\widetilde{\mathfrak{d}}_*^k(\widetilde{a}\widetilde{f}_0 - af_0)| + r^{-1}|\widetilde{\mathfrak{d}}_*^k(\widetilde{r}-r)| + |\widetilde{\mathfrak{d}}_*^k(\widetilde{b}_* + e_3(u)+e_3(s))| \\
&\qquad\qquad + |\widetilde{\mathfrak{d}}_*^k(\widetilde{\nu}(\widetilde{r}) + e_3(u))|\Big).
\end{aligned} \tag{8.218}$$

We now control the terms on the RHS of (8.218). Note first the have have on $\Sigma_*^{(extend)}$, in view of the initialization of the PG frame of ${}^{(ext)}\mathcal{M}$,

$$\lambda' = 1, \qquad f' = \frac{a}{r}f_0, \qquad \underline{f}' = -\frac{(\nu(r)-b_*)}{1-\frac{1}{4}b_*\frac{a^2}{r^2}|f_0|^2}\frac{a}{r}f_0.$$

Also, we have $r=s$ on $\Sigma_*^{(extend)}$, and hence, using also $e_4(u)=0$ and $e_4(s)=1$, and the fact that ν is tangent to $\Sigma_*^{(extend)}$,

$$\begin{aligned} 0 &= \nu(u+r)=\nu(u)+\nu(s)=e_3(u)+b_*e_4(u)+e_3(s)+b_*e_4(s)\\ &= e_3(u)+e_3(s)+b_*,\\ \nu(r) &= \nu(s)=e_3(s)+b_*e_4(s)=e_3(s)+b_*, \end{aligned}$$

and hence, we deduce

$$b_*=-e_3(u)-e_3(s), \qquad \nu(r)=-e_3(u) \quad \text{on} \quad \Sigma_*^{(extend)}.$$

In view of the above, this yields

$$\begin{aligned} &\lambda'-1=0, \qquad f'-\frac{a}{r}f_0=0,\\ &\underline{f}'+\frac{e_3(s)}{1+\frac{1}{4}(e_3(u)+e_3(s))\frac{a^2}{r^2}|f_0|^2}\frac{a}{r}f_0=0 \quad \text{on} \quad \Sigma_*^{(extend)}. \end{aligned}$$

Next, using Corollary 2.13, we have the following transport equations for $(f',\underline{f}',\lambda')$

$$\begin{aligned} \nabla_{\lambda'^{-1}e_4'}f'+\frac{1}{2}\kappa f' &= -f'\cdot\widehat{\chi}+E_1(f',\Gamma),\\ \lambda'^{-1}e_4'(\log\lambda') &= 2f'\cdot\zeta+E_2(f',\Gamma),\\ \nabla_{\lambda'^{-1}e_4'}\underline{f}'+\frac{1}{2}\kappa\underline{f}' &= 2\nabla'(\log\lambda')+2\underline{\omega}f'-\underline{f}'\cdot\widehat{\chi}-\underline{f}'\cdot\nabla'f\\ &\quad +E_3(f',\underline{f}',\Gamma), \end{aligned}$$

where $E_1(f',\Gamma)$ and $E_2(f',\Gamma)$ contain expressions of the type $O(\Gamma f'^2)$ with no derivatives, and $E_3(f',\underline{f}',\Gamma)$ contain expressions of the type $O(\Gamma(f',\underline{f}')^2)$ with no derivatives. Together with the control of Step 1 on $\mathcal{M}^{(extend)}$, we infer, using also the extension $\nabla_4 f_0=0$ of f_0 from $\Sigma_*^{(extend)}$ to $\widetilde{\mathcal{R}}$, for $k\le k_*-3$,

$$\sup_{\widetilde{\mathcal{R}}}\left(r^{-1}\left|\mathfrak{d}^k e_4'\left(r\left(f'-\frac{a}{r}f_0\right)\right)\right|+\left|\mathfrak{d}^k e_4'\left(\log(\lambda')\right)\right|\right)\lesssim\frac{1}{r^3}+\frac{1}{r}|\mathfrak{d}^{\le k}\Gamma_g|\lesssim\frac{1}{r^3},$$

and, for $k\le k_*-4$,

$$\begin{aligned} &\sup_{\widetilde{\mathcal{R}}} r^{-1}\left|\mathfrak{d}^k e_4'\left(r\left(\underline{f}'+\frac{e_3(s)}{1+\frac{1}{4}(e_3(u)+e_3(s))\frac{a^2}{r^2}|f_0|^2}\frac{a}{r}f_0-2r\nabla'(\log(\lambda'))\right)\right)\right|\\ &\lesssim\frac{1}{r^3}+\frac{1}{r}|\mathfrak{d}^{\le k}\Gamma_b|\lesssim\frac{1}{r^3}+\frac{\epsilon_0}{r^2}. \end{aligned}$$

Together with the above identities on $\Sigma_*^{(extend)}$, we integrate from $\Sigma_*^{(extend)}$ and obtain, using the dominant condition for r on $\widetilde{\mathcal{R}}$, for $k\leq k_*-3$,

$$\sup_{\widetilde{\mathcal{R}}} r\left|\mathfrak{d}^k\left(f'-\frac{a}{r}f_0, \lambda'-1\right)\right| \lesssim \frac{\Delta_{ext}}{r^2}\lesssim\frac{\epsilon_0\Delta_{ext}}{r}\lesssim\epsilon_0\delta_{ext}$$

and for $k\leq k_*-4$

$$\begin{aligned}\sup_{\widetilde{\mathcal{R}}} r\left|\mathfrak{d}^k\left(\underline{f}'+\frac{e_3(s)}{1+\frac{1}{4}(e_3(u)+e_3(s))\frac{a^2}{r^2}|f_0|^2}\frac{a}{r}f_0\right)\right| &\lesssim \frac{\Delta_{ext}}{r^2}+\frac{\epsilon_0\Delta_{ext}}{r}\\ &\lesssim \frac{\epsilon_0\Delta_{ext}}{r}\lesssim\epsilon_0\delta_{ext}.\end{aligned}$$

Together with (8.218), we infer, for $k\leq k_*-5$,

$$\begin{aligned}&\sup_{\widetilde{\Sigma}_*} r\left|\widetilde{\mathfrak{d}}_*^k(f''',\underline{f}''',\lambda'''-1)\right|\\ &\lesssim \epsilon_0\delta_{ext}+\sup_{\widetilde{\Sigma}_*}\Big(|\widetilde{\mathfrak{d}}_*^k(\widetilde{a}\widetilde{f}_0-af_0)|+r^{-1}|\widetilde{\mathfrak{d}}_*^k(\widetilde{r}-r)|\\ &\quad+|\widetilde{\mathfrak{d}}_*^k(\widetilde{b}_*+e_3(u)+e_3(s))|+|\widetilde{\mathfrak{d}}_*^k(\widetilde{\nu}(\widetilde{r})+e_3(u))|\Big).\end{aligned}\tag{8.219}$$

Step 15. Next, we focus on the control of the terms on the RHS of (8.219). To ease the notations, we introduce the scalar $\mathfrak{b}_*$ on $\widetilde{\Sigma}_*$ given by

$$\mathfrak{b}_* := \widetilde{b}_*+e_3(u)+e_3(s).\tag{8.220}$$

Lemma 8.13 yields, in view of the definition of $\mathfrak{b}_*$ and the control for $(f,\underline{f},\lambda)$ provided by (8.217), for any scalar function h, any sphere $\widetilde{S}\subset\widetilde{\Sigma}_*$, and any $k\leq k_*-5$,

$$\begin{aligned}&\|\widetilde{\mathfrak{d}}_*^k h\|_{\mathfrak{h}_{s_{max}-j}(\widetilde{S})}\\ &\lesssim \sup_{\widetilde{\mathcal{R}}}\Big(|\widehat{\mathfrak{d}}^{\leq k}h|+\epsilon_0\delta_{ext}|\mathfrak{d}^{\leq k}h|\Big)+\|\widetilde{\mathfrak{d}}_*^{\leq k-1}\mathfrak{b}_*\|_{L^2(\widetilde{S})}\sup_{\widetilde{\mathcal{R}}}|\mathfrak{d}^{\leq k}h|.\end{aligned}\tag{8.221}$$

We first estimate $\widetilde{r}-r$ on $\widetilde{\Sigma}_*$. Recall from (8.184) that we have in particular, for $k\leq k_*-4$,

$$\sup_{\widetilde{\mathcal{R}}} r^2\left|\widehat{\mathfrak{d}}^k\left(\check{\kappa}\right)\right| \lesssim \frac{\epsilon_0}{r}\Delta_{ext}\lesssim\epsilon_0\delta_{ext}.$$

Together with the GCM condition $\widecheck{\widetilde{\kappa}} = 0$ and (8.221), we infer, for $k \leq k_* - 5$,

$$\sup_{\widetilde{\Sigma}_*} r \left\| \widetilde{\mathfrak{d}}_*^k \left(\widecheck{\widetilde{\kappa}} - \widecheck{\kappa} \right) \right\|_{L^2(\widetilde{S})} \lesssim \epsilon_0 \delta_{ext} + \epsilon_0 \sup_{\widetilde{\Sigma}_*} r^{-1} \| \widetilde{\mathfrak{d}}_*^{\leq k-1} \mathfrak{b}_* \|_{L^2(\widetilde{S})}.$$

Now, we have

$$\widecheck{\widetilde{\kappa}} - \widecheck{\kappa} = \widetilde{\kappa} - \kappa + \frac{2}{\widetilde{r}} - \frac{2}{r} = \widetilde{\kappa} - \kappa - \frac{2(\widetilde{r} - r)}{r\widetilde{r}}$$

so that

$$\widetilde{r} - r = \frac{r\widetilde{r}}{2} \left(\widetilde{\kappa} - \kappa - \left(\widecheck{\widetilde{\kappa}} - \widecheck{\kappa} \right) \right)$$

and hence, using the above estimate for $\widecheck{\widetilde{\kappa}} - \widecheck{\kappa}$, we have, for $k \leq k_* - 5$,

$$\sup_{\widetilde{\Sigma}_*} r^{-1} \left\| \widetilde{\mathfrak{d}}_*^k (\widetilde{r} - r) \right\|_{L^2(\widetilde{S})} \lesssim \epsilon_0 \delta_{ext} + \sup_{\widetilde{\Sigma}_*} r^2 \left| \widetilde{\mathfrak{d}}_*^k (\widetilde{\kappa} - \kappa) \right| + \epsilon_0 \sup_{\widetilde{\Sigma}_*} r^{-1} \| \widetilde{\mathfrak{d}}_*^{\leq k-1} \mathfrak{b}_* \|_{L^2(\widetilde{S})}.$$

Using the change of frame formula for $\widetilde{\kappa}$, together with the control (8.217) for $(f, \underline{f}, \lambda)$ and the control of Step 1 on $\mathcal{M}^{(extend)}$, we deduce for $k \leq k_* - 6$

$$\sup_{\widetilde{\Sigma}_*} r^{-1} \left\| \widetilde{\mathfrak{d}}_*^k (\widetilde{r} - r) \right\|_{L^2(\widetilde{S})} \lesssim \epsilon_0 \delta_{ext} + \epsilon_0 \sup_{\widetilde{\Sigma}_*} r^{-1} \| \widetilde{\mathfrak{d}}_*^{\leq k-1} \mathfrak{b}_* \|_{L^2(\widetilde{S})}. \tag{8.222}$$

Next, we control $\mathfrak{b}_* = \widetilde{b}_* + e_3(u) + e_3(s)$. First, note that we have

$$\widetilde{\nu}(\widetilde{r} - r) = \widetilde{\nu}(\widetilde{r}) - \widetilde{e}_3(r) - \widetilde{b}_* \widetilde{e}_4(r).$$

Together with the control of $\widetilde{r} - r$ in (8.222), the fact that $\widetilde{\nu}$ is tangent to $\widetilde{\Sigma}_*$, the change of frame formulas, the control (8.217) for $(f, \underline{f}, \lambda)$ and the control of Step 1 on $\mathcal{M}^{(extend)}$, we deduce for $k \leq k_* - 7$,

$$\sup_{\widetilde{\Sigma}_*} r^{-1} \left\| \widetilde{\mathfrak{d}}_*^k \left(\widetilde{\nu}(\widetilde{r}) - e_3(r) - \widetilde{b}_* e_4(r) \right) \right\|_{L^2(\widetilde{S})} \lesssim \epsilon_0 \delta_{ext} + \epsilon_0 \sup_{\widetilde{\Sigma}_*} r^{-1} \| \widetilde{\mathfrak{d}}_*^{\leq k-1} \mathfrak{b}_* \|_{L^2(\widetilde{S})}.$$

The control (8.187) of $r - s$ and (8.221) allows us to replace r by s in the above formula which yields, since $e_4(s) = 1$, for $k \leq k_* - 7$,

$$\sup_{\widetilde{\Sigma}_*} r^{-1}\left\|\widetilde{\mathfrak{d}}_*^k\left(\widetilde{\nu}(\widetilde{r})-e_3(s)-\widetilde{b}_*\right)\right\|_{L^2(\widetilde{S})} \lesssim \epsilon_0\delta_{ext}+\epsilon_0\sup_{\widetilde{\Sigma}_*} r^{-1}\|\widetilde{\mathfrak{d}}_*^{\leq k-1}\mathfrak{b}_*\|_{L^2(\widetilde{S})}.$$

Since $\widetilde{\nu}(\widetilde{u}+\widetilde{r})=0$ on $\widetilde{\Sigma}_*$, we infer for $k\leq k_*-7$

$$\sup_{\widetilde{\Sigma}_*} r^{-1}\left\|\widetilde{\mathfrak{d}}_*^k\left(\widetilde{b}_*+\widetilde{\nu}(\widetilde{u})+e_3(s)\right)\right\|_{L^2(\widetilde{S})} \lesssim \epsilon_0\delta_{ext}+\epsilon_0\sup_{\widetilde{\Sigma}_*} r^{-1}\|\widetilde{\mathfrak{d}}_*^{\leq k-1}\mathfrak{b}_*\|_{L^2(\widetilde{S})}.$$

Now, as part of the construction of $\widetilde{\Sigma}_*$, the following transversality conditions on $\widetilde{\Sigma}_*$ are assumed, see Theorem 4.1 in [50] (restated here in Theorem 8.11),

$$\widetilde{\xi}=\widetilde{\omega}=0,\qquad \underline{\widetilde{\eta}}=-\widetilde{\zeta},\qquad \widetilde{e}_4(\widetilde{r})=1,\qquad \widetilde{e}_4(\widetilde{u})=0. \tag{8.223}$$

We infer

$$\widetilde{\nu}(\widetilde{u}) = \widetilde{e}_3(\widetilde{u})+\widetilde{b}_*\widetilde{e}_4(\widetilde{u})=\widetilde{e}_3(\widetilde{u})$$

and hence, for $k\leq k_*-7$,

$$\sup_{\widetilde{\Sigma}_*} r^{-1}\left\|\widetilde{\mathfrak{d}}_*^k\left(\widetilde{b}_*+\widetilde{e}_3(\widetilde{u})+e_3(s)\right)\right\|_{L^2(\widetilde{S})} \lesssim \epsilon_0\delta_{ext}+\epsilon_0\sup_{\widetilde{\Sigma}_*} r^{-1}\|\widetilde{\mathfrak{d}}_*^{\leq k-1}\mathfrak{b}_*\|_{L^2(\widetilde{S})}.$$

Also, using again the transversality conditions (8.223), we have

$$\widetilde{\nabla}(\widetilde{e}_3(\widetilde{u}))=(\widetilde{\zeta}-\widetilde{\eta})\widetilde{e}_3(\widetilde{u}).$$

Similarly, we have for the outgoing geodesic foliation of $\widetilde{\mathcal{R}}$

$$\nabla(e_3(u))=(\zeta-\eta)e_3(u).$$

Together with the change of frame formulas, the control (8.217) for $(f,\underline{f},\lambda)$ and the control of Step 1 on $\mathcal{M}^{(extend)}$, we deduce, for $k\leq k_*-6$,

$$\sup_{\widetilde{\Sigma}_*}\left|\widetilde{\mathfrak{d}}_*^k\left(\widetilde{\nabla}(e_3(u))-(\widetilde{\zeta}-\widetilde{\eta})e_3(u)\right)\right| \lesssim r^{-1}\epsilon_0\delta_{ext}.$$

Subtracting to the above identity for $\widetilde{e}_3(\widetilde{u})$, we infer for $k\leq k_*-6$

$$\sup_{\widetilde{\Sigma}_*}\left|\widetilde{\mathfrak{d}}_*^k\left(\widetilde{\nabla}(\widetilde{e}_3(\widetilde{u})-e_3(u))-(\widetilde{\zeta}-\widetilde{\eta})(\widetilde{e}_3(\widetilde{u})-e_3(u))\right)\right| \lesssim r^{-1}\epsilon_0\delta_{ext}.$$

Together with the above estimate for $\widetilde{b}_*+\widetilde{e}_3(\widetilde{u})+e_3(s)$, we infer, for $k\leq k_*-8$,

$$
\begin{aligned}
& \sup_{\widetilde{\Sigma}_*} \left\| \widetilde{\mathfrak{d}}_*^k \left(\widetilde{\nabla} \left(\widetilde{b}_* + e_3(u) + e_3(s) \right) - (\widetilde{\zeta} - \widetilde{\eta}) \left(\widetilde{b}_* + e_3(u) + e_3(s) \right) \right) \right\|_{L^2(\widetilde{S})} \\
\lesssim \quad & \epsilon_0 \delta_{ext} + \epsilon_0 \sup_{\widetilde{\Sigma}_*} r^{-1} \|\widetilde{\mathfrak{d}}_*^{\leq k-1} \mathfrak{b}_*\|_{L^2(\widetilde{S})}.
\end{aligned}
$$

Using the change of frame formulas, the control (8.217) for $(f, \underline{f}, \lambda)$ and the control of Step 1 on $\mathcal{M}^{(extend)}$ to control $\widetilde{\zeta}$ and $\widetilde{\eta}$, we infer in view of $\mathfrak{b}_* = \widetilde{b}_* + e_3(u) + e_3(s)$, for $k \leq k_* - 9$,

$$
\sup_{\widetilde{\Sigma}_*} \left\| \widetilde{\mathfrak{d}}_*^k \widetilde{\nabla} (\mathfrak{b}_*) \right\|_{\mathfrak{h}_1(\widetilde{S})} \quad \lesssim \quad \epsilon_0 \delta_{ext} + \epsilon_0 \sup_{\widetilde{\Sigma}_*} r^{-1} \left\| \widetilde{\mathfrak{d}}_*^k (\mathfrak{b}_*) \right\|_{\mathfrak{h}_1(\widetilde{S})}.
$$

Also, recall that we have obtained in Step 4

$$
\overline{e_3(u) + e_3(s)} \quad = \quad 1 + \frac{2m}{r} \text{ on } \Sigma_*^{(extend)},
$$

where $\overline{e_3(u) + e_3(s)}$ denotes the average of $e_3(u) + e_3(s)$ on the spheres of the u-foliation. Since $b_* = -(e_3(u) + e_3(s))$ on $\Sigma_*^{(extend)}$, we infer, since $\nu = e_3 + b_* e_4$ is tangent to $\Sigma_*^{(extend)}$,

$$
(e_3 - (e_3(u) + e_3(s))\, e_4)^k \left(\overline{e_3(u) + e_3(s)} - \left(1 + \frac{2m}{r} \right) \right) \quad = \quad 0 \text{ on } \Sigma_*^{(extend)}.
$$

Arguing as for (8.188), we propagate forward in e_4, and using the bounds of Step 1 on $\mathcal{M}^{(extend)}$, we infer, for $k \leq k_* - 3$,

$$
\begin{aligned}
\sup_{\widetilde{\mathcal{R}}} \left| (e_3 - (e_3(u) + e_3(s))\, e_4)^k \left(\overline{e_3(u) + e_3(s)} - \left(1 + \frac{2m}{r} \right) \right) \right| \quad & \lesssim \quad \frac{\epsilon_0}{r} \Delta_{ext} \\
& \lesssim \quad \epsilon_0 \delta_{ext}.
\end{aligned}
$$

On the other hand, by our GCM condition on $\widetilde{\Sigma}_*$ for $\widetilde{b}_*$, we have

$$
\overline{\widetilde{b}_*} = -1 - \frac{2\widetilde{m}}{\widetilde{r}} \quad \text{on} \quad \widetilde{\Sigma}_*,
$$

where $\overline{\widetilde{b}_*}$ denotes the average of $\widetilde{b}_*$ on the spheres of $\widetilde{\Sigma}_*$, and hence, we have on $\widetilde{\Sigma}_*$

$$
\left(\widetilde{e}_3 + \widetilde{b}_* \widetilde{e}_4 \right)^k \left(\overline{\widetilde{b}_*} + \left(1 + \frac{2\widetilde{m}}{\widetilde{r}} \right) \right) \quad = \quad 0.
$$

Subtracting this identity to the previous estimate, using the change of frame formulas, the control (8.217) for $(f, \underline{f}, \lambda)$ and the control of Step 1 on $\mathcal{M}^{(extend)}$, we infer, for $k\le k_*-6$,

$$\sup_{\widetilde{\Sigma}_*}\left|\widetilde{\nu}^k\left(\overline{\widetilde{b}_*}+\overline{e_3(u)+e_3(s)}\right)\right| \lesssim \epsilon_0\delta_{ext}+r^{-1}|\widetilde{m}-m|+r^{-2}\sup_{\widetilde{\Sigma}_*}|\widetilde{\nu}^{\le k}(\widetilde{r}-r)| +\epsilon_0\sup_{\widetilde{\Sigma}_*}|\widetilde{\nu}^{\le k}(\widetilde{b}_*+e_3(u)+e_3(s))|.$$

Recall that the averages in $\overline{\widetilde{b}_*}$ and $\overline{e_3(u)+e_3(s)}$ are taken respectively on the spheres foliating $\widetilde{\Sigma}_*$ and $\widetilde{\mathcal{R}}$. Using the fourth item of Proposition 8.3 to compare the average of $e_3(u)+e_3(s)$ on both types of spheres, together with the control (8.217) for $(f,\underline{f},\lambda)$ and the control of Step 1 on $\mathcal{M}^{(extend)}$, we obtain, for $k\le k_*-6$,

$$\sup_{\widetilde{\Sigma}_*}\left|\widetilde{\nu}^k\left(\overline{\widetilde{b}_*+e_3(u)+e_3(s)}\right)\right| \lesssim \epsilon_0\delta_{ext}+r^{-1}|\widetilde{m}-m|+r^{-2}\sup_{\widetilde{\Sigma}_*}|\widetilde{\nu}^{\le k}(\widetilde{r}-r)| +\epsilon_0\sup_{\widetilde{\Sigma}_*}|\widetilde{\nu}^{\le k}(\widetilde{b}_*+e_3(u)+e_3(s))|,$$

where the average is, from now on, only taken w.r.t. to the spheres foliating $\widetilde{\Sigma}_*$. Recalling the definition $\mathfrak{b}_*=\widetilde{b}_*+e_3(u)+e_3(s)$, we have thus obtained, for $k\le k_*-6$,

$$\sup_{\widetilde{\Sigma}_*}\left|\widetilde{\nu}^k\left(\overline{\mathfrak{b}_*}\right)\right|\lesssim \epsilon_0\delta_{ext}+r^{-1}|\widetilde{m}-m|+r^{-2}\sup_{\widetilde{\Sigma}_*}|\widetilde{\nu}^{\le k}(\widetilde{r}-r)|+\epsilon_0\sup_{\widetilde{\Sigma}_*}|\widetilde{\nu}^{\le k}\mathfrak{b}_*|.$$

Together with (8.210) for $\widetilde{m}-m$, (8.222) for $\widetilde{r}-r$, and Sobolev, we deduce, for $k\le k_*-8$,

$$\sup_{\widetilde{\Sigma}_*}\left|\widetilde{\nu}^k\left(\overline{\mathfrak{b}_*}\right)\right| \lesssim \epsilon_0\delta_{ext}+\epsilon_0\sup_{\widetilde{\Sigma}_*}r^{-1}\left\|\widetilde{\mathfrak{d}}_*^{\le k}\left(\mathfrak{b}_*\right)\right\|_{\mathfrak{h}_2(\widetilde{S})}.$$

Using Corollary 5.32 and Sobolev, we obtain, for $k\le k_*-8$,

$$\begin{aligned}\sup_{\widetilde{\Sigma}_*}\left|\overline{\widetilde{\nu}^k(\mathfrak{b}_*)}\right| &\lesssim \epsilon_0\delta_{ext}+\epsilon_0\sup_{\widetilde{\Sigma}_*}|\widetilde{\nu}^{\le k}\mathfrak{b}_*|+\epsilon_0\sup_{\widetilde{\Sigma}_*}r^{-1}\left\|\widetilde{\mathfrak{d}}_*^{\le k}\left(\mathfrak{b}_*\right)\right\|_{\mathfrak{h}_2(\widetilde{S})}\\ &\lesssim \epsilon_0\delta_{ext}+\epsilon_0\sup_{\widetilde{\Sigma}_*}r^{-1}\left\|\widetilde{\mathfrak{d}}_*^{\le k}\left(\mathfrak{b}_*\right)\right\|_{\mathfrak{h}_2(\widetilde{S})}.\end{aligned}$$

Thus, we have obtained so far on $\widetilde{\Sigma}_*$, for $k\le k_*-9$,

$$\sup_{\widetilde{\Sigma}_*}\left\|\widetilde{\mathfrak{d}}_*^k\widetilde{\nabla}\left(\mathfrak{b}_*\right)\right\|_{\mathfrak{h}_1(\widetilde{S})} \lesssim \epsilon_0\delta_{ext}+\epsilon_0\sup_{\widetilde{\Sigma}_*}r^{-1}\left\|\widetilde{\mathfrak{d}}_*^k\left(\mathfrak{b}_*\right)\right\|_{\mathfrak{h}_1(\widetilde{S})}.$$

and, for $k\leq k_*-8$,

$$\sup_{\widetilde{\Sigma}_*}\left|\overline{\widetilde{\nu}^k(\mathfrak{b}_*)}\right| \lesssim \epsilon_0\delta_{ext}+\epsilon_0\sup_{\widetilde{\Sigma}_*}r^{-1}\left\|\widetilde{\mathfrak{d}}_*^{\leq k}(\mathfrak{b}_*)\right\|_{\mathfrak{h}_2(\widetilde{S})}.$$

Using again Poincaré inequality, we infer, for $k\leq k_*-9$,

$$\begin{aligned}
r^{-1}\left\|\widetilde{\mathfrak{d}}_*^k\mathfrak{b}_*\right\|_{\mathfrak{h}_2(\widetilde{S})} &\lesssim |\overline{\widetilde{\nu}^k(\mathfrak{b}_*)}|+\left\|\widetilde{\mathfrak{d}}_*^k\widetilde{\nabla}\mathfrak{b}_*\right\|_{\mathfrak{h}_1(\widetilde{S})}\\
&\lesssim \epsilon_0\sup_{\widetilde{\Sigma}_*}r^{-1}\left\|\widetilde{\mathfrak{d}}_*^{\leq k}(\mathfrak{b}_*)\right\|_{\mathfrak{h}_2(\widetilde{S})}+\epsilon_0\delta_{ext}.
\end{aligned}$$

For ϵ_0 small enough, we infer, for $k\leq k_*-9$,

$$\sup_{\widetilde{S}\subset\widetilde{\Sigma}_*}\left(r^{-1}\left\|\widetilde{\mathfrak{d}}_*^k\mathfrak{b}_*\right\|_{\mathfrak{h}_2(\widetilde{S})}\right) \lesssim \epsilon_0\delta_{ext}.$$

Using Sobolev, and recalling the definition of $\mathfrak{b}_*$, we infer, for $k\leq k_*-9$,

$$\sup_{\widetilde{\Sigma}_*}\left|\widetilde{\mathfrak{d}}_*^k(\widetilde{b}_*+e_3(u)+e_3(s))\right| \lesssim \epsilon_0\delta_{ext}.$$

Also, (8.222), together with the above control of $\mathfrak{b}_*$ and Sobolev, implies, for $k\leq k_*-9$,

$$\sup_{\widetilde{\Sigma}_*}r^{-1}|\widetilde{\mathfrak{d}}_*^k(\widetilde{r}-r)| \lesssim \epsilon_0\delta_{ext}. \tag{8.224}$$

Next, we control $\widetilde{\nu}(\widetilde{r})+e_3(u)$. We have

$$\begin{aligned}
\widetilde{\nu}(\widetilde{r})+e_3(u) &= \widetilde{\nu}(\widetilde{r}-r)+\widetilde{\nu}(r)+e_3(u)\\
&= \widetilde{\nu}(\widetilde{r}-r)+\widetilde{e}_3(r)+\widetilde{b}_*\widetilde{e}_4(r)+e_3(u)\\
&= \widetilde{\nu}(\widetilde{r}-r)+e_3(r)+\widetilde{b}_*e_4(r)+e_3(u)+(\widetilde{e}_3(r)-e_3(r))\\
&\quad +\widetilde{b}_*(\widetilde{e}_4(r)-e_4(r))\\
&= \widetilde{\nu}(\widetilde{r}-r)+e_3(r-s)+e_3(s)+\widetilde{b}_*+\widetilde{b}_*e_4(r-s)+e_3(u)\\
&\quad +(\widetilde{e}_3(r)-e_3(r))+\widetilde{b}_*(\widetilde{e}_4(r)-e_4(r))\\
&= \left(\widetilde{b}_*+e_3(s)+e_3(u)\right)+\widetilde{\nu}(\widetilde{r}-r)+e_3(r-s)+\widetilde{b}_*e_4(r-s)\\
&\quad +(\widetilde{e}_3(r)-e_3(r))+\widetilde{b}_*(\widetilde{e}_4(r)-e_4(r)).
\end{aligned}$$

Together with

- the above control of $\widetilde{b}_* + e_3(u) + e_3(s)$ on $\widetilde{\Sigma}_*$,
- the above control of $\widetilde{r} - r$ on $\widetilde{\Sigma}_*$ and the fact that $\widetilde{\nu}$ is tangent to $\widetilde{\Sigma}_*$,
- the control (8.187) of $e_3(r-s)$ and $e_4(r-s)$ on $\widetilde{\mathcal{R}}$, together with (8.221),
- the change of frame formulas, the control (8.217) for $(f, \underline{f}, \lambda)$ and the control of Step 1 on $\mathcal{M}^{(extend)}$, which yields the control of $\widetilde{e}_3(r) - e_3(r)$ and $\widetilde{e}_4(r) - e_4(r)$,

we deduce, for $k \leq k_* - 9$,

$$\sup_{\widetilde{\Sigma}_*} \left|\widetilde{\mathfrak{d}}_*^k(\widetilde{\nu}(\widetilde{r}) + e_3(u))\right| \lesssim \epsilon_0\delta_{ext}.$$

The above estimates, together with (8.219), yield for $\leq k_* - 9$,

$$\sup_{\widetilde{\Sigma}_*} r\left|\widetilde{\mathfrak{d}}_*^k(f''', \underline{f}''', \lambda''' - 1)\right| \lesssim \epsilon_0\delta_{ext} + \sup_{\widetilde{\Sigma}_*}|\widetilde{\mathfrak{d}}_*^k(\widetilde{a}\widetilde{f}_0 - af_0)|. \tag{8.225}$$

Step 16. Next, we focus on the control of the RHS of (8.225). To this end, we first control $\widetilde{a}$. Recall that we have in view of (8.186)

$$\begin{aligned}\sup_{\widetilde{\mathcal{R}}\cap\{u\geq u_*\}} r^5\left(\left|\frac{1}{|S|}\int_S \operatorname{curl}\beta J^{(0)} - \frac{2am}{r^5}\right| + \frac{1}{|S|}\left|\int_S \operatorname{curl}\beta J^{(\pm)}\right|\right) &\lesssim \frac{\epsilon_0}{r}\Delta_{ext}\\ &\lesssim \epsilon_0\delta_{ext}.\end{aligned}$$

Also, recall that $\overset{\circ}{S} = S(\overset{\circ}{u}, \overset{\circ}{s})$ is the sphere of the foliation of $\mathcal{R}\cap\{u \geq u_*\}$ which shares the same south pole a $\widetilde{S}_*$. Relying on Corollary 5.9 in [40] (see also Proposition 8.3 here), we have, in view of the control (8.234) of the deformation map $\Psi : \overset{\circ}{S} \to \widetilde{S}_*$, for $p = 0, +, -$,

$$\begin{aligned}&\left|\int_{\widetilde{S}_*} \operatorname{curl}\beta J^{(p)} - \int_{\overset{\circ}{S}} \operatorname{curl}\beta J^{(p)}\right|\\ &\lesssim r\epsilon_0\delta_{ext}\Bigg(\sup_{\widetilde{\mathcal{R}}}|\not{\mathfrak{d}}^{\leq 1}(\operatorname{curl}\beta J^{(p)})| + r\sup_{\widetilde{\mathcal{R}}}(|\nabla_3(\operatorname{curl}\beta J^{(p)})|\\ &\qquad +|\nabla_4(\operatorname{curl}\beta J^{(p)})|)\Bigg)\\ &\lesssim \epsilon_0\delta_{ext}\sup_{\widetilde{\mathcal{R}}}\left(|\mathfrak{d}^{\leq 2}\beta| + r|\mathfrak{d}\nabla_3\beta| + r|\mathfrak{d}\beta|(|\nabla_3 J^{(p)}| + |\nabla_4 J^{(p)}|\right).\end{aligned}$$

Together with the control of β, the fact that $e_4(J^{(p)}) = 0$, and the control (8.201) for $e_3(J^{(p)})$, we deduce

$$\max_{p=0,+,-}\left|\int_{\widetilde{S}_*}\mathrm{curl}\,\beta J^{(p)}-\int_{\overset{\circ}{S}}\mathrm{curl}\,\beta J^{(p)}\right| \lesssim \frac{\epsilon_0\delta_{ext}}{r^3}.$$

Using the control of $\widetilde{r}_*-\overset{\circ}{r}$ in Proposition 8.3, we infer

$$\max_{p=0,+,-}\widetilde{r}^5\left|\frac{1}{|\widetilde{S}_*|}\int_{\widetilde{S}_*}\mathrm{curl}\,\beta J^{(p)}-\frac{1}{|\overset{\circ}{S}|}\int_{\overset{\circ}{S}}\mathrm{curl}\,\beta J^{(p)}\right| \lesssim \epsilon_0\delta_{ext}.$$

Plugging in the above, using the fact that $\overset{\circ}{S}\subset\widetilde{\mathcal{R}}\cap\{u\geq u_*\}$, and using again the control of $\widetilde{r}_*-\overset{\circ}{r}$ in Proposition 8.3, we obtain

$$\widetilde{r}^5\left(\left|\frac{1}{|\widetilde{S}_*|}\int_{\widetilde{S}_*}\mathrm{curl}\,\beta J^{(0)}-\frac{2am}{\widetilde{r}^5}\right|+\frac{1}{|\widetilde{S}_*|}\left|\int_{\widetilde{S}_*}\mathrm{curl}\,\beta J^{(\pm)}\right|\right) \lesssim \epsilon_0\delta_{ext}.$$

Together with (8.210), this yields

$$\widetilde{r}^5\left(\left|\frac{1}{|\widetilde{S}_*|}\int_{\widetilde{S}_*}\mathrm{curl}\,\beta J^{(0)}-\frac{2a\widetilde{m}}{\widetilde{r}^5}\right|+\frac{1}{|\widetilde{S}_*|}\left|\int_{\widetilde{S}_*}\mathrm{curl}\,\beta J^{(\pm)}\right|\right) \lesssim \epsilon_0\delta_{ext}.$$

Next, in view of the change of frame formulas, we have, using also the control of Step 1 on $\mathcal{M}^{(extend)}$,

$$\begin{aligned}&\left|\widetilde{\mathrm{curl}}\,\widetilde{\beta}-\mathrm{curl}\,\beta\right|\\ \lesssim\;&\left|\widetilde{\mathrm{curl}}\left(\lambda\left(\beta+\frac{3}{2}(f\rho+\ {}^*f\ {}^*\rho)+\frac{1}{2}\alpha\cdot\underline{f}+\cdots\right)\right)-\mathrm{curl}\,\beta\right|\\ \lesssim\;& r^{-4}|\widetilde{\mathfrak{d}}^{\leq 1}(f,\underline{f},\lambda-1)|\end{aligned}$$

and hence, using the control (8.217) for $(f,\underline{f},\lambda)$, we deduce on $\widetilde{\Sigma}_*$

$$\left|\widetilde{\mathrm{curl}}\,\widetilde{\beta}-\mathrm{curl}\,\beta\right| \lesssim r^{-5}\epsilon_0\delta_{ext}.$$

Plugging in the above, we infer

$$\widetilde{r}^5\left(\left|\frac{1}{|\widetilde{S}_*|}\int_{\widetilde{S}_*}\widetilde{\mathrm{curl}}\,\widetilde{\beta}J^{(0)}-\frac{2a\widetilde{m}}{\widetilde{r}^5}\right|+\frac{1}{|\widetilde{S}_*|}\left|\int_{\widetilde{S}_*}\widetilde{\mathrm{curl}}\,\widetilde{\beta}J^{(\pm)}\right|\right) \lesssim \epsilon_0\delta_{ext}.$$

Next, in view of Corollary 7.2 of [41] (restated here as Corollary 8.6), there exists a canonical basis of $\ell=1$ modes on $\widetilde{S}_*$ in the sense of Definition 3.10

of [41] (recalled here in Definition 5.3), which we denote by $J_0^{(p,\widetilde{S}_*)}$, such that

$$\max_{p=0,+,-}\left\|J_0^{(p,\widetilde{S}_*)}-J^{(p)}\right\|_{\mathfrak{h}_{k_*-4}(\widetilde{S}_*)}\lesssim\epsilon_0\delta_{ext}.$$

Also, recall that $\widetilde{J}^{(p)}=J^{(p,\widetilde{S}_*)}$ on $\widetilde{S}_*$, where $J^{(p,\widetilde{S}_*)}$ is in general another canonical basis of $\ell=1$ modes on $\widetilde{S}_*$. In view of Definition 5.3, note that the canonical basis of $\ell=1$ modes on $\widetilde{S}_*$ are unique modulo isometries of $\mathbb{S}^2$, i.e. there exists $O\in O(3)$ such that

$$J^{(p,\widetilde{S}_*)}=\sum_{q=0,+,-}O_{pq}J_0^{(q,\widetilde{S}_*)},\qquad p=0,+,-. \tag{8.226}$$

Remark 8.43. *In general, we have $O\neq I$ in (8.226). In fact, the role of O corresponds in Step 6 to the application of Corollary 8.8 which ensures that the following holds on $\widetilde{S}_*$ w.r.t. the canonical basis of $\ell=1$ modes $J^{(p,\widetilde{S}_*)}$, see (8.204),*

$$(\widetilde{curl}\,\widetilde{\beta})_{\ell=1,\pm}=0,$$

which corresponds to fixing the axis of $\widetilde{S}_*$. Note that this condition (and hence the axis of $\widetilde{S}_*$) is preserved by multiplying the basis $J^{(p,\widetilde{S}_*)}$ by $O=-I$ or by any O fixing $J^{(0,\widetilde{S}_*)}$, so that we may assume in (8.226) that O satisfies

$$O_{00}\geq 0,\qquad O_{++}\geq 0,\qquad O_{+-}=0,\qquad O_{--}\geq 0. \tag{8.227}$$

Since $\widetilde{J}^{(p)}=J^{(p,\widetilde{S}_*)}$ on $\widetilde{S}_*$, we infer

$$\max_{p=0,+,-}\left\|\widetilde{J}^{(p)}-\sum_{q=0,+,-}O_{pq}J^{(q)}\right\|_{\mathfrak{h}_{k_*-4}(\widetilde{S}_*)}\lesssim\epsilon_0\delta_{ext}. \tag{8.228}$$

where O satisfies (8.227). Plugging (8.228) in the above, we deduce

$$\widetilde{r}^5\Bigg(\left|\frac{1}{|\widetilde{S}_*|}\sum_{q=0,+,-}O_{0q}\int_{\widetilde{S}_*}\widetilde{\mathrm{curl}}\,\widetilde{\beta}\widetilde{J}^{(q)}-\frac{2a\widetilde{m}}{\widetilde{r}^5}\right| +\frac{1}{|\widetilde{S}_*|}\left|\sum_{q=0,+,-}O_{\pm q}\int_{\widetilde{S}_*}\widetilde{\mathrm{curl}}\,\widetilde{\beta}\widetilde{J}^{(p)}\right|\Bigg)\lesssim\epsilon_0\delta_{ext}.$$

On the other hand, we have in view of (8.204) and the fact that $\widetilde{J}^{(p)}=J^{(p,\widetilde{S}_*)}$

on $\widetilde{S}_*$

$$\frac{1}{|\widetilde{S}_*|}\int_{\widetilde{S}_*}\widetilde{\mathrm{curl}}\,\widetilde{\beta}\widetilde{J}^{(0)}\frac{2\widetilde{a}\widetilde{m}}{\widetilde{r}^5}, \qquad \frac{1}{|\widetilde{S}_*|}\int_{\widetilde{S}_*}\widetilde{\mathrm{curl}}\,\widetilde{\beta}\widetilde{J}^{(\pm)} = 0.$$

Plugging in the previous estimate, we obtain

$$\widetilde{m}\Big(|O_{00}\widetilde{a}-a|+|\widetilde{a}|\,|O_{\pm0}|\Big) \lesssim \epsilon_0\delta_{ext},$$

and since $|\widetilde{m}-m|\lesssim\delta_{ext}\epsilon_0$ and $|m-m_0|\lesssim\epsilon_0$, we may divide by $\widetilde{m}$ and hence

$$|O_{00}\widetilde{a}-a|+|\widetilde{a}|\,|O_{+0}|+|\widetilde{a}|\,|O_{-0}| \lesssim \epsilon_0\delta_{ext}. \tag{8.229}$$

Since $O\in O(3)$, we have $\sum_p O_{p0}^2=1$ which together with (8.229) implies

$$(\widetilde{a})^2 = (\widetilde{a})^2\left(\sum_p O_{p0}^2\right) = (a+O(\epsilon_0\delta_{ext}))^2+O(\epsilon_0^2\delta_{ext}^2)\lesssim a^2+\epsilon_0^2\delta_{ext}^2.$$

In particular, we infer the following estimate for $\widetilde{a}$

$$|\widetilde{a}|\lesssim\sqrt{\epsilon_0\delta_{ext}} \quad \text{if} \quad |a|\le\sqrt{\epsilon_0\delta_{ext}}. \tag{8.230}$$

This allows us to control, in the case $|a|\le\sqrt{\epsilon_0\delta_{ext}}$, the change of frame coefficients $(f''',\underline{f}''',\lambda''')$ introduced in Step 14 and corresponding to the change from the outgoing PG frame (e_4',e_3',e_1',e_2') of ${}^{(ext)}\mathcal{M}$ extended to the spacetime $\mathcal{M}^{(extend)}$ and the PG frame $(\widetilde{e}_4',\widetilde{e}_3',\widetilde{e}_1',\widetilde{e}_2')$. Indeed, (8.230) and (8.225) yield, for $\le k_*-9$,

$$\sup_{\widetilde{\Sigma}_*} r\left|\widetilde{\mathfrak{d}}_*^k(f''',\underline{f}''',\lambda'''-1)\right| \lesssim \sqrt{\epsilon_0\delta_{ext}} \quad \text{if} \quad |a|\le\sqrt{\epsilon_0\delta_{ext}}. \tag{8.231}$$

Step 17. Next, we focus on controlling the RHS of (8.225) in the case $|a|>\sqrt{\epsilon_0\delta_{ext}}$. In this case, we have from (8.229) and the fact that $\sum_p O_{p0}^2=1$

$$(\widetilde{a})^2 = (\widetilde{a})^2\left(\sum_p O_{p0}^2\right) = (a+O(\epsilon_0\delta_{ext}))^2+O(\epsilon_0^2\delta_{ext}^2) = a^2+O(\epsilon_0^{\frac{3}{2}}\delta_{ext}^{\frac{3}{2}})$$

and hence $\widetilde{a}\ge\frac{1}{2}\sqrt{\epsilon_0\delta_{ext}}$. Thus, dividing (8.229) by $\widetilde{a}$, we obtain

$$\left|O_{00}-\frac{a}{\widetilde{a}}\right|+|O_{+0}|+|O_{-0}| \lesssim \sqrt{\epsilon_0\delta_{ext}}.$$

Together with the fact that $\sum_p O_{p0}^2 = 1$, and recalling also that $O_{00} \geq 0$ in view of (8.227), we infer

$$|\widetilde{a} - a| \lesssim \sqrt{\epsilon_0 \delta_{ext}} \quad \text{if} \quad |a| > \sqrt{\epsilon_0 \delta_{ext}} \tag{8.232}$$

and

$$|O_{00} - 1| + |O_{+0}| + |O_{-0}| \lesssim \sqrt{\epsilon_0 \delta_{ext}}.$$

Also, since $O \in O(3)$, we also have

$$0 = \sum_p O_{p+} O_{p0} = O_{0+} + O(\sqrt{\epsilon_0 \delta_{ext}}), \quad 0 = \sum_p O_{p-} O_{p0} = O_{0-} + O(\sqrt{\epsilon_0 \delta_{ext}}),$$

and hence

$$|O_{0+}| + |O_{0-}| \lesssim \sqrt{\epsilon_0 \delta_{ext}}.$$

Together with the fact that $O_{+-} = 0$ and $O_{--} \geq 0$ in view of (8.227), and since $\sum_p O_{p-}^2 = 1$, we infer

$$|O_{--} - 1| \lesssim \sqrt{\epsilon_0 \delta_{ext}}.$$

Finally $O_{++} \geq 0$ in view of (8.227), since we have obtained above that $|O_{0+}| \lesssim \sqrt{\epsilon_0 \delta_{ext}}$, and since $\sum_p O_{p-}^2 = 1$ and $\sum_p O_{p+} O_{p-} = 0$, we infer

$$|O_{++} - 1| + |O_{-+}| \lesssim \sqrt{\epsilon_0 \delta_{ext}}.$$

We have thus obtained

$$|O - I| \lesssim \sqrt{\epsilon_0 \delta_{ext}},$$

which together with (8.228) implies

$$\max_{p=0,+,-} r^{-1} \left\| \widetilde{J}^{(p)} - J^{(p)} \right\|_{\mathfrak{h}_{k_*-4}(\widetilde{S}_*)} \lesssim \sqrt{\epsilon_0 \delta_{ext}} \quad \text{if} \quad |a| > \sqrt{\epsilon_0 \delta_{ext}}.$$

Next, we control $\widetilde{J}^{(p)} - J^{(p)}$ for $p = 0, +, -$ on $\widetilde{\Sigma}_*$. Recall that we have $\widetilde{\nu}(\widetilde{J}^{(p)}) = 0$ along $\widetilde{\Sigma}_*$. We infer

$$\widetilde{\nu}\left(\widetilde{J}^{(p)} - J^{(p)}\right) = -\widetilde{\nu}(J^{(p)}) = -\widetilde{e}_4(J^{(p)}) - \widetilde{b}_* \widetilde{e}_3(J^{(p)}).$$

Using the change of frame formulas, and the fact that $e_4(J^{(p)}) = 0$, we obtain

$$\begin{aligned}
&\widetilde{\nu}\left(\widetilde{J}^{(p)} - J^{(p)}\right)\\
= \;& -\lambda\left(f\cdot\nabla + \frac{1}{4}|f|^2 e_3\right)J^{(p)}\\
&-\widetilde{b}_*\lambda^{-1}\left(\left(1+\frac{1}{2}f\cdot\underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right)e_3 + \left(\underline{f}+\frac{1}{4}|\underline{f}|^2\right)\cdot\nabla\right)J^{(p)}.
\end{aligned}$$

Together with the control (8.217) for $(f, \underline{f}, \lambda)$, and (8.201) for $e_3(J^{(p)})$, we infer

$$\sup_{\widetilde{\Sigma}_*}\left|\widetilde{\mathfrak{d}}_*^{k_*-5}\widetilde{\nu}\left(\widetilde{J}^{(p)} - J^{(p)}\right)\right| \lesssim \frac{\epsilon_0\delta_{ext}}{ru_*} + \frac{\epsilon_0\delta_{ext}}{r^2}.$$

Integrating along $\widetilde{\Sigma}_*$ from $\widetilde{S}_*$, using the above estimate on $\widetilde{S}_*$ and Sobolev, and using the dominant condition on $\widetilde{r}$ on $\widetilde{\Sigma}_*$, we infer

$$\max_{p=0,+,-}\sup_{\widetilde{\Sigma}_*}\left|\widetilde{\mathfrak{d}}_*^{k_*-6}\left(\widetilde{J}^{(p)} - J^{(p)}\right)\right| \lesssim \sqrt{\epsilon_0\delta_{ext}} \quad \text{if} \quad |a| > \sqrt{\epsilon_0\delta_{ext}}. \tag{8.233}$$

Next, we control $\widetilde{f}_0 - f_0$ on $\widetilde{\Sigma}_*$. First, recall from Lemma 5.61 that the following identity holds on S_*

$$\nabla J^{(0)} = -\frac{1}{re^{\phi}}\,{}^*f_0.$$

Next, recalling that we have extended ϕ, $J^{(0)}$ and f_0 first to $\Sigma_*^{(extend)}$ by $\nu(\phi) = 0$, $\nabla_\nu f_0 = 0$ and $\nu(J^{(0)}) = 0$, and then to $\widetilde{\mathcal{R}}$ by $e_4(\phi) = 0$, $\nabla_4 f_0 = 0$ and $e_4(J^{(0)}) = 0$, we have on $\Sigma_*^{(extend)}$

$$\nabla_\nu\left(r\left(\nabla J^{(0)} + \frac{1}{re^{\phi}}\,{}^*f_0\right)\right) = \Gamma_b\,\mathfrak{d}\!\!/^{\leq 1}J^{(0)}$$

and on $\widetilde{\mathcal{R}}$

$$\nabla_4\left(r\left(\nabla J^{(0)} + \frac{1}{re^{\phi}}\,{}^*f_0\right)\right) = \Gamma_g\,\mathfrak{d}\!\!/^{\leq 1}J^{(0)}.$$

Integrating first from S_*, where the above identity holds, to $\Sigma_*^{(extend)}(u \geq u_*)$, and then from $\Sigma_*^{(extend)}(u \geq u_*)$ to $\widetilde{\mathcal{R}}(u \geq u_*)$, we infer, using also the control of Step 1 on $\mathcal{M}^{(extend)}$, for $k \le k_* - 3$,

$$\sup_{\widetilde{\mathcal{R}}(u \ge u_*)} \left| \mathfrak{d}^k \left(\nabla J^{(0)} + \frac{1}{re^{\phi}} {}^* f_0 \right) \right| \lesssim \frac{\epsilon_0}{r^3 u_*^{\frac{1}{2}+\delta_{dec}}} \Delta_{ext} + \frac{\epsilon_0}{r^2 u_*^{1+\delta_{dec}}} \delta_{ext} \lesssim \frac{\epsilon_0}{r^2 u_*^{1+\delta_{dec}}} \delta_{ext}.$$

In particular, since the GCM sphere $\widetilde{S}_*$ constructed in Step 6 is included in $\widetilde{\mathcal{R}}(u \ge u_*)$, we infer, for $k \le k_* - 3$,

$$\sup_{\widetilde{S}_*} \left| \mathfrak{d}^k \left(\nabla J^{(0)} + \frac{1}{re^{\phi}} {}^* f_0 \right) \right| \lesssim \frac{\epsilon_0}{r^2 u_*^{1+\delta_{dec}}} \delta_{ext}.$$

Together with the control (8.224) for $\widetilde{r} - r$, the change of frame formulas, the control (8.217) for $(f, \underline{f}, \lambda)$ and the control of Step 1 on $\mathcal{M}^{(extend)}$, we infer, for $k \le k_* - 6$,

$$\sup_{\widetilde{S}_*} \left| \widetilde{\mathfrak{d}}^k \left(\widetilde{\nabla} J^{(0)} + \frac{1}{\widetilde{r} e^{\phi}} {}^* f_0 \right) \right| \lesssim \frac{\epsilon_0}{r^2 u_*^{1+\delta_{dec}}} \delta_{ext}.$$

Also, recall that $\overset{\circ}{S} = S(\overset{\circ}{u}, \overset{\circ}{s})$ is the sphere of the foliation of $\mathcal{R} \cap \{u \ge u_*\}$ which shares the same south pole a $\widetilde{S}_*$. As a byproduct of the construction of the GCM sphere[191] $\widetilde{S}_*$ in Step 6 we have, see Corollary 7.2 of [41] (stated here as Corollary 8.6), the deformation map $\Psi : \overset{\circ}{S} \to S$ is given by

$$\Psi(\overset{\circ}{u}, \overset{\circ}{s}, x^1, x^2) = (\overset{\circ}{u} + U(x^1, x^2), \overset{\circ}{s} + S(x^1, x^2)),$$

where the scalar function U and S on $\overset{\circ}{S}$ satisfy

$$r^{-1} \|(U, S)\|_{\mathfrak{h}_{k_*-2}(\overset{\circ}{S})} \lesssim \epsilon_0 \delta_{ext}. \tag{8.234}$$

The above control of the deformation (U, S), together with the estimate (8.197), (8.198) and (8.199) that hold on any sphere S of $\mathcal{R} \cap \{u \ge u_*\}$, and hence in particular on $\overset{\circ}{S}$, imply corresponding estimates with the metric g on $\overset{\circ}{S}$, and the scalar functions ϕ and $J^{(p)}$ on $\overset{\circ}{S}$ replaced with the corresponding analogs $(\Psi^{-1})^\# g$, $\phi \circ \Psi^{-1}$ and $J^{(p)} \circ \Psi^{-1}$ on $\widetilde{S}_*$. We may thus apply

[191] Based on Theorem 7.3 and Corollary 7.7 of [41] (restated here as Theorem 8.7 and Corollary 8.8).

Proposition 4.15 in [41] (restated here as Proposition 5.5) which yields on $\widetilde{S}_*$ the estimate

$$r^{-1}\|\widetilde{\phi}-\phi\circ\Psi^{-1}\|_{\mathfrak{h}_{k_*-3}(\widetilde{S}_*)}\lesssim\frac{\epsilon_0}{r^2}\Delta_{ext}\lesssim\frac{\epsilon_0}{r}\delta_{ext}.$$

Together with the above form of Ψ, the above control of (U,S), and the control of ϕ provided by (8.198), we infer

$$r^{-1}\|\widetilde{\phi}-\phi\|_{\mathfrak{h}_{k_*-3}(\widetilde{S}_*)} \quad\lesssim\quad \frac{\epsilon_0}{r}\delta_{ext}.$$

Together with Sobolev, we deduce from the above, for $k\leq k_*-6$,

$$\sup_{\widetilde{S}_*}\left|\widetilde{\mathfrak{d}}^k\left(\widetilde{\nabla}J^{(0)}+\frac{1}{\widetilde{r}e^{\widetilde{\phi}}}\,{}^*f_0\right)\right| \quad\lesssim\quad \frac{\epsilon_0}{r^2}\delta_{ext}.$$

On the other hand, in view of Lemma 5.61 applied to the GCM sphere $\widetilde{S}_*$, the following identity holds on $\widetilde{S}_*$

$$\widetilde{\nabla}\widetilde{J}^{(0)}=-\frac{1}{\widetilde{r}e^{\widetilde{\phi}}}\,{}^*\widetilde{f}_0.$$

We infer, for $k\leq k_*-6$,

$$\sup_{\widetilde{S}_*}\left|\widetilde{\mathfrak{d}}^k\left(\widetilde{\nabla}(\widetilde{J}^{(0)}-J^{(0)})+\frac{1}{\widetilde{r}e^{\widetilde{\phi}}}\,{}^*(\widetilde{f}_0-f_0)\right)\right| \quad\lesssim\quad \frac{\epsilon_0}{r^2}\delta_{ext}.$$

Together with the above control of $\widetilde{J}^{(p)}-J^{(p)}$ on $\widetilde{\Sigma}_*$, we infer, for $k\leq k_*-7$,

$$\sup_{\widetilde{S}_*}\left|\widetilde{\mathfrak{d}}^k\left(\widetilde{f}_0-f_0\right)\right| \quad\lesssim\quad \sqrt{\epsilon_0\delta_{ext}}.$$

Next, we control $\nabla_3 f_0$ on $\widetilde{\mathcal{R}}$. Recall that we have $\nabla_4 f_0=0$ on $\widetilde{\mathcal{R}}$ and $\nabla_\nu f_0=0$ on $\Sigma_*^{(extend)}$. Since $\nu = e_3+b_*e_4$, we infer $\nabla_3 f_0 = 0$ on $\Sigma_*^{(extend)}$. Since $\nabla_4 f_0=0$ on $\widetilde{\mathcal{R}}$, integrating from $\Sigma_*^{(extend)}$, we easily infer on $\widetilde{\mathcal{R}}$, for $k\leq k_*-3$,

$$\sup_{\widetilde{\mathcal{R}}}\left|\mathfrak{d}^k\nabla_3 f_0\right| \quad\lesssim\quad \frac{\epsilon_0}{r^2}\Delta_{ext}\lesssim\frac{\epsilon_0}{r}\delta_{ext}.$$

Also, using the change of frame formulas and the fact that $\nabla_4 f_0=0$, we have on $\widetilde{\Sigma}_*$

$$\widetilde{\nabla}_{\widetilde{\nu}}f_0 \quad=\quad \widetilde{\nabla}_{\widetilde{e}_3}f_0+\widetilde{b}_*\widetilde{\nabla}_{\widetilde{e}_4}f_0$$

$$= \lambda^{-1}\left(\left(1+\frac{1}{2}f\cdot\underline{f}+\frac{1}{16}|f|^2|\underline{f}|^2\right)\nabla_3+\left(\underline{f}+\frac{1}{4}|\underline{f}|^2 f\right)\cdot\nabla\right)f_0$$
$$+\widetilde{b}_*\lambda\left(f\cdot\nabla+\frac{1}{4}|f|^2\nabla_3\right)f_0.$$

Together with the above estimate for $\nabla_3 f_0$ and the control (8.217) for $(f,\underline{f},\lambda)$, we infer, for $k\le k_*-5$,

$$\sup_{\widetilde{\Sigma}_*}\left|\mathfrak{d}^k\widetilde{\nabla}_{\widetilde{\nu}}f_0\right| \lesssim \frac{\epsilon_0}{r}\delta_{ext}.$$

Since $\widetilde{\nabla}_{\widetilde{\nu}}\widetilde{f}_0=0$, we deduce, for $k\le k_*-5$,

$$\sup_{\widetilde{\Sigma}_*}\left|\mathfrak{d}^k\widetilde{\nabla}_{\widetilde{\nu}}\left(\widetilde{f}_0-f_0\right)\right| \lesssim \frac{\epsilon_0}{r}\delta_{ext}.$$

Integrating from $\widetilde{S}_*$ and using the above control on $\widetilde{S}_*$ for $\widetilde{f}_0-f_0$, as well as the dominant condition for r on $\widetilde{\mathcal{R}}$, we deduce, for $k\le k_*-7$,

$$\sup_{\widetilde{\Sigma}_*}\left|\mathfrak{d}^k\left(\widetilde{f}_0-f_0\right)\right| \lesssim \sqrt{\epsilon_0\delta_{ext}} \quad \text{if} \quad |a|>\sqrt{\epsilon_0\delta_{ext}}. \tag{8.235}$$

We are now in position to control, in the case $|a|>\sqrt{\epsilon_0\delta_{ext}}$, the change of frame coefficients $(f''',\underline{f}''',\lambda''')$ introduced in Step 14 and corresponding to the change from the outgoing PG frame (e'_4,e'_3,e'_1,e'_2) of ${}^{(ext)}\mathcal{M}$ extended to the spacetime $\mathcal{M}^{(extend)}$ and the PG frame $(\widetilde{e}'_4,\widetilde{e}'_3,\widetilde{e}'_1,\widetilde{e}'_2)$. Recall (8.225) that holds for $\le k_*-9$

$$\sup_{\widetilde{\Sigma}_*} r\left|\widetilde{\mathfrak{d}}^k_*(f''',\underline{f}''',\lambda'''-1)\right| \lesssim \epsilon_0\delta_{ext}+\sup_{\widetilde{\Sigma}_*}|\widetilde{\mathfrak{d}}^k_*(\widetilde{a}\widetilde{f}_0-af_0)|.$$

In view of the above estimates for $\widetilde{a}-a$ and $\widetilde{f}_0-f_0$, we infer, $\le k_*-9$,

$$\sup_{\widetilde{\Sigma}_*} r\left|\widetilde{\mathfrak{d}}^k_*(f''',\underline{f}''',\lambda'''-1)\right| \lesssim \sqrt{\epsilon_0\delta_{ext}} \quad \text{if} \quad |a|>\sqrt{\epsilon_0\delta_{ext}}. \tag{8.236}$$

We conclude this step with the control of $\widetilde{\mathfrak{J}}-\mathfrak{J}$ on $\widetilde{\Sigma}_*$ in the case $|a|>\sqrt{\epsilon_0\delta_{ext}}$. Recall from Step 14 that $(f',\underline{f}',\lambda')$ denote the change of frame coefficients introduced in Step 14 and corresponding to the change from the outgoing geodesic frame (e_4,e_3,e_1,e_2) of $\mathcal{M}^{(extend)}$ to the outgoing PG frame (e'_4,e'_3,e'_1,e'_2) of ${}^{(ext)}\mathcal{M}$ extended to the spacetime $\mathcal{M}^{(extend)}$. Recall from (5.130) and (3.34) that $\mathfrak{J}$ satisfies in $\mathcal{M}^{(extend)}$, and hence in $\widetilde{\mathcal{R}}$, the following identities

$$\mathfrak{J} = \frac{1}{|q|}\left(f_0 + i\,{}^*f_0\right) \quad \text{on} \quad \Sigma_*^{(extend)}, \qquad \nabla_{e_4'}\mathfrak{J} = -\frac{1}{q}\mathfrak{J} \quad \text{on} \quad \widetilde{\mathcal{R}}.$$

Using the change of frame formulas, we have on $\widetilde{\mathcal{R}}$

$$\begin{aligned}
\nabla_{e_4'}\mathfrak{J} &= \nabla_{\lambda'\left(e_4+f'\cdot\nabla+\frac{1}{4}|f'|^2e_3\right)}\mathfrak{J}\\
&= \nabla_4\mathfrak{J} + (\lambda'-1)\nabla_4\mathfrak{J} + \lambda' f'\cdot\nabla\mathfrak{J} + \frac{\lambda'}{4}|f'|^2\nabla_3\mathfrak{J}
\end{aligned}$$

and hence

$$\nabla_4\mathfrak{J} - \frac{1}{q}\mathfrak{J} = -(\lambda'-1)\nabla_4\mathfrak{J} - \lambda' f'\cdot\nabla\mathfrak{J} - \frac{\lambda'}{4}|f'|^2\nabla_3\mathfrak{J}.$$

Recall also that we have derived the following control for (f',λ') in Step 14, for $k \leq k_*-3$,

$$\sup_{\widetilde{\mathcal{R}}} r\left|\mathfrak{d}^k\left(f' - \frac{a}{r}f_0, \lambda'-1\right)\right| \lesssim \epsilon_0\delta_{ext}.$$

Together with the control of $\mathfrak{J}$ in $\mathcal{M}^{(extend)}$, we infer, for $k \leq k_*-3$,

$$\sup_{\widetilde{\mathcal{R}}} r^3\left|\mathfrak{d}^k\left(\nabla_4\mathfrak{J} - \frac{1}{q}\mathfrak{J}\right)\right| \lesssim \epsilon_0\delta_{ext}.$$

Also, recall that we have $\nabla_4 f_0 = 0$ in $\widetilde{\mathcal{R}}$ and hence

$$\begin{aligned}
\nabla_4\left(|q|\mathfrak{J} - (f_0 + i\,{}^*f_0)\right) &= |q|\left(\nabla_4 + \frac{\nabla_4(|q|)}{|q|}\right)\mathfrak{J} = |q|\left(\nabla_4 + \frac{r}{|q|^2}\right)\mathfrak{J}\\
&= |q|\left(\nabla_4 + \frac{1}{q}\right)\mathfrak{J} - \frac{ia\cos\theta}{|q|}\mathfrak{J}.
\end{aligned}$$

Together with the above estimate for $\nabla_4\mathfrak{J} - \frac{1}{q}\mathfrak{J}$, we infer, for $k \leq k_*-3$,

$$\sup_{\widetilde{\mathcal{R}}} r^2\left|\mathfrak{d}^k\nabla_4\left(|q|\mathfrak{J} - (f_0 + i\,{}^*f_0)\right)\right| \lesssim 1.$$

Integrating from $\Sigma_*^{(extend)}$ where $|q|\mathfrak{J} - (f_0 + i\,{}^*f_0) = 0$, we infer, for $k \leq k_*-3$,

$$\sup_{\widetilde{\mathcal{R}}}\left|\mathfrak{d}^k\left(|q|\mathfrak{J}-(f_0+i\,{}^*f_0)\right)\right| \lesssim \frac{1}{r^2}\Delta_{ext}\lesssim\frac{1}{r}\delta_{ext}.$$

Together with the dominant condition for r on $\widetilde{\mathcal{R}}$, we obtain, for $k\leq k_*-3$,

$$\sup_{\widetilde{\mathcal{R}}} r\left|\mathfrak{d}^k\left(\mathfrak{J}-\frac{1}{|q|}(f_0+i\,{}^*f_0)\right)\right| \lesssim \epsilon_0\delta_{ext}.$$

On the other hand, we have on $\widetilde{\Sigma}_*$

$$\widetilde{\mathfrak{J}} = \frac{1}{|\widetilde{q}|}\left(\widetilde{f}_0+i\,{}^*\widetilde{f}_0\right).$$

Since $\widetilde{\Sigma}_*\subset\widetilde{\mathcal{R}}$, this yields, for $k\leq k_*-3$,

$$\begin{aligned}
&\sup_{\widetilde{\Sigma}_*} r\left|\widetilde{\mathfrak{d}}^k_*\left(\widetilde{\mathfrak{J}}-\mathfrak{J}\right)\right|\\
\lesssim\;& \epsilon_0\delta_{ext}+\sup_{\widetilde{\Sigma}_*} r\left|\widetilde{\mathfrak{d}}^k_*\left(\frac{1}{|\widetilde{q}|}\left(\widetilde{f}_0+i\,{}^*\widetilde{f}_0\right)-\frac{1}{|q|}(f_0+i\,{}^*f_0)\right)\right|\\
\lesssim\;& \epsilon_0\delta_{ext}+\sup_{\widetilde{\Sigma}_*}\left(\left|\widetilde{\mathfrak{d}}^k_*\left(\widetilde{f}_0-f_0\right)\right|+r^{-1}\left|\widetilde{\mathfrak{d}}^k_*(\widetilde{r}-r)\right|+r^{-1}\left|\widetilde{\mathfrak{d}}^k_*\left(\widetilde{J}^{(0)}-J^{(0)}\right)\right|\right).
\end{aligned}$$

In view of the estimates for $\widetilde{r}-r$ in Step 15, and the above estimates for $\widetilde{f}_0-f_0$ and $\widetilde{J}^{(0)}-J^{(0)}$ in the case $|a|>\sqrt{\epsilon_0\delta_{ext}}$, we infer, for $\leq k_*-7$,

$$\sup_{\widetilde{\Sigma}_*} r\left|\widetilde{\mathfrak{d}}^k_*\left(\widetilde{\mathfrak{J}}-\mathfrak{J}\right)\right| \lesssim \sqrt{\epsilon_0\delta_{ext}} \quad\text{if}\quad |a|>\sqrt{\epsilon_0\delta_{ext}}. \tag{8.237}$$

Finally, we have obtained in this step, for $\leq k_*-9$,

$$\begin{aligned}
&\sup_{\widetilde{\Sigma}_*}\Bigg(r\left|\widetilde{\mathfrak{d}}^k_*(f''',\underline{f}''',\lambda'''-1)\right|+|\widetilde{a}-a|\\
&+\left|\widetilde{\mathfrak{d}}^k_*\left(\widetilde{J}^{(0)}-J^{(0)}\right)\right|+r\left|\widetilde{\mathfrak{d}}^k_*\left(\widetilde{\mathfrak{J}}-\mathfrak{J}\right)\right|\Bigg) \lesssim \sqrt{\epsilon_0\delta_{ext}} \quad\text{if}\quad |a|>\sqrt{\epsilon_0\delta_{ext}}.
\end{aligned}$$

Together with the control (8.210) for $\widetilde{r}-r$ and $\widetilde{m}-m$, (8.230) and (8.231) for the case $|a|\leq\sqrt{\epsilon_0\delta_{ext}}$, and possibly reducing the size of $\delta_{ext}>0$ (which can be chosen arbitrarily small), we deduce, for $\leq k_*-9$,

$$\sup_{\widetilde{\Sigma}_*}\widetilde{u}^{1+\delta_{dec}}\Bigg(r\left|\widetilde{\mathfrak{d}}^k_*(f''',\underline{f}''',\lambda'''-1)\right|+\left|\widetilde{\mathfrak{d}}^k_*(\widetilde{r}-r)\right|+|\widetilde{m}-m|$$

$$+|\widetilde{a}-a|+\left|\widetilde{\mathfrak{d}}_*^k\left(\widetilde{a}\widetilde{J}^{(0)}-aJ^{(0)}\right)\right|+r\left|\widetilde{\mathfrak{d}}_*^k\left(\widetilde{a}\widetilde{\mathfrak{J}}-a\mathfrak{J}\right)\right|\Bigg) \lesssim \epsilon_0. \tag{8.238}$$

Step 18. We now control the outgoing PG structure initialized on $\widetilde{\Sigma}_*$. We denote by ${}^{(ext)}\widetilde{\mathcal{M}}$ the region covered by this outgoing PG structure. For convenience, we change our notation. From now on:

- (e_4, e_3, e_1, e_2) denotes the outgoing PG frame of ${}^{(ext)}\mathcal{M}$ extended to the spacetime $\mathcal{M}^{(extend)}$,
- $(\widetilde{e}_4, \widetilde{e}_3, \widetilde{e}_1, \widetilde{e}_2)$ denotes the outgoing PG frame initialized on $\widetilde{\Sigma}_*$,
- $(f, \underline{f}, \lambda)$ denote the transition coefficients from the outgoing PG frame (e_4, e_3, e_1, e_2) to the outgoing PG frame $(\widetilde{e}_4, \widetilde{e}_3, \widetilde{e}_1, \widetilde{e}_2)$.

In view of (8.238), using the above new notations for $(f, \underline{f}, \lambda)$, and noticing that the structure equations in the $\widetilde{e}_4$ direction for the outgoing PG structure initialized on $\widetilde{\Sigma}_*$ allow to recover the $\widetilde{e}_4$ derivatives (which are transversal to $\widetilde{\Sigma}_*$), we have, for $\leq k_* - 9$,

$$\sup_{\widetilde{\Sigma}_*}\widetilde{u}^{1+\delta_{dec}}\Bigg(r\left|\mathfrak{d}^k(f,\underline{f},\lambda-1)\right|+\left|\mathfrak{d}^k\left(\widetilde{r}-r\right)\right|+|\widetilde{m}-m|$$
$$+|\widetilde{a}-a|+\left|\mathfrak{d}^k\left(\widetilde{a}\widetilde{J}^{(0)}-aJ^{(0)}\right)\right|+r\left|\mathfrak{d}^k\left(\widetilde{a}\widetilde{\mathfrak{J}}-a\mathfrak{J}\right)\right|\Bigg) \lesssim \epsilon_0. \tag{8.239}$$

Also (e_4, e_3, e_1, e_2), as discussed in Step 1 to Step 3, satisfies

$$\mathfrak{N}^{(Dec)}_{k_*-3}(\mathcal{M}^{(extend)}) \lesssim \epsilon_0. \tag{8.240}$$

We introduce the notations

$$F := f + i\,{}^*f, \qquad \underline{F} := \underline{f} + i\,{}^*\underline{f}.$$

Since (e_4, e_3, e_1, e_2) and $(\widetilde{e}_4, \widetilde{e}_3, \widetilde{e}_1, \widetilde{e}_2)$ are outgoing PG frames, we have

$$\Xi = 0, \qquad \omega = 0, \qquad \underline{H} + Z = 0, \qquad \widetilde{\Xi} = 0, \qquad \widetilde{\omega} = 0, \qquad \widetilde{\underline{H}} + \widetilde{Z} = 0.$$

In view of Corollary 2.14, we have the following transport equations

$$\begin{aligned}
\nabla_{\lambda^{-1}\widetilde{e}_4}(qF) &= E_4(f,\Gamma),\\
\lambda^{-1}\nabla_{\widetilde{e}_4}(\log\lambda) &= 2f\cdot\zeta + E_2(f,\Gamma),\\
\nabla_{\lambda^{-1}\widetilde{e}_4}\left[q\Big(\underline{F}-2q\widetilde{\mathcal{D}}(\log\lambda)+e_3(r)F\Big)\right] &= -3q^2\widetilde{\mathcal{D}}\,(f\cdot\zeta)
\end{aligned}$$

$$+E_5(\widetilde{\nabla}^{\leq 1}f, \underline{f}, \widetilde{\nabla}^{\leq 1}\lambda, \mathbf{D}^{\leq 1}\Gamma),$$

where E_2, E_4 and E_5 are given in Corollary 2.14. Integrating these transport equations from $\widetilde{\Sigma}_*$ in the order they appear, using the control in (8.239) for $(f, \underline{f}, \lambda)$ on $\widetilde{\Sigma}_*$, and together with the control (8.240) for the Ricci coefficients of the foliation of $\mathcal{M}^{(extend)}$, we obtain, for $\leq k_* - 9$,

$$\sup_{{}^{(ext)}\widetilde{\mathcal{M}}} \Big(\widetilde{r}\,\widetilde{u}^{\frac{1}{2}+\delta_{dec}} + \widetilde{u}^{1+\delta_{dec}}\Big)\Big(|\mathfrak{d}^k(f, \log(\lambda))| + |\mathfrak{d}^{k-1}\underline{f}|\Big) \lesssim \epsilon_0. \tag{8.241}$$

Also, we have

$$\widetilde{e}_4(\widetilde{r} - \lambda^{-1}r) \;=\; 1 - \lambda^{-1}\widetilde{e}_4(r) + \lambda^{-1}\widetilde{e}_4(\log(\lambda)).$$

Using the change of frame formula and the above transport equation for $\log(\lambda)$, we infer

$$\begin{aligned}\widetilde{e}_4(\widetilde{r} - \lambda^{-1}r) &= 1 - \left(e_4 + f\cdot\nabla + \frac{1}{4}|f|^2 e_3\right) r + \frac{3}{2} f\cdot\zeta + E_2(f, \Gamma)\\ &= -\frac{1}{4}|f|^2 e_3(r) + \frac{3}{2} f\cdot\zeta + E_2(f, \Gamma).\end{aligned}$$

Integrating from $\widetilde{\Sigma}_*$ where $\widetilde{r} - r$ is under control in view of (8.239), and using the control (8.241) for f and λ as well as the control (8.240) for the foliation of $\mathcal{M}^{(extend)}$, we infer, for $\leq k_* - 9$,

$$\sup_{{}^{(ext)}\widetilde{\mathcal{M}}} \widetilde{u}^{1+\delta_{dec}} \left|\mathfrak{d}^k(\widetilde{r} - r)\right| \;\lesssim\; \epsilon_0. \tag{8.242}$$

Also, we have

$$\begin{aligned}\widetilde{e}_4(\widetilde{a}\widetilde{J}^{(0)} - aJ^{(0)}) &= -\widetilde{e}_4(aJ^{(0)}) = -a\lambda\left(e_4 + f\cdot\nabla + \frac{1}{4}|f|^2 e_3\right) J^{(0)}\\ &= -a\lambda\left(f\cdot\nabla + \frac{1}{4}|f|^2 e_3\right) J^{(0)}\end{aligned}$$

and

$$\begin{aligned}\nabla_{\widetilde{e}_4}(\widetilde{a}\widetilde{q}\widetilde{\mathfrak{J}} - aq\mathfrak{J}) &= -\nabla_{\widetilde{e}_4}(aq\mathfrak{J}) = -a\lambda\left(\nabla_4 + f\cdot\nabla + \frac{1}{4}|f|^2\nabla_3\right)(q\mathfrak{J})\\ &= -a\lambda\left(f\cdot\nabla + \frac{1}{4}|f|^2\nabla_3\right)(q\mathfrak{J}).\end{aligned}$$

Integrating from $\widetilde{\Sigma}_*$ where $\widetilde{a}\widetilde{J}^{(0)} - aJ^{(0)}$ and $\widetilde{a}\widetilde{\mathfrak{J}} - a\mathfrak{J}$ are under control in view of (8.239), and using the control (8.241) for f and λ as well as the control (8.240) for the foliation of $\mathcal{M}^{(extend)}$, we infer, for $\leq k_* - 9$,

$$\sup_{{}^{(ext)}\widetilde{\mathcal{M}}} \widetilde{u}^{1+\delta_{dec}}\left(\left|\mathfrak{d}^k(\widetilde{a}\widetilde{J}^{(0)} - aJ^{(0)})\right| + r\left|\mathfrak{d}^k(\widetilde{a}\widetilde{\mathfrak{J}} - a\mathfrak{J})\right|\right) \lesssim \epsilon_0. \tag{8.243}$$

Then, using the outgoing PG structure of ${}^{(ext)}\widetilde{\mathcal{M}}$, we initialize

- the ingoing PG structure of ${}^{(int)}\widetilde{\mathcal{M}}$ on $\mathcal{T} = \{\widetilde{r} = r_0\}$,
- the ingoing PG structure of ${}^{(top)}\widetilde{\mathcal{M}}$ on $\{\widetilde{u} = \widetilde{u}_*\}$,

as in Section 3.2.5. Using the control of $(f, \underline{f}, \lambda)$, $\widetilde{r} - r$, $\widetilde{a}\widetilde{J}^{(0)} - aJ^{(0)}$ and $\widetilde{a}\widetilde{\mathfrak{J}} - a\mathfrak{J}$ induced on $\{\widetilde{r} = r_0\}$ and $\{\widetilde{u} = \widetilde{u}_*\}$ by (8.241), (8.242) and (8.243), and using the analog in the $\widetilde{e}_3$ direction for ingoing PG structures of the above transport equation in the $\widetilde{e}_4$ direction for outgoing PG structures, we obtain for ${}^{(int)}\widetilde{\mathcal{M}}$ and $\leq k_* - 10$

$$\begin{aligned}\sup_{{}^{(int)}\widetilde{\mathcal{M}}} \underline{\widetilde{u}}^{1+\delta_{dec}}\Big(&|\mathfrak{d}^k(\underline{f}, \log(\lambda))| + |\mathfrak{d}^{k-1}f| + |\mathfrak{d}^k(\widetilde{r} - r)| \\ &+ |\mathfrak{d}^k(\widetilde{a}\widetilde{J}^{(0)} - aJ^{(0)})| + |\mathfrak{d}^k(\widetilde{a}\widetilde{\mathfrak{J}} - a\mathfrak{J})|\Big) \lesssim \epsilon_0,\end{aligned} \tag{8.244}$$

and a similar estimate for ${}^{(top)}\widetilde{\mathcal{M}}$.

Let now

$$\widetilde{\mathcal{M}} := {}^{(ext)}\widetilde{\mathcal{M}} \cup {}^{(int)}\widetilde{\mathcal{M}} \cup {}^{(top)}\widetilde{\mathcal{M}}.$$

Then, in view of (8.241)–(8.244), the control of $\widetilde{a} - a$ and $\widetilde{m} - m$ in (8.239), and (8.240), and using the transformation formulas of Proposition 2.12, and well as the definition of the linearized quantities based on a, m, r, $aJ^{(0)} = a\cos\theta$ and $a\mathfrak{J}$, we deduce

$$\mathfrak{N}^{(Dec)}_{k_*-12}(\widetilde{\mathcal{M}}) \lesssim \epsilon_0.$$

In particular, since $k_* = k_{small} + 20$, we infer

$$\mathfrak{N}^{(Dec)}_{k_{small}}(\widetilde{\mathcal{M}}) \lesssim \epsilon_0$$

which concludes the proof of Theorem M7.

9. TOP ORDER ESTIMATES (THEOREM M8)

The goal of this chapter is to prove Theorem M8, i.e. to improve our bootstrap assumptions on boundedness on Σ_*, and for the PG structures of ${}^{(ext)}\mathcal{M}$, ${}^{(int)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$. Now, while the PG structures we have studied so far are perfectly adequate for deriving decay estimates, they are deficient in terms of loss of derivatives and thus inadequate for deriving boundedness estimates for the top derivatives of the Ricci coefficients. Hence, we cannot rely on PG structures in the proof of Theorem M8, and will instead rely on the PT structures introduced in Section 2.8. Once boundedness estimates for top order derivatives are obtained for the PT structures, see Theorem 9.44, they will induce the improvement of our bootstrap assumptions on boundedness on Σ_*, and for the PG structures of ${}^{(ext)}\mathcal{M}$, ${}^{(int)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$, see Section 9.4.3.

Remark 9.1. *Recall that we in fact only prove part of Theorem M8 in this paper. Indeed, the estimates for the curvature components in a global frame*[192]*, see Theorem 9.65, are done in Part III of [28].*

9.1. Principal temporal structures in $\mathcal{M}$

Before introducing the temporal structure of $\mathcal{M}$, we recall the main definitions of outgoing and ingoing temporal structures, see Section 2.8.

9.1.1. Outgoing PT structures

We recall Definitions 2.74 and 2.75 and Lemma 2.76 of Section 2.8.1.

Definition 9.2. *An outgoing PT structure $\{(e_3, e_4, \mathcal{H}), r, \theta, \mathfrak{J}\}$ on $\mathcal{M}$ consists of a null pair (e_3, e_4), the induced horizontal structure $\mathcal{H}$, functions (r, θ), and a horizontal 1-form $\mathfrak{J}$ such that the following hold true:*

1. *e_4 is geodesic.*
2. *We have*

$$e_4(r) = 1, \qquad e_4(\theta) = 0, \qquad \nabla_4(q\mathfrak{J}) = 0, \qquad q = r + ai\cos\theta.$$

3. *We have*

$$\underline{H} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J}.$$

[192]The global frame used for the curvature estimates is constructed from the PT frames in section 9.6.1.

An extended outgoing PT structure possesses, in addition, a scalar function u verifying $e_4(u) = 0$.

Definition 9.3. *An outgoing PT initial data set consists of a hypersurface Σ transversal to e_4 together with a null pair (e_3, e_4), the induced horizontal structure $\mathcal{H}$, scalar functions (r, θ), and a horizontal 1-form $\mathfrak{J}$, all defined on Σ.*

Lemma 9.4. *Any outgoing PT initial data set, as in Definition 2.75, can be locally extended to an outgoing PT structure.*

9.1.2. Ingoing PT structures We recall Definitions 2.83 and 2.84 and Lemma 2.85 of Section 2.8.5.

Definition 9.5. *An ingoing PT structure $\{(e_3, e_4, \mathcal{H}), r, \theta, \mathfrak{J}\}$ on $\mathcal{M}$ consists of a null pair (e_3, e_4), the induced horizontal structure $\mathcal{H}$, functions (r, θ), and a horizontal 1-form $\mathfrak{J}$ such that the following hold true:*

1. *e_3 is geodesic.*
2. *We have*

$$e_3(r) = -1, \qquad e_3(\theta) = 0, \qquad \nabla_3(\overline{q}\mathfrak{J}) = 0, \qquad q = r + ai\cos\theta.$$

3. *We have*

$$H = \frac{aq}{|q|^2}\mathfrak{J}.$$

An extended ingoing PT structure possesses, in addition, a function $\underline{u}$ verifying $e_3(\underline{u}) = 0$.

Definition 9.6. *An ingoing PT initial data set consists of a hypersurface Σ transversal to e_3 together with a null pair (e_3, e_4), the induced horizontal structure $\mathcal{H}$, scalar functions (r, θ), and a horizontal 1-form $\mathfrak{J}$, all defined on Σ.*

Lemma 9.7. *Any ingoing PT initial data set, as in Definition 2.84, can be locally extended to an ingoing PT structure.*

9.1.3. Definition of the PT structures in $\mathcal{M}$ Let $\mathcal{M}$ denote our admissible GCM spacetime introduced in Section 3.2. We decompose $\mathcal{M}$ as follows

$$\mathcal{M} \quad = \quad {}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M}' \cup {}^{(top)}\mathcal{M}',$$

where

- ${}^{(ext)}\mathcal{M}$ is covered by an outgoing PT structure initialized on Σ_* as will be made precise below,
- ${}^{(int)}\mathcal{M}'$ is covered by an ingoing PT structure initialized on $\mathcal{T} = \{r = r_0\}$ as will be made precise below,
- ${}^{(top)}\mathcal{M}'$ is covered by an ingoing PT structure initialized on $\{u = u'_*\}$ for some $u'_* \in [u_* - 5, u_* - 4]$ as will be made precise below.

Remark 9.8. *Recall that ${}^{(ext)}\mathcal{M}$ is covered by an outgoing PG structure. A priori, the outgoing PT structure introduced above covers a different region ${}^{(ext)}\mathcal{M}'$. The fact that these two regions coincide, i.e. ${}^{(ext)}\mathcal{M}' = {}^{(ext)}\mathcal{M}$, is due to the fact that ${}^{(ext)}\mathcal{M}$ is defined purely in terms of (u, r), and that the functions (u, r) for the outgoing PT frame coincide with the ones of the outgoing PG frame in ${}^{(ext)}\mathcal{M}$, see Lemma 9.24.*

Remark 9.9. *The constant u'_* involved in the definition ${}^{(top)}\mathcal{M}'$ and its associated ingoing PT structure will be fixed in Section 9.2.8.*

9.1.3.1. Initialization of the outgoing PT structure of ${}^{(ext)}\mathcal{M}$ Recall from Section 3.2.3 that the GCM boundary Σ_* comes together with a null frame, as well as scalar functions (u, r, θ) all defined on Σ_*. Also, recall the definition of the 1-form f_0 on Σ_*, see Definition 5.56,

$$(f_0)_1 = 0, \quad (f_0)_2 = \sin\theta, \quad \text{on} \quad S_*, \qquad \nabla_\nu f_0 = 0 \quad \text{on} \quad \Sigma_*,$$

where, on S_*, we consider the orthonormal basis (e_1, e_2) of S_* given by (5.129).

Next, recall that the outgoing PT structure of ${}^{(ext)}\mathcal{M}$ is initialized on Σ_*. In view of Lemma 9.4, it suffices to prescribe the corresponding outgoing PT initial data set, as in Definition 9.3, on Σ_*. We make the following choice of outgoing PT initial data set $\{(e_3, e_4, \mathcal{H}), r, \theta, \mathfrak{J}\}$ on Σ_*:

1. The frame (e_1, e_2, e_3, e_4), consisting of the null pair (e_3, e_4) and the horizontal structure $\mathcal{H}$, is obtained from the null frame attached to Σ_* by the change of frame formula with frame coefficients $(f, \underline{f}, \lambda)$ given by

$$\lambda = 1, \qquad f = \frac{a}{r} f_0, \qquad \underline{f} = \frac{a\Upsilon}{r} f_0. \tag{9.1}$$

2. The functions (r, θ) coincide on Σ_* with the ones of Σ_*.
3. The complex horizontal 1-form $\mathfrak{J}$ is given on Σ_* by

$$\mathfrak{J} = \frac{1}{r}(f_0 + i\,{}^*f_0). \tag{9.2}$$

Remark 9.10. *Let $(\widetilde{f}, \underline{\widetilde{f}}, \widetilde{\lambda})$ denote the coefficients involved in the initialization of the outgoing PG frame of ${}^{(ext)}\mathcal{M}$ on Σ_*, i.e. corresponding to the change from the frame of Σ_* to the outgoing PG frame of ${}^{(ext)}\mathcal{M}$. Recall from Section 3.2.5.1 that $(\widetilde{f}, \underline{\widetilde{f}}, \widetilde{\lambda})$ is given on Σ_* by*

$$\widetilde{\lambda} = 1, \qquad \widetilde{f} = \frac{a}{r} f_0, \qquad \underline{\widetilde{f}} = -\frac{(\nu(r) - b_*)}{1 - \frac{1}{4} b_* \frac{a^2 (\sin\theta)^2}{r^2}} \frac{a}{r} f_0.$$

Note in particular that $\widetilde{\lambda} = \lambda$ and $\widetilde{f} = f$, but $\underline{\widetilde{f}} \neq \underline{f}$ so that the outgoing PG frame and the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ are initialized differently on Σ_*. Indeed, the initialization used for the PG frame on Σ_* would lead to a loss of one derivative and hence cannot be used for the PT frame.

9.1.3.2. Initialization of the ingoing PT structure of ${}^{(int)}\mathcal{M}'$ Recall that the ingoing PT structure of ${}^{(int)}\mathcal{M}'$ is initialized on the timelike hypersurface $\mathcal{T}$. We denote by $\{(e_3, e_4, \mathcal{H}), r, \theta, \mathfrak{J}\}$ the outgoing PT initial data set induced by the outgoing PT structure of ${}^{(ext)}\mathcal{M}$ on $\mathcal{T} = \{r = r_0\}$. In view of Lemma 9.7, it suffices to prescribe the ingoing PT initial data set, as in Definition 9.6, corresponding to the PT structure of ${}^{(int)}\mathcal{M}'$ on $\mathcal{T}$. We make the following choice of ingoing PT initial data set $\{(e'_3, e'_4, \mathcal{H}'), r', \theta', \mathfrak{J}'\}$ on $\mathcal{T}$:

1. The frame (e'_1, e'_2, e'_3, e'_4), consisting of the null pair (e'_3, e'_4) and the horizontal structure $\mathcal{H}'$, is obtained from the null frame (e_1, e_2, e_3, e_4) of the outgoing PT structure of ${}^{(ext)}\mathcal{M}$ on $\mathcal{T}$ by

$$e'_a = e_a, \ a = 1, 2, \qquad e'_4 = \frac{\Delta}{|q|^2} e_4, \qquad e'_3 = \frac{|q|^2}{\Delta} e_3. \tag{9.3}$$

2. The functions (r', θ') coincide on $\mathcal{T}$ with the functions (r, θ).
3. The complex horizontal 1-form $\mathfrak{J}'$ coincides on $\mathcal{T}$ with the complex horizontal 1-form $\mathfrak{J}$.
4. The function $\underline{u}'$ of the extended PT structure, i.e. such that $e'_3(\underline{u}') = 0$, coincides on $\mathcal{T}$ with u.

Definition 9.11. *We define ${}^{(int)}\mathcal{M}' \subset \mathcal{M} \setminus {}^{(ext)}\mathcal{M}$ to be the region covered by the maximally extended[193] ingoing PT structure initialized on $\mathcal{T} = \{r = r_0\}$ by the outgoing PT structure of ${}^{(ext)}\mathcal{M}$ as above.*

9.1.3.3. Initialization of the ingoing PT structure of ${}^{(top)}\mathcal{M}'$ Recall that the ingoing PT structure of ${}^{(top)}\mathcal{M}'$ is initialized on the hypersurface $\{u = u'_*\}$

[193] That is the null ingoing geodesics generated by the PT structure end up either at $\mathcal{A}$ or ${}^{(top)}\Sigma$.

of ${}^{(ext)}\mathcal{M}$. We denote by $\{(e_3, e_4, \mathcal{H}), r, \theta, \mathfrak{J}\}$ the outgoing PT initial data set induced by the outgoing PT structure of ${}^{(ext)}\mathcal{M}$ on $\{u = u'_*\}$. In view of Lemma 9.7, it suffices to prescribe the ingoing PT initial data set, as in Definition 9.6, corresponding to the PT structure of ${}^{(top)}\mathcal{M}'$ on $\{u = u'_*\}$. We make the following choice of ingoing PT initial data set $\{(e''_3, e''_4, \mathcal{H}''), r'', \theta'', \mathfrak{J}''\}$ on $\{u = u'_*\}$:

1. The frame $(e''_1, e''_2, e''_3, e''_4)$, consisting of the null pair (e''_3, e''_4) and the horizontal structure $\mathcal{H}''$, is obtained from the null frame (e_1, e_2, e_3, e_4) of the outgoing PT structure of ${}^{(ext)}\mathcal{M}$ on $\{u = u'_*\}$ by

$$e''_a = e_a, \; a = 1, 2, \qquad e''_4 = \frac{\Delta}{|q|^2} e_4, \qquad e''_3 = \frac{|q|^2}{\Delta} e_3. \tag{9.4}$$

2. The functions (r'', θ'') coincide on $\{u = u'_*\}$ with the functions (r, θ).
3. The complex horizontal 1-form $\mathfrak{J}''$ coincides on $\{u = u'_*\}$ with the complex horizontal 1-form $\mathfrak{J}$.
4. The function $\underline{u}''$ of the extended PT structure, i.e. such that $e''_3(\underline{u}'') = 0$, satisfies on $\{u = u'_*\}$

$$\underline{u}'' = u + 2\int_{r_0}^{r} \frac{\tilde{r}^2 + a^2}{\tilde{r}^2 - 2m\tilde{r} + a^2} d\tilde{r}.$$

Definition 9.12. *We define ${}^{(top)}\mathcal{M}' \subset \mathcal{M}$ to be the region covered by the maximally extended*[194] *ingoing PT structure initialized on $\{u = u'_*\}$ by the outgoing PT structure of ${}^{(ext)}\mathcal{M}$ as above.*

Remark 9.13. *Since $u_* - 5 \leq u'_* \leq u_* - 4$, note that ${}^{(top)}\mathcal{M}'$ includes ${}^{(top)}\mathcal{M}$, as well as a small portion of ${}^{(int)}\mathcal{M}$, ${}^{(int)}\mathcal{M}'$ and ${}^{(ext)}\mathcal{M}$.*

9.2. Outgoing PT structure of ${}^{(ext)}\mathcal{M}$

9.2.1. Null structure equations for the PT frame of ${}^{(ext)}\mathcal{M}$ We recall below Proposition 2.77 for outgoing PT structures which applies in particular to the outgoing PT structure of ${}^{(ext)}\mathcal{M}$.

Proposition 9.14. *Consider an outgoing PT structure. Then, the equations in the e_4 direction for the Ricci coefficients of the outgoing PT frame take the form*

$$\nabla_4 trX + \frac{1}{2}(trX)^2 = -\frac{1}{2}\widehat{X}\cdot\overline{\widehat{X}},$$

[194] That is the null ingoing geodesics generated by the PT structure end up either at ${}^{(top)}\Sigma$ or $\mathcal{A}$.

$$
\begin{aligned}
\nabla_4\widehat{X}+\Re(trX)\widehat{X} &= -A,\\
\nabla_4 tr\underline{X}+\frac{1}{2}trX\,tr\underline{X} &= -\mathcal{D}\cdot\left(\frac{aq}{|q|^2}\overline{\mathfrak{J}}\right)+\frac{a^2}{|q|^2}|\mathfrak{J}|^2+2\overline{P}-\frac{1}{2}\widehat{X}\cdot\overline{\widehat{\underline{X}}},\\
\nabla_4\widehat{\underline{X}}+\frac{1}{2}trX\,\widehat{\underline{X}} &= -\mathcal{D}\widehat{\otimes}\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)+\frac{a^2(\overline{q})^2}{|q|^4}\mathfrak{J}\widehat{\otimes}\mathfrak{J}-\frac{1}{2}\overline{tr\underline{X}}\widehat{X},\\
\nabla_4 Z+\frac{1}{2}trXZ &= -\frac{1}{2}trX\frac{a\overline{q}}{|q|^2}\mathfrak{J}-\frac{1}{2}\widehat{X}\cdot\left(\overline{Z}+\frac{aq}{|q|^2}\overline{\mathfrak{J}}\right)-B,\\
\nabla_4\underline{\Xi} &= -\nabla_3\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)-\frac{1}{2}\overline{tr\underline{X}}\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}+H\right)\\
&\quad -\frac{1}{2}\widehat{\underline{X}}\cdot\left(\frac{aq}{|q|^2}\overline{\mathfrak{J}}+\overline{H}\right)-\underline{B},\\
\nabla_4 H &= -\frac{1}{2}\overline{trX}\left(H+\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)-\frac{1}{2}\widehat{X}\cdot\left(\overline{H}+\frac{aq}{|q|^2}\overline{\mathfrak{J}}\right)-B,\\
\nabla_4\underline{\omega} &= \left(\eta+\Re\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)\right)\cdot\zeta+\eta\cdot\Re\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)+\rho.
\end{aligned}
$$

9.2.2. Other transport equations in the e_4 direction

Lemma 9.15. *We have*

$$
\begin{aligned}
\nabla_4\mathcal{D}\cos\theta+\frac{1}{2}trX\mathcal{D}\cos\theta &= -\frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}\cos\theta,\\
\nabla_4\mathcal{D}r+\frac{1}{2}trX\mathcal{D}r &= -\frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}r+(Z+\underline{H}),\\
\nabla_4\mathcal{D}u+\frac{1}{2}trX\mathcal{D}u &= -\frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}u,
\end{aligned}
$$

and

$$
\begin{aligned}
e_4(e_3(\cos\theta)) &= -\Re\Big((H-\underline{H})\cdot\overline{\mathcal{D}}(\cos\theta)\Big),\\
e_4(e_3(r)) &= -2\underline{\omega}-\Re\Big((H-\underline{H})\cdot\overline{\mathcal{D}}r\Big),\\
e_4(e_3(u)) &= -\Re\Big((H-\underline{H})\cdot\overline{\mathcal{D}}u\Big).
\end{aligned}
$$

Proof. Straightforward verification. □

9.2.3. Linearized quantities for the outgoing PT frame Recall the definition of the linearized quantities in the PT frame, see Definition 2.79.

Definition 9.16. *We consider the following renormalizations, for given constants (a, m),*

$$\widetilde{trX} := trX - \frac{2}{q}, \qquad \widetilde{tr\underline{X}} := tr\underline{X} + \frac{2q\Delta}{|q|^4},$$
$$\check{Z} := Z - \frac{a\overline{q}}{|q|^2}\mathfrak{J}, \qquad \check{H} := H - \frac{aq}{|q|^2}\mathfrak{J}, \tag{9.5}$$
$$\check{\underline{\omega}} := \underline{\omega} - \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right), \qquad \check{P} := P + \frac{2m}{q^3},$$

as well as

$$\widetilde{e_3(r)} := e_3(r) + \frac{\Delta}{|q|^2}, \qquad \mathcal{D}\widetilde{(\cos\theta)} := \mathcal{D}(\cos(\theta)) - i\mathfrak{J},$$
$$\widetilde{\mathcal{D}}u := \mathcal{D}u - a\mathfrak{J}, \qquad \widetilde{e_3(u)} := e_3(u) - \frac{2(r^2+a^2)}{|q|^2}, \tag{9.6}$$

and

$$\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}} := \overline{\mathcal{D}}\cdot\mathfrak{J} - \frac{4i(r^2+a^2)\cos\theta}{|q|^4}, \qquad \widetilde{\nabla_3\mathfrak{J}} := \nabla_3\mathfrak{J} - \frac{\Delta q}{|q|^4}\mathfrak{J}. \tag{9.7}$$

9.2.4. Definition of the notations Γ_b and Γ_g for error terms

Definition 9.17. *The set of all linearized quantities is of the form* $\Gamma_g \cup \Gamma_b$ *with* Γ_g, Γ_b *defined as follows.*

1. *The set* Γ_g *with*

$$\Gamma_g = \left\{\widetilde{trX}, \quad \widehat{X}, \quad \check{Z}, \quad \widetilde{tr\underline{X}}, \quad r^{-1}\nabla(r), \quad r\check{P}, \quad rB, \quad rA\right\}.$$

2. *The set* $\Gamma_b = \Gamma_{b,1} \cup \Gamma_{b,2} \cup \Gamma_{b,3}$ *with*

$$\Gamma_{b,1} = \left\{\check{H}, \quad \widehat{\underline{X}}, \quad \check{\underline{\omega}}, \quad \underline{\Xi}, \quad r\underline{B}, \quad \underline{A}\right\},$$
$$\Gamma_{b,2} = \left\{r^{-1}\widetilde{e_3(r)}, \quad \mathcal{D}\widetilde{(\cos\theta)}, \quad e_3(\cos\theta), \quad \widetilde{\mathcal{D}}u, \quad r^{-1}\widetilde{e_3(u)}\right\},$$
$$\Gamma_{b,3} = \left\{r\,\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}, \quad r\,\mathcal{D}\widehat{\otimes}\mathfrak{J}, \quad r\,\widetilde{\nabla_3\mathfrak{J}}\right\}.$$

9.2.5. Linearized equations for outgoing PT structures Recall the convention $O(r^{-p})$ introduced earlier in Definition 6.3.

Definition 9.18 (Order of magnitude notation)**.** *Throughout this chapter, we will be using the notation $O(r^{-p})$ to denote:*

1. *a scalar function depending only on (r,θ) which is smooth and such that*
$$r^p|(r\partial_r,\partial_\theta)^k O(r^{-p})|\lesssim 1 \quad for \quad k\geq 0 \quad and \quad r\geq r_0,$$
2. *a 1-form of the type $O(r^{-p+1})\mathfrak{J}$ where $O(r^{-p+1})$ denotes a scalar function as above,*
3. *a symmetric traceless 2-tensor of the type $O(r^{-p+2})\mathfrak{J}\widehat{\otimes}\mathfrak{J}$ where $O(r^{-p+2})$ denotes a scalar function as above.*

The following proposition provides the linearized null structure equations for the outgoing PT frame of ${}^{(ext)}\mathcal{M}$.

Proposition 9.19. *In an outgoing PT frame, the linearized null structure equations in the e_4 direction are*

$$\begin{aligned}
\nabla_4\widetilde{trX}+\frac{2}{q}\widetilde{trX} &= \Gamma_g\cdot\Gamma_g,\\
\nabla_4\widehat{X}+\Re\left(\frac{2}{q}\right)\widehat{X} &= -A+\Gamma_g\cdot\Gamma_g,\\
\nabla_4\check{Z}+\frac{1}{q}\check{Z} &= -\frac{a\overline{q}}{|q|^2}\widetilde{trX}\mathfrak{J}-\frac{aq}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X}-B+\Gamma_g\cdot\Gamma_g,\\
\nabla_4\check{H}+\frac{1}{\overline{q}}\check{H} &= -\frac{ar}{|q|^2}\overline{\widetilde{trX}}\mathfrak{J}-\frac{ar}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X}-B+\Gamma_b\cdot\Gamma_g,\\
\nabla_4\underline{\check{\omega}} &= \Re\left(\check{P}\right)+\frac{ar}{|q|^2}\Re\left(\mathfrak{J}\cdot\overline{\check{Z}}\right)+\frac{2a}{|q|^2}\Re\left(q\overline{\mathfrak{J}}\cdot\check{H}\right)+\Gamma_g\cdot\Gamma_g,\\
\nabla_4\widetilde{tr\underline{X}}+\frac{1}{q}\widetilde{tr\underline{X}} &= 2\overline{\check{P}}+\frac{q\Delta}{|q|^4}\widetilde{trX}-\frac{aq}{|q|^2}\widetilde{\mathcal{D}\cdot\overline{\mathfrak{J}}}+\frac{a}{\overline{q}^2}\mathcal{D}(r)\cdot\overline{\mathfrak{J}}\\
&\quad -\frac{ia^2}{\overline{q}^2}\widetilde{\mathcal{D}(\cos\theta)}\cdot\overline{\mathfrak{J}}+\Gamma_b\cdot\Gamma_g,\\
\nabla_4\widehat{\underline{X}}+\frac{1}{q}\widehat{\underline{X}} &= -\frac{a\overline{q}}{|q|^2}\mathcal{D}\widehat{\otimes}\mathfrak{J}+\frac{a}{q^2}\mathcal{D}(r)\widehat{\otimes}\mathfrak{J}+\frac{ia^2}{q^2}\widetilde{\mathcal{D}(\cos\theta)}\widehat{\otimes}\mathfrak{J}+\frac{q\Delta}{|q|^4}\widehat{X}\\
&\quad +\Gamma_b\cdot\Gamma_g,\\
\nabla_4\underline{\Xi} &= \frac{\overline{q}\Delta}{|q|^4}\check{H}-\frac{ar}{|q|^2}\overline{\widetilde{tr\underline{X}}}\mathfrak{J}-\frac{ar}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{\underline{X}}-\underline{B}-\frac{a\overline{q}}{|q|^2}\widetilde{\nabla_3\mathfrak{J}}\\
&\quad +\frac{a}{q^2}\left(\widetilde{e_3(r)}+iae_3(\cos\theta)\right)\mathfrak{J}+\Gamma_b\cdot\Gamma_b.
\end{aligned}$$

Proof. See appendix D.1. Compare also with the proof of Lemma 6.15 in the outgoing PG frame. □

Remark 9.20. *Note that the equations for $\nabla_4\widecheck{tr\underline{X}}$, $\nabla_4\widehat{\underline{X}}$, and $\nabla_4\underline{\Xi}$ in Proposition 9.19 do not lose derivative, unlike the corresponding ones in the outgoing PG frame (compare with Lemma 6.15).*

9.2.6. Other linearized equations The following lemma follows immediately from Lemma 9.15, the definition of the linearized quantities, and the definition of Γ_g and Γ_b.

Lemma 9.21. *We have*

$$\begin{aligned}
\nabla_4\widecheck{\mathcal{D}\cos\theta} + \frac{1}{q}\widecheck{\mathcal{D}\cos\theta} &= O(r^{-1})\widecheck{trX} + \frac{i}{2}\overline{\mathfrak{J}}\cdot\widehat{X} + \Gamma_b\cdot\Gamma_g,\\
\nabla_4\mathcal{D}r + \frac{1}{q}\mathcal{D}r &= \check{Z} + r\Gamma_g\cdot\Gamma_g,\\
\nabla_4\widecheck{\mathcal{D}u} + \frac{1}{q}\widecheck{\mathcal{D}u} &= O(r^{-1})\widecheck{trX} - \frac{a}{2}\overline{\mathfrak{J}}\cdot\widehat{X} + \Gamma_b\cdot\Gamma_g,
\end{aligned}$$

and

$$\begin{aligned}
e_4(e_3(\cos\theta)) &= -\Im\Big(\overline{\mathfrak{J}}\cdot\check{H}\Big) - \Re\left(\frac{2ar}{|q|^2}\mathfrak{J}\cdot\overline{\widecheck{\mathcal{D}(\cos\theta)}}\right) + \Gamma_b\cdot\Gamma_b,\\
e_4(\widecheck{e_3(r)}) &= -2\underline{\check{\omega}} - \Re\left(\frac{2ar}{|q|^2}\mathfrak{J}\cdot\overline{\mathcal{D}}r\right) + r\Gamma_b\cdot\Gamma_g,\\
e_4(\widecheck{e_3(u)}) &= -\Re\Big(a\overline{\mathfrak{J}}\cdot\check{H}\Big) - \Re\left(\frac{2ar}{|q|^2}\mathfrak{J}\cdot\widecheck{\overline{\mathcal{D}}u}\right) + \Gamma_b\cdot\Gamma_b.
\end{aligned}$$

Proof. Straightforward verification. Compare it also with the proof of Lemma 6.16. □

Lemma 9.22. *We have*

$$\begin{aligned}
\nabla_4\mathcal{D}\widehat{\otimes}\mathfrak{J} + \frac{2}{q}\mathcal{D}\widehat{\otimes}\mathfrak{J} &= O(r^{-1})B + O(r^{-2})\widecheck{trX} + O(r^{-2})\widehat{X}\\
&\quad + O(r^{-2})\check{Z} + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)},\\
\nabla_4\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}} + \Re\left(\frac{2}{q}\right)\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}} &= O(r^{-1})B + O(r^{-2})\widecheck{trX} + O(r^{-2})\widehat{X} + O(r^{-2})\check{Z}\\
&\quad + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)},\\
\nabla_4\widecheck{\nabla_3\mathfrak{J}} + \frac{1}{q}\widecheck{\nabla_3\mathfrak{J}} &= O(r^{-1})\check{P} + O(r^{-3})\widecheck{e_3(r)} + O(r^{-3})e_3(\cos\theta)\\
&\quad + O(r^{-2})\underline{\check{\omega}} + O(r^{-2})\check{\underline{H}} + O(r^{-2})\widecheck{\nabla\mathfrak{J}}.
\end{aligned}$$

Proof. Straightforward verification[195]. Compare also with the proof of Lemma 6.17. □

Remark 9.23 (Triangular structure of the main equations)**.** *We can order the linearized quantities appearing in the equations of Proposition 9.19, Lemma 9.21 and Lemma 9.22 as follows*

$$\widecheck{trX},\ \widehat{X},\ \check{Z},\ \check{H},\ \widecheck{\mathcal{D}\cos\theta},\ \underline{\check{\omega}},\ \mathcal{D}r,\ \widecheck{\mathcal{D}u},\ e_3(\cos\theta),$$
$$\widecheck{e_3(r)},\ \widecheck{e_3(u)},\ \mathcal{D}\widehat{\otimes}\mathfrak{J},\ \widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}},\ \widecheck{\nabla_3\mathfrak{J}},\ \widecheck{tr\underline{X}},\ \underline{\widehat{X}},\ \underline{\Xi},$$

and note that the transport equation for each one of them depends[196] *only on the previous components of the sequence. This triangular structure is essential for estimating the terms one by one. This crucial fact will be used in Section 9.8 to estimate the Ricci coefficients of the PT frame of* $^{(ext)}\mathcal{M}$*.*

9.2.7. Comparison between the PT and PG structures of $^{(ext)}\mathcal{M}$

The following lemma compares the outgoing PG and PT structures of $^{(ext)}\mathcal{M}$.

Lemma 9.24. *Let* $\{(e_3, e_4, \mathcal{H}), r, \theta, \mathfrak{J}\}$ *denote the extended outgoing PG structure of* $^{(ext)}\mathcal{M}$*, and let* $\{(e_3', e_4', \mathcal{F}'), r', \theta', \mathfrak{J}'\}$ *denote the extended outgoing PT structure of* $^{(ext)}\mathcal{M}$*. Also, let* $(f, \underline{f}, \lambda)$ *denote the transition coefficients from the PG frame to the PT one. Then, the following identities hold in* $^{(ext)}\mathcal{M}$*:*

1. *We have* $f = 0$ *and* $\lambda = 1$*. In particular, we have*
$$e_4' = e_4.$$
2. *We have*
$$u' = u, \qquad r' = r, \qquad \theta' = \theta, \qquad \mathfrak{J}' = \mathfrak{J}.$$
3. *Let* $(f', \underline{f}', \lambda')$ *denote the transition coefficients from the PT frame to the PG one. Then, we have*
$$\lambda' = 1, \quad f' = 0, \quad \underline{f}' = -\underline{f}.$$
4. *We have, in view of Definition 2.66 for linearized outgoing PG quantities,*

[195] Note that the proof must be adapted from the case of a PG frame, i.e. $\underline{H} = -Z$, to the case of a PT frame, i.e. $\underline{H} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J}$. This introduces a slight change in the commutators $[\nabla_4, \nabla]$ and $[\nabla_4, \nabla_3]$.

[196] At the linear level and excluding curvature terms.

$$\nabla_4 \underline{f} = 2\check{\zeta}.$$

5. *With the notation $\underline{F}' = \underline{f}' + i\,{}^*\underline{f}'$, we have, in view of Definition 9.16 for linearized outgoing PT quantities,*

$$\nabla'_4\underline{F}' + \frac{1}{2}trX'\underline{F}' = -2\check{Z}' - \underline{F}'\cdot\widehat{\chi}'.$$

Proof. See Section D.2. □

9.2.8. The choice of the constant u'_* Recall from Section 9.1.3 that the ingoing PT structure of ${}^{(top)}\mathcal{M}'$ in initialized from the outgoing PT structure of ${}^{(ext)}\mathcal{M}$ on the hypersurface $\{u = u'_*\}$ of ${}^{(ext)}\mathcal{M}$ for $u'_* \in [u_* - 5, u_* - 4]$. We are now ready to make a specific choice of u'_*. First, we introduce the following notation

$$(9.8)\qquad \begin{aligned} |\check{R}|^2_{w,k} &:= r^{3+\delta_B}|\mathfrak{d}_*^{\leq k}(A,B)|^2 + r^{3-\delta_B}|\mathfrak{d}_*^{\leq k}\check{P}|^2 + r^{1-\delta_B}|\mathfrak{d}_*^{\leq k}\underline{B}|^2,\\ |\check{\Gamma}|^2_{w,k} &:= r^2|\not{\mathfrak{d}}^{\leq k}\Gamma_g|^2 + |\not{\mathfrak{d}}^{\leq k}\Gamma_b|^2, \end{aligned}$$

where $(A, B, \check{P}, \underline{B})$ denote linearized curvature components w.r.t. the the outgoing PT frame of ${}^{(ext)}\mathcal{M}$, Γ_g and Γ_b are defined w.r.t. the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ as in Definition 9.17, $\mathfrak{d}_*$ denote weighted derivatives tangential to the hypersurface $\{u = u'_*\}$, and $\not{\mathfrak{d}}$ denote weighted derivatives tangential to the sphere $\Sigma_* \cap \{u = u'_*\}$.

With the notations in (9.8), we choose u'_* such that we have

$$(9.9)\qquad \begin{aligned} &\int_{\{u=u'_*\}}|\check{R}|^2_{w,k_{large}+7} + \int_{\Sigma_*\cap\{u=u'_*\}}|\check{\Gamma}|^2_{w,k_{large}+7}\\ &= \inf_{u_*-5\leq u_1\leq u_*-4}\left(\int_{\{u=u_1\}}|\check{R}|^2_{w,k_{large}+7} + \int_{\Sigma_*\cap\{u=u_1\}}|\check{\Gamma}|^2_{w,k_{large}+7}\right). \end{aligned}$$

9.3. Ingoing PT structures of ${}^{(int)}\mathcal{M}'$ and ${}^{(top)}\mathcal{M}'$

9.3.1. Linearized quantities in an ingoing PT frame Given an extended ingoing PT structure $\{(e_3, e_4, \mathcal{H}), r, \theta, \underline{u}, \mathfrak{J}\}$ hold true:

1. We have

$$\underline{\xi} = \underline{\omega} = 0,\quad e_3(r) = -1,\quad e_3(\underline{u}) = e_3(\theta) = 0,\qquad \nabla_3(\overline{q}\mathfrak{J}) = 0.$$

In addition, we have

$$H = \frac{aq}{|q|^2}\mathfrak{J}.$$

2. The quantities

$$\widehat{X}, \quad \underline{\widehat{X}}, \quad \Xi, \quad A, \quad B, \quad \underline{B}, \quad \underline{A}, \quad \mathcal{D}r, \quad e_4(\cos\theta), \quad \mathcal{D}\widehat{\otimes}\mathfrak{J},$$

vanish in Kerr and therefore are small in perturbations.

We renormalize below all other quantities, not vanishing in Kerr[197], by subtracting their Kerr(a,m) values.

Definition 9.25. *We consider the following renormalizations, for given constants* (a,m)*,*

$$\begin{aligned}
\widetilde{trX} &:= trX - \frac{2\overline{q}\Delta}{|q|^4}, & \widetilde{tr\underline{X}} &:= tr\underline{X} + \frac{2}{\overline{q}},\\
\check{Z} &:= Z - \frac{aq}{|q|^2}\mathfrak{J}, & \underline{\check{H}} &:= \underline{H} + \frac{a\overline{q}}{|q|^2}\mathfrak{J},\\
\check{\omega} &:= \omega + \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right), & \check{P} &:= P + \frac{2m}{q^3},
\end{aligned} \tag{9.10}$$

as well as

$$\begin{aligned}
\widetilde{e_4(r)} &:= e_4(r) - \frac{\Delta}{|q|^2}, & \widetilde{\mathcal{D}(\cos\theta)} &:= \mathcal{D}(\cos(\theta)) - i\mathfrak{J},\\
\widetilde{\mathcal{D}\underline{u}} &:= \mathcal{D}\underline{u} - a\mathfrak{J}, & \widetilde{e_4(\underline{u})} &:= e_4(\underline{u}) - \frac{2(r^2+a^2)}{|q|^2},
\end{aligned} \tag{9.11}$$

and

$$\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}} := \overline{\mathcal{D}}\cdot\mathfrak{J} - \frac{4i(r^2+a^2)\cos\theta}{|q|^4}, \quad \widetilde{\nabla_4\mathfrak{J}} := \nabla_4\mathfrak{J} + \frac{\Delta\overline{q}}{|q|^4}\mathfrak{J}. \tag{9.12}$$

9.3.2. Definition of the notations Γ_b and Γ_g for error terms

Definition 9.26. *The set of all linearized quantities is of the form* $\Gamma_g \cup \Gamma_b$ *with* Γ_g, Γ_b *defined as follows.*

1. *The set* $\Gamma_g = \Gamma_{g,1} \cup \Gamma_{g,2}$ *with*

[197] Since $H = \frac{aq}{|q|^2}\mathfrak{J}$, H does not need to be included in Definition 9.25.

$$\Gamma_{g,1} = \Big\{ \Xi, \quad \check{\omega}, \quad \widecheck{trX}, \quad \widehat{X}, \quad \check{Z}, \quad \widecheck{\underline{H}}, \quad \widecheck{tr\underline{X}}, \quad r\check{P}, \quad rB, \quad rA \Big\},$$

$$\Gamma_{g,2} = \Big\{ r^{-1}\nabla(r), \quad \widecheck{re_4(r)}, \quad \widecheck{re_4(\underline{u})}, \quad re_4(\cos\theta), \quad r^2\widecheck{\nabla_4\mathfrak{J}} \Big\}.$$

2. The set Γ_b *with*

$$\Gamma_b = \Big\{ \widehat{\underline{X}}, \quad r\underline{B}, \quad \underline{A}, \quad \widecheck{\mathcal{D}(\cos\theta)}, \quad \widecheck{\mathcal{D}\underline{u}}, \quad r\,\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}, \quad r\,\mathcal{D}\widehat{\otimes}\mathfrak{J} \Big\}.$$

9.3.3. Main linearized equations for ingoing PT structures

Proposition 9.27. *In an ingoing PT frame, the linearized null structure equations in the* e_3 *direction are*

$$\begin{aligned}
\nabla_3(\widecheck{tr\underline{X}}) - \frac{2}{q}\widecheck{tr\underline{X}} &= \Gamma_b\cdot\Gamma_b,\\
\nabla_3\widehat{\underline{X}} - \frac{2r}{|q|^2}\widehat{\underline{X}} &= -\underline{A} + \Gamma_b\cdot\Gamma_b,\\
\nabla_3\check{Z} - \frac{1}{q}\check{Z} &= -\underline{B} + O(r^{-2})\widehat{\underline{X}} + O(r^{-2})\widecheck{tr\underline{X}} + \Gamma_b\cdot\Gamma_g,\\
\nabla_3\widecheck{\underline{H}} - \frac{1}{q}\widecheck{\underline{H}} &= \underline{B} + O(r^{-2})\widehat{\underline{X}} + O(r^{-2})\widecheck{tr\underline{X}} + \Gamma_b\cdot\Gamma_g,\\
\nabla_3\widecheck{trX} - \frac{1}{q}\widecheck{trX} &= 2\check{P} + O(r^{-1})\widecheck{tr\underline{X}} + O(r^{-1})\widecheck{\mathcal{D}\cdot\overline{\mathfrak{J}}} + O(r^{-3})\mathcal{D}r\\
&\quad + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} + \Gamma_b\cdot\Gamma_g,\\
\nabla_3\widehat{X} - \frac{1}{q}\widehat{X} &= O(r^{-1})\mathcal{D}\widehat{\otimes}\mathfrak{J} + O(r^{-4})\widecheck{\mathcal{D}(\cos\theta)} + O(r^{-1})\widehat{\underline{X}} + \Gamma_b\cdot\Gamma_g,\\
\nabla_3\check{\omega} &= \Re(\check{P}) + O(r^{-2})\check{Z} + O(r^{-2})\widecheck{\underline{H}} + \Gamma_b\cdot\Gamma_g,\\
\nabla_3\Xi &= O(r^{-1})\widecheck{\underline{H}} + O(r^{-2})\widecheck{trX} + O(r^{-2})\widehat{X} + B + O(r^{-1})\widecheck{\nabla_4\mathfrak{J}}\\
&\quad + O(r^{-3})\widecheck{e_4(r)} + O(r^{-3})e_4(\cos\theta) + \Gamma_b\cdot\Gamma_g.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\nabla_3\widecheck{\mathcal{D}\cos\theta} - \frac{1}{q}\widecheck{\mathcal{D}\cos\theta} &= O(r^{-1})\widecheck{tr\underline{X}} + O(r^{-1})\widehat{\underline{X}} + \Gamma_b\cdot\Gamma_b,\\
\nabla_3\mathcal{D}r - \frac{1}{q}\mathcal{D}r &= \check{Z} + r\Gamma_b\cdot\Gamma_g,\\
\nabla_3\widecheck{\mathcal{D}\underline{u}} - \frac{1}{q}\widecheck{\mathcal{D}\underline{u}} &= O(r^{-1})\widecheck{tr\underline{X}} + O(r^{-1})\widehat{\underline{X}} + \Gamma_b\cdot\Gamma_b,
\end{aligned}$$

$$
\begin{aligned}
e_3(e_4(\cos\theta)) &= O(r^{-1})\widecheck{\underline{H}} + O(r^{-2})\mathcal{D}\widecheck{(\cos\theta)} + \Gamma_g\cdot\Gamma_g,\\
e_3(\widecheck{e_4(r)}) &= -2\widecheck{\omega} + O(r^{-2})\mathcal{D}r + r\Gamma_g\cdot\Gamma_g,\\
e_3(\widecheck{e_4(\underline{u})}) &= O(r^{-1})\widecheck{\underline{H}} + O(r^{-2})\widecheck{\mathcal{D}u} + \Gamma_g\cdot\Gamma_g,\\
\nabla_3\mathcal{D}\widehat{\otimes}\mathfrak{J} - \frac{2}{q}\mathcal{D}\widehat{\otimes}\mathfrak{J} &= O(r^{-1})\underline{B} + O(r^{-2})\widecheck{tr\underline{X}} + O(r^{-2})\widehat{\underline{X}} + O(r^{-2})\widecheck{Z}\\
& +O(r^{-3})\mathcal{D}\widecheck{(\cos\theta)},\\
\nabla_3\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}} - \Re\left(\frac{2}{\overline{q}}\right)\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}} &= O(r^{-1})\underline{B} + O(r^{-2})\widecheck{tr\underline{X}} + O(r^{-2})\widehat{\underline{X}} + O(r^{-2})\widecheck{Z}\\
& +O(r^{-3})\mathcal{D}\widecheck{(\cos\theta)},\\
\nabla_3\widecheck{\nabla_4\mathfrak{J}} - \frac{1}{q}\widecheck{\nabla_4\mathfrak{J}} &= O(r^{-1})\widecheck{P} + O(r^{-3})\widecheck{e_4(r)} + O(r^{-3})e_4(\cos\theta)\\
& +O(r^{-2})\widecheck{\omega} + O(r^{-2})\widecheck{\underline{H}} + O(r^{-2})\widecheck{\nabla\mathfrak{J}}.
\end{aligned}
$$

Proof. Straightforward verification. Compare with the proof of Proposition 9.19, and Lemmas 9.21 and 9.22 for the corresponding equations for outgoing PT structures. Compare also with Lemma 7.2, Lemma 7.3 and Lemma 7.4 for the corresponding equations for ingoing PG structures. □

In order to deal with the trapping in the region ${}^{(int)}\mathcal{M}'$, we will need additional equations provided in the proposition below.

Proposition 9.28. *In the ingoing PT structure of ${}^{(int)}\mathcal{M}'$, the linearized Codazzi of $\widehat{\underline{X}}$ takes the following schematic form*[198]

$$
\overline{\mathcal{D}}\cdot\widehat{\underline{X}} = \mathcal{D}\widecheck{\overline{tr\underline{X}}} + \Gamma_b + \Gamma_b\cdot\Gamma_b.
$$

Also, we can write the Bianchi identities in ${}^{(int)}\mathcal{M}'$ schematically in the form

$$
\begin{aligned}
&\nabla_3 A - \frac{1}{2}\mathcal{D}\widehat{\otimes}B = \Gamma_b + \Gamma_b\cdot\Gamma_b, && \nabla_4 B - \frac{1}{2}\overline{\mathcal{D}}\cdot A = \Gamma_b + \Gamma_b\cdot\Gamma_b,\\
&\nabla_3 B - \mathcal{D}\overline{\widecheck{P}} = \Gamma_b + \Gamma_b\cdot\Gamma_b, && \nabla_4 P - \frac{1}{2}\mathcal{D}\cdot\overline{B} = \Gamma_b + \Gamma_b\cdot\Gamma_b,\\
&\nabla_3 P + \frac{1}{2}\overline{\mathcal{D}}\cdot\underline{B} = \Gamma_b + \Gamma_b\cdot\Gamma_b, && \nabla_4\underline{B} + \mathcal{D}\widecheck{P} = \Gamma_b + \Gamma_b\cdot\Gamma_b,\\
&\nabla_3\underline{B} + \frac{1}{2}\overline{\mathcal{D}}\cdot\underline{A} = \Gamma_b + \Gamma_b\cdot\Gamma_b, && \nabla_4\underline{A} + \frac{1}{2}\mathcal{D}\widehat{\otimes}\underline{B} = \Gamma_b + \Gamma_b\cdot\Gamma_b.
\end{aligned}
$$

[198]Note that $r \leq r_0$ in ${}^{(int)}\mathcal{M}'$ so that powers of r do not matter. As a consequence, we may simply denote linear terms by Γ_b and nonlinear terms by $\Gamma_b\cdot\Gamma_b$.

9.3.4. The scalar function τ on $\mathcal{M}$ We introduce on $\mathcal{M}$ a scalar function τ with the following properties.

Proposition 9.29. *There exists a scalar function τ defined on $\mathcal{M}$ such that:*

1. *We have on $\mathcal{M}$*
$$\mathbf{g}(\mathbf{D}\tau, \mathbf{D}\tau) \leq -\frac{m^2}{8r^2}, \tag{9.13}$$
so that the level sets of τ are spacelike and asymptotically null.
2. *The future boundary ${}^{(top)}\Sigma$ of $\mathcal{M}$ is given by*
$${}^{(top)}\Sigma = \{\tau = u_*\} \tag{9.14}$$
and $\tau \leq u_*$ on $\mathcal{M}$.
3. *Denoting, on each level set of $\underline{u}$ in ${}^{(top)}\mathcal{M}'(r \geq r_0)$, by $r_+(\underline{u})$ the maximal value of r and by $r_-(\underline{u})$ the minimal value of r, we have*[199]
$$0 \leq r_+(\underline{u}) - r_-(\underline{u}) \leq 2(2m+1). \tag{9.15}$$
4. *In ${}^{(top)}\mathcal{M}'(r \leq r_0)$, τ satisfies*
$$u_* - 2(m+2) \leq \tau \leq u_*. \tag{9.16}$$
5. *In $\mathcal{M}(r \leq r_0)$, τ satisfies*
$$\begin{aligned} e_4(\tau) &= \frac{2(r^2+a^2) - \frac{m^2}{r^2}\Delta}{|q|^2} + r^{-1}\Gamma_g, \\ e_3(\tau) &= \frac{m^2}{r^2}, \quad \nabla(\tau) = a\Re(\mathfrak{J}) + \Gamma_b. \end{aligned} \tag{9.17}$$

Proof. See Section D.3. □

Remark 9.30. *In view of the third property and the fourth property of Proposition 9.29, ${}^{(top)}\mathcal{M}'$ is in fact a local existence type region.*

Remark 9.31. *The fact that $r_+(\underline{u}) - r_-(\underline{u}) \leq 2(2m+1)$ in ${}^{(top)}\mathcal{M}'(r \geq r_0)$ in view of Proposition 9.29 is crucial to recover the Ricci coefficients in ${}^{(top)}\mathcal{M}'(r \geq r_0)$. In particular, it is crucial to control $\widehat{\underline{X}}$ from $\underline{A}$ as we have schematically*

[199]Note that (9.15) depends on the choice of ${}^{(top)}\Sigma$ and hence on the choice of τ.

$$|\underline{\widehat{X}}| \lesssim |\underline{\widehat{X}}_{|_{u=u'_*}}| + (r_+(\underline{u}) - r_-(\underline{u}))|\underline{A}|$$

so that we indeed need $r_+(\underline{u}) - r_-(\underline{u}) \lesssim 1$.

Definition 9.32. *Given a level hypersurface* $\Sigma(\tau)$ *of* τ, *we denote*[200]

$$N_\tau := -\mathbf{g}^{\alpha\beta}\partial_\beta\tau\partial_\alpha, \qquad \widehat{N}_\tau := \frac{1}{\sqrt{|\mathbf{g}(N_\tau, N_\tau)|}}N_\tau, \tag{9.18}$$

so that $\widehat{N}_\tau$ *is the future unit normal to* $\Sigma(\tau)$.

We will consider a subregion ${}^{(int)}\mathcal{M}'_*$ of ${}^{(int)}\mathcal{M}'$ defined as follows.

Definition 9.33. *Let* τ_* *the supremum*[201] *of the values of* τ *such that* $\Sigma(\tau)\cap {}^{(int)}\mathcal{M}' \neq \emptyset$ *and* $\Sigma(\tau)$ *does not intersect* $\{\underline{u} = u_*\}$. *We denote by* ${}^{(int)}\mathcal{M}'_*$ *the subset of* ${}^{(int)}\mathcal{M}'$ *with* $\tau \leq \tau_*$, *i.e.*

$$ {}^{(int)}\mathcal{M}'_* := {}^{(int)}\mathcal{M}' \cap \{\tau \leq \tau_*\}. \tag{9.19}$$

Remark 9.34. *Note that the region* ${}^{(int)}\mathcal{M}'$ *has as future boundary the hypersurface* $\{\underline{u} = u_*\}$ *which is not spacelike, while the region* ${}^{(int)}\mathcal{M}'_* \subset {}^{(int)}\mathcal{M}'$ *has as future boundary the hypersurface* $\{\tau = \tau_*\}$ *which is spacelike in view of Proposition 9.29.*

9.4. Control of top order derivatives in the PT frame

In this section, we state our main result concerning the control of top order derivatives in the PT frame, see Theorem 9.44. As a consequence, this yields the control of the PG frame and concludes the proof of Theorem M8, see Section 9.4.3.

9.4.1. Main norms We introduce here the main norms appearing in the statement of our main PT-Theorem in Section 9.4.2.

9.4.1.1. Norms on Σ_*

Definition 9.35. *We define the following PT-Ricci coefficients norms on* Σ_*

[200] Note from Proposition 9.29 that

$$\mathbf{g}(N_\tau, N_\tau) = \mathbf{g}(\mathbf{D}\tau, \mathbf{D}\tau) \leq -\frac{m^2}{8r^2} < 0.$$

[201] Note that $u_* - 2(m+1) \leq \tau_* < u_*$ in view of Proposition 9.29.

(9.20)

$$
\begin{aligned}
{}^\star\mathfrak{G}_k^2 &:= \int_{\Sigma_*} \Big(r^2\big|\mathfrak{d}^{\leq k}(\widehat{X}, \widetilde{trX}, \check{Z}, \widetilde{tr\underline{X}})\big|^2 + \big|\mathfrak{d}^{\leq k}(\widehat{\underline{X}}, \check{H}, \check{\underline{\omega}}, \underline{\Xi})\big|^2\Big)\\
&+ \int_{\Sigma_*} \Big(\big|\mathfrak{d}^{\leq k}\widetilde{\mathcal{D}\cos\theta}\big|^2 + \big|\mathfrak{d}^{\leq k}\mathcal{D}r\big|^2 + \big|\mathfrak{d}^{\leq k}e_3(\cos\theta)\big|^2 + r^{-2}\big|\mathfrak{d}^{\leq k}\widetilde{e_3(r)}\big|^2\Big)\\
&+ \int_{\Sigma_*} r^2\Big(|\mathfrak{d}^{\leq k}\mathcal{D}\widehat{\otimes}\mathfrak{J}|^2 + \big|\mathfrak{d}^{\leq k}\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}\big|^2 + \big|\mathfrak{d}^{\leq k}\widetilde{\nabla_3\mathfrak{J}}\big|^2\Big)\\
&+ \int_{\Sigma_*} \left(\left|\mathfrak{d}_*^{\leq k}\left(b_* + 1 + \frac{2m}{r}\right)\right| + \left|\mathfrak{d}_*^{\leq k}\left(\nu(r)+2\right)\right|\right),
\end{aligned}
$$

where $\widetilde{trX}$, $\widehat{X}$, $\check{Z}$, $\check{H}$, $\widetilde{tr\underline{X}}$, $\widehat{\underline{X}}$, $\check{\underline{\omega}}$, $\underline{\Xi}$ are the linearized Ricci coefficients of the outgoing PT frame of ${}^{(ext)}\mathcal{M}$, $\nu = e_3 + b_* e_4$ is tangent to Σ_* with (e_3, e_4) denoting the null pair attached to Σ_*, and $\mathfrak{d}_*$ corresponds weighted derivatives tangent to Σ_*, i.e. $\mathfrak{d}_* = (\nabla_\nu, \not{\mathfrak{d}})$ with $\not{\mathfrak{d}}$ associated to Σ_*.

Definition 9.36. *We define the curvature norm on Σ_**

$$
\begin{aligned}
{}^\star\mathfrak{R}_k^2 \;:=\; \int_{\Sigma_*}\Big[& r^{4+\delta_B}\left(|\mathfrak{d}^{\leq k}A|^2 + |\mathfrak{d}^{\leq k}B|^2\right) + r^4|\mathfrak{d}^{\leq k}\check{P}|^2\\
&+ r^2|\mathfrak{d}^{\leq k}\underline{B}|^2 + |\mathfrak{d}^{\leq k}\underline{A}|^2\Big],
\end{aligned}
\tag{9.21}
$$

where A, B, $\check{P}$, $\underline{B}$, $\underline{A}$ denote the linearized curvature components relative to the outgoing PT frame of ${}^{(ext)}\mathcal{M}$.

9.4.1.2. Norms of ${}^{(ext)}\mathcal{M}$

Definition 9.37. *We define the following norms for the PT-Ricci coefficients of ${}^{(ext)}\mathcal{M}$*

$$
\begin{aligned}
{}^{(ext)}\mathfrak{G}_k^2 \;:=\; & \sup_{\lambda\geq r_0}\int_{r=\lambda}\Big(r^2\big|\mathfrak{d}^{\leq k}(\widehat{X}, \widetilde{trX}, \check{Z})\big|^2 + r^{2-\delta_B}|\mathfrak{d}^{\leq k}\widetilde{tr\underline{X}}|^2\\
&\qquad + \big|\mathfrak{d}^{\leq k}(\widehat{\underline{X}}, \check{H}, \check{\underline{\omega}})\big|^2 + r^{-\delta_B}|\mathfrak{d}^{\leq k}\underline{\Xi}|^2\Big)\\
&+ \sup_{\lambda\geq r_0}\int_{r=\lambda}\Big(\big|\mathfrak{d}^{\leq k}\widetilde{\mathcal{D}\cos\theta}\big|^2 + \big|\mathfrak{d}^{\leq k}\mathcal{D}r\big|^2 + \big|\mathfrak{d}^{\leq k}e_3(\cos\theta)\big|^2\\
&\qquad + r^{-2}\big|\mathfrak{d}^{\leq k}\widetilde{e_3(r)}\big|^2\Big)\\
&+ \sup_{r\geq r_0}\int_{r=\lambda} r^2\Big(|\mathfrak{d}^{\leq k}\mathcal{D}\widehat{\otimes}\mathfrak{J}|^2 + \big|\mathfrak{d}^{\leq k}\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}\big|^2 + \big|\mathfrak{d}^{\leq k}\widetilde{\nabla_3\mathfrak{J}}\big|^2\Big),
\end{aligned}
\tag{9.22}
$$

where $\widetilde{trX}$, $\widehat{X}$, $\check{Z}$, $\check{H}$, $\widetilde{tr\underline{X}}$, $\widehat{\underline{X}}$, $\check{\underline{\omega}}$, $\underline{\Xi}$ are the linearized Ricci coefficients of the outgoing PT frame of ${}^{(ext)}\mathcal{M}$.

Definition 9.38. *We define the following norms for the PT curvature coefficients in* ${}^{(ext)}\mathcal{M}$

$$
{}^{(ext)}\mathfrak{R}_k^2 \;:=\; \int_{{}^{(ext)}\mathcal{M}} \Big(r^{3+\delta_B} |\mathfrak{d}^{\le k}(A,B)|^2 + r^{3-\delta_B} \big(|\mathfrak{d}^{\le k}\check{P}|^2 + r^{-2}|\mathfrak{d}^{\le k}\underline{B}|^2 + r^{-4}|\mathfrak{d}^{\le k}\underline{A}|^2 \big) \Big), \tag{9.23}
$$

where A, B, $\check{P}$, $\underline{B}$, $\underline{A}$ *denote the linearized curvature components relative to the outgoing PT frame of* ${}^{(ext)}\mathcal{M}$.

9.4.1.3. Norms of ${}^{(int)}\mathcal{M}'$ Recall that $r \le r_0$ on ${}^{(int)}\mathcal{M}'$. We thus discard r-weights in the Ricci and curvature norms of ${}^{(int)}\mathcal{M}'$ introduced below.

Definition 9.39. *We define the following norms for the PT-Ricci coefficients of* ${}^{(int)}\mathcal{M}'$

$$
{}^{(int)}\mathfrak{G}_k^2 := \int_{{}^{(int)}\mathcal{M}'} \big|\mathfrak{d}^{\le k}\check{\Gamma}\big|^2,
$$

where $\check{\Gamma}$ *denotes the set of all linearized Ricci and metric coefficients with respect to the ingoing PT frame of* ${}^{(int)}\mathcal{M}'$, *i.e.*

$$
\check{\Gamma} \;:=\; \Big\{ \widecheck{tr\underline{X}}, \widehat{\underline{X}}, \check{Z}, \widecheck{\underline{H}}, \widecheck{\mathcal{D}\cos\theta}, \check{\omega}, \mathcal{D}r, \widecheck{\mathcal{D}u}, e_4(\cos\theta), \widecheck{e_4(r)}, \widecheck{e_4(\underline{u})}, \widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}, \mathcal{D}\widehat{\otimes}\mathfrak{J}, \widecheck{\nabla_4\mathfrak{J}}, \widecheck{trX}, \widehat{X}, \Xi \Big\}.
$$

For the curvature norms in ${}^{(int)}\mathcal{M}'$, we rely in particular on the scalar function τ introduced in Section 9.3.4. We also introduce the following vectorfield[202] in ${}^{(int)}\mathcal{M}'$

$$
\widehat{R} := \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2} e_4 - \frac{\Delta}{r^2+a^2} e_3 \right). \tag{9.24}
$$

Definition 9.40. *We define the following norms for the curvature coefficients in* ${}^{(int)}\mathcal{M}'$

$$
{}^{(int)}\mathfrak{R}_k^2 \;=\; \int_{{}^{(int)}\mathcal{M}'} \Big(\big|\nabla_{\widehat{R}}\mathfrak{d}^{\le k-1}\check{R}\big|^2 + |\mathfrak{d}^{\le k-1}\check{R}|^2 \Big) + \sup_{\tau} \int_{{}^{(int)}\mathcal{M}'\cap\Sigma(\tau)} |\mathfrak{d}^{\le k}\check{R}|^2,
$$

where $\check{R} = \{A, B, \check{P}, \underline{B}, \underline{A}\}$ *is the set of all linearized curvature coefficients w.r.t. the ingoing PT frame of* ${}^{(int)}\mathcal{M}'$. *The derivative* $\nabla_{\widehat{R}}$ *is taken with respect to the vectorfield* $\widehat{R}$ *defined in* (9.24).

[202]In Kerr, we have $\widehat{R} = \frac{\Delta}{r^2+a^2}\partial_r$ in Boyer Lindquist coordinates.

9.4.1.4. Norms of ${}^{(top)}\mathcal{M}'$ In the norms on ${}^{(top)}\mathcal{M}'$ introduced below, we separate ${}^{(top)}\mathcal{M}'$ in ${}^{(top)}\mathcal{M}'(r \leq r_0)$ and ${}^{(top)}\mathcal{M}'(r \geq r_0)$, and we discard r-weights in the region ${}^{(top)}\mathcal{M}'(r \leq r_0)$.

Definition 9.41. *We define the following norms for the Ricci coefficients in ${}^{(top)}\mathcal{M}'$.*

$$
{}^{(top)}\mathfrak{G}_k^2 := \int_{{}^{(top)}\mathcal{M}'(r\leq r_0)} |\mathfrak{d}^{\leq k}\check{\Gamma}|^2 + ({}^{(top)}\mathfrak{G}_k^{\geq r_0})^2 \tag{9.25}
$$

where $\check{\Gamma}$ denotes the set of all linearized Ricci and metric coefficients with respect to the ingoing PT frame of ${}^{(top)}\mathcal{M}'$ as above, and where

(9.26)

$$
\begin{aligned}
({}^{(top)}\mathfrak{G}_k^{\geq r_0})^2 := \sup_{\underline{u}_1\geq u'_*} \int_{{}^{(top)}\mathcal{M}'_{r_0,\underline{u}_1}} & \Big(r^2\big|\mathfrak{d}^{\leq k}(\Xi, \check{\omega}, \widehat{X}, \widetilde{trX}, \check{Z}, \check{\underline{H}})\big|^2 \\
& + r^{2-\delta_B}|\mathfrak{d}^{\leq k}\widetilde{tr\underline{X}}| + \big|\mathfrak{d}^{\leq k}\widehat{\underline{X}}\big|^2\Big) \\
+ \sup_{\underline{u}_1\geq u'_*} \int_{{}^{(top)}\mathcal{M}'_{r_0,\underline{u}_1}} & \Big(\big|\mathfrak{d}^{\leq k}\widetilde{\mathcal{D}\cos\theta}\big|^2 + \big|\mathfrak{d}^{\leq k}\mathcal{D}r\big|^2\Big) \\
+ \sup_{\underline{u}_1\geq u'_*} \int_{{}^{(top)}\mathcal{M}'_{r_0,\underline{u}_1}} & \Big(r^4\big|\mathfrak{d}^{\leq k}e_4(\cos\theta)\big|^2 + r^4\big|\mathfrak{d}^{\leq k}\widetilde{e_4(r)}\big|^2\Big) \\
+ \sup_{\underline{u}_1\geq u'_*} \int_{{}^{(top)}\mathcal{M}'_{r_0,\underline{u}_1}} & r^2\Big(|\mathfrak{d}^{\leq k}\mathcal{D}\widehat{\otimes}\mathfrak{J}|^2 + |\mathfrak{d}^{\leq k}\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}|^2\Big) + r^6\big|\mathfrak{d}^{\leq k}\widetilde{\nabla_4\mathfrak{J}}\big|^2\Big),
\end{aligned}
$$

with Ξ, $\check{\omega}$, $\widetilde{trX}$, $\widehat{X}$, $\check{Z}$, $\check{\underline{H}}$, $\widetilde{tr\underline{X}}$, $\widehat{\underline{X}}$ the linearized Ricci coefficients of the ingoing PT frame of ${}^{(top)}\mathcal{M}'$, and with the notation

$$
{}^{(top)}\mathcal{M}'_{r_0,\underline{u}_1} := {}^{(top)}\mathcal{M}'(r \geq r_0) \cap \{\underline{u}_1 \leq \underline{u} \leq \underline{u}_1 + 1\}. \tag{9.27}
$$

For the curvature norms in ${}^{(top)}\mathcal{M}'$, we rely in particular on the scalar function τ introduced in Section 9.3.4.

Definition 9.42. *We define the following norms for the curvature coefficients in ${}^{(top)}\mathcal{M}'$.*

$$
\begin{aligned}
{}^{(top)}\mathfrak{R}_k^2 := & \int_{{}^{(top)}\mathcal{M}(r\geq r_0)} \Big(r^{3+\delta_B}|\mathfrak{d}^{\leq k}(A,B)|^2 + r^{3-\delta_B}|\mathfrak{d}^{\leq k}\check{P}|^2\Big) \\
& + \sup_{\underline{u}_1\geq u'_*} \int_{{}^{(top)}\mathcal{M}_{r_0,\underline{u}_1}} \Big(r^2|\mathfrak{d}^{\leq k}\underline{B}|^2 + |\mathfrak{d}^{\leq k}\underline{A}|^2\Big) \\
& + \sup_{\tau} \int_{{}^{(top)}\mathcal{M}'(r\leq r_0)\cap\Sigma(\tau)} |\mathfrak{d}^{\leq k}\check{R}|^2,
\end{aligned} \tag{9.28}
$$

where $\check{R}$ *is the set of all linearized curvature coefficients w.r.t. the ingoing PT frame of* ${}^{(top)}\mathcal{M}'$ *as above, and* A, B, $\check{P}$, $\underline{B}$, $\underline{A}$ *denote the linearized curvature components relative to the ingoing PT frame of* ${}^{(top)}\mathcal{M}'$*.*

9.4.1.5. Global norms We define the global norms for the PT frames of $\mathcal{M}$ as follows

$$\begin{aligned} \mathfrak{G}_k &= {}^\star\mathfrak{G}_k + {}^{(ext)}\mathfrak{G}_k + {}^{(int)}\mathfrak{G}_k + {}^{(top)}\mathfrak{G}, \\ \mathfrak{R}_k &= {}^\star\mathfrak{R}_k + {}^{(ext)}\mathfrak{R}_k + {}^{(int)}\mathfrak{R}_k + {}^{(top)}\mathfrak{R}. \end{aligned} \tag{9.29}$$

9.4.1.6. Initial data norms Given $u, \underline{u}$ defined respectively relative to the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ and the ingoing PT frame of ${}^{(int)}\mathcal{M}'$, we define

$$\mathcal{B}_1 := \{u = 1\}, \qquad \underline{\mathcal{B}}_1 := \{\underline{u} = 1\}. \tag{9.30}$$

Definition 9.43. *We define the following initial data norms on* $\mathcal{B}_1 \cup \underline{\mathcal{B}}_1$

(9.31)

$$\begin{aligned} {}^{(PT-ext)}\mathfrak{I}_k &:= \sup_{S\subset\mathcal{B}_1} r^{\frac{5}{2}+\delta_B}\Big(\|\mathfrak{d}^k\, {}^{(ext)}A\|_{L^2(S)} + \|\mathfrak{d}^k\, {}^{(ext)}B\|_{L^2(S)}\Big) \\ &\quad + \sup_{S\subset\mathcal{B}_1}\Big(r^2\|\mathfrak{d}^k\, {}^{(ext)}\check{P}\|_{L^2(S)} + r\|\mathfrak{d}^k\, {}^{(ext)}\underline{B}\|_{L^2(S)} + \|\mathfrak{d}^k\, {}^{(ext)}\underline{A}\|_{L^2(S)}\Big), \\ {}^{(PT-int)}\mathfrak{I}_k &:= \sup_{S\subset\underline{\mathcal{B}}_1}\Big(\|\mathfrak{d}^k\, {}^{(int)}A\|_{L^2(S)} + \|\mathfrak{d}^k\, {}^{(int)}B\|_{L^2(S)}\Big) \\ &\quad + \sup_{S\subset\underline{\mathcal{B}}_1}\Big(\|\mathfrak{d}^k\, {}^{(int)}\check{P}\|_{L^2(S)} + r\|\mathfrak{d}^k\, {}^{(int)}\underline{B}\|_{L^2(S)} + \|\mathfrak{d}^k\, {}^{(int)}\underline{A}\|_{L^2(S)}\Big), \end{aligned}$$

where the linearized curvature components are taken respectively w.r.t. the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ *on* $\mathcal{B}_1$ *and the ingoing PT frame of* ${}^{(int)}\mathcal{M}'$ *on* $\underline{\mathcal{B}}_1$*. We also set*

$${}^{(PT)}\mathfrak{I}_k := {}^{(PT-ext)}\mathfrak{I}_k + {}^{(PT-int)}\mathfrak{I}_k. \tag{9.32}$$

9.4.2. Statement of the Main PT-Theorem We are now ready to state the main result of this chapter on the control of the PT structures of $\mathcal{M}$.

Theorem 9.44 (Main PT-Theorem)**.** *Consider a GCM admissible spacetime verifying the initial data assumptions of the Main Theorem in Section 3.4.3, i.e.*

$$\mathfrak{I}_{k_{large}+10} \le \epsilon_0, \qquad {}^{(ext)}\mathfrak{I}_3 \le \epsilon_0^2.$$

Then, relative to the global norms defined in Section 9.4.1 for the PT frames *of $\mathcal{M}$, we have the following bounds*

$$\mathfrak{G}_k + \mathfrak{R}_k \lesssim \epsilon_0, \qquad k \leq k_{large} + 7. \tag{9.33}$$

9.4.3. Proof of Theorem M8 Using Theorem 9.44, we are ready to prove Theorem M8, stated in Section 3.7.2. We proceed in several steps.

Step 1. Let $(f, \underline{f}, \lambda)$ denote the transition coefficients corresponding to the change from the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ to the outgoing PG frame of ${}^{(ext)}\mathcal{M}$. Also, we denote the quantities corresponding to the outgoing PT frame without primes, and the quantities corresponding to the outgoing PG frame with primes. In view of Lemma 9.24, the following identities hold in ${}^{(ext)}\mathcal{M}$:

1. We have $f = 0$ and $\lambda = 1$. In particular, we have
$$e_4' = e_4.$$
2. We have
$$r' = r, \qquad \theta' = \theta, \qquad q' = q, \qquad \mathfrak{J}' = \mathfrak{J}.$$
3. With the notation $\underline{F} = \underline{f} + i\,{}^*\underline{f}$, we have
$$\nabla_4\underline{F} + \frac{1}{2}\mathrm{tr}X\underline{F} \;\; = \;\; -2\check{Z} - \underline{F}\cdot\widehat{\chi}.$$

Also, note from the initialization of the outgoing PG frame of ${}^{(ext)}\mathcal{M}$ on Σ_* of Section 3.2.5, and the initialization of the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ on Σ_* of Section 9.1.3, that we have on Σ_*

$$\underline{f} \;\; = \;\; -\frac{(\nu(r) - b_*)}{1 - \frac{1}{4}b_*\frac{a^2(\sin\theta)^2}{r^2}}\frac{a}{r}f_0 - \frac{a\Upsilon}{r}f_0.$$

We infer on Σ_*

$$\underline{f} \;\; = \;\; -\frac{a}{r}\left(\frac{(\nu(r)+2) - \left(b_* + 1 + \frac{2m}{r}\right)}{1 - \frac{1}{4}b_*\frac{a^2(\sin\theta)^2}{r^2}} + O(r^{-2})\right)f_0. \tag{9.34}$$

Step 2. We first control the $L^2(\Sigma_*)$ norm of $\check{H}'$ on Σ_* as this quantity is part of the boundedness norm for the outgoing PG frame of ${}^{(ext)}\mathcal{M}$, see (3.37). In view of the transformation formula for η' in Proposition 2.12, together with the fact that $f = 0$ and $\lambda = 1$, we have

$$\eta' \;=\; \eta + \frac{1}{4}\underline{f}\mathrm{tr}\,\chi - \frac{1}{4}\,{}^*\underline{f}\,{}^{(a)}\mathrm{tr}\chi + \frac{1}{2}\underline{f}\cdot\widehat{\chi}.$$

Since $r' = r$, $\theta' = \theta$, $q' = q$, and $\mathfrak{J}' = \mathfrak{J}$ in ${}^{(ext)}\mathcal{M}$, we infer

$$\check{\eta}' \;=\; \check{\eta} + \frac{1}{4}\underline{f}\mathrm{tr}\,\chi - \frac{1}{4}\,{}^*\underline{f}\,{}^{(a)}\mathrm{tr}\chi + \frac{1}{2}\underline{f}\cdot\widehat{\chi}$$

and hence

$$\|\mathfrak{d}^{\leq k}\check{H}'\|_{L^2(\Sigma_*)} \;\lesssim\; \|\mathfrak{d}^{\leq k}\check{H}\|_{L^2(\Sigma_*)} + \left\|\mathfrak{d}^{\leq k}\left(\underline{f}(\mathrm{tr}\,\chi,\,{}^{(a)}\mathrm{tr}\chi,\widehat{\chi})\right)\right\|_{L^2(\Sigma_*)}.$$

Together with the control of the PT frames provided by Theorem 9.44, we infer

$$\|\mathfrak{d}^{\leq k_{large}+7}\check{H}'\|_{L^2(\Sigma_*)} \;\lesssim\; \epsilon_0 + \left\|r^{-1}\mathfrak{d}^{\leq k_{large}+7}\underline{f}\right\|_{L^2(\Sigma_*)}.$$

Using the formula for $\underline{f}$ on Σ_* of Step 1, and using again the control of the PT frames provided by Theorem 9.44, we obtain

$$\|\mathfrak{d}^{\leq k_{large}+7}\check{H}'\|_{L^2(\Sigma_*)} \;\lesssim\; \epsilon_0 + \left\|\mathfrak{d}^{\leq k_{large}+7}O(r^{-3})\right\|_{L^2(\Sigma_*)}.$$

In view of the dominance condition (3.50) for r on Σ_*, we deduce

$$\|\mathfrak{d}^{\leq k_{large}+7}\check{H}'\|_{L^2(\Sigma_*)} \;\lesssim\; \epsilon_0 \tag{9.35}$$

which yields the desired behavior for $\check{H}'$ on Σ_*.

Step 3. The remaining estimates for the PG frames of $\mathcal{M}$ being all in sup norm, we first derive sup norms estimates for the PT frames of $\mathcal{M}$. In view of the control of the PT frames provided by Theorem 9.44, together with Sobolev and the trace theorem, we obtain, for $k \leq k_{large}+4$,

$$\begin{aligned}
&\sup_{{}^{(ext)}\mathcal{M}}\left\{r^2|\mathfrak{d}^{\leq k}\Gamma_g| + r|\mathfrak{d}^{\leq k}\Gamma_b| + r^{\frac{7}{2}+\frac{\delta_B}{2}}\left(|\mathfrak{d}^{\leq k}A| + |\mathfrak{d}^{\leq k}B|\right)\right\}\\
&+\sup_{{}^{(top)}\mathcal{M}'}\left\{r^2|\mathfrak{d}^{\leq k}\Gamma_g| + r|\mathfrak{d}^{\leq k}\Gamma_b| + r^{\frac{7}{2}+\frac{\delta_B}{2}}\left(|\mathfrak{d}^{\leq k}A| + |\mathfrak{d}^{\leq k}B|\right)\right\}\\
&+\sup_{{}^{(int)}\mathcal{M}'}\left\{|\mathfrak{d}^{\leq k}\Gamma_g| + |\mathfrak{d}^{\leq k}\Gamma_b|\right\}\\
&\lesssim \epsilon_0,
\end{aligned} \tag{9.36}$$

where in each case, (Γ_g, Γ_b) is defined w.r.t. the linearized quantities in the PT frame of the corresponding region.

Remark 9.45. *In view of the definition of* ${}^{(ext)}\mathfrak{G}_k$ *and* ${}^{(top)}\mathfrak{G}_k$, (9.36) *holds a priori only for* $\Gamma_g \setminus \{\widecheck{tr\underline{X}}\}$ *and* $\Gamma_b \setminus \{\underline{\Xi}\}$, *while* $\widecheck{tr\underline{X}}$ *and* $\underline{\Xi}$ *satisfy a priori the following weaker estimates (in terms of powers of* r*) in* ${}^{(ext)}\mathcal{M}$ *and* ${}^{(top)}\mathcal{M}$, *for* $k \leq k_{large} + 4$,

$$\sup_{{}^{(ext)}\mathcal{M}} \left(r^{2-\frac{\delta_B}{2}} |\mathfrak{d}^{\leq k} \widecheck{tr\underline{X}}| + r^{1-\frac{\delta_B}{2}} |\mathfrak{d}^{\leq k} \underline{\Xi}| \right) + \sup_{{}^{(top)}\mathcal{M}} r^{2-\frac{\delta_B}{2}} |\mathfrak{d}^{\leq k} \widecheck{tr\underline{X}}| \lesssim \epsilon_0.$$

To recover the claimed estimates for $\widecheck{tr\underline{X}}$ *and* $\underline{\Xi}$ *of* (9.36) *in* ${}^{(ext)}\mathcal{M}$, *it suffices to integrate the transport equations for* $\nabla_4 \widecheck{tr\underline{X}}$ *and* $\nabla_4 \underline{\Xi}$ *of Proposition 9.19, using the control provided by* (9.36) *for* $\Gamma_g \setminus \{\widecheck{tr\underline{X}}\}$ *and* $\Gamma_b \setminus \{\underline{\Xi}\}$ *for the RHS, and the control of* $\widecheck{tr\underline{X}}$ *and* $\underline{\Xi}$ *on* Σ_* *(which has no loss in* r*). Finally, integrating along the* e_3 *direction in* ${}^{(top)}\mathcal{M}$ *the equation for* $\nabla_3 \widecheck{tr\underline{X}}$ *yields the desired claim* (9.36) *also for* $\widecheck{tr\underline{X}}$ *in* ${}^{(top)}\mathcal{M}$ *so that* (9.36) *indeed holds true.*

We will also need a sharper estimate for $\check{Z}$ on ${}^{(ext)}\mathcal{M}$. Recall from Proposition 9.19 that $\check{Z}$, in the outgoing PT frame of ${}^{(ext)}\mathcal{M}$, satisfies on ${}^{(ext)}\mathcal{M}$

$$\nabla_4 \check{Z} + \frac{1}{q}\check{Z} = -\frac{a\overline{q}}{|q|^2}\widecheck{\mathrm{tr}X}\mathfrak{J} - \frac{aq}{|q|^2}\overline{\mathfrak{J}} \cdot \widehat{X} - B + \Gamma_g \cdot \Gamma_g.$$

We infer from the above control, on Σ_*,

$$\left|\mathfrak{d}^{\leq k_{large}+4}(q\check{Z})\right| \lesssim \frac{\epsilon_0}{r^{\frac{5}{2}}}.$$

Integrating from Σ_*, we infer the following improved bound on ${}^{(ext)}\mathcal{M}$

$$\left|\mathfrak{d}^{\leq k_{large}+4}\check{Z}\right| \lesssim \frac{\epsilon_0}{r_*^2} + \frac{\epsilon_0}{r^{\frac{5}{2}}}. \tag{9.37}$$

Step 4. Next, we estimate $\underline{f}$ on ${}^{(ext)}\mathcal{M}$. First, in view of the identity for $\underline{f}$ derived on Σ_* in Step 1, together with the control of Step 3 for the PT frame of ${}^{(ext)}\mathcal{M}$, and the dominant condition for r on Σ_*, we have

$$\sup_{\Sigma_*} r|\mathfrak{d}^{\leq k_{large}+4}\underline{f}| \lesssim \epsilon_0.$$

Also, recall from Step 1 the following transport equation on ${}^{(ext)}\mathcal{M}$

$$\nabla_4 \underline{F} + \frac{1}{2}\mathrm{tr}X\underline{F} = -2\check{Z} - \underline{F} \cdot \widehat{\chi},$$

where $\underline{F} = \underline{f} + i\, {}^*\underline{f}$. Together with the control of the PT frames provided by Step 3, we infer

$$\nabla_4\underline{F} + \frac{1}{2}\mathrm{tr}X\underline{F} \;=\; -2\check{\underline{Z}} - \underline{F}\cdot\widehat{\chi}.$$

In view of the bounds provided by Step 3 for the outgoing PT frame of ${}^{(ext)}\mathcal{M}$, using in particular the improved bound for $\check{\underline{Z}}$, we infer, on ${}^{(ext)}\mathcal{M}$,

$$\left|\mathfrak{d}^{\leq k_{large}+4}\nabla_4(q\underline{F})\right| \;=\; \frac{\epsilon_0}{r_*} + \frac{\epsilon_0}{r^{\frac{3}{2}}} + \frac{\epsilon_0}{r^2}\left|\mathfrak{d}^{\leq k_{large}+4}(q\underline{F})\right|.$$

Integrating from Σ_*, and using the above control for $\underline{f}$ on Σ_*, we infer, for ϵ_0 small enough,

$$\left|\mathfrak{d}^{\leq k_{large}+4}\underline{F}\right| \;\lesssim\; \frac{r_*-r}{rr_*}\epsilon_0 + \frac{\epsilon_0}{r^{\frac{3}{2}}}$$

and hence

$$\sup_{{}^{(ext)}\mathcal{M}} r\left|\mathfrak{d}^{\leq k_{large}+4}\underline{f}\right| \;\lesssim\; \epsilon_0. \tag{9.38}$$

Step 5. Next, let $(f', \underline{f}', \lambda')$ denote the transition coefficients corresponding to the change from the ingoing PT frame of ${}^{(top)}\mathcal{M}'$ to the ingoing PG frame of ${}^{(top)}\mathcal{M}$. Since ${}^{(top)}\mathcal{M} \subset {}^{(top)}\mathcal{M}'$, $(f', \underline{f}', \lambda')$ are defined on ${}^{(top)}\mathcal{M}$. In view of

1. The fact that $\{u = u_*\} = {}^{(top)}\mathcal{M} \cap {}^{(ext)}\mathcal{M}$.
2. The fact that $\lambda = 1$, $f = 0$, and $\underline{f}$ is controlled on ${}^{(ext)}\mathcal{M}$ in Step 4, where $(f, \underline{f}, \lambda)$ denote the transition coefficients corresponding to the change from the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ to the outgoing PG frame of ${}^{(ext)}\mathcal{M}$.
3. The initialization of the ingoing PG frame of ${}^{(top)}\mathcal{M}$ from the ingoing PG frame of ${}^{(ext)}\mathcal{M}$ in Section 3.2.5.
4. The control in ${}^{(ext)}\mathcal{M} \cap {}^{(top)}\mathcal{M}'$ of the change of frame coefficients between the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ and the ingoing PT frame of ${}^{(top)}\mathcal{M}'$ which results from
   (a) the initialization of the ingoing PT frame of ${}^{(top)}\mathcal{M}'$ from the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ on $\{u = u'_*\}$ in Section 9.1.3,
   (b) the transport equations for transition coefficients involving PT frames in Section 2.8.4,

we easily obtain

$$\sup_{\{u=u_*\}} r\left|\mathfrak{d}^{\leq k_{large}+4}(f', \underline{f}', \log(\lambda'))\right| \lesssim \epsilon_0.$$

Next, using the analog of Corollary 2.14 for ingoing foliations, and the fact that both the ingoing PT frame of ${}^{(top)}\mathcal{M}'$ and the ingoing PG frame of ${}^{(top)}\mathcal{M}$ verify $\underline{\xi} = \underline{\omega} = 0$, we obtain the following transport equations

$$\begin{aligned}\nabla_{\lambda'^{-1}e_3'}\underline{F}' + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\underline{F}' &= -\widehat{\underline{\chi}}\cdot\underline{F}' + \underline{E}_1(\underline{f}', \Gamma),\\ \lambda'^{-1}\nabla_3'(\log\lambda') &= \underline{f}'\cdot(-\zeta-\underline{\eta}) + \underline{E}_2(\underline{f}', \Gamma),\end{aligned}$$

where $\underline{F}' = \underline{f}' + i\,{}^*\underline{f}'$, and where the Ricci coefficients appearing are the ones of the ingoing PT frame of ${}^{(top)}\mathcal{M}'$. Integrating both transport equations from $\{u = u_*\}$ starting with the one for $\underline{f}'$, using the control of the ingoing PT frame of ${}^{(top)}\mathcal{M}'$ provided by Step 3, and the above control for $(\underline{f}', \lambda')$ on $\{u = u_*\}$, we obtain

$$\sup_{{}^{(top)}\mathcal{M}} r\left|\mathfrak{d}^{\leq k_{large}+4}(\underline{f}', \log(\lambda'))\right| \lesssim \epsilon_0.$$

Also, using again the analog of Corollary 2.14 for ingoing foliation, we have

$$\begin{aligned}\nabla_{\lambda'^{-1}e_3'}F' + \frac{1}{2}\mathrm{tr}\underline{X}F' &= -2(H-Z) + 2\mathcal{D}'(\log\lambda') + 2\omega F'\\ &\quad + E_3(\nabla'^{\leq 1}\underline{f}', f', \Gamma, \lambda'^{-1}\underline{\chi}'),\end{aligned}$$

where $F' = f' + i\,{}^*f'$, and where the Ricci coefficients appearing are the ones of the ingoing PT frame of ${}^{(top)}\mathcal{M}'$. Now, we have $Z - H = \check{Z}$ in view of the fact that $H = \frac{aq}{|q|^2}\mathfrak{J}$ for ingoing PT foliations, and hence

$$\nabla_{\lambda'^{-1}e_3'}F' + \frac{1}{2}\mathrm{tr}\underline{X}F' = -2\check{Z} + 2\mathcal{D}'(\log\lambda') + 2\omega F' + E_3(\nabla'^{\leq 1}\underline{f}', f', \Gamma, \lambda'^{-1}\underline{\chi}').$$

Integrating the transport equations from $\{u = u_*\}$, using the control of the ingoing PT frame of ${}^{(top)}\mathcal{M}'$ provided by Step 3, the above control of $(\underline{f}', \lambda')$ on ${}^{(top)}\mathcal{M}$, and the above control for f' on $\{u = u_*\}$, we obtain

$$\sup_{{}^{(top)}\mathcal{M}} r\left|\mathfrak{d}^{\leq k_{large}+3}f'\right| \lesssim \epsilon_0.$$

Step 6. We are now ready to conclude the proof of Theorem M8. First, let $(f'', \underline{f}'', \lambda'')$ denote the transition coefficients corresponding to the change from the ingoing PT frame of ${}^{(int)}\mathcal{M}'$ to the ingoing PG frame of ${}^{(int)}\mathcal{M}$. Starting from the timelike hypersurface ${}^{(ext)}\mathcal{M} \cap {}^{(int)}\mathcal{M} = \{r = r_0\} \cap \{u \leq u_*\}$, and arguing as in Step 5, one easily obtains

$$\sup_{{}^{(int)}\mathcal{M}} \Big(\Big|\mathfrak{d}^{\leq k_{large}+4}(\underline{f}'', \log(\lambda''))\Big| + \Big|\mathfrak{d}^{\leq k_{large}+3} f''\Big| \Big) \lesssim \epsilon_0.$$

Next, relying on:

1. the fact that $\lambda = 1$ and $f = 0$ on ${}^{(ext)}\mathcal{M}$ according to Step 1,
2. the control of $\underline{f}$ on ${}^{(ext)}\mathcal{M}$ in Step 4,
3. the control of $(f', \underline{f}', \lambda')$ on ${}^{(top)}\mathcal{M}$ in Step 5,
4. the above control of $(f'', \underline{f}'', \lambda'')$ on ${}^{(int)}\mathcal{M}$,
5. the sup norm control of the PT structures of $\mathcal{M}$ in Step 3,
6. the change of frame formulas of Proposition 2.12,

we infer, for $k \leq k_{large} + 2$,

$$\begin{aligned}
& \sup_{{}^{(ext)}\mathcal{M}} \Big\{ r^2|\mathfrak{d}^{\leq k}\Gamma_g'| + r|\mathfrak{d}^{\leq k}\Gamma_b'| + r^{\frac{7}{2}+\frac{\delta_B}{2}}\left(|\mathfrak{d}^{\leq k}A'| + |\mathfrak{d}^{\leq k}B'|\right)\Big\} \\
& + \sup_{{}^{(top)}\mathcal{M}} \Big\{ r^2|\mathfrak{d}^{\leq k}\Gamma_g'| + r|\mathfrak{d}^{\leq k}\Gamma_b'| + r^{\frac{7}{2}+\frac{\delta_B}{2}}\left(|\mathfrak{d}^{\leq k}A'| + |\mathfrak{d}^{\leq k}B'|\right)\Big\} \\
& + \sup_{{}^{(int)}\mathcal{M}} \Big\{ |\mathfrak{d}^{\leq k}\Gamma_g'| + |\mathfrak{d}^{\leq k}\Gamma_b'| \Big\} \\
\lesssim\ & \epsilon_0 + \sup_{{}^{(ext)}\mathcal{M}} r\left|\mathfrak{d}^{\leq k+1}\Big(r' - r, \cos(\theta') - \cos(\theta), q' - q, r(\mathfrak{J}' - \mathfrak{J})\Big)\right| \\
& + \sup_{{}^{(top)}\mathcal{M}} r\left|\mathfrak{d}^{\leq k+1}\Big(r' - r, \cos(\theta') - \cos(\theta), q' - q, r(\mathfrak{J}' - \mathfrak{J})\Big)\right| \\
& + \sup_{{}^{(int)}\mathcal{M}} \left|\mathfrak{d}^{\leq k+1}\Big(r' - r, \cos(\theta') - \cos(\theta), q' - q, \mathfrak{J}' - \mathfrak{J}\Big)\right|
\end{aligned}$$

where in each case, (Γ_g', Γ_b') is defined w.r.t. the linearized quantities in the PG frame of the corresponding region, and $r' - r$, $\cos(\theta') - \cos(\theta)$, $q' - q$ and $\mathfrak{J}' - \mathfrak{J}$ correspond to the difference between the un-primed quantity in the PT frame and the primed quantity in the PG frame of the corresponding region. In view of the definition of the combined sup norm $\mathfrak{N}_k^{(Sup)}$ for the PG structures of ${}^{(ext)}\mathcal{M}$, ${}^{(int)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$, see Section 3.3.5, and using also the estimate for $\check{H}'$ on Σ_* derived in Step 2, this yields

$$\mathfrak{N}^{(Sup)}_{k_{large}+2} \lesssim \epsilon_0$$

$$
\begin{aligned}
&+\sup_{{}^{(ext)}\mathcal{M}} r\left|\mathfrak{d}^{\leq k_{large}+3}\Big(r'-r, \cos(\theta')-\cos(\theta), q'-q, r(\mathfrak{J}'-\mathfrak{J})\Big)\right|\\
&+\sup_{{}^{(top)}\mathcal{M}} r\left|\mathfrak{d}^{\leq k_{large}+3}\Big(r'-r, \cos(\theta')-\cos(\theta), q'-q, r(\mathfrak{J}'-\mathfrak{J})\Big)\right|\\
&+\sup_{{}^{(int)}\mathcal{M}} \left|\mathfrak{d}^{\leq k_{large}+3}\Big(r'-r, \cos(\theta')-\cos(\theta), q'-q, \mathfrak{J}'-\mathfrak{J}\Big)\right|.
\end{aligned}
$$

Next, recall from Step 1 that we have on ${}^{(ext)}\mathcal{M}$

$$
r'=r, \qquad \theta'=\theta, \qquad q'=q, \qquad \mathfrak{J}'=\mathfrak{J}.
$$

Hence, we obtain

$$
\begin{aligned}
\mathfrak{N}^{(Sup)}_{k_{large}+2} \lesssim\ & \epsilon_0\\
&+\sup_{{}^{(top)}\mathcal{M}} r\left|\mathfrak{d}^{\leq k_{large}+3}\Big(r'-r, \cos(\theta')-\cos(\theta), q'-q, r(\mathfrak{J}'-\mathfrak{J})\Big)\right|\\
&+\sup_{{}^{(int)}\mathcal{M}} \left|\mathfrak{d}^{\leq k_{large}+3}\Big(r'-r, \cos(\theta')-\cos(\theta), q'-q, \mathfrak{J}'-\mathfrak{J}\Big)\right|.
\end{aligned}
$$

It thus remains to control the quantities $r'-r$, $\cos(\theta')-\cos(\theta)$, $q'-q$ and $\mathfrak{J}'-\mathfrak{J}$ on ${}^{(top)}\mathcal{M}$ and ${}^{(int)}\mathcal{M}$. Denoting in each region by (e_1', e_2', e_3', e_4') the corresponding ingoing PG frame, and by (e_1, e_2, e_3, e_4) the corresponding ingoing PT frame, we have

$$
\begin{aligned}
&e_3(r)=-1, \qquad e_3(\theta)=0, \qquad e_3(q)=-1, \qquad \nabla_3(\overline{q}\mathfrak{J})=0,\\
&e_3'(r')=-1, \qquad e_3'(\theta')=0, \qquad e_3'(q')=-1, \qquad \nabla_3'(\overline{q'}\mathfrak{J}')=0.
\end{aligned}
$$

Expressing e_3' on the frame (e_1, e_2, e_3, e_4) using the fame transformation formulas, using the control of $(f', \underline{f}', \lambda')$ on ${}^{(top)}\mathcal{M}$ in Step 5, and the above control of $(f'', \underline{f}'', \lambda'')$ on ${}^{(int)}\mathcal{M}$, we easily infer

$$
\begin{aligned}
&\sup_{{}^{(top)}\mathcal{M}} r\left|\mathfrak{d}^{\leq k_{large}+3}\Big(e_3'(r'-r), e_3'(\cos(\theta')-\cos(\theta)), e_3'(q'-q), \nabla_3'(q'(\mathfrak{J}'-\mathfrak{J}))\Big)\right|\\
&+\sup_{{}^{(int)}\mathcal{M}} \left|\mathfrak{d}^{\leq k_{large}+3}\Big(e_3'(r'-r), e_3'(\cos(\theta')-\cos(\theta)), e_3'(q'-q), \nabla_3'(q'(\mathfrak{J}'-\mathfrak{J}))\Big)\right|\\
&\lesssim \epsilon_0.
\end{aligned}
$$

Together with the fact that we have on ${}^{(ext)}\mathcal{M}$

$$
r'=r, \qquad \theta'=\theta, \qquad q'=q, \qquad \mathfrak{J}'=\mathfrak{J},
$$

we infer, integrating the transport equations in e_3' from ${}^{(ext)}\mathcal{M} \cap {}^{(top)}\mathcal{M} = \{u = u_*\}$ to ${}^{(top)}\mathcal{M}$ and from ${}^{(ext)}\mathcal{M} \cap {}^{(int)}\mathcal{M} = \{r = r_0\}$ to ${}^{(int)}\mathcal{M}$,

$$\sup_{{}^{(top)}\mathcal{M}} r\left|\mathfrak{d}^{\leq k_{large}+3}\Big(r'-r, \cos(\theta')-\cos(\theta), q'-q, r(\mathfrak{J}'-\mathfrak{J})\Big)\right| + \sup_{{}^{(int)}\mathcal{M}}\left|\mathfrak{d}^{\leq k_{large}+3}\Big(r'-r, \cos(\theta')-\cos(\theta), q'-q, \mathfrak{J}'-\mathfrak{J}\Big)\right| \lesssim \epsilon_0.$$

Plugging in the above, we deduce

$$\mathfrak{N}^{(Sup)}_{k_{large}+2} \lesssim \epsilon_0$$

as desired. This concludes the proof of Theorem M8.

9.4.4. Bootstrap assumptions for the Main PT-Theorem The non-linear error terms in the proof of Theorem 9.44 may all be controlled by the standard estimate

$$\begin{aligned} \mathrm{Err}_{k_{large}+7} &\lesssim \left(\mathfrak{R}_{\frac{k_{large}+7}{2}} + \mathfrak{G}_{\frac{k_{large}+7}{2}}\right)(\mathfrak{R}_{k_{large}+7} + \mathfrak{G}_{k_{large}+7}) \\ &\lesssim \epsilon(\mathfrak{R}_{k_{large}+7} + \mathfrak{G}_{k_{large}+7}). \end{aligned} \tag{9.39}$$

The systematic use of (9.39) would result in carrying the term $\epsilon(\mathfrak{R}_{k_{large}+7} + \mathfrak{G}_{k_{large}+7})$ on the RHS of all estimates throughout sections 9.6–9.10. Thus, to lighten notations, it will be convenient to make instead the following assumption which, by abuse of language, we call **BA-T** bootstrap assumptions[203]:

BA-PT. Relative to the global norms defined in Section 9.4.1 for the PT frames of $\mathcal{M}$, we have

$$\mathfrak{G}_k + \mathfrak{R}_k \leq \epsilon, \qquad k \leq k_{large} + 7. \tag{9.40}$$

9.4.5. Control of the initial data in the PT frames Recall that the control of the initial data for the PG frames of $\mathcal{M}$ is provided by Theorem M0, see Section 3.7.1. We will need a sharper[204] analog for the PT frames of $\mathcal{M}$ which we state below.

[203] In view of (9.40), the error terms appearing in sections 9.6–9.10 will be simply estimated by $O(\epsilon^2)$ and hence by $O(\epsilon_0)$.

[204] Sharper in terms of derivatives. Indeed, the conclusions of Theorem M0 hold for $k \leq k_{large} - 2$ while the ones of Theorem M0-PT hold for $k \leq k_{large} + 7$.

Theorem 9.46 (Theorem M0-PT)**.** *Assume that the initial data layer $\mathcal{L}_0$, as defined in Section 3.1, satisfies*

$$\mathfrak{I}_{k_{large}+10} \leq \epsilon_0, \qquad {}^{(ext)}\mathfrak{I}_3 \leq \epsilon_0^2.$$

Then, under the bootstrap assumptions **BA-PT***, relative to the initial data norms defined in Section 9.4.1 for the PT frames of $\mathcal{M}$, we have*

$$ {}^{(PT)}\mathfrak{I}_k \lesssim \epsilon_0, \qquad k \leq k_{large} + 7. \tag{9.41}$$

The proof of Theorem M0-PT is postponed to Section 9.5.

9.4.6. Control of low derivatives of the PT frame The following lemma will allow us to initiate an iterative procedure in Section 9.4.7.

Lemma 9.47. *Relative to the global norms defined in Section 9.4.1 for the PT frames of $\mathcal{M}$, we have the following bounds*

$$\mathfrak{G}_{k_{small}-1} + \mathfrak{R}_{k_{small}-1} \lesssim \epsilon_0. \tag{9.42}$$

In addition, we have, for $k \leq k_{small} - 1$,

$$\begin{aligned}&\sup_{{}^{(ext)}\mathcal{M}\cup{}^{(top)}\mathcal{M}'(r\geq r_0)} \Big\{\big(ru^{\frac{1}{2}+\delta_{dec}} + u^{1+\frac{3\delta_{dec}}{4}}\big)|\mathfrak{d}^{\leq k}\Gamma_g| + ru^{1+\frac{3\delta_{dec}}{4}}|\mathfrak{d}^{\leq k}\Gamma_b|\Big\}\\ &\qquad + \sup_{{}^{(top)}\mathcal{M}'(r\leq r_0)\cup{}^{(int)}\mathcal{M}'} \underline{u}^{1+\frac{3\delta_{dec}}{4}}\Big\{|\mathfrak{d}^{\leq k}\Gamma_g| + |\mathfrak{d}^{\leq k}\Gamma_b|\Big\} \lesssim \epsilon_0,\end{aligned} \tag{9.43}$$

where in each case, (Γ_g, Γ_b) is defined w.r.t. the linearized quantities in the PT frame of the corresponding region.

Proof. The PG structures of $\mathcal{M}$ satisfy in view of Theorem M7, stated in Section 3.7.2, the following decay estimates, for $k \leq k_{small}$,

$$\begin{aligned}&\sup_{{}^{(ext)}\mathcal{M}\cup{}^{(top)}\mathcal{M}(r\geq r_0)} \Big\{\big(r^2u^{\frac{1}{2}+\delta_{dec}} + ru^{1+\delta_{dec}}\big)|\mathfrak{d}^{\leq k}\Gamma_g| + ru^{1+\delta_{dec}}|\mathfrak{d}^{\leq k}\Gamma_b|\Big\}\\ &\qquad + \sup_{{}^{(top)}\mathcal{M}(r\leq r_0)\cup{}^{(int)}\mathcal{M}} \underline{u}^{1+\delta_{dec}}\Big\{|\mathfrak{d}^{\leq k}\Gamma_g| + |\mathfrak{d}^{\leq k}\Gamma_b|\Big\} \lesssim \epsilon_0,\end{aligned} \tag{9.44}$$

where in each case, (Γ_g, Γ_b) is defined w.r.t. the linearized quantities in the PG frame of the corresponding region.

The proof is similar is spirit to the proof of Theorem M8 in Section 9.4.3. Here, instead of transferring the control of the PT frames to the PG frames of $\mathcal{M}$, we instead transfer the above control of the PG frames to the PT frames. We proceed as follows:

1. We start with the control in ${}^{(ext)}\mathcal{M}$. Let $(f, \underline{f}, \lambda)$ denote the transition coefficients corresponding to the change from the outgoing PG frame of ${}^{(ext)}\mathcal{M}$ to the outgoing PT frame of ${}^{(ext)}\mathcal{M}$. Also, we denote the quantities corresponding to the outgoing PG frame without primes, and the quantities corresponding to the outgoing PT frame with primes. In view of Lemma 9.24, the following identities hold in ${}^{(ext)}\mathcal{M}$:
   (a) We have $f = 0$ and $\lambda = 1$. In particular, we have
   $$e_4' = e_4.$$
   (b) We have
   $$r' = r, \qquad \theta' = \theta, \qquad q' = q, \qquad \mathfrak{J}' = \mathfrak{J}. \tag{9.45}$$
   (c) We have
   $$\nabla_4 \underline{f} \quad = \quad 2\check{\zeta}. \tag{9.46}$$

   Also, note from the initialization of the outgoing PG frame of ${}^{(ext)}\mathcal{M}$ on Σ_* of Section 3.2.5, and the initialization of the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ on Σ_* of Section 9.1.3, that we have on Σ_*
   $$\underline{f} \quad = \quad \frac{a}{r}\left(\frac{(\nu(r)+2) - (b_* + 1 + \frac{2m}{r})}{1 - \frac{1}{4}b_*\frac{a^2(\sin\theta)^2}{r^2}} + O(r^{-2})\right) f_0. \tag{9.47}$$
2. Next, as in Step 4 of the proof of Theorem M8 in Section 9.4.3, we control $\underline{f}$ using the above transport equation for $\underline{f}$ and the above initialization on Σ_*. Based on the above control of the outgoing PG structure of ${}^{(ext)}\mathcal{M}$, we easily obtain
   $$\sup_{{}^{(ext)}\mathcal{M}} \left(ru^{\frac{1}{2}+\delta_{dec}} + u^{1+\frac{3\delta_{dec}}{4}}\right)|\mathfrak{d}^{\leq k_{small}}\underline{f}| \quad \lesssim \quad \epsilon_0, \tag{9.48}$$
   where we used in particular the fact that $r^{1+\frac{\delta_{dec}}{2}}u^{1+\frac{3\delta_{dec}}{4}}|\mathfrak{d}^{\leq k_{small}}\check{\zeta}| \lesssim \epsilon_0$.
3. Next, let $(f', \underline{f}', \lambda')$ denote the transition coefficients corresponding to the change from the ingoing PG frame of ${}^{(top)}\mathcal{M}$ to the ingoing PT

frame of ${}^{(top)}\mathcal{M}'$. As in Step 5 of the proof of Theorem M8 in Section 9.4.3, we control $(f', \underline{f}', \lambda')$. We now rely on the analog for ingoing PT structures of the transport equations of Corollary 2.81. Note that these transport equations do not lose derivatives[205] and we obtain

$$\begin{aligned}
&\sup_{{}^{(top)}\mathcal{M}'(r\geq r_0)} \left(ru^{\frac{1}{2}+\delta_{dec}} + u^{1+\frac{3\delta_{dec}}{4}}\right)|\mathfrak{d}^{\leq k_{small}}(f', \underline{f}', \log(\lambda'))| \\
&+ \sup_{{}^{(top)}\mathcal{M}'(r\leq r_0)} \underline{u}^{1+\frac{3\delta_{dec}}{4}}|\mathfrak{d}^{\leq k_{small}}(f', \underline{f}', \log(\lambda'))| \;\lesssim\; \epsilon_0. \qquad (9.49)
\end{aligned}$$

4. Next, let $(f'', \underline{f}'', \lambda'')$ denote the transition coefficients corresponding to the change from the ingoing PG frame of ${}^{(int)}\mathcal{M}$ to the ingoing PT frame of ${}^{(int)}\mathcal{M}'$. As in Step 6 of the proof of Theorem M8 in Section 9.4.3, we control $(f'', \underline{f}'', \lambda'')$, now relying on the analog for ingoing PT structures of the transport equations of Corollary 2.81, and obtain

$$\sup_{{}^{(int)}\mathcal{M}'} \underline{u}^{1+\frac{3\delta_{dec}}{4}}|\mathfrak{d}^{\leq k_{small}}(f'', \underline{f}'', \log(\lambda'')) \;\lesssim\; \epsilon_0. \qquad (9.50)$$

5. As in Step 6 of the proof of Theorem M8 in Section 9.4.3, we then rely on the above control of the various change of frame coefficients, the above control of the PG structures of $\mathcal{M}$, and the change of frame formulas of Proposition 2.12 to infer, for $k \leq k_{small} - 1$,

$$\begin{aligned}
&\sup_{{}^{(ext)}\mathcal{M}\cup{}^{(top)}\mathcal{M}'(r\geq r_0)} \left\{\left(ru^{\frac{1}{2}+\delta_{dec}} + u^{1+\frac{3\delta_{dec}}{4}}\right)|\mathfrak{d}^{\leq k}\Gamma'_g| + ru^{1+\frac{3\delta_{dec}}{4}}|\mathfrak{d}^{\leq k}\Gamma'_b|\right\} \\
&+ \sup_{{}^{(top)}\mathcal{M}'(r\leq r_0)\cup{}^{(int)}\mathcal{M}'} \underline{u}^{1+\frac{3\delta_{dec}}{4}}\left\{|\mathfrak{d}^{\leq k}\Gamma'_g| + |\mathfrak{d}^{\leq k}\Gamma'_b|\right\} \\
&\lesssim \epsilon_0,
\end{aligned}$$

where in each case, (Γ'_g, Γ'_b) is defined w.r.t. the linearized quantities in the PT frame of the corresponding region. The weights in r, u and $\underline{u}$ are enough to take care of the spacetime integrations in the global norms defined in Section 9.4.1 for the PT frames of $\mathcal{M}$, and we finally obtain the following desired bounds for the PT frames of $\mathcal{M}$

$$\mathfrak{G}_{k_{small}-1} + \mathfrak{R}_{k_{small}-1} \;\lesssim\; \epsilon_0.$$

This concludes the proof of Lemma 9.47. □

[205] Unlike their analogs for PG structures in Corollary 2.14.

9.4.7. Iterative procedure for the proof of the Main PT-Theorem First, recall our bootstrap assumptions **BA-PT** which are assumed throughout the proof of Theorem 9.44

$$\mathfrak{G}_{k_{large}+7} + \mathfrak{R}_{k_{large}+7} \leq \epsilon. \tag{9.51}$$

For J in the range $k_{small} - 1 \leq J \leq k_{large} + 5$, we also make the following iteration assumption

$$\mathfrak{G}_J + \mathfrak{R}_J \lesssim \epsilon_0 + L_*(J), \tag{9.52}$$

with $L_*(k)$ given by

$$L_*^2(k) := L_*^2(k, u_*') = \int_{\{u=u_*'\}} |\check{R}|^2_{w,k} + \int_{\Sigma_* \cap \{u=u_*'\}} |\check{\Gamma}|^2_{w,k} \tag{9.53}$$

where we recall the notation (9.8)

$$\begin{aligned} |\check{R}|^2_{w,k} &:= r^{3+\delta_B}|\mathfrak{d}_*^{\leq k}(A,B)|^2 + r^{3-\delta_B}|\mathfrak{d}_*^{\leq k}\check{P}|^2 + r^{1-\delta_B}|\mathfrak{d}_*^{\leq k}\underline{B}|^2, \\ |\check{\Gamma}|^2_{w,k} &:= r^2|\not{\mathfrak{d}}^{\leq k}\Gamma_g|^2 + |\not{\mathfrak{d}}^{\leq k}\Gamma_b|^2. \end{aligned} \tag{9.54}$$

In (9.54), $(A, B, \check{P}, \underline{B})$ denote linearized curvature components w.r.t. the the outgoing PT frame of ${}^{(ext)}\mathcal{M}$, Γ_g and Γ_b are defined w.r.t. the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ as in Definition 9.17, $\mathfrak{d}_*$ denote weighted derivatives tangential to the hypersurface $\{u = u_*'\}$, and $\not{\mathfrak{d}}$ denote weighted derivatives tangential to the sphere $\Sigma_* \cap \{u = u_*'\}$.

Remark 9.48. *In view of* (9.42), *i.e.*

$$\mathfrak{G}_{k_{small}-1} + \mathfrak{R}_{k_{small}-1} \lesssim \epsilon_0,$$

(9.52) *holds for* $J = k_{small} - 1$.

We now state the main sequence of estimates which will allow us to prove Theorem 9.46 in the next section.

Theorem 9.49. *Let J such that $k_{small} - 1 \leq J \leq k_{large} + 6$. Under the iteration assumption* (9.52), *we have the following estimate in $\mathcal{M}$ for the global PT curvature norm of Section 9.4.1*

$$\begin{aligned} \mathfrak{R}_{J+1} \lesssim\ & r_0^{-\frac{\delta_B}{2}}\left({}^{(top)}\mathfrak{G}^{\geq r_0}_{J+1} + {}^{(ext)}\mathfrak{G}_{J+1}\right) + r_0^{21+\delta_B}\epsilon_J \\ & + r_0^{\frac{21}{2}+\frac{\delta_B}{2}}\left(\sqrt{\epsilon_{J+1}}\sqrt{\epsilon_J} + \epsilon_0\right) + \sqrt{|a|}r_0^{3+\frac{\delta_B}{2}}\mathfrak{G}_{J+1} \end{aligned}$$

$$+r_0^{\frac{39}{8}+\frac{\delta_B}{2}}\epsilon_{J+1}^{\frac{3}{4}}\Big(\epsilon_0+\sqrt{\epsilon_J}\sqrt{\epsilon_{J+1}}\Big)^{\frac{1}{4}}+r_0^{\frac{39}{7}+\frac{4\delta_B}{7}}\epsilon_{J+1}^{\frac{6}{7}}\epsilon_J^{\frac{1}{7}} \tag{9.55}$$

where the constant in $\lesssim$ is independent of r_0 and where $\epsilon_J := \mathfrak{G}_J + \mathfrak{R}_J$.

Proof. The proof is given in Section 9.6. □

Proposition 9.50. *Let J such that $k_{small}-1 \leq J \leq k_{large}+6$. The following estimate holds true on Σ_* for the Ricci and metric coefficients of the outgoing PT frame of ${}^{(ext)}\mathcal{M}$*

$$^\star\mathfrak{G}_{J+1} \lesssim {}^\star\mathfrak{R}_{J+1}+\epsilon_0, \tag{9.56}$$

where the constant in $\lesssim$ is independent of r_0.

Proof. The proof is given in Section 9.7, see Proposition 9.67. □

Proposition 9.51. *Let J such that $k_{small}-1 \leq J \leq k_{large}+6$. The following estimates hold true for the Ricci and metric coefficients of the outgoing PT frame of ${}^{(ext)}\mathcal{M}$*

$${}^{(ext)}\mathfrak{G}_{J+1} \lesssim {}^\star\mathfrak{G}_{J+1}+{}^{(ext)}\mathfrak{R}_{J+1}+\epsilon_0, \tag{9.57}$$

where the constant in $\lesssim$ is independent of r_0.

Proof. The proof is given in Section 9.8, see Proposition 9.82. □

Proposition 9.52. *Let J such that $k_{small}-1 \leq J \leq k_{large}+6$. The following estimates hold true for the Ricci and metric coefficients of the ingoing PT frame of ${}^{(int)}\mathcal{M}'$*

$${}^{(int)}\mathfrak{G}_{J+1} \lesssim {}^{(ext)}\mathfrak{G}_{J+1}+{}^{(int)}\mathfrak{R}_{J+1}+\epsilon_0. \tag{9.58}$$

Proof. The proof is given in Section 9.9, see Proposition 9.86. □

Proposition 9.53. *Let J such that $k_{small}-1 \leq J \leq k_{large}+6$. The following estimates hold true for the Ricci and metric coefficients of the ingoing PT frame of ${}^{(top)}\mathcal{M}'$*

$${}^{(top)}\mathfrak{G}_{J+1} \lesssim \epsilon_0+L_*(J+1)+{}^{(top)}\mathfrak{R}_{J+1}. \tag{9.59}$$

In particular, we have

$${}^{(top)}\mathfrak{G}^{\geq r_0}_{J+1} \lesssim \epsilon_0+L_*(J+1)+{}^{(top)}\mathfrak{R}_{J+1} \tag{9.60}$$

in which case the constant in $\lesssim$ is independent of r_0.

Proof. The proof is given in Section 9.10. □

The following is a corollary of the above propositions.

Corollary 9.54. *Let J such that $k_{small}-1\leq J\leq k_{large}+6$. Let*

$$\begin{aligned}\widetilde{\epsilon}_{J+1} &:= r_0^{21+\delta_B}\epsilon_J + r_0^{\frac{21}{2}+\frac{\delta_B}{2}}\Big(\sqrt{\epsilon_{J+1}}\sqrt{\epsilon_J}+\epsilon_0\Big)+\sqrt{|a|}r_0^{3+\frac{\delta_B}{2}}\mathfrak{G}_{J+1}\\ &\quad +r_0^{\frac{39}{8}+\frac{\delta_B}{2}}\epsilon_{J+1}^{\frac{3}{4}}\Big(\epsilon_0+\sqrt{\epsilon_J}\sqrt{\epsilon_{J+1}}\Big)^{\frac{1}{4}}+r_0^{\frac{39}{7}+\frac{4\delta_B}{7}}\epsilon_{J+1}^{\frac{6}{7}}\epsilon_J^{\frac{1}{7}}. \qquad (9.61)\end{aligned}$$

Then:

1. *We have*

$$\mathfrak{R}_{J+1}+{}^\star\mathfrak{G}_{J+1} \lesssim r_0^{-\delta_B}L_*(J+1)+\widetilde{\epsilon}_{J+1}, \qquad (9.62)$$

   where the constant in $\lesssim$ is independent of r_0.

2. *We have*

$$\mathfrak{G}_{J+1} \lesssim L_*(J+1)+\widetilde{\epsilon}_{J+1}. \qquad (9.63)$$

Proof. We start with the first estimate. In view of Theorem 9.49, and Propositions 9.50, 9.51 and 9.53, we have

$$\mathfrak{R}_{J+1}+{}^\star\mathfrak{G}_{J+1}\lesssim r_0^{-\delta_B}\left({}^{(top)}\mathfrak{G}^{\geq r_0}_{J+1}+{}^{(ext)}\mathfrak{G}_{J+1}\right)+\widetilde{\epsilon}_{J+1}$$

and

$${}^{(top)}\mathfrak{G}^{\geq r_0}_{J+1}+{}^{(ext)}\mathfrak{G}_{J+1} \lesssim \mathfrak{R}_{J+1}+{}^\star\mathfrak{G}_{J+1}+\epsilon_0+L_*(J+1)$$

where the constant in $\lesssim$ is independent of r_0 in both inequalities. Plugging the second inequality in the first, we infer

$$\mathfrak{R}_{J+1}+{}^\star\mathfrak{G}_{J+1} \lesssim r_0^{-\delta_B}\Big(\mathfrak{R}_{J+1}+{}^\star\mathfrak{G}_{J+1}\Big)+r_0^{-\delta_B}L_*(J+1)+\widetilde{\epsilon}_{J+1}$$

where the constant in $\lesssim$ is independent of r_0. For r_0 large enough, we may absorb the first term on the RHS, and we obtain

$$\mathfrak{R}_{J+1}+{}^\star\mathfrak{G}_{J+1} \lesssim r_0^{-\delta_B}L_*(J+1)+\widetilde{\epsilon}_{J+1},$$

where the constant in $\lesssim$ is independent of r_0, which is the first desired estimate.

Next, we focus on the second estimate of the corollary. In view of Propositions 9.50, 9.51, 9.52 and 9.53, we have

$$\mathfrak{G}_{J+1} \lesssim \mathfrak{R}_{J+1} + \epsilon_0 + L_*(J+1).$$

Plugging the first estimate of the corollary proved above to control the term $\mathfrak{R}_{J+1}$ in the RHS, we infer

$$\mathfrak{G}_{J+1} \lesssim L_*(J+1) + \widetilde{\epsilon}_{J+1}$$

which is the second desired estimate. This concludes the proof of Corollary 9.54. □

9.4.8. End of the proof of the Main PT-Theorem

Step 1. As mentioned in Remark 9.48, the estimate (9.42) trivially implies the iteration assumption (9.52) with $J = k_{small} - 1$.

Step 2. We assume that the iteration assumption (9.52) holds for any fixed J such that $k_{small} - 1 \leq J \leq k_{large} + 5$, i.e.

$$\mathfrak{G}_J + \mathfrak{R}_J \lesssim \epsilon_0 + L_*(J).$$

Plugging the iteration assumption, for J in the range $k_{small} - 1 \leq J \leq k_{large} + 5$, in the estimates of Corollary 9.54, we deduce

$$\mathfrak{R}_{J+1} + \mathfrak{G}_{J+1} \lesssim L_*(J+1) + \widetilde{\epsilon}_{J+1}.$$

Recalling the definition of $\widetilde{\epsilon}_{J+1}$ in (9.61), we infer

$$\begin{aligned}
\mathfrak{R}_{J+1} + \mathfrak{G}_{J+1} &\lesssim \epsilon_0 + \mathfrak{G}_J + \mathfrak{R}_J + L_*(J+1) + \sqrt{|a|} r_0^{3+\frac{\delta_B}{2}} \mathfrak{G}_{J+1} \\
&\lesssim \epsilon_0 + L_*(J+1) + \sqrt{|a|} r_0^{3+\frac{\delta_B}{2}} \mathfrak{G}_{J+1},
\end{aligned}$$

and hence, for $|a|$ small enough,

$$\mathfrak{R}_{J+1} + \mathfrak{G}_{J+1} \lesssim \epsilon_0 + L_*(J+1),$$

where the constant in $\lesssim$ depends on r_0. Thus, the iteration assumption (9.52) holds for $J+1$. This implies that the iteration assumption (9.52) holds true for any J in the range $k_{small} - 1 \leq J \leq k_{large} + 6$, i.e.

$$\mathfrak{R}_J + \mathfrak{G}_J \lesssim \epsilon_0 + L_*(J), \qquad k_{small} - 1 \leq J \leq k_{large} + 6. \tag{9.64}$$

In particular, we have for $J = k_{large} + 6$

$$\mathfrak{R}_{k_{large}+6} + \mathfrak{G}_{k_{large}+6} \lesssim \epsilon_0 + L_*(k_{large} + 6). \tag{9.65}$$

Step 3. In order to control the term $L_*(k_{large} + 6)$ on the RHS of (9.65), we rely on the following interpolation lemma.

Lemma 9.55. *For any $k_{small} - 1 \leq k \leq k_{large} + 7$, we have*

$$\begin{aligned} L_*(k) &\lesssim (\epsilon_0 + L_*(k_{small} - 1))^{\frac{k_{large}+7-k}{k_{large}+7-(k_{small}-1)}} \\ &\quad \times (\epsilon_0 + \mathfrak{R}_{k_{large}+7} + L_*(k_{large} + 7))^{\frac{k-(k_{small}-1)}{k_{large}+7-(k_{small}-1)}}. \end{aligned} \tag{9.66}$$

Proof. We prove the lemma by iteration. Assume for $k_{small} - 1 < p \leq k_{large} + 5$ the following iteration assumption

$$\begin{aligned} L_*(k) &\lesssim (\epsilon_0 + L_*(k_{small} - 1))^{\frac{p-k}{p-(k_{small}-1)}} \\ &\quad \times (\epsilon_0 + L_*(p))^{\frac{k-(k_{small}-1)}{p-(k_{small}-1)}}, \quad k_{small} - 1 \leq k \leq p. \end{aligned}$$

The iteration assumption holds trivially in the case $p = k_{small}$.

Next, assume that the iteration assumption holds for some p in the range $k_{small} - 1 < p \leq k_{large} + 5$. We now consider whether the iteration assumption holds for $p + 1$. Recall from (9.53) that

$$L_*^2(p) = \int_{\{u=u'_*\}} |\check{R}|^2_{w,p} + \int_{\Sigma_* \cap \{u=u'_*\}} |\check{\Gamma}|^2_{w,p}.$$

In view of the definition (9.8), and noting that $|\check{R}|_{w,k}$ only involves derivatives tangential to $\{u = u'_*\}$, while $|\check{\Gamma}|_{w,k}$ contains only derivatives tangential to $\Sigma_* \cap \{u = u'_*\}$, we may integrate by parts once, which yields

$$L_*^2(p) \lesssim L_*(p-1)L_*(p+1) + |\mathcal{B}_p|,$$

where the boundary term $\mathcal{B}_p$ corresponds to a product of curvature terms integrated on the boundary of $\{u = u'_*\}$ where one curvature term has $p - 1$ derivatives and the other p derivatives. Using the trace theorem, we easily derive the following estimate

$$|\mathcal{B}_p| \lesssim \sqrt{\mathfrak{R}_{p-1}\mathfrak{R}_p}\sqrt{\mathfrak{R}_p\mathfrak{R}_{p+1}} = \sqrt{\mathfrak{R}_{p-1}}\mathfrak{R}_p\sqrt{\mathfrak{R}_{p+1}}.$$

Now, recall from (9.64) that we have obtained

$$\mathfrak{R}_k \lesssim \epsilon_0 + L_*(k), \qquad k \leq k_{large} + 6.$$

Since $p \leq k_{large} + 5$, we have $p + 1 \leq k_{large} + 6$ and we infer

$$|\mathcal{B}_p| \quad \lesssim \quad \sqrt{\epsilon_0 + L_*(p-1)}(\epsilon_0 + L_*(p))\sqrt{\epsilon_0 + L_*(p+1)}.$$

Plugging in the above bound for $L_*^2(p)$, we infer

$$L_*^2(p) \quad \lesssim \quad (\epsilon_0 + L_*(p-1))(\epsilon_0 + L_*(p+1)).$$

Also, applying the iteration assumption with $k = p - 1$, we have

$$L_*(p-1) \quad \lesssim \quad (\epsilon_0 + L_*(k_{small} - 1))^{\frac{1}{p-(k_{small}-1)}}(\epsilon_0 + L_*(p))^{\frac{p-1-(k_{small}-1)}{p-(k_{small}-1)}}$$

and hence

$$\begin{aligned} L_*^2(p) \quad &\lesssim \quad (\epsilon_0 + L_*(k_{small} - 1))^{\frac{1}{p-(k_{small}-1)}}(\epsilon_0 + L_*(p))^{\frac{p-1-(k_{small}-1)}{p-(k_{small}-1)}} \\ &\quad\quad \times(\epsilon_0 + L_*(p+1)), \end{aligned}$$

or

$$L_*(p) \quad \lesssim \quad (\epsilon_0 + L_*(k_{small} - 1))^{\frac{1}{p+1-(k_{small}-1)}}(\epsilon_0 + L_*(p+1))^{\frac{p-(k_{small}-1)}{p+1-(k_{small}-1)}}.$$

Plugging in the iteration assumption, we infer, for any $k_{small} - 1 \leq k \leq p$,

$$\begin{aligned} L_*(k) \quad &\lesssim \quad (\epsilon_0 + L_*(k_{small} - 1))^{\frac{p-k}{p-(k_{small}-1)}}(\epsilon_0 + L_*(p))^{\frac{k-(k_{small}-1)}{p-(k_{small}-1)}} \\ &\lesssim \quad (\epsilon_0 + L_*(k_{small} - 1))^{\frac{p+1-k}{p+1-(k_{small}-1)}}(\epsilon_0 + L_*(p+1))^{\frac{k-(k_{small}-1)}{p+1-(k_{small}-1)}} \end{aligned}$$

which, together with the fact that the case $k = p + 1$ trivially holds, implies the iteration assumption for p replaced by $p+1$. We deduce that the iteration assumption holds true for any $k_{small} - 1 < p \leq k_{large} + 6$. In particular, for $p = k_{large} + 6$, we infer that

$$\begin{aligned} L_*(k) \quad &\lesssim \quad (\epsilon_0 + L_*(k_{small} - 1))^{\frac{k_{large}+6-k}{k_{large}+6-(k_{small}-1)}} \\ &\quad\quad \times(\epsilon_0 + L_*(k_{large} + 6))^{\frac{k-(k_{small}-1)}{k_{large}+6-(k_{small}-1)}} \end{aligned}$$

for any $k_{small} - 1 \leq k \leq k_{large} + 6$.

We still need to go from $p = k_{large} + 6$ to $p = k_{large} + 7$. To this end, we proceed as above and obtain

$$\begin{aligned} L_*^2(k_{large}+6) &\lesssim (\epsilon_0 + L_*(k_{large}+5))(\epsilon_0 + L_*(k_{large}+7)) + |\mathcal{B}_{k_{large}+6}| \\ &\lesssim (\epsilon_0 + L_*(k_{large}+5))(\epsilon_0 + L_*(k_{large}+7)) \\ &\quad + \sqrt{\mathfrak{R}_{k_{large}+5}}\,\mathfrak{R}_{k_{large}+6}\sqrt{\mathfrak{R}_{k_{large}+7}}. \end{aligned}$$

Using (9.64) for $J = k_{large} + 5$ and $J = k_{large} + 6$, we infer

$$\begin{aligned} L_*^2(k_{large}+6) &\lesssim (\epsilon_0 + L_*(k_{large}+5))(\epsilon_0 + L_*(k_{large}+7)) \\ &\quad + \sqrt{\epsilon_0 + L_*(k_{large}+5)}\,(\epsilon_0 + L_*(k_{large}+6))\sqrt{\mathfrak{R}_{k_{large}+7}} \end{aligned}$$

and hence

$$L_*^2(k_{large}+6) \lesssim (\epsilon_0 + L_*(k_{large}+5))(\epsilon_0 + \mathfrak{R}_{k_{large}+7} + L_*(k_{large}+7)).$$

Then, in view of the above, we infer

$$\begin{aligned} L_*^2(k_{large}+6) &\lesssim (\epsilon_0 + L_*(k_{small}-1))^{\frac{1}{k_{large}+6-(k_{small}-1)}} \\ &\quad \times(\epsilon_0 + L_*(k_{large}+6))^{1-\frac{1}{k_{large}+6-(k_{small}-1)}} \\ &\quad \times(\epsilon_0 + \mathfrak{R}_{k_{large}+7} + L_*(k_{large}+7)) \end{aligned}$$

or

$$\begin{aligned} (L_*(k_{large}+6)) &\lesssim (\epsilon_0 + L_*(k_{small}-1))^{\frac{1}{k_{large}+7-(k_{small}-1)}} \\ &\quad \times(\epsilon_0 + \mathfrak{R}_{k_{large}+7} + L_*(k_{large}+7))^{1-\frac{1}{k_{large}+7-(k_{small}-1)}}. \end{aligned}$$

Plugging in the above, we obtain for any $k_{small} - 1 \le k \le k_{large} + 6$

$$\begin{aligned} L_*(k) &\lesssim (\epsilon_0 + L_*(k_{small}-1))^{\frac{k_{large}+6-k}{k_{large}+6-(k_{small}-1)}} \\ &\quad \times(\epsilon_0 + L_*(k_{large}+6))^{\frac{k-(k_{small}-1)}{k_{large}+6-(k_{small}-1)}} \\ &\lesssim (\epsilon_0 + L_*(k_{small}-1))^{\frac{k_{large}+7-k}{k_{large}+7-(k_{small}-1)}} \\ &\quad \times(\epsilon_0 + \mathfrak{R}_{k_{large}+7} + L_*(k_{large}+7))^{\frac{k-(k_{small}-1)}{k_{large}+7-(k_{small}-1)}}. \end{aligned}$$

This is the stated estimate for $k_{small} - 1 \le k \le k_{large} + 6$. Since the case $k = k_{large} + 7$ trivially holds, this concludes the proof of Lemma 9.55. ☐

Step 4. Applying Lemma 9.55 with $k = k_{large} + 6$, we have

$$\begin{aligned} L_*(k_{large}+6) &\lesssim (\epsilon_0 + L_*(k_{small}-1))^{\frac{1}{k_{large}+7-(k_{small}-1)}} \\ &\quad \times(\epsilon_0 + \mathfrak{R}_{k_{large}+7} + L_*(k_{large}+7))^{1-\frac{1}{k_{large}+7-(k_{small}-1)}}. \end{aligned}$$

Also, in view of (9.42), we have

$$L_*(k_{small}-1) \lesssim \epsilon_0$$

and hence

$$\begin{aligned} L_*(k_{large}+6) &\lesssim \epsilon_0^{\frac{1}{k_{large}+7-(k_{small}-1)}} \\ &\quad \times(\epsilon_0 + \mathfrak{R}_{k_{large}+7} + L_*(k_{large}+7))^{1-\frac{1}{k_{large}+7-(k_{small}-1)}}. \end{aligned}$$

Plugging in (9.65), we deduce

$$\begin{aligned} &\mathfrak{R}_{k_{large}+6} + \mathfrak{G}_{k_{large}+6} \\ \lesssim\ & \epsilon_0 + L_*(k_{large}+6) \\ \lesssim\ & \epsilon_0 + \epsilon_0^{\frac{1}{k_{large}+7-(k_{small}-1)}}(\epsilon_0 + \mathfrak{R}_{k_{large}+7} + L_*(k_{large}+7))^{1-\frac{1}{k_{large}+7-(k_{small}-1)}}. \end{aligned}$$

We also use the first estimate of Corollary 9.54 with $J = k_{large} + 6$, i.e.

$$\mathfrak{R}_{k_{large}+7} + {}^\star\mathfrak{G}_{k_{large}+7} \lesssim r_0^{-\delta_B} L_*(k_{large}+7) + \widetilde{\epsilon}_{k_{large}+7},$$

where the constant in $\lesssim$ is independent of r_0.

We now estimate $L_*(k_{large}+7)$. Recall from (9.53) that

$$L_*^2(k_{large}+7) = \int_{\{u=u'_*\}} |\check{R}|^2_{w,k_{large}+7} + \int_{\Sigma_*\cap\{u=u'_*\}} |\check{\Gamma}|^2_{w,k_{large}+7}.$$

Also, recall from (9.9) that

$$\begin{aligned} &\int_{\{u=u'_*\}} |\check{R}|^2_{w,k_{large}+7} + \int_{\Sigma_*\cap\{u=u'_*\}} |\check{\Gamma}|^2_{w,k_{large}+7} \\ =\ & \inf_{u*-5\le u_1\le u*-4}\left(\int_{\{u=u_1\}} |\check{R}|^2_{w,k_{large}+7} + \int_{\Sigma_*\cap\{u=u_1\}} |\check{\Gamma}|^2_{w,k_{large}+7}\right). \end{aligned}$$

We infer

$$L_*^2(k_{large}+7)$$

$$\lesssim \int_{u_1=u_*-5}^{u_*-4}\left(\int_{\{u=u_1\}}|\check{R}|^2_{w,k_{large}+7}+\int_{\Sigma_*\cap\{u=u_1\}}|\check{\Gamma}|^2_{w,k_{large}+7}\right)du_1.$$

In view of the definition of $|\check{R}|_{w,k}$ and $|\check{\Gamma}|_{w,k}$ in (9.54), and the definition of $\mathfrak{R}_k$ and ${}^\star\mathfrak{G}_k$ in Section 9.4.1, we deduce

$$L_*(k_{large}+7) \ \lesssim\ \mathfrak{R}_{k_{large}+7}+{}^\star\mathfrak{G}_{k_{large}+7}$$

where the constant in $\lesssim$ is independent of r_0. Plugging in the above, we obtain

$$\mathfrak{R}_{k_{large}+7}+{}^\star\mathfrak{G}_{k_{large}+7} \ \lesssim\ r_0^{-\delta_B}\left(\mathfrak{R}_{k_{large}+7}+{}^\star\mathfrak{G}_{k_{large}+7}\right)+\widetilde{\epsilon}_{k_{large}+7},$$

where the constant in $\lesssim$ is independent of r_0, which immediately implies, for r_0 large enough,

$$\mathfrak{R}_{k_{large}+7}+{}^\star\mathfrak{G}_{k_{large}+7}+L_*(k_{large}+7) \ \lesssim\ \widetilde{\epsilon}_{k_{large}+7}.$$

Together with the estimate of Corollary 9.54 with $J=k_{large}+6$, i.e.

$$\mathfrak{G}_{k_{large}+7} \ \lesssim\ L_*(k_{large}+7)+\widetilde{\epsilon}_{k_{large}+7},$$

we infer

$$\mathfrak{R}_{k_{large}+7}+\mathfrak{G}_{k_{large}+7} \ \lesssim\ \widetilde{\epsilon}_{k_{large}+7}.$$

In view of the the definition (9.61) of $\widetilde{\epsilon}_{J+1}$, we immediately infer

$$\mathfrak{R}_{k_{large}+7}+\mathfrak{G}_{k_{large}+7} \ \lesssim\ \sqrt{|a|}\mathfrak{G}_{k_{large}+7}+\epsilon_0+\mathfrak{R}_{k_{large}+6}+\mathfrak{G}_{k_{large}+6}$$

where the constant in $\lesssim$ now depends on r_0. For $|a|$ small enough, we infer

$$\mathfrak{R}_{k_{large}+7}+\mathfrak{G}_{k_{large}+7} \ \lesssim\ \epsilon_0+\mathfrak{R}_{k_{large}+6}+\mathfrak{G}_{k_{large}+6}.$$

On the other hand, we have obtained

$$\begin{aligned}&\mathfrak{R}_{k_{large}+6}+\mathfrak{G}_{k_{large}+6}\\ \lesssim\ &\epsilon_0+\epsilon_0^{\frac{1}{k_{large}+7-(k_{small}-1)}}\left(\epsilon_0+\mathfrak{R}_{k_{large}+7}+L_*(k_{large}+7)\right)^{1-\frac{1}{k_{large}+7-(k_{small}-1)}}\\ \lesssim\ &\epsilon_0+\epsilon_0^{\frac{1}{k_{large}+7-(k_{small}-1)}}\left(\epsilon_0+\mathfrak{R}_{k_{large}+7}+{}^\star\mathfrak{G}_{k_{large}+7}\right)^{1-\frac{1}{k_{large}+7-(k_{small}-1)}}.\end{aligned}$$

Plugging in the above, we deduce

$$\mathfrak{R}_{k_{large}+7}+\mathfrak{G}_{k_{large}+7} \lesssim \epsilon_0+\epsilon_0^{\frac{1}{k_{large}+7-(k_{small}-1)}} \times(\epsilon_0+\mathfrak{R}_{k_{large}+7}+\mathfrak{G}_{k_{large}+7})^{1-\frac{1}{k_{large}+7-(k_{small}-1)}}.$$

We deduce

$$\mathfrak{R}_{k_{large}+7}+\mathfrak{G}_{k_{large}+7} \lesssim \epsilon_0$$

as desired. This concludes the proof of Theorem 9.44.

9.5. Proof of Theorem 9.46

As in the proof of Theorem M0, see Section 8.3, we divide the proof in 24 steps, denoted here by primes, i.e. 1'–24', which can thus be compared with Steps 1–24 of Section 8.3.

Remark 9.56. *Steps 1'–19' differ little from Steps 1–19. The main difference occurs with Steps 20'–24' where the properties of the PT frames will be important. More generally, as the conclusions of Theorem M0 hold for* $k\leq k_{large}-2$ *while the ones of Theorem 9.46 hold for* $k\leq k_{large}+7$, *only estimates involving top regularity will differ.*

Steps 1'–7'. Steps 1'–7' are identical to Steps 1–7. More precisely, we propagate from S_* along Σ_*, relative to the integrable frame of Σ_*, the $\ell=1$ modes of $\mathrm{div}\,\beta$, $\mathrm{curl}\,\beta$, $\check{\rho}$ and $\check{\underline{\kappa}}$, use the GCM assumptions on S_*, and arrive at the estimate (8.79) of Proposition 8.22, i.e.

$$\sup_{\Sigma_*}\Bigg(r^5|(\mathrm{div}\,\beta)_{\ell=1}|+r^5|(\mathrm{curl}\,\beta)_{\ell=1,\pm}| +r^5\left|(\mathrm{curl}\,\beta)_{\ell=1,0}-\frac{2am}{r^5}\right|\Bigg)\lesssim\epsilon_0, \tag{9.67}$$

$$\sup_{\Sigma_*}\Big(r^3|(\check{\rho})_{\ell=1}|+r^2|(\check{\underline{\kappa}})_{\ell=1}|\Big)\lesssim\epsilon_0.$$

Steps 8'–16'. Next, as in Steps 8–16, $\mathcal{C}_1'$ denotes the portion of the past directed outgoing, geodesic, null cone initialized on the sphere $S_1'=S_1$ of Σ_* and restricted to $r'\geq\delta_*\epsilon_0^{-1}$. Recall also that $\widetilde{r}$ denotes the volume radius for the outgoing geodesic foliation of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ constructed in Section 8.2, while r'

is the are radius of the spheres $S' \subset \mathcal{C}_1'$. In fact all quantities associated to the outgoing geodesic foliation of $\mathcal{C}_1'$ are denoted by primes, while the quantities associated to the outgoing geodesic foliation of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ are denoted by tildes.

As in the proof of Theorem M0, we rely on the estimates (8.79), the GCM conditions on $S_1' = S_1$ and the following bootstrap assumptions.

Local Bootstrap Assumptions:

1. Along $\mathcal{C}_1'$, we have

$$\sup_{S'\subset\mathcal{C}_1'}\Big(\|f\|_{\mathfrak{h}_4(S')} + (r')^{-1}\|(\underline{f}, \log\lambda)\|_{\mathfrak{h}_4(S')}\Big) \leq \epsilon. \tag{9.68}$$

2. On $S_1' = S_1$, we assume

$$\|f\|_{\mathfrak{h}_{k_{large}+7}(S_1')} + r^{-1}\|(\underline{f}, \log(\lambda))\|_{\mathfrak{h}_{k_{large}+7}(S_1')} \leq \epsilon. \tag{9.69}$$

3. In the case $a_0 \neq 0$, we make the following assumption[206], on $S_1' = S_1$, on the difference between the basis of $\ell = 1$ modes $J^{(p)}$ of Σ_*, and the basis of $\ell = 1$ modes $\widetilde{J}^{(p)}$ of ${}^{(ext)}\widetilde{\mathcal{L}_0}$

$$\max_{p=0,+,-}\Big\|\mathfrak{d}_*^{\leq k}(J^{(p)} - \widetilde{J}^{(p)})\Big\|_{L^2(S_1')} \leq \epsilon, \text{ for all } k \leq k_{large} + 7. \tag{9.70}$$

Remark 9.57. *Note that (9.69) and (9.70) hold for $k \leq k_{large} + 7$, while their analogs in Theorem M0 hold only for $k \leq k_{large}$.*

The proof of Steps 8'–16' is then completely analogous to the one of Steps 8–16, with k_{large} being replaced by $k_{large} + 7$ in view of Remark 9.57. In particular, we obtain the following analog of (8.111)

$$\begin{aligned}\sup_{k\leq k_{large}+8}\Big(\|\mathfrak{d}^k f\|_{L^2(S_1')} + r^{-1}\|\mathfrak{d}^k(\underline{f}, \log\lambda)\|_{L^2(S_1')}&\\ +\|\mathfrak{d}^{\leq k-1}\nabla_\nu'(\underline{f}, \log\lambda)\|_{L^2(S_1')}\Big) &\lesssim \epsilon_0,\\ \sup_{S_1'}\left(\left|\frac{m}{m_0} - 1\right| + \left|\frac{r'}{r} - 1\right|\right) &\lesssim \epsilon_0,\end{aligned} \tag{9.71}$$

which improves the bootstrap assumption (9.69). Also, we obtain the following analog of (8.118)

$$|a| \;\lesssim\; \epsilon_0 \quad \text{if} \quad a_0 = 0, \tag{9.72}$$

[206] Recall that $\mathfrak{d}_*$ refers to the properly normalized tangential derivatives along Σ_*.

and the following analog of (8.125)

$$|a - a_0| + \max_{p=0,+,-} \sup_{S'_1} \Big| J^{(p)} - J'^{(p)} \Big| \lesssim \epsilon_0 \quad \text{if} \quad a_0 \neq 0. \tag{9.73}$$

Step 17'. The main difference with Step 17 is that we now rely on L^2 norms rather than sup norms. In particular, we rely on

$$\|\mathfrak{d}_*^{\leq k}\Gamma'_b\|_{L^2(\Sigma_*)} \leq \epsilon, \quad k \leq k_{large} + 7, \tag{9.74}$$

which follows immediately from bootstrap assumptions **BA-PT**, see (9.40).

As in Step 17 of the proof of Theorem M0, using Lemmas 8.37, 5.61 and 5.62 and simple elliptic estimates, we derive first the estimates, for $k \leq k_{large} + 6$,

$$\begin{aligned}
&\|\phi\|_{\mathfrak{h}_{k+2}(S_*)} \lesssim \|\Gamma'_b\|_{\mathfrak{h}_k(S_*)},\\
&\max_{p=0,+,-} \|r'\widetilde{\nabla' J'^{(p)}}\|_{\mathfrak{h}_{k+2}(S_*)} \lesssim \|\phi\|_{\mathfrak{h}_{k+2}(S_*)},\\
&\|r'\nabla f'_0 - J'^{(0)} \in \|_{\mathfrak{h}_{k+1}(S_*)} + \|r'\nabla f'_\pm + J'^{(\pm)}\delta\|_{\mathfrak{h}_{k+1}(S_*)} \lesssim \|\phi\|_{\mathfrak{h}_{k+2}(S_*)}.
\end{aligned}$$

Now, in view of (9.74) and the trace theorem, we have, for $k \leq k_{large} + 6$,

$$\|\Gamma'_b\|_{\mathfrak{h}_k(S_*)} \lesssim \|\mathfrak{d}_*^{\leq k+1}\Gamma'_b\|_{L^2(\Sigma_*)} \lesssim \epsilon.$$

Hence, for $k + 1 \leq k_{large} + 7$,

$$\begin{aligned}
&\|\phi\|_{\mathfrak{h}_{k+2}(S_*)} \lesssim \epsilon,\\
&\max_{p=0,+,-} \|r'\widetilde{\nabla' J'^{(p)}} \in \|_{\mathfrak{h}_{k+2}(S_*)} \lesssim \epsilon,\\
&\|r'\nabla f'_0 - J'^{(0)} \in \|_{\mathfrak{h}_{k+1}(S_*)} + \|r'\nabla f'_\pm + J'^{(\pm)}\delta\|_{\mathfrak{h}_{k+1}(S_*)} \lesssim \epsilon.
\end{aligned}$$

Next, on Σ_*, we have by Lemma 5.66,

$$\begin{aligned}
\nabla_\nu\Big[r'\nabla f'_0 - J'^{(0)} \in \Big] &= \Gamma'_b \cdot \not{\mathfrak{d}}^{\leq 1} f'_0,\\
\nabla_\nu\Big[r'\nabla f'_\pm + J'^{(\pm)}\delta\Big] &= \Gamma'_b \cdot \not{\mathfrak{d}}^{\leq 1} f'_\pm,\\
\nabla_\nu\Big[r'\widetilde{\nabla' J'^{(p)}}\Big] &= \Gamma'_b \cdot \not{\mathfrak{d}}^{\leq 1} J'^{(p)}, \quad p = 0, +, -.
\end{aligned}$$

Integrating from S_*, we obtain for $k \leq k_{large} + 7$

$$\|r'\nabla f'_0 - J'^{(0)} \in \|_{L^\infty_u \mathfrak{h}_k(S')} \lesssim \|r'\nabla f'_0 - J'^{(0)} \in \|_{\mathfrak{h}_k(S_*)} + \sqrt{u_*}\|\mathfrak{d}_*^k\Gamma'_b\|_{L^2(\Sigma_*)},$$

$$\begin{aligned}\|r'\nabla f'_\pm + J'^{(\pm)}\delta\|_{L^\infty_u\mathfrak{h}_k(S')} &\lesssim \|r'\nabla f'_\pm + J'^{(\pm)}\delta\|_{\mathfrak{h}_k(S_*)} + \sqrt{u_*}\|\mathfrak{d}^k_*\Gamma'_b\|_{L^2(\Sigma_*)},\\ \|r'\widetilde{\nabla' J'^{(p)}}\|_{L^\infty_u\mathfrak{h}_k(S')} &\lesssim \|r'\widetilde{\nabla' J'^{(p)}}\|_{\mathfrak{h}_k(S_*)}\\ &\quad +\sqrt{u_*}\|\mathfrak{d}^k_*\Gamma'_b\|_{L^2(\Sigma_*)}, \quad p=0,+,-.\end{aligned}$$

Together with the above control on S_*, the control (9.74) for Γ'_b, and the dominance condition for r on Σ_*, we deduce, for all $k \leq k_{large}+7$,

$$\begin{aligned}&\|r'\nabla' f'_0 - J'^{(0)} \in \|_{L^\infty_u\mathfrak{h}_k(S')} + \|r'\nabla f'_\pm + J'^{(\pm)}\delta\|_{L^\infty_u\mathfrak{h}_k(S')}\\ &\qquad + \max_{p=0,+,-}\|r'\widetilde{\nabla' J'^{(p)}}\|_{L^\infty_u\mathfrak{h}_k(S')} \lesssim \sqrt{u_*}\epsilon \lesssim r\epsilon_0.\end{aligned}$$

In view of the above, using the definition of $\widetilde{\nabla' J'^{(0)}}$, see Definition 5.57, we infer on S'_1, for all $k \leq k_{large}+7$,

$$\begin{aligned}\left\|\nabla' f'_0 - \frac{J'^{(0)}}{r'}\in\right\|_{\mathfrak{h}_k(S'_1)} + \left\|\nabla' f'_\pm + \frac{J'^{(p)}}{r'}\delta\right\|_{\mathfrak{h}_k(S'_1)} &\lesssim \epsilon_0,\\ \left\|\nabla' J'^{(0)} + \frac{1}{r'}\,{}^*f'_0\right\|_{\mathfrak{h}_k(S'_1)} + \left\|\nabla' J'^{(\pm)} - \frac{1}{r'}f'_\pm\right\|_{\mathfrak{h}_k(S'_1)} &\lesssim \epsilon_0.\end{aligned}$$

On the other hand, from the control of ${}^{(ext)}\widetilde{\mathcal{L}_0}$, the change of frame formula for ∇', the control (9.71) for the change of frame coefficients $(f, \underline{f}, \lambda)$ from ${}^{(ext)}\widetilde{\mathcal{L}_0}$ to Σ_*, and the control of $r'-r$ on S'_1 also provided by (9.71), we have, for all $k \leq k_{large}+7$,

$$\begin{aligned}\left\|\nabla' f_0 - \frac{J^{(0)}}{r'}\in\right\|_{\mathfrak{h}_k(S'_1)} + \left\|\nabla' f_\pm + \frac{J^{(p)}}{r'}\delta\right\|_{\mathfrak{h}_k(S'_1)} &\lesssim \epsilon_0,\\ \left\|\nabla' J^{(0)} + \frac{1}{r'}\,{}^*f_0\right\|_{\mathfrak{h}_k(S'_1)} + \left\|\nabla' J^{(\pm)} - \frac{1}{r'}f_\pm\right\|_{\mathfrak{h}_k(S'_1)} &\lesssim \epsilon_0,\end{aligned}$$

and hence, for all $k \leq k_{large}+7$,

$$\begin{aligned}&\left\|\nabla'(f'_0 - f_0) - \frac{J'^{(0)} - J^{(0)}}{r'}\in\right\|_{\mathfrak{h}_k(S'_1)}\\ &+\left\|\nabla'(f'_\pm - f_\pm) + \frac{J'^{(\pm)} - J^{(\pm)}}{r'}\delta\right\|_{\mathfrak{h}_k(S'_1)} \lesssim \epsilon_0,\\ &\left\|\nabla'(J'^{(0)} - J^{(0)}) + \frac{1}{r'}\,{}^*(f'_0 - f_0)\right\|_{\mathfrak{h}_k(S'_1)}\end{aligned}$$

$$+\left\|\nabla'(J'^{(\pm)} - J^{(\pm)}) - \frac{1}{r'}\,{}^*(f'_\pm - f_\pm)\right\|_{\mathfrak{h}_k(S'_1)} \lesssim \epsilon_0.$$

We deduce, for all $k \leq k_{large} + 7$,

$$r^{-1}\|f'_0 - f_0\|_{\mathfrak{h}_{k+1}(S'_1)} + r^{-1}\left\|J'^{(0)} - J^{(0)}\right\|_{\mathfrak{h}_{k+1}(S'_1)} \lesssim \epsilon_0 + \|J'^{(0)} - J^{(0)}\|_{L^2(S'_1)}$$

and

$$\begin{aligned} & r^{-1}\|f'_\pm - f_\pm\|_{\mathfrak{h}_{k+1}(S'_1)} + r^{-1}\left\|J'^{(\pm)} - J^{(\pm)}\right\|_{\mathfrak{h}_{k+1}(S'_1)} \\ \lesssim\ & \epsilon_0 + r^{-1}\|J'^{(\pm)} - J^{(\pm)}\|_{L^2(S'_1)}. \end{aligned}$$

Together with (9.73), we obtain in the case $a_0 \neq 0$, for all $k \leq k_{large} + 7$,

$$\max_{p=0,+,-}\left(r^{-1}\left\|f'_p - f_p\right\|_{\mathfrak{h}_{k+1}(S'_1)} + r^{-1}\left\|J'^{(p)} - J^{(p)}\right\|_{\mathfrak{h}_{k+1}(S'_1)}\right) \lesssim \epsilon_0.$$

In view of the fact that $\nabla_\nu f'_p = 0$ for $p = 0, +, -$ on Σ_*, and in view of the control of ${}^{(ext)}\widetilde{\mathcal{L}_0}$, and hence of f_p for $p = 0, +, -$, we deduce for $a_0 \neq 0$, for all $k \leq k_{large} + 8$,

$$\max_{p=0,+,-}\left(r^{-1}\left\|\mathfrak{d}_*^{\leq k}(f'_p - f_p)\right\|_{L^2(S'_1)} + r^{-1}\left\|\mathfrak{d}_*^{\leq k}(J'^{(p)} - J^{(p)})\right\|_{L^2(S'_1)}\right) \lesssim \epsilon_0.$$

In particular, the above improves[207] the bootstrap assumption (9.70) on $J'^{(p)} - J^{(p)}$ for $p = 0, +, -$.

Remark 9.58. *Recall that Step 18 in the proof of Theorem M0 focuses on the control of $\nu(r')$ and b_* which are involved in the change from the frame of Σ_* to the PG frame of ${}^{(ext)}\mathcal{M}$. Since the change between the frame of Σ_* and the PT frame of ${}^{(ext)}\mathcal{M}$ involves neither $\nu(r')$ nor b_*, we note that there is no need for Step 18'.*

Step 19'. We consider the following change of frames coefficients:

- $(f, \underline{f}, \lambda)$ are the change of frame coefficients from the outgoing geodesic frame of ${}^{(ext)}\widetilde{\mathcal{L}_0}$ to the frame of Σ_*. $(f, \underline{f}, \lambda)$ satisfies according to (9.71)

$$\sup_{k\leq k_{large}+8}\Big(\|\mathfrak{d}^k f\|_{L^2(S'_1)} + r^{-1}\|\mathfrak{d}^k(\underline{f}, \log\lambda)\|_{L^2(S'_1)}$$

[207] Both with respect to the smallness constant and to regularity.

$$+\|\mathfrak{d}^{\leq k-1}\nabla'_\nu(\underline{f}, \log\lambda)\|_{L^2(S'_1)}\Big) \lesssim \epsilon_0.$$

- $(f', \underline{f}', \lambda')$ are the change of frame coefficients from the outgoing PG frame of ${}^{(ext)}\mathcal{L}_0$ to the outgoing geodesic frame of ${}^{(ext)}\widetilde{\mathcal{L}_0}$. In view of Proposition 8.20, we have

$$\sup_{{}^{(ext)}\widetilde{\mathcal{L}_0}} r\left|\mathfrak{d}^{\leq k_{large}+8}\left(f' + \frac{a_0}{r}f_0, \underline{f}' + \frac{a_0\Upsilon}{r}f_0, \log\lambda'\right)\right| \lesssim \epsilon_0.$$

- $(f'', \underline{f}'', \lambda'')$ are the change of frame coefficients from the frame of Σ_* to the outgoing PT frame of ${}^{(ext)}\mathcal{M}$. We assume here that $(f'', \underline{f}'', \lambda'')$ satisfies by the initialization of the PT structure on Σ_*, see Section 9.1.3,

$$\lambda'' = 1, \qquad f'' = \frac{a}{r'}f'_0, \qquad \underline{f}'' = \frac{a\Upsilon'}{r'}f'_0.$$

We now consider the change of frame coefficients $(f''', \underline{f}''', \lambda''')$ from the outgoing PG frame of ${}^{(ext)}\mathcal{L}_0$ to the outgoing PT frame of ${}^{(ext)}\mathcal{M}$. In view of:

- the above estimates for $(f, \underline{f}, \lambda)$ and $(f', \underline{f}', \lambda')$,
- the above formula for $(f'', \underline{f}'', \lambda'')$,
- the control for $r - r'$ and $m - m_0$ given by (9.71),
- the control of a in (9.72) in the case $a_0 = 0$,
- the control for $a - a_0$ in (9.73) and the control for $f'_0 - f_0$ in Step 17' in the case $a_0 \neq 0$,

we infer the following estimates, for all $k \leq k_{large} + 8$,

$$\|\mathfrak{d}^{\leq k} f'''\|_{L^2(S'_1)} + r^{-1}\|\mathfrak{d}^{\leq k}(\underline{f}''', \log\lambda''')\|_{L^2(S'_1)} \lesssim \epsilon_0 + \frac{1}{r}.$$

Together with the dominance condition for r on Σ_*, we infer

$$\|\mathfrak{d}^{\leq k} f'''\|_{L^2(S'_1)} + r^{-1}\|\mathfrak{d}^{\leq k}(\underline{f}''', \log\lambda''')\|_{L^2(S'_1)} \lesssim \epsilon_0, \qquad k \leq k_{large} + 8.$$

Step 20'. In this step, (e_1, e_2, e_3, e_4) denotes the outgoing PG frame of ${}^{(ext)}\mathcal{L}_0$, and (e'_1, e'_2, e'_3, e'_4) denotes the outgoing PT frame of ${}^{(ext)}\mathcal{M}$. From now on, $(f, \underline{f}, \lambda)$ denotes the change of frame coefficients[208] from the outgoing PG

[208] Denoted in the previous step by $(f''', \underline{f}''', \lambda''')$.

frame of ${}^{(ext)}\mathcal{L}_0$ to the outgoing PG frame of ${}^{(ext)}\mathcal{M}$. In view of Step 19', we have on S'_1

$$\|\mathfrak{d}^{\leq k} f\|_{L^2(S'_1)} + r^{-1}\|\mathfrak{d}^{\leq k}(\underline{f}, \log\lambda)\|_{L^2(S'_1)} \lesssim \epsilon_0, \quad k \leq k_{large}+8.$$

Let

$$F = f + i\,{}^*f.$$

Since

$$\Xi' = 0, \qquad \omega' = 0, \qquad \Xi = 0, \qquad \omega = 0, \qquad \underline{\eta} = -\zeta,$$

we have, in view of Corollary 2.81,

$$\begin{aligned} \nabla_{\lambda^{-1}e'_4}(\overline{q}F) &= E_4(f, \Gamma), \\ \lambda^{-1}\nabla'_4(\log\lambda) &= 2f\cdot\zeta + E_2(f, \Gamma). \end{aligned}$$

We integrate the above transport equations for F and λ from S'_1. In view of the control for (f, λ) on S'_1 derived in Step 19', and in view of the assumptions on the initial data layer norm we infer, for all $k \leq k_{large}+8$,

$$\sup_{S'\subset\{u'=1\}}\Big(\|\mathfrak{d}'^{\leq k} f\|_{L^2(S')} + r^{-1}\|\mathfrak{d}'^{\leq k}\log\lambda\|_{L^2(S')}\Big) \lesssim \epsilon_0, \tag{9.75}$$

where[209] u' denotes from now on the scalar function of the PT structure of ${}^{(ext)}\mathcal{M}$. In particular, we have by construction $S'_1 = \Sigma_*\cap\{u'=1\}$, $e'_4(u') = 0$ and $\{u'=1\} \subset {}^{(ext)}\mathcal{L}_0$.

Step 21'. In this step, we estimate $r'-r$, $J'^{(0)} - J^{(0)}$ and $\mathfrak{J}' - \mathfrak{J}$, as well as A', $\widetilde{\mathrm{tr} X}'$ and $\widehat{X}'$. First, since $J'^{(0)}$ is propagated from Σ_* by $e'_4(J'^{(0)}) = 0$, and using the change of frame formula between the PG frame of ${}^{(ext)}\mathcal{L}_0$ and the PT frame of ${}^{(ext)}\mathcal{M}$, we infer

$$e'_4(J'^{(0)} - J^{(0)}) \;=\; -\lambda\left(f\cdot\nabla + \frac{1}{4}|f|^2 e_3\right)J^{(0)}.$$

Together with the control of f and λ of Step 20', and the control of ${}^{(ext)}\mathcal{L}_0$, we infer, for $k \leq k_{large}+8$,

$$\sup_{S'\subset\{u'=1\}} r\|\mathfrak{d}^{\leq k}(e'_4(J'^{(0)} - J^{(0)}))\|_{L^2(S')} \lesssim \epsilon_0.$$

[209] Note that u' is associated both to the PT and to the PG structure of ${}^{(ext)}\mathcal{M}$ in view of Lemma 9.24.

Integrating from S'_1 where $J'^{(0)} - J^{(0)}$ is under control in view of Step 17', we infer, for all $k \leq k_{large} + 8$,

$$\sup_{S' \subset \{u'=1\}} r^{-1} \|\mathfrak{d}^{\leq k}(J'^{(0)} - J^{(0)})\|_{L^2(S')} \lesssim \epsilon_0. \tag{9.76}$$

Next, we control $\widetilde{\mathrm{tr}X}'$. To this end, we also need to control A' and $\widehat{X}'$. First, note that the change of frame formula for A', the control of the foliation of ${}^{(ext)}\mathcal{L}_0$, the control of f and λ of Step 20', and the fact that the transformation formula for A' does not depend on $\underline{f}$ implies, since $k \leq k_{large} + 8$,

$$\sup_{S' \subset \{u'=1\}} r^{\frac{5}{2}+\delta_B} \|\mathfrak{d}^{\leq k} A'\|_{L^2(S')} \lesssim \epsilon_0. \tag{9.77}$$

Then, using the control of $(f, \underline{f}, \lambda)$ on S'_1 derived in Step 19', the control of the initial data layer, and the change of frame formulas between the PG frame of ${}^{(ext)}\mathcal{L}_0$ and the PT frame of ${}^{(ext)}\mathcal{M}$ on S'_1, we infer, for all $k \leq k_{large} + 7$,

$$r\Big(\|\mathfrak{d}^{\leq k}\widetilde{\mathrm{tr}X}'\|_{L^2(S'_1)} + \|\mathfrak{d}^{\leq k}\widehat{X}'\|_{L^2(S'_1)}\Big) \lesssim \epsilon_0.$$

Then, propagating Raychadhuri for $\widetilde{\mathrm{tr}X}'$ and the null structure equation for $\nabla'_4\widehat{X}'$ from S'_1 where $\widetilde{\mathrm{tr}X}'$ and $\widehat{X}'$ are under control in view of the above estimate, we infer, using also the above control of A',

$$\sup_{S' \subset \{u'=1\}} r\Big(\|\mathfrak{d}^{\leq k}\widetilde{\mathrm{tr}X}'\|_{L^2(S')} + \|\mathfrak{d}^{\leq k}\widehat{X}'\|_{L^2(S')}\Big) \lesssim \epsilon_0, \qquad k \leq k_{large} + 7.$$

Next, we notice

$$\mathrm{tr}X' - \lambda\mathrm{tr}X = \frac{2}{q'} - \frac{2}{q} + (\lambda - 1)\mathrm{tr}X + \widetilde{\mathrm{tr}X}' - \widetilde{\mathrm{tr}X}$$

so that

$$q' - q = \frac{qq'}{2}\left(-\lambda\Big(\lambda^{-1}\mathrm{tr}X' - \mathrm{tr}X\Big) + (\lambda - 1)\mathrm{tr}X + \widetilde{\mathrm{tr}X}' - \widetilde{\mathrm{tr}X}\right).$$

Together with the control of Step 20' for f and λ, the above control for $J'^{(0)} - J^{(0)}$ and $\widetilde{\mathrm{tr}X}'$, the control of $a - a_0$ in (9.72) (9.73), the control of the foliation ${}^{(ext)}\mathcal{L}_0$, and the fact that $q = r + ia_0 J^{(0)}$ and $q' = r' + iaJ'^{(0)}$ we infer, for all $k \leq k_{large} + 7$.

$$\sup_{S'\subset\{u'=1\}} r^{-1}\left\|\mathfrak{d}^{\leq k}\left(\frac{r'}{r}-1+\frac{qq'}{2r}\Big(\lambda^{-1}\mathrm{tr}X'-\mathrm{tr}X\Big)\right)\right\|_{L^2(S')} \lesssim \epsilon_0.$$

Moreover, from the change of frame formulas for $\mathrm{tr}\,\chi$ and ${}^{(a)}\mathrm{tr}\chi$ we have, schematically,

$$\lambda^{-1}\mathrm{tr}X'-\mathrm{tr}X = r'^{-1}\mathfrak{d}'f+\Gamma\cdot f+f\cdot\underline{f}\cdot\Gamma+f\cdot\underline{f}\cdot\mathrm{tr}X'+ \text{ l.o.t.}$$

Together with the control of Step 20' for f, the above control for $\widetilde{\mathrm{tr}X}'$ and the control of the foliation ${}^{(ext)}\mathcal{L}_0$ we deduce, for all $k\leq k_{large}+7$,

$$\begin{aligned}\sup_{S'\subset\{u'=1\}} r^{-1}\left\|\mathfrak{d}^{\leq k}\left(\frac{r'}{r}-1\right)\right\|_{L^2(S')}\\ \lesssim \epsilon_0+\epsilon_0\sup_{S'\subset\{u'=1\}} r^{-1}\left\|\mathfrak{d}^{\leq k}\underline{f}\right\|_{L^2(S')}.\end{aligned} \tag{9.78}$$

Next, recall the definition of $\mathfrak{J}'$ on Σ_*, see Definition 3.8,

$$\mathfrak{J}' = \frac{1}{|q'|}(f_0'+i\,{}^*f_0')=\frac{1}{\sqrt{r'^2+a^2(J'^{(0)})^2}}(f_0'+i\,{}^*f_0').$$

Together with the control of $r-r'$ and $m-m_0$ given by (9.71), the control of $a-a_0$ in (9.72) (9.73), the control for $f_0'-f_0$ and $J'^{(0)}-J^{(0)}$ in Step 17', and the control of ${}^{(ext)}\mathcal{L}_0$, we infer

$$\|\mathfrak{d}_*^{\leq k}(\mathfrak{J}'-\mathfrak{J})\|_{L^2(S_1')} \lesssim \epsilon_0+\frac{1}{r}.$$

Together with the dominance condition for r on Σ_*, we infer, for all $k\leq k_{large}+7$,

$$\|\mathfrak{d}_*^{\leq k}(\mathfrak{J}'-\mathfrak{J})\|_{L^2(S_1')} \lesssim \epsilon_0.$$

Together with the identity

$$\begin{aligned}q'\mathfrak{J}'-q\mathfrak{J} &= q'(\mathfrak{J}'-\mathfrak{J})+(q'-q)\mathfrak{J}\\ &= q'(\mathfrak{J}'-\mathfrak{J})+\Big(r'-r+i(aJ'^{(0)}-a_0J^{(0)})\Big)\mathfrak{J},\end{aligned}$$

the control of $r'-r$ given by (9.71), the control of $J'^{(0)}-J^{(0)}$ in Step 17' and the control of $a-a_0$ in (9.72) (9.73), we obtain

$$r^{-1}\|\mathfrak{d}_*^{\leq k}(q'\mathfrak{J}'-q\mathfrak{J})\|_{L^2(S_1')} \lesssim \epsilon_0.$$

Also, recall that $\mathfrak{J}'$ and $\mathfrak{J}$ satisfy in $\{u'=1\}$

$$\nabla'_4\mathfrak{J}' = -\frac{1}{q'}\mathfrak{J}', \qquad \nabla_4\mathfrak{J} = -\frac{1}{q}\mathfrak{J},$$

and hence

$$\nabla'_4(q'\mathfrak{J}') = 0, \qquad \nabla_4(q\mathfrak{J}) = 0.$$

We infer

$$\nabla_{\lambda^{-1}4'}(q'\mathfrak{J}') = 0, \qquad \nabla_{\lambda^{-1}4'}(q\mathfrak{J}) = \left(f\cdot\nabla + \frac{1}{4}|f|^2 e_3\right)(q\mathfrak{J}).$$

Together with the control of f and λ of Step 20', (see estimate (9.75)), and the control of ${}^{(ext)}\mathcal{L}_0$, we obtain, for $k\leq k_{large}+8$,

$$\sup_{S'\subset\{u'=1\}} r\|\mathfrak{d}^{\leq k}(\nabla'_4(q'\mathfrak{J}'-q\mathfrak{J}))\|_{L^2(S')} \quad\lesssim\quad \epsilon_0.$$

Integrating from S'_1 where $q'\mathfrak{J}'-q\mathfrak{J}$ is under control in view of the above we infer, for all $k\leq k_{large}+7$,

$$\sup_{S'\subset\{u'=1\}} \|\mathfrak{d}^{\leq k}(q'\mathfrak{J}'-q\mathfrak{J})\|_{L^2(S')} \quad\lesssim\quad \epsilon_0.$$

Using again the above identity for $q'\mathfrak{J}'-q\mathfrak{J}$, the above control of $J'^{(0)}-J^{(0)}$ and $r'-r$ and the control of $a-a_0$ in (9.72) (9.73), we deduce for all $k\leq k_{large}+7$,

$$\sup_{S'\subset\{u'=1\}} \|\mathfrak{d}^{\leq k}(\mathfrak{J}'-\mathfrak{J})\|_{L^2(S')} \lesssim \epsilon_0 + \epsilon_0 \sup_{S'\subset\{u'=1\}} r^{-1}\left\|\mathfrak{d}^{\leq k}\underline{f}\right\|_{L^2(S')}. \tag{9.79}$$

Step 22'. To estimate $\underline{f}$ we make use of the last equation in Corollary 2.81, and the fact that $\underline{H}+\frac{a\overline{q}}{|q|^2}\mathfrak{J} = -\check{Z}$, see Definition 2.66. For $\underline{F}=\underline{f}+i\,{}^*\underline{f}$, we obtain

$$\nabla_{\lambda^{-1}e'_4}\underline{F} \quad=\quad -2\left(\frac{a\overline{q'}}{|q'|^2}\mathfrak{J}' - \frac{a\overline{q}}{|q|^2}\mathfrak{J}\right) + 2\check{Z} - \frac{1}{2}\overline{\mathrm{tr}X}\underline{F} - F\cdot\underline{\widehat{\chi}} + E_6(f,\underline{f},\Gamma).$$

Together with the control of $a-a_0$ in (9.72) (9.73), the control of $J'^{(0)}-J^{(0)}$, $r'-r$ and $\mathfrak{J}'-\mathfrak{J}$ of Step 21', and the control on the Ricci coefficients of the PG frame of ${}^{(ext)}\mathcal{L}_0$, we obtain for, all $k\leq k_{large}+7$,

$$\sup_{S'\subset\{u'=1\}} r\left\|\mathfrak{d}^{\leq k}(\lambda^{-1}\nabla_{e'_4}\underline{f})\right\|_{L^2(S')} \lesssim \quad \epsilon_0 + \epsilon_0 \sup_{S'\subset\{u'=1\}} r^{-1}\left\|\mathfrak{d}^{\leq k}\underline{f}\right\|_{L^2(S')}.$$

Together with the above, and the control of λ of Step 20', see estimate (9.75), we infer

$$\sup_{S'\subset\{u'=1\}} r\left\|\mathfrak{d}^{\leq k}\nabla_{e_4'}\underline{f}\right\|_{L^2(S')} \lesssim \epsilon_0+\epsilon_0\sup_{S'\subset\{u'=1\}} r^{-1}\left\|\mathfrak{d}^{\leq k}\underline{f}\right\|_{L^2(S')}.$$

Integrating from S_1' where $\underline{f}$ is under control in view of Step 19', we infer

$$\sup_{S'\subset\{u'=1\}} r^{-1}\left\|\mathfrak{d}^{\leq k}\underline{f}\right\|_{L^2(S')} \lesssim \epsilon_0+\epsilon_0\sup_{S'\subset\{u'=1\}} r^{-1}\left\|\mathfrak{d}^{\leq k}\underline{f}\right\|_{L^2(S')}$$

and hence, for all $k\leq k_{large}+7$,

$$\sup_{S'\subset\{u'=1\}} r^{-1}\left\|\mathfrak{d}^{\leq k}\underline{f}\right\|_{L^2(S')} \lesssim \epsilon_0.$$

Together with the control of f and λ of Step 20', we have finally obtained, for all $k\leq k_{large}+7$,

$$\sup_{S'\subset\{u'=1\}}\Big(\|\mathfrak{d}^{\leq k}f\|_{L^2(S')}+r^{-1}\|\mathfrak{d}^{\leq k}(\log(\lambda),\underline{f})\|_{L^2(S')}\Big) \lesssim \epsilon_0. \tag{9.80}$$

Also, together with the estimates (9.79) and (9.78) for $r'-r$ and $\mathfrak{J}'-\mathfrak{J}$ of Step 21', we obtain for all $k\leq k_{large}+7$,

$$\sup_{S'\subset\{u'=1\}}\left(r^{-1}\left\|\mathfrak{d}^{\leq k}\left(\frac{r'}{r}-1\right)\right\|_{L^2(S')}+\|\mathfrak{d}^{\leq k}(\mathfrak{J}'-\mathfrak{J})\|_{L^2(S')}\right)\lesssim\epsilon_0. \tag{9.81}$$

Step 23'. Let $(f',\underline{f}',\lambda')$ denote the change of frame coefficients from the ingoing PG frame of ${}^{(int)}\mathcal{L}_0$ to the ingoing PT frame of ${}^{(int)}\mathcal{M}'$. From

- the estimates of Step 22' on $\{u'=1\}$,
- the fact that ${}^{(int)}\mathcal{M}'\cap{}^{(ext)}\mathcal{M}=\{r'=r_0\}$,
- the fact that $\{u'=1\}\cap\{\underline{u}'=1\}$ is included in ${}^{(ext)}\mathcal{L}_0\cap{}^{(int)}\mathcal{L}_0$,
- the initialization of the frame of ${}^{(int)}\mathcal{M}'$ as an explicit renormalization of the frame of ${}^{(ext)}\mathcal{M}$ on $\{r'=r_0\}$,
- the control in ${}^{(ext)}\mathcal{L}_0\cap{}^{(int)}\mathcal{L}_0$ of the difference between the frame of ${}^{(int)}\mathcal{L}_0$ and an explicit renormalization of the frame of ${}^{(ext)}\mathcal{L}_0$,

we easily infer, using also $\underline{u}'=u'$ on $\{r=r_0\}$, for all $k\leq k_{large}+7$,

$$\sup_{S'\subset\{r'=r_0\}\cap\{\underline{u}'=1\}}\Big(\|\mathfrak{d}^{\leq k}(f',\log(\lambda'),\underline{f}')\|_{L^2(S')}\Big) \lesssim \epsilon_0.$$

Next, we proceed as in Step 20', exchanging the role of e_3 and e_4, and we propagate along e_3 the above estimate to $\{\underline{u}' = 1\}$ for $\underline{f}'$ and λ'. We also propagate the control of Step 21' for $J'^{(0)} - J^{(0)}$ on $\{u' = 1\}$, and hence on its boundary $\{r' = r_0\}$ to $\{\underline{u}' = 1\}$. Also one propagates the control of Step 22' for $r - r'$ on $\{u' = 1\}$, and hence on its boundary $\{r' = r_0\}$ to $\{\underline{u}' = 1\}$ using the transport equation[210]

$$\begin{aligned} e_3'(r'-r) &= 1 - \lambda'\left(e_3 + \underline{f}'^a e_a + \frac{1}{4}|\underline{f}'|^2 e_4\right) r \\ &= -(\lambda'-1) + \underline{f}' \cdot \nabla(r) + \frac{1}{4}|\underline{f}'|^2 e_4(r). \end{aligned}$$

Then, one propagates $\mathfrak{J}' - \mathfrak{J}$ similarly to Step 21'. Finally, we propagate f' similarly to Step 22'. We finally obtain

$$\sup_{S' \subset \{\underline{u}'=1\}} \left(\|\mathfrak{d}^{\leq k}(f', \underline{f}', \log \lambda')\|_{L^2(S')} \right) \lesssim \epsilon_0, \qquad k \leq k_{large} + 7$$

and

$$\sup_{S' \subset \{\underline{u}'=1\}} \left\|\mathfrak{d}^{\leq j+1}\left(J'^{(0)} - J^{(0)}, r' - r, \mathfrak{J}' - \mathfrak{J}\right)\right\|_{L^2(S')} \lesssim \epsilon_0, \qquad k \leq k_{large} + 7.$$

Step 24'. Note that the desired estimates[211] for $m - m_0$ and $a - a_0$ have been obtained respectively in Step 13' and in (9.72) (9.73). To conclude the proof of Theorem 9.46, it remains to control $k \leq k_{karge} + 7$ derivatives, with suitable r-weights and $O(\epsilon_0)$ smallness constant, of A, B, $\check{P}$, $\underline{B}$ and $\underline{A}$ in $\{u' = 1\} \cup \{\underline{u}' = 1\}$. This follows from:

- the control of $(f, \underline{f}, \lambda)$ on $\{u' = 1\}$ derived in Step 22',
- the control of $(f', \underline{f}', \lambda')$ on $\{\underline{u}' = 1\}$ derived in Step 23',
- the fact that $(f, \underline{f}, \lambda)$ denote the change of frame coefficients from the outgoing PT frame of ${}^{(ext)}\mathcal{L}_0$ to the outgoing PT frame of ${}^{(ext)}\mathcal{M}$, and

[210]Note that we could not have used this transport equation in ${}^{(ext)}\mathcal{L}_0$ in view of the lack of decay in r for $\lambda - 1$. This is why we avoided this transport equation in Step 21' and used instead the control of $\widetilde{\mathrm{tr}X}'$. On the other hand, r is bounded in ${}^{(int)}\mathcal{L}_0$ so that one can simply rely on the transport equation for $e_3'(r'-r)$ in ${}^{(int)}\mathcal{L}_0$. This is in fact crucial: proceeding as in Step 21' would only lead to the control $k_{large} + 6$ derivatives of $r' - r$, and hence to a loss of at least one derivatives in Theorem 9.46.

[211]Since they only require a small number of derivatives their proof has in fact already been obtained in Theorem M0.

the fact that $(f', \underline{f}', \lambda')$ denote the change of frame coefficients from the ingoing PT frame of ${}^{(int)}\mathcal{L}_0$ to the ingoing PT frame of ${}^{(int)}\mathcal{M}'$,
- the change of frame formulas for the curvature components,
- in the particular case of the estimate for $\check{P}$, the fact that

$$P' - P \quad = \quad -\frac{2m}{q'^3} + \frac{2m_0}{q^3} + \check{P}' - \check{P},$$

together with the control of $m - m_0$ derived in Step 13', the control of $a - a_0$ in (9.72) (9.73), the control of $J'^{(0)} - J^{(0)}$ in Step 21' and 23', and the control of $r' - r$ in Steps 22' and 23',
- the assumptions for the curvature components of the foliations of ${}^{(ext)}\mathcal{L}_0$ and ${}^{(int)}\mathcal{L}_0$.

This concludes the proof of Theorem 9.46.

9.6. Control of curvature

The goal of this section is to prove Theorem 9.49 on the control of curvature coefficients. We first construct a global frame that will be used to perform the curvature estimates on $\mathcal{M}$.

9.6.1. Construction of a global null frame The construction of the global null frame in this section is analogous to the one in section 4.5, the main difference being that we cannot lose derivatives in the context of the proof of Theorem 9.49.

In order to produce a global frame on $\mathcal{M}$, we will proceed in several steps. First, we extend the ingoing PT structure of ${}^{(int)}\mathcal{M}'$ slightly inside ${}^{(ext)}\mathcal{M}$.

Lemma 9.59. *We may extend the ingoing PT structure of ${}^{(int)}\mathcal{M}'$ into the region*

$$\mathcal{R}_{(1)} := {}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1) \cap \{\underline{u} \leq u_*\}. \tag{9.82}$$

Furthermore:

- *we have, for $0 \leq k \leq k_{large} + 7$,*

$$\int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^k({}^{(int)}\check{\Gamma})\right|^2 + \int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^k({}^{(int)}\check{R})\right|^2 \quad \lesssim \quad \mathfrak{G}_k^2 + \mathfrak{R}_k^2,$$

where ${}^{(int)}\check{\Gamma}$ and ${}^{(int)}\check{R}$ denote respectively the linearized Ricci coefficients and curvature components of the ingoing PT structure of ${}^{(int)}\mathcal{M}'$ extended into $\mathcal{R}_{(1)}$,

- *we have, for* $1 \le k \le k_{large}+8$,

$$\int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\le k}\left(f, \underline{f}, \log\left(\frac{|{}^{(ext)}q|^2}{{}^{(ext)}\Delta}\lambda\right)\right)\right|^2 \lesssim \mathfrak{G}_{k-1}^2 + \mathfrak{R}_{k-1}^2,$$

  where $(f, \underline{f}, \lambda)$ *denotes the change of frame coefficients from the outgoing PT frame of* ${}^{(ext)}\mathcal{M}$ *to the ingoing PT frame of* ${}^{(int)}\mathcal{M}'$ *extended into* $\mathcal{R}_{(1)}$,
- *we have, for* $1 \le k \le k_{large}+8$,

$$\int_{\mathcal{R}_{(1)}\cap {}^{(top)}\mathcal{M}'} \left|\mathfrak{d}^{\le k}\left(f, \underline{f}, \log\lambda\right)\right|^2 \lesssim \mathfrak{G}_{k-1}^2 + \mathfrak{R}_{k-1}^2,$$

  where $(f, \underline{f}, \lambda)$ *denotes the change of frame coefficients from the ingoing PT frame of* ${}^{(int)}\mathcal{M}'$ *to the ingoing PT frame of* ${}^{(top)}\mathcal{M}'$,
- *we have, for* $1 \le k \le k_{large}+8$,

$$\begin{aligned}\int_{\mathcal{R}_{(1)}} \Bigg(\left|\mathfrak{d}^{\le k}\left({}^{(int)}r - {}^{(ext)}r\right)\right|^2 &\\ + \left|\mathfrak{d}^{\le k}\left(\cos({}^{(int)}\theta) - \cos({}^{(ext)}\theta)\right)\right|^2 &\\ + \left|\mathfrak{d}^{\le k}\left({}^{(int)}\mathfrak{J} - {}^{(ext)}\mathfrak{J}\right)\right|^2\Bigg) &\lesssim \mathfrak{G}_{k-1}^2 + \mathfrak{R}_{k-1}^2,\end{aligned}$$

  and

$$\begin{aligned}\int_{\mathcal{R}_{(1)}\cap {}^{(top)}\mathcal{M}'} \Bigg(\left|\mathfrak{d}^{\le k}\left({}^{(int)}r - {}^{(top)}r, \underline{u} - {}^{(top)}\underline{u}\right)\right|^2 &\\ + \left|\mathfrak{d}^{\le k}\left(\cos({}^{(int)}\theta) - \cos({}^{(top)}\theta)\right)\right|^2 &\\ + \left|\mathfrak{d}^{\le k}\left({}^{(int)}\mathfrak{J} - {}^{(top)}\mathfrak{J}\right)\right|^2\Bigg) &\lesssim \mathfrak{G}_{k-1}^2 + \mathfrak{R}_{k-1}^2,\end{aligned}$$

  where $({}^{(ext)}r, {}^{(ext)}\theta, {}^{(ext)}\mathfrak{J})$ *is associated with outgoing PT structure of* ${}^{(ext)}\mathcal{M}$, *and where* $({}^{(int)}r, \underline{u}, {}^{(int)}\theta, {}^{(int)}\mathfrak{J})$ *and* $({}^{(top)}r, {}^{(top)}\underline{u}, {}^{(top)}\theta, {}^{(top)}\mathfrak{J})$ *are associated respectively with the ingoing PT structures of* ${}^{(int)}\mathcal{M}'$ *and* ${}^{(top)}\mathcal{M}'$.

Proof. See Section 9.6.3. □

Remark 9.60. *Along level hypersurfaces of* $\underline{u}$, *in the region* $r \sim r_0$ *on* ${}^{(ext)}\mathcal{M}$, *we have*

$$\frac{du}{dr} = \frac{e_3(u)}{e_3(r)} = -2 + O(r_0^{-1}).$$

In particular, since $\underline{u} = u$ *on* $\mathcal{T} = \{r = r_0\}$, *we infer*

$$u \geq (u_* - 1) - 2 + O(r_0^{-1}) > u_* - \frac{7}{2}$$
$$\text{on} \quad {}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1) \cap \{u_* - 1 \leq \underline{u} \leq u_*\}$$

and hence

$${}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1) \cap \{u_* - 1 \leq \underline{u} \leq u_*\} \subset {}^{(ext)}\mathcal{M}(u \geq u_* - 7/2).$$

Since ${}^{(top)}\mathcal{M}' \cap {}^{(ext)}\mathcal{M} = {}^{(ext)}\mathcal{M}(u \geq u'_*)$, *and since* u'_* *is chosen such that* $u'_* \leq u_* - 4$, *we also have*

$${}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1) \cap \{u_* - 1 \leq \underline{u} \leq u_*\} \subset {}^{(top)}\mathcal{M}'.$$

We now glue the ingoing PT frame of ${}^{(int)}\mathcal{M}'$, extended slightly into ${}^{(ext)}\mathcal{M}$ in Lemma 9.59, to the ingoing PT frame of ${}^{(top)}\mathcal{M}'$ in the matching region

$$\text{(9.83)} \qquad \text{Match}_1 \;:=\; {}^{(int)}\mathcal{M}'(\underline{u} \geq u_* - 1) \cup \Big({}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1) \cap \{u_* - 1 \leq \underline{u} \leq u_*\}\Big).$$

Lemma 9.61. *There exists a frame* (e'_4, e'_3, e'_1, e'_2) *on*

$${}^{(int)}\mathcal{M}' \cup \big({}^{(top)}\mathcal{M}' \setminus {}^{(ext)}\mathcal{M}\big) \cup {}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1) \cup {}^{(ext)}\mathcal{M}(u \geq u_* - 1),$$

as well as a pair of scalar functions $(r', J'^{(0)})$, *and a complex 1-form* $\mathfrak{J}'$, *such that:*

(a) In

$$\Big(\big({}^{(ext)}\mathcal{M}(u \geq u_* - 1) \cup {}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1)\big) \setminus \{\underline{u} \leq u_*\}\Big) \cup \Big({}^{(top)}\mathcal{M}' \setminus \big({}^{(int)}\mathcal{M}' \cup {}^{(ext)}\mathcal{M}\big)\Big)$$

we have

$$(e'_4, e'_3, e'_1, e'_2) = ({}^{(top)}e_4, {}^{(top)}e_3, {}^{(top)}e_1, {}^{(top)}e_2),$$

as well as $r' = {}^{(top)}r$, $J'^{(0)} = \cos({}^{(top)}\theta)$, *and* $\mathfrak{J}' = {}^{(top)}\mathfrak{J}$, *where the* *triplet* $({}^{(top)}r, {}^{(top)}\theta, {}^{(top)}\mathfrak{J})$ *is associated with the ingoing PT structure of* ${}^{(top)}\mathcal{M}'$.

(b) *In*

$$\Big({}^{(int)}\mathcal{M}' \cup {}^{(ext)}\mathcal{M}({}^{(ext)}r \le r_0 + 1)\Big) \cap \{\underline{u} \le u_* - 1\},$$

we have

$$(e'_4, e'_3, e'_1, e'_2) = ({}^{(int)}e_4, {}^{(int)}e_3, {}^{(int)}e_1, {}^{(int)}e_2),$$

as well as $r' = {}^{(int)}r$, $J'^{(0)} = \cos({}^{(int)}\theta)$, *and* $\mathfrak{J}' = {}^{(int)}\mathfrak{J}$, *where we recall that the ingoing PT structure of* ${}^{(int)}\mathcal{M}'$ *has been extended slightly into* ${}^{(ext)}\mathcal{M}$ *in Lemma 9.59.*

(c) *In the matching region, we have, for* $0 \le k \le k_{large} + 7$,

$$\int_{Match_1} \Big|\mathfrak{d}^k(\check{\Gamma}')\Big|^2 + \int_{Match_1} \Big|\mathfrak{d}^k(\check{R}')\Big|^2 \lesssim \mathfrak{R}_k^2 + \mathfrak{G}_k^2,$$

where $\check{\Gamma}'$ *and* $\check{R}'$ *are the one associated to the frame* (e'_4, e'_3, e'_1, e'_2) *and renormalized using* $(r', J'^{(0)}, \mathfrak{J}')$.

(d) *In the matching region, we also have, for* $1 \le k \le k_{large} + 8$,

$$\int_{Match_1} \Big|\mathfrak{d}^{\le k}(f, \underline{f}, \log\lambda)\Big|^2 \quad \lesssim \quad \mathfrak{R}_{k-1}^2 + \mathfrak{G}_{k-1}^2,$$

where $(f, \underline{f}, \lambda)$ *denotes*

- *either the change of frame coefficients from the ingoing null frame* $({}^{(top)}e_4, {}^{(top)}e_3, {}^{(top)}e_1, {}^{(top)}e_2)$ *to* (e'_4, e'_3, e'_1, e'_2),
- *or the one from* $({}^{(int)}e_4, {}^{(int)}e_3, {}^{(int)}e_1, {}^{(int)}e_2)$ *to* (e'_4, e'_3, e'_1, e'_2).

(e) *In the matching region, we have, for* $1 \le k \le k_{large} + 8$,

$$\int_{Match_1} \Big|\mathfrak{d}^{\le k}\big(r' - {}^{(int)}r,\ J'^{(0)} - \cos({}^{(int)}\theta),\ \mathfrak{J}' - {}^{(int)}\mathfrak{J}\big)\Big|^2 \lesssim \mathfrak{R}_{k-1}^2 + \mathfrak{G}_{k-1}^2.$$

Proof. See Section 9.6.4. ☐

Finally, we glue a renormalization of the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ to the frame of Lemma 9.61 in the matching region

$$\text{Match}_2 \quad := \quad {}^{(ext)}\mathcal{M}(u \ge u_* - 1) \cup {}^{(ext)}\mathcal{M}({}^{(ext)}r \le r_0 + 1). \tag{9.84}$$

Remark 9.62. *Note that the frame of Lemma 9.61 is defined on*

$$^{(int)}\mathcal{M}' \cup \big({}^{(top)}\mathcal{M}' \setminus {}^{(ext)}\mathcal{M}\big) \cup Match_2.$$

In particular, the outgoing PT frame of $^{(ext)}\mathcal{M}$ *and the frame of Lemma 9.61 are both defined in* $Match_2$.

The following lemme provides the desired global frame on $\mathcal{M}$.

Lemma 9.63. *Let* (e_4', e_3', e_1', e_2') *be the frame of Lemma 9.61. There exists a global null frame* $({}^{(glo)}e_4, {}^{(glo)}e_3, {}^{(glo)}e_1, {}^{(glo)}e_2)$ *defined on* $\mathcal{M}$*, as well as a pair of scalar functions* $({}^{(glo)}r, {}^{(glo)}J^{(0)})$*, and a complex 1-form* $^{(glo)}\mathfrak{J}$*, such that:*

(a) In $^{(ext)}\mathcal{M} \setminus Match_2$*, we have*

$$\begin{aligned} & \big({}^{(glo)}e_4, {}^{(glo)}e_3, {}^{(glo)}e_1, {}^{(glo)}e_2\big) \\ = & \big({}^{(ext)}\lambda {}^{(ext)}e_4, {}^{(ext)}\lambda^{-1} {}^{(ext)}e_3, {}^{(ext)}e_1, {}^{(ext)}e_2\big) \end{aligned}$$

where $^{(ext)}\lambda := \frac{^{(ext)}\Delta}{|^{(ext)}q|^2}$*, as well as* $^{(glo)}r = {}^{(ext)}r$*,* $^{(glo)}J^{(0)} = \cos({}^{(ext)}\theta)$*, and* $^{(glo)}\mathfrak{J} = {}^{(ext)}\mathfrak{J}$.

(b) In $^{(int)}\mathcal{M}' \cup \big({}^{(top)}\mathcal{M}' \setminus {}^{(ext)}\mathcal{M}\big)$*, we have*

$$\big({}^{(glo)}e_4, {}^{(glo)}e_3, {}^{(glo)}e_1, {}^{(glo)}e_2\big) = (e_4', e_3', e_1', e_2'),$$

as well as $^{(glo)}r = r'$*,* $^{(glo)}J^{(0)} = J'^{(0)}$*, and* $^{(glo)}\mathfrak{J} = \mathfrak{J}'$.

(c) In the matching region, we have, for $0 \le k \le k_{large} + 7$*,*

$$^{(glo)}\mathfrak{G}_{k,Match_2} + {}^{(glo)}\mathfrak{R}_{k,Match_2} \lesssim \mathfrak{R}_k + \mathfrak{G}_k,$$

where

$$\begin{aligned} \big({}^{(glo)}\mathfrak{R}_{k,Match_2}\big)^2 \;:=\; & \int_{Match_2} \Big(r^{3+\delta_B} |\mathfrak{d}^k({}^{(glo)}A, {}^{(glo)}B)|^2 \\ & + r^{3-\delta_B} |\mathfrak{d}^{k\,(glo)}\check{P}|^2 + r^{1-\delta_B} |\mathfrak{d}^{k\,(glo)}\underline{B}|^2 \\ & + r^{-1-\delta_B} |\mathfrak{d}^{k\,(glo)}\underline{A}|^2 \Big) \end{aligned}$$

and

$$\begin{aligned} \big({}^{(glo)}\mathfrak{G}_{k,Match_2}\big)^2 \;:=\; & \int_{Match_2({}^{(ext)}r \le r_0+1)} |\mathfrak{d}^{k\,(glo)}\check{\Gamma}|^2 \\ & + \sup_{\underline{u}_1 \ge u_*} \int_{Match_{2,\underline{u}_1}} \Big(r^2 \Big|\mathfrak{d}^k\big({}^{(glo)}\Gamma_g \setminus \{\widetilde{{}^{(glo)}tr\underline{X}}\}\big)\Big|^2 \end{aligned}$$

$$+r^{2-\delta_B}\left|\mathfrak{d}^k\big(\widetilde{{}^{(glo)}tr\underline{X}}\big)\right|^2+\left|\mathfrak{d}^k\big({}^{(glo)}\Gamma_b\setminus\{{}^{(glo)}\underline{\Xi}\}\big)\right|^2$$
$$+r^{-\delta_B}\left|\mathfrak{d}^k\big({}^{(glo)}\underline{\Xi}\big)\right|^2\Bigg),$$

with the notation

$$Match_{2,\underline{u}_1} \quad := \quad Match_2\cap\{\underline{u}_1\leq\underline{u}\leq\underline{u}_1+1\}, \tag{9.85}$$

and with ${}^{(glo)}\Gamma_g$ *and* ${}^{(glo)}\Gamma_b$ *only containing linearized metric and Ricci coefficients, but not curvature components.*

(d) In the matching region, we have, for $1\leq k\leq k_{large}+8$,

$$\sup_{\underline{u}_1\geq u_*}\int_{Match_{2,\underline{u}_1}}\left(\left|\mathfrak{d}^{\leq k}(f,\log\lambda)\right|^2+r^{-\delta_B}\left|\mathfrak{d}^{\leq k}\underline{f}\right|^2\right)$$
$$+\int_{Match_2({}^{(ext)}r\leq r_0+1)}\left|\mathfrak{d}^{\leq k}(f,\underline{f},\log\lambda)\right|^2 \quad\lesssim\quad \mathfrak{R}^2_{k-1}+\mathfrak{G}^2_{k-1},$$

where $(f,\underline{f},\lambda)$ *denotes*

- *either the change of frame coefficients from* (e_4',e_3',e_1',e_2') *to the global frame* $({}^{(glo)}e_4,{}^{(glo)}e_3,{}^{(glo)}e_1,{}^{(glo)}e_2)$,
- *or the one from* $({}^{(ext)}\lambda{}^{(ext)}e_4,{}^{(ext)}\lambda^{-1}{}^{(ext)}e_3,{}^{(ext)}e_1,{}^{(ext)}e_2)$ *to the global frame* $({}^{(glo)}e_4,{}^{(glo)}e_3,{}^{(glo)}e_1,{}^{(glo)}e_2)$.

(e) In the matching region, we have, for $1\leq k\leq k_{large}+8$,

$$\sup_{\underline{u}_1\geq u_*}\int_{Match_{2,\underline{u}_1}}\left(r^{-2}\left|\mathfrak{d}^{\leq k}\left({}^{(glo)}r-r'\right)\right|^2+\left|\mathfrak{d}^{\leq k}\left({}^{(glo)}J^{(0)}-J'^{(0)}\right)\right|^2\right.$$
$$\left.+r^2\left|\mathfrak{d}^{\leq k}\left({}^{(glo)}\mathfrak{J}-\mathfrak{J}'\right)\right|^2\right)$$
$$+\int_{Match_2({}^{(ext)}r\leq r_0+1)}\left|\mathfrak{d}^{\leq k}\left({}^{(glo)}r-r',{}^{(glo)}J^{(0)}-J'^{(0)},{}^{(glo)}\mathfrak{J}-\mathfrak{J}'\right)\right|^2$$
$$\lesssim\quad\mathfrak{R}^2_{k-1}+\mathfrak{G}^2_{k-1},$$

and

$$\sup_{\underline{u}_1\geq u_*}\int_{Match_{2,\underline{u}_1}}\left(r^{-2}\left|\mathfrak{d}^{\leq k}\left({}^{(glo)}r-{}^{(ext)}r\right)\right|^2\right.$$
$$\left.+\left|\mathfrak{d}^{\leq k}\left({}^{(glo)}J^{(0)}-\cos({}^{(ext)}\theta)\right)\right|^2+r^2\left|\mathfrak{d}^{\leq k}\left({}^{(glo)}\mathfrak{J}-{}^{(ext)}\mathfrak{J}\right)\right|^2\right)$$

$$+\int_{Match_2({}^{(ext)}r\leq r_0+1)}\Big|\mathfrak{d}^{\leq k}\Big({}^{(glo)}r-{}^{(ext)}r,{}^{(glo)}J^{(0)}-\cos({}^{(ext)}\theta),$$
$$ {}^{(glo)}\mathfrak{J}-{}^{(ext)}\mathfrak{J}\Big)\Big|^2$$
$$\lesssim\ \mathfrak{R}_{k-1}^2+\mathfrak{G}_{k-1}^2.$$

Proof. See Section 9.6.5. ☐

Remark 9.64. *By construction, the global frame of Lemma 9.63 coincides with the ingoing PT frame of ${}^{(int)}\mathcal{M}'$ in ${}^{(int)}\mathcal{M}'(\underline{u}\leq u_*-1)$. In particular, we infer from (9.43) that the global frame of Lemma 9.63 satisfies, for $k\leq k_{small}-1$,*

$$\sup_{{}^{(int)}\mathcal{M}'(\underline{u}\leq u_*-1)}\underline{u}^{1+\frac{3\delta_{dec}}{4}}\Big\{|\mathfrak{d}^{\leq k\,(glo)}\check{\Gamma}|+|\mathfrak{d}^{\leq k\,(glo)}\check{R}|\Big\}\ \lesssim\epsilon_0. \tag{9.86}$$

The decay in ${}^{(int)}\mathcal{M}'(\underline{u}\leq u_*-1)$ for $\mathfrak{d}^{\leq k_{small}-1\,(glo)}\check{\Gamma}$ and $\mathfrak{d}^{\leq k_{small}-1\,(glo)}\check{R}$ at an integrable rate, provided by (9.86), is used in the wave estimates of [28] to control nonlinear terms in the trapped region[212].

9.6.2. Proof of Theorem 9.49

The proof of Theorem 9.49 relies on the following theorem which provides the control of curvature components in the global frame of Lemma 9.63.

Theorem 9.65 (Control of Curvature)**.** *Let J such that $J\leq k_{large}+6$. Under the iteration assumption (9.52), we have the following estimate in $\mathcal{M}$*

$$\begin{aligned}{}^{(glo)}\mathfrak{R}_{J+1}\ \lesssim\ & r_0^{-\frac{\delta_B}{2}}\,{}^{(glo)}\mathfrak{G}^{\geq r_0}_{J+1}+r_0^{21+\delta_B}\,{}^{(glo)}\epsilon_J\\ &+r_0^{\frac{21}{2}+\frac{\delta_B}{2}}\Big(\sqrt{{}^{(glo)}\mathfrak{G}_{J+1}}\sqrt{{}^{(glo)}\epsilon_J}+\epsilon_0\Big)+\sqrt{|a|}r_0^{3+\frac{\delta_B}{2}}\,{}^{(glo)}\mathfrak{G}_{J+1}\\ &+r_0^{\frac{39}{8}+\frac{\delta_B}{2}}\,{}^{(glo)}\mathfrak{G}^{\frac{3}{4}}_{J+1}\Big(\epsilon_0+\sqrt{{}^{(glo)}\epsilon_J}\sqrt{{}^{(glo)}\mathfrak{G}_{J+1}}\Big)^{\frac{1}{4}}\\ &+r_0^{\frac{39}{7}+\frac{4\delta_B}{7}}\,{}^{(glo)}\mathfrak{G}^{\frac{6}{7}}_{J+1}\,{}^{(glo)}\epsilon_J^{\frac{1}{7}},\end{aligned}\tag{9.87}$$

where the constant in $\lesssim$ is independent of r_0, where ${}^{(glo)}\epsilon_J={}^{(glo)}\mathfrak{G}_J+{}^{(glo)}\mathfrak{R}_J$, and where ${}^{(glo)}\mathfrak{R}_k$ and ${}^{(glo)}\mathfrak{G}_k$ are the analog for the global frame of Lemma 9.63 of the norms of Section 9.4.1 for the PT structures of $\mathcal{M}$.

[212] The trapped region is in fact located in ${}^{(int)}\mathcal{M}'\cup{}^{(top)}\mathcal{M}'$. Note that the part of the trapped region in $\mathcal{M}\setminus{}^{(int)}\mathcal{M}'(\underline{u}\leq u_*-1)$ is harmless for wave estimates as it lies inside a local existence type region.

Remark 9.66. *Theorem 9.65 is proved in a separate paper, see Theorem 13.6.3 in [28]. The proof relies on energy, Morawetz and r^p weighted estimates for the Bianchi system.*

We are now ready to prove Theorem 9.49.

Proof of Theorem 9.49. According to Theorem 9.65, we have, for J such that $J \leq k_{large}+6$,

$$\begin{aligned}
{}^{(glo)}\mathfrak{R}_{J+1} \lesssim\ & r_0^{-\frac{\delta_B}{2}}\,{}^{(glo)}\mathfrak{G}^{\geq r_0}_{J+1} + r_0^{21+\delta_B}\,{}^{(glo)}\epsilon_J \\
& + r_0^{\frac{21}{2}+\frac{\delta_B}{2}}\left(\sqrt{{}^{(glo)}\mathfrak{G}_{J+1}}\sqrt{{}^{(glo)}\epsilon_J}+\epsilon_0\right) + \sqrt{|a|}\,r_0^{3+\frac{\delta_B}{2}}\,{}^{(glo)}\mathfrak{G}_{J+1} \\
& + r_0^{\frac{39}{8}+\frac{\delta_B}{2}}\,{}^{(glo)}\mathfrak{G}^{\frac{3}{4}}_{J+1}\left(\epsilon_0+\sqrt{{}^{(glo)}\epsilon_J}\sqrt{{}^{(glo)}\mathfrak{G}_{J+1}}\right)^{\frac{1}{4}} \\
& + r_0^{\frac{39}{7}+\frac{4\delta_B}{7}}\,{}^{(glo)}\mathfrak{G}^{\frac{6}{7}}_{J+1}\,{}^{(glo)}\epsilon_J^{\frac{1}{7}},
\end{aligned}$$

where the constant in $\lesssim$ is independent of r_0. Since we have, in view of properties (a), (b) and (c) of Lemma 9.63, for any $k \leq k_{large}+7$,

$${}^{(glo)}\mathfrak{G}_k + {}^{(glo)}\mathfrak{R}_k \lesssim \mathfrak{R}_k + \mathfrak{G}_k, \qquad {}^{(glo)}\mathfrak{G}^{\geq r_0}_k \lesssim \mathfrak{R}_k + {}^{(top)}\mathfrak{G}^{\geq r_0}_k + {}^{(ext)}\mathfrak{G}_k,$$

we deduce, for J such that $J \leq k_{large}+6$,

$$\begin{aligned}
{}^{(glo)}\mathfrak{R}_{J+1} \lesssim\ & r_0^{-\frac{\delta_B}{2}}\left({}^{(top)}\mathfrak{G}^{\geq r_0}_{J+1} + {}^{(ext)}\mathfrak{G}_{J+1} + \mathfrak{R}_{J+1}\right) + r_0^{21+\delta_B}\epsilon_J \\
& + r_0^{\frac{21}{2}+\frac{\delta_B}{2}}\left(\sqrt{\epsilon_{J+1}}\sqrt{\epsilon_J}+\epsilon_0\right) + \sqrt{|a|}\,r_0^{3+\frac{\delta_B}{2}}\left(\mathfrak{R}_{J+1}+\mathfrak{G}_{J+1}\right) \\
& + r_0^{\frac{39}{8}+\frac{\delta_B}{2}}\epsilon^{\frac{3}{4}}_{J+1}\left(\epsilon_0+\sqrt{\epsilon_J}\sqrt{\epsilon_{J+1}}\right)^{\frac{1}{4}} + r_0^{\frac{39}{7}+\frac{4\delta_B}{7}}\epsilon^{\frac{6}{7}}_{J+1}\epsilon_J^{\frac{1}{7}}
\end{aligned}$$

where the constant in $\lesssim$ is independent of r_0. In particular, for r_0 large enough and $|a| \ll r_0$ small enough, we may absorb the terms in $\mathfrak{R}_{J+1}$ on the RHS which yields, for J such that $J \leq k_{large}+6$,

$$\begin{aligned}
{}^{(glo)}\mathfrak{R}_{J+1} \lesssim\ & r_0^{-\frac{\delta_B}{2}}\left({}^{(top)}\mathfrak{G}^{\geq r_0}_{J+1} + {}^{(ext)}\mathfrak{G}_{J+1}\right) + r_0^{21+\delta_B}\epsilon_J \\
& + r_0^{\frac{21}{2}+\frac{\delta_B}{2}}\left(\sqrt{\epsilon_{J+1}}\sqrt{\epsilon_J}+\epsilon_0\right) + \sqrt{|a|}\,r_0^{3+\frac{\delta_B}{2}}\mathfrak{G}_{J+1} \\
& + r_0^{\frac{39}{8}+\frac{\delta_B}{2}}\epsilon^{\frac{3}{4}}_{J+1}\left(\epsilon_0+\sqrt{\epsilon_J}\sqrt{\epsilon_{J+1}}\right)^{\frac{1}{4}} + r_0^{\frac{39}{7}+\frac{4\delta_B}{7}}\epsilon^{\frac{6}{7}}_{J+1}\epsilon_J^{\frac{1}{7}}
\end{aligned} \tag{9.88}$$

where the constant in $\lesssim$ is independent of r_0.

Next, let $(f, \underline{f}, \lambda)$ denote the change of frame coefficients:

- from the ingoing PT frame of ${}^{(int)}\mathcal{M}'$ to the global frame of Lemma 9.63 in ${}^{(int)}\mathcal{M}'$,
- from the ingoing PT frame of ${}^{(top)}\mathcal{M}'$ to the global frame of Lemma 9.63 in ${}^{(top)}\mathcal{M}'$,
- from $({}^{(ext)}\lambda^{(ext)}e_4, {}^{(ext)}\lambda^{-1(ext)}e_3, {}^{(ext)}e_1, {}^{(ext)}e_2)$ to the global frame of Lemma 9.63, where ${}^{(ext)}\lambda = \frac{{}^{(ext)}\Delta}{|{}^{(ext)}q|^2}$, in ${}^{(ext)}\mathcal{M}$.

Then, according to Lemma 9.63, we have

$$f = 0, \qquad \underline{f} = 0, \qquad \lambda = 1 \quad \text{in} \quad \mathcal{M} \setminus \text{Match}, \tag{9.89}$$

and, for $1 \leq k \leq k_{large} + 8$,

$$\begin{aligned}\sup_{\underline{u}_1 \geq u_*} \int_{\text{Match}_{\underline{u}_1}} \left(\left|\mathfrak{d}^{\leq k}(f, \log\lambda)\right|^2 + r^{-\delta_B}\left|\mathfrak{d}^{\leq k}\underline{f}\right|^2 \right) & \\ + \int_{\text{Match}(r \leq r_0+1)} \left|\mathfrak{d}^{\leq k}(f, \underline{f}, \log\lambda)\right|^2 \;\; \lesssim \;\; \mathfrak{R}_{k-1}^2 + \mathfrak{G}_{k-1}^2, &\end{aligned} \tag{9.90}$$

where

$$\begin{aligned}\text{Match} \;\; = \;\; & {}^{(int)}\mathcal{M}'(\underline{u} \geq u_* - 1) \cup {}^{(ext)}\mathcal{M}(u \geq u_* - 1) \\ & \cup {}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1)\end{aligned}$$

and

$$\text{Match}_{\underline{u}_1} \;\; := \;\; \text{Match} \cap \{\underline{u}_1 \leq \underline{u} \leq \underline{u}_1 + 1\}.$$

Now, using the change of frame formulas of Proposition 2.12, we have

$$\begin{aligned}\lambda^{-2(glo)}\alpha = \alpha + (f\widehat{\otimes}\beta - {}^*f\widehat{\otimes}{}^*\beta) + \left(f\widehat{\otimes}f - \frac{1}{2}{}^*f\widehat{\otimes}{}^*f\right)\rho + \frac{3}{2}(f\widehat{\otimes}{}^*f)\;{}^*\rho & \\ + \;\text{l.o.t.}, & \\ \lambda^{2(glo)}\underline{\alpha} = \underline{\alpha} - (\underline{f}\widehat{\otimes}\underline{\beta} - {}^*\underline{f}\widehat{\otimes}{}^*\underline{\beta}) + (\underline{f}\widehat{\otimes}\underline{f} - \frac{1}{2}{}^*\underline{f}\widehat{\otimes}{}^*\underline{f})\rho + \frac{3}{2}(\underline{f}\widehat{\otimes}{}^*\underline{f})\;{}^*\rho & \\ + \;\text{l.o.t.} &\end{aligned}$$

Together with the control of $(f, \underline{f}, \lambda)$ provided by (9.89) and (9.90), and using the bootstrap assumption (9.40) for $\mathfrak{G}_k$ and $\mathfrak{R}_k$, we infer, for $J \leq k_{large} + 6$,

$$\mathfrak{R}_{J+1}[A] + \mathfrak{R}_{J+1}[\underline{A}] \;\; \lesssim \;\; {}^{(glo)}\mathfrak{R}_{J+1} + \epsilon^2$$

and, for $1 \leq k \leq J+1$,

$$\int_{\mathcal{M}(r\geq r_0)} r^{5+\delta_B}|\mathfrak{d}^{\leq k-1}\nabla_3 A|^2 \lesssim \int_{\mathcal{M}(r\geq r_0)} r^{5+\delta_B}|\mathfrak{d}^{\leq k-1}\nabla_{{}^{(glo)}e_3}{}^{(glo)}A|^2 + \epsilon^4.$$

Using Bianchi for $\nabla_{{}^{(glo)}e_3}{}^{(glo)}A$, we have, for $1 \leq k \leq J+1$,

$$\int_{\mathcal{M}(r\geq r_0)} r^{5+\delta_B}|\mathfrak{d}^{\leq k-1}\nabla_{{}^{(glo)}e_3}{}^{(glo)}A|^2 \lesssim ({}^{(glo)}\mathfrak{R}_{J+1})^2 + ({}^{(glo)}\mathfrak{G}_J)^2 + \epsilon^4.$$

In view of the above, we deduce

$$\mathfrak{R}_{J+1}[A] + \mathfrak{R}_{J+1}[\underline{A}] \lesssim {}^{(glo)}\mathfrak{R}_{J+1} + \epsilon_0 \tag{9.91}$$

and

$$\int_{\mathcal{M}(r\geq r_0)} r^{5+\delta_B}|\mathfrak{d}^{\leq k-1}\nabla_{{}^{(glo)}e_3}{}^{(glo)}A|^2 \lesssim ({}^{(glo)}\mathfrak{R}_{J+1})^2 + (\mathfrak{G}_J)^2 + \epsilon_0^2. \tag{9.92}$$

Also, using again the change of frame formulas of Proposition 2.12, we have

$$\begin{aligned}
{}^{(glo)}\rho &= \rho + \underline{f}\cdot\beta - f\cdot\underline{\beta} + \frac{3}{2}\rho(f\cdot\underline{f}) - \frac{3}{2}\,{}^*\rho(f\wedge\underline{f}) + \text{ l.o.t.},\\
{}^{(glo)}\,{}^*\rho &= {}^*\rho - \underline{f}\cdot{}^*\beta - f\cdot{}^*\underline{\beta} + \frac{3}{2}\,{}^*\rho(f\cdot\underline{f}) + \frac{3}{2}\rho(f\wedge\underline{f}) + \text{ l.o.t.}
\end{aligned}$$

Together with the control of $(f, \underline{f}, \lambda)$ provided by (9.89) and (9.90), and using the bootstrap assumption (9.40) for $\mathfrak{G}_k$ and $\mathfrak{R}_k$, we infer, for $J \leq k_{large}+6$,

$$\begin{aligned}
\mathfrak{R}_{J+1}[\check{P}] \lesssim\; & {}^{(glo)}\mathfrak{R} + \epsilon^2 + \sup_\tau \int_{\text{Match}(r\leq r_0)\cap\Sigma(\tau)} \left|\mathfrak{d}^{\leq J+1}\left(\frac{1}{({}^{(glo)}q)^3} - \frac{1}{q^3}\right)\right|^2\\
& + \int_{\text{Match}(r\geq r_0)} r^{3-\delta_B}\left|\mathfrak{d}^{\leq J+1}\left(\frac{1}{({}^{(glo)}q)^3} - \frac{1}{q^3}\right)\right|^2
\end{aligned}$$

where we have also used the fact that ${}^{(glo)}r = r$ and ${}^{(glo)}J^{(0)} = \cos(\theta)$ on $\mathcal{M}\setminus$Match in view of Properties (a) and (b) of Lemma 9.63. Using in particular a trace theorem for the integral on Match$(r \leq r_0) \cap \Sigma(\tau)$, we infer

$$\begin{aligned}
\mathfrak{R}_{J+1}[\check{P}] \lesssim\; & {}^{(glo)}\mathfrak{R} + \epsilon_0\\
& + \int_{\text{Match}(r\geq r_0)} r^{-5-\delta_B}\left|\mathfrak{d}^{\leq J+1}\left({}^{(glo)}r - r, {}^{(glo)}J^{(0)} - \cos(\theta)\right)\right|^2
\end{aligned}$$

$$+\left(\int_{\mathrm{Match}(r\leq r_0)}\left|\mathfrak{d}^{\leq J+1}\left({}^{(glo)}r-r,{}^{(glo)}J^{(0)}-\cos(\theta)\right)\right|^2\right)^{\frac{1}{2}}$$
$$\times\left(\int_{\mathrm{Match}(r\leq r_0)}\left|\mathfrak{d}^{\leq J+2}\left({}^{(glo)}r-r,{}^{(glo)}J^{(0)}-\cos(\theta)\right)\right|^2\right)^{\frac{1}{2}}.$$

Together with property (e) of Lemma 9.63, we deduce

$$\mathfrak{R}_{J+1}[\check{P}] \lesssim {}^{(glo)}\mathfrak{R}_{J+1}+\epsilon_0+\sqrt{\epsilon_J}\sqrt{\epsilon_{J+1}}.$$

In view of (9.91), this yields

$$\mathfrak{R}_{J+1}[A]+\mathfrak{R}_{J+1}[\underline{A}]+\mathfrak{R}_{J+1}[\check{P}]\lesssim {}^{(glo)}\mathfrak{R}_{J+1}+\epsilon_0+\sqrt{\epsilon_J}\sqrt{\epsilon_{J+1}}. \tag{9.93}$$

Next, we use Bianchi identities to derive

$$\begin{aligned}\mathfrak{R}_{J+1}[B]+\mathfrak{R}_{J+1}[\underline{B}] \lesssim\ & \mathfrak{R}_{J+1}[A]+\mathfrak{R}_{J+1}[\underline{A}]+\mathfrak{R}_{J+1}[\check{P}]\\ &+\left(\int_{\mathcal{M}(r\geq r_0)}r^{5+\delta_B}|\mathfrak{d}^{\leq J}\nabla_{{}^{(glo)}e_3}{}^{(glo)}A|^2\right)^{\frac{1}{2}}\\ &+\mathfrak{G}_J+\mathfrak{R}_J+\epsilon^2.\end{aligned}$$

Together with (9.93) and (9.92), we deduce, for $J\leq k_{large}+6$,

$$\mathfrak{R}_{J+1} \lesssim {}^{(glo)}\mathfrak{R}_{J+1}+\epsilon_0+\mathfrak{R}_J+\sqrt{\epsilon_J}\sqrt{\epsilon_{J+1}}.$$

In view of (9.88), this yields, for $J\leq k_{large}+6$,

$$\begin{aligned}&\mathfrak{R}_{J+1}\\ \lesssim\ & r_0^{-\frac{\delta_B}{2}}\left({}^{(top)}\mathfrak{G}^{\geq r_0}_{J+1}+{}^{(ext)}\mathfrak{G}_{J+1}\right)+r_0^{21+\delta_B}\epsilon_J+r_0^{\frac{21}{2}+\frac{\delta_B}{2}}\left(\sqrt{\epsilon_{J+1}}\sqrt{\epsilon_J}+\epsilon_0\right)\\ &+\sqrt{|a|}r_0^{3+\frac{\delta_B}{2}}\mathfrak{G}_{J+1}+r_0^{\frac{39}{8}+\frac{\delta_B}{2}}\epsilon_{J+1}^{\frac{3}{4}}\left(\epsilon_0+\sqrt{\epsilon_J}\sqrt{\epsilon_{J+1}}\right)^{\frac{1}{4}}+r_0^{\frac{39}{7}+\frac{4\delta_B}{7}}\epsilon_{J+1}^{\frac{6}{7}}\epsilon_J^{\frac{1}{7}}\end{aligned}$$

as stated. This concludes the proof of Theorem 9.49. □

9.6.3. Proof of Lemma 9.59 To simplify the notations, in this section, we denote

- by (e_4, e_3, e_1, e_2) the outgoing PT frame of ${}^{(ext)}\mathcal{M}$,
- by (u, r, θ) and by $\mathfrak{J}$ respectively the triplet of scalar functions and the complex 1-form associated to the outgoing PT structure of ${}^{(ext)}\mathcal{M}$,

- by (e_4', e_3', e_1', e_2') the ingoing PT frame of ${}^{(int)}\mathcal{M}'$ slightly extended into ${}^{(ext)}\mathcal{M}$,
- by $(\underline{u}, r', \theta')$ and by $\mathfrak{J}'$ respectively the triplet of scalar functions and the complex 1-form associated to the ingoing PT structure of ${}^{(int)}\mathcal{M}'$ slightly extended into ${}^{(ext)}\mathcal{M}$,
- by $(f, \underline{f}, \lambda)$ the change of frame from (e_4, e_3, e_1, e_2) to (e_4', e_3', e_1', e_2').

Recall from Section 9.1.3 that we have, in view of the initialization of the ingoing PT structure of ${}^{(int)}\mathcal{M}'$ on $\mathcal{T} = \{r = r_0\}$ from the outgoing PT structure of ${}^{(ext)}\mathcal{M}$,

$$\underline{u} = u,\ r' = r,\ \theta' = \theta,\ \mathfrak{J}' = \mathfrak{J},\ f = \underline{f} = 0,\ \lambda = \frac{\Delta}{|q|^2} \text{ on } \{r = r_0\}. \tag{9.94}$$

Step 1. We start with a first control for $(f, \underline{f}, \lambda)$ and $(r' - r, \cos(\theta') - \cos\theta, \mathfrak{J}' - \mathfrak{J})$. To this end, we introduce, as in the proof of Proposition 4.8, the following auxiliary transformation

$$\begin{aligned}
e_3' &= \lambda'\left(e_3 + (\underline{f}')^b e_b + \frac{1}{4}|\underline{f}'|^2 e_3\right),\\
e_a' &= \left(\delta_a^b + \frac{1}{2}(f')_a(\underline{f}')^b\right)e_b + \frac{1}{2}(f')_a e_3 + \left(\frac{1}{2}(\underline{f}')_a + \frac{1}{8}|\underline{f}'|^2(f')_a\right)e_4,\\
e_4' &= (\lambda')^{-1}\Bigg(\left(1 + \frac{1}{2}f'\cdot\underline{f}' + \frac{1}{16}|f'|^2|\underline{f}'|^2\right)e_4 + \left((f')^b + \frac{1}{4}|f'|^2(\underline{f}')^b\right)e_b\\
&\qquad + \frac{1}{4}|f'|^2 e_3\Bigg),
\end{aligned}$$

where $\lambda' > 0$ is a scalar and $(f', \underline{f}')$ are 1-forms. In view of (4.34), it suffices to control $(f', \underline{f}', \lambda')$ in order to control $(f, \underline{f}, \lambda)$. Note also that (9.94) and (4.34) imply

$$f' = \underline{f}' = 0, \quad \lambda' = \frac{|q|^2}{\Delta} \quad \text{on} \quad \{r = r_0\}. \tag{9.95}$$

Now, let

$$F' := f' + i\,{}^*f', \qquad \underline{F}' := \underline{f}' + i\,{}^*\underline{f}'.$$

In view of (4.36), and Step 3 of Section 4.4.4 for the equation for the second equation, we have the following transport equations for $(F', \underline{F}', \lambda')$

(9.96)
$$
\begin{aligned}
\nabla_{(\lambda')^{-1}e_3'}\underline{F}' + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\underline{F}' + 2\underline{\omega}\,\underline{F}' =& \Gamma_b + \Gamma_b\cdot\underline{F}' + E_1(\underline{f}',\Gamma),\\
(\lambda')^{-1}\nabla_3'\left(\log\left(\frac{\Delta}{|q|^2}\lambda'\right)\right) =& \Gamma_b + O(r^{-2})\underline{F}' + \Gamma_b\cdot\underline{F}' + E_2(\underline{f}',\Gamma)\\
&+\frac{|q|^2}{\Delta}\left(\underline{f}'\cdot\nabla + \frac{1}{4}|\underline{f}'|^2 e_3\right)\left(\frac{\Delta}{|q|^2}\right),\\
(\lambda')^{-1}\nabla_3' F' - 2\underline{\omega}F' =& \frac{2aq'}{|q'|^2}\mathfrak{J}' - \frac{2aq}{|q|^2}\mathfrak{J} + \Gamma_b + O(r^{-1})\underline{F}' + \Gamma_g\cdot\underline{F}'\\
&+E_3(f',\underline{f}',\Gamma),
\end{aligned}
$$

where $E_1(f',\Gamma)$ and $E_2(f',\Gamma)$ contain expressions of the type $O(\Gamma(\underline{f}')^2)$ with no derivatives, and $E_3(f',\underline{f}',\Gamma)$ contains expressions of the type $O(\Gamma(f',\underline{f}')^2)$ with no derivatives. Also, since $e_3'(r)=1$, $e_3'(\cos\theta)=0$ and $\nabla_3'\mathfrak{J}' = \frac{1}{\overline{q'}}\mathfrak{J}'$, we may proceed as in Step 4 of the proof of Proposition 4.8 and obtain

(9.97)
$$
\begin{aligned}
e_3'(r'-r) =& -\lambda^{-1}\left(\lambda - \frac{\Delta}{|q|^2}\right) + r\Gamma_b + O(1)\underline{f}\cdot(f,\underline{f})\\
&+r\Gamma_g\cdot\underline{f},\\
e_3'(\cos(\theta')-\cos(\theta)) =& \Gamma_b + O(r^{-1})\underline{f} + \Gamma_b\cdot\underline{f}\cdot(f,\underline{f}),\\
\nabla_3'(\mathfrak{J}'-\mathfrak{J}) + \frac{1}{\overline{q'}}(\mathfrak{J}'-\mathfrak{J}) =& r^{-1}\Gamma_b + O(r^{-2})\lambda^{-1}\left(\lambda - \frac{\Delta}{|q|^2}\right)\\
&+O(r^{-3})(r'-r) + O(r^{-3})(\cos(\theta')-\cos\theta)\\
&+O(r^{-2})\underline{f} + \Gamma_b\cdot(f,\underline{f}) + O(r^{-2})\underline{f}\cdot(f,\underline{f}).
\end{aligned}
$$

In view of (9.94) and (9.95), we have

$$
\widetilde{\mathfrak{d}}^k\left(f',\underline{f}',\log\left(\frac{\Delta}{|q|^2}\lambda'\right), r'-r, \theta'-\theta, \mathfrak{J}'-\mathfrak{J}\right) = 0, \quad k\le k_{large}+8,
$$

where $\widetilde{\mathfrak{d}}$ denotes tangential derivatives to $\mathcal{T}$. Since $\mathfrak{d}$ is generated by $\widetilde{\mathfrak{d}}$ and e_3', together with the transport equations in (9.96) and (9.97), we infer, for $1\le k\le k_{large}+8$,

$$
\int_{\mathcal{T}}\left|\mathfrak{d}^{\le k}\left(f',\underline{f}',\log\left(\frac{\Delta}{|q|^2}\lambda'\right), r'-r, \theta'-\theta, \mathfrak{J}'-\mathfrak{J}\right)\right|^2 \lesssim \int_{\mathcal{T}}\left|\mathfrak{d}^{\le k-1}\check{\Gamma}\right|^2
$$

which implies, for $1\le k\le k_{large}+8$,

$$(9.98)\qquad \int_{\mathcal{T}}\left|\mathfrak{d}^{\leq k}\left(f', \underline{f}', \log\left(\frac{\Delta}{|q|^2}\lambda'\right), r'-r, \theta'-\theta, \mathfrak{J}'-\mathfrak{J}\right)\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_{k-1}^2.$$

We now integrate the transport equations in (9.96) and (9.97) from $\mathcal{T}$ in the following order, taking advantage of a triangular structure:

$$\underline{F}', \quad \frac{\Delta}{|q|^2}\lambda', \quad r'-r, \quad \cos(\theta')-\cos(\theta), \quad \mathfrak{J}'-\mathfrak{J}, \quad F'.$$

Together with the control on $\mathcal{T}$ provided by (9.98), and since we have $\mathcal{R}_{(1)} \subset {}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0+1)$, we easily infer, for $k \leq k_{large}+7$,

$$(9.99)\qquad \int_{\mathcal{R}_{(1)}}\left|\mathfrak{d}^{\leq k}\left(f', \underline{f}', \log\left(\frac{\Delta}{|q|^2}\lambda'\right), r'-r, \cos(\theta')-\cos(\theta), \mathfrak{J}'-\mathfrak{J}\right)\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_k^2.$$

Together with (4.34), this implies, for $k \leq k_{large}+7$,

$$(9.100)\qquad \int_{\mathcal{R}_{(1)}}\left|\mathfrak{d}^{\leq k}\left(f, \underline{f}, \log\left(\frac{|q|^2}{\Delta}\lambda\right)\right)\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_k^2.$$

Notice, in view of (9.99) and (9.100), that we still need to derive the stated control for $(f, \underline{f}, \lambda)$ and $(r'-r, \cos(\theta')-\cos\theta, \mathfrak{J}'-\mathfrak{J})$ for $k = k_{large}+8$. This will be achieved in Step 4.

Step 2. Next, we control $\check{R}'$ and obtain a first control for $\check{\Gamma}'$. We have, in view of the change of frame formulas of Proposition 2.12,

$$\begin{aligned}
&\int_{\mathcal{R}_{(1)}}\left|\mathfrak{d}^{\leq k}\check{R}'\right|^2\\
&\lesssim \int_{\mathcal{R}_{(1)}}\left|\mathfrak{d}^{\leq k}\check{R}\right|^2 + \int_{\mathcal{R}_{(1)}}\left|\mathfrak{d}^{\leq k}\left(f, \underline{f}, \log\left(\frac{|q|^2}{\Delta}\lambda\right), r'-r, \cos(\theta')-\cos(\theta)\right)\right|^2\\
&\lesssim {}^{(ext)}\mathfrak{R}_k^2 + \int_{\mathcal{R}_{(1)}}\left|\mathfrak{d}^{\leq k}\left(f, \underline{f}, \log\left(\frac{|q|^2}{\Delta}\lambda\right), r'-r, \cos(\theta')-\cos(\theta)\right)\right|^2
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathcal{R}_{(1)}}\left|\mathfrak{d}^{\leq k}\check{\Gamma}'\right|^2\\
&\lesssim \int_{\mathcal{R}_{(1)}}\left|\mathfrak{d}^{\leq k}\check{\Gamma}\right|^2 + \int_{\mathcal{R}_{(1)}}\left|\mathfrak{d}^{\leq k+1}\left(f, \underline{f}, \log\left(\frac{|q|^2}{\Delta}\lambda\right), r'-r, \cos(\theta')-\cos(\theta)\right)\right|^2
\end{aligned}$$

$$\lesssim {}^{(ext)}\mathfrak{G}_k^2 + \int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k+1}\left(f, \underline{f}, \log\left(\frac{|q|^2}{\Delta}\lambda\right), r'-r, \cos(\theta')-\cos(\theta)\right)\right|^2$$

which together with (9.99) and (9.100) implies, for $1 \leq k \leq k_{large}+7$,

$$\int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\check{\Gamma}'\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_k^2, \quad \int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k}\check{R}'\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_k^2 + {}^{(ext)}\mathfrak{R}_k^2. \tag{9.101}$$

(9.101) provides the stated control for $\check{R}'$, but we still need to derive the stated control for $\check{\Gamma}'$ for $k = k_{large}+7$. This will be achieved in Step 3.

Step 3. In this step, we obtain the stated control for $\check{\Gamma}'$. To this end, we integrate the transport equations for the linearized equations of the ingoing PT structure of ${}^{(int)}\mathcal{M}'$ provided by Proposition 9.27 from $\mathcal{T}$ in the following order, taking advantage of a triangular structure:

$$\widecheck{\mathrm{tr}\underline{X}},\ \widehat{\underline{X}},\ \check{Z},\ \widecheck{\underline{H}},\ \check{\omega},\ \widecheck{\mathcal{D}\cos\theta},\ \mathcal{D}r,\ \mathcal{D}\widehat{\otimes}\mathfrak{J},\ \widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}},\ \widecheck{\mathcal{D}\underline{u}},\ \widecheck{\mathrm{tr}X},\ \widehat{X},$$
$$e_4(\cos\theta),\ \widecheck{e_4(r)},\ \widecheck{e_4(\underline{u})},\ \widecheck{\nabla_4\mathfrak{J}},\ \Xi.$$

Since $\mathcal{R}_{(1)} \subset {}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0+1)$, we easily infer, for $k \leq k_{large}+7$,

$$\int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k}\check{\Gamma}'\right|^2 \lesssim \int_{\mathcal{T}} \left|\mathfrak{d}^{\leq k}\check{\Gamma}'\right|^2 + \int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k}\check{R}'\right|^2.$$

Together with the control for $\check{R}'$ provided by (9.101), we deduce, for $k \leq k_{large}+7$,

$$\int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k}\check{\Gamma}'\right|^2 \lesssim \int_{\mathcal{T}} \left|\mathfrak{d}^{\leq k}\check{\Gamma}'\right|^2 + {}^{(ext)}\mathfrak{G}_k^2 + {}^{(ext)}\mathfrak{R}_k^2. \tag{9.102}$$

In view of (9.102), we need to control $\check{\Gamma}'$ on $\mathcal{T}$. According to the proof of Lemma 9.100, we have the following estimates on $\mathcal{T}$ for the ingoing PT structure of ${}^{(int)}\mathcal{M}'$

$$\int_{\mathcal{T}} \left|\mathfrak{d}^{\leq k}\check{\Gamma}'\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_k^2 + \int_{\mathcal{T}}\left(\left|\mathfrak{d}^{\leq k-1}\check{R}'\right|^2 + \left|\mathfrak{d}^{\leq k-1}\check{\Gamma}'\right|^2\right), \quad 1 \leq k \leq k_{large}+7,$$

which by iteration implies

$$\int_{\mathcal{T}} \left|\mathfrak{d}^{\leq k}\check{\Gamma}'\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_k^2 + \int_{\mathcal{T}}\left(\left|\mathfrak{d}^{\leq k-1}\check{R}'\right|^2 + \left|\check{\Gamma}'\right|^2\right), \qquad 1 \leq k \leq k_{large}+7.$$

Using the trace theorem, this yields

$$\int_{\mathcal{T}}\left|\mathfrak{d}^{\leq k}\check{\Gamma}'\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_k^2+\int_{\mathcal{R}_{(1)}}\left(\left|\mathfrak{d}^{\leq k}\check{R}'\right|^2+\left|\mathfrak{d}^{\leq 1}\check{\Gamma}'\right|^2\right), \qquad 1\leq k\leq k_{large}+7,$$

which implies, in view of (9.101),

$$\int_{\mathcal{T}}\left|\mathfrak{d}^{\leq k}\check{\Gamma}'\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_k^2+{}^{(ext)}\mathfrak{R}_k^2, \qquad 1\leq k\leq k_{large}+7.$$

Together with (9.102), we deduce, for $k\leq k_{large}+7$,

$$\int_{\mathcal{R}_{(1)}}\left|\mathfrak{d}^{\leq k}\check{\Gamma}'\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_k^2+{}^{(ext)}\mathfrak{R}_k^2,$$

which is the stated estimate for $\check{\Gamma}'$.

Step 4. In this step, we improve the control of $(f,\underline{f},\lambda)$ and $(r'-r,\cos(\theta')-\cos\theta,\mathfrak{J}'-\mathfrak{J})$ derived in Step 1 which misses the case $k=k_{large}+8$. First, as a consequence of the change of frame formulas of Proposition 2.12 for Ricci coefficients, we have, for $1\leq k\leq k_{large}+8$,

$$\begin{aligned}
&\int_{\mathcal{R}_{(1)}}\left|\mathfrak{d}^{\leq k}\left(f,\underline{f},\log\left(\frac{|q|^2}{\Delta}\lambda\right)\right)\right|^2\\
\lesssim\;& \int_{\mathcal{R}_{(1)}}\left|\mathfrak{d}^{\leq k-1}\check{\Gamma}'\right|^2+\int_{\mathcal{R}_{(1)}}\left|\mathfrak{d}^{\leq k-1}\check{\Gamma}\right|^2\\
&+\int_{\mathcal{R}_{(1)}}\left|\mathfrak{d}^{\leq k-1}\left(r'-r,\cos(\theta')-\cos(\theta),\mathfrak{J}'-\mathfrak{J}\right)\right|^2\\
&+\int_{\mathcal{R}_{(1)}}\left|\mathfrak{d}^{\leq k-1}\left(f,\underline{f},\log\left(\frac{|q|^2}{\Delta}\lambda\right)\right)\right|^2.
\end{aligned}$$

Together with (9.99), (9.100) and the control for $\check{\Gamma}'$ in Step 3, we infer, for $1\leq k\leq k_{large}+8$,

$$\int_{\mathcal{R}_{(1)}}\left|\mathfrak{d}^{\leq k}\left(f,\underline{f},\log\left(\frac{|q|^2}{\Delta}\lambda\right)\right)\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_{k-1}^2+{}^{(ext)}\mathfrak{R}_{k-1}^2$$

which is the stated control of $(f,\underline{f},\lambda)$.

It remains to improve the control of $(r'-r,\cos(\theta')-\cos\theta,\mathfrak{J}'-\mathfrak{J})$ derived in Step 1. We have, for $1\leq k\leq k_{large}+8$,

$$
\begin{aligned}
&\int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k}\left(r'-r, \cos(\theta')-\cos(\theta), \mathfrak{J}'-\mathfrak{J}\right)\right|^2 \\
\lesssim\ & \int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\left(\mathcal{D}'(r'), \widecheck{e_4'(r')}, \widecheck{\mathcal{D}'\cos(\theta')}, e_4'(\cos\theta'), \mathcal{D}'\widehat{\otimes}\mathfrak{J}', \widecheck{\overline{\mathcal{D}'}\cdot\mathfrak{J}'}, \widecheck{\nabla_4'\mathfrak{J}'}\right)\right|^2 \\
&+\int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\left(\mathcal{D}(r), \widecheck{e_3(r)}, \widecheck{\mathcal{D}\cos(\theta)}, e_3(\cos\theta), \mathcal{D}\widehat{\otimes}\mathfrak{J}, \widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}, \widecheck{\nabla_3\mathfrak{J}}\right)\right|^2 \\
&+\int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\left(r'-r, \cos(\theta')-\cos(\theta), \mathfrak{J}'-\mathfrak{J}\right)\right|^2
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k}\left(r'-r, \cos(\theta')-\cos(\theta), \mathfrak{J}'-\mathfrak{J}\right)\right|^2 \\
\lesssim\ & \int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\check{\Gamma}'\right|^2 + \int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\check{\Gamma}\right|^2 \\
&+\int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\left(r'-r, \cos(\theta')-\cos(\theta), \mathfrak{J}'-\mathfrak{J}\right)\right|^2.
\end{aligned}
$$

Together with (9.99) and the control for $\check{\Gamma}'$ in Step 3, we infer, for $1 \leq k \leq k_{large}+8$,

$$
\int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k}\left(r'-r, \cos(\theta')-\cos(\theta), \mathfrak{J}'-\mathfrak{J}\right)\right|^2 \lesssim\ {}^{(ext)}\mathfrak{G}_{k-1}^2 + {}^{(ext)}\mathfrak{R}_{k-1}^2
$$

which is the stated control of $(r'-r, \cos(\theta')-\cos\theta, \mathfrak{J}'-\mathfrak{J})$.

Step 5. So far, we have proved all statements involving comparisons between the extension of the ingoing PT structure of ${}^{(int)}\mathcal{M}'$ into ${}^{(ext)}\mathcal{M}$ and the outgoing PT structure of ${}^{(ext)}\mathcal{M}$. It thus remains to prove the statements involving comparisons between the extension of the ingoing PT structure of ${}^{(int)}\mathcal{M}'$ into ${}^{(ext)}\mathcal{M}$ and the ingoing PT structure of ${}^{(top)}\mathcal{M}'$. To this end, it suffices to compare the ingoing PT structure of ${}^{(top)}\mathcal{M}'$ and the outgoing PT structure of ${}^{(ext)}\mathcal{M}$ in a region including $\mathcal{R}_{(1)} \cap {}^{(top)}\mathcal{M}'$. We denote

- by $(e_4'', e_3'', e_1'', e_2'')$ the ingoing PT frame of ${}^{(top)}\mathcal{M}'$,
- by $(\underline{u}'', r'', \theta'')$ and by $\mathfrak{J}''$ respectively the triplet of scalar functions and the complex 1-form associated to the ingoing PT structure of ${}^{(top)}\mathcal{M}'$,
- by $(f'', \underline{f}'', \lambda'')$ the change of frame from (e_4, e_3, e_1, e_2) to $(e_4'', e_3'', e_1'', e_2'')$,
- by $\widetilde{\mathcal{R}}_{(1)}$ the spacetime region

$$
\widetilde{\mathcal{R}}_{(1)} = {}^{(ext)}\mathcal{M} \cap {}^{(top)}\mathcal{M}'(\underline{u}'' \leq u_* + 1)
$$

which satisfies $\mathcal{R}_{(1)} \cap {}^{(top)}\mathcal{M}' \subset \widetilde{\mathcal{R}}_{(1)}$.

Then, arguing as in Step 1 above, and integrating the transport equations in e_3'' from $\{u = u'_*\}$ where the ingoing PT structure of ${}^{(top)}\mathcal{M}'$ is initialized from the outgoing PT structure of ${}^{(ext)}\mathcal{M}$, we obtain, for $k \leq k_{large} + 7$,

$$\int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k}\left(f'', \underline{f}'', \log\left(\frac{\Delta}{|q|^2}\lambda''\right), r'' - r, \cos(\theta'') - \cos(\theta), \mathfrak{J}'' - \mathfrak{J}\right)\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_k^2.$$

Next, arguing as in Step 4 above, we infer, for $1 \leq k \leq k_{large} + 8$,

$$\begin{aligned}
& \int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k}\left(f'', \underline{f}'', \log\left(\frac{|q|^2}{\Delta}\lambda''\right)\right)\right|^2 \\
\lesssim & \int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\check{\Gamma}''\right|^2 + \int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\check{\Gamma}\right|^2 \\
& + \int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\left(r'' - r, \cos(\theta'') - \cos(\theta), \mathfrak{J}'' - \mathfrak{J}\right)\right|^2 \\
& + \int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\left(f'', \underline{f}'', \log\left(\frac{|q|^2}{\Delta}\lambda''\right)\right)\right|^2
\end{aligned}$$

and hence, for $1 \leq k \leq k_{large} + 8$,

$$\int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k}\left(f'', \underline{f}'', \log\left(\frac{|q|^2}{\Delta}\lambda''\right)\right)\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_{k-1}^2 + {}^{(top)}\mathfrak{G}_{k-1}^2.$$

Similarly, arguing as in Step 4 above, we infer, for $1 \leq k \leq k_{large} + 8$,

$$\int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k}\left(r'' - r, \cos(\theta'') - \cos(\theta), \mathfrak{J}'' - \mathfrak{J}\right)\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_{k-1}^2 + {}^{(top)}\mathfrak{G}_{k-1}^2.$$

Step 6. To conclude the proof of Lemma 9.59, it remains to control $\underline{u}'' - \underline{u}$ in the spacetime region $\mathcal{R}_{(1)} \cap {}^{(top)}\mathcal{M}'$. We first control $\underline{u}'' - \tilde{u}$ where

$$\tilde{u} := u + 2\int_{r_0}^{r} \frac{\tilde{r}^2 + a^2}{\tilde{r}^2 - 2m\tilde{r} + a^2} d\tilde{r}.$$

Note that

$$e_4(\tilde{u}) = \frac{2(r^2 + a^2)}{\Delta}, \quad \nabla(\tilde{u}) = \nabla(u) + \frac{2(r^2 + a^2)}{\Delta}\nabla(r),$$

$$e_3(\tilde{u}) = e_3(u) + \frac{2(r^2+a^2)}{\Delta} e_3(r),$$

which implies

$$e_4(\tilde{u}) = \frac{2(r^2+a^2)}{\Delta}, \quad \mathcal{D}(\tilde{u}) = a\mathfrak{J} + \check{\Gamma}, \quad e_3(\tilde{u}) = \check{\Gamma}.$$

We rewrite it as

$$e_4(\tilde{u}) = \frac{2(r^2+a^2)}{\Delta}, \quad \widetilde{\mathcal{D}(\tilde{u})} = \check{\Gamma}, \quad e_3(\tilde{u}) = \check{\Gamma},$$

where

$$\widetilde{\mathcal{D}(\tilde{u})} := \mathcal{D}(\tilde{u}) - a\mathfrak{J}.$$

We have, for $1 \leq k \leq k_{large} + 8$,

$$\begin{aligned}
&\int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k}\left(\underline{u}'' - \tilde{u}\right)\right|^2 \\
\lesssim\ & \int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\left(\widetilde{\mathcal{D}''(\underline{u}'')}, \widetilde{e_4''(\underline{u})}\right)\right|^2 + \int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\left(\widetilde{\mathcal{D}(\tilde{u})}, e_3(\tilde{u})\right)\right|^2 \\
& + \int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\left(\underline{u}'' - \tilde{u}\right)\right|^2 \\
\lesssim\ & \int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\check{\Gamma}''\right|^2 + \int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\check{\Gamma}\right|^2 + \int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\left(\underline{u}'' - \underline{u}\right)\right|^2 \\
\lesssim\ & {}^{(top)}\mathfrak{G}_{k-1} + {}^{(ext)}\mathfrak{G}_{k-1} + \int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\left(\underline{u}'' - \tilde{u}\right)\right|^2.
\end{aligned}$$

By iteration, we deduce, for $1 \leq k \leq k_{large} + 8$,

$$\int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k}\left(\underline{u}'' - \tilde{u}\right)\right|^2 \lesssim {}^{(top)}\mathfrak{G}_{k-1} + {}^{(ext)}\mathfrak{G}_{k-1} + \int_{\widetilde{\mathcal{R}}_{(1)}} \left|\underline{u}'' - \tilde{u}\right|^2.$$

Also, we have

$$\begin{aligned}
e_3''(\underline{u}'' - \tilde{u}) &= -e_3''(\tilde{u}) \\
&= (\lambda'')^{-1}\Bigg(\left(1 + \frac{1}{2}f'' \cdot \underline{f}'' + \frac{1}{16}|f''|^2|\underline{f}''|^2\right) e_3 \\
&\quad + \left((\underline{f}'')^b + \frac{1}{4}|\underline{f}''|^2 (f'')^b\right) e_b + \frac{1}{4}|\underline{f}''|^2 e_4\Bigg)\tilde{u}
\end{aligned}$$

$$= \quad (\lambda'')^{-1}\Big(\check{\Gamma} + O(r^{-1})\underline{f}'' + O(1)|\underline{f}''|^2 + O(r^{-1})|\underline{f}''|^2\underline{f}''\Big).$$

Together with the above control of $(f'', \underline{f}'', \lambda'')$, we infer

$$\begin{aligned}\int_{\widetilde{\mathcal{R}}_{(1)}} |e_3''(\underline{u}'' - \tilde{u})|^2 &\lesssim {}^{(ext)}\mathfrak{G}_0^2 + \int_{\widetilde{\mathcal{R}}_{(1)}} \left|\left(f'', \underline{f}'', \log\left(\frac{|q|^2}{\Delta}\lambda''\right)\right)\right|^2 \\ &\lesssim {}^{(ext)}\mathfrak{G}_0^2 + {}^{(top)}\mathfrak{G}_0^2.\end{aligned}$$

Integrating this transport equation from $\{u = u'_*\}$ where $\underline{u}'' = \tilde{u}$ in view of Section 9.1.3, we infer

$$\int_{\widetilde{\mathcal{R}}_{(1)}} |\underline{u}'' - \tilde{u}|^2 \lesssim {}^{(ext)}\mathfrak{G}_0^2 + {}^{(top)}\mathfrak{G}_0^2.$$

Plugging in the above, we deduce, for $1 \leq k \leq k_{large} + 8$,

$$\int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k}\left(\underline{u}'' - \tilde{u}\right)\right|^2 \lesssim {}^{(top)}\mathfrak{G}_{k-1} + {}^{(ext)}\mathfrak{G}_{k-1}.$$

Arguing as above for $\underline{u}'' - \tilde{u}$, we estimate $\underline{u} - \tilde{u}$ and obtain first, for $1 \leq k \leq k_{large} + 8$,

$$\int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k}\left(\underline{u} - \tilde{u}\right)\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_{k-1} + {}^{(ext)}\mathfrak{R}_{k-1} + \int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\left(\underline{u}'' - \underline{u}\right)\right|^2,$$

where we have used in particular the control for the extension of the ingoing PT structure of ${}^{(int)}\mathcal{M}'$ into $\mathcal{R}_{(1)}$ derived in Steps 1 to 4. By iteration, we deduce, for $1 \leq k \leq k_{large} + 8$,

$$\int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k}\left(\underline{u} - \tilde{u}\right)\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_{k-1} + {}^{(ext)}\mathfrak{R}_{k-1} + \int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k-1}\left(\underline{u} - \tilde{u}\right)\right|^2.$$

Also, arguing again as above for $\underline{u}'' - \tilde{u}$, we obtain

$$\int_{\mathcal{R}_{(1)}} |e_3(\underline{u} - \tilde{u})|^2 \lesssim {}^{(ext)}\mathfrak{G}_0^2 + {}^{(ext)}\mathfrak{R}_0^2,$$

where we have used in particular the control for the extension of the ingoing PT structure of ${}^{(int)}\mathcal{M}'$ into $\mathcal{R}_{(1)}$ derived in Steps 1 to 4. Integrating this transport equation from $\mathcal{T}$ where $\underline{u} = u$ in view of Section 9.1.3, and where $u = \tilde{u}$, we infer

$$\int_{\mathcal{R}_{(1)}} |\underline{u} - \tilde{u}|^2 \lesssim {}^{(ext)}\mathfrak{G}_0^2 + {}^{(ext)}\mathfrak{R}_0^2.$$

Plugging in the above, we deduce, for $1 \leq k \leq k_{large} + 8$,

$$\int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k}\left(\underline{u} - \tilde{u}\right)\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_{k-1} + {}^{(ext)}\mathfrak{R}_{k-1}.$$

In particular, since $\mathcal{R}_{(1)} \cap {}^{(top)}\mathcal{M}' \subset \widetilde{R}_{(1)}$, we deduce, for $1 \leq k \leq k_{large} + 8$,

$$\begin{aligned}\int_{\mathcal{R}_{(1)}\cap{}^{(top)}\mathcal{M}'} \left|\mathfrak{d}^{\leq k}\left(\underline{u}'' - \underline{u}\right)\right|^2 &\lesssim \int_{\mathcal{R}_{(1)}} \left|\mathfrak{d}^{\leq k}\left(\underline{u} - \tilde{u}\right)\right|^2 + \int_{\widetilde{\mathcal{R}}_{(1)}} \left|\mathfrak{d}^{\leq k}\left(\underline{u}'' - \tilde{u}\right)\right|^2\\ &\lesssim \mathfrak{G}_{k-1} + \mathfrak{R}_{k-1}\end{aligned}$$

as stated. This concludes the proof of Lemma 9.59.

9.6.4. Proof of Lemma 9.61 To simplify the notations, in this section, we denote

- by (e_4, e_3, e_1, e_2) the ingoing PT frame of ${}^{(int)}\mathcal{M}'$ slightly extended into ${}^{(ext)}\mathcal{M}$ in Lemma 9.59,
- by $(\underline{u}, r, \theta)$ and by $\mathfrak{J}$ respectively the triplet of scalar functions and the complex 1-form associated to the ingoing PT structure of ${}^{(int)}\mathcal{M}'$,
- by (e_4', e_3', e_1', e_2') the ingoing PT structure of ${}^{(top)}\mathcal{M}'$,
- by $(\underline{u}', r', \theta')$ and by $\mathfrak{J}'$ respectively the triplet of scalar functions and the complex 1-form associated to the ingoing PT structure of ${}^{(top)}\mathcal{M}'$,
- by $(f, \underline{f}, \lambda)$ the change of frame coefficients from the frame (e_4, e_3, e_1, e_2) to (e_4', e_3', e_1', e_2').

Recall the definition of the matching region in (9.83)

$$\begin{aligned}\text{Match}_1 &= {}^{(int)}\mathcal{M}'(\underline{u} \geq u_* - 1)\\ &\cup\Big({}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1) \cap \{u_* - 1 \leq \underline{u} \leq u_*\}\Big).\end{aligned}$$

Note that

$$\text{Match}_1 \subset {}^{(int)}\mathcal{M}' \cup \Big({}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1) \cap \{\underline{u} \leq u_*\}\Big),$$

and that, in view of the definition of ${}^{(top)}\mathcal{M}'$, Remark 9.60 and the control of $\underline{u}' - \underline{u}$ in Lemma 9.59, we also have[213]

[213]Lemma 9.59 only provides the control of $\underline{u}' - \underline{u}$ on $\text{Match}_1 \cap {}^{(ext)}\mathcal{M}$, but one can immediately extend the control to $\text{Match}_1 \cap {}^{(int)}\mathcal{M}'$ by the same method.

$$\text{Match}_1 \subset {}^{(top)}\mathcal{M}'.$$

In particular, the above mentioned frames, scalars and complex 1-forms exist on Match_1. Let ψ be a smooth cut-off function of $\underline{u}$ such that $\psi = 0$ for $\underline{u} \leq u_* - 1$ and $\psi = 1$ for $\underline{u} \geq u_*$. Then, we define the null frame $(e''_4, e''_3, e''_1, e''_2)$ on

$${}^{(int)}\mathcal{M} \cup {}^{(top)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1) \cup {}^{(ext)}\mathcal{M}(u \geq u_* - 1),$$

and the quantities $(r'', J''^{(0)}, \mathfrak{J}'')$ as follows:

- In

$${}^{(top)}\mathcal{M} \cup \Big(({}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1) \cup {}^{(ext)}\mathcal{M}(u \geq u_* - 1)) \setminus \{\underline{u} \leq u_*\}\Big),$$

  we have

$$(e''_4, e''_3, e''_1, e''_2) = (e'_4, e'_3, e'_1, e'_2), \quad r'' = r', \quad J''^{(0)} = \cos(\theta'), \quad \mathfrak{J}'' = \mathfrak{J}'.$$

- In

$$\Big({}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}({}^{(ext)}r \leq r_0 + 1)\Big) \cap \{\underline{u} \leq u_* - 1\},$$

  we have

$$(e''_4, e''_3, e''_1, e''_2) = (e_4, e_3, e_1, e_2), \quad r'' = r, \quad J''^{(0)} = \cos(\theta), \quad \mathfrak{J}'' = \mathfrak{J}.$$

- In the matching region Match_1, $(e''_4, e''_3, e''_1, e''_2)$ is defined from the frame (e_4, e_3, e_1, e_2) using the change of frame coefficients $(f', \underline{f}', \lambda')$ with

$$f' = \psi(\underline{u})f, \qquad \underline{f}' = \psi(\underline{u})\underline{f}, \qquad \lambda' = 1 - \psi(\underline{u}) + \psi(\underline{u})\lambda,$$

  where we recall that $(f, \underline{f}, \lambda)$ denotes the coefficients corresponding to the change of frame from (e_4, e_3, e_1, e_2) to (e'_4, e'_3, e'_1, e'_2).
- In the matching region Match_1, r'', $J''^{(0)}$ and $\mathfrak{J}''$ are defined by

$$r'' = \psi(\underline{u})r' + (1 - \psi(\underline{u}))r, \quad J''^{(0)} = \psi(\underline{u})\cos(\theta') + (1 - \psi(\underline{u}))\cos\theta,$$
$$\mathfrak{J}'' = \psi(\underline{u})\mathfrak{J}' + (1 - \psi(\underline{u}))\mathfrak{J}.$$

In view of the above definitions, properties (a) and (b) of Lemma 9.61 are immediate. Also, in view of Lemma 9.59, we have, for $1 \leq k \leq k_{large} + 8$,

$$\int_{\text{Match}_1 \cap {}^{(ext)}\mathcal{M}} \left|\mathfrak{d}^{\leq k}\left(f, \underline{f}, \log(\lambda)\right)\right|^2 \lesssim \mathfrak{G}^2_{k-1} + \mathfrak{R}^2_{k-1}.$$

Arguing as in Lemma 9.59, one easily extends this estimate to $\text{Match}_1 \cap {}^{(int)}\mathcal{M}'$ to obtain, for $1 \leq k \leq k_{large} + 8$,

$$\int_{\text{Match}_1} \left|\mathfrak{d}^{\leq k}\left(f, \underline{f}, \log(\lambda)\right)\right|^2 \lesssim \mathfrak{G}_{k-1}^2 + \mathfrak{R}_{k-1}^2.$$

In view of the definition of $(f', \underline{f}', \lambda')$, we deduce, for $1 \leq k \leq k_{large} + 8$,

$$\int_{\text{Match}_1} \left|\mathfrak{d}^{\leq k}\left(f', \underline{f}', \log(\lambda')\right)\right|^2 \lesssim \mathfrak{G}_{k-1}^2 + \mathfrak{R}_{k-1}^2.$$

Also, if $(f'', \underline{f}'', \lambda'')$ denotes the coefficients of the change of frame from (e_4', e_3', e_1', e_2') to $(e_4'', e_3'', e_1'', e_2'')$, we easily obtain from the above control of $(f', \underline{f}', \lambda')$ and $(f, \underline{f}, \lambda)$, for $1 \leq k \leq k_{large} + 8$,

$$\int_{\text{Match}_1} \left|\mathfrak{d}^{\leq k}\left(f'', \underline{f}'', \log(\lambda'')\right)\right|^2 \lesssim \mathfrak{G}_{k-1}^2 + \mathfrak{R}_{k-1}^2$$

which concludes the proof of property (d) of Lemma 9.61.

Next, in view of Lemma 9.59, we have, for $1 \leq k \leq k_{large} + 8$,

$$\int_{\text{Match}_1 \cap {}^{(ext)}\mathcal{M}} \left|\mathfrak{d}^{\leq k}\left(r' - r,\ \cos(\theta') - \cos(\theta),\ \mathfrak{J}' - \mathfrak{J}\right)\right|^2 \lesssim \mathfrak{G}_{k-1}^2 + \mathfrak{R}_{k-1}^2.$$

Arguing as in Lemma 9.59, one easily extends this estimate to $\text{Match}_1 \cap {}^{(int)}\mathcal{M}'$ to obtain, for $1 \leq k \leq k_{large} + 8$,

$$\int_{\text{Match}_1} \left|\mathfrak{d}^{\leq k}\left(r' - r,\ \cos(\theta') - \cos(\theta),\ \mathfrak{J}' - \mathfrak{J}\right)\right|^2 \lesssim \mathfrak{G}_{k-1}^2 + \mathfrak{R}_{k-1}^2.$$

In view of the definition of r'', $J''^{(0)}$ and $\mathfrak{J}''$ in Match_1, we deduce, for $1 \leq k \leq k_{large} + 8$,

$$\int_{\text{Match}_1} \left|\mathfrak{d}^{\leq k}\left(r'' - r,\ J''^{(0)} - \cos(\theta),\ \mathfrak{J}'' - \mathfrak{J}\right)\right|^2 \lesssim \mathfrak{G}_{k-1}^2 + \mathfrak{R}_{k-1}^2$$

which concludes the proof of property (e) of Lemma 9.61.

Finally, the change of frame formulas of Proposition 2.12, the above control of the change of frame coefficients $(f', \underline{f}', \lambda')$, and the control of $(r'' - r,\ J''^{(0)} - \cos(\theta),\ \mathfrak{J}'' - \mathfrak{J})$ yields, for $k \leq k_{large} + 7$,

$$\int_{\text{Match}_1} \left|\mathfrak{d}^{\leq k}\left(\check{\Gamma}'', \check{R}''\right)\right|^2 \lesssim \mathfrak{G}_k^2 + \mathfrak{R}_k^2 + \int_{\text{Match}_1} \left|\mathfrak{d}^{\leq k+1}\left(f', \underline{f}', \log(\lambda')\right)\right|^2$$

$$
\begin{aligned}
&+\int_{\mathrm{Match}_1}\left|\mathfrak{d}^{\leq k}\left(r''-r,\ J''^{(0)}-\cos(\theta),\ \mathfrak{J}''-\mathfrak{J}\right)\right|^2\\
\lesssim\ & \mathfrak{G}_k^2+\mathfrak{R}_k^2
\end{aligned}
$$

which is property (c). This concludes the proof of Lemma 9.61.

9.6.5. Proof of Lemma 9.63 To simplify the notations, in this section, we denote

- by (e_4, e_3, e_1, e_2) the frame of Lemma 9.61,
- by $(r, J^{(0)})$ and by $\mathfrak{J}$ respectively the pair of scalar functions and the complex 1-form of Lemma 9.61,
- by (e_4', e_3', e_1', e_2') the null frame associated to the outgoing PT structure of ${}^{(ext)}\mathcal{M}$,
- by (u, r', θ') and by $\mathfrak{J}'$ respectively the triplet of scalar functions and the complex 1-form associated to the outgoing PT structure of ${}^{(ext)}\mathcal{M}$.

Recall the definition of the matching region in (9.84)

$$
\mathrm{Match}_2 \quad = \quad {}^{(ext)}\mathcal{M}(u\geq u_*-1)\cup {}^{(ext)}\mathcal{M}({}^{(ext)}r\leq r_0+1).
$$

Note that the above mentioned frames, scalars and complex 1-forms exist on Match_2. Also, in view of the change of frame formulas of Proposition 2.12, and the control of (e_4, e_3, e_1, e_2) provided by Lemma 9.59 and Lemma 9.61, we have, for $k\leq k_{large}+7$,

$$
\begin{aligned}
&\int_{\mathrm{Match}_2}\left(r^{3+\delta_B}|\mathfrak{d}^k(A,B)|^2+r^{3-\delta_B}|\mathfrak{d}^k\check{P}|^2+r^{1-\delta_B}|\mathfrak{d}^k\underline{B}|^2+r^{-1-\delta_B}|\mathfrak{d}^k\underline{A}|^2\right)\\
&+\int_{\mathrm{Match}_2({}^{(ext)}r\leq r_0+1)}|\mathfrak{d}^k\check{\Gamma}|^2+\sup_{\underline{u}_1\geq u_*}\int_{\mathrm{Match}_{2,\underline{u}_1}}\left\{r^2\left|\mathfrak{d}^k(\Gamma_g\setminus\{\widetilde{\mathrm{tr}\underline{X}}\})\right|^2\right.\\
&\left.+r^{2-\delta_B}\left|\mathfrak{d}^k(\widetilde{\mathrm{tr}\underline{X}})\right|^2+\left|\mathfrak{d}^k(\Gamma_b\setminus\{\underline{\Xi}\})\right|^2+r^{-\delta_B}\left|\mathfrak{d}^k(\underline{\Xi})\right|^2\right\}\\
\lesssim&\mathfrak{R}_k+\mathfrak{G}_k,
\end{aligned}
$$

where

$$
\mathrm{Match}_{2,\underline{u}_1} \quad = \quad \mathrm{Match}_2\cap\{\underline{u}_1\leq\underline{u}\leq\underline{u}_1+1\}.
$$

We define the following null frame $(e_4'', e_3'', e_1'', e_2'')$ on ${}^{(ext)}\mathcal{M}$

$$
e_4''=\frac{\Delta'}{|q'|^2}e_4',\quad e_3''=\frac{|q'|^2}{\Delta'}e_3',\quad e_a'=e_a,\ a=1,2.
$$

We define the linearized quantities (Γ''_g, Γ''_b) using (r', θ') and $\mathfrak{J}'$, with the ingoing normalization. Let also $(f, \underline{f}, \lambda)$ denote the coefficients corresponding to the change of frame from (e_4, e_3, e_1, e_2) to $(e''_4, e''_3, e''_1, e''_2)$. In view of the control of the change of frame coefficients in Proposition 3.26, Lemma 9.59 and Lemma 9.61, we have, for $1 \leq k \leq k_{large} + 8$,

$$\int_{\mathrm{Match}_2({}^{(ext)}r \leq r_0+1)} \left|\mathfrak{d}^{\leq k}(f, \underline{f}, \log\lambda)\right|^2 \lesssim \mathfrak{R}^2_{k-1} + \mathfrak{G}^2_{k-1}. \tag{9.103}$$

Also, in view of Lemma 9.59 and Lemma 9.61, we have, for $1 \leq k \leq k_{large}+8$,

$$\begin{aligned} &\int_{\mathrm{Match}_2({}^{(ext)}r \leq r_0+1)} \left|\mathfrak{d}^{\leq k}\Big(r' - r, J^{(0)} - \cos(\theta'), \mathfrak{J}' - \mathfrak{J}\Big)\right|^2 \\ &\lesssim \mathfrak{R}^2_{k-1} + \mathfrak{G}^2_{k-1}. \end{aligned} \tag{9.104}$$

Next, we complete the estimates (9.103) and (9.104) by considering the spacetime region $\mathrm{Match}_2(\underline{u} \geq u_*)$[214] where, in view of property (a) of Lemma 9.61, (e_4, e_3, e_1, e_2) coincides with the ingoing PT frame of ${}^{(top)}\mathcal{M}'$, and $(r, J^{(0)} = \cos(\theta))$ and by $\mathfrak{J}$ are the scalar functions and complex 1-form associated to the ingoing PT structure of ${}^{(top)}\mathcal{M}'$. It thus suffices to compare the ingoing PT structure of ${}^{(top)}\mathcal{M}'$ with the outgoing PT structure of ${}^{(ext)}\mathcal{M}$ on $\mathrm{Match}_2(\underline{u} \geq u_*)$. First, as a consequence of the change of frame formulas of Proposition 2.12 for Ricci coefficients, we have, for $1 \leq k \leq k_{large} + 8$,

$$\begin{aligned} &\sup_{\underline{u}_1 \geq u_*} \int_{\mathrm{Match}_{2,\underline{u}_1}} \left(\left|\mathfrak{d}^{\leq k}(f, \log\lambda)\right|^2 + r^{-\delta_B}\left|\mathfrak{d}^{\leq k}\underline{f}\right|^2\right) \\ \lesssim\ & \mathfrak{G}^2_{k-1} + \sup_{\underline{u}_1 \geq u_*} \int_{\mathrm{Match}_{2,\underline{u}_1}} \Big(r^{-2}\left|\mathfrak{d}^{\leq k-1}(r' - r)\right|^2 \\ &\qquad + \left|\mathfrak{d}^{\leq k-1}\Big(J^{(0)} - \cos(\theta')\Big)\right|^2 + r^2\left|\mathfrak{d}^{\leq k-1}(\mathfrak{J}' - \mathfrak{J})\right|^2\Big) \\ &+ \sup_{\underline{u}_1 \geq u_*} \int_{\mathrm{Match}_{2,\underline{u}_1}} \left(\left|\mathfrak{d}^{\leq k-1}(f, \log\lambda)\right|^2 + r^{-\delta_B}\left|\mathfrak{d}^{\leq k-1}\underline{f}\right|^2\right). \end{aligned}$$

Also, we have, for $1 \leq k \leq k_{large} + 8$,

$$\begin{aligned} \sup_{\underline{u}_1 \geq u_*} \int_{\mathrm{Match}_{2,\underline{u}_1}} &\Big(r^{-2}\left|\mathfrak{d}^{\leq k}(r' - r)\right|^2 + \left|\mathfrak{d}^{\leq k}\Big(J^{(0)} - \cos(\theta')\Big)\right|^2 \\ &+ r^2\left|\mathfrak{d}^{\leq k}(\mathfrak{J}' - \mathfrak{J})\right|^2\Big) \end{aligned}$$

[214] Notice that $\mathrm{Match}_2({}^{(ext)}r \leq r_0 + 1) \cup \mathrm{Match}_2(\underline{u} \geq u_*) = \mathrm{Match}_2$.

$$\lesssim \mathfrak{G}_{k-1}^2 + \sup_{\underline{u}_1\geq u_*}\int_{\mathrm{Match}_{2,\underline{u}_1}}\Big(r^{-2}\left|\mathfrak{d}^{\leq k-1}(r'-r)\right|^2 + \left|\mathfrak{d}^{\leq k-1}\Big(J^{(0)}-\cos(\theta')\Big)\right|^2 + r^2\left|\mathfrak{d}^{\leq k-1}(\mathfrak{J}'-\mathfrak{J})\right|^2\Big).$$

By iteration, we deduce, for $1 \leq k \leq k_{large}+8$,

$$\begin{aligned}
&\sup_{\underline{u}_1\geq u_*}\int_{\mathrm{Match}_{2,\underline{u}_1}}\Big(\left|\mathfrak{d}^{\leq k}(f,\log\lambda)\right|^2 + r^{-\delta_B}\left|\mathfrak{d}^{\leq k}\underline{f}\right|^2\Big)\\
&+\sup_{\underline{u}_1\geq u_*}\int_{\mathrm{Match}_{2,\underline{u}_1}}\Big(r^{-2}\left|\mathfrak{d}^{\leq k}(r'-r)\right|^2 + \left|\mathfrak{d}^{\leq k}\Big(J^{(0)}-\cos(\theta')\Big)\right|^2 + r^2\left|\mathfrak{d}^{\leq k}(\mathfrak{J}'-\mathfrak{J})\right|^2\Big)\\
\lesssim\;& \mathfrak{G}_{k-1}^2 + \sup_{\underline{u}_1\geq u_*}\int_{\mathrm{Match}_{2,\underline{u}_1}}\Big(|(f,\log\lambda)|^2 + r^{-\delta_B}\left|\underline{f}\right|^2\Big)\\
&+\sup_{\underline{u}_1\geq u_*}\int_{\mathrm{Match}_{2,\underline{u}_1}}\Big(r^{-2}|r'-r|^2 + \left|J^{(0)}-\cos(\theta')\right|^2 + r^2|\mathfrak{J}'-\mathfrak{J}|^2\Big).
\end{aligned}\tag{9.105}$$

Next, we control the RHS of (9.105). Recalling that (e_4, e_3, e_1, e_2) coincides with the ingoing PT frame of ${}^{(top)}\mathcal{M}'$, and that $(r, J^{(0)} = \cos(\theta))$ and by $\mathfrak{J}$ are the scalar functions and complex 1-form associated to the ingoing PT structure of ${}^{(top)}\mathcal{M}'$, we may thus estimate $(f, \underline{f}, \lambda)$ and $(r'-r, \cos(\theta')-\cos\theta, \mathfrak{J}'-\mathfrak{J})$ on ${}^{(ext)}\mathcal{M}\cap{}^{(top)}\mathcal{M}' = {}^{(ext)}\mathcal{M}(u\geq u'_*)$. We then proceed by deriving transport equations along e_3 as in Step 1 of Section 9.6.3. Integrating these transport equations from $u = u'_*$ where

$$f=0,\quad \underline{f}=0,\quad \lambda=1,\quad r'=r,\quad \theta'=\theta,\quad \mathfrak{J}'=\mathfrak{J},$$

in view of Section 9.1.3, we easily infer

$$\begin{aligned}
&\sup_{\underline{u}_1\geq u_*}\int_{\mathrm{Match}_{2,\underline{u}_1}}\Big(|(f,\log\lambda)|^2 + r^{-\delta_B}\left|\underline{f}\right|^2\Big)\\
&+\sup_{\underline{u}_1\geq u_*}\int_{\mathrm{Match}_{2,\underline{u}_1}}\Big(r^{-2}|r'-r|^2 + |\cos(\theta)-\cos(\theta')|^2 + r^2|\mathfrak{J}'-\mathfrak{J}|^2\Big) \lesssim \mathfrak{G}_0^2,
\end{aligned}$$

where the above estimate is first derived on ${}^{(ext)}\mathcal{M}(u\geq u'_*)$ and then restricted to $\mathrm{Match}_2(\underline{u}\geq u_*)$. Plugging in (9.105), we deduce, for $1\leq k\leq k_{large}+8$,

$$\sup_{\underline{u}_1\geq u_*}\int_{\mathrm{Match}_{2,\underline{u}_1}}\Big(\left|\mathfrak{d}^{\leq k}(f,\log\lambda)\right|^2 + r^{-\delta_B}\left|\mathfrak{d}^{\leq k}\underline{f}\right|^2\Big)$$

$$
\begin{aligned}
&+\sup_{\underline{u}_1\geq u_*}\int_{\mathrm{Match}_{2,\underline{u}_1}}\Bigg(r^{-2}\left|\mathfrak{d}^{\leq k}(r'-r)\right|^2+\left|\mathfrak{d}^{\leq k}\left(J^{(0)}-\cos(\theta')\right)\right|^2\\
&\qquad\qquad +r^2\left|\mathfrak{d}^{\leq k}(\mathfrak{J}'-\mathfrak{J})\right|^2\Bigg)\\
&\lesssim\ \mathfrak{G}^2_{k-1}.
\end{aligned}
$$

Together with (9.103) and (9.104), we infer, for $1\leq k\leq k_{large}+8$,

$$
\begin{aligned}
&\sup_{\underline{u}_1\geq u_*}\int_{\mathrm{Match}_{2,\underline{u}_1}}\left(\left|\mathfrak{d}^{\leq k}(f,\log\lambda)\right|^2+r^{-\delta_B}\left|\mathfrak{d}^{\leq k}\underline{f}\right|^2\right)\\
&\qquad +\int_{\mathrm{Match}_2(\,^{(ext)}r\leq r_0+1)}\left|\mathfrak{d}^{\leq k}(f,\underline{f},\log\lambda)\right|^2\ \lesssim\ \mathfrak{R}^2_{k-1}+\mathfrak{G}^2_{k-1}
\end{aligned}
\tag{9.106}
$$

and

$$
\begin{aligned}
&\sup_{\underline{u}_1\geq u_*}\int_{\mathrm{Match}_{2,\underline{u}_1}}\Bigg(r^{-2}\left|\mathfrak{d}^{\leq k}(r'-r)\right|^2+\left|\mathfrak{d}^{\leq k}\left(J^{(0)}-\cos(\theta')\right)\right|^2\\
&+r^2\left|\mathfrak{d}^{\leq k}(\mathfrak{J}'-\mathfrak{J})\right|^2\Bigg)\\
&+\int_{\mathrm{Match}_2(\,^{(ext)}r\leq r_0+1)}\left|\mathfrak{d}^{\leq k}\left(r'-r, J^{(0)}-\cos(\theta'),\mathfrak{J}'-\mathfrak{J}\right)\right|^2\\
&\lesssim\ \mathfrak{R}^2_{k-1}+\mathfrak{G}^2_{k-1}.
\end{aligned}
\tag{9.107}
$$

We are now ready to construct the global frame of Lemma 9.63. Let ψ be a smooth cut-off function of (r',u) such that $\psi=0$ on $^{(ext)}\mathcal{M}\setminus\mathrm{Match}_2$, $\psi=1$ for $\mathcal{M}\setminus{}^{(ext)}\mathcal{M}$, and such that ψ only depends on u for $r'\geq r_0+1$. Then, we define the global null frame $(e_4''',e_3''',e_1''',e_2''')$ of $\mathcal{M}$ and the quantities $(r''',J'''^{(0)},\mathfrak{J}''')$ as follows

- In $^{(ext)}\mathcal{M}\setminus\mathrm{Match}_2$, we have

$$
(e_4''',e_3''',e_1''',e_2''')=(e_4'',e_3'',e_1'',e_2''),\quad r'''=r',\quad J'''^{(0)}=\cos(\theta'),\quad \mathfrak{J}'''=\mathfrak{J}'.
$$

- In $\mathcal{M}\setminus{}^{(ext)}\mathcal{M}$, we have

$$
(e_4''',e_3''',e_1''',e_2''')=(e_4,e_3,e_1,e_2),\quad r'''=r,\quad J'''^{(0)}=J^{(0)},\quad \mathfrak{J}'''=\mathfrak{J}.
$$

- In the matching region Match_2, the frame $(e_4''',e_3''',e_1''',e_2''')$ is defined from (e_4,e_3,e_1,e_2) using the change of frame coefficients $(f',\underline{f}',\lambda')$ with

$$
f'=\psi(r',u)f,\qquad \underline{f}'=\psi(r',u)\underline{f},\qquad \lambda'=1-\psi(r',u)+\psi(r',u)\lambda,
$$

where we recall that $(f, \underline{f}, \lambda)$ denotes the coefficients corresponding to the change of frame from (e_4, e_3, e_1, e_2) to $(e''_4, e''_3, e''_1, e''_2)$.

- In the matching region Match_1, r''', $J'''^{(0)}$ and $\mathfrak{J}'''$ are defined by

$$\begin{aligned} r''' &= \psi(r', u)r' + (1-\psi(r', u))r, \\ J'''^{(0)} &= \psi(r', u)\cos(\theta') + (1-\psi(r', u))J^{(0)}, \\ \mathfrak{J}''' &= \psi(r', u)\mathfrak{J}' + (1-\psi(r', u))\mathfrak{J}. \end{aligned}$$

In view of the above definitions, properties (a) and (b) of Lemma 9.61 are immediate. Also, using the definition of $(f', \underline{f}', \lambda')$, the fact that ψ only depends on[215] u for $r' \geq r_0+1$ and the control of $(f, \underline{f}, \lambda)$ in (9.106), we have, for $1 \leq k \leq k_{large}+8$,

$$\begin{aligned} \sup_{\underline{u}_1 \geq u_*} \int_{\text{Match}_{2,\underline{u}_1}} \left(\left|\mathfrak{d}^{\leq k}(f', \log(\lambda'))\right|^2 + r^{-\delta_B}\left|\mathfrak{d}^{\leq k}\underline{f}'\right|^2 \right) \\ + \int_{\text{Match}_2({}^{(ext)}r \leq r_0+1)} \left|\mathfrak{d}^{\leq k}(f', \underline{f}', \log(\lambda'))\right|^2 \lesssim \mathfrak{R}^2_{k-1} + \mathfrak{G}^2_{k-1}. \end{aligned}$$

Also, if $(f'', \underline{f}'', \lambda'')$ denotes the coefficients of the change of frame from (e'_4, e'_3, e'_1, e'_2) to $(e''_4, e''_3, e''_1, e''_2)$, we easily obtain from the above control of $(f', \underline{f}', \lambda')$ and $(f, \underline{f}, \lambda)$, for $1 \leq k \leq k_{large}+8$,

$$\begin{aligned} \sup_{\underline{u}_1 \geq u_*} \int_{\text{Match}_{2,\underline{u}_1}} \left(\left|\mathfrak{d}^{\leq k}(f'', \log(\lambda''))\right|^2 + r^{-\delta_B}\left|\mathfrak{d}^{\leq k}\underline{f}''\right|^2 \right) \\ + \int_{\text{Match}_2({}^{(ext)}r \leq r_0+1)} \left|\mathfrak{d}^{\leq k}(f'', \underline{f}'', \log(\lambda''))\right|^2 \lesssim \mathfrak{R}^2_{k-1} + \mathfrak{G}^2_{k-1} \end{aligned}$$

which concludes the proof of property (d) of Lemma 9.63.

Next, in view of the definition of r''', $J'''^{(0)}$ and $\mathfrak{J}'''$ in Match_1, and the control of $r'-r$, $\cos(\theta') - J^{(0)}$ and $\mathfrak{J}' - \mathfrak{J}$ in (9.107), we have, for $1 \leq k \leq k_{large}+8$,

$$\begin{aligned} \sup_{\underline{u}_1 \geq u_*} \int_{\text{Match}_{2,\underline{u}_1}} \Big(r^{-2}\left|\mathfrak{d}^{\leq k}\left(r''' - r\right)\right|^2 + \left|\mathfrak{d}^{\leq k}\left(J'''^{(0)} - J^{(0)}\right)\right|^2 \\ + r^2\left|\mathfrak{d}^{\leq k}\left(\mathfrak{J}''' - \mathfrak{J}\right)\right|^2 \Big) \\ + \int_{\text{Match}_2({}^{(ext)}r \leq r_0+1)} \left|\mathfrak{d}^{\leq k}\Big(r''' - r, J'''^{(0)} - J^{(0)}, \mathfrak{J}''' - \mathfrak{J}\Big)\right|^2 \end{aligned}$$

---

[215]Note in particular that we have $|\mathfrak{d}(\psi)| \lesssim |\mathfrak{d}(u)| \lesssim 1$ for $r' \geq r_0+1$.

$$\lesssim \quad \mathfrak{R}_{k-1}^2 + \mathfrak{G}_{k-1}^2$$

which concludes the proof of property (e) of Lemma 9.63. Together with the change of frame formulas of Proposition 2.12, the above control of the change of frame coefficients $(f', \underline{f}', \lambda')$, and the above control for $(\Gamma_b, \Gamma_g, A, B)$, we infer, for $k \leq k_{large} + 7$,

$$\begin{aligned}
&\int_{\text{Match}_2} \Big(r^{3+\delta_B} |\mathfrak{d}^k (A''', B''')|^2 + r^{3-\delta_B} |\mathfrak{d}^k \check{P}'''|^2 + r^{1-\delta_B} |\mathfrak{d}^k \underline{B}'''|^2 \\
&\qquad + r^{-1-\delta_B} |\mathfrak{d}^k \underline{A}'''|^2 \Big) + \int_{\text{Match}_2({}^{(ext)}r \leq r_0 + 1)} |\mathfrak{d}^k \check{\Gamma}'''|^2 \\
&\qquad + \sup_{\underline{u}_1 \geq u_*} \int_{\text{Match}_{2,\underline{u}_1}} \Bigg\{ r^2 \left| \mathfrak{d}^k (\Gamma_g''' \setminus \{\widetilde{\mathrm{tr}\underline{X}}'''\}) \right|^2 \\
&+ r^{2-\delta_B} \left| \mathfrak{d}^k (\widetilde{\mathrm{tr}\underline{X}}''') \right|^2 + \left| \mathfrak{d}^k (\Gamma_b''' \setminus \{\underline{\Xi}'''\}) \right|^2 + r^{-\delta_B} \left| \mathfrak{d}^k (\underline{\Xi}''') \right|^2 \Bigg\} \lesssim \mathfrak{R}_k + \mathfrak{G}_k,
\end{aligned}$$

which is property (c) of Lemma 9.63. This concludes the proof of Lemma 9.63.

9.7. Control of the PT-Ricci coefficients on Σ_*

The goal of this section is to provide the proof of Proposition 9.50. For convenience, we restate the result below.

Proposition 9.67. *The Ricci and metric coefficients of the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ verify the following estimates on Σ_*, for all $k \leq k_{large} + 7$,*

$$^\star\mathfrak{G}_k \lesssim \epsilon_0 + {}^\star\mathfrak{R}_k. \tag{9.108}$$

Recall, see Definition 5.56, the tangential 1-form f_0 on Σ_* given by

$$(f_0)_1 = 0, \quad (f_0)_2 = \sin\theta, \quad \text{on} \quad S_*, \qquad \nabla_\nu f_0 = 0,$$

where, on S_*, we consider the orthonormal basis (e_1, e_2) of S_* given by (5.129).

Also, recall from Section 9.1.3 that we initialize the PT frame of ${}^{(ext)}\mathcal{M}$ from the integrable frame on Σ_*, by relying on the change of frame formula with the transition coefficients

$$\lambda = 1, \qquad f = \frac{a}{r} f_0, \qquad \underline{f} = \frac{a\Upsilon}{r} f_0.$$

In order to prove Proposition 9.67, we proceed as follows:

1. We first derive analogous estimates for the integrable frame of Σ_* in Section 9.7.1.
2. Then, we derive estimates for the 1-form f_0 on Σ_* in Section 9.7.2.
3. Finally, we conclude the proof of Proposition 9.67 in Section 9.7.3 by using the transition coefficients

$$\lambda = 1, \qquad f = \frac{a}{r} f_0, \qquad \underline{f} = \frac{a\Upsilon}{r} f_0,$$

the transformation formulas to pass from the integrable frame of Σ_* to the PT frame of ${}^{(ext)}\mathcal{M}$ on Σ_*, and the control of the integrable of Σ_* and of the 1-form f_0.

9.7.1. Control of the integrable frame of Σ_* Recall that on Σ_*, the Ricci coefficients of the integrable frame of Σ_* verify the following GCM conditions

$$\widecheck{\mathrm{tr}\chi} = 0, \qquad \widecheck{\mathrm{tr}\underline{\chi}} = \sum_p \underline{C}_p J^{(p)}, \qquad \check{\mu} = \sum_p M_p J^{(p)},$$
$$(\mathrm{div}\,\eta)_{\ell=1} = 0, \qquad (\mathrm{div}\,\underline{\xi})_{\ell=1} = 0, \qquad \overline{b_*} = -1 - \frac{2m}{r},$$

where $\underline{C}_p$, M_p, $p = 0, +, -$, are functions of r along Σ_*, and where $\overline{b_*}$ denotes the average of b_* on the spheres foliating Σ_*.

Additionally, the following holds on S_*

$$(\widecheck{\mathrm{tr}\underline{\chi}})_{\ell=1} = 0, \quad (\mathrm{div}\,\beta)_{\ell=1} = 0, \quad (\mathrm{curl}\,\beta)_{\ell=1,\pm} = 0, \quad (\mathrm{curl}\,\beta)_{\ell=1,0} = \frac{2am}{r^5}.$$

Also, remember that the integrable frame of Σ_* satisfies the following transversality conditions

$$\xi = 0, \qquad \omega = 0, \qquad \underline{\eta} = -\zeta, \qquad e_4(r) = 1, \qquad e_4(u) = 0, \tag{9.109}$$

where r denotes the area radius of the u-foliation of Σ_*, and we have $u + r = c_{\Sigma_*}$ on Σ_*, where c_{Σ_*} is a constant.

Remark 9.68. *Note that the transversality conditions above are compatible with an outgoing geodesic foliation initialized on Σ_*. Note also that they must be specified to make sense of $\underline{\xi}$ and η, and hence to make sense of the GCM conditions $(div\,\eta)_{\ell=1} = 0$ and $(div\,\underline{\xi})_{\ell=1} = 0$ on Σ_*.*

To state the main result of this section we need to introduce the following norms.

Definition 9.69. *We define the following Ricci coefficients norms on Σ_* relative to the integrable frame of Σ_**

$$
\begin{aligned}
{}^{(\Sigma_*)}\mathfrak{G}_k^2 &:= \int_{\Sigma_*} r^2|\mathfrak{d}_*^{\leq k+1}(\widehat{\chi}, \widecheck{tr\chi}, \zeta, \widecheck{tr\underline{\chi}})|^2 + |\mathfrak{d}_*^{\leq k+1}\underline{\widehat{\chi}}| + |\not{\mathfrak{d}}^{\leq 1}\mathfrak{d}_*^{\leq k}(\eta, \underline{\check{\omega}}, \underline{\xi})|^2 \\
&+ \int_{\Sigma_*} r^{-2}|\not{\mathfrak{d}}^{\leq 2}\mathfrak{d}_*^{\leq k}(\widecheck{\nu(r)}, \widecheck{\nu(u)}, \check{b}_*)|^2,
\end{aligned}
$$

where $\widecheck{tr\chi}$, $\widehat{\chi}$, ζ, η, $\widecheck{tr\underline{\chi}}$, $\underline{\widehat{\chi}}$, $\underline{\check{\omega}}$, $\underline{\xi}$ are the Ricci coefficients of the integrable frame of Σ_*, and $\nu = e_3 + b_* e_4$ is tangent to Σ_*.

Definition 9.70. *We define the curvature norm on Σ_*, relative to the integrable frame of Σ_*,*

$$
{}^{(\Sigma_*)}\mathfrak{R}_k^2 := \int_{\Sigma_*} \left[r^{4+\delta}|\mathfrak{d}^{\leq k}(\alpha, \beta)|^2 + r^4|\mathfrak{d}^{\leq k}(\check{\rho}, \widecheck{{}^*\rho})|^2 + r^2|\mathfrak{d}^{\leq k}\underline{\beta}|^2 + |\mathfrak{d}^{\leq k}\underline{\alpha}|^2 \right],
$$

where α, β, ρ, ${}^*\rho$, $\underline{\beta}$, $\underline{\alpha}$ denote the curvature components relative to the integrable frame of Σ_*.

The following proposition, which provides the control of the Ricci coefficients of the integrable frame of Σ_*, is the main result of this section.

Proposition 9.71. *The following estimates hold true for the integrable frame of Σ_**

$$
{}^{(\Sigma_*)}\mathfrak{G}_k \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k \leq k_{large} + 7. \tag{9.110}
$$

Proof. We rely on $L^2(\Sigma_*)$-estimates for the main quantities, with the exception of $\ell = 1$ modes and averages which are estimated in $L^2_u L^\infty(S)$. Also, we rely on the following local bootstrap assumptions

$$
{}^{(\Sigma_*)}\mathfrak{G}_k \leq \epsilon, \qquad k \leq k_{large} + 7. \tag{9.111}
$$

Finally, we rely repeatedly on the results of Sections 5.1 and 5.2.

Step 1. The estimate for $\widecheck{\mathrm{tr}\chi}$ follows immediately from our GCM conditions according to which we have $\widecheck{\mathrm{tr}\chi} = 0$ on Σ_*.

Step 2. Since we have $\mathrm{tr}\,\chi = \frac{2}{r}$ and $\mathrm{tr}\,\underline{\chi} = -\frac{2\Upsilon}{r}$ on S_* according to our GCM conditions, we infer from the Gauss equation

$$
K = -\rho - \frac{1}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}} = -\rho + \frac{\Upsilon}{r^2} + \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}
$$

$$= \frac{1}{r^2} - \check{\rho} + \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}} \quad \text{on} \quad S_*.$$

We deduce, using the trace theorem,

$$\begin{aligned} r^2\left\|K - \frac{1}{r^2}\right\|_{\mathfrak{h}_{k-1}(S_*)} &\lesssim r^2\|\check{\rho}\|_{\mathfrak{h}_{k-1}(S_*)} + \|\widehat{\chi}\cdot\underline{\widehat{\chi}}\|^2_{\mathfrak{h}_{k-1}(S_*)} \\ &\lesssim \epsilon^2 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k \le k_{large}+7. \end{aligned}$$

Next, recall that ϕ denotes the uniformization factor of S_*, see (2.48) in Section 2.5.3. Using the effective uniformization theorem, see Theorem 5.2,

$$\|\phi\|_{\mathfrak{h}_{k+1}(S_*)} \lesssim r^2\left\|K - \frac{1}{r^2}\right\|_{\mathfrak{h}_{k-1}(S_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k \le k_{large}+7,$$

we deduce

$$\|\phi\|_{\mathfrak{h}_{k+1}(S_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad \text{for all } k \le k_{large}+7. \tag{9.112}$$

Step 3. Using the formula $\Delta J^{(p)} + \frac{2}{r^2}J^{(p)} = \frac{2}{r^2}(1-e^{-2\phi})J^{(p)}$, see Lemma 5.36 and its proof, we infer from the control of ϕ provided by Step 2

$$\left\|\left(\Delta J^{(p)} + \frac{2}{r^2}J^{(p)}\right)\right\|_{\mathfrak{h}_{k+1}(S_*)} \lesssim \frac{\epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k}{r^2}, \text{ for all } k \le k_{large}+7. \tag{9.113}$$

Step 4. Recall from Lemma 5.37 that we have

$$\nabla_\nu\Big[(r^2\Delta+2)J^{(p)}\Big] = O(\mathfrak{d}^{\le 1}\Gamma_b).$$

Integrating from S_* along Σ_* and using (9.111) as well as the control on S_* of Step 3, we infer

$$\sup_{S\subset\Sigma_*} r^2\left\|\left(\Delta J^{(p)} + \frac{2}{r^2}J^{(p)}\right)\right\|_{\mathfrak{h}_{k-1}(S)} \lesssim \epsilon_0 + \sqrt{u_*}\epsilon \lesssim \sqrt{u_*}\epsilon, \quad k \le k_{large}+7.$$

Then, proceeding as in the proof of Lemma 5.37, we have

$$\left\|{\not{d}}_2^*{\not{d}}_1^*J^{(p)}\right\|_{\mathfrak{h}_k(S)} \lesssim \left\|\left(\Delta J^{(p)} + \frac{2}{r^2}J^{(p)}\right)\right\|_{\mathfrak{h}_k(S)} + r^{-1}\|\Gamma_g\|_{\mathfrak{h}_k(S)},$$

where by ${\not{d}}_1^*J^{(p)}$, we mean either ${\not{d}}_1^*(J^{(p)},0)$ or ${\not{d}}_1^*(0,J^{(p)})$. Together with the above estimate and (9.111) yields

$$\sup_{S\subset\Sigma_*} r^2 \left\| \not{d}_2^* \not{d}_1^* J^{(p)} \right\|_{\mathfrak{h}_{k-1}(S)} \lesssim \sqrt{u_*}\epsilon, \qquad k\leq k_{large}+7.$$

Together with the dominance condition for r on Σ_*, we deduce

$$\begin{aligned}
&\sup_{S\subset\Sigma_*} r\left(\left\|\left(\Delta J^{(p)}+\frac{2}{r^2}J^{(p)}\right)\right\|_{\mathfrak{h}_{k-1}(S)}+\left\|\not{d}_2^*\not{d}_1^*J^{(p)}\right\|_{\mathfrak{h}_{k-1}(S)}\right)\\
\lesssim\ & \frac{\sqrt{u_*}\epsilon}{r}\lesssim\epsilon_0, \qquad k\leq k_{large}+7.
\end{aligned}$$

Using $\nu(J^{(p)})=0$, we recover ν derivatives as well, and hence, for $k\leq k_{large}+7$,

$$\sup_{S\subset\Sigma_*} r\left(\left\|\mathfrak{d}_*^{k-1}\left(\Delta J^{(p)}+\frac{2}{r^2}J^{(p)}\right)\right\|_{L^2(S)}+\left\|\mathfrak{d}_*^{k-1}\not{d}_2^*\not{d}_1^*J^{(p)}\right\|_{L^2(S)}\right)\lesssim\epsilon_0.$$

Step 5. Next, differentiating Codazzi for $\widehat{\chi}$ by div , and integrating against $J^{(p)}$, we infer, see Steps 1 of the proof of Proposition 5.48 in Section 5.4.2,

$$(\mathrm{div}\,\not{d}_2\widehat{\chi})_{\ell=1} \;=\; \frac{1}{r}(\mathrm{div}\,\zeta)_{\ell=1}-(\mathrm{div}\,\beta)_{\ell=1}+r^{-3}\int_S \not{\mathfrak{d}}^{\leq 1}(\widehat{\chi}\cdot\zeta)J^{(p)}.$$

This then yields, after after integration by parts,

$$(\mathrm{div}\,\zeta)_{\ell=1} \;=\; r(\mathrm{div}\,\beta)_{\ell=1}+r^{-2}\int_S\widehat{\chi}\cdot\not{d}_2^*\not{d}_1^*J^{(p)}+r^{-3}\int_S\not{\mathfrak{d}}^{\leq 1}(\widehat{\chi}\cdot\zeta)J^{(p)}.$$

Together with (9.111) and the control for $\not{d}_2^*\,\not{d}_1^*J^{(p)}$ of Step 4, we infer, using integration by parts on S for angular derivatives to avoid loss of derivatives, for $k\leq k_{large}+7$,

$$\begin{aligned}
r^3\left(\int_{u=1}^{u_*}\left|\nu^{\leq k+1}((\mathrm{div}\,\zeta)_{\ell=1})\right|^2\right)^{\frac{1}{2}} \;\lesssim\;& \epsilon^2+{}^{(\Sigma_*)}\mathfrak{R}_k\\
&+r\left(\int_{u=1}^{u_*}\left|\int_S\nabla_\nu^{k+1}\beta\cdot\not{\mathfrak{d}}J^{(p)}\right|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $\nu=e_3+b_*e_4$, and using the Bianchi identities for $\nabla_3\beta$ and $\nabla_4\beta$, as well as the commutator Lemma 5.20, we have

$$\nabla_\nu^{k+1}\beta \;=\; \nabla(\nabla_\nu^k\check{\rho})+{}^*\nabla(\nabla_\nu^k\;{}^*\rho)-b_*\mathrm{div}\,(\nabla_\nu^k\alpha)+\text{ l.o.t.}$$

Plugging in the above and integrating the angular derivatives by parts to avoid a loss of derivatives, we infer

$$r^3\left(\int_{u=1}^{u_*}\left|\nu^{\leq k+1}((\mathrm{div}\,\zeta)_{\ell=1})\right|^2\right)^{\frac{1}{2}}\lesssim\epsilon^2+{}^{(\Sigma_*)}\mathfrak{R}_k,\quad k\leq k_{large}+7,$$

and hence

$$r^3\left(\int_{u=1}^{u_*}\left|\nu^{\leq k+1}((\mathrm{div}\,\zeta)_{\ell=1})\right|^2\right)^{\frac{1}{2}}\lesssim\epsilon_0+{}^{(\Sigma_*)}\mathfrak{R}_k,\quad k\leq k_{large}+7.$$

Step 6. Next, recall that we have already established an estimate for $(\widecheck{\mathrm{tr}\underline{\chi}})_{\ell=1}$ on Σ_* in (8.86), i.e.

$$\sup_{\Sigma_*}r^2u^{1+\delta_{dec}}|(\underline{\check{\kappa}})_{\ell=1}|\quad\lesssim\quad\epsilon_0.$$

To derive an estimate for higher order derivatives in ν, we rely on the following equation of Corollary 5.41

$$\begin{aligned}
&\nu\left(\int_S\left(\Delta\widecheck{\mathrm{tr}\underline{\chi}}+\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right)J^{(p)}\right)\\
=\;&O(r^{-3})\int_S\underline{\check{\kappa}}J^{(p)}+O(r^{-2})\int_S\mathrm{div}\,\zeta J^{(p)}+O(r^{-1})\int_S\mathrm{div}\,\underline{\beta}J^{(p)}\\
&+O(r^{-2})\int_S\check{\rho}J^{(p)}+O(r^{-1})\int_S\mathrm{div}\,\beta J^{(p)}+r^{-1}\int_S\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\,\mathfrak{d}^{\leq 1}\Gamma_b\\
&+r^{-2}\int_S\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_b)J^{(p)},
\end{aligned}$$

which yields

$$\begin{aligned}
&\nu\left(\int_S\left(\Delta\underline{\check{\kappa}}+\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right)J^{(p)}\right)\\
=\;&O(r^{-1})\int_S\mathrm{div}\,\underline{\beta}J^{(p)}+O(r^{-3})\int_S J^{(p)}\,\mathfrak{d}^{\leq 1}\Gamma_g\\
&+r^{-1}\int_S\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\,\mathfrak{d}^{\leq 1}\Gamma_b+r^{-2}\int_S\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_b)J^{(p)}.
\end{aligned}$$

Differentiating in ν, using integration by parts on S for angular derivatives to avoid loss of derivatives, relying on the estimates of Step 4 and Step 5, and using (9.111), we deduce, for $k\leq k_{large}+7$,

$$\begin{aligned}
r^2\left(\int_{u=1}^{u_*}\left|\nu^{\leq k+1}((\widecheck{\mathrm{tr}\underline{\chi}})_{\ell=1})\right|^2\right)^{\frac{1}{2}}\quad\lesssim\quad& r^{-1}\epsilon+\epsilon^2+{}^{(\Sigma_*)}\mathfrak{R}_k\\
&+\left(\int_{u=1}^{u_*}\left|\int_S\nabla_\nu^{k+1}\underline{\beta}\cdot\mathfrak{d}J^{(p)}\right|^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$+\left(\int_{u=1}^{u_*}\left|\int_S \nabla_\nu^{k+1}\beta\cdot \not{\mathfrak{d}} J^{(p)}\right|^2\right)^{\frac{1}{2}}.$$

Next, as in Step 5, we use the fact that $\nu = e_3 + b_* e_4$ and the Bianchi identities, as well as the commutator Lemma 5.20, to derive

$$\begin{aligned}\nabla_\nu^{k+1}\beta &= \nabla(\nabla_\nu^k\check{\rho}) + {}^*\nabla(\nabla_\nu^k\, {}^*\rho) - b_*\mathrm{div}\,(\nabla_\nu^k\alpha) + \text{ l.o.t.},\\ \nabla_\nu^{k+1}\underline{\beta} &= -\mathrm{div}\,(\nabla_\nu^k\underline{\alpha}) - b_*\nabla(\nabla_\nu^k\check{\rho}) + b_*\, {}^*\nabla(\nabla_\nu^k\, {}^*\rho) + \text{ l.o.t.}\end{aligned}$$

Plugging in the above and integrating the angular derivatives by parts to avoid a loss of derivatives, we infer

$$r^2\left(\int_{u=1}^{u_*}\left|\nu^{\leq k+1}((\widecheck{\mathrm{tr}\underline{\chi}})_{\ell=1})\right|^2\right)^{\frac{1}{2}} \lesssim r^{-1}\epsilon + \epsilon^2 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k\leq k_{large}+7.$$

Together with the dominance condition for r on Σ_*, we obtain

$$r^2\left(\int_{u=1}^{u_*}\left|\nu^{\leq k+1}((\widecheck{\mathrm{tr}\underline{\chi}})_{\ell=1})\right|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k\leq k_{large}+7.$$

Step 7. We are now ready to control $\widecheck{\mathrm{tr}\underline{\chi}}$.

Step 7a. First, we control $\overline{\widecheck{\mathrm{tr}\underline{\chi}}}$. Recall the definition of the Hawking mass m_H

$$\frac{2m_H}{r} = 1 + \frac{1}{16\pi}\int_S \mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}.$$

Since we have $\mathrm{tr}\,\chi = \frac{2}{r}$ by our GCM conditions on Σ_*, we infer on Σ_*

$$\overline{\mathrm{tr}\,\underline{\chi}} = -\frac{2\left(1-\frac{2m_H}{r}\right)}{r},$$

where $\overline{\mathrm{tr}\,\underline{\chi}}$ denotes the average of $\mathrm{tr}\,\underline{\chi}$ on S. In particular, we infer

$$\overline{\widecheck{\mathrm{tr}\underline{\chi}}} = -\frac{4(m-m_H)}{r^2}.$$

Since we have by Lemma 5.51 and by Proposition 5.52

$$\nu(m_H) = \int_S \not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b), \qquad \sup_{\Sigma_*} u^{1+2\delta_{dec}}|m_H - m| \lesssim \epsilon_0,$$

we infer, together with (9.111), and integrating the angular derivatives by parts to avoid a loss of derivatives,

$$\left(\int_{u=1}^{u_*}\left|\nu^{\leq k+1}(m_H-m)\right|^2\right)^{\frac{1}{2}}+r^2\left(\int_{u=1}^{u_*}\left|\nu^{\leq k+1}(\overline{\widecheck{\mathrm{tr}\underline{\chi}}})\right|^2\right)^{\frac{1}{2}}$$
$$\lesssim \epsilon_0, \quad k\leq k_{large}+7.$$

Step 7b. Next, we control $\nu^k(\underline{C}_0)$ and $\nu^k(\underline{C}^{(p)})$. In view of our GCM conditions for $\widecheck{\mathrm{tr}\underline{\chi}}$ and the fact that $\nu(J^{(p)})=0$, we have

$$\begin{aligned}\nu^k(\widecheck{\mathrm{tr}\underline{\chi}}) &= \nu^k(\underline{C}_0)+\sum_p\nu^k(\underline{C}^{(p)})J^{(p)}=\overline{\nu^k(\underline{C}_0)}+\sum_p\overline{\nu^k(\underline{C}^{(p)})}J^{(p)}+h_k,\\ h_k &:= \left(\nu^k(\underline{C}_0)-\overline{\nu^k(\underline{C}_0)}\right)+\sum_p\left(\nu^k(\underline{C}^{(p)})-\overline{\nu^k(\underline{C}^{(p)})}\right)J^{(p)}.\end{aligned}$$

Using a Poincaré inequality on S, the fact that $\nabla(\underline{C}_0)=0$ and $\nabla(\underline{C}^{(p)})=0$, (9.111), and the following commutator formula of Lemma 5.20

$$[\nabla_\nu,\nabla]f = -\frac{2}{r}\nabla f+\Gamma_b\cdot\nabla_\nu f+r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}f,$$

we easily obtain

$$r\|h_k\|_{L^2(\Sigma_*)}\lesssim\epsilon^2\lesssim\epsilon_0,\quad k\leq k_{large}+8.$$

Then, coming back to the above identity for $\nu^k(\widecheck{\mathrm{tr}\underline{\chi}})$, multiplying it respectively with 1 or $J^{(p)}$, and integrating on S, we easily obtain, using the properties of $J^{(p)}$ in Lemma 5.37, for $k\leq k_{large}+8$,

$$r^2\left(\int_{u=1}^{u_*}\left|\nu^k(\underline{C}_0,\underline{C}^{(p)})\right|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0+r^2\left(\int_{u=1}^{u_*}\left|\nu^k(\overline{\widecheck{\mathrm{tr}\underline{\chi}}},\,(\widecheck{\mathrm{tr}\underline{\chi}})_{\ell=1})\right|^2\right)^{\frac{1}{2}}.$$

Together with the above estimate for $\overline{\widecheck{\mathrm{tr}\underline{\chi}}}$, and the one of Step 6 for $(\widecheck{\mathrm{tr}\underline{\chi}})_{\ell=1}$, we infer

$$r^2\left(\int_{u=1}^{u_*}\left|\nu^{k+1}(\underline{C}_0,\underline{C}^{(p)})\right|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0+{}^{(\Sigma_*)}\mathfrak{R}_k,\quad k\leq k_{large}+7.$$

Step 7c. We finally control $\mathfrak{d}_*^k\widecheck{\mathrm{tr}\underline{\chi}}$. First, the above identity for $\nu^k(\widecheck{\mathrm{tr}\underline{\chi}})$ and the above control of $\nu^k(\underline{C}_0)$ and $\nu^k(\underline{C}^{(p)})$ implies

$$r\|\nu^{k+1}\widecheck{\mathrm{tr}\underline{\chi}}\|_{L^2(\Sigma_*)}\lesssim\epsilon_0+{}^{(\Sigma_*)}\mathfrak{R}_k,\qquad k\leq k_{large}+7.$$

Next, to recover the other derivatives, we rely on Corollary 5.38 which yields

$$\|\nu^k(\widecheck{\mathrm{tr}\underline{\chi}})\|_{\mathfrak{h}_{j+2}(S)} \lesssim r^2\|\,{\not{d}}_2^*\,{\not{d}}_1^*\nu^k(\widecheck{\mathrm{tr}\underline{\chi}})\|_{\mathfrak{h}_j(S)} + r\big|(\nu^k(\widecheck{\mathrm{tr}\underline{\chi}}))_{\ell=1}\big| + r\big|\overline{\nu^k(\widecheck{\mathrm{tr}\underline{\chi}})}\big|.$$

Together with the above estimate for $\overline{\widecheck{\mathrm{tr}\underline{\chi}}}$, and the one of Step 6 for $(\widecheck{\mathrm{tr}\underline{\chi}})_{\ell=1}$, we infer

$$\begin{aligned} r\|\,\mathfrak{\not{d}}^{\le j+2}\nu^k(\widecheck{\mathrm{tr}\underline{\chi}})\|_{L^2(\Sigma_*)} &\lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k \\ &\quad + r^3\|\,\mathfrak{\not{d}}^{\le j}\,{\not{d}}_2^*\,{\not{d}}_1^*\nu^k(\widecheck{\mathrm{tr}\underline{\chi}})\|_{L^2(\Sigma_*)}, \quad k+j\le k_{large}+6. \end{aligned}$$

Also, differentiating the above identity for $\nu^k(\widecheck{\mathrm{tr}\underline{\chi}})$, and since $\nabla(\underline{C}_0)=0$ and $\nabla(\underline{C}^{(p)})=0$, we have

$${\not{d}}_2^*\,{\not{d}}_1^*\nu^k(\widecheck{\mathrm{tr}\underline{\chi}}) = [\,{\not{d}}_2^*\,{\not{d}}_1^*,\nu^k]\underline{C}_0 + \sum_p[\,{\not{d}}_2^*\,{\not{d}}_1^*,\nu^k]\underline{C}^{(p)}J^{(p)} + \sum_p \nu^k(\underline{C}^{(p)})\,{\not{d}}_2^*\,{\not{d}}_1^*J^{(p)}.$$

Using the above commutator formula for $[\nabla_\nu,\nabla]$, (9.111), and the control for ${\not{d}}_2^*\,{\not{d}}_1^*J^{(p)}$ of Step 4, we infer

$$r^3\|\,\mathfrak{\not{d}}^{\le j}\,{\not{d}}_2^*\,{\not{d}}_1^*\nu^k(\widecheck{\mathrm{tr}\underline{\chi}})\|_{L^2(\Sigma_*)} \lesssim \epsilon^2\lesssim\epsilon_0, \quad k+j\le k_{large}+6,$$

and hence

$$r\|\,\mathfrak{\not{d}}^{\le j+2}\nu^k(\widecheck{\mathrm{tr}\underline{\chi}})\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k+j\le k_{large}+6.$$

Together with the above estimate for $\nu^k(\widecheck{\mathrm{tr}\underline{\chi}})$, we deduce

$$r\|\mathfrak{d}_*^{\le k+1}\widecheck{\mathrm{tr}\underline{\chi}}\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k\le k_{large}+7. \tag{9.114}$$

Step 8. In view of the definition of μ, we have

$$\check{\mu} = -\mathrm{div}\,\zeta - \check{\rho} + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}}.$$

Together with (9.111) and the control of $\nu^k(\mathrm{div}\,\zeta)_{\ell=1}$ in Step 5, we infer

$$\begin{aligned} & r^3\left(\int_{u=1}^{u_*}\big|\nu^{\le k+1}((\check{\mu})_{\ell=1})\big|^2\right)^{1/2} \\ \lesssim\ & \epsilon_0+\epsilon^2+{}^{(\Sigma_*)}\mathfrak{R}_k + r\left(\int_{u=1}^{u_*}\left|\int_S\nu^{k+1}(\check{\rho})J^{(p)}\right|^2\right)^{1/2}, \quad k\le k_{large}+7. \end{aligned}$$

Next, as in Step 5, we use the fact that $\nu = e_3 + b_* e_4$ and the Bianchi identities, as well as the commutator Lemma 5.20, to derive

$$\nabla_\nu^{k+1}\check{\rho} = -\mathrm{div}\,(\nabla_\nu^k\underline{\beta}) + b_*\mathrm{div}\,(\nabla_\nu^k\beta) + \text{ l.o.t.}$$

Plugging in the above and integrating the angular derivatives by parts to avoid a loss of derivatives, we infer

$$r^3\left(\int_{u=1}^{u_*}\left|\nu^{\leq k+1}\big((\check{\underline{\mu}})_{\ell=1}\big)\right|^2\right)^{1/2} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k \leq k_{large}+7.$$

Step 9. In view of the proof of Proposition 5.52, the average of $\check{\underline{\mu}}$ is given by

$$\overline{\check{\underline{\mu}}} = \frac{2(m_H - m)}{r^3},$$

which together with the control of $m_H - m$ in Step 7a implies

$$r^3\left(\int_{u=1}^{u_*}\left|\nu^{\leq k+1}(\overline{\check{\underline{\mu}}})\right|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0, \quad k \leq k_{large}+7.$$

Also recall the GCM condition for μ on Σ_*

$$\check{\underline{\mu}} = M_0 + \sum_p M_p J^{(p)}.$$

In view of the above formula for $\check{\underline{\mu}}$ on Σ_*, the above control of $\overline{\check{\underline{\mu}}}$, and the estimate for $(\check{\underline{\mu}})_{\ell=1}$ in Step 8, the control of $\check{\underline{\mu}}$ is completely analogous to the one of $\widecheck{\mathrm{tr}\underline{\chi}}$ in Step 7, and we infer the corresponding estimate

$$r^2\|\mathfrak{d}_*^{\leq k+1}\check{\underline{\mu}}\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k \leq k_{large}+7. \tag{9.115}$$

Step 10. Next, recall that we have

$$\begin{aligned}\mathrm{div}\,\zeta &= -\check{\underline{\mu}} - \check{\rho} + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}},\\ \mathrm{curl}\,\zeta &= {}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}}.\end{aligned}$$

In view of (9.111) and the control of $\check{\underline{\mu}}$ in Step 9, as well as the commutator Lemma 5.20, we infer

$$r^2\|\,\not{d}_1(\mathfrak{d}_*^{\leq k}\zeta)\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + \epsilon^2 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k \leq k_{large}+7,$$

which together with the elliptic estimate of Lemma 5.27 implies

$$r\|\not{\mathfrak{d}}^{\leq 1}\mathfrak{d}_*^{\leq k}\zeta\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k\leq k_{large}+7.$$

To control $\nabla_\nu^{k+1}\zeta$, we come back to the above system and differentiate it w.r.t. ∇_ν^{k+1}. We obtain

$$\not{d}_1\left(\nabla_\nu^{k+1}\zeta\right) = \nabla_\nu^{k+1}(-\check{\rho}, {}^*\rho)+h_{k+1}$$

where h_{k+1} satisfies, in view of (9.111) and the control of $\check{\mu}$ in Step 9, as well as the commutator Lemma 5.20,

$$r^2\|h_{k+1}\|_{L^2(\Sigma_*)} \lesssim \epsilon_0+\epsilon^2\lesssim\epsilon_0, \quad k\leq k_{large}+7.$$

Next, as in Step 5, we use the fact that $\nu=e_3+b_*e_4$ and the Bianchi identities, as well as the commutator Lemma 5.20, to derive

$$\nabla_\nu^{k+1}(-\check{\rho}, {}^*\rho) = \Big(\text{div}\,(\nabla_\nu^k\underline{\beta})-b_*\text{div}\,(\nabla_\nu^k\beta), -\text{curl}\,(\nabla_\nu^k\underline{\beta})-b_*\text{curl}\,(\nabla_\nu^k\beta)\Big) + \text{l.o.t.}$$

Plugging in the above, and using the above estimate for h_{k+1}, we infer

$$r\|\nabla_\nu^{k+1}\zeta\|_{L^2(\Sigma_*)} \lesssim r\|(r\not{d}_1)^{-1}\not{\mathfrak{d}}\nabla_\nu^k(\underline{\beta},\beta)\|_{L^2(\Sigma_*)}+\epsilon_0, \quad k\leq k_{large}+7.$$

Using the ellipticity of $(r\not{d}_1)^{-1}\not{\mathfrak{d}}$ on the spheres S, this yields

$$r\|\nabla_\nu^{k+1}\zeta\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k\leq k_{large}+7.$$

Together with the above estimate for $\not{\mathfrak{d}}^{\leq 1}\mathfrak{d}_*^k\zeta$, we deduce

$$r\|\mathfrak{d}_*^{\leq k+1}\zeta\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k\leq k_{large}+7. \tag{9.116}$$

Step 11. Using Codazzi for $\widehat{\chi}$, (9.111), the control of Step 1 for $\widecheck{\text{tr}\chi}$, and the control of Step 10 for ζ, as well as the commutator Lemma 5.20, we have

$$r^2\|\not{d}_2(\mathfrak{d}_*^{\leq k}\widehat{\chi})\|_{L^2(\Sigma_*)} \lesssim \epsilon_0+\epsilon^2+{}^{(\Sigma_*)}\mathfrak{R}_k, \quad k\leq k_{large}+7,$$

which together with the elliptic estimate of Lemma 5.27 implies

$$r\|\not{\mathfrak{d}}^{\leq 1}(\mathfrak{d}_*^{\leq k}\widehat{\chi})\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k\leq k_{large}+7. \tag{9.117}$$

Note that this is not yet the desired control for $\widehat{\chi}$ as we still need to recover $\nabla_\nu^{k+1}\widehat{\chi}$ which will be done in Step 15.

Step 12. Using Codazzi for $\widehat{\underline{\chi}}$, (9.111), the control of Step 7 for $\widecheck{\mathrm{tr}\underline{\chi}}$, and the control of Step 10 for ζ, as well as the commutator Lemma 5.20, we have

$$r\|\,\not{d}_2(\mathfrak{d}_*^{\leq k}\widehat{\underline{\chi}})\|_{L^2(\Sigma_*)} \lesssim \epsilon_0+\epsilon^2+{}^{(\Sigma_*)}\mathfrak{R}_k, \quad k\leq k_{large}+7,$$

which together with the elliptic estimate of Lemma 5.27 implies

$$\|\,\not{\mathfrak{d}}^{\leq 1}(\mathfrak{d}_*^{\leq k}\widehat{\underline{\chi}})\|_{L^2(\Sigma_*)} \lesssim \epsilon_0+{}^{(\Sigma_*)}\mathfrak{R}_k, \quad k\leq k_{large}+7. \tag{9.118}$$

Note that this is not yet the desired control for $\widehat{\underline{\chi}}$ as we still need to recover $\nabla_\nu^{k+1}\widehat{\underline{\chi}}$ which will be done in Step 15.

Step 13. Recall the equation for $\not{d}_2\not{d}_2^*\eta$ derived in Proposition 5.22

$$2\not{d}_2\not{d}_2^*\eta = -\nabla_3\nabla\check{\kappa}-\frac{2}{r}\nabla_3\zeta-\frac{2}{r}\underline{\beta}+r^{-2}\not{\mathfrak{d}}^{\leq 1}\Gamma_g+r^{-1}\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

We infer

$$\not{d}_2\not{d}_2^*(\mathfrak{d}_*^{\leq k}\eta) = \not{\mathfrak{d}}^{\leq 1}(h_k), \quad k\leq k_{large}+7,$$

where h_k satisfies in view of (9.111), the estimate of Step 1 for $\widecheck{\mathrm{tr}\chi}$ and of Step 10 for ζ, and the commutator Lemma 5.20,

$$r^2\|h_k\|_{L^2(\Sigma_*)} \lesssim \epsilon_0+r^{-1}\epsilon+\epsilon^2+{}^{(\Sigma_*)}\mathfrak{R}_k, \quad k\leq k_{large}+7.$$

Using the ellipticity of $(r\not{d}_2)^{-1}\not{\mathfrak{d}}^{\leq 1}$ on the spheres S, and the elliptic estimate for $\not{d}_2^*$ of Lemma 5.28, we infer

$$\|\,\not{\mathfrak{d}}^{\leq 1}(\mathfrak{d}_*^{\leq k}\eta)\|_{L^2(\Sigma_*)} \lesssim r^2\|h_k\|_{L^2(\Sigma_*)}+r^2|(\not{d}_1\nabla_\nu^{\leq k}\eta)_{\ell=1}|, \quad k\leq k_{large}+7.$$

Together with the above estimate for h_k and the dominance condition for r on Σ_*, this yields

$$\|\,\not{\mathfrak{d}}^{\leq 1}(\mathfrak{d}_*^{\leq k}\eta)\|_{L^2(\Sigma_*)} \lesssim \epsilon_0+{}^{(\Sigma_*)}\mathfrak{R}_k+r^2|(\not{d}_1\nabla_\nu^{\leq k}\eta)_{\ell=1}|, \quad k\leq k_{large}+7.$$

Since $\not{d}_1=(\mathrm{div}\,,\mathrm{curl}\,)$, and using the null structure equation $\mathrm{curl}\,\eta = {}^*\rho-\frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}}$, (9.111), and the commutator Lemma 5.20, we infer

$$\|\,\not{\mathfrak{d}}^{\leq 1}(\mathfrak{d}_*^{\leq k}\eta)\|_{L^2(\Sigma_*)} \lesssim \epsilon_0+{}^{(\Sigma_*)}\mathfrak{R}_k+r^2|(\mathrm{div}\,\nabla_\nu^{\leq k}\eta)_{\ell=1}|, \quad k\leq k_{large}+7.$$

Together with the GCM condition $(\operatorname{div}\eta)_{\ell=1} = 0$ on Σ_*, the fact that ν is tangent to Σ_*, (9.111), and Corollary 5.32, we finally obtain

$$\|\not{\mathfrak{d}}^{\leq 1}(\mathfrak{d}_*^{\leq k}\eta)\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k \leq k_{large}+7. \tag{9.119}$$

Step 14. Recall the equation for $\not{d}_2\not{d}_2^*\underline{\xi}$ derived in Proposition 5.22

$$2\not{d}_2\not{d}_2^*\underline{\xi} = -\nabla_3\nabla\underline{\check{\kappa}} - \frac{2}{r}\nabla_3\zeta - \frac{2}{r}\underline{\beta} + r^{-2}\not{\mathfrak{d}}^{\leq 1}\Gamma_g + r^{-1}\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

We infer

$$\not{d}_2\not{d}_2^*(\mathfrak{d}_*^{\leq k}\underline{\xi}) = \not{\mathfrak{d}}^{\leq 1}(h_k), \quad k \leq k_{large}+7,$$

where h_k satisfies in view of (9.111), the estimate of Step 7 for $\widecheck{\operatorname{tr}\underline{\chi}}$ and of Step 10 for ζ, and the commutator Lemma 5.20,

$$r^2\|h_k\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + r^{-1}\epsilon + \epsilon^2 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k \leq k_{large}+7.$$

Since $\underline{\xi}$ satisfies $\operatorname{curl}\underline{\xi} = \Gamma_b\cdot\Gamma_b$ and the GCM condition $(\operatorname{div}\underline{\xi})_{\ell=1} = 0$ on Σ_*, we proceed exactly as in Step 13 and obtain

$$\|\not{\mathfrak{d}}^{\leq 1}(\mathfrak{d}_*^{\leq k}\underline{\xi})\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k \leq k_{large}+7. \tag{9.120}$$

Step 15. We may now complete the estimates for $\widehat{\chi}$ and $\underline{\widehat{\chi}}$. Recall from Step 11 and Step 12 that we have

$$r\|\not{\mathfrak{d}}^{\leq 1}(\mathfrak{d}_*^{\leq k}\widehat{\chi})\|_{L^2(\Sigma_*)} + \|\not{\mathfrak{d}}^{\leq 1}(\mathfrak{d}_*^{\leq k}\underline{\widehat{\chi}})\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k \leq k_{large}+7,$$

so that we still need to recover $\nabla_\nu^{k+1}\widehat{\chi}$ and $\nabla_\nu^{k+1}\underline{\widehat{\chi}}$. In view of the fact that $\nu = e_3 + b_*e_4$, the null structure equations for $\nabla_3\widehat{\chi}$, $\nabla_3\underline{\widehat{\chi}}$, $\nabla_4\widehat{\chi}$, and $\nabla_4\underline{\widehat{\chi}}$, (9.111), the estimate of Step 10 for ζ, of Step 13 for η and of Step 14 for $\underline{\xi}$, we have

$$r\|\nabla_\nu^{k+1}\widehat{\chi}\|_{L^2(\Sigma_*)} + \|\nabla_\nu^{k+1}\underline{\widehat{\chi}}\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + \epsilon^2 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k \leq k_{large}+7.$$

Together with the above estimates, we deduce

$$r\|\mathfrak{d}_*^{\leq k+1}\widehat{\chi}\|_{L^2(\Sigma_*)} + \|\mathfrak{d}_*^{\leq k+1}\underline{\widehat{\chi}}\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k \leq k_{large}+7.$$

Step 16. We have

$$\nabla_a(e_3(r)) = [e_a, e_3]r = (\zeta_a - \eta_a)e_3(r) - \underline{\xi}_a e_4(r) = (\zeta_a - \eta_a)e_3(r) - \underline{\xi}_a.$$

Together with the estimate of Step 10 for ζ, of Step 13 for η and of Step 14 for $\underline{\xi}$, the commutator Lemma 5.20, and a Poincaré inequality, we derive

$$r^{-1}\|\not{\mathfrak{d}}^{\leq 2}\not{\mathfrak{d}}\mathfrak{d}_*^{k-1}e_3(r))\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k\leq k_{large}+7,$$

and

$$r^{-1}\|\not{\mathfrak{d}}^{\leq 2}(\nu^k(e_3(r)) - \overline{\nu^k(e_3(r))})\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k\leq k_{large}+7.$$

Step 17. We have

$$\nabla_a(e_3(u)) = [e_a, e_3]u = (\zeta_a - \eta_a)e_3(u) - \underline{\xi}_a e_4(u) = (\zeta_a - \eta_a)e_3(u).$$

Together with the estimate of Step 10 for ζ and of Step 13 for η, the commutator Lemma 5.20, and a Poincaré inequality, we derive

$$r^{-1}\|\not{\mathfrak{d}}^{\leq 2}\not{\mathfrak{d}}\mathfrak{d}_*^{k-1}e_3(u))\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k\leq k_{large}+7,$$

and

$$r^{-1}\|\not{\mathfrak{d}}^{\leq 2}(\nu^k(e_3(u)) - \overline{\nu^k(e_3(u))})\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k\leq k_{large}+7.$$

Step 18. Since $u + r = c_{\Sigma_*}$ on Σ_*, and since ν is tangent to Σ_*, we have $\nu(r+u) = 0$ and hence

$$0 = \nu(u+r) = e_3(u+r) + b_*e_4(u+r) = e_3(u) + e_3(r) + b_*,$$

so that $b_* = -e_3(u) - e_3(r)$. In view of the above estimates for $e_3(u) - \overline{e_3(u)}$ and $e_3(r) - \overline{e_3(r)}$, we deduce

$$r^{-1}\|\not{\mathfrak{d}}^{\leq 2}\not{\mathfrak{d}}\mathfrak{d}_*^{k-1}b_*)\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k\leq k_{large}+7,$$

and

$$r^{-1}\|\not{\mathfrak{d}}^{\leq 2}(\nu^k(b_*) - \overline{\nu^k(b_*)})\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k\leq k_{large}+7.$$

Step 19. Since $\overline{b_*} = -1 - \frac{2m}{r}$ on Σ_*, and since ν is tangent to Σ_*, we have for any k

$$\nu^k\left(\overline{b_* + 1 + \frac{2m}{r}}\right) = 0.$$

Using Corollary 5.32, we deduce

$$\overline{\nu^k\left(b_*+1+\frac{2m}{r}\right)} \quad = \quad \nu^{\leq k}(r^2\Gamma_b\cdot\Gamma_b)$$

and hence

$$\left(\int_{u=1}^{u_*}\left|\overline{\nu^k\left(b_*+1+\frac{2m}{r}\right)}\right|^2\right)^{1/2} \lesssim \epsilon^2 \lesssim \epsilon_0, \qquad 1\leq k\leq k_{large}+7.$$

Also, we have, for any $k\geq 1$,

$$\nu^k\left(b_*+1+\frac{2m}{r}\right) \quad = \quad \nu^k(b_*)+O(r^{-1-k}),$$

and hence, in view of the dominant condition for r on Σ_*, we infer

$$\left|\nu^k\left(b_*+1+\frac{2m}{r}\right)-\nu^k(b_*)\right| \quad \lesssim \quad \epsilon_0 u^{-1-\delta_{dec}}, \quad 1\leq k\leq k_{large}+7.$$

This yields

$$\left(\int_{u=1}^{u_*}|\overline{\nu^k(b_*)}|^2\right)^{1/2} \lesssim \epsilon_0, \qquad 1\leq k\leq k_{large}+7.$$

Plugging back in the bounds of Step 18, we deduce

$$r^{-1}\|\not{\mathfrak{d}}^{\leq 2}\nu^k(b_*))\|_{L^2(\Sigma_*)} \lesssim \epsilon_0+{}^{(\Sigma_*)}\mathfrak{R}_k, \qquad 1\leq k\leq k_{large}+7.$$

Together with the dominance in r, this yields

$$r^{-1}\|\not{\mathfrak{d}}^{\leq 2}\nu^k(\check{b}_*))\|_{L^2(\Sigma_*)} \lesssim \epsilon_0+{}^{(\Sigma_*)}\mathfrak{R}_k, \qquad 1\leq k\leq k_{large}+7.$$

Since the case $k=0$ follows from Theorem M3, we thus have

$$r^{-1}\|\not{\mathfrak{d}}^{\leq 2}\nu^k(\check{b}_*))\|_{L^2(\Sigma_*)} \lesssim \epsilon_0+{}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k\leq k_{large}+7.$$

Plugging back in the bounds of Step 18, we deduce

$$r^{-1}\|\not{\mathfrak{d}}^{\leq 2}\mathfrak{d}_*^k\check{b}_*\|_{L^2(\Sigma_*)} \quad \lesssim \quad \epsilon_0+{}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k\leq k_{large}+7. \tag{9.121}$$

Step 20. We make use of Lemma 5.31, the definition of $\nu = e_4 + b_* e_4$, and the GCM condition for $\operatorname{tr}\chi$ on Σ_*, to deduce

$$\begin{aligned} e_3(r) + b_* = \nu(r) &= \frac{re_3(u)}{2}\overline{\frac{1}{e_3(u)}(\operatorname{tr}\underline{\chi} + b_*\operatorname{tr}\chi)} \\ &= \frac{re_3(u)}{2}\overline{\frac{1}{e_3(u)}\left(-\frac{2\Upsilon}{r} + \widecheck{\operatorname{tr}\underline{\chi}} + \frac{2}{r}b_*\right)} \\ &= -2e_3(u)\overline{\frac{1}{e_3(u)}} + \frac{re_3(u)}{2}\overline{\frac{1}{e_3(u)}\left(\widecheck{\operatorname{tr}\underline{\chi}} + \frac{2}{r}\check{b}_*\right)} \end{aligned}$$

and hence

$$e_3(r) = -\Upsilon + 2\left(1 - e_3(u)\overline{\frac{1}{e_3(u)}}\right) - \check{b}_* + \frac{re_3(u)}{2}\overline{\frac{1}{e_3(u)}\left(\widecheck{\operatorname{tr}\underline{\chi}} + \frac{2}{r}\check{b}_*\right)}.$$

Together with the control of $\widecheck{\operatorname{tr}\underline{\chi}}$ in Step 7, the control of $\check{b}_*$ in Step 19, and the dominant condition for r on Σ_*, we infer, for $1 \le k \le k_{large} + 7$,

$$\left(\int_{u=1}^{u_*} \left|\overline{\nu^k(e_3(r))}\right|^2\right)^{\frac{1}{2}} \lesssim \left(\int_{u=1}^{u_*} \left|\overline{\nu^k\left(1 - e_3(u)\overline{\frac{1}{e_3(u)}}\right)}\right|^2\right)^{\frac{1}{2}} + \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k.$$

Since $e_3(u) = 2 + \widecheck{e_3(u)}$, using (9.111), we obtain, for $1 \le k \le k_{large} + 7$,

$$\left(\int_{u=1}^{u_*} \left|\overline{\nu^k(e_3(r))}\right|^2\right)^{\frac{1}{2}} \lesssim \left(\int_{u=1}^{u_*} \left|\overline{\nu^k\left(\widecheck{e_3(u)} - \overline{\widecheck{e_3(u)}}\right)}\right|^2\right)^{\frac{1}{2}} + \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k.$$

Together with Corollary 5.32, we infer, for $1 \le k \le k_{large} + 7$,

$$\begin{aligned} \left(\int_{u=1}^{u_*} \left|\overline{\nu^k(e_3(r))}\right|^2\right)^{\frac{1}{2}} \lesssim{}& \left(\int_{u=1}^{u_*} \left\|\nu^k\left(\widecheck{e_3(u)}\right) - \overline{\nu^k\left(\widecheck{e_3(u)}\right)}\right\|^2_{L^\infty(S)}\right)^{\frac{1}{2}} + \epsilon_0 \\ &+ {}^{(\Sigma_*)}\mathfrak{R}_k, \end{aligned}$$

and since $k \ge 1$ and $e_3(u) = 2 + \widecheck{e_3(u)}$, this yields, for $1 \le k \le k_{large} + 7$,

$$\begin{aligned} \left(\int_{u=1}^{u_*} \left|\overline{\nu^k(e_3(r))}\right|^2\right)^{\frac{1}{2}} \lesssim{}& \left(\int_{u=1}^{u_*} \left\|\nu^k\left(e_3(u)\right) - \overline{\nu^k\left(e_3(u)\right)}\right\|^2_{L^\infty(S)}\right)^{\frac{1}{2}} + \epsilon_0 \\ &+ {}^{(\Sigma_*)}\mathfrak{R}_k. \end{aligned}$$

Together with the control of $\nu^k e_3(u) - \overline{\nu^k e_3(u)}$ in Step 17, we infer, for $1 \leq k \leq k_{large}+7$,

$$\left(\int_{u=1}^{u_*} \left|\overline{\nu^k(e_3(r))}\right|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k.$$

In view of the control of $e_3(r)$ in Step 16, this yields

$$r^{-1}\|\,\mathfrak{d}\!\!/^{\leq 2}(\nu^k(e_3(r)))\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad 1 \leq k \leq k_{large}+7.$$

Since the case $k=0$ is treated by Theorem M3, since $e_3(r) = -\Upsilon + \widecheck{e_3(r)}$, and using the dominant condition on r, we obtain

$$r^{-1}\|\,\mathfrak{d}\!\!/^{\leq 2}(\nu^k\widecheck{e_3(r)})\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + \frac{1}{r} + {}^{(\Sigma_*)}\mathfrak{R}_k \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k \leq k_{large}+7.$$

Using again the control of $e_3(r)$ in Step 16, we deduce

$$r^{-1}\|\,\mathfrak{d}\!\!/^{\leq 2}\mathfrak{d}_*^k\widecheck{e_3(r)})\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k \leq k_{large}+7.$$

Finally, since $\nu(r) = e_3(r) + b_*$, and since $\nu(u+r) = 0$ along Σ_*, we infer, together with the control of b_* in Step 19,

$$r^{-1}\left\|\,\mathfrak{d}\!\!/^{\leq 2}\mathfrak{d}_*^k(\widecheck{\nu(r)},\, \widecheck{\nu(u)})\right\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k \leq k_{large}+7.$$

Step 21. It remains to control $\underline{\widecheck{\omega}}$. Recall from Proposition 5.18 the following linearized null structure equation

$$\nabla_3\zeta - \frac{\Upsilon}{r}\zeta = -\underline{\beta} - 2\nabla\underline{\widecheck{\omega}} + \tfrac{\Upsilon}{r}(\eta+\zeta) + \tfrac{1}{r}\underline{\xi} + \tfrac{2m}{r^2}(\zeta-\eta) + \Gamma_b\cdot\Gamma_b.$$

The control of ζ in Step 10, of η in Step 13, of $\underline{\xi}$ in Step 14, and (9.111) implies

$$\left\|\,\mathfrak{d}\!\!/\mathfrak{d}_*^k\underline{\widecheck{\omega}}\right\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k \leq k_{large}+7.$$

Also, using the null structure equations for $\nabla_3\widecheck{\mathrm{tr}\chi}$ and $\nabla_4\widecheck{\mathrm{tr}\chi}$, and the fact that $\nu = e_3 + b_* e_4$, we have

$$\nabla_\nu\widecheck{\mathrm{tr}\chi} = 2\mathrm{div}\,\eta + 2\widecheck{\rho} - \frac{1}{r}\underline{\widecheck{\kappa}} + \frac{4}{r}\underline{\widecheck{\omega}} + \frac{2}{r^2}\widecheck{e_3(r)} + \Gamma_b\cdot\Gamma_g.$$

Together with the control of $\widecheck{\mathrm{tr}\chi}$ in Step 1, of $\widecheck{\mathrm{tr}\underline{\chi}}$ in Step 7, of η in Step 13, of $\widecheck{e_3(r)}$ in Step 20, and (9.111), we infer

$$\left\|\nu^k\underline{\check{\omega}}\right\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k\leq k_{large}+7.$$

In view of the above estimate for $\not{\mathfrak{d}}\mathfrak{d}_*^k\underline{\check{\omega}}$, this implies

$$\left\|\not{\mathfrak{d}}^{\leq 1}\mathfrak{d}_*^k\underline{\check{\omega}}\right\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k\leq k_{large}+7. \tag{9.122}$$

In view of Steps 1–21, and in view of the definition of the norm ${}^{(\Sigma_*)}\mathfrak{G}_k$, we have obtained

$${}^{(\Sigma_*)}\mathfrak{G}_k \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad k\leq k_{large}+7,$$

which is an improvement of the local bootstrap assumptions (9.111). This concludes the proof of Proposition 9.71. □

9.7.2. Control of $J^{(0)}$, f_0 and $\mathfrak{J}$ on Σ_* In view of Section 3.2.3, θ initialized on S_* as in Section 2.5.3, and propagated to Σ_* by $\nu(\theta)=0$. Recall also that

$$J^{(0)} = \cos\theta$$

so that $\nu(J^{(0)})=0$ along Σ_*.

Moreover, the tangential 1-form f_0 on Σ_* given by, see Definition 5.56,

$$(f_0)_1=0, \quad (f_0)_2=\sin\theta, \quad \text{on} \quad S_*, \qquad \nabla_\nu f_0=0,$$

where, on S_*, we consider the orthonormal basis (e_1,e_2) of S_* given by (5.129).

Finally, note that the complex horizontal 1-form $\mathfrak{J}$ introduced in Definition 3.8 verifies

$$\mathfrak{J} = \frac{1}{|q|}\left(f_0 + i\,{}^*f_0\right) \quad \text{on} \quad \Sigma_*.$$

Recall from Section 9.1.3 that we initialize the PT frame of ${}^{(ext)}\mathcal{M}$ from the integrable frame on Σ_*, by relying on the change of frame formula with the transition coefficients

$$\lambda=1, \qquad f=\frac{a}{r}f_0, \qquad \underline{f}=\frac{a\Upsilon}{r}f_0.$$

Thus, in order to control the PT frame of ${}^{(ext)}\mathcal{M}$ on Σ_* in the next section, we first need to control the 1-form f_0. We also need to control $J^{(0)} = \cos\theta$ and $\mathfrak{J}$ as these quantities are involved in the definition of the linearized quantities corresponding to the PT frame of ${}^{(ext)}\mathcal{M}$. The following lemma provides the control of $J^{(0)}$, f_0 and $\mathfrak{J}$ on Σ_*.

Lemma 9.72. *The following estimates holds true on Σ_*, for all $k \leq k_{large}+7$,*

$$\begin{aligned}
&\|\mathfrak{d}_*^{\leq k}\widetilde{\nabla J^{(0)}}\|_{L^2(\Sigma_*)} + \|\mathfrak{d}_*^{\leq k} div(f_0)\|_{L^2(\Sigma_*)}\\
&+\|\mathfrak{d}_*^{\leq k}\widetilde{curl(f_0)}\|_{L^2(\Sigma_*)} + \|\mathfrak{d}_*^{\leq k}\nabla\widehat{\otimes} f_0\|_{L^2(\Sigma_*)}\\
&+\|r\mathfrak{d}_*^{\leq k}\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}\|_{L^2(\Sigma_*)} + \|r\mathfrak{d}_*^{\leq k}\mathcal{D}\widehat{\otimes}\mathfrak{J}\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k,
\end{aligned} \tag{9.123}$$

where, see Definitions 5.57 and 5.58,

$$\begin{aligned}
&\widetilde{\nabla J^{(0)}} := \nabla J^{(0)} + \frac{1}{r}\,{}^*f_0, \qquad \widetilde{curl(f_0)} := curl(f_0) - \frac{2}{r}\cos\theta,\\
&\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}} := \overline{\mathcal{D}}\cdot\mathfrak{J} - \frac{4i(r^2+a^2)\cos\theta}{|q|^4}.
\end{aligned} \tag{9.124}$$

Proof. We first derive estimates on S_*. We make use of Lemmas 5.61 and 5.62 according to which we have on S_*

$$\begin{aligned}
\widetilde{\nabla J^{(0)}} &= -\frac{1}{r}(e^{-\phi}-1)\,{}^*f_0, \qquad \text{div}\,(f_0) = f_0\cdot\nabla\phi,\\
\text{curl}\,(f_0) &= \frac{2}{re^{\phi}}\cos\theta - f_0\wedge\nabla\phi, \qquad \nabla\widehat{\otimes}f_0 = \begin{pmatrix} f_0\cdot\nabla\phi & f_0\wedge\nabla\phi\\ f_0\wedge\nabla\phi & -f_0\cdot\nabla\phi\end{pmatrix}.
\end{aligned}$$

We also use the following estimate on S_*

$$\|\phi\|_{\mathfrak{h}_{k+1}(S_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad \text{for all } k\leq k_{large}+7,$$

derived in Step 2 of the proof of Proposition 9.71. The above control of ϕ on S_* and the above identities on S_* implies immediately

$$\begin{aligned}
& r\left\|\left(\widetilde{\nabla J^{(0)}},\, \text{div}\,(f_0),\, \widetilde{\text{curl}\,(f_0)},\, \nabla\widehat{\otimes}f_0\right)\right\|_{\mathfrak{h}_k(S_*)}\\
\lesssim\;& \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad \text{for all } k\leq k_{large}+7,
\end{aligned}$$

which is equivalent to

$$\left\|r\widetilde{\nabla J^{(0)}}\right\|_{\mathfrak{h}_k(S_*)} + \|r\nabla f_0 - \cos\theta\in\|_{\mathfrak{h}_k(S_*)}$$

$$\lesssim \quad \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \qquad \text{for all } k \le k_{large} + 7.$$

Next, using $\nu(J^{(0)}) = 0$ and $\nabla_\nu f_0 = 0$, we have along Σ_*, see Lemma 5.66,

$$\nabla_\nu\Big[r\widetilde{\nabla J^{(0)}}\Big] = \Gamma_b \cdot \not{\mathfrak{d}}^{\le 1} J^{(0)}, \qquad \nabla_\nu\Big[r\nabla f_0 - \cos\theta \in\Big] \quad = \quad \Gamma_b \not{\mathfrak{d}}^{\le 1} f_0.$$

Commuting with $\not{\mathfrak{d}}$ and integrating from S_*, where $r\widetilde{\nabla J^{(0)}}$ and $r\nabla f_0 - \cos\theta \in$ are under control in view of the above, and making use of Proposition 9.71 to control Γ_b, we deduce on Σ_*

$$\begin{aligned}
&\sup_{S\subset\Sigma_*}\left(\Big\|\not{\mathfrak{d}}^{\le k}(r\widetilde{\nabla J^{(0)}})\Big\|_{L^2(S)} + \|\not{\mathfrak{d}}^{\le k}(r\nabla f_0 - \cos\theta \in)\|_{L^2(S)}\right)\\
\lesssim\quad & \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k + \int_1^{u_*}\|\not{\mathfrak{d}}^{\le k}(\Gamma_b \not{\mathfrak{d}}^{\le 1} f_0)\|_{L^2(S)}\\
\lesssim\quad & \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k + \sqrt{u_*}\|\not{\mathfrak{d}}^{\le k}(\Gamma_b \not{\mathfrak{d}}^{\le 1} f_0)\|_{L^2(\Sigma_*)}\\
\lesssim\quad & \sqrt{u_*}\Big(\epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k\Big).
\end{aligned}$$

Thus, using the dominant condition for r on Σ_*, we infer, for $k \le k_{large} + 7$,

$$\begin{aligned}
&\sup_{S\subset\Sigma_*}\left(\Big\|\not{\mathfrak{d}}^{\le k}\widetilde{\nabla J^{(0)}}\Big\|_{L^2(S)} + \Big\|\not{\mathfrak{d}}^{\le k}\left(\nabla f_0 - \frac{\cos\theta}{r}\in\right)\Big\|_{L^2(S)}\right)\\
\lesssim\quad & u_*^{-\frac{1}{2}-\delta_{dec}}\big(\epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k\big).
\end{aligned}$$

As a consequence, since $\delta_{dec} > 0$, we have

$$\begin{aligned}
&\Big\|\not{\mathfrak{d}}^{\le k}\widetilde{\nabla J^{(0)}}\Big\|_{L^2(\Sigma_*)} + \Big\|\not{\mathfrak{d}}^{\le k}\left(\nabla f_0 - \frac{\cos\theta}{r}\in\right)\Big\|_{L^2(\Sigma_*)}\\
\lesssim\quad & \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k \le k_{large} + 7.
\end{aligned}$$

Using again $\nabla_\nu[r\widetilde{\nabla J^{(0)}}] = \Gamma_b \cdot \not{\mathfrak{d}}^{\le 1} J^{(0)}$ and $\nabla_\nu[r\nabla f_0 - \cos\theta \in] = \Gamma_b \not{\mathfrak{d}}^{\le 1} f_0$, together with the control of Γ_b provided by Proposition 9.71, we may recover the ∇_ν derivatives and obtain

$$\begin{aligned}
&\Big\|\mathfrak{d}_*^{\le k}\widetilde{\nabla J^{(0)}}\Big\|_{L^2(\Sigma_*)} + \Big\|\mathfrak{d}_*^{\le k}\left(\nabla f_0 - \frac{\cos\theta}{r}\in\right)\Big\|_{L^2(\Sigma_*)}\\
\lesssim\quad & \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k, \quad k \le k_{large} + 7,
\end{aligned}$$

or equivalently, for $k \le k_{large} + 7$,

$$\begin{aligned}
&\|\mathfrak{d}_*^{\leq k}\widecheck{\nabla J^{(0)}}\|_{L^2(\Sigma_*)}+\|\mathfrak{d}_*^{\leq k}\text{div}\,(f_0)\|_{L^2(\Sigma_*)}\\
&+\|\mathfrak{d}_*^{\leq k}\widecheck{\text{curl}\,(f_0)}\|_{L^2(\Sigma_*)}+\|\mathfrak{d}_*^{\leq k}\nabla\widehat{\otimes}f_0\|_{L^2(\Sigma_*)} \lesssim \epsilon_0+{}^{(\Sigma_*)}\mathfrak{R}_k
\end{aligned}$$

as stated.

Finally, we control $\mathfrak{J}$. Recall that we have on Σ_*

$$\mathfrak{J}=\frac{1}{|q|}\left(f_0+i\,{}^*f_0\right)=\frac{1}{r}\left(f_0+i\,{}^*f_0\right)+O(r^{-3})\left(f_0+i\,{}^*f_0\right).$$

We infer on Σ_*

$$\begin{aligned}
\overline{\mathcal{D}}\cdot\mathfrak{J} &= \frac{1}{r}(\nabla-i\,{}^*\nabla)\cdot(f_0+i\,{}^*f_0)+O(r^{-4})=\frac{2}{r}\left(\text{div}\,(f_0)+i\text{curl}\,(f_0)\right)\\
&= \frac{4i\cos\theta}{r^2}+\frac{2}{r}\Big(\text{div}\,(f_0)+i\widecheck{\text{curl}\,(f_0)}\Big)+O(r^{-4})\\
&= \frac{4i(r^2+a^2)\cos\theta}{|q|^4}+\frac{2}{r}\Big(\text{div}\,(f_0)+i\widecheck{\text{curl}\,(f_0)}\Big)+O(r^{-4}),
\end{aligned}$$

and hence

$$\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}} = \frac{2}{r}\Big(\text{div}\,(f_0)+i\widecheck{\text{curl}\,(f_0)}\Big)+O(r^{-4}),$$

as well as

$$\begin{aligned}
\mathcal{D}\widehat{\otimes}\mathfrak{J} &= \frac{1}{r}(\nabla+i\,{}^*\nabla)\widehat{\otimes}\left(f_0+i\,{}^*f_0\right)+O(r^{-4})\\
&= \frac{2}{r}\left(\nabla\widehat{\otimes}f_0+i\,{}^*(\nabla\widehat{\otimes}f_0)\right)+O(r^{-4}).
\end{aligned}$$

Together with the above control of $\text{div}\,(f_0)$, $\widecheck{\text{curl}\,(f_0)}$ and $\nabla\widehat{\otimes}f_0$, and the dominant condition for r on Σ_*, we immediately infer, for all $k\leq k_{large}+7$,

$$\begin{aligned}
\|r\mathfrak{d}_*^{\leq k}\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}\|_{L^2(\Sigma_*)}+\|r\mathfrak{d}_*^{\leq k}\mathcal{D}\widehat{\otimes}\mathfrak{J}\|_{L^2(\Sigma_*)} &\lesssim \epsilon_0+{}^{(\Sigma_*)}\mathfrak{R}_k+r^{-2}\sqrt{u_*}\\
&\lesssim \epsilon_0+{}^{(\Sigma_*)}\mathfrak{R}_k.
\end{aligned}$$

This concludes the proof of Lemma 9.72. □

For convenience, we introduce the following notation.

Definition 9.73. *We denote by* ${}^{(\Sigma_*)}\Gamma_b$ *and* ${}^{(\Sigma_*)}\Gamma_g$ *the set of linearized quantities below*

$$\begin{aligned}
{}^{(\Sigma_*)}\Gamma_b &:= {}^{(\Sigma_*)}\Gamma_{b,1} \cup {}^{(\Sigma_*)}\Gamma_{b,2},\\
{}^{(\Sigma_*)}\Gamma_{b,1} &:= \Big\{\eta,\ \widehat{\underline{\chi}},\ \underline{\check{\omega}},\ \underline{\xi}, r^{-1}\widecheck{\nu(r)},\ r^{-1}\widecheck{\nu(u)},\ r^{-1}\check{b}_*\Big\},\\
{}^{(\Sigma_*)}\Gamma_{b,2} &:= \Big\{\widecheck{\nabla J^{(0)}},\ div(f_0),\ \widecheck{curl(f_0)},\ \nabla\widehat{\otimes} f_0,\ r\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}},\ \mathcal{D}\widehat{\otimes}\mathfrak{J}\Big\},\\
{}^{(\Sigma_*)}\Gamma_g &:= \Big\{\widehat{\chi},\ \widecheck{tr\chi},\ \zeta,\ \widecheck{tr\underline{\chi}}\Big\},
\end{aligned}$$

where $\widecheck{tr\chi}$, $\widehat{\chi}$, ζ, η, $\widecheck{tr\underline{\chi}}$, $\widehat{\underline{\chi}}$, $\underline{\check{\omega}}$, $\underline{\xi}$ *are the Ricci coefficients of the integrable frame of* Σ_*.

Corollary 9.74. *We have on* Σ_**, for* $k \leq k_{large} + 7$*,*

$$\|\mathfrak{d}_*^{\leq k}({}^{(\Sigma_*)}\Gamma_b)\|_{L^2(\Sigma_*)} + \|r\mathfrak{d}_*^{\leq k}({}^{(\Sigma_*)}\Gamma_g)\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^{(\Sigma_*)}\mathfrak{R}_k.$$

Proof. This is an immediate consequence of Proposition 9.71 and Lemma 9.72. □

9.7.3. Proof of Proposition 9.67 Throughout this section, to avoid confusion between the integrable frame of Σ_* and the outgoing PT frame of ${}^{(ext)}\mathcal{M}$:

- the frame and quantities associated to the integrable frame of Σ_* are denoted without prime,
- the frame and quantities associated to the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ are denoted with prime.

Recall that we have in the outgoing PT frame on ${}^{(ext)}\mathcal{M}$

$$\xi' = 0, \qquad \omega' = 0, \qquad \underline{H}' = -\frac{a\overline{q'}}{|q'|^2}\mathfrak{J}', \qquad e_4'(r) = 1, \qquad e_4'(u) = e_4'(\theta) = 0,$$

while the integrable frame of Σ_* satisfies the transversality conditions (9.109), i.e.

$$\xi = 0, \qquad \omega = 0, \qquad \underline{\eta} = -\zeta, \qquad e_4(r) = 1, \qquad e_4(u) = 0.$$

Recall also that $\mathfrak{J}'$ satisfies on ${}^{(ext)}\mathcal{M}$

$$\nabla_4'\mathfrak{J}' = -\frac{1}{q'}\mathfrak{J}'.$$

We denote:

- by $(f, \underline{f}, \lambda)$ the transition coefficients from the integrable frame E of Σ_* to the outgoing PT frame E' of ${}^{(ext)}\mathcal{M}$,
- by $(f', \underline{f}', \lambda')$ the transition coefficients of the reverse transformation, i.e. the transition coefficients from the frame E' to the frame E,

and recall from Section 9.1.3 that the initialization of the PT frame of ${}^{(ext)}\mathcal{M}$ given by the following choice for $(f, \underline{f}, \lambda)$ on Σ_*

$$\lambda = 1, \qquad f = \frac{a}{r} f_0, \qquad \underline{f} = \frac{a\Upsilon}{r} f_0,$$

and that r', u', θ' and $\mathfrak{J}'$ are initialized on Σ_* by

$$r' = r, \qquad u' = u, \qquad \theta' = \theta, \qquad \mathfrak{J}' = \mathfrak{J}.$$

Remark 9.75. *We note that, in view of the transformation formulas of Proposition 2.12 in the particular case of the transformation from the integrable frame of Σ_* to the outgoing PT frame of ${}^{(ext)}\mathcal{M}$, and in view of the form of the transition coefficients $(f, \underline{f}, \lambda)$ on Σ_* recalled above, the curvature norm ${}^{(\Sigma_*)}\mathfrak{R}_k$ of Definition 9.70 and the curvature norm ${}^\star\mathfrak{R}_k$ of Definition 9.36 are equivalent.*

Our bootstrap assumption **BA-PT**, see (9.40), implies in particular for the Ricci and metric coefficients of the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ on Σ_*

$$ {}^\star\mathfrak{G}_k \leq \epsilon, \qquad k \leq k_{large} + 7. \tag{9.125}$$

To control the PT frame on Σ_*, we will rely on the reverse transformation, i.e. from the PT frame of ${}^{(ext)}\mathcal{M}$ to the integrable frame of Σ_*. This is done in the following lemma.

Lemma 9.76. *Consider the change of frame coefficients $(f', \underline{f}', \lambda')$ from the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ to the integrable frame of Σ_*. Then, we have*[216] *on Σ_**

$$\lambda' = 1 + O(r^{-2}), \quad f' = -\frac{a}{r}\Big(1 + O(r^{-2})\Big) f_0, \quad \underline{f}' = -\frac{a\Upsilon}{r}\Big(1 + O(r^{-2})\Big) f_0.$$

Proof. Recall that the change of frame coefficients from the integrable frame of Σ_* to the PT frame of ${}^{(ext)}\mathcal{M}$ are given by

$$\lambda = 1, \qquad f = \frac{a}{r} f_0, \qquad \underline{f} = \frac{a\Upsilon}{r} f_0.$$

[216] See Definition 9.18 for our notation $O(r^{-p})$.

In view of equation (2.8) of Lemma 2.10, we have

$$\lambda' = \lambda^{-1}\left(1+\frac{1}{2}f\cdot\underline{f}+\frac{1}{16}|f|^2|\underline{f}|^2\right),$$
$$f'_a = -\frac{\lambda}{1+\frac{1}{2}f\cdot\underline{f}+\frac{1}{16}|f|^2|\underline{f}|^2}\left(f_a+\frac{1}{4}|f|^2\underline{f}_a\right),$$
$$\underline{f}'_a = -\lambda^{-1}\left(\underline{f}_a+\frac{1}{4}|\underline{f}|^2 f_a\right),$$

from which the lemma easily follows. □

Lemma 9.77. *Let*

$$\widetilde{curl\,(f')} := curl\,(f') + \frac{2a}{r^2}\cos\theta, \qquad \widetilde{curl\,(\underline{f}')} := curl\,(\underline{f}') + \frac{2a\Upsilon}{r^2}\cos\theta,$$
$$\widetilde{\nabla_\nu f'} := \nabla_\nu f' + \frac{2a}{r^2}f_0, \qquad \widetilde{\nabla_\nu \underline{f}'} := \nabla_\nu \underline{f}' + \frac{2a}{r^2}f_0.$$

Then, we have, for all $k\leq k_{large}+7$,

$$\begin{aligned}
&\|r\mathfrak{d}_*^{\leq k+1}\log(\lambda')\|_{L^2(\Sigma_*)} + \|r\mathfrak{d}_*^{\leq k}div\,(f')\|_{L^2(\Sigma_*)} + \|r\mathfrak{d}_*^{\leq k}\widetilde{curl\,(f')}\|_{L^2(\Sigma_*)}\\
&+\|r\mathfrak{d}_*^{\leq k}\nabla\widehat{\otimes}f'\|_{L^2(\Sigma_*)} + \|r\mathfrak{d}_*^{\leq k}\widetilde{\nabla_\nu f'}\|_{L^2(\Sigma_*)} + \|r\mathfrak{d}_*^{\leq k}div\,(\underline{f}')\|_{L^2(\Sigma_*)}\\
&+\|r\mathfrak{d}_*^{\leq k}\widetilde{curl\,(\underline{f}')}\|_{L^2(S_*)} + \|r\mathfrak{d}_*^{\leq k}\nabla\widehat{\otimes}\underline{f}'\|_{L^2(\Sigma_*)} + \|r\mathfrak{d}_*^{\leq k}\widetilde{\nabla_\nu \underline{f}'}\|_{L^2(\Sigma_*)}\\
\lesssim\ & \epsilon_0 + {}^\star\mathfrak{R}_k.
\end{aligned}$$

Proof. Recall that we have

$$\lambda' = 1+O(r^{-2}), \quad f' = -\frac{a}{r}\Big(1+O(r^{-2})\Big)f_0, \quad \underline{f}' = -\frac{a\Upsilon}{r}\Big(1+O(r^{-2})\Big)f_0.$$

We infer, using in particular $\nabla(r)=0$,

$$\begin{aligned}
&\nabla\lambda' = O(r^{-3}), \quad \mathrm{div}\,(f') = -\frac{a}{r}\mathrm{div}\,(f_0)+O(r^{-4}),\\
&\mathrm{curl}\,(f') = -\frac{a}{r}\mathrm{curl}\,(f_0)+O(r^{-4}), \quad \nabla\widehat{\otimes}f' = -\frac{a}{r}\nabla\widehat{\otimes}f_0+O(r^{-4}),\\
&\mathrm{div}\,(\underline{f}') = -\frac{a\Upsilon}{r}\mathrm{div}\,(f_0)+O(r^{-4}), \quad \mathrm{curl}\,(\underline{f}') = -\frac{a\Upsilon}{r}\mathrm{curl}\,(f_0)+O(r^{-4}),\\
&\nabla\widehat{\otimes}\underline{f}' = -\frac{a\Upsilon}{r}\nabla\widehat{\otimes}f_0+O(r^{-4}).
\end{aligned}$$

Together with the control of f_0 in Lemma 9.72, the dominant condition for r on Σ_*, the definition of $\widetilde{\mathrm{curl}\,(f')}$ and $\widetilde{\mathrm{curl}\,(\underline{f}')}$, and Remark 9.75 on the equivalence between the norms ${}^{(\Sigma_*)}\mathfrak{R}_k$ and ${}^\star\mathfrak{R}_k$, we deduce, for all $k \leq k_{large}+7$,

$$\begin{aligned}
& \|r\mathfrak{d}_*^{\leq k}\nabla(\lambda')\|_{L^2(\Sigma_*)} + \|r\mathfrak{d}_*^{\leq k}\mathrm{div}\,(f')\|_{L^2(\Sigma_*)} + \|r\mathfrak{d}_*^{\leq k}\widetilde{\mathrm{curl}\,(f')}\|_{L^2(\Sigma_*)} \\
& +\|r\mathfrak{d}_*^{\leq k}\nabla\widehat{\otimes} f'\|_{L^2(\Sigma_*)} + \|r\mathfrak{d}_*^{\leq k}\mathrm{div}\,(\underline{f}')\|_{L^2(\Sigma_*)} + \|r\mathfrak{d}_*^{\leq k}\widetilde{\mathrm{curl}\,(\underline{f}')}\|_{L^2(S_*)} \\
& +\|r\mathfrak{d}_*^{\leq k}\nabla\widehat{\otimes}\underline{f}'\|_{L^2(\Sigma_*)} \\
\lesssim\ & \epsilon_0 + {}^\star\mathfrak{R}_k.
\end{aligned}$$

Also, using again

$$\lambda' = 1+O(r^{-2}), \quad f' = -\frac{a}{r}\Big(1+O(r^{-2})\Big)f_0, \quad \underline{f}' = -\frac{a\Upsilon}{r}\Big(1+O(r^{-2})\Big)f_0,$$

and since $\nabla_\nu f_0 = 0$ and $\nu(\theta)=0$ on Σ_*, and $\nu(r) = -2+\widetilde{\nu(r)}$, we obtain

$$\begin{aligned}
& \nu(\lambda') = O(r^{-3})\left(-2+\widetilde{\nu(r)}\right), \quad \nabla_\nu f' = \frac{a}{r^2}\Big(1+O(r^{-2})\Big)\left(-2+\widetilde{\nu(r)}\right)f_0, \\
& \nabla_\nu \underline{f}' = \frac{a}{r^2}\Big(1+O(r^{-1})\Big)\left(-2+\widetilde{\nu(r)}\right)f_0,
\end{aligned}$$

and hence

$$\begin{aligned}
\nu(\lambda') &= O(r^{-3})\left(-2+\widetilde{\nu(r)}\right), \\
\widetilde{\nabla_\nu f'} &= \frac{a}{r^2}\widetilde{\nu(r)}f_0 + O(r^{-4})\left(-2+\widetilde{\nu(r)}\right)f_0, \\
\widetilde{\nabla_\nu \underline{f}'} &= \frac{a}{r^2}\widetilde{\nu(r)}f_0 + O(r^{-3})\left(-2+\widetilde{\nu(r)}\right)f_0.
\end{aligned}$$

Together with the control of $\widetilde{\nu(r)}$ derived in Proposition 9.71, the control of f_0 in Lemma 9.72, the dominant condition for r on Σ_*, and Remark 9.75 on the equivalence between the norms ${}^{(\Sigma_*)}\mathfrak{R}_k$ and ${}^\star\mathfrak{R}_k$, we infer, for all $k \leq k_{large}+7$,

$$\|r\mathfrak{d}_*^{\leq k}\nu(\lambda')\|_{L^2(\Sigma_*)} + \|r\mathfrak{d}_*^{\leq k}\widetilde{\nabla_\nu f'}\|_{L^2(\Sigma_*)} + \|r\mathfrak{d}_*^{\leq k}\widetilde{\nabla_\nu \underline{f}'}\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^\star\mathfrak{R}_k.$$

This concludes the proof of Lemma 9.77. □

9.7.3.1. Proof of Proposition 9.67 We are now ready to prove Proposition 9.67.

Step 1. We start with the following lemma.

Lemma 9.78. *For all $k \leq k_{large}+7$, we have*

$$\begin{aligned}
& \|\mathfrak{d}_*^{\leq k}\mathcal{D}'(r')\|_{L^2(\Sigma_*)} + \|\mathfrak{d}_*^{\leq k}\widetilde{\mathcal{D}'(u')}\|_{L^2(\Sigma_*)} + \|\mathfrak{d}_*^{\leq k}\mathcal{D}'(\widetilde{\cos(\theta')})\|_{L^2(\Sigma_*)}\\
& +\|r^{-1}\mathfrak{d}_*^{\leq k}\widetilde{e_3'(r')}\|_{L^2(\Sigma_*)} + \|r^{-1}\mathfrak{d}_*^{\leq k}\widetilde{e_3'(u')}\|_{L^2(\Sigma_*)} + \|\mathfrak{d}_*^{\leq k}e_3'(\cos(\theta'))\|_{L^2(\Sigma_*)}\\
\lesssim\ & \epsilon_0 + {}^\star\mathfrak{R}_k.
\end{aligned}$$

Proof. Since $r' = r$ and $u' = u$ on Σ_* by initialization, and since ∇ is tangent to Σ_*, we infer $\nabla(r') = \nabla(u') = 0$ on Σ_*. Using the change of frame transformation, and using the fact that $\lambda = 1$, $\nabla(r') = 0$ and $\nabla(u') = 0$ on Σ_*, we have on Σ_*,

$$\begin{aligned}
e_4'(r') &= e_4(r') + \frac{1}{4}|f|^2 e_3(r'),\\
e_4'(u') &= e_4(u') + \frac{1}{4}|f|^2 e_3(u'),\\
e_4'(\cos(\theta')) &= e_4(\cos(\theta')) + f\cdot\nabla\cos(\theta) + \frac{1}{4}|f|^2 e_3(\cos(\theta')).
\end{aligned}$$

Also, since we have $e_4'(r') = 1$, $e_4'(\theta') = 0$, $e_4'(u') = 0$ in ${}^{(ext)}\mathcal{M}$, since $\nu = e_3 + b_* e_4$, and since $\nu(\theta') = 0$ (since $\theta' = \theta$ on Σ_*, $\nu(\theta) = 0$ and ν is tangent to Σ_*), we infer

$$\begin{aligned}
e_4(r') &= \frac{1 - \frac{1}{4}|f|^2\nu(r)}{1 - \frac{1}{4}|f|^2 b_*},\\
e_4(u') &= -\frac{\frac{1}{4}|f|^2\nu(u)}{1 - \frac{1}{4}|f|^2 b_*},\\
e_4(\cos(\theta')) &= -\frac{f\cdot\nabla\cos\theta}{1 - \frac{1}{4}|f|^2 b_*}.
\end{aligned}$$

In view of the choice of f, we infer

$$\begin{aligned}
e_4(r') &= 1 + O(r^{-2}) + O(r^{-2})(\check{b}_*, \widetilde{\nu(r)}),\\
e_4(u') &= O(r^{-2}) + O(r^{-2})\widetilde{\nu(u)} + O(r^{-4})\check{b}_*,\\
e_4(\cos(\theta')) &= O(r^{-2}) + O(r^{-1})\widetilde{\mathcal{D}\cos\theta} + O(r^{-4})\check{b}_*,
\end{aligned}$$

and hence, denoting for convenience ${}^{(\Sigma_*)}\Gamma_b$ and ${}^{(\Sigma_*)}\Gamma_g$ of Definition 9.73 simply by Γ_b and Γ_g, we obtain

$$\begin{aligned}
& e_4(r') = 1 + O(r^{-2}) + r^{-1}\Gamma_b, \qquad e_4(u') = O(r^{-2}) + r^{-1}\Gamma_b,\\
& e_4(\cos(\theta')) = O(r^{-2}) + r^{-1}\Gamma_b.
\end{aligned}$$

Using again $\nu = e_3 + b_* e_4$, and the fact that ν is tangent to Σ_* so that we have $\nu(r'-r) = 0$, $\nu(u'-u) = 0$ and $\nu(\theta') = 0$, we infer that

$$e_3(r') = -\Upsilon + O(r^{-2}) + r\Gamma_b, \qquad e_3(u') = 2 + O(r^{-2}) + r\Gamma_b,$$
$$e_3(\cos(\theta')) = O(r^{-2}) + r^{-1}\Gamma_b.$$

Using the above identities on Σ_* for $e_4(r')$, $e_4(u')$, $e_4(\cos(\theta'))$ and $e_3(r')$, $e_3(u')$, $e_3(\cos(\theta'))$, together with the change of frame transformation, the choice of λ, f and $\underline{f}$, and the fact that $\nabla(r') = \nabla(u') = 0$ on Σ_*, we obtain

$$\begin{aligned}
\nabla'(r') &= \frac{1}{2}\left(\frac{a}{r}e_3(r') + \frac{a\Upsilon}{r}e_4(r')\right) f_0 + O(r^{-3}) = \Gamma_b + O(r^{-3}),\\
\nabla'(u') &= \frac{1}{2}\left(\frac{a}{r}e_3(u') + \frac{a\Upsilon}{r}e_4(u')\right) f_0 + O(r^{-3}) = \frac{a}{r}f_0 + \Gamma_b + O(r^{-3}),\\
\nabla'(\cos(\theta')) &= \nabla(\cos\theta) + r^{-2}\Gamma_b + O(r^{-3}) = -\frac{1}{r}\,{}^*f_0 + \Gamma_b + O(r^{-3}),
\end{aligned}$$

and

$$\begin{aligned}
e_3'(r') &= (1+O(r^{-2}))e_3(r') + O(r^{-2})e_4(r') = -\Upsilon + O(r^{-2}) + r\Gamma_b,\\
e_3'(u') &= (1+O(r^{-2}))e_3(u') + O(r^{-2})e_4(u') = 2 + O(r^{-2}) + r\Gamma_b,\\
e_3'(\cos(\theta')) &= (1+O(r^{-2}))e_3(\cos(\theta')) + \frac{a\Upsilon}{r}f_0\cdot\nabla\cos(\theta')\\
&\quad + O(r^{-2})e_4(\cos(\theta'))\\
&= O(r^{-2}) + r^{-1}\Gamma_b.
\end{aligned}$$

Since we have by definition of $\mathfrak{J}$ on Σ_*

$$\mathfrak{J} = \frac{1}{|q|}(f_0 + i\,{}^*f_0) = \frac{1}{r}(f_0 + i\,{}^*f_0) + O(r^{-3})(f_0 + i\,{}^*f_0),$$

we deduce

$$\mathcal{D}'(r),\ \widetilde{\mathcal{D}'(u)},\ \widetilde{\mathcal{D}'(\cos\theta)} = O(r^{-3}) + \Gamma_b, \qquad \widetilde{e_3'(r)},\ \widetilde{e_3'(u)}, = O(r^{-2}) + r\Gamma_b,$$
$$e_3'(\cos\theta) = O(r^{-2}) + r^{-1}\Gamma_b.$$

The proof of Lemma 9.78 then follows from the control of Γ_b provided by Corollary 9.74, and the dominant condition for r on Σ_*. □

Step 2. Next, we consider the following lemma.

Lemma 9.79. *For all $k \leq k_{large}+7$, we have*

$$\|r\mathfrak{d}_*^{\leq k}\mathcal{D}'\widehat{\otimes}\mathfrak{J}'\|_{L^2(\Sigma_*)} + \|r\mathfrak{d}_*^{\leq k}\widetilde{\overline{\mathcal{D}'}\cdot\mathfrak{J}'}\|_{L^2(\Sigma_*)} + \|r\mathfrak{d}_*^{\leq k}\widetilde{\nabla_3'\mathfrak{J}'}\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^\star\mathfrak{R}_k.$$

Proof. The proof is in the same spirit as the one of Lemma 9.78. First, since $\nabla_4'\mathfrak{J}' = -\frac{1}{q'}\mathfrak{J}'$, since $q'=q$ and $\mathfrak{J}'=\mathfrak{J}$ on Σ_*, and since $\nabla\mathfrak{J}'=\nabla\mathfrak{J}$ we have, using the transformation formulas

$$\begin{aligned}
-\frac{1}{q'}\mathfrak{J}' &= \nabla_4'\mathfrak{J}' = \nabla_4\mathfrak{J}' + f\cdot\nabla\mathfrak{J} + O(r^{-2})\nabla_3\mathfrak{J}' \\
&= \nabla_4\mathfrak{J}' + f\cdot\nabla\mathfrak{J} + O(r^{-2})\nabla_\nu\mathfrak{J} + O(r^{-2})\nabla_4\mathfrak{J}' \\
&= (1+O(r^{-2}))\nabla_4\mathfrak{J}' + O(r^{-3}) + r^{-3}\Gamma_b
\end{aligned}$$

where we have denoted for convenience ${}^{(\Sigma_*)}\Gamma_b$ and ${}^{(\Sigma_*)}\Gamma_g$ of Definition 9.73 simply by Γ_b and Γ_g. Hence, we have on Σ_*

$$\nabla_4\mathfrak{J}' = -\frac{1}{q}\mathfrak{J} + O(r^{-3}) + r^{-3}\Gamma_b.$$

Also, since $\nabla_\nu\mathfrak{J}' = \nabla_\nu\mathfrak{J}$ on Σ_*, and since $\nu = e_3 + b_* e_4$, we have

$$\begin{aligned}
\nabla_3\mathfrak{J}' &= \nabla_\nu\mathfrak{J} - b_*\nabla_4\mathfrak{J}' \\
&= \frac{1}{r}\nabla_\nu(r\mathfrak{J}) - \frac{\nu(r)}{r}\mathfrak{J} + \frac{b_*}{q}\mathfrak{J} + O(r^{-3}) + r^{-3}\Gamma_b \\
&= \frac{1}{r}\mathfrak{J} + O(r^{-3}) + r^{-1}\Gamma_b,
\end{aligned}$$

where we have used in particular $\nabla_\nu(r\mathfrak{J}) = \nabla_\nu(f_0 + i\,{}^*f_0) = 0$ on Σ_*.

Using the above identities on Σ_* for $\nabla_4\mathfrak{J}'$ and $\nabla_3\mathfrak{J}'$, together with the change of frame transformation, the choice of λ, f and $\underline{f}$, and the fact that $\nabla(r') = \nabla(u') = 0$ on Σ_*, we obtain

$$\begin{aligned}
\mathcal{D}'\mathfrak{J}' &= \mathcal{D}\mathfrak{J} + O(r^{-3}) + r^{-2}\Gamma_b, \\
\nabla_3'\mathfrak{J}' &= \nabla_3\mathfrak{J}' + O(r^{-3}) + r^{-2}\Gamma_b = \frac{1}{r}\mathfrak{J} + O(r^{-3}) + r^{-1}\Gamma_b.
\end{aligned}$$

We infer

$$\begin{aligned}
\mathcal{D}'\widehat{\otimes}\mathfrak{J}' &= \mathcal{D}\widehat{\otimes}\mathfrak{J} + O(r^{-3}) + r^{-2}\Gamma_b = r^{-1}\Gamma_b, \\
\overline{\mathcal{D}'}\cdot\mathfrak{J}' &= \overline{\mathcal{D}}\cdot\mathfrak{J} + O(r^{-3}) + r^{-2}\Gamma_b = \frac{4i(r^2+a^2)\cos\theta}{|q|^4} + O(r^{-3}) + r^{-1}\Gamma_b,
\end{aligned}$$

$$\nabla'_3\mathfrak{J}' = \frac{1}{r}\mathfrak{J} + O(r^{-3}) + r^{-1}\Gamma_b = \frac{\Delta q}{|q|^4}\mathfrak{J} + O(r^{-3}) + r^{-1}\Gamma_b,$$

and hence

$$\mathcal{D}'\widehat{\otimes}\mathfrak{J}' = r^{-1}\Gamma_b, \qquad \widetilde{\mathcal{D}'}\cdot\mathfrak{J}' = O(r^{-3}) + r^{-1}\Gamma_b, \qquad \widetilde{\nabla'_3\mathfrak{J}'} = O(r^{-3}) + r^{-1}\Gamma_b.$$

The proof of Lemma 9.79 then follows from the control of Γ_b provided by Corollary 9.74, and the dominant condition for r on Σ_*. □

Step 3. We prove the following lemma.

Lemma 9.80. *We have, for all $k \leq k_{large} + 7$,*

$$\|\mathfrak{d}_*^{\leq k}(\underline{\widehat{X}}', \check{H}', \underline{\Xi}', \underline{\check{\omega}}')\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^\star\mathfrak{R}_k.$$

Proof. In this lemma, we use the following notation

$$\check{\Gamma}' := \Big\{r\widetilde{\mathrm{tr}X}', r\widehat{X}', r\check{Z}', r\widetilde{\mathrm{tr}\underline{X}}', \check{H}', \underline{\widehat{X}}', \underline{\omega}', \underline{\Xi}'\Big\},$$

i.e. $\check{\Gamma}'$ contain all the linearized Ricci coefficients of the PT frame of ${}^{(ext)}\mathcal{M}$. Also, we denote for convenience ${}^{(\Sigma_*)}\Gamma_b$ and ${}^{(\Sigma_*)}\Gamma_g$ of Definition 9.73 simply by Γ_b and Γ_g. In particular, we have in view of (9.125)

$$\|\mathfrak{d}_*^{\leq k}\check{\Gamma}'\|_{L^2(\Sigma_*)} \leq \epsilon, \qquad k \leq k_{large} + 7,$$

and in view of Corollary 9.74

$$\|r\mathfrak{d}_*^{\leq k}\Gamma_g\|_{L^2(\Sigma_*)} + \|\mathfrak{d}_*^{\leq k}\Gamma_b\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^\star\mathfrak{R}_k, \qquad k \leq k_{large} + 7.$$

With these notations, recall from Lemma 5.82 the following consequence of the change of frame formulas of Proposition 2.12

$$\begin{aligned}
\underline{\widehat{\chi}} &= \underline{\widehat{\chi}}' + \nabla\widehat{\otimes}\underline{f}' + O(r^{-3}) + r^{-1}\check{\Gamma}' + r^{-2}\Gamma_b,\\
\underline{\xi} - b_*\zeta &= \underline{\xi}' - b_*\zeta' + \frac{1}{2}\nabla_\nu\underline{f}' + \frac{1}{4}\mathrm{tr}\,\underline{\chi}'\,\underline{f}' + \frac{b_*}{4}\mathrm{tr}\,\underline{\chi}'f' + O(r^{-3}) + r^{-1}\check{\Gamma}'\\
&\quad + r^{-2}\Gamma_b,\\
\underline{\omega} &= \underline{\omega}' + \frac{1}{2}\nu(\log\lambda') + O(r^{-3}) + r^{-1}\check{\Gamma}' + r^{-2}\Gamma_b,\\
\eta &= \eta' + \frac{1}{2}\nabla_\nu f' + \frac{1}{4}\underline{f}'\mathrm{tr}\,\chi' + \frac{b_*}{4}\mathrm{tr}\,\chi' f' + O(r^{-3}) + r^{-1}\check{\Gamma}' + r^{-2}\Gamma_b.
\end{aligned}$$

Together with the form of $(f', \underline{f}', \lambda')$ in Lemma 9.76 and the asymptotic for large r of the Kerr values of the PT frame, we infer

$$\begin{aligned}
\underline{\widehat{\chi}}' &= \Gamma_b + \nabla\widehat{\otimes}\underline{f}' + O(r^{-3}) + r^{-1}\check{\Gamma}',\\
\underline{\xi}' &= \Gamma_b + \frac{1}{2}\widecheck{\nabla_\nu \underline{f}'} + O(r^{-3}) + r^{-1}\check{\Gamma}',\\
\underline{\check{\omega}}' &= \Gamma_b + \frac{1}{2}\nu(\log\lambda') + O(r^{-3}) + r^{-1}\check{\Gamma}',\\
\check{\eta}' &= \Gamma_b + \frac{1}{2}\widecheck{\nabla_\nu f'} + O(r^{-3}) + r^{-1}\check{\Gamma}'.
\end{aligned}$$

Lemma 9.80 is then an immediate consequence of the above identities, the above control of $\check{\Gamma}'$ and Γ_b, the control of $\nabla\widehat{\otimes}\underline{f}'$, $\nabla_\nu f'$, $\nabla_\nu \underline{f}'$ and $\nu(\log\lambda')$ provided by Lemma 9.77, and the dominant condition for r on Σ_*. □

Step 4. We prove the following lemma.

Lemma 9.81. *The following estimates hold true, for $k \le k_{large} + 7$,*

$$\|r\mathfrak{d}_*^{\le k}(\widecheck{trX}', \widehat{X}', \widecheck{tr\underline{X}}', \check{Z}')\|_{L^2(\Sigma_*)} \lesssim \epsilon_0 + {}^\star\mathfrak{R}_k. \tag{9.126}$$

Proof. We use again the notations $\check{\Gamma}'$, Γ_b and Γ_g of Lemma 9.80. Then, we have in view of Lemma 5.82 the following consequence of the change of frame formulas of Proposition 2.12

$$\begin{aligned}
\mathrm{tr}\,\chi &= \mathrm{tr}\,\chi' + \mathrm{div}\,(f') + O(r^{-3}) + O(r^{-1})\check{\eta}' + r^{-2}\check{\Gamma}' + r^{-2}\Gamma_b,\\
0 &= {}^{(a)}\mathrm{tr}\chi' + \mathrm{curl}\,(f') + O(r^{-3}) + O(r^{-1})\check{\eta}' + r^{-2}\check{\Gamma}' + r^{-2}\Gamma_b,\\
\widehat{\chi} &= \widehat{\chi}' + \nabla\widehat{\otimes} f' + O(r^{-3}) + O(r^{-1})\check{\eta}' + r^{-2}\check{\Gamma}' + r^{-2}\Gamma_b,\\
\mathrm{tr}\,\underline{\chi} &= \mathrm{tr}\,\underline{\chi}' + \mathrm{div}\,(\underline{f}') + O(r^{-3}) + O(r^{-1})\underline{\xi}' + r^{-2}\check{\Gamma}' + r^{-2}\Gamma_b,\\
0 &= {}^{(a)}\mathrm{tr}\underline{\chi}' + \mathrm{curl}\,(\underline{f}') + O(r^{-3}) + O(r^{-1})\underline{\xi}' + r^{-2}\check{\Gamma}' + r^{-2}\Gamma_b,
\end{aligned}$$

and

$$\begin{aligned}
\zeta &= \zeta' - \nabla(\log\lambda') - \frac{1}{4}\mathrm{tr}\,\underline{\chi}' f' + \frac{1}{4}\underline{f}'\mathrm{tr}\,\chi' + \frac{1}{4}\underline{f}'\mathrm{div}\,(f') + \frac{1}{4}\,{}^*\underline{f}'\mathrm{curl}\,(f')\\
&\quad + O(r^{-1})(\underline{\check{\omega}}', \underline{\widehat{\chi}}') + r^{-2}\check{\Gamma}' + O(r^{-3}) + r^{-1}\Gamma_g.
\end{aligned}$$

Together with the form of $(f', \underline{f}', \lambda')$ in Lemma 9.76 and the asymptotic for large r of the Kerr values of the PT frame, we infer

$$\widecheck{\mathrm{tr}\chi}' = \Gamma_g + \mathrm{div}\,(f') + O(r^{-3}) + O(r^{-1})\check{\eta}' + r^{-2}\check{\Gamma}',$$

$$\begin{aligned}
\widetilde{{}^{(a)}\mathrm{tr}\chi'} &= \widetilde{\mathrm{curl}\,(f')} + O(r^{-3}) + O(r^{-1})\check{\eta}' + r^{-2}\check{\Gamma}' + r^{-2}\Gamma_b,\\
\widehat{\chi}' &= \Gamma_g + \nabla\widehat{\otimes} f' + O(r^{-3}) + O(r^{-1})\check{\eta}' + r^{-2}\check{\Gamma}',
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}\,\underline{\chi}' &= \Gamma_g + \mathrm{div}\,(\underline{f}') + O(r^{-3}) + O(r^{-1})\underline{\xi}' + r^{-2}\check{\Gamma}' + r^{-2}\Gamma_b,\\
\widetilde{{}^{(a)}\mathrm{tr}\underline{\chi}'} &= \widetilde{\mathrm{curl}\,(\underline{f}')} + O(r^{-3}) + O(r^{-1})\underline{\xi}' + r^{-2}\check{\Gamma}' + r^{-2}\Gamma_b,
\end{aligned}$$

and

$$\begin{aligned}
\check{\zeta}' &= \Gamma_g - \nabla(\log\lambda') + \frac{1}{4}\underline{f}'\mathrm{div}\,(f') + \frac{1}{4}\,{}^*\underline{f}'\widetilde{\mathrm{curl}\,(f')}\\
&\quad + O(r^{-1})(\underline{\check{\omega}}', \underline{\widehat{\chi}}') + r^{-2}\check{\Gamma}' + O(r^{-3}).
\end{aligned}$$

Lemma 9.81 is then an immediate consequence of the above identities, the control of $(\underline{\widehat{\chi}}', \check{\eta}', \underline{\Xi}', \underline{\check{\omega}}')$ in Lemma 9.80, the control of $\check{\Gamma}'$ and Γ_b recalled in Lemma 9.80, the control of $\nabla\widehat{\otimes}\underline{f}'$, $\nabla_\nu f'$, $\nabla_\nu \underline{f}'$ and $\nu(\log\lambda')$ provided by Lemma 9.77, and the dominant condition for r on Σ_*. $\square$

The proof of Proposition 9.67 follows immediately from Lemmas 9.78–9.81 above.

9.8. Control of the PT-Ricci coefficients in ${}^{(ext)}\mathcal{M}$

The goal of this section is to provide the proof of Proposition 9.51. For convenience, we restate the result below.

Proposition 9.82. *Relative to the PT frame of ${}^{(ext)}\mathcal{M}$ we have*

$$ {}^{(ext)}\mathfrak{G}_k \lesssim \epsilon_0 + {}^\star\mathfrak{G}_k + {}^{(ext)}\mathfrak{R}_k, \qquad k \le k_{large} + 7. \tag{9.127}$$

To prove Proposition 9.82, we rely in particular on our bootstrap assumption **BA-PT**, see (9.40), which implies for the Ricci and metric coefficients of the outgoing PT frame of ${}^{(ext)}\mathcal{M}$

$$ {}^{(ext)}\mathfrak{G}_k \le \epsilon, \qquad k \le k_{large} + 7. \tag{9.128}$$

Also, we make use of the transport lemmas derived in the next section.

9.8.1. Transport lemmas

In what follows the weighted derivatives $\not{\mathfrak{d}} = (r\nabla)$ and $\mathfrak{d} = (r\nabla, r\nabla_4, \nabla_3)$ are defined with respect to the outgoing PT frame of ${}^{(ext)}\mathcal{M}$. We revisit Proposition 6.47 and state below its L^2 version using the following norms

$$\|f\|_2(u,r) := \|f\|_{L^2\big(S(u,r)\big)}, \qquad \|f\|_{2,k}(u,r) := \sum_{i=0}^{k} \|\mathfrak{d}^i f\|_2(u,r).$$

Proposition 9.83. *Let U and F anti-selfdual k-tensors. Assume that U verifies one of the following equations, for a real constant c,*

$$\nabla_4 U + \frac{c}{q} U = F \tag{9.129}$$

or

$$\nabla_4 U + \Re\left(\frac{c}{q}\right) U = F. \tag{9.130}$$

In both cases we derive, for any $r_0 \le r \le r_* = r_*(u)$ at fixed u, with $1 \le u \le u_*$, in ${}^{(ext)}\mathcal{M}$

$$r^{c-1}\|U\|_{2,k}(u,r) \lesssim \quad r_*^{c-1}\|U\|_{2,k}(u,r_*) + \int_r^{r_*} \lambda^{c-1}\|F\|_{2,k}(u,\lambda)d\lambda. \tag{9.131}$$

Proof. The proof is completely analogous of the one of Proposition 6.47 where one simply has to replace $L^\infty(S(u,r))$ based norms by by $L^2(S(u,r))$ based norms. □

Corollary 9.84. *If U verifies (9.129) or (9.130), we obtain, for any $C > c - \frac{1}{2}$, and for any $r_1 \ge r_0$,*

$$\sup_{\lambda \ge r_1} \int_{r=\lambda} \lambda^{2(c-1)} \big|\mathfrak{d}^{\le k} U\big|^2 \quad \lesssim \quad \int_{\Sigma_*} r^{2(c-1)} \big|\mathfrak{d}^{\le k} U\big|^2 + \int_{{}^{(ext)}\mathcal{M}(r \ge r_1)} r^{2C} \big|\mathfrak{d}^{\le k} F\big|^2.$$

Proof. Apply (9.131) for $r_0 \le r_1 \le r \le r_* = r_*(u)$, $1 \le u \le u_*$ with $r_*(u) = c_* - u$ on Σ_*, take the square and integrate in u to derive

$$\begin{aligned} &\int_{r=\lambda} \lambda^{2(c-1)} \big|\mathfrak{d}^{\le k} U\big|^2 \\ \lesssim \quad & \int_{\Sigma_*} r^{2(c-1)} \big|\mathfrak{d}^{\le k} U\big|^2 + \int_1^{u_*} du \left(\int_\lambda^{r_*} \lambda'^{(c-1)} \|F\|_{2,k}(u,\lambda')d\lambda'\right)^2. \end{aligned}$$

By Cauchy-Schwartz, we have for any $C > c - \frac{1}{2}$

$$\begin{aligned} &\left(\int_\lambda^{r_*} \lambda'^{(c-1)} \|F\|_{2,k}(u,\lambda')d\lambda'\right)^2 \\ \lesssim \quad & \left(\int_\lambda^{r_*} \lambda'^{2C} \|F\|_{2,k}(u,\lambda')d\lambda'\right)\left(\int_\lambda^{r_*} \lambda'^{2c-2-2C} d\lambda'\right) \end{aligned}$$

$$\lesssim \int_\lambda^{r_*} \lambda'^{2C} \|F\|^2_{2,k}(u,\lambda')d\lambda'.$$

Hence, we infer for any $C > c - \frac{1}{2}$, and for any $r_1 \geq r_0$,

$$\sup_{\lambda\geq r_1}\int_{r=\lambda}\lambda^{2(c-1)}\big|\mathfrak{d}^{\leq k}U\big|^2 \lesssim \int_{\Sigma_*} r^{2(c-1)}\big|\mathfrak{d}^{\leq k}U\big|^2 + \int_{{}^{(ext)}\mathcal{M}(r\geq r_1)} r^{2C}\big|\mathfrak{d}^{\leq k}F\big|^2$$

as stated. □

9.8.2. Proof of Proposition 9.82 We estimate the Ricci and metric coefficients of the outgoing PT structure of ${}^{(ext)}\mathcal{M}$ in the following order

$$\widetilde{\mathrm{tr}X},\ \widehat{X},\ \mathcal{D}\cos\theta,\ \check{Z},\ \mathcal{D}r,\ \check{H},\ \widetilde{e_3(r)},\ \check{\underline{\omega}},\ \mathcal{D}\widehat{\otimes}\mathfrak{J},\ \widetilde{\mathcal{D}\cdot\overline{\mathfrak{J}}},\ e_3(\cos\theta),\ \widetilde{e_3\mathfrak{J}},\ \underline{\Xi},$$

making use of the triangular structure of the linearized equations for outgoing PT structures derived in sections 9.2.5 and 9.2.6, the transport lemmas of the previous section, and the bootstrap assumption (9.128).

Step 1. Estimates for $\widetilde{\mathrm{tr}X}$.

We apply Corollary 9.84 to the following equation, see Proposition 9.19,

$$\nabla_4\widetilde{\mathrm{tr}X} + \frac{2}{q}\widetilde{\mathrm{tr}X} = \Gamma_g\cdot\Gamma_g,$$

with $c = 2$ and $C = 2$ and derive, for $k \leq k_{large} + 7$,

$$\begin{aligned}\sup_{\lambda\geq r_0}\int_{r=\lambda}\lambda^2\big|\mathfrak{d}^{\leq k}\widetilde{\mathrm{tr}X}\big|^2 &\lesssim \int_{\Sigma_*} r^2\big|\mathfrak{d}^{\leq k}\widetilde{\mathrm{tr}X}\big|^2 + \int_{{}^{(ext)}\mathcal{M}} r^4\big|\mathfrak{d}^{\leq k}(\Gamma_g\cdot\Gamma_g)\big|^2\\ &\lesssim ({}^\star\mathfrak{G}_k)^2 + \int_{{}^{(ext)}\mathcal{M}} r^4\big|\mathfrak{d}^{\leq k}(\Gamma_g\cdot\Gamma_g)\big|^2.\end{aligned}$$

Note that, in view of our bootstrap assumptions (9.128) on Γ_g,

$$\big|\mathfrak{d}^{\leq k}(\Gamma_g\cdot\Gamma_g)\big|^2 \lesssim \big|\mathfrak{d}^{\leq k/2}(\Gamma_g)\big|^2\big|\mathfrak{d}^{\leq k}(\Gamma_g)\big|^2 \lesssim \epsilon^2 r^{-4}\big|\mathfrak{d}^{\leq k}(\Gamma_g)\big|^2.$$

Hence

$$\begin{aligned}\int_{{}^{(ext)}\mathcal{M}} r^4\big|\mathfrak{d}^{\leq k}(\Gamma_g\cdot\Gamma_g)\big|^2 &\lesssim \epsilon^2\int_{{}^{(ext)}\mathcal{M}}\|\Gamma_g\|^2_{2,k} \lesssim \epsilon^2\int_{r_0}^{r_*}\int_{r=\lambda}\big|\mathfrak{d}^{\leq k}(\Gamma_g)\big|^2\\ &= \epsilon^2\int_{r_0}^{r_*}\lambda^{-2}\int_{r=\lambda}\lambda^2\big|\mathfrak{d}^{\leq k}(\Gamma_g)\big|^2\end{aligned}$$

$$
\begin{aligned}
&\lesssim \epsilon^2 \sup_{\lambda\geq r_0}\int_{r=\lambda}\lambda^2\big|\mathfrak{d}^{\leq k}(\Gamma_g)\big|^2\\
&\lesssim \epsilon^2\mathfrak{G}_k^2 \lesssim \epsilon^4 \lesssim \epsilon_0^2.
\end{aligned}
$$

Thus, we deduce

$$
\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}\lambda^2\big|\mathfrak{d}^{\leq k}\widetilde{\mathrm{tr} X}\big|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0+{}^\star\mathfrak{G}_k, \qquad k\leq k_{large}+7. \tag{9.132}
$$

Step 2. Estimates for $\widehat{X}$.

We apply Corollary 9.84 to the following equation, see Proposition 9.19,

$$
\nabla_4\widehat{X}+\Re\left(\frac{2}{q}\right)\widehat{X} = -A+\Gamma_g\cdot\Gamma_g,
$$

with $c=2$ and $C=3/2+\delta_B/2$ and derive, for $k\leq k_{large}+7$,

$$
\begin{aligned}
\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}\lambda^2\big|\mathfrak{d}^{\leq k}\widehat{X}\big|^2\right) \lesssim\; & \int_{\Sigma_*}r^2\big|\mathfrak{d}^{\leq k}\widehat{X}\big|^2+\int_{{}^{(ext)}\mathcal{M}}r^{3+\delta_B}\big|\mathfrak{d}^{\leq k}A\big|^2\\
&+\int_{{}^{(ext)}\mathcal{M}}r^{3+\delta_B}\big|\mathfrak{d}^{\leq k}(\Gamma_g\cdot\Gamma_g)\big|^2.
\end{aligned}
$$

Recalling the definition of ${}^\star\mathfrak{G}_k$ and ${}^{(ext)}\mathfrak{R}_k$, and proceeding as in Step 1 with the quadratic term, we deduce

$$
\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}\lambda^2\big|\mathfrak{d}^{\leq k}\widehat{X}\big|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0+{}^\star\mathfrak{G}_k+{}^{(ext)}\mathfrak{R}_k, \qquad k\leq k_{large}+7.
$$

Step 3. Estimate for $\mathcal{D}\cos\theta$.

We apply Corollary 9.84 to the following equation, see Lemma 9.21,

$$
\nabla_4\widetilde{\mathcal{D}\cos\theta}+\frac{1}{q}\widetilde{\mathcal{D}\cos\theta} = O(r^{-1})\widetilde{\mathrm{tr} X}+O(r^{-1})\widehat{X}+\Gamma_b\cdot\Gamma_g,
$$

with $c=1$ and $C=1$. Using the estimate of Step 1 for $\widetilde{\mathrm{tr} X}$, the estimate of Step 2 for $\widehat{X}$, and treating the quadratic terms as before, we easily derive

$$
\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}\big|\mathfrak{d}^{\leq k}\widetilde{\mathcal{D}\cos\theta}\big|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0+{}^\star\mathfrak{G}_k+{}^{(ext)}\mathfrak{R}_k, \qquad k\leq k_{large}+7.
$$

Step 4. Estimate for $\check{Z}$.

We apply[217] Proposition 9.83 to the following equation, see Proposition 9.19,

$$\begin{aligned}\nabla_4 \check{Z} + \frac{1}{q}\check{Z} &= F,\\ F &= O(r^{-2})\widetilde{\mathrm{tr} X} + O(r^{-2})\widehat{X} - B + \Gamma_g\cdot\Gamma_g,\end{aligned}$$

with $c = 1$ and derive, for $k \leq k_{large} + 7$,

$$\|\check{Z}\|_{2,k}(r,u) \lesssim \|\check{Z}\|_{2,k}(r_*,u) + \int_r^{r_*} \|F\|(\lambda,u)\|_{2,k}(u,\lambda)d\lambda,$$

and hence, for $k \leq k_{large} + 7$,

$$\begin{aligned}\|\check{Z}\|^2_{2,k}(r,u) &\lesssim \|\check{Z}\|^2_{2,k}(r_*,u) + \left(\int_r^{r_*} \|F\|(\lambda,u)\|_{2,k}(u,\lambda)d\lambda\right)^2\\ &\lesssim \|\check{Z}\|^2_{2,k}(r_*,u) + r^{-2-\delta_B}\left(\int_r^{r_*} \lambda^{3+\delta_B}\|F\|(\lambda,u)\|^2_{2,k}(u,\lambda)d\lambda\right).\end{aligned}$$

In view of the form of F, this yields, for $k \leq k_{large} + 7$,

$$\begin{aligned}\|\check{Z}\|^2_{2,k}(r,u) \lesssim \|\check{Z}\|^2_{2,k}(r_*,u) + r^{-2-\delta_B}\int_r^{r_*}\Big(\lambda^{3+\delta_B}\|B\|^2_{2,k}(u,\lambda)\\ + \|(\widetilde{\mathrm{tr} X},\widehat{X})\|^2_{2,k}(u,\lambda) + \epsilon^2\|\Gamma_g\|^2_{2,k}(u,\lambda)\Big).\end{aligned} \tag{9.133}$$

Multiplying by r^2, and integrating in u, we derive, for $k \leq k_{large} + 7$,

$$\begin{aligned}&\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}\lambda^2|\mathfrak{d}^{\leq k}\check{Z}|^2\right)^{\frac{1}{2}}\\ &\lesssim \int_{\Sigma_*} r^2|\mathfrak{d}^{\leq k}\check{Z}|^2 + \int_{{}^{(ext)}\mathcal{M}}\Big(r^{3+\delta_B}|\mathfrak{d}^{\leq k}B|^2 + |\mathfrak{d}^{\leq k}(\widetilde{\mathrm{tr} X},\widehat{X})|^2 + \epsilon^2|\mathfrak{d}^{\leq k}\Gamma_g|^2\Big).\end{aligned}$$

Using the estimate of Step 1 for $\widetilde{\mathrm{tr} X}$ and the estimate of Step 2 for $\widehat{X}$, we easily derive

$$\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}\lambda^2|\mathfrak{d}^{\leq k}\check{Z}|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0 + {}^\star\mathfrak{G}_k + {}^{(ext)}\mathfrak{R}_k, \qquad k \leq k_{large} + 7.$$

Step 5. Estimates for $\mathcal{D}r$.

[217] In order to obtain an estimate that will be useful in Step 5, see (9.133), we proceed differently compared to previous steps.

We apply Proposition 9.83 to the following equation, see Lemma 9.21,

$$\nabla_4 \mathcal{D}r + \frac{1}{q}\mathcal{D}r = \check{Z} + r\Gamma_g \cdot \Gamma_g$$

with $c = 1$ and deduce, for $k \leq k_{large} + 7$,

$$\begin{aligned}\|\mathcal{D}r\|_{2,k}(u,r) &\lesssim \|\mathcal{D}r\|_{2,k}(u,r_*) + \int_r^{r_*}\|\check{Z}\|_{2,k}(u,\lambda)d\lambda \\ &+ \int_r^{r_*}\lambda\|\Gamma_g\cdot\Gamma_g\|_{2,k}(u,\lambda)d\lambda.\end{aligned}$$

Squaring and integrating in u, controlling the error term as before using (9.128), we derive, for $k \leq k_{large} + 7$,

$$\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}\left|\mathfrak{d}^{\leq k}\mathcal{D}r\right|^2\right) \lesssim {}^\star\mathfrak{G}_k^2 + \sup_{r\geq r_0}\int_{u=1}^{u_*}\left(\int_r^{r_*}\|\check{Z}\|_{2,k}(u,\lambda)d\lambda\right)^2 + \epsilon_0^2.$$

Now, recall (9.133)

$$\begin{aligned}\|\check{Z}\|_{2,k}(r,u) \lesssim \|\check{Z}\|_{2,k}(r_*,u) + r^{-1-\frac{\delta_B}{2}}\Bigg(\int_r^{r_*}\Big(\lambda^{3+\delta_B}\|B\|^2_{2,k}(u,\lambda) \\ + \|(\widetilde{\mathrm{tr}X},\widehat{X})\|^2_{2,k}(u,\lambda) + \epsilon^2\|\Gamma_g\|^2_{2,k}(u,\lambda)\Big)\Bigg)^{\frac{1}{2}}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\int_r^{r_*}\|\check{Z}\|_{2,k}(\lambda,u)d\lambda &\lesssim r_*\|\check{Z}\|_{2,k}(r_*,u) + \Bigg(\int_r^{r_*}\Big(\lambda^{3+\delta_B}\|B\|^2_{2,k}(u,\lambda) \\ &+\|(\widetilde{\mathrm{tr}X},\widehat{X})\|^2_{2,k}(u,\lambda) + \epsilon^2\|\Gamma_g\|^2_{2,k}(u,\lambda)\Big)\Bigg)^{\frac{1}{2}}\end{aligned}$$

and thus, using the estimate of Step 1 for $\widetilde{\mathrm{tr}X}$ and the estimate of Step 2 for $\widehat{X}$, we obtain, for $k \leq k_{large} + 7$,

$$\int_{u=1}^{u_*}\left(\int_r^{r_*}\|\check{Z}\|_{2,k}(\lambda,u)d\lambda\right)^2 \lesssim {}^\star\mathfrak{G}_k^2 + {}^{(ext)}\mathfrak{R}_k^2 + \epsilon_0^2.$$

Plugging in the above estimate for $\mathcal{D}r$, we infer, for $k \leq k_{large}+7$,

$$\begin{aligned}\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}|\mathfrak{d}^{\leq k}\mathcal{D}r|^2\right) &\lesssim {}^\star\mathfrak{G}_k^2+\sup_{r\geq r_0}\int_{u=1}^{u_*}\left(\int_r^{r_*}\|\check{Z}\|_{2,k}(u,\lambda)d\lambda\right)^2+\epsilon_0^2\\ &\lesssim {}^\star\mathfrak{G}_k^2+{}^{(ext)}\mathfrak{R}_k^2+\epsilon_0^2,\end{aligned}$$

i.e.

$$\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}|\mathfrak{d}^{\leq k}\mathcal{D}r|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0+{}^\star\mathfrak{G}_k+{}^{(ext)}\mathfrak{R}_k, \qquad k\leq k_{large}+7.$$

Step 6. Estimates for $\check{H}$.

We apply Corollary 9.84 to the following equation, see Proposition 9.19,

$$\nabla_4\check{H}+\frac{1}{q}\check{H} = O(r^{-2})\widecheck{\mathrm{tr}X}+O(r^{-2})\widehat{X}-B+\Gamma_b\cdot\Gamma_g,$$

with $c=1$ and $C=1$ and derive, for $k\leq k_{large}+7$,

$$\begin{aligned}&\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}|\mathfrak{d}^{\leq k}\check{H}|^2\right)\\ &\lesssim \int_{\Sigma_*}|\mathfrak{d}^{\leq k}\check{H}|^2+\int_{{}^{(ext)}\mathcal{M}}r^2\left|\mathfrak{d}^{\leq k}\Big(O(r^{-2})\widecheck{\mathrm{tr}X}+O(r^{-2})\widehat{X}-B+\Gamma_b\cdot\Gamma_g\Big)\right|^2.\end{aligned}$$

Using the estimate of Step 1 for $\widecheck{\mathrm{tr}X}$ and the estimate of Step 2 for $\widehat{X}$, and proceeding as in Step 1 with the quadratic term, we deduce

$$\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}|\mathfrak{d}^{\leq k}\check{H}|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0+{}^\star\mathfrak{G}_k+{}^{(ext)}\mathfrak{R}_k, \qquad k\leq k_{large}+7.$$

Step 7. Estimate for $e_3(\cos\theta)$.

We apply Proposition 9.83 to the following equation, see Lemma 9.21,

$$e_4(e_3(\cos\theta)) = O(r^{-1})\check{H}+O(r^{-2})\widecheck{\mathcal{D}(\cos\theta)}+\Gamma_b\cdot\Gamma_b,$$

with $c=0$ and deduce, for $k\leq k_{large}+7$,

$$r^{-1}\|e_3(\cos\theta)\|_{2,k}(u,r)\lesssim r_*^{-1}\|e_3(\cos\theta)\|_{2,k}(u,r_*)+\int_r^{r_*}\lambda^{-1}\|F\|_{2,k}(u,\lambda)d\lambda,$$

with

$$F = O(r^{-1})\check{H}+O(r^{-2})\widecheck{\mathcal{D}(\cos\theta)}+\Gamma_b\cdot\Gamma_b.$$

Multiplying by r, squaring, and integrating in u, we deduce, for $r\geq r_0$ and $k\leq k_{large}+7$,

$$\begin{aligned}
&\int_{u=1}^{u_*}\|e_3(\cos\theta)\|^2_{2,k}(u,r)\\
\lesssim\;& \int_{\Sigma_*}\left|\mathfrak{d}^{\leq k}e_3(\cos\theta)\right|^2+\int_{u=1}^{u_*}\left(r\int_r^{r_*}\lambda^{-1}\|F\|_{2,k}(u,\lambda)d\lambda\right)^2\\
\lesssim\;& ({}^\star\mathfrak{G}_k)^2+\int_{u=1}^{u_*}\left(r\int_r^{r_*}\lambda^{-1}\|F\|_{2,k}(u,\lambda)d\lambda\right)^2.
\end{aligned}$$

Also

$$\begin{aligned}
&\int_r^{r_*}\lambda^{-1}\|F\|_{2,k}(u,\lambda)d\lambda\\
\lesssim\;& \int_r^{r_*}\lambda^{-2}\|(\check{H},\widetilde{\mathcal{D}(\cos\theta)})\|_{2,k}(u,\lambda)d\lambda+\epsilon\int_r^{r_*}\lambda^{-2}\|\Gamma_b\|_{2,k}(u,\lambda)d\lambda\\
\lesssim\;& r^{-\frac{1}{2}}\left(\int_r^{r_*}\lambda^{-2}\|(\check{H},\widetilde{\mathcal{D}(\cos\theta)})\|^2_{2,k}(u,\lambda)d\lambda\right)^{\frac{1}{2}}\\
&+\epsilon r^{-\frac{1}{2}}\left(\int_r^{r_*}\lambda^{-2}\|\Gamma_b\|^2_{2,k}(u,\lambda)d\lambda\right)^{\frac{1}{2}}
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_{u=1}^{u_*}\left(r\int_r^{r_*}\lambda^{-1}\|F\|_{2,k}(u,\lambda)d\lambda\right)^2\\
\lesssim\;& r\int_{u=1}^{u_*}du\left(\int_r^{r_*}\lambda^{-2}\|(\check{H},\widetilde{\mathcal{D}(\cos\theta)})\|^2_{2,k}(u,\lambda)d\lambda\right)\\
&+\epsilon^2 r\int_{u=1}^{u_*}du\left(\int_r^{r_*}\lambda^{-2}\|\Gamma_b\|^2_{2,k}(u,\lambda)d\lambda\right)\\
\lesssim\;& r\left(\int_r^{r_*}\lambda^{-2}d\lambda\right)\sup_{r\geq r_0}\left[\int_{r=\lambda}\left|\mathfrak{d}^{\leq k}(\check{H},\widetilde{\mathcal{D}(\cos\theta)})\right|^2+\epsilon^2\int_{r=\lambda}\left|\mathfrak{d}^{\leq k}\Gamma_b\right|^2\right]\\
\lesssim\;& \sup_{r\geq r_0}\left[\int_{r=\lambda}\left|\mathfrak{d}^{\leq k}(\check{H},\widetilde{\mathcal{D}(\cos\theta)})\right|^2+\epsilon^2\int_{r=\lambda}\left|\mathfrak{d}^{\leq k}\Gamma_b\right|^2\right].
\end{aligned}$$

Together with (9.128), the estimate of Step 3 for $\widetilde{\mathcal{D}(\cos\theta)})$ and the estimate of Step 6 for $\check{H}$, we infer, for $k\leq k_{large}+7$,

$$\int_{u=1}^{u_*}\left(r\int_r^{r_*}\lambda^{-1}\|F\|_{2,k}(u,\lambda)d\lambda\right)^2\;\lesssim\;\left(\epsilon_0+{}^\star\mathfrak{G}_k+{}^{(ext)}\mathfrak{R}_k\right)^2.$$

Plugging in the above, we infer

$$\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}\left|\mathfrak{d}^{\leq k}e_3(\cos\theta)\right|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0+{}^\star\mathfrak{G}_k+{}^{(ext)}\mathfrak{R}_k, \qquad k\leq k_{large}+7.$$

Step 8. Estimate for $\check{\underline{\omega}}$.

We start with the following equation, see Proposition 9.19,

$$\begin{aligned}\nabla_4\check{\underline{\omega}} &= \Re\left(\check{P}\right)+F,\\ F &= O(r^{-2})\check{Z}+O(r^{-2})\check{H}+\Gamma_g\cdot\Gamma_g.\end{aligned}$$

By Proposition 9.83, we deduce, for $k\leq k_{large}+7$,

$$r^{-1}\|\underline{\omega}\|_{2,k}(r,u) \lesssim r_*^{-1}\|\underline{\omega}\|_{2,k}(r_*,u)+\int_r^{r_*}\lambda^{-1}\|\check{P}\|_{2,k}+\int_r^{r_*}\lambda^{-1}\|F\|_{2,k}.$$

By Cauchy-Schwartz, we infer

(9.134)

$$\begin{aligned}r^{-2}\|\underline{\omega}\|^2_{2,k}(r,u) &\lesssim r_*^{-2}\|\underline{\omega}\|^2_{2,k}(r_*,u)+r^{-4+\delta_B}\left(\int_r^{r_*}\lambda^{3-\delta_B}\|\check{P}\|^2_{2,k}d\lambda\right)\\ &\quad+r^{-3}\left(\int_r^{r_*}\lambda^{-2}\|(\check{H},\check{Z})\|^2_{2,k}d\lambda+\epsilon^2\int_r^{r_*}\lambda^{-2}\|\Gamma_g\|^2_{2,k}d\lambda\right).\end{aligned}$$

Multiplying by r^2, integrating in u, using (9.128), the estimate of Step 4 for $\check{Z}$ and the estimate of Step 6 for $\check{H}$, we easily derive

$$\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}\left|\mathfrak{d}^{\leq k}\check{\underline{\omega}}\right|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0+{}^\star\mathfrak{G}_k+{}^{(ext)}\mathfrak{R}_k, \qquad k\leq k_{large}+7.$$

Step 9. Estimates for $\widetilde{e_3(r)}$.

In view of Lemma 9.21, we have

$$\begin{aligned}e_4(\widetilde{e_3(r)}) &= -2\check{\underline{\omega}}+F,\\ F &= O(r^{-2})\mathcal{D}r+r\Gamma_b\cdot\Gamma_g.\end{aligned}$$

Proceeding as above we deduce

$$r^{-1}\|\widetilde{e_3(r)}\|_{2,k} \lesssim r_*^{-1}\|\widetilde{e_3(r)}\|_{2,k}(r,_*u)+\int_r^{r_*}\lambda^{-1}\|\check{\underline{\omega}}\|_{2,k}(u,\lambda)d\lambda$$

$$+\int_r^{r_*}\lambda^{-1}\|F\|_{2,k}(u,\lambda)d\lambda$$

and hence, squaring and integrating in u,

$$\begin{aligned} r^{-2}\int_{\lambda=r}\left|\mathfrak{d}^{\leq k}(\widetilde{e_3(r)})\right|^2 &\lesssim ({}^\star\mathfrak{G}_k)^2+\int_{u=1}^{u_*}\left(\int_r^{r_*}\lambda^{-1}\|\check{\underline{\omega}}\|_{2,k}(u,\lambda)d\lambda\right)^2\\ &+\int_{u=1}^{u_*}\left(\int_r^{r_*}\lambda^{-1}\|F\|_{2,k}(u,\lambda)d\lambda\right)^2. \end{aligned}$$

Using Cauchy-Schwarz, (9.128), and the estimate of Step 5 for $\mathcal{D}r$, we infer

$$r^{-2}\int_{\lambda=r}\left|\mathfrak{d}^{\leq k}(\widetilde{e_3(r)})\right|^2 \lesssim ({}^\star\mathfrak{G}_k)^2+\epsilon_0^2+\int_{u=1}^{u_*}\left(\int_r^{r_*}\lambda^{-1}\|\check{\underline{\omega}}\|_{2,k}(u,\lambda)d\lambda\right)^2.$$

The term in $\check{\underline{\omega}}$ is the more dangerous as it could lead to a logarithmic divergence. We estimate it using the more precise estimate for $\check{\underline{\omega}}$ in (9.134). Thus,

$$\begin{aligned} &\int_r^{r_*}\lambda^{-1}\|\check{\underline{\omega}}\|_{2,k}(u,\lambda)d\lambda\\ \lesssim\ & \frac{r_*-r}{r_*}\|\check{\underline{\omega}}\|_{2,k}(r_*,u)+r^{-1+\frac{\delta_B}{2}}\left(\int_r^{r_*}\lambda^{3-\delta_B}\|\check{P}\|^2_{2,k}d\lambda\right)^{1/2}\\ &+r^{-\frac12}\left(\int_r^{r_*}\lambda^{-2}\|(\check{H},\check{Z})\|^2_{2,k}d\lambda+\epsilon^2\int_r^{r_*}\lambda^{-2}\|\Gamma_g\|^2_{2,k}d\lambda\right)^{\frac12}\\ \lesssim\ & \|\check{\underline{\omega}}\|_{2,k}(r_*,u)+\left(\int_r^{r_*}\lambda^{3-\delta_B}\|\check{P}\|^2_{2,k}d\lambda\right)^{1/2}\\ &+\left(\int_r^{r_*}\lambda^{-2}\|(\check{H},\check{Z})\|^2_{2,k}d\lambda+\epsilon^2\int_r^{r_*}\lambda^{-2}\|\Gamma_g\|^2_{2,k}d\lambda\right)^{\frac12}. \end{aligned}$$

Squaring, integrating in u, using (9.128), and the estimate of Step 4 for $\check{Z}$ and the estimate of Step 6 for $\check{H}$, we easily derive

$$\int_{u=1}^{u_*}\left(\int_r^{r_*}\lambda^{-1}\|\check{\underline{\omega}}\|_{2,k}(u,\lambda)d\lambda\right)^2\lesssim\left({}^\star\mathfrak{G}_k+{}^{(ext)}\mathfrak{R}_k+\epsilon_0\right)^2,\quad k\leq k_{large}+7.$$

In view of the above, we infer

$$\sup_{\lambda\geq r_0}\lambda^{-2}\left(\int_{r=\lambda}\left|\mathfrak{d}^{\leq k}(\widetilde{e_3(r)})\right|^2\right)^{\frac12} \lesssim \epsilon_0+{}^\star\mathfrak{G}_k+{}^{(ext)}\mathfrak{R}_k,\quad k\leq k_{large}+7.$$

Step 10. Estimates for $\mathcal{D}\widehat{\otimes}\mathfrak{J}$ and $\widetilde{\mathcal{D}\cdot\overline{\mathfrak{J}}}$.

We make use of equations, see Lemma 9.22,

$$\begin{aligned}
\nabla_4\mathcal{D}\widehat{\otimes}\mathfrak{J}+\frac{2}{q}\mathcal{D}\widehat{\otimes}\mathfrak{J} &= O(r^{-1})B+O(r^{-2})\widetilde{\mathrm{tr}X}+O(r^{-2})\widehat{X}\\
&\quad +O(r^{-2})\check{Z}+O(r^{-3})\widetilde{\mathcal{D}(\cos\theta)},\\
\nabla_4\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}+\Re\left(\frac{2}{q}\right)\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}} &= O(r^{-1})B+O(r^{-2})\widetilde{\mathrm{tr}X}+O(r^{-2})\widehat{X}+O(r^{-2})\check{Z}\\
&\quad +O(r^{-3})\widetilde{\mathcal{D}(\cos\theta)}.
\end{aligned}$$

Using Corollary 9.84 with $c=2$ and $C=2$, the estimates of Steps 1–4 for $\widetilde{\mathrm{tr}X}$, $\widehat{X}$, $\widetilde{\mathcal{D}(\cos\theta)}$ and $\check{Z}$, we easily derive

$$\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}\lambda^2\left|\mathfrak{d}^{\leq k}(\mathcal{D}\widehat{\otimes}\mathfrak{J},\,\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}})\right|^2\right)^{\frac{1}{2}}\lesssim\epsilon_0+{}^\star\mathfrak{G}_k+{}^{(ext)}\mathfrak{R}_k,\quad k\leq k_{large}+7.$$

Step 11. Estimates for $\widetilde{\mathrm{tr}\underline{X}}$.

We make use of the following equation, see Proposition 9.19,

$$\begin{aligned}
\nabla_4\widetilde{\mathrm{tr}\underline{X}}+\frac{1}{q}\widetilde{\mathrm{tr}\underline{X}} &= 2\overline{\check{P}}+F,\\
F &= O(r^{-1})\widetilde{\mathrm{tr}X}+O(r^{-1})\widetilde{\mathcal{D}\cdot\overline{\mathfrak{J}}}+O(r^{-3})\mathcal{D}(r)\\
&\quad +O(r^{-3})\widetilde{\mathcal{D}(\cos\theta)}+\Gamma_b\cdot\Gamma_g.
\end{aligned}$$

Using Proposition 9.83, we infer

$$\|\widetilde{\mathrm{tr}\underline{X}}\|_{2,k}(u,r)\lesssim\|\widetilde{\mathrm{tr}\underline{X}}\|_{2,k}(u,r_*)+\int_r^{r_*}\|\check{P}\|_{2,k}(u,\lambda)d\lambda+\int_r^{r_*}\|F\|_{2,k}(u,\lambda)d\lambda.$$

Now, we have, for $k\leq k_{large}+7$,

$$\begin{aligned}
\int_r^{r_*}\|\check{P}\|_{2,k}(u,\lambda)d\lambda &\lesssim r^{-1+\frac{\delta_B}{2}}\left(\int_r^{r_*}\lambda^{3-\delta_B}\|\check{P}\|^2_{2,k}(u,\lambda)d\lambda\right)^{\frac{1}{2}},\\
\int_{u=1}^{u_*}\left(\int_r^{r_*}\|\check{P}\|_{2,k}(u,\lambda)d\lambda\right)^2 &\lesssim r^{-2+\delta_B}\left({}^{(ext)}\mathfrak{R}_k\right)^2.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\int_r^{r_*}\|F\|_{2,k}(u,\lambda)d\lambda\\
\lesssim\;& r^{-1+\frac{\delta_B}{2}}\left(\int_r^{r_*}\lambda^{3-\delta_B}\|F\|^2_{2,k}(u,\lambda)d\lambda\right)^{\frac{1}{2}}
\end{aligned}$$

$$\lesssim \quad r^{-1+\frac{\delta_B}{2}}\Bigg(\int_r^{r_*}\lambda^{1-\delta_B}\Big(\|(\widecheck{\mathrm{tr}X},\widecheck{\mathcal{D}\cdot\overline{\mathfrak{J}}})\|_{2,k}(u,\lambda)$$
$$+\lambda^{-2}\|(\mathcal{D}r,\widecheck{\mathcal{D}(\cos\theta)})\|_{2,k}(u,\lambda)+\lambda^2\|\Gamma_b\cdot\Gamma_g\|_{2,k}(u,\lambda)\Big)^2\Bigg)^{\frac{1}{2}}.$$

Integrating in u, using (9.128), and the estimates of Steps 1, 3, 5 and 10 respectively for $\widecheck{\mathrm{tr}X}$, $\widecheck{\mathcal{D}(\cos\theta)}$, $\mathcal{D}r$ and $\widecheck{\mathcal{D}\cdot\overline{\mathfrak{J}}}$, we easily derive, for $k\leq k_{large}+7$,

$$\int_{u=1}^{u_*}\left(\int_r^{r_*}\|F\|_{2,k}(u,\lambda)d\lambda\right)^2 \quad\lesssim\quad r^{-2+\delta_B}({}^\star\mathfrak{G}+{}^{(ext)}\mathfrak{R}_k+\epsilon_0)^2.$$

We deduce

$$\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}\lambda^{2-\delta_B}\big|\mathfrak{d}^{\leq k}\widecheck{\mathrm{tr}\underline{X}}\big|^2\right)^{\frac{1}{2}}\lesssim\epsilon_0+{}^\star\mathfrak{G}_k+{}^{(ext)}\mathfrak{R}_k,\qquad k\leq k_{large}+7.$$

Step 12. Estimates for $\widehat{\underline{X}}$.

We make use of the following equation, see Proposition 9.19,

$$\nabla_4\widehat{\underline{X}}+\frac{1}{q}\widehat{\underline{X}} \quad=\quad O(r^{-1})\mathcal{D}\widehat{\otimes}\mathfrak{J}+O(r^{-3})\mathcal{D}r+O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)}+O(r^{-1})\widehat{X}$$
$$+\Gamma_b\cdot\Gamma_g.$$

Using Corollary 9.84 with $c=1$ and $C=1$, (9.128), and the estimates of Steps 2, 3, 5 and 10 respectively for $\widehat{X}$, $\widecheck{\mathcal{D}(\cos\theta)}$, $\mathcal{D}r$ and $\widecheck{\mathcal{D}\cdot\overline{\mathfrak{J}}}$, we easily derive

$$\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}\big|\mathfrak{d}^{\leq k}\widehat{\underline{X}}\big|^2\right)^{\frac{1}{2}}\lesssim\epsilon_0+{}^\star\mathfrak{G}_k+{}^{(ext)}\mathfrak{R}_k,\qquad k\leq k_{large}+7.$$

Step 13. Estimates for $\widecheck{\nabla_3\mathfrak{J}}$.

We make use of the following equation, see Lemma 9.22,

$$\nabla_4\widecheck{\nabla_3\mathfrak{J}}+\frac{1}{q}\widecheck{\nabla_3\mathfrak{J}} \quad=\quad F,$$
$$F \quad=\quad O(r^{-1})\check{P}+O(r^{-3})\widecheck{e_3(r)}+O(r^{-3})e_3(\cos\theta)+O(r^{-2})\underline{\check{\omega}}$$
$$+O(r^{-2})\check{H}+O(r^{-2})\widecheck{\nabla\mathfrak{J}}.$$

Using Proposition 9.83, we infer

$$\|\widecheck{\nabla_3\mathfrak{J}}\|_{2,k}(u,r) \lesssim \|\widecheck{\nabla_3\mathfrak{J}}\|_{2,k}(u,r_*) + \int_r^{r_*}\|F\|_{2,k}(u,\lambda)d\lambda.$$

Now, we have

$$\begin{aligned}\int_r^{r_*}\|F\|_{2,k}(u,\lambda)d\lambda \lesssim\; & r^{-2+\frac{\delta_B}{2}}\left(\int_r^{r_*} r^{3-\delta_{dec}}\|\check{P}\|^2_{2,k}(u,\lambda)d\lambda\right)^{\frac{1}{2}}\\ & +r^{-\frac{1}{2}}\Bigg(\int_r^{r_*} r^{-2}\Big(\|(\check{\underline{\omega}}, \check{H}, \widecheck{\nabla\mathfrak{J}})\|^2_{2,k}(u,\lambda)\\ & +r^{-2}\|(\widecheck{e_3(r)}, e_3(\cos\theta))\|^2_{2,k}(u,\lambda)\Big)d\lambda\Bigg)^{\frac{1}{2}}.\end{aligned}$$

Squaring, integrating in u, using (9.128), and the estimates of Steps 6–10 respectively for $\check{H}$, $e_3(\cos\theta)$, $\check{\underline{\omega}}$, $\widecheck{e_3(r)}$ and $\widecheck{\nabla\mathfrak{J}}$, we easily derive

$$\int_{u=1}^{u_*}\left(\int_r^{r_*}\|F\|_{2,k}(u,\lambda)d\lambda\right)^2 \lesssim r^{-2}({}^\star\mathfrak{G} + {}^{(ext)}\mathfrak{R}_k + \epsilon_0)^2.$$

We deduce

$$\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}\lambda^2\left|\mathfrak{d}^{\leq k}\widecheck{\nabla_3\mathfrak{J}}\right|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0 + {}^\star\mathfrak{G}_k + {}^{(ext)}\mathfrak{R}_k, \qquad k\leq k_{large}+7.$$

Step 14. Estimates for $\underline{\Xi}$.

We make use of the following equation, see Proposition 9.19,

$$\begin{aligned}\nabla_4\underline{\Xi} &= F,\\ F &= O(r^{-1})\check{H} + O(r^{-2})\widecheck{\mathrm{tr}\underline{X}} + O(r^{-2})\widehat{\underline{X}} - \underline{B} + O(r^{-1})\widecheck{\nabla_3\mathfrak{J}}\\ &\quad +O(r^{-3})\widecheck{e_3(r)} + O(r^{-3})e_3(\cos\theta) + \Gamma_b\cdot\Gamma_b.\end{aligned}$$

Using Proposition 9.83, we infer

$$r^{-1}\|\underline{\Xi}\|_{2,k}(u,r) \lesssim r_*^{-1}\|\underline{\Xi}\|_{2,k}(u,r_*) + \int_r^{r_*}\lambda^{-1}\|F\|_{2,k}(u,\lambda)d\lambda.$$

Now, we have

$$\begin{aligned}&\int_r^{r_*}\|F\|_{2,k}(u,\lambda)d\lambda\\ \lesssim\; & r^{-1+\frac{\delta_B}{2}}\left(\int_r^{r_*}\lambda^{1-\delta_B}\|\underline{B}\|^2_{2,k}(u,\lambda)d\lambda\right)^{\frac{1}{2}}\end{aligned}$$

$$+r^{-\frac{1}{2}}\left(\int_r^{r_*}\lambda^{-2}\Big(\|(\check{H},\widecheck{\nabla_3\mathfrak{J}})\|^2_{2,k}(u,\lambda)+\lambda^{-2}\|(\widecheck{\mathrm{tr}\underline{X}},\widehat{\underline{X}})\|^2_{2,k}(u,\lambda)\right.$$
$$\left.+\lambda^{-4}\|(\widecheck{e_3(r)},e_3(\cos\theta))\|^2_{2,k}(u,\lambda)\Big)\right)^{\frac{1}{2}}.$$

Squaring, integrating in u, using (9.128), and the estimates of Steps 6, 7, 9, 11, 12, 13 respectively for $\check{H}$, $e_3(\cos\theta)$, $\widecheck{e_3(r)}$, $\widecheck{\mathrm{tr}\underline{X}}$, $\widehat{\underline{X}}$ and $\widecheck{\nabla_3\mathfrak{J}}$, we easily derive

$$\int_{u=1}^{u_*}\left(\int_r^{r_*}\|F\|_{2,k}(u,\lambda)d\lambda\right)^2 \quad\lesssim\quad r^{-2+\delta_B}\big({}^\star\mathfrak{G}+{}^{(ext)}\mathfrak{R}_k+\epsilon_0\big)^2.$$

We deduce

$$\sup_{\lambda\geq r_0}\left(\int_{r=\lambda}\lambda^{-\delta_B}\big|\mathfrak{d}^{\leq k}\Xi\big|^2\right)^{\frac{1}{2}}\lesssim\epsilon_0+{}^\star\mathfrak{G}_k+{}^{(ext)}\mathfrak{R}_k,\qquad k\leq k_{large}+7.$$

Gathering the estimates derived in Steps 1–14, we infer, in view of the definition of ${}^{(ext)}\mathfrak{G}_k$,

$${}^{(ext)}\mathfrak{G}_k\lesssim\epsilon_0+{}^\star\mathfrak{G}_k+{}^{(ext)}\mathfrak{R}_k,\qquad k\leq k_{large}+7,$$

as stated. This concludes the proof of Proposition 9.82.

9.9. Control of the PT-Ricci coefficients in ${}^{(int)}\mathcal{M}'$

Since we will not need to refer to the old region ${}^{(int)}\mathcal{M}$, defined w.r.t. the PG frame, we drop the prime of ${}^{(int)}\mathcal{M}'$ in this section.

Recall the following norms on ${}^{(int)}\mathcal{M}$ introduced in Section 9.4.1, for $k\leq k_{large}+7$,

$$\begin{aligned}
{}^{(int)}\mathfrak{G}^2_k &= \int_{{}^{(int)}\mathcal{M}}\big|\mathfrak{d}^{\leq k}\check{\Gamma}\big|^2,\\
{}^{(int)}\mathfrak{R}^2_k &= \int_{{}^{(int)}\mathcal{M}}\Big(\big|\nabla_{\widehat{R}}\mathfrak{d}^{\leq k-1}\check{R}\big|^2+|\mathfrak{d}^{\leq k-1}\check{R}|^2\Big)+\sup_\tau\int_{{}^{(int)}\mathcal{M}\cap\Sigma(\tau)}|\mathfrak{d}^{\leq k}\check{R}|^2,
\end{aligned}$$

where $\check{\Gamma}$ denotes the set of all linearized Ricci and metric coefficients with respect to the ingoing PT frame of ${}^{(int)}\mathcal{M}$, i.e.

$$\check{\Gamma}\quad=\quad\Big\{\widecheck{\mathrm{tr}\underline{X}},\widehat{\underline{X}},\check{Z},\check{\underline{H}},\widecheck{\mathcal{D}\cos\theta},\check{\omega},\mathcal{D}r,\widecheck{\mathcal{D}u},e_4(\cos\theta),\widecheck{e_4(r)},\widecheck{e_4(\underline{u})},$$

$$\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}, \mathcal{D}\widehat{\otimes}\mathfrak{J}, \widetilde{\nabla_4\mathfrak{J}}, \widetilde{\mathrm{tr}X}, \widehat{X}, \Xi\Big\},$$

where $\check{R}$ denotes the set of all linearized curvature components with respect to the ingoing PT frame of ${}^{(int)}\mathcal{M}$, i.e.

$$\check{R} = \{A, B, \check{P}, \underline{B}, \underline{A}\},$$

and where the vectorfield $\widehat{R}$ in ${}^{(int)}\mathcal{M}$ is given by, see (9.24),

$$\widehat{R} = \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4 - \frac{\Delta}{r^2+a^2}e_3\right).$$

Remark 9.85. *Recall that $r \leq r_0$ in ${}^{(int)}\mathcal{M}$ so that r is uniformly bounded in that region. In particular, all components have the same behavior in ${}^{(int)}\mathcal{M}$ which is reflected in the definition of the norms ${}^{(int)}\mathfrak{G}_k$ and ${}^{(int)}\mathfrak{R}_k$.*

The goal of this section is to provide the proof of Proposition 9.52. For convenience, we restate the result below.

Proposition 9.86 (Control of $\check{\Gamma}$ in ${}^{(int)}\mathcal{M}$)**.** *Relative to the PT frame of ${}^{(int)}\mathcal{M}$, we have*

$$^{(int)}\mathfrak{G}_k \lesssim \epsilon_0 + {}^{(ext)}\mathfrak{G}_k + {}^{(int)}\mathfrak{R}_k, \qquad k \leq k_{large}+7.$$

To prove Proposition 9.86, we rely in particular on our bootstrap assumption **BA-PT**, see (9.40), which implies for the Ricci and metric coefficients of the ingoing PT frame of ${}^{(int)}\mathcal{M}$

$$^{(int)}\mathfrak{G}_k \leq \epsilon, \qquad k \leq k_{large}+7. \tag{9.135}$$

9.9.1. Preliminaries Recall that the following identities hold for an ingoing PT structure

$$\underline{\xi} = 0, \quad \underline{\omega} = 0, \quad H = \frac{aq}{|q|^2}\mathfrak{J}, \quad e_3(r) = -1, \quad e_3(\underline{u}) = e_3(\theta) = 0, \quad \nabla_3\mathfrak{J} = \frac{1}{q}\mathfrak{J}.$$

In view of Definition 9.25 and the notation $\check{\Gamma}$, we have

$$e_4(r) = \frac{\Delta}{|q|^2} + \check{\Gamma}, \quad \nabla(r) = \check{\Gamma}, \quad e_4(\underline{u}) = \frac{2(r^2+a^2)}{|q|^2} + \check{\Gamma}, \quad \mathcal{D}\underline{u} = a\mathfrak{J} + \check{\Gamma}.$$

Also, note from the definition of $\widehat{R}$ that

$$\widehat{R}(r) = \frac{\Delta}{r^2+a^2} + \check{\Gamma}, \qquad e_4 = \frac{2(r^2+a^2)}{|q|^2}\widehat{R} + \frac{\Delta}{|q|^2}e_3.$$

9.9.1.1. The region ${}^{(int)}\mathcal{M}_*$ We recall the scalar function τ constructed in Proposition 9.29 whose level sets are uniformly spacelike in ${}^{(int)}\mathcal{M}$, and the region

$$ {}^{(int)}\mathcal{M}_* = {}^{(int)}\mathcal{M} \cap \{\tau \le \tau_*\}, \tag{9.136}$$

see Definition 9.33. In particular, the boundary of ${}^{(int)}\mathcal{M}_*$ is given by

$$\partial\, {}^{(int)}\mathcal{M}_* = \mathcal{A} \cup {}^{(int)}\Sigma_* \cup \mathcal{T} \cup \{\underline{u} = 1\} \tag{9.137}$$

where $\mathcal{A}$ and

$$ {}^{(int)}\Sigma_* = {}^{(int)}\mathcal{M} \cap \{\tau = \tau_*\} \tag{9.138}$$

are strictly spacelike, while $\mathcal{T} = \{r = r_0\}$ and $\{\underline{u} = 1\}$ are timelike, with $\{\underline{u} = 1\}$ included in the initial data layer.

9.9.1.2. A simple computation The following simple computation will be useful in Lemma 9.93.

Lemma 9.87. *For any function f, we have*

$$\boldsymbol{Div}(fe_3) \;=\; e_3(f) - \frac{2r}{|q|^2}f + f\check{\Gamma}.$$

Proof. We have, with $\pi^{(3)}$ the deformation tensor of e_3,

$$\mathbf{Div}(e_3) = \frac{1}{2}\mathrm{tr}\pi^{(3)} = \frac{1}{2}\left(\delta_{ab}\pi^{(3)}_{ab} - \pi^{(3)}_{34}\right).$$

Observe that

$$\begin{aligned}
\pi^{(3)}_{34} &= \mathbf{g}(\mathbf{D}_3e_3, e_4) + \mathbf{g}(\mathbf{D}_4e_3, e_3) = 0,\\
\pi^{(3)}_{ab} &= \mathbf{g}(\mathbf{D}_ae_3, e_b) + \mathbf{g}(\mathbf{D}_be_3, e_a) = \underline{\chi}_{ab} + \underline{\chi}_{ba} = \mathrm{tr}\,\underline{\chi}\delta_{ab} + 2\widehat{\underline{\chi}}_{ab}.
\end{aligned}$$

This yields

$$\mathbf{Div}(e_3) = \mathrm{tr}\,\underline{\chi} = -\frac{2r}{|q|^2} + \check{\Gamma}.$$

Thus,

$$\mathbf{Div}(fe_3) = f\mathbf{Div}(e_3) + e_3(f) = -\frac{2r}{|q|^2}f + f\check{\Gamma} + e_3(f)$$

as stated. □

9.9.1.3. Commutation formulas

Lemma 9.88. *Given U a horizontal tensor in ${}^{(int)}\mathcal{M}$, we have*[218]

$$\begin{aligned}
[\nabla_3, \nabla]U &= O(1)\nabla U + \check{\Gamma}\mathfrak{d}U + \Big(O(1) + \check{\Gamma} + \check{R}\Big)U, \\
[\nabla_3, \nabla_4]U &= O(1)\nabla U + O(1)\nabla_3 U + \check{\Gamma}\mathfrak{d}U + \Big(O(1) + \check{\Gamma} + \check{R}\Big)U.
\end{aligned}$$

Proof. In view of Lemma 2.2, we have in ${}^{(int)}\mathcal{M}$, for a real horizontal tensor in ${}^{(int)}\mathcal{M}$,

$$\begin{aligned}
[\nabla_3, \nabla_b]U &= -\underline{\chi}_{bc}\nabla_c U + (\eta_b - \zeta_b)\nabla_3 U + \underline{\xi}_b\nabla_4 U + \Big(O(1) + \check{\Gamma} + \check{R}\Big)U, \\
[\nabla_4, \nabla_3]U &= 2(\underline{\eta}_b - \eta_b)\nabla_b U + 2\omega\nabla_3 U - 2\underline{\omega}\nabla_4 U + \Big(O(1) + \check{\Gamma} + \check{R}\Big)U.
\end{aligned}$$

Since, in view of the identities for ingoing PT structures, we have $\underline{\omega} = 0$, $\underline{\xi} = 0$, and $\zeta - \eta = \check{\zeta}$, we infer

$$\begin{aligned}
[\nabla_3, \nabla]U &= O(1)\nabla U + \check{\Gamma}\mathfrak{d}U + \Big(O(1) + \check{\Gamma} + \check{R}\Big)U, \\
[\nabla_4, \nabla_3]U &= O(1)\nabla U + O(1)\nabla_3 U + \check{\Gamma}\mathfrak{d}U + \Big(O(1) + \check{\Gamma} + \check{R}\Big)U,
\end{aligned}$$

as stated. □

Corollary 9.89. *Given U a complex horizontal tensor in ${}^{(int)}\mathcal{M}$ we have*

$$[\nabla_3, \nabla_{\widehat{R}}]U = O(1)\nabla U + O(1)\nabla_3 U + O(1)\nabla_{\widehat{R}}U + \check{\Gamma}\mathfrak{d}U + \Big(O(1) + \check{\Gamma} + \check{R}\Big)U.$$

Proof. Recall that we have

$$\widehat{R} = \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4 - \frac{\Delta}{r^2+a^2}e_3\right),$$

so that

[218] Recall that $r \le r_0$ in ${}^{(int)}\mathcal{M}$ so that weights in r do not matter and are hence dropped.

$$
\begin{aligned}
[\nabla_3, \nabla_{\widehat{R}}] &= \frac{1}{2}\left(e_3\left(\frac{|q|^2}{r^2+a^2}\right)\nabla_4 - e_3\left(\frac{\Delta}{r^2+a^2}\right)\nabla_3\right) + \frac{1}{2}\frac{|q|^2}{r^2+a^2}[\nabla_3, \nabla_4]\\
&= O(1)\nabla_{\widehat{R}} + O(1)\nabla_3 + O(1)[\nabla_3, \nabla_4]
\end{aligned}
$$

and the corollary follows immediately from the formula for $[\nabla_3, \nabla_4]$ in Lemma 9.88. □

9.9.2. Commuted structure equations In this section, we rely on the commutation formulas of Lemma 9.88 and Corollary 9.89 to derive the equations for higher order derivatives of the equations for the Ricci coefficients and metric coefficients of the ingoing PT structure of ${}^{(int)}\mathcal{M}$. We start with the following proposition concerning the schematic form of the uncommuted equations in ${}^{(int)}\mathcal{M}$.

Proposition 9.90. *In the PT structure of ${}^{(int)}\mathcal{M}$, the equations of Proposition 9.27 for the Ricci and metric coefficients take the following form*[219]*:*

1. *$\widehat{\underline{X}}$ satisfies a transport equation of the following form*

$$
\nabla_3\widehat{\underline{X}} = O(1)\widehat{X} - \underline{A} + \check{\Gamma}\cdot\check{\Gamma}. \tag{9.139}
$$

2. *If Φ is among the quantities $\widecheck{tr\underline{X}}$, $\widehat{X}$, $\widecheck{\mathcal{D}\cos\theta}$, $\mathcal{D}r$, $e_4(\cos\theta)$, $\widecheck{e_4(r)}$, then Φ satisfies a transport equation of the following form*

$$
\nabla_3\Phi = O(1)\Phi + \check{\Gamma}_0[\Phi] + \check{\Gamma}\cdot\check{\Gamma}, \tag{9.140}
$$

where

$$
\begin{aligned}
\check{\Gamma}_0[\widecheck{tr\underline{X}}] &= 0, & \check{\Gamma}_0[\widehat{X}] &= \left(\mathcal{D}\widehat{\otimes}\mathfrak{J},\, \widecheck{\mathcal{D}(\cos\theta)},\, \widehat{\underline{X}}\right),\\
\check{\Gamma}_0[\widecheck{\mathcal{D}\cos\theta}] &= \left(\widecheck{tr\underline{X}},\, \widehat{\underline{X}}\right), & \check{\Gamma}_0[\mathcal{D}r] &= \check{Z},\\
\check{\Gamma}_0[e_4(\cos\theta)] &= \left(\widecheck{\underline{H}}, \widecheck{\mathcal{D}(\cos\theta)}\right), & \check{\Gamma}_0[\widecheck{e_4(r)}] &= \left(\check{\omega}, \mathcal{D}r\right).
\end{aligned}
$$

3. *If Φ is among the quantities $\check{Z}$, $\widecheck{\underline{H}}$, $\widecheck{trX}$, $\check{\omega}$, Ξ, $\mathcal{D}\widehat{\otimes}\mathfrak{J}$, $\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}$, $\widecheck{\nabla_4\mathfrak{J}}$, then Φ satisfies a transport equation of the following form*

$$
\nabla_3\Phi = O(1)\Phi + \check{\Gamma}_0[\Phi] + \check{R}_0[\Phi] + \check{\Gamma}\cdot\check{\Gamma}, \tag{9.141}
$$

where $\check{R}_0[\Phi]$ is a curvature component among the list

[219] Recall that $r \le r_0$ in ${}^{(int)}\mathcal{M}$ so that weights in r do not matter and are hence dropped.

$$\check{R}_0[\Phi] \;=\; \Big\{B,\ \check{P},\ \underline{B}\Big\}, \tag{9.142}$$

and

$$\begin{aligned}
\check{\Gamma}_0[\check{Z}] &= \Big(\widehat{\underline{X}},\ \widecheck{tr\underline{X}}\Big),\\
\check{\Gamma}_0[\widecheck{\underline{H}}] &= \Big(\widehat{\underline{X}},\ \widecheck{tr\underline{X}}\Big),\\
\check{\Gamma}_0[\widecheck{trX}] &= \Big(\widecheck{tr\underline{X}},\ \widecheck{\mathcal{D}\cdot\overline{\mathfrak{J}}},\ \mathcal{D}r,\ \widecheck{\mathcal{D}(\cos\theta)}\Big),\\
\check{\Gamma}_0[\check{\omega}] &= \Big(\check{Z},\ \widecheck{\underline{H}}\Big),\\
\check{\Gamma}_0[\Xi] &= \Big(\widecheck{\underline{H}},\ \widecheck{trX},\ \widehat{X},\ \widecheck{\nabla_4\mathfrak{J}},\ \widecheck{e_4(r)},\ e_4(\cos\theta)\Big),\\
\check{\Gamma}_0[\mathcal{D}\widehat{\otimes}\mathfrak{J}] &= \Big(\widecheck{tr\underline{X}},\ \widehat{\underline{X}},\ \check{Z},\ \widecheck{\mathcal{D}(\cos\theta)}\Big),\\
\check{\Gamma}_0[\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}] &= \Big(\widecheck{tr\underline{X}},\ \widehat{\underline{X}},\ \check{Z},\ \widecheck{\mathcal{D}(\cos\theta)}\Big),\\
\check{\Gamma}_0[\widecheck{\nabla_4\mathfrak{J}}] &= \Big(\widecheck{e_4(r)},\ e_4(\cos\theta),\ \check{\omega},\ \widecheck{\underline{H}},\ \widecheck{\nabla\mathfrak{J}}\Big).
\end{aligned} \tag{9.143}$$

In addition, the linearized Codazzi equation for $\widehat{\underline{X}}$ in Proposition 9.28

$$\overline{\mathcal{D}}\cdot\widehat{\underline{X}} \;=\; \mathcal{D}\widecheck{\overline{tr\underline{X}}} + \check{R} + \check{\Gamma} + \check{\Gamma}\cdot\check{\Gamma}.$$

Proof. This follows immediately from Proposition 9.27. □

We now commute the equations of Proposition 9.90.

Proposition 9.91. *Let $1 \le k \le k_{large}$ an integer. In the PT structure of ${}^{(int)}\mathcal{M}$, the commutation of the equation of Proposition 9.90 for the Ricci and metric coefficients take the following form:*

1. *$\widehat{\underline{X}}$ satisfies a transport equation of the following form*

$$\begin{aligned}
\nabla_3(\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\widehat{\underline{X}}) \;=\;& O(1)\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\widehat{\underline{X}} + O(1)\nabla\mathfrak{d}^{k-1}\widehat{\underline{X}}\\
&+\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\check{R} + \mathfrak{d}^{\le k-1}(\check{R},\check{\Gamma}) + \mathfrak{d}^{\le k}(\check{\Gamma}\cdot\check{\Gamma}).
\end{aligned} \tag{9.144}$$

2. *If Φ is among the quantities $\widecheck{tr\underline{X}}$, $\widehat{X}$, $\widecheck{\mathcal{D}\cos\theta}$, $\mathcal{D}r$, $e_4(\cos\theta)$, $\widecheck{e_4(r)}$, then Φ satisfies a transport equation of the following form*

$$\begin{aligned}
\nabla_3(\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi) \;=\;& O(1)\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi + O(1)\nabla\mathfrak{d}^{k-1}\Phi + \check{\Gamma}_k[\Phi]\\
&+\mathfrak{d}^{\le k-1}(\check{R},\check{\Gamma}) + \mathfrak{d}^{\le k}(\check{\Gamma}\cdot\check{\Gamma}),
\end{aligned} \tag{9.145}$$

and

$$
\begin{aligned}
\nabla_3(\nabla\mathfrak{d}^{k-1}\Phi) \;&=\; O(1)\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi + O(1)\nabla\mathfrak{d}^{k-1}\Phi + \check{\Gamma}_k[\Phi]\\
&\quad +\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma}) + \mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma}),
\end{aligned}
\tag{9.146}
$$

where

$$
\begin{aligned}
\check{\Gamma}_k[\widecheck{tr\underline{X}}] &= 0, & \check{\Gamma}_k[\widehat{X}] &= \mathfrak{d}^k\Big(\mathcal{D}\widehat{\otimes}\mathfrak{J},\, \widecheck{\mathcal{D}(\cos\theta)},\, \widehat{\underline{X}}\Big),\\
\check{\Gamma}_k[\widecheck{\mathcal{D}\cos\theta}] &= \mathfrak{d}^k\Big(\widecheck{tr\underline{X}},\, \widehat{\underline{X}}\Big), & \check{\Gamma}_k[\mathcal{D}r] &= \mathfrak{d}^k\check{Z},\\
\check{\Gamma}_k[e_4(\cos\theta)] &= \mathfrak{d}^k\Big(\widecheck{\underline{H}}, \widecheck{\mathcal{D}(\cos\theta)}\Big), & \check{\Gamma}_k[\widecheck{e_4(r)}] &= \mathfrak{d}^k\Big(\check{\omega}, \mathcal{D}r\Big).
\end{aligned}
$$

3. *If* Φ *is among the quantities* $\check{Z}$, $\widecheck{\underline{H}}$, $\widecheck{trX}$, $\check{\omega}$, Ξ, $\mathcal{D}\widehat{\otimes}\mathfrak{J}$, $\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}$, $\widecheck{\nabla_4\mathfrak{J}}$, *then* Φ *satisfies transport equations of the following form*

$$
\begin{aligned}
\nabla_3(\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi) \;&=\; O(1)\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi + O(1)\nabla\mathfrak{d}^{k-1}\Phi + \check{\Gamma}_k[\Phi]\\
&\quad +\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\check{R} + \mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma}) + \mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma}),
\end{aligned}
\tag{9.147}
$$

and

$$
\begin{aligned}
\nabla_3(\nabla\mathfrak{d}^{k-1}\Phi + \mathfrak{d}^{k-1}\check{R}) \;&=\; O(1)\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi + O(1)\nabla\mathfrak{d}^{k-1}\Phi + \check{\Gamma}_k[\Phi]\\
&\quad +\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\check{R} + \mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma}) + \mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma}),
\end{aligned}
\tag{9.148}
$$

where

$$
\begin{aligned}
\check{\Gamma}_k[\check{Z}] &= \mathfrak{d}^k\Big(\widehat{\underline{X}},\, \widecheck{tr\underline{X}}\Big),\\
\check{\Gamma}_k[\widecheck{\underline{H}}] &= \mathfrak{d}^k\Big(\widehat{\underline{X}},\, \widecheck{tr\underline{X}}\Big),\\
\check{\Gamma}_k[\widecheck{trX}] &= \mathfrak{d}^k\Big(\widecheck{tr\underline{X}},\, \widecheck{\mathcal{D}\cdot\overline{\mathfrak{J}}},\, \mathcal{D}r,\, \widecheck{\mathcal{D}(\cos\theta)}\Big),\\
\check{\Gamma}_k[\check{\omega}] &= \mathfrak{d}^k\Big(\check{Z},\, \widecheck{\underline{H}}\Big),\\
\check{\Gamma}_k[\Xi] &= \mathfrak{d}^k\Big(\widecheck{\underline{H}},\, \widecheck{trX},\, \widehat{X},\, \widecheck{\nabla_4\mathfrak{J}},\, \widecheck{e_4(r)},\, e_4(\cos\theta)\Big),\\
\check{\Gamma}_k[\mathcal{D}\widehat{\otimes}\mathfrak{J}] &= \mathfrak{d}^k\Big(\widecheck{tr\underline{X}},\, \widehat{\underline{X}},\, \check{Z},\, \widecheck{\mathcal{D}(\cos\theta)}\Big),\\
\check{\Gamma}_k[\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}] &= \mathfrak{d}^k\Big(\widecheck{tr\underline{X}},\, \widehat{\underline{X}},\, \check{Z},\, \widecheck{\mathcal{D}(\cos\theta)}\Big),\\
\check{\Gamma}_k[\widecheck{\nabla_4\mathfrak{J}}] &= \mathfrak{d}^k\Big(\widecheck{e_4(r)},\, e_4(\cos\theta),\, \check{\omega},\, \widecheck{\underline{H}},\, \widecheck{\nabla\mathfrak{J}}\Big).
\end{aligned}
\tag{9.149}
$$

In addition, we have

$$\overline{\mathcal{D}}\cdot(\mathfrak{d}^{k-1}\widehat{\underline{X}}) \quad = \quad \mathfrak{d}^k\widecheck{tr\underline{X}}+\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k-1}(\check{\Gamma}\cdot\check{\Gamma}). \tag{9.150}$$

Proof. We focus on (9.148) as the other commuted equations follow immediately from the corresponding uncommuted equation in Proposition 9.90 and the commutation formulas of Lemma 9.88 and Corollary 9.89.

Recall that the case of (9.148), Φ satisfies according to Proposition 9.90 the following uncommuted equation

$$\nabla_3\Phi \quad = \quad O(1)\Phi+\check{\Gamma}_0[\Phi]+\check{R}_0[\Phi]+\check{\Gamma}\cdot\check{\Gamma},$$

where $\check{R}_0[\Phi]$ is a curvature component among the list $\{B,\ \check{P},\ \underline{B}\}$. Thus, using the commutation formulas of Lemma 9.88 and Corollary 9.89, we immediately infer

$$\begin{aligned}\nabla_3(\nabla\mathfrak{d}^{k-1}\Phi) \quad = \quad & O(1)\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi+O(1)\nabla\mathfrak{d}^{k-1}\Phi+O(1)\nabla_3\mathfrak{d}^{k-1}\Phi+\check{\Gamma}_k[\Phi]\\ &+\mathfrak{d}^{k-1}\nabla\check{R}_0[\Phi]+\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma}).\end{aligned}$$

Also, using

$$\nabla_3(\mathfrak{d}^{k-1}\Phi)=\mathfrak{d}^{\leq k-1}\check{\Gamma}+\mathfrak{d}^{\leq k-1}\check{R}+\mathfrak{d}^{\leq k-1}(\check{\Gamma}\cdot\check{\Gamma}),$$

we may get rid of the term $O(1)\nabla_3\mathfrak{d}^{k-1}\Phi$ on the RHS and obtain

$$\begin{aligned}\nabla_3(\nabla\mathfrak{d}^{k-1}\Phi) \quad = \quad & O(1)\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi+O(1)\nabla\mathfrak{d}^{k-1}\Phi+\check{\Gamma}_k[\Phi]\\ &+\mathfrak{d}^{k-1}\nabla\check{R}_0[\Phi]+\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma}),\end{aligned}$$

where the only term of the right-hand side which does not agree with (9.148) is $\mathfrak{d}^{k-1}\nabla\check{R}_0[\Phi]$.

Next, in view of Proposition 9.28, we may rewrite some of the linearized Bianchi identities in the following schematic form

$$\begin{aligned}\mathcal{D}\widehat{\otimes}B &= \nabla_3\check{R}+\check{R}+\check{\Gamma}+\check{R}\cdot\check{\Gamma}, & \mathcal{D}\cdot\overline{B} &= \nabla_4\check{R}+\check{R}+\check{\Gamma}+\check{R}\cdot\check{\Gamma},\\ \mathcal{D}\overline{\check{P}} &= \nabla_3\check{R}+\check{R}+\check{\Gamma}+\check{R}\cdot\check{\Gamma}, & \mathcal{D}\check{P} &= \nabla_4\check{R}+\check{R}+\check{\Gamma}+\check{R}\cdot\check{\Gamma},\\ \overline{\mathcal{D}}\cdot\underline{B} &= \nabla_3\check{R}+\check{R}+\check{\Gamma}+\check{R}\cdot\check{\Gamma}, & \mathcal{D}\widehat{\otimes}\underline{B} &= \nabla_4\check{R}+\check{R}+\check{\Gamma}+\check{R}\cdot\check{\Gamma}.\end{aligned}$$

Now, note that we have

$$\nabla\check{\rho}=\frac{1}{2}\Big(\Re(\mathcal{D}\check{P})+\Re(\mathcal{D}\overline{\check{P}})\Big), \qquad \nabla\widecheck{{}^*\rho}=\frac{1}{2}\Big(\Im(\mathcal{D}\check{P})-\Im(\mathcal{D}\overline{\check{P}})\Big),$$

so that $\mathcal{D}\check{P}$ and $\mathcal{D}\overline{\check{P}}$ generate any angular derivative of $\check{P}$. Thus, in view of the above schematic linearized Bianchi identities, we have

$$\nabla\check{P} = \nabla_3\check{R} + \nabla_4\check{R} + \check{R} + \check{\Gamma} + \check{R}\cdot\check{\Gamma}.$$

Also, note that for a complex anti-selfdual 1-form $F = f + i\,{}^*f$, we have

$$\mathrm{div}\,(f) = \frac{1}{2}\Re(\overline{\mathcal{D}}\cdot F), \qquad \mathrm{curl}\,(f) = \frac{1}{2}\Im(\overline{\mathcal{D}}\cdot F), \qquad \nabla\widehat{\otimes}f = \Re(\mathcal{D}\widehat{\otimes}F),$$

and

$$\nabla_a f_b = \frac{1}{2}\mathrm{div}\,(f)\delta_{ab} + \frac{1}{2}\mathrm{curl}\,(f)\in_{ab} + \frac{1}{2}(\nabla\widehat{\otimes}\beta)_{ab},$$

so that $\overline{\mathcal{D}}\cdot F$ and $\mathcal{D}\widehat{\otimes}F$ generate any angular derivative of F. Thus, in view of the above schematic linearized Bianchi identities, we have

$$\begin{aligned}
\nabla B &= \nabla_3\check{R} + \nabla_4\check{R} + \check{R} + \check{\Gamma} + \check{R}\cdot\check{\Gamma},\\
\nabla\underline{B} &= \nabla_3\check{R} + \nabla_4\check{R} + \check{R} + \check{\Gamma} + \check{R}\cdot\check{\Gamma}.
\end{aligned}$$

Since $\check{R}_0[\Phi]$ is a curvature component among the list $\{B, \check{P}, \underline{B}\}$, we infer from the above identities for $\nabla\check{P}$, ∇B and $\nabla\underline{B}$ that we have

$$\nabla\check{R}_0[\Phi] = \nabla_3\check{R} + \nabla_4\check{R} + \check{R} + \check{\Gamma} + \check{R}\cdot\check{\Gamma}.$$

Also, since e_4 is in the span of $\widehat{R}$ and e_3, we infer

$$\nabla\check{R}_0[\Phi] = \nabla_3\check{R} + \nabla_{\widehat{R}}\check{R} + \check{R} + \check{\Gamma} + \check{R}\cdot\check{\Gamma}.$$

Next, recall from the above that we have

$$\begin{aligned}
\nabla_3(\nabla\mathfrak{d}^{k-1}\Phi) &= O(1)\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi + O(1)\nabla\mathfrak{d}^{k-1}\Phi + \check{\Gamma}_k[\Phi]\\
&\quad +\mathfrak{d}^{k-1}\nabla\check{R}_0[\Phi] + \mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma}) + \mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma}).
\end{aligned}$$

Plugging the above identity for $\nabla\check{R}_0[\Phi]$, we infer

$$\begin{aligned}
\nabla_3(\nabla\mathfrak{d}^{k-1}\Phi) &= O(1)\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi + O(1)\nabla\mathfrak{d}^{k-1}\Phi + \check{\Gamma}_k[\Phi]\\
&\quad +\mathfrak{d}^{k-1}\nabla_3\check{R} + \mathfrak{d}^{k-1}\nabla_{\widehat{R}}\check{R} + \mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma}) + \mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma}).
\end{aligned}$$

Using again the commutation formulas of Lemma 9.88 and Corollary 9.89, we infer

$$\nabla_3(\nabla\mathfrak{d}^{k-1}\Phi) = O(1)\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi + O(1)\nabla\mathfrak{d}^{k-1}\Phi + \check{\Gamma}_k[\Phi]$$

$$+\nabla_3\mathfrak{d}^{k-1}\check{R}+\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\check{R}+\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma}),$$

and hence

$$\begin{aligned}\nabla_3(\nabla\mathfrak{d}^{k-1}\Phi+\mathfrak{d}^{k-1}\check{R}) &= O(1)\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi+O(1)\nabla\mathfrak{d}^{k-1}\Phi+\check{\Gamma}_k[\Phi]\\ &\quad +\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\check{R}+\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma}),\end{aligned}$$

as stated in (9.148). This concludes the proof of Proposition 9.91. □

9.9.3. Estimates for transport equations We shall make use of the following divergence lemma in ${}^{(int)}\mathcal{M}_*$.

Lemma 9.92 (Divergence Lemma in ${}^{(int)}\mathcal{M}_*$). *Consider a vectorfield X in ${}^{(int)}\mathcal{M}$ and the region ${}^{(int)}\mathcal{M}_*={}^{(int)}\mathcal{M}\cap\{\tau\leq\tau_*\}$. We have,*

$$\begin{aligned}&\int_{{}^{(int)}\Sigma_*}\mathbf{g}(X,N)+\int_{\mathcal{T}}\mathbf{g}(X,N)+\int_{\{\underline{u}=1\}}\mathbf{g}(X,N)+\int_{\mathcal{A}}\mathbf{g}(X,N)\\ =\ &\int_{{}^{(int)}\mathcal{M}_*}\boldsymbol{Div}(X),\end{aligned}$$

where N is the exterior unit normal to each portion of the boundary.

Proof. This follows immediately from the standard divergence lemma and the fact that the region ${}^{(int)}\mathcal{M}_*$ has the boundary $\partial\,{}^{(int)}\mathcal{M}_*=\mathcal{A}\cup{}^{(int)}\Sigma_*\cup\mathcal{T}\cup\{\underline{u}=1\}$. □

Next we state an application of the divergence theorem to transport equations.

Lemma 9.93. *Suppose Φ and F are two anti-selfdual horizontal tensors of the same type satisfying*

$$\nabla_3\Phi=F. \tag{9.151}$$

Then, we have on ${}^{(int)}\mathcal{M}_*$, for any real number p,

$$\begin{aligned}&\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|q|^p|\Phi|^2|\mathbf{g}(e_3,N)|+p\int_{{}^{(int)}\mathcal{M}_*}r|q|^{p-2}|\Phi|^2\\ \leq\ &\int_{{}^{(int)}\mathcal{M}_*}r^{-1}|q|^{p+2}|F|^2+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|q|^{p-2}|\Phi|^2|\mathbf{g}(e_3,N)|.\end{aligned} \tag{9.152}$$

Remark 9.94. *Recall that $r\leq r_0$ on ${}^{(int)}\mathcal{M}$ so that powers of r are a priori irrelevant. They will be however useful in Proposition 9.95 to produce* *an arbitrary large positive bulk term on the LHS, by choosing p large enough in* (9.152)*, to allow to absorb terms on the RHS. This is analogous to the use of a Gronwall lemma for the transport equation* (9.151)*.*

Proof. Let p a real number. In view of Lemma 9.87, we have

$$\mathbf{Div}(|q|^p|\Phi|^2 e_3) = e_3(|q|^p|\Phi|^2) - \frac{2r}{|q|^2}|q|^p|\Phi|^2 + |q|^p|\Phi|^2\check{\Gamma}.$$

Also, we have, since $e_3(|q|) = -\frac{r}{|q|}$,

$$e_3(|q|^p|\Phi|^2) = -pr|q|^{p-2}|\Phi|^2 + 2|q|^p\Re(\overline{\Phi}\cdot\nabla_3\Phi)$$

so that for Φ satisfying (9.151), we have

$$e_3(|q|^p|\Phi|^2) = -pr|q|^{p-2}|\Phi|^2 + 2|q|^p\Re(\overline{\Phi}\cdot F).$$

We deduce

$$\mathbf{Div}(|q|^p|\Phi|^2 e_3) = -(p+2)r|q|^{p-2}|\Phi|^2 + 2|q|^p\Re(\overline{\Phi}\cdot F) + |q|^p|\Phi|^2\check{\Gamma}$$

and hence, using in particular $\check{\Gamma} = O(\epsilon)$,

$$\mathbf{Div}(|q|^p|\Phi|^2 e_3) \leq -\Big(p+1+O(\epsilon)\Big)r|q|^{p-2}|\Phi|^2 + r^{-1}|q|^{p+2}|F|^2.$$

Integrating on ${}^{(int)}\mathcal{M}_*$, we infer, for $\epsilon > 0$ small enough,

$$\begin{aligned}&\int_{^{(int)}\Sigma_*}|q|^p|\Phi|^2\mathbf{g}(e_3,N) + \int_{\mathcal{A}}|q|^p|\Phi|^2\mathbf{g}(e_3,N) + p\int_{^{(int)}\mathcal{M}_*} r|q|^{p-2}|\Phi|^2\\ \leq&\int_{^{(int)}\mathcal{M}_*} r^{-1}|q|^{p+2}|F|^2 + \int_{\mathcal{T}}|q|^p|\Phi|^2|\mathbf{g}(e_3,N)| + \int_{\{\underline{u}=1\}}|q|^p|\Phi|^2|\mathbf{g}(e_3,N)|.\end{aligned}$$

Since both ${}^{(int)}\Sigma_*$ and $\mathcal{A}$ are spacelike, and taking the orientation into account, we have $\mathbf{g}(e_3,N) > 0$ in both cases, and hence

$$\begin{aligned}&\int_{^{(int)}\Sigma_*}|q|^p|\Phi|^2|\mathbf{g}(e_3,N)| + \int_{\mathcal{A}}|q|^p|\Phi|^2|\mathbf{g}(e_3,N)| + p\int_{^{(int)}\mathcal{M}_*} r^{-1}|q|^{p-2}|\Phi|^2\\ \leq&\int_{^{(int)}\mathcal{M}_*} r^{-1}|q|^{p+2}|F|^2 + \int_{\mathcal{T}}|q|^p|\Phi|^2|\mathbf{g}(e_3,N)| + \int_{\{\underline{u}=1\}}|q|^p|\Phi|^2|\mathbf{g}(e_3,N)|\end{aligned}$$

as stated. □

We now apply Lemma 9.93 to the transport equations of Proposition 9.91.

Proposition 9.95. *If Φ is any quantity among $\check{\Gamma}\setminus\{\widehat{\underline{X}}\}$, i.e. Φ is any linearized Ricci or metric coefficient of the ingoing PT structure of ${}^{(int)}\mathcal{M}$ except $\widehat{\underline{X}}$, then Φ satisfies, for $k\leq k_{large}+7$,*

$$\begin{aligned}\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|\mathfrak{d}^k\Phi|^2+\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\Phi|^2 \quad\lesssim\quad & \Big(\epsilon_0+{}^{(int)}\mathfrak{R}_k+{}^{(int)}\mathfrak{G}_{k-1}\Big)^2\\ &+\int_{{}^{(int)}\mathcal{M}_*}|\check{\Gamma}_k[\Phi]|^2+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|\mathfrak{d}^k\Phi|^2.\end{aligned}\tag{9.153}$$

Proof. We start with the case where Φ is among the quantities $\widecheck{\mathrm{tr}\underline{X}}$, $\widehat{X}$, $\widecheck{\mathcal{D}\cos\theta}$, $\mathcal{D}r$, $e_4(\cos\theta)$, $\widecheck{e_4(r)}$. Then, according to Proposition 9.91. Φ satisfies a transport equations of the following form

$$\begin{aligned}\nabla_3(\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi) &= O(1)\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi+O(1)\nabla\mathfrak{d}^{k-1}\Phi+\check{\Gamma}_k[\Phi]\\ &\quad+\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma}),\end{aligned}$$

and

$$\begin{aligned}\nabla_3(\nabla\mathfrak{d}^{k-1}\Phi) &= O(1)\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi+O(1)\nabla\mathfrak{d}^{k-1}\Phi+\check{\Gamma}_k[\Phi]\\ &\quad+\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma}).\end{aligned}$$

Then, applying (9.152), we obtain

$$\begin{aligned}&\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|q|^p\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi|^2+|\nabla\mathfrak{d}^{k-1}\Phi|^2\Big)|\mathbf{g}(e_3,N)|\\ &+p\int_{{}^{(int)}\mathcal{M}_*}r|q|^{p-2}\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi|^2+|\nabla\mathfrak{d}^{k-1}\Phi|^2\Big)\\ \leq\;&\int_{{}^{(int)}\mathcal{M}_*}O(1)r^{-1}|q|^{p+2}\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi|^2+|\nabla\mathfrak{d}^{k-1}\Phi|^2\Big)\\ &+\int_{{}^{(int)}\mathcal{M}_*}r^{-1}|q|^{p+2}\Big|\check{\Gamma}_k[\Phi]+\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma})\Big|^2\\ &+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|q|^{p-2}|\mathfrak{d}^k\Phi|^2|\mathbf{g}(e_3,N)|.\end{aligned}$$

Note that $O(1)$ depends only on k and the equation of Φ, and hence not on p. Together with the fact that $r_+-\delta_{\mathcal{H}}\leq r\leq r_0$ in ${}^{(int)}\mathcal{M}$, there exists thus a constant C independent of p such that

$$\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|q|^p\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi|^2+|\nabla\mathfrak{d}^{k-1}\Phi|^2\Big)|\mathbf{g}(e_3,N)|$$

$$
\begin{aligned}
& +p\int_{{}^{(int)}\mathcal{M}_*} r|q|^{p-2}\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi|^2+|\nabla\mathfrak{d}^{k-1}\Phi|^2\Big)\\
\leq\; & C\int_{{}^{(int)}\mathcal{M}_*} r|q|^{p-2}\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi|^2+|\nabla\mathfrak{d}^{k-1}\Phi|^2\Big)\\
& +\int_{{}^{(int)}\mathcal{M}_*} r^{-1}|q|^{p+2}\Big|\check{\Gamma}_k[\Phi]+\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma})\Big|^2\\
& +\int_{\mathcal{T}\cup\{\underline{u}=1\}} |q|^{p-2}|\mathfrak{d}^k\Phi|^2|\mathbf{g}(e_3,N)|.
\end{aligned}
$$

In particular, taking p large enough, we may absorb the first term on the RHS by the LHS and obtain, using again the fact that $r_+-\delta_{\mathcal{H}}\leq r\leq r_0$ in ${}^{(int)}\mathcal{M}$,

$$
\begin{aligned}
& \int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi|^2+|\nabla\mathfrak{d}^{k-1}\Phi|^2\Big)|\mathbf{g}(e_3,N)|\\
& +\int_{{}^{(int)}\mathcal{M}_*}\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi|^2+|\nabla\mathfrak{d}^{k-1}\Phi|^2\Big)\\
\lesssim\; & \int_{{}^{(int)}\mathcal{M}_*}\Big|\check{\Gamma}_k[\Phi]+\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma})\Big|^2\\
& +\int_{\mathcal{T}\cup\{\underline{u}=1\}}\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi|^2+|\nabla\mathfrak{d}^{k-1}\Phi|^2\Big)|\mathbf{g}(e_3,N)|.
\end{aligned}
$$

Also, using in particular Proposition 9.29 for the boundary ${}^{(int)}\Sigma_*=\{\tau=\tau_*\}$, as well as the fact that $\mathcal{A}=\{r=r_+-\delta_{\mathcal{H}}\}$ and $\mathcal{T}\{r=r_0\}$, we infer

$$
|\mathbf{g}(e_3,N)|\lesssim 1\quad\text{on}\quad \mathcal{T}\cup\{\underline{u}=1\},\qquad |\mathbf{g}(e_3,N)|\gtrsim 1\quad\text{on}\quad {}^{(int)}\Sigma_*\cup\mathcal{A}
$$

which yields

$$
\begin{aligned}
& \int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi|^2+|\nabla\mathfrak{d}^{k-1}\Phi|^2\Big)\\
& +\int_{{}^{(int)}\mathcal{M}_*}\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi|^2+|\nabla\mathfrak{d}^{k-1}\Phi|^2\Big)\\
\lesssim\; & \int_{{}^{(int)}\mathcal{M}_*}\Big|\check{\Gamma}_k[\Phi]+\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma})\Big|^2+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|\mathfrak{d}^k\Phi|^2.
\end{aligned}
$$

Next, we estimate the first term on the RHS. In view of the definition of ${}^{(int)}\mathfrak{R}_k$ and ${}^{(int)}\mathfrak{G}_k$, and using the bootstrap assumption (9.135), we have, for $k\leq k_{large}+7$,

$$
\int_{{}^{(int)}\mathcal{M}_*}\Big|\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma})\Big|^2\;\lesssim\;\Big(\epsilon^2+{}^{(int)}\mathfrak{R}_k+{}^{(int)}\mathfrak{G}_{k-1}\Big)^2
$$

$$\lesssim \quad \left(\epsilon_0 + {}^{(int)}\mathfrak{R}_k + {}^{(int)}\mathfrak{G}_{k-1}\right)^2.$$

Plugging in the above, we deduce, for $k \leq k_{large} + 7$,

$$\begin{aligned}
&\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}} \left(|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi|^2 + |\nabla\mathfrak{d}^{k-1}\Phi|^2\right)\\
&+\int_{{}^{(int)}\mathcal{M}_*} \left(|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi|^2 + |\nabla\mathfrak{d}^{k-1}\Phi|^2\right)\\
\lesssim \quad & \left(\epsilon_0 + {}^{(int)}\mathfrak{R}_k + {}^{(int)}\mathfrak{G}_{k-1}\right)^2 + \int_{{}^{(int)}\mathcal{M}_*} |\check{\Gamma}_k[\Phi]|^2 + \int_{\mathcal{T}\cup\{\underline{u}=1\}} |\mathfrak{d}^k\Phi|^2.
\end{aligned}$$

Now, note that

$$|\mathfrak{d}^k\Phi| \quad \lesssim \quad |\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi| + |\nabla\mathfrak{d}^{k-1}\Phi| + |\nabla_3^k\Phi| + |\mathfrak{d}^{\leq k-1}\Phi|$$

and hence, for $k \leq k_{large} + 7$,

$$\begin{aligned}
&\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}} |\mathfrak{d}^k\Phi|^2 + \int_{{}^{(int)}\mathcal{M}_*} |\mathfrak{d}^k\Phi|^2\\
\lesssim \quad & \left(\epsilon_0 + {}^{(int)}\mathfrak{R}_k + {}^{(int)}\mathfrak{G}_{k-1}\right)^2 + \int_{{}^{(int)}\mathcal{M}_*} |\check{\Gamma}_k[\Phi]|^2 + \int_{\mathcal{T}\cup\{\underline{u}=1\}} |\mathfrak{d}^k\Phi|^2\\
&+\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}} |\nabla_3^k\Phi|^2 + \int_{{}^{(int)}\mathcal{M}_*} |\nabla_3^k\Phi|^2
\end{aligned}$$

so that it still remains to control $\nabla_3^k\Phi$. Using $\nabla_3\Phi = \check{\Gamma} + \check{R}$, and iterating, we have

$$\nabla_3^k\Phi = \check{\Gamma} + \mathfrak{d}^{\leq k-1}\check{R},$$

and hence, in view of the definition of ${}^{(int)}\mathfrak{R}_k$ and ${}^{(int)}\mathfrak{G}_k$, and the trace theorem for $\check{\Gamma}$, we have

$$\begin{aligned}
&\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}} |\nabla_3^k\Phi|^2 + \int_{{}^{(int)}\mathcal{M}_*} |\nabla_3^k\Phi|^2\\
\lesssim \quad & \int_{{}^{(int)}\Sigma_*\cup\mathcal{A}} |\check{\Gamma} + \mathfrak{d}^{\leq k-1}\check{R}|^2 + \int_{{}^{(int)}\mathcal{M}_*} |\check{\Gamma} + \mathfrak{d}^{\leq k-1}\check{R}|^2\\
\lesssim \quad & {}^{(int)}\mathfrak{R}_k^2 + \int_{{}^{(int)}\mathcal{M}_*} |\mathfrak{d}^{\leq 1}\check{\Gamma}|^2 \lesssim {}^{(int)}\mathfrak{G}_1^2 + {}^{(int)}\mathfrak{R}_k^2.
\end{aligned}$$

Since ${}^{(int)}\mathfrak{G}_1 \lesssim \epsilon_0$ in view of (9.42), we infer

$$\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}} |\nabla_3^k\Phi|^2 + \int_{{}^{(int)}\mathcal{M}_*} |\nabla_3^k\Phi|^2 \quad \lesssim \quad \epsilon_0^2 + {}^{(int)}\mathfrak{R}_k^2.$$

Plugging in the above estimate for $\mathfrak{d}^k\Phi$, we infer, for $k\leq k_{large}+7$,

$$\begin{aligned}
&\int_{^{(int)}\Sigma_*\cup\mathcal{A}}|\mathfrak{d}^k\Phi|^2+\int_{^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\Phi|^2\\
\lesssim\;& \left(\epsilon_0+{}^{(int)}\mathfrak{R}_k+{}^{(int)}\mathfrak{G}_{k-1}\right)^2+\int_{^{(int)}\mathcal{M}_*}|\check{\Gamma}_k[\Phi]|^2+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|\mathfrak{d}^k\Phi|^2
\end{aligned}$$

as stated.

It remains to treat the case where Φ is among the quantities $\check{Z}$, $\widecheck{\underline{H}}$, $\widecheck{\mathrm{tr}X}$, $\check{\omega}$, Ξ, $\mathcal{D}\widehat{\otimes}\mathfrak{J}$, $\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}$, $\widecheck{\nabla_4\mathfrak{J}}$. Then, according to Proposition 9.91. Φ satisfies a transport equations of the following form

$$\begin{aligned}
\nabla_3(\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi) &= O(1)\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi+O(1)\nabla\mathfrak{d}^{k-1}\Phi+\check{\Gamma}_k[\Phi]\\
&\quad+\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\check{R}+\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma}),
\end{aligned}$$

and

$$\begin{aligned}
\nabla_3(\nabla\mathfrak{d}^{k-1}\Phi+\mathfrak{d}^{k-1}\check{R}) &= O(1)\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\Phi+O(1)\nabla\mathfrak{d}^{k-1}\Phi+\check{\Gamma}_k[\Phi]\\
&\quad+\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\check{R}+\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma}).
\end{aligned}$$

Proceeding exactly as in the first case, we obtain

$$\begin{aligned}
&\int_{^{(int)}\Sigma_*\cup\mathcal{A}}|\mathfrak{d}^k\Phi+\mathfrak{d}^{k-1}\check{R}|^2+\int_{^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\Phi+\mathfrak{d}^{k-1}\check{R}|^2\\
\lesssim\;& \int_{^{(int)}\mathcal{M}_*}|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\check{R}|^2+\left(\epsilon_0+{}^{(int)}\mathfrak{R}_k+{}^{(int)}\mathfrak{G}_{k-1}\right)^2+\int_{^{(int)}\mathcal{M}_*}|\check{\Gamma}_k[\Phi]|^2\\
&+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|\mathfrak{d}^k\Phi|^2.
\end{aligned}$$

Since we have, in view of the definition of ${}^{(int)}\mathfrak{R}_k$,

$$\int_{^{(int)}\Sigma_*\cup\mathcal{A}}|\mathfrak{d}^{k-1}\check{R}|^2+\int_{^{(int)}\mathcal{M}_*}|\mathfrak{d}^{k-1}\check{R}|^2+\int_{^{(int)}\mathcal{M}_*}|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\check{R}|^2\lesssim({}^{(int)}\mathfrak{R}_k)^2,$$

we infer, for $k\leq k_{large}+7$,

$$\begin{aligned}
&\int_{^{(int)}\Sigma_*\cup\mathcal{A}}|\mathfrak{d}^k\Phi|^2+\int_{^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\Phi|^2\\
\lesssim\;& \left(\epsilon_0+{}^{(int)}\mathfrak{R}_k+{}^{(int)}\mathfrak{G}_{k-1}\right)^2+\int_{^{(int)}\mathcal{M}_*}|\check{\Gamma}_k[\Phi]|^2+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|\mathfrak{d}^k\Phi|^2
\end{aligned}$$

as stated. This concludes the proof of Proposition 9.95. $\square$

9.9.4. Non-integrable Hodge estimates Note that Proposition 9.95 does not apply to $\widehat{\underline{X}}$. In this section, we provide estimate for $\widehat{\underline{X}}$ relying in particular on Codazzi. To this end, we start by considering Hodge type systems of the form

$$\overline{\mathcal{D}}\cdot U = H \tag{9.154}$$

in ${}^{(int)}\mathcal{M}$ where U is an anti-selfdual horizontal symmetric traceless 2-tensor, and H is a anti-selfdual horizontal 1-form. We prove the following non-integrable version of elliptic-Hodge estimates.

Proposition 9.96. *Given U is an anti-selfdual horizontal symmetric traceless 2-tensor, we have the estimate*

$$\begin{aligned}\int_{{}^{(int)}\mathcal{M}_*}|\nabla U|^2 \lesssim & \int_{{}^{(int)}\mathcal{M}_*}\Big(|\overline{\mathcal{D}}\cdot U|^2+|U||\mathfrak{d}U|\Big)\\ &+\int_{\partial\,{}^{(int)}\mathcal{M}_*}\Big(|\nabla_3 U|+|\nabla_{\widehat{R}}U|+|\overline{\mathcal{D}}\cdot U|+|U|\Big)|U|.\end{aligned}\tag{9.155}$$

Proof. We rely on the following identity.

Lemma 9.97. *Given a horizontal structure, and given U an anti-selfdual horizontal symmetric traceless 2-tensor, we have the following point-wise identity*[220]

$$\begin{aligned}|\nabla U|^2+{}^{(h)}K|U|^2 =& 2|\overline{\mathcal{D}}\cdot U|^2+\frac{1}{2}\Big(\Big({}^{(a)}tr\chi\nabla_3+{}^{(a)}tr\underline{\chi}\nabla_4\Big)\,{}^*U\Big)\cdot U\\ &+\nabla_a\Big(\nabla^a U\cdot U-(\overline{\mathcal{D}}\cdot U)_b U^{ab}\Big).\end{aligned}\tag{9.156}$$

Proof. See Proposition 2.1.47 in [28]. □

We rewrite Lemma 9.97 schematically in ${}^{(int)}\mathcal{M}$.

Corollary 9.98. *We have for an anti-selfdual horizontal symmetric traceless 2-tensor U in ${}^{(int)}\mathcal{M}$*

$$\begin{aligned}|\nabla U|^2 =& 2|\overline{\mathcal{D}}\cdot U|^2+O(1)|U||\mathfrak{d}U|\\ &+\mathbf{D}_\alpha\Big(\mathbf{D}^a U\cdot U-(\overline{\mathcal{D}}\cdot U)_b\cdot U^{\alpha b}\Big).\end{aligned}\tag{9.157}$$

[220] Here ${}^{(h)}K=-\frac{1}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}-\frac{1}{4}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}-\rho+\Gamma_g\cdot\Gamma_b$ is a non-integrable version of the Gauss curvature appearing in section 2.1.4 of [28].

Proof. In view of the following simple calculation for an anti-selfdual 1-form V

$$\mathbf{D}^\alpha V_\alpha = \nabla^a V_a + (\eta + \underline{\eta}) \cdot V,$$

this follows immediately from Lemma 9.97. □

We next integrate (9.157) in ${}^{(int)}\mathcal{M}_*$ and apply the divergence lemma to derive

$$\begin{aligned}\int_{{}^{(int)}\mathcal{M}_*} |\nabla U|^2 &\lesssim \int_{{}^{(int)}\mathcal{M}_*} \Big(|\overline{\mathcal{D}} \cdot U|^2 + |U||\mathfrak{d} U|\Big)\\ &+ \int_{\partial\, {}^{(int)}\mathcal{M}_*} N_\alpha \Big(\mathbf{D}^\alpha U \cdot U - (\overline{\mathcal{D}} \cdot U)_b \cdot U^{\alpha b}\Big)\end{aligned}$$

where $\partial\, {}^{(int)}\mathcal{M}_* = {}^{(int)}\Sigma_* \cup \mathcal{A} \cup \{\underline{u} = 1\} \cup \mathcal{T}$ and N the corresponding normal. In order to decompose the first boundary term on the RHS, i.e. $N_\alpha \mathbf{D}^\alpha U \cdot U$, we introduce the vectorfields[221]

$$\widehat{e}_b = e_b + \left(e_b(r) - \frac{e_b(\underline{u})}{e_4(\underline{u})} e_4(r)\right) e_3 - \frac{e_b(\underline{u})}{e_4(\underline{u})} e_4, \quad b = 1, 2,$$

and note that with this definition, we have

$$\widehat{e}_1(\underline{u}) = 0, \qquad \widehat{e}_1(r) = 0, \qquad \widehat{e}_2(\underline{u}) = 0, \qquad \widehat{e}_2(r) = 0,$$

so that $(\widehat{e}_1, \widehat{e}_2)$ spans the tangent space of the spheres $S(\underline{u}, r)$. We decompose

$$\begin{aligned}N_\alpha \mathbf{D}^\alpha U \cdot U =& -\frac{1}{2}\mathbf{g}(N, e_4)\mathbf{D}_3 U \cdot U - \frac{1}{2}\mathbf{g}(N, e_3)\mathbf{D}_4 U \cdot U + \mathbf{g}(N, e_b)\mathbf{D}_b U \cdot U\\ =& O(1)\nabla_3 U \cdot U + O(1)\nabla_4 U \cdot U + \frac{1}{2}\mathbf{g}(N, e_b)\widehat{e}_b(U \cdot U).\end{aligned}$$

Now, since $\partial\, {}^{(int)}\mathcal{M}_* = {}^{(int)}\Sigma_* \cup \mathcal{A} \cup \{\underline{u} = 1\} \cup \mathcal{T}$, and since ${}^{(int)}\Sigma_* = \{\tau = \tau_*\}$, $\mathcal{A} = \{r = r_+ - \delta_{\mathcal{H}}\}$, $\mathcal{T} = \{r = r_0\}$, and since $\tau = \underline{u} + f(r)$ for some scalar function f by construction, see the proof of Proposition 9.29, all parts of $\partial\, {}^{(int)}\mathcal{M}_*$ are foliated by spheres $S(\underline{u}, r)$. In particular, we may integrate the term $g(N, e_b)\widehat{e}_b(U \cdot U)$ by parts on $\partial\, {}^{(int)}\mathcal{M}_*$ and hence

$$\begin{aligned}&\int_{\partial\, {}^{(int)}\mathcal{M}_*} N_\alpha \mathbf{D}^\alpha U \cdot U\\ =& \int_{\partial\, {}^{(int)}\mathcal{M}_*} \Big(O(1)|\nabla_3 U||U| + O(1)|\nabla_4 U||U| + O(1)|U|^2\Big)\end{aligned}$$

[221]Recall that $e_4(\underline{u}) = \frac{2(r^2+a^2)}{|q|^2} + \check{\Gamma}$ so that we may divide by $e_4(\underline{u})$.

$$= \int_{\partial\, {}^{(int)}\mathcal{M}_*} \Big(O(1)|\nabla_3 U||U| + O(1)|\nabla_{\widehat{R}} U||U| + O(1)|U|^2 \Big).$$

Plugging in the above, we infer

$$\begin{aligned}\int_{{}^{(int)}\mathcal{M}_*} |\nabla U|^2 \;\lesssim\; & \int_{{}^{(int)}\mathcal{M}_*} \Big(|\overline{\mathcal{D}} \cdot U|^2 + |U||\mathfrak{d} U| \Big) \\ & + \int_{\partial\, {}^{(int)}\mathcal{M}_*} \Big(|\nabla_3 U| + |\nabla_{\widehat{R}} U| + |\overline{\mathcal{D}} \cdot U| + |U| \Big) |U|\end{aligned}$$

as stated. This completes the proof of Proposition 9.96. □

We are now ready to derive and estimate for $\widehat{\underline{X}}$ on ${}^{(int)}\mathcal{M}_*$.

Proposition 9.99. *$\widehat{\underline{X}}$ satisfies, for $k \le k_{large} + 7$,*

$$\begin{aligned}(9.158) \qquad & \int_{{}^{(int)}\mathcal{M}_*} |\mathfrak{d}^k \widehat{\underline{X}}|^2 \\ \lesssim\; & \int_{{}^{(int)}\mathcal{M}_*} |\mathfrak{d}^k \widetilde{tr \underline{X}}|^2 + \int_{{}^{(int)}\Sigma_* \cup \mathcal{A}} \Big(|\mathfrak{d}^k \widetilde{tr\underline{X}}|^2 + |\mathfrak{d}^{\le k-1}(\check{\Gamma} \setminus \{\widehat{\underline{X}}\})|^2 \Big) \\ & + \int_{\mathcal{T} \cup \{\underline{u}=1\}} |\mathfrak{d}^{\le k} \check{\Gamma}|^2 + \Big(\epsilon_0 + {}^{(int)}\mathfrak{R}_k + {}^{(int)}\mathfrak{G}_{k-1} \Big)^2.\end{aligned}$$

Proof. Recall from Proposition 9.91 that $\widehat{\underline{X}}$ satisfies the following equation

$$\begin{aligned}\nabla_3(\nabla_{\widehat{R}} \mathfrak{d}^{k-1} \widehat{\underline{X}}) \;=\; & O(1)\nabla_{\widehat{R}} \mathfrak{d}^{k-1} \widehat{\underline{X}} + O(1)\nabla \mathfrak{d}^{k-1} \widehat{\underline{X}} \\ & + \nabla_{\widehat{R}} \mathfrak{d}^{k-1} \check{R} + \mathfrak{d}^{\le k-1}(\check{R}, \check{\Gamma}) + \mathfrak{d}^{\le k}(\check{\Gamma} \cdot \check{\Gamma}).\end{aligned}$$

Then, applying (9.152), we obtain

$$\begin{aligned}& \int_{{}^{(int)}\Sigma_* \cup \mathcal{A}} |q|^p |\nabla_{\widehat{R}} \mathfrak{d}^{k-1} \widehat{\underline{X}}|^2 |\mathbf{g}(e_3, N)| + p \int_{{}^{(int)}\mathcal{M}_*} r|q|^{p-2} |\nabla_{\widehat{R}} \mathfrak{d}^{k-1} \widehat{\underline{X}}|^2 \\ \le\; & \int_{{}^{(int)}\mathcal{M}_*} O(1) r^{-1} |q|^{p+2} |\nabla_{\widehat{R}} \mathfrak{d}^{k-1} \widehat{\underline{X}}|^2 + \int_{{}^{(int)}\mathcal{M}_*} O(1) r^{-1} |q|^{p+2} |\nabla \mathfrak{d}^{k-1} \widehat{\underline{X}}|^2 \\ & + \int_{{}^{(int)}\mathcal{M}_*} r^{-1} |q|^{p+2} \Big| \nabla_{\widehat{R}} \mathfrak{d}^{k-1} \check{R} + \mathfrak{d}^{\le k-1}(\check{R}, \check{\Gamma}) + \mathfrak{d}^{\le k}(\check{\Gamma} \cdot \check{\Gamma}) \Big|^2 \\ & + \int_{\mathcal{T} \cup \{\underline{u}=1\}} |q|^{p-2} |\nabla_{\widehat{R}} \mathfrak{d}^{k-1} \widehat{\underline{X}}|^2 |\mathbf{g}(e_3, N)|.\end{aligned}$$

Next, we choose p large enough to absorb the first term on the RHS. Proceeding as in the proof of Proposition 9.95, we obtain

$$\int_{{}^{(int)}\Sigma_* \cup \mathcal{A}} |\nabla_{\widehat{R}} \mathfrak{d}^{k-1} \widehat{\underline{X}}|^2 + \int_{{}^{(int)}\mathcal{M}_*} |\nabla_{\widehat{R}} \mathfrak{d}^{k-1} \widehat{\underline{X}}|^2$$

$$\begin{aligned}
&\lesssim \int_{{}^{(int)}\mathcal{M}_*}|\nabla\mathfrak{d}^{k-1}\underline{\widehat{X}}|^2+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\underline{\widehat{X}}|^2\\
&\quad+\int_{{}^{(int)}\mathcal{M}_*}\Big|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\check{R}+\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma})\Big|^2.
\end{aligned}$$

In view of the definition of ${}^{(int)}\mathfrak{R}_k$ and ${}^{(int)}\mathfrak{G}_k$, and using the bootstrap assumption (9.135), we infer, for $k\leq k_{large}+7$,

$$\begin{aligned}
&\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\underline{\widehat{X}}|^2+\int_{{}^{(int)}\mathcal{M}_*}|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\underline{\widehat{X}}|^2\\
\lesssim\ &\int_{{}^{(int)}\mathcal{M}_*}|\nabla\mathfrak{d}^{k-1}\underline{\widehat{X}}|^2+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\underline{\widehat{X}}|^2\\
&+\Big(\epsilon_0+{}^{(int)}\mathfrak{R}_k+{}^{(int)}\mathfrak{G}_{k-1}\Big)^2.
\end{aligned}$$

Next, we focus on controlling the first term on the RHS of the above estimate, i.e. the one involving $\nabla\mathfrak{d}^{k-1}\underline{\widehat{X}}$. We use Proposition 9.96 with $U=\mathfrak{d}^{k-1}\underline{\widehat{X}}$ which yields

$$\begin{aligned}
&\int_{{}^{(int)}\mathcal{M}_*}|\nabla\mathfrak{d}^{k-1}\underline{\widehat{X}}|^2\\
&\lesssim\int_{{}^{(int)}\mathcal{M}_*}\Big(|\overline{\mathcal{D}}\cdot\mathfrak{d}^{k-1}\underline{\widehat{X}}|^2+|\mathfrak{d}^{k-1}\underline{\widehat{X}}||\mathfrak{d}^{k}\underline{\widehat{X}}|\Big)\\
&\quad+\int_{\partial\,{}^{(int)}\mathcal{M}_*}\Big(|\nabla_3\mathfrak{d}^{k-1}\underline{\widehat{X}}|+|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\underline{\widehat{X}}|+|\overline{\mathcal{D}}\cdot\mathfrak{d}^{k-1}\underline{\widehat{X}}|+|\mathfrak{d}^{k-1}\underline{\widehat{X}}|\Big)|\mathfrak{d}^{k-1}\underline{\widehat{X}}|.
\end{aligned}$$

Also, recall from Proposition 9.91 that $\underline{\widehat{X}}$ satisfies the following equation

$$\begin{aligned}
\nabla_3\mathfrak{d}^{k-1}\underline{\widehat{X}} &= \mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k-1}(\check{\Gamma}\cdot\check{\Gamma}),\\
\overline{\mathcal{D}}\cdot(\mathfrak{d}^{k-1}\underline{\widehat{X}}) &= \mathfrak{d}^k\widetilde{\mathrm{tr}\underline{X}}+\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k-1}(\check{\Gamma}\cdot\check{\Gamma}).
\end{aligned}$$

Plugging in the above, we deduce

$$\begin{aligned}
&\int_{{}^{(int)}\mathcal{M}_*}|\nabla\mathfrak{d}^{k-1}\underline{\widehat{X}}|^2\\
\lesssim\ &\int_{{}^{(int)}\mathcal{M}_*}\Big(\Big|\mathfrak{d}^k\widetilde{\mathrm{tr}\underline{X}}+\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k-1}(\check{\Gamma}\cdot\check{\Gamma})\Big|^2+|\mathfrak{d}^{k-1}\underline{\widehat{X}}||\mathfrak{d}^k\underline{\widehat{X}}|\Big)\\
&+\int_{\partial\,{}^{(int)}\mathcal{M}_*}\Big(|\mathfrak{d}^k\widetilde{\mathrm{tr}\underline{X}}|+|\mathfrak{d}^{\leq k-1}(\check{R},\check{\Gamma})+\mathfrak{d}^{\leq k-1}(\check{\Gamma}\cdot\check{\Gamma})|+|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\underline{\widehat{X}}|\\
&\qquad+|\mathfrak{d}^{k-1}\underline{\widehat{X}}|\Big)|\mathfrak{d}^{k-1}\underline{\widehat{X}}|.
\end{aligned}$$

In view of the definition of ${}^{(int)}\mathfrak{R}_k$ and ${}^{(int)}\mathfrak{G}_k$, and using the bootstrap assumption (9.135), we infer, for $k \leq k_{large}+7$,

$$
\begin{aligned}
&\int_{{}^{(int)}\mathcal{M}_*} |\nabla \mathfrak{d}^{k-1}\widehat{\underline{X}}|^2 \\
\lesssim\ & \int_{{}^{(int)}\mathcal{M}_*} \Big(|\mathfrak{d}^k \widetilde{\mathrm{tr}\underline{X}}|^2 + |\mathfrak{d}^{k-1}\widehat{\underline{X}}||\mathfrak{d}^k\widehat{\underline{X}}|\Big)\\
&+\int_{\partial\,{}^{(int)}\mathcal{M}_*} \Big(|\mathfrak{d}^k \widetilde{\mathrm{tr}\underline{X}}|^2 + |\mathfrak{d}^{\leq k-1}(\check{\Gamma}\setminus\{\widehat{\underline{X}}\})|^2\Big)\\
&+\int_{\partial\,{}^{(int)}\mathcal{M}_*} \Big(|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\widehat{\underline{X}}| + |\mathfrak{d}^{k-1}\widehat{\underline{X}}|\Big)|\mathfrak{d}^{k-1}\widehat{\underline{X}}|\\
&+\Big(\epsilon_0 + {}^{(int)}\mathfrak{R}_k + {}^{(int)}\mathfrak{G}_{k-1}\Big)^2.
\end{aligned}
$$

Also, using the trace theorem, we have

$$
\int_{\partial\,{}^{(int)}\mathcal{M}_*} |\mathfrak{d}^{k-1}\widehat{\underline{X}}|^2 \ \lesssim\ \int_{{}^{(int)}\mathcal{M}_*} |\mathfrak{d}^{\leq k-1}\widehat{\underline{X}}||\mathfrak{d}^{\leq k}\widehat{\underline{X}}|.
$$

Together with

$$
\int_{{}^{(int)}\mathcal{M}_*} |\mathfrak{d}^{\leq k-1}\widehat{\underline{X}}||\mathfrak{d}^{\leq k}\widehat{\underline{X}}| \ \lesssim\ {}^{(int)}\mathfrak{G}_{k-1}\left(\int_{{}^{(int)}\mathcal{M}_*} |\mathfrak{d}^k\widehat{\underline{X}}|^2\right)^{\frac{1}{2}} + {}^{(int)}\mathfrak{G}_{k-1}^2,
$$

we infer

$$
\begin{aligned}
&\int_{{}^{(int)}\mathcal{M}_*} |\nabla \mathfrak{d}^{k-1}\widehat{\underline{X}}|^2 \\
\lesssim\ & \int_{{}^{(int)}\mathcal{M}_*} |\mathfrak{d}^k \widetilde{\mathrm{tr}\underline{X}}|^2 + {}^{(int)}\mathfrak{G}_{k-1}\left(\int_{{}^{(int)}\mathcal{M}_*} |\mathfrak{d}^k\widehat{\underline{X}}|^2\right)^{\frac{1}{2}}\\
&+({}^{(int)}\mathfrak{G}_{k-1})^{\frac{1}{2}}\left(\int_{{}^{(int)}\mathcal{M}_*} |\mathfrak{d}^k\widehat{\underline{X}}|^2\right)^{\frac{1}{4}}\left(\int_{\partial\,{}^{(int)}\mathcal{M}_*} |\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\widehat{\underline{X}}|^2\right)^{\frac{1}{2}}\\
&+\int_{\partial\,{}^{(int)}\mathcal{M}_*} \Big(|\mathfrak{d}^k \widetilde{\mathrm{tr}\underline{X}}|^2 + |\mathfrak{d}^{\leq k-1}(\check{\Gamma}\setminus\{\widehat{\underline{X}}\})|^2\Big)\\
&+\Big(\epsilon_0 + {}^{(int)}\mathfrak{R}_k + {}^{(int)}\mathfrak{G}_{k-1}\Big)^2
\end{aligned}
$$

and hence, for $k \leq k_{large}+7$,

$$
\int_{{}^{(int)}\mathcal{M}_*} |\nabla \mathfrak{d}^{k-1}\widehat{\underline{X}}|^2
$$

$$\begin{aligned}
&\lesssim \int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widecheck{\mathrm{tr}\underline{X}}|^2+{}^{(int)}\mathfrak{G}_{k-1}\left(\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widehat{\underline{X}}|^2\right)^{\frac{1}{2}}\\
&+({}^{(int)}\mathfrak{G}_{k-1})^{\frac{1}{2}}\left(\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widehat{\underline{X}}|^2\right)^{\frac{1}{4}}\left(\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\widehat{\underline{X}}|^2\right)^{\frac{1}{2}}\\
&+\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}\Big(|\mathfrak{d}^k\widecheck{\mathrm{tr}\underline{X}}|^2+|\mathfrak{d}^{\leq k-1}(\check{\Gamma}\setminus\{\widehat{\underline{X}}\})|^2\Big)\\
&+\Big(\epsilon_0+{}^{(int)}\mathfrak{R}_k+{}^{(int)}\mathfrak{G}_{k-1}\Big)^2+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|\mathfrak{d}^{\leq k}\check{\Gamma}|^2.
\end{aligned}$$

Next, recall the following above estimate, for $k\leq k_{large}+7$,

$$\begin{aligned}
&\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\widehat{\underline{X}}|^2+\int_{{}^{(int)}\mathcal{M}_*}|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\widehat{\underline{X}}|^2\\
\lesssim&\int_{{}^{(int)}\mathcal{M}_*}|\nabla\mathfrak{d}^{k-1}\widehat{\underline{X}}|^2+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\widehat{\underline{X}}|^2\\
&+\Big(\epsilon_0+{}^{(int)}\mathfrak{R}_k+{}^{(int)}\mathfrak{G}_{k-1}\Big)^2.
\end{aligned}$$

Plugging the above estimate for $\nabla\mathfrak{d}^{k-1}\widehat{\underline{X}}$, we obtain, for $k\leq k_{large}+7$,

$$\begin{aligned}
&\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\widehat{\underline{X}}|^2+\int_{{}^{(int)}\mathcal{M}_*}|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\widehat{\underline{X}}|^2+\int_{{}^{(int)}\mathcal{M}_*}|\nabla\mathfrak{d}^{k-1}\widehat{\underline{X}}|^2\\
\lesssim&\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widecheck{\mathrm{tr}\underline{X}}|^2+{}^{(int)}\mathfrak{G}_{k-1}\left(\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widehat{\underline{X}}|^2\right)^{\frac{1}{2}}\\
&+({}^{(int)}\mathfrak{G}_{k-1})^{\frac{1}{2}}\left(\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widehat{\underline{X}}|^2\right)^{\frac{1}{4}}\left(\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\widehat{\underline{X}}|^2\right)^{\frac{1}{2}}\\
&+\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}\Big(|\mathfrak{d}^k\widecheck{\mathrm{tr}\underline{X}}|^2+|\mathfrak{d}^{\leq k-1}(\check{\Gamma}\setminus\{\widehat{\underline{X}}\})|^2\Big)\\
&+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|\mathfrak{d}^{\leq k}\check{\Gamma}|^2+\Big(\epsilon_0+{}^{(int)}\mathfrak{R}_k+{}^{(int)}\mathfrak{G}_{k-1}\Big)^2.
\end{aligned}$$

We use the boundary term for $\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\widehat{\underline{X}}$ on the LHS to absorb the corresponding term on the RHS and infer

$$\begin{aligned}
&\int_{{}^{(int)}\mathcal{M}_*}|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\widehat{\underline{X}}|^2+\int_{{}^{(int)}\mathcal{M}_*}|\nabla\mathfrak{d}^{k-1}\widehat{\underline{X}}|^2\\
\lesssim&\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widecheck{\mathrm{tr}\underline{X}}|^2+{}^{(int)}\mathfrak{G}_{k-1}\left(\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widehat{\underline{X}}|^2\right)^{\frac{1}{2}}\\
&+\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}\Big(|\mathfrak{d}^k\widecheck{\mathrm{tr}\underline{X}}|^2+|\mathfrak{d}^{\leq k-1}(\check{\Gamma}\setminus\{\widehat{\underline{X}}\})|^2\Big)
\end{aligned}$$

$$+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|\mathfrak{d}^{\leq k}\check{\Gamma}|^2+\Big(\epsilon_0+{}^{(int)}\mathfrak{R}_k+{}^{(int)}\mathfrak{G}_{k-1}\Big)^2.$$

Using the fact that $\widehat{\underline{X}}$ satisfies

$$\nabla_3\mathfrak{d}^{k-1}\widehat{\underline{X}} \;=\; \mathfrak{d}^{\leq k-1}\check{\Gamma}+\mathfrak{d}^{\leq k-1}\check{R}+\mathfrak{d}^{\leq k-1}(\check{\Gamma}\cdot\check{\Gamma}),$$

and using again the definition of ${}^{(int)}\mathfrak{R}_k$ and ${}^{(int)}\mathfrak{G}_k$, and the bootstrap assumption (9.135), we infer, for $k\leq k_{large}+7$,

$$\begin{aligned}
&\int_{{}^{(int)}\mathcal{M}_*}|\nabla_{\widehat{R}}\mathfrak{d}^{k-1}\widehat{\underline{X}}|^2+\int_{{}^{(int)}\mathcal{M}_*}|\nabla\mathfrak{d}^{k-1}\widehat{\underline{X}}|^2+\int_{{}^{(int)}\mathcal{M}_*}|\nabla_3\mathfrak{d}^{k-1}\widehat{\underline{X}}|^2\\
\lesssim\;&\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widetilde{\mathrm{tr}\underline{X}}|^2+{}^{(int)}\mathfrak{G}_{k-1}\left(\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widehat{\underline{X}}|^2\right)^{\frac12}\\
&+\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}\Big(|\mathfrak{d}^k\widetilde{\mathrm{tr}\underline{X}}|^2+|\mathfrak{d}^{\leq k-1}(\check{\Gamma}\setminus\{\widehat{\underline{X}}\})|^2\Big)\\
&+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|\mathfrak{d}^{\leq k}\check{\Gamma}|^2+\Big(\epsilon_0+{}^{(int)}\mathfrak{R}_k+{}^{(int)}\mathfrak{G}_{k-1}\Big)^2.
\end{aligned}$$

Since ∇, $\nabla_{\widehat{R}}$ and ∇_3 span all derivatives, we deduce

$$\begin{aligned}
\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widehat{\underline{X}}|^2\;\lesssim\;&\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widetilde{\mathrm{tr}\underline{X}}|^2+{}^{(int)}\mathfrak{G}_{k-1}\left(\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widehat{\underline{X}}|^2\right)^{\frac12}\\
&+\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}\Big(|\mathfrak{d}^k\widetilde{\mathrm{tr}\underline{X}}|^2+|\mathfrak{d}^{\leq k-1}(\check{\Gamma}\setminus\{\widehat{\underline{X}}\})|^2\Big)\\
&+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|\mathfrak{d}^{\leq k}\check{\Gamma}|^2+\Big(\epsilon_0+{}^{(int)}\mathfrak{R}_k+{}^{(int)}\mathfrak{G}_{k-1}\Big)^2.
\end{aligned}$$

We may absorb the second term on the RHS by the LHS and obtain, for $k\leq k_{large}+7$,

$$\begin{aligned}
&\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widehat{\underline{X}}|^2\\
\lesssim\;&\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widetilde{\mathrm{tr}\underline{X}}|^2+\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}\Big(|\mathfrak{d}^k\widetilde{\mathrm{tr}\underline{X}}|^2+|\mathfrak{d}^{\leq k-1}(\check{\Gamma}\setminus\{\widehat{\underline{X}}\})|^2\Big)\\
&+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|\mathfrak{d}^{\leq k}\check{\Gamma}|^2+\Big(\epsilon_0+{}^{(int)}\mathfrak{R}_k+{}^{(int)}\mathfrak{G}_{k-1}\Big)^2
\end{aligned}$$

as stated. This concludes the proof of Proposition 9.99. □

9.9.5. Estimates for the PT frame of $^{(int)}\mathcal{M}'$ on $\mathcal{T}$ The following lemma provides the control of the ingoing PT structure of $^{(int)}\mathcal{M}$ on $\mathcal{T}$ from the outgoing PT structure of $^{(ext)}\mathcal{M}$.

Lemma 9.100. *The following estimates hold on $\mathcal{T}$ for the ingoing PT structure of $^{(int)}\mathcal{M}$*

$$\int_{\mathcal{T}}\left|\mathfrak{d}^{\leq k}\,{}^{(int)}\check{\Gamma}\right|^2 \lesssim \left(\epsilon_0 + {}^{(ext)}\mathfrak{G}_k + {}^{(int)}\mathfrak{R}_k\right)^2, \qquad k\leq k_{large}+7.$$

Proof. The proof is the analog for the PT frames of $^{(ext)}\mathcal{M}$ and $^{(int)}\mathcal{M}$ of the one of Lemma 7.7 for the PG frames of $^{(ext)}\mathcal{M}$ and $^{(int)}\mathcal{M}$. To simplify the notations, in this proof, we denote:

- by (e_4, e_3, e_1, e_2) the outgoing PT frame of $^{(ext)}\mathcal{M}$, with all quantities associated to the outgoing PT structure of $^{(ext)}\mathcal{M}$ being unprimed,
- by (e'_4, e'_3, e'_1, e'_2) the ingoing PT frame of $^{(int)}\mathcal{M}$, with all quantities associated to the ingoing PT structure of $^{(int)}\mathcal{M}$ being primed.

Recall that $^{(ext)}\mathcal{M}\cap{}^{(int)}\mathcal{M} = \mathcal{T} = \{r=r_0\}$. In view of the above notations, and the initialization of the ingoing PT structure of $^{(int)}\mathcal{M}$ from the outgoing PG structure of $^{(ext)}\mathcal{M}$ on $\mathcal{T}$, see Section 9.1.3, we have

$$\underline{u}=u, \qquad r'=r, \qquad \theta'=\theta, \qquad \mathfrak{J}'=\mathfrak{J} \qquad \text{on} \quad \mathcal{T},$$

and

$$e'_4=\lambda e_4, \qquad e'_3=\lambda^{-1}e_3, \qquad e'_a=e_a,\ \ a=1,2, \quad \text{on} \quad \mathcal{T},$$

where λ is given by

$$\lambda=\frac{\Delta}{|q|^2}.$$

Based on this initialization we derive in particular, for any tangent vector X on $\mathcal{T}$,

$$\begin{aligned}
\mathbf{g}(\mathbf{D}_X e'_4, e'_a) &= \mathbf{g}\left(\mathbf{D}_X(\lambda e_4), e_a\right) = \lambda\mathbf{g}\left(\mathbf{D}_X e_4, e_a\right),\\
\mathbf{g}(\mathbf{D}_X e'_3, e'_a) &= \mathbf{g}\left(\mathbf{D}_X\left(\lambda^{-1}e_3\right), e_a\right) = \lambda^{-1}\mathbf{g}\left(\mathbf{D}_X e_3, e_a\right),\\
\mathbf{g}(\mathbf{D}_X e'_4, e'_3) &= \mathbf{g}\left(\mathbf{D}_X(\lambda e_4), \lambda^{-1}e_3\right) = -2X(\log\lambda)+\mathbf{g}\left(\mathbf{D}_X e_4, e_3\right).
\end{aligned}$$

Note that any such tangent vector X is a linear combination of the following three tangent directions to $\mathcal{T}$,

$$X_1 = e_1 - e_1(r)e_4, \qquad X_2 = e_2 - e_2(r)e_4, \qquad X_3 = e_3 - e_3(r)e_4.$$

Also, note that we have

$$\nabla(r) = \check{\Gamma}, \qquad e_3(r) = -\lambda + \check{\Gamma},$$

so that

$$X_1 = e_1 + \check{\Gamma}e_4, \qquad X_2 = e_2 + \check{\Gamma}e_4, \qquad X_3 = e_3 + \lambda e_4 + \check{\Gamma}e_4,$$

as well as

$$X_1 = e_1' + \check{\Gamma}e_4', \qquad X_2 = e_2' + \check{\Gamma}e_4', \qquad X_3 = \lambda e_3' + e_4' + \check{\Gamma}e_4'.$$

We infer

$$\begin{aligned}
\chi'_{ba} &= \lambda\chi_{ba} + \check{\Gamma},\\
\underline{\chi}'_{ba} &= \lambda^{-1}\underline{\chi}_{ba} + \check{\Gamma},\\
\zeta'_b &= -e_b(\log\lambda) + \zeta_b + \check{\Gamma},
\end{aligned}$$

and

$$\begin{aligned}
\lambda\eta'_a + \xi'_a &= \lambda\eta_a + \lambda^2\xi + \check{\Gamma},\\
\lambda\underline{\xi}'_a + \underline{\eta}'_a &= \lambda^{-1}\underline{\xi}_a + \underline{\eta}_a + \check{\Gamma},\\
-\lambda\underline{\omega}' + \omega' &= -\frac{1}{2}e_3(\log\lambda) - \underline{\omega} + \lambda\omega + \check{\Gamma}.
\end{aligned}$$

Together with

- the fact that, by the PT gauge choices, $\underline{\xi}' = 0$, $\underline{\omega}' = 0$, $H' = \frac{aq'}{|q'|^2}$, $\xi = 0$, $\omega = 0$, $\underline{H} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J}$,
- the form of λ,
- the fact that $r' = r$, $\theta' = \theta$, and $\mathfrak{J}' = \mathfrak{J}$ on $\mathcal{T}$,
- the definition of the linearized quantities for the PT frame in ${}^{(int)}\mathcal{M}$ and ${}^{(ext)}\mathcal{M}$,

we infer[222] on $\mathcal{T}$

[222] See Lemma 7.6 for more precise formulas concerning the analog situation for PG structures.

$$\widetilde{\mathrm{tr}X}', \quad \widehat{X}', \quad \widetilde{\mathrm{tr}\underline{X}}', \quad \underline{\widehat{X}}', \quad \check{Z}', \quad \Xi', \quad \underline{\check{H}}', \quad \omega' = \check{\Gamma}.$$

In view of the definition of ${}^{(ext)}\mathfrak{G}_k$, we have

$$\int_{\mathcal{T}} |\mathfrak{d}^{\leq k}\check{\Gamma}|^2 \lesssim {}^{(ext)}\mathfrak{G}_k^2.$$

We thus infer

$$\int_{\mathcal{T}} \Big|\widetilde{\mathfrak{d}}^{\leq k}\Big(\widetilde{\mathrm{tr}X}', \widehat{X}', \widetilde{\mathrm{tr}\underline{X}}', \underline{\widehat{X}}', \check{Z}', \Xi', \underline{\check{H}}', \omega'\Big)\Big|^2 \lesssim {}^{(ext)}\mathfrak{G}_k^2,$$

where $\widetilde{\mathfrak{d}}^{\leq k}$ denote tangential derivatives to $\mathcal{T}$. Since $\mathfrak{d}$ is generated by e_3' and $\widetilde{\mathfrak{d}}$, we obtain

$$\begin{aligned}
&\int_{\mathcal{T}} \Big|\mathfrak{d}^{\leq k}\Big(\widetilde{\mathrm{tr}X}', \widehat{X}', \widetilde{\mathrm{tr}\underline{X}}', \underline{\widehat{X}}', \check{Z}', \Xi', \underline{\check{H}}', \omega'\Big)\Big|^2 \\
\lesssim\ & {}^{(ext)}\mathfrak{G}_k^2 + \int_{\mathcal{T}} \Big(|\mathfrak{d}^{k-1}\nabla_3'\check{\Gamma}'|^2 + |\mathfrak{d}^{k-1}\check{\Gamma}'|^2\Big).
\end{aligned}$$

Concerning the linearized metric coefficients, we consider

$$\underline{u} = u, \qquad r' = r, \qquad \theta' = \theta, \qquad \mathfrak{J}' = \mathfrak{J} \qquad \text{on} \quad \mathcal{T},$$

and apply to both sides of these identities the tangential vectorfields X to $\mathcal{T}$ as above. Then, proceeding similarly to the linearized Ricci coefficients[223], we obtain on $\mathcal{T}$ relations between the linearized metric coefficients on ${}^{(int)}\mathcal{M}$ and ${}^{(ext)}\mathcal{M}$, which then, together with the above control for linearized Ricci coefficients, yields

$$\int_{\mathcal{T}} \Big|\mathfrak{d}^{\leq k}\check{\Gamma}'\Big|^2 \lesssim {}^{(ext)}\mathfrak{G}_k^2 + \int_{\mathcal{T}} \Big(|\mathfrak{d}^{k-1}\nabla_3'\check{\Gamma}'|^2 + |\mathfrak{d}^{k-1}\check{\Gamma}'|^2\Big).$$

Then, since the PT structure equations take the form $\nabla_3\check{\Gamma} = \check{\Gamma} + \check{R} + \check{\Gamma}\cdot\check{\Gamma}$, and using the bootstrap assumption (9.135), we infer, for $k \leq k_{large} + 7$,

$$\int_{\mathcal{T}} \Big|\mathfrak{d}^{\leq k}\check{\Gamma}'\Big|^2 \lesssim {}^{(ext)}\mathfrak{G}_k^2 + \int_{\mathcal{T}} \Big(|\mathfrak{d}^{\leq k-1}\check{R}'|^2 + |\mathfrak{d}^{\leq k-1}\check{\Gamma}'|^2\Big).$$

Together with the trace theorem and the definition of ${}^{(int)}\mathfrak{R}_k$, we obtain, for $k \leq k_{large} + 7$,

$$\int_{\mathcal{T}} \Big|\mathfrak{d}^{\leq k}\check{\Gamma}'\Big|^2 \lesssim {}^{(ext)}\mathfrak{G}_k^2 + {}^{(int)}\mathfrak{R}_k^2 + \int_{\mathcal{T}} |\mathfrak{d}^{\leq k-1}\check{\Gamma}'|^2.$$

[223]See Lemma 7.6 for the analog situation for PG structures.

Arguing by iteration on k, we deduce, for $k \leq k_{large}+7$,

$$\int_{\mathcal{T}}\left|\mathfrak{d}^{\leq k}\check{\Gamma}'\right|^2 \lesssim {}^{(ext)}\mathfrak{G}_k^2 + {}^{(int)}\mathfrak{R}_k^2 + \int_{\mathcal{T}}|\check{\Gamma}'|^2.$$

Together with (9.42), we infer, for $k \leq k_{large}+7$,

$$\int_{\mathcal{T}}\left|\mathfrak{d}^{\leq k}\check{\Gamma}'\right|^2 \lesssim \left(\epsilon_0 + {}^{(ext)}\mathfrak{G}_k + {}^{(int)}\mathfrak{R}_k\right)^2$$

as desired. □

9.9.6. Proof of Proposition 9.86 We are now ready to control ${}^{(int)}\mathfrak{G}_k$ for $k \leq k_{large}+7$.

9.9.6.1. Iteration assumption Note first from (9.42) that we have

$$(9.159)\qquad {}^{(int)}\mathfrak{G}_{k_{small}-2} + \left(\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|\mathfrak{d}^{\leq k_{small}-2}(\check{\Gamma}\setminus\{\widehat{\underline{X}}\})|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0.$$

This allows us to prove Proposition 9.86 by iteration. To this end, consider the following iteration assumption

$$(9.160)\qquad {}^{(int)}\mathfrak{G}_{k-1} + \left(\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|\mathfrak{d}^{\leq k-1}(\check{\Gamma}\setminus\{\widehat{\underline{X}}\})|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0 + {}^{(ext)}\mathfrak{G}_{k-1} + {}^{(int)}\mathfrak{R}_{k-1}.$$

In view of (9.159), (9.160) holds for $k = k_{small}-1$. From now on, we assume (9.160) for $k_{small} \leq k \leq k_{large}+7$. The proof of Proposition 9.86 will follow from proving (9.160) for k replaced by $k+1$.

9.9.6.2. Control on ${}^{(int)}\mathcal{M}\setminus{}^{(int)}\mathcal{M}_*$ and on $1 \leq \underline{u} \leq 2$ We start with controlling the solution in $\{1 \leq \underline{u} \leq 2\}$ and in ${}^{(int)}\mathcal{M}\setminus{}^{(int)}\mathcal{M}_*$. Note from the definition of ${}^{(int)}\mathcal{M}_* = \{\tau = \tau_*\}$ and τ_*, see Definition 9.33, and the properties of the scalar function τ constructed in Proposition 9.29 that we have

$$(9.161)\qquad {}^{(int)}\mathcal{M}\setminus{}^{(int)}\mathcal{M}_* \subset \{u_* - 4(m+1) \leq \underline{u} \leq u_*\}.$$

We thus need in this step to control the solution on regions of the type

$$(9.162)\qquad \mathcal{R}_{\underline{u}_1,\underline{u}_2} := \{\underline{u}_1 \leq \underline{u} \leq \underline{u}_2\},\ 1 \leq \underline{u}_1 < \underline{u}_1+1 \leq \underline{u}_2 \leq u_*,\quad \underline{u}_2 - \underline{u}_1 \leq 4(m+1).$$

We use the following lemma.

Lemma 9.101. *Let Φ satisfying on ${}^{(int)}\mathcal{M}$ an equation of the type*

$$\nabla_3(\mathfrak{d}^k\Phi) \quad = \quad O(1)\mathfrak{d}^k\Phi + \mathfrak{d}^{\leq k-1}\check{\Gamma} + \mathfrak{d}^{\leq k}\check{R} + \mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma}). \tag{9.163}$$

Then, Φ satisfies on regions $\mathcal{R}_{\underline{u}_1,\underline{u}_2}$ as in (9.162), for $k \leq k_{large}+7$,

$$\int_{\mathcal{R}_{\underline{u}_1,\underline{u}_2}} |\mathfrak{d}^k\Phi|^2 \lesssim \int_{\mathcal{T}} |\mathfrak{d}^k\Phi|^2 + \Big(\epsilon_0 + {}^{(int)}\mathfrak{G}_{k-1} + {}^{(int)}\mathfrak{R}_k\Big)^2. \tag{9.164}$$

Proof. Let $\underline{u}_1 \leq \underline{u} \leq \underline{u}_2$. We integrate the transport equation in r along the level set of $\underline{u}$ which yields, after applying Gronwall lemma and using the fact that $r \leq r_0$ in ${}^{(int)}\mathcal{M}$,

$$\begin{aligned}
&\sup_{r_+-\delta_H\leq r\leq r_0}\int_{S(\underline{u},r)}|\mathfrak{d}^k\Phi|^2\\
\lesssim\ & \int_{S(\underline{u},r_0)}|\mathfrak{d}^k\Phi|^2 + \int_{r_+-\delta_{\mathcal{H}}}^{r_0}\int_{S(r,\underline{u})}\Big|\mathfrak{d}^{\leq k-1}\check{\Gamma} + \mathfrak{d}^{\leq k}\check{R} + \mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma})\Big|^2.
\end{aligned}$$

Since $r \leq r_0$ in ${}^{(int)}\mathcal{M}$, we infer

$$\begin{aligned}
&\int_{r_+-\delta_{\mathcal{H}}}^{r_0}\int_{S(\underline{u},r)}|\mathfrak{d}^k\Phi|^2\\
\lesssim\ & \int_{S(\underline{u},r_0)}|\mathfrak{d}^k\Phi|^2 + \int_{r_+-\delta_{\mathcal{H}}}^{r_0}\int_{S(r,\underline{u})}\Big|\mathfrak{d}^{\leq k-1}\check{\Gamma} + \mathfrak{d}^{\leq k}\check{R} + \mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma})\Big|^2.
\end{aligned}$$

Integrating in $\underline{u}$ for $\underline{u} \in (\underline{u}_1, \underline{u}_2)$, we infer

$$\int_{\mathcal{R}_{\underline{u}_1,\underline{u}_2}}|\mathfrak{d}^k\Phi|^2 \quad\lesssim\quad \int_{\mathcal{T}}|\mathfrak{d}^k\Phi|^2 + \int_{\mathcal{R}_{\underline{u}_1,\underline{u}_2}}\Big|\mathfrak{d}^{\leq k-1}\check{\Gamma} + \mathfrak{d}^{\leq k}\check{R} + \mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma})\Big|^2.$$

Then, using the definition of ${}^{(int)}\mathfrak{R}_k$ and ${}^{(int)}\mathfrak{G}_k$, and the bootstrap assumption (9.135), we have, for $k \leq k_{large}+7$,

$$\begin{aligned}
&\int_{\mathcal{R}_{\underline{u}_1,\underline{u}_2}}\Big|\mathfrak{d}^{\leq k-1}\check{\Gamma} + \mathfrak{d}^{\leq k}\check{R} + \mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma})\Big|^2\\
\lesssim\ & {}^{(int)}\mathfrak{G}_{k-1}^2 + \epsilon_0^2 + \Big(\max_{\mathcal{R}_{\underline{u}_1,\underline{u}_2}}\tau - \max_{\mathcal{R}_{\underline{u}_1,\underline{u}_2}}\tau\Big)\sup_\tau\int_{\Sigma(\tau)\cap{}^{(int)}\mathcal{M}}|\mathfrak{d}^{\leq k}\check{R}|^2\\
\lesssim\ & \Big(\epsilon_0 + {}^{(int)}\mathfrak{G}_{k-1} + {}^{(int)}\mathfrak{R}\Big)^2
\end{aligned}$$

where we have also use the fact that $\underline{u}_2 - \underline{u}_1 \lesssim 1$ by the definition of $\mathcal{R}_{\underline{u}_1,\underline{u}_2}$ and that the variation of τ is controlled by the variation of $\underline{u}$ on ${}^{(int)}\mathcal{M}$ in view of Proposition 9.29. We thus infer, for $k \leq k_{large} + 7$,

$$\int_{\mathcal{R}_{\underline{u}_1,\underline{u}_2}} |\mathfrak{d}^k\Phi|^2 \lesssim \int_{\mathcal{T}} |\mathfrak{d}^k\Phi|^2 + \Big(\epsilon_0 + {}^{(int)}\mathfrak{G}_{k-1} + {}^{(int)}\mathfrak{R}_k\Big)^2$$

as desired. This concludes the proof of Lemma 9.101. $\square$

The set of all linearized Ricci and metric coefficients $\check{\Gamma}$ satisfies, in view of the equations in the ingoing PT structure,

$$\nabla_3(\mathfrak{d}^k\check{\Gamma}) \;=\; O(1)\mathfrak{d}^k\Phi + \mathfrak{d}^{\leq k-1}\check{\Gamma} + \mathfrak{d}^{\leq k}\check{R} + \mathfrak{d}^{\leq k}(\check{\Gamma}\cdot\check{\Gamma}).$$

We may thus apply Lemma 9.101 which yields on regions $\mathcal{R}_{\underline{u}_1,\underline{u}_2}$ as in (9.162), for $k \leq k_{large} + 7$,

$$\int_{\mathcal{R}_{\underline{u}_1,\underline{u}_2}} |\mathfrak{d}^k\check{\Gamma}|^2 \;\lesssim\; \int_{\mathcal{T}} |\mathfrak{d}^k\check{\Gamma}|^2 + \Big(\epsilon_0 + {}^{(int)}\mathfrak{G}_{k-1} + {}^{(int)}\mathfrak{R}_k\Big)^2.$$

Together with the iteration assumption (9.160) and the control on $\mathcal{T}$ for $\check{\Gamma}$ provided by Lemma 9.100, we infer, for $k \leq k_{large} + 7$,

$$\int_{\mathcal{R}_{\underline{u}_1,\underline{u}_2}} |\mathfrak{d}^k\check{\Gamma}|^2 \;\lesssim\; \Big(\epsilon_0 + {}^{(ext)}\mathfrak{G}_k + {}^{(int)}\mathfrak{R}_k\Big)^2. \tag{9.165}$$

In particular, since ${}^{(int)}\mathcal{M} \setminus {}^{(int)}\mathcal{M}_*$ is included in $\mathcal{R}_{\underline{u}_1,\underline{u}_2}$ with $\underline{u}_1 = u_* - 4(m+1)$ and $\underline{u}_2 = u_*$ in view of (9.161), (9.165) yields

$$\begin{aligned}\int_{{}^{(int)}\mathcal{M}\setminus {}^{(int)}\mathcal{M}_*} |\mathfrak{d}^k\check{\Gamma}|^2 \\ \lesssim \Big(\epsilon_0 + {}^{(ext)}\mathfrak{G}_k + {}^{(int)}\mathfrak{R}_k\Big)^2, \quad k \leq k_{large} + 7.\end{aligned} \tag{9.166}$$

Also, choosing $\underline{u}_1 = 1$ and $\underline{u}_2 = 2$, we have in view of (9.165)

$$\int_{1\leq\underline{u}\leq 2} |\mathfrak{d}^k\check{\Gamma}|^2 \;\lesssim\; \Big(\epsilon_0 + {}^{(ext)}\mathfrak{G}_k + {}^{(int)}\mathfrak{R}_k\Big)^2, \quad k \leq k_{large} + 7. \tag{9.167}$$

Let $1 \leq \underline{u}_0 \leq 2$ such that

$$\int_{\underline{u}=\underline{u}_0} |\mathfrak{d}^k\check{\Gamma}|^2 \;=\; \inf_{1\leq\widetilde{\underline{u}}\leq 2} \int_{\underline{u}=\widetilde{\underline{u}}} |\mathfrak{d}^k\check{\Gamma}|^2.$$

Together with (9.167), this yields

$$\begin{aligned}\int_{\underline{u}=\underline{u}_0}|\mathfrak{d}^k\check{\Gamma}|^2 &\leq \int_1^2\int_{\underline{u}=\widetilde{\underline{u}}}|\mathfrak{d}^k\check{\Gamma}|^2 d\widetilde{\underline{u}}\\ &\lesssim \int_{1\leq\underline{u}\leq 2}|\mathfrak{d}^k\check{\Gamma}|^2\\ &\lesssim \left(\epsilon_0+{}^{(ext)}\mathfrak{G}_k+{}^{(int)}\mathfrak{R}_k\right)^2\end{aligned}$$

so that

$$\int_{\underline{u}=\underline{u}_0}|\mathfrak{d}^k\check{\Gamma}|^2 \lesssim \left(\epsilon_0+{}^{(ext)}\mathfrak{G}_k+{}^{(int)}\mathfrak{R}_k\right)^2, \quad k\leq k_{large}+7. \tag{9.168}$$

In view of (9.168), we should:

- reduce ${}^{(int)}\mathcal{M}_*$ to ${}^{(int)}\mathcal{M}_*\cap\{\underline{u}\geq\underline{u}_0\}$ so that the boundary terms on $\{\underline{u}=\underline{u}_0\}$ in the integration by parts of Propositions 9.95 and 9.99 are under control,
- control the remaining region $\{1\leq\underline{u}\leq\underline{u}_0\}$ thanks to (9.167).

For simplicity, we pretend that $\underline{u}_0=1$, i.e. we assume from now by a slight abuse that we have (9.168) with $\underline{u}_0=1$ and hence

$$\int_{\underline{u}=1}|\mathfrak{d}^k\check{\Gamma}|^2 \lesssim \left(\epsilon_0+{}^{(ext)}\mathfrak{G}_k+{}^{(int)}\mathfrak{R}_k\right)^2, \quad k\leq k_{large}+7. \tag{9.169}$$

9.9.6.3. Control on ${}^{(int)}\mathcal{M}_*$ Note that (9.166) provides the desired control of $\check{\Gamma}$ on ${}^{(int)}\mathcal{M}\setminus{}^{(int)}\mathcal{M}_*$. It thus remains to control $\check{\Gamma}$ on ${}^{(int)}\mathcal{M}_*$. We start with the following corollary of Propositions 9.95 and 9.99.

Corollary 9.102. *Φ is any quantity among $\check{\Gamma}\setminus\{\widehat{\underline{X}}\}$, i.e. Φ is any linearized Ricci or metric coefficient of the ingoing PT structure of ${}^{(int)}\mathcal{M}$, then Φ satisfies, for $k\leq k_{large}+7$,*

$$\begin{aligned}\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|\mathfrak{d}^k\Phi|^2+\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\Phi|^2 &\lesssim \left(\epsilon_0+{}^{(int)}\mathfrak{R}_k+{}^{(ext)}\mathfrak{G}_k\right)^2\\ &\quad+\int_{{}^{(int)}\mathcal{M}_*}|\check{\Gamma}_k[\Phi]|^2.\end{aligned} \tag{9.170}$$

Moreover, the remaining component $\widehat{\underline{X}}$ satisfies, for $k\leq k_{large}+7$,

$$\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widehat{\underline{X}}|^2 \lesssim \int_{{}^{(int)}\mathcal{M}_*}|\check{\Gamma}_k[\widehat{\underline{X}}]|^2+\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|\check{\Gamma}_k[\widehat{\underline{X}}]|^2$$

$$+\Big(\epsilon_0 + {}^{(int)}\mathfrak{R}_k + {}^{(ext)}\mathfrak{G}_k\Big)^2 \tag{9.171}$$

with the notation

$$\check{\Gamma}_k[\widehat{\underline{X}}] \quad := \quad \mathfrak{d}^k \widecheck{tr\underline{X}}. \tag{9.172}$$

Proof. Consider first the case of Φ in $\check{\Gamma}\setminus\{\widehat{\underline{X}}\}$. Then, according to Proposition 9.95, Φ satisfies, for $k\le k_{large}+7$,

$$\begin{aligned}
&\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|\mathfrak{d}^k\Phi|^2 + \int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\Phi|^2\\
\lesssim\ & \Big(\epsilon_0 + {}^{(int)}\mathfrak{R}_k + {}^{(int)}\mathfrak{G}_{k-1}\Big)^2 + \int_{{}^{(int)}\mathcal{M}_*}|\check{\Gamma}_k[\Phi]|^2 + \int_{\mathcal{T}\cup\{\underline{u}=1\}}|\mathfrak{d}^k\check{\Gamma}|^2.
\end{aligned}$$

Together with the iteration assumption (9.160), the control on $\mathcal{T}$ for $\check{\Gamma}$ provided by Lemma 9.100, and the control of $\check{\Gamma}$ on $\underline{u}=1$ provided by (9.169), we infer, for $k\le k_{large}+7$,

$$\begin{aligned}
&\int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|\mathfrak{d}^k\Phi|^2 + \int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\Phi|^2\\
\lesssim\ & \Big(\epsilon_0 + {}^{(int)}\mathfrak{R}_k + {}^{(ext)}\mathfrak{G}_k\Big)^2 + \int_{{}^{(int)}\mathcal{M}_*}|\check{\Gamma}_k[\Phi]|^2
\end{aligned}$$

as stated.

Next, we focus on the estimate for $\widehat{\underline{X}}$. According to Proposition 9.99, $\widehat{\underline{X}}$ satisfies, for $k\le k_{large}+7$,

$$\begin{aligned}
&\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widehat{\underline{X}}|^2\\
\lesssim\ & \int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widecheck{\mathrm{tr}\underline{X}}|^2 + \int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}\Big(|\mathfrak{d}^k\widecheck{\mathrm{tr}\underline{X}}|^2 + |\mathfrak{d}^{\le k-1}(\check{\Gamma}\setminus\{\widehat{\underline{X}}\})|^2\Big)\\
&+\int_{\mathcal{T}\cup\{\underline{u}=1\}}|\mathfrak{d}^{\le k}\check{\Gamma}|^2 + \Big(\epsilon_0 + {}^{(int)}\mathfrak{R}_k + {}^{(int)}\mathfrak{G}_{k-1}\Big)^2.
\end{aligned}$$

Together with the iteration assumption (9.160), the control on $\mathcal{T}$ for $\check{\Gamma}$ provided by Lemma 9.100, and the control of $\check{\Gamma}$ on $\underline{u}=1$ provided by (9.169), we infer, for $k\le k_{large}+7$,

$$\begin{aligned}
\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widehat{\underline{X}}|^2 \ \lesssim\ & \int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widecheck{\mathrm{tr}\underline{X}}|^2 + \int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|\mathfrak{d}^k\widecheck{\mathrm{tr}\underline{X}}|^2\\
&+\Big(\epsilon_0 + {}^{(int)}\mathfrak{R}_k + {}^{(ext)}\mathfrak{G}_k\Big)^2.
\end{aligned}$$

Together with the notation $\check{\Gamma}_k[\widehat{\underline{X}}] = \mathfrak{d}^k\widecheck{\mathrm{tr}X}$, we deduce, for $k \leq k_{large}+7$,

$$\int_{{}^{(int)}\mathcal{M}_*}|\mathfrak{d}^k\widehat{\underline{X}}|^2 \lesssim \int_{{}^{(int)}\mathcal{M}_*}|\check{\Gamma}_k[\widehat{\underline{X}}]|^2 + \int_{{}^{(int)}\Sigma_*\cup\mathcal{A}}|\check{\Gamma}_k[\widehat{\underline{X}}]|^2 + \Big(\epsilon_0 + {}^{(int)}\mathfrak{R}_k + {}^{(ext)}\mathfrak{G}_k\Big)^2$$

as stated. This concludes the proof of Corollary 9.102. □

Recall from Proposition 9.91 and Corollary 9.102 that $\check{\Gamma}_k[\Phi]$ given for each component of $\check{\Gamma}$ by

$$
\begin{aligned}
\check{\Gamma}_k[\widecheck{\mathrm{tr}\underline{X}}] &= 0,\\
\check{\Gamma}_k[\widehat{\underline{X}}] &= \mathfrak{d}^k\widecheck{\mathrm{tr}\underline{X}}, \qquad \check{\Gamma}_k[\widecheck{\mathcal{D}\cos\theta}] = \mathfrak{d}^k\Big(\widecheck{\mathrm{tr}\underline{X}},\ \widehat{\underline{X}}\Big),\\
\check{\Gamma}_k[\check{Z}] &= \mathfrak{d}^k\Big(\widehat{\underline{X}},\ \widecheck{\mathrm{tr}\underline{X}}\Big), \qquad \check{\Gamma}_k[\widecheck{\underline{H}}] = \mathfrak{d}^k\Big(\widehat{\underline{X}},\ \widecheck{\mathrm{tr}\underline{X}}\Big),\\
\check{\Gamma}_k[\mathcal{D}r] &= \mathfrak{d}^k\check{Z}, \qquad \check{\Gamma}_k[e_4(\cos\theta)] = \mathfrak{d}^k\Big(\widecheck{\underline{H}}, \widecheck{\mathcal{D}(\cos\theta)}\Big),\\
\check{\Gamma}_k[\check{\omega}] &= \mathfrak{d}^k\Big(\check{Z},\ \widecheck{\underline{H}}\Big),\\
\check{\Gamma}_k[\mathcal{D}\widehat{\otimes}\mathfrak{J}] &= \mathfrak{d}^k\Big(\widecheck{\mathrm{tr}\underline{X}},\ \widehat{\underline{X}},\ \check{Z},\ \widecheck{\mathcal{D}(\cos\theta)}\Big),\\
\check{\Gamma}_k[\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}] &= \mathfrak{d}^k\Big(\widecheck{\mathrm{tr}\underline{X}},\ \widehat{\underline{X}},\ \check{Z},\ \widecheck{\mathcal{D}(\cos\theta)}\Big),\\
\check{\Gamma}_k[\widecheck{\mathrm{tr}X}] &= \mathfrak{d}^k\Big(\widecheck{\mathrm{tr}\underline{X}},\ \widecheck{\mathcal{D}\cdot\overline{\mathfrak{J}}},\ \mathcal{D}r,\ \widecheck{\mathcal{D}(\cos\theta)}\Big),\\
\check{\Gamma}_k[\widehat{X}] &= \mathfrak{d}^k\Big(\mathcal{D}\widehat{\otimes}\mathfrak{J},\ \widecheck{\mathcal{D}(\cos\theta)},\ \widehat{\underline{X}}\Big),\\
\check{\Gamma}_k[\widecheck{e_4(r)}] &= \mathfrak{d}^k\Big(\check{\omega}, \mathcal{D}r\Big),\\
\check{\Gamma}_k[\widecheck{\nabla_4\mathfrak{J}}] &= \mathfrak{d}^k\Big(\widecheck{e_4(r)},\ e_4(\cos\theta),\ \check{\omega},\ \widecheck{\underline{H}},\ \widecheck{\nabla\mathfrak{J}}\Big),\\
\check{\Gamma}_k[\Xi] &= \mathfrak{d}^k\Big(\widecheck{\underline{H}},\ \widecheck{\mathrm{tr}X},\ \widehat{X},\ \widecheck{\nabla_4\mathfrak{J}},\ \widecheck{e_4(r)},\ e_4(\cos\theta)\Big).
\end{aligned}
\tag{9.173}
$$

We now proceed as follows to control $\check{\Gamma}$ on ${}^{(int)}\mathcal{M}_*$. We use (9.170) to control all the components in $\check{\Gamma}\setminus\{\widehat{\underline{X}}\}$ and (9.171) to control $\widehat{\underline{X}}$. The components of $\check{\Gamma}$ are then recovered in the following order, suggested by the triangular structure in (9.173),

$$
\begin{aligned}
&\widecheck{\mathrm{tr}\underline{X}},\ \widehat{\underline{X}},\ \widecheck{\mathcal{D}\cos\theta},\ \check{Z},\ \widecheck{\underline{H}},\ \mathcal{D}r,\ e_4(\cos\theta),\ \check{\omega},\\
&\mathcal{D}\widehat{\otimes}\mathfrak{J},\ \widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}},\ \widecheck{\mathrm{tr}X},\ \widehat{X},\ \widecheck{e_4(r)},\ \widecheck{\nabla_4\mathfrak{J}},\ \Xi.
\end{aligned}
$$

The crucial point of this triangular structure is that at each step, when estimating a component Φ on $\check{\Gamma}$, all the components in the term $\check{\Gamma}_k[\Phi]$ on the RHS have been already estimated. We easily obtain, for $k \leq k_{large}+7$,

$$\int_{^{(int)}\Sigma_*\cup\mathcal{A}} |\mathfrak{d}^k(\check{\Gamma}\setminus\{\underline{\widehat{X}}\})|^2 + \int_{^{(int)}\mathcal{M}_*} |\mathfrak{d}^k\check{\Gamma}|^2 \lesssim \left(\epsilon_0 + {}^{(int)}\mathfrak{R}_k + {}^{(ext)}\mathfrak{G}_k\right)^2.$$

Together with the control on ${}^{(int)}\mathcal{M}\setminus{}^{(int)}\mathcal{M}_*$ provided by (9.166), we infer

$$\int_{^{(int)}\Sigma_*\cup\mathcal{A}} |\mathfrak{d}^k(\check{\Gamma}\setminus\{\underline{\widehat{X}}\})|^2 + \int_{^{(int)}\mathcal{M}} |\mathfrak{d}^k\check{\Gamma}|^2 \lesssim \left(\epsilon_0 + {}^{(int)}\mathfrak{R}_k + {}^{(ext)}\mathfrak{G}_k\right)^2.$$

Together with iteration assumption (9.160), and in view of the definition of ${}^{(int)}\mathfrak{G}_k$, we deduce

$${}^{(int)}\mathfrak{G}_k + \left(\int_{^{(int)}\Sigma_*\cup\mathcal{A}} |\mathfrak{d}^{\leq k}(\check{\Gamma}\setminus\{\underline{\widehat{X}}\})|^2\right)^{\frac{1}{2}} \lesssim \epsilon_0 + {}^{(int)}\mathfrak{R}_k + {}^{(ext)}\mathfrak{G}_k.$$

This is the iteration assumption (9.160) with k replaced by $k+1$. We have thus obtained

$${}^{(int)}\mathfrak{G}_k \lesssim \epsilon_0 + {}^{(int)}\mathfrak{R}_k + {}^{(ext)}\mathfrak{G}_k, \quad k \leq k_{large}+7.$$

This concludes the proof of Proposition 9.86.

9.10. Control of the PT-Ricci coefficients in ${}^{(top)}\mathcal{M}'$

The goal of this section is to provide the proof of Proof of Proposition 9.53 recalled in Proposition 9.103 below. Since we will not need to refer to the old region ${}^{(top)}\mathcal{M}$, defined w.r.t. the PG frame, we drop the prime of ${}^{(top)}\mathcal{M}'$ in this section.

9.10.1. Preliminaries Recall the following norm for the linearized Ricci and metric coefficients on ${}^{(top)}\mathcal{M}$ introduced in Section 9.4.1, for $k \leq k_{large}+7$,

$${}^{(top)}\mathfrak{G}_k^2 = \int_{^{(top)}\mathcal{M}(r\leq r_0)} |\mathfrak{d}^{\leq k}\check{\Gamma}|^2 + \left({}^{(top)}\mathfrak{G}_k^{\geq r_0}\right)^2$$

where $\check{\Gamma}$ denotes the set of all linearized Ricci and metric coefficients with respect to the ingoing PT frame of ${}^{(top)}\mathcal{M}$, and where

$$
\begin{aligned}
\left({}^{(top)}\mathfrak{G}_k^{\geq r_0}\right)^2 := &\sup_{\underline{u}_1\geq u'_*}\int_{{}^{(top)}\mathcal{M}_{r_0,\underline{u}_1}}\Big(r^2\big|\mathfrak{d}^{\leq k}(\Xi,\check{\omega},\widehat{X},\widecheck{\mathrm{tr}X},\check{Z},\widecheck{\underline{H}})\big|^2\\
&\qquad\qquad + r^{2-\delta_B}|\mathfrak{d}^{\leq k}\widecheck{\mathrm{tr}\underline{X}}| + \big|\mathfrak{d}^{\leq k}\widehat{\underline{X}}\big|^2\Big)\\
&+\sup_{\underline{u}_1\geq u'_*}\int_{{}^{(top)}\mathcal{M}_{r_0,\underline{u}_1}}\Big(\big|\mathfrak{d}^{\leq k}\widecheck{\mathcal{D}\cos\theta}\big|^2+\big|\mathfrak{d}^{\leq k}\mathcal{D}r\big|^2\Big)\\
&+\sup_{\underline{u}_1\geq u'_*}\int_{{}^{(top)}\mathcal{M}_{r_0,\underline{u}_1}}\Big(r^4\big|\mathfrak{d}^{\leq k}e_4(\cos\theta)\big|^2+r^4\big|\mathfrak{d}^{\leq k}\widecheck{e_4(r)}\big|^2\Big)\\
&+\sup_{\underline{u}_1\geq u'_*}\int_{{}^{(top)}\mathcal{M}_{r_0,\underline{u}_1}} r^2\Big(\big|\mathfrak{d}^{\leq k}\mathcal{D}\widehat{\otimes}\mathfrak{J}\big|^2+\big|\mathfrak{d}^{\leq k}\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}\big|^2\\
&\qquad\qquad + r^6\big|\mathfrak{d}^{\leq k}\widecheck{\nabla_4\mathfrak{J}}\big|^2\Big),
\end{aligned}
$$

with Ξ, $\check{\omega}$, $\widecheck{\mathrm{tr}X}$, $\widehat{X}$, $\check{Z}$, $\widecheck{\underline{H}}$, $\widecheck{\mathrm{tr}\underline{X}}$, $\widehat{\underline{X}}$ the linearized Ricci coefficients of the ingoing PT frame of ${}^{(top)}\mathcal{M}$, and with the notation

$$
{}^{(top)}\mathcal{M}_{r_0,\underline{u}_1} \quad := \quad {}^{(top)}\mathcal{M}(r\geq r_0)\cap\{\underline{u}_1\leq \underline{u}\leq\underline{u}_1+1\}.
$$

For the curvature norms in ${}^{(top)}\mathcal{M}$, we rely in particular on the scalar function τ introduced in Section 9.3.4 and define

$$
\begin{aligned}
{}^{(top)}\mathfrak{R}_k^2 := &\int_{{}^{(top)}\mathcal{M}(r\geq r_0)}\Big(r^{3+\delta_B}|\mathfrak{d}^{\leq k}(A,B)|^2+r^{3-\delta_B}|\mathfrak{d}^{\leq k}\check{P}|^2\Big)\\
&+\sup_{\underline{u}_1\geq u'_*}\int_{{}^{(top)}\mathcal{M}_{r_0,\underline{u}_1}}\Big(r^2|\mathfrak{d}^{\leq k}\underline{B}|^2+|\mathfrak{d}^{\leq k}\underline{A}|^2\Big)\\
&+\sup_{\tau}\int_{{}^{(top)}\mathcal{M}(r\leq r_0)\cap\Sigma(\tau)}|\mathfrak{d}^{\leq k}\check{R}|^2,
\end{aligned}
$$

where $\check{R}$ is the set of all linearized curvature coefficients w.r.t. the ingoing PT frame of ${}^{(top)}\mathcal{M}$, and A, B, $\check{P}$, $\underline{B}$, $\underline{A}$ denote the linearized curvature components relative to the ingoing PT frame of ${}^{(top)}\mathcal{M}$.

Also, recall the definition of $L_*(k)$, see (9.53),

$$
L_*^2(k) = \int_{\{u=u'_*\}}|\check{R}|^2_{w,k}+\int_{\Sigma_*\cap\{u=u'_*\}}|\check{\Gamma}|^2_{w,k} \tag{9.174}
$$

where, see (9.54),

$$
\begin{aligned}
|\check{R}|^2_{w,k} &:= r^{3+\delta_B}|\mathfrak{d}_*^{\leq k}(A,B)|^2+r^{3-\delta_B}|\mathfrak{d}_*^{\leq k}\check{P}|^2+r^{1-\delta_B}|\mathfrak{d}_*^{\leq k}\underline{B}|^2,\\
|\check{\Gamma}|^2_{w,k} &:= r^2|\not{\mathfrak{d}}^{\leq k}\Gamma_g|^2+|\not{\mathfrak{d}}^{\leq k}\Gamma_b|^2.
\end{aligned} \tag{9.175}
$$

In (9.175), $(A, B, \check{P}, \underline{B})$ denote linearized curvature components w.r.t. the the outgoing PT frame of ${}^{(ext)}\mathcal{M}$, Γ_g and Γ_b are defined w.r.t. the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ as in Definition 9.17, $\mathfrak{d}_*$ denote weighted derivatives tangential to the hypersurface $\{u = u'_*\}$, and $\not{\mathfrak{d}}$ denote weighted derivatives tangential to the sphere $\Sigma_* \cap \{u = u'_*\}$.

The goal of Section 9.10 is to provide the proof of Proof of Proposition 9.53. For convenience, we restate the result below.

Proposition 9.103. *Relative to the ingoing PT frame of ${}^{(top)}\mathcal{M}$, we have*

$$ {}^{(top)}\mathfrak{G}_k \lesssim \epsilon_0 + L_*(k) + {}^{(top)}\mathfrak{R}_k, \quad k \leq k_{large} + 7. \tag{9.176}$$

In particular, we have

$$ {}^{(top)}\mathfrak{G}_k^{\geq r_0} \lesssim \epsilon_0 + L_*(k) + {}^{(top)}\mathfrak{R}_k, \quad k \leq k_{large} + 7, \tag{9.177}$$

in which case the constant in $\lesssim$ is independent of r_0.

To prove Proposition 9.103, we rely in particular on our bootstrap assumption **BA-PT**, see (9.40), which implies for the Ricci and metric coefficients of the ingoing PT frame of ${}^{(top)}\mathcal{M}$

$$ {}^{(top)}\mathfrak{G}_k \leq \epsilon, \qquad k \leq k_{large} + 7. \tag{9.178}$$

9.10.2. Control on the hypersurface $\{u = u'_*\}$ Our goal in this section is to control the linearized Ricci and metric coefficients of the ingoing PT frame of ${}^{(top)}\mathcal{M}$ on $\{u = u'_*\}$. We start with the following transport lemma along the hypersurface $\{u = u'_*\}$ of ${}^{(ext)}\mathcal{M}$.

Lemma 9.104. *Let U and F anti-selfdual k-tensors on $\{u = u'_*\}$. Assume that U verifies one of the following equations, for a real constant c,*

$$ \nabla_4 U + \frac{c}{q} U = F \tag{9.179}$$

or

$$ \nabla_4 U + \Re\left(\frac{c}{q}\right) U = F. \tag{9.180}$$

In both cases we derive, for any $r_0 \leq r \leq r_* = r_*(u'_*)$

$$ r^{c-1}\|\widetilde{\mathfrak{d}}^{\leq k} U\|_{L^2(S(u'_*,r))} \lesssim r_*^{c-1}\|\widetilde{\mathfrak{d}}^{\leq k} U\|_{L^2(S(u'_*,r_*))} \tag{9.181}$$

$$+\int_r^{r_*}\lambda^{c-1}\|\widetilde{\mathfrak{d}}^{\leq k}F\|_{L^2(S(u'_*,\lambda))}d\lambda,$$

where $\widetilde{\mathfrak{d}}$ *denotes weighted derivatives tangential to* $\{u = u'_*\}$, *i.e.*

$$\widetilde{\mathfrak{d}} \quad := \quad \left\{r\nabla_4,\, r\left(\nabla_1 - \frac{1}{e_3(u)}e_1(u)\nabla_3\right),\, r\left(\nabla_2 - \frac{1}{e_3(u)}e_2(u)\nabla_3\right)\right\}.$$

Proof. This follows from a straightforward adaptation of Proposition 9.83 which is our main transport lemma in ${}^{(ext)}\mathcal{M}$. □

Next, we control the linearized Ricci and metric coefficients of the outgoing PT frame of ${}^{(ext)}\mathcal{M}$ on the hypersurface $\{u = u'_*\}$.

Proposition 9.105. *Along the hypersurface* $\{u = u'_*\}$*, the outgoing PT frame of* ${}^{(ext)}\mathcal{M}$ *satisfies the following estimates, for* $k \leq k_{large} + 7$*,*

(9.182)
$$\sup_{S\subset\{u=u'_*\}}\left(r^2\|\widetilde{\mathfrak{d}}^{\leq k}(\Gamma_g\setminus\{\widecheck{tr\underline{X}}\})\|^2_{L^2(S)} + r^{2-\delta_B}\|\widetilde{\mathfrak{d}}^{\leq k}\widecheck{tr\underline{X}}\|^2_{L^2(S)}\right)$$
$$+\sup_{S\subset\{u=u'_*\}}\left(\|\widetilde{\mathfrak{d}}^{\leq k}(\Gamma_b\setminus\{\underline{\Xi}\})\|^2_{L^2(S)} + r^{-\delta_B}\|\widetilde{\mathfrak{d}}^{\leq k}\underline{\Xi}\|^2_{L^2(S)}\right) \lesssim \epsilon_0^2 + L_*^2(k)$$

where the constant in the definition of $\lesssim$ *is independent of* r_0.

Proof. We proceed as follows:

1. We rely on the equations for the linearized Ricci and metric coefficients of the PT structure of ${}^{(ext)}\mathcal{M}$, see Proposition 9.19 and Lemmas 9.21 and 9.22, and the control of transport equations along e_4 provided by Lemma 9.104.
2. We integrate the transport equations in the order consistent with the triangular structure of the system, i.e., as in Section 9.8.2, we estimate the linearized Ricci and metric coefficients of the outgoing PT structure of ${}^{(ext)}\mathcal{M}$ in the following order

$$\widecheck{\mathrm{tr}X},\ \widehat{X},\ \mathcal{D}\cos\theta,\ \check{Z},\ \mathcal{D}r,\ \check{H},\ \widecheck{e_3(r)},\ \underline{\check{\omega}},\ \mathcal{D}\widehat{\otimes}\mathfrak{J},\ \widecheck{\overline{\mathcal{D}}\cdot\overline{\mathfrak{J}}},$$
$$e_3(\cos\theta),\ \widecheck{e_3\mathfrak{J}},\ \underline{\Xi}.$$

3. We also make use of the bootstrap assumption (9.128) for the outgoing PT frame of ${}^{(ext)}\mathcal{M}$.

As a result, we obtain the control of the linearized Ricci and metric coefficients of the outgoing PT structure of ${}^{(ext)}\mathcal{M}$ by the following two contributions:

- their restriction to $S(u'_*, r_*) = \Sigma_* \cap \{u = u'_*\}$,
- r^p weighed norms of curvature along $\{u = u_*\}$,

which are precisely in the form of $L_*(k)$, see (9.174). The proof is very similar to the one in Section 9.8.2, and in fact simpler. We leave the details to the reader. □

We are now ready to state the main result of this section on the control of the linearized Ricci and metric coefficients of the ingoing PT frame of ${}^{(top)}\mathcal{M}$ on $\{u = u'_*\}$.

Proposition 9.106. *Along the hypersurface $\{u = u'_*\}$, the ingoing PT frame of ${}^{(top)}\mathcal{M}$ satisfies the following estimates, for $k \leq k_{large} + 7$,*

(9.183)
$$\sup_{S\subset\{u=u'_*\}}\Big(r^2\|\mathfrak{d}^{\leq k}(\Gamma_g\setminus\{\widetilde{tr\underline{X}}\})\|^2_{L^2(S)} + r^{2-\delta_B}\|\mathfrak{d}^{\leq k}\widetilde{tr\underline{X}}\|^2_{L^2(S)}\Big) + \sup_{S\subset\{u=u'_*\}}\|\mathfrak{d}^{\leq k}\Gamma_b\|^2_{L^2(S)} \lesssim \epsilon_0^2 + L_*^2(k)$$

where the constant in the definition of $\lesssim$ is independent of r_0.

Proof. The proof is the analog for the PT frames of ${}^{(ext)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$ of the one of Lemma 7.14 for the PG frames of ${}^{(ext)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$. To simplify the notations, in this proof, we denote:

- by (e_4, e_3, e_1, e_2) the outgoing PT frame of ${}^{(ext)}\mathcal{M}$, with all quantities associated to the outgoing PT structure of ${}^{(ext)}\mathcal{M}$ being unprimed,
- by (e'_4, e'_3, e'_1, e'_2) the ingoing PT frame of ${}^{(top)}\mathcal{M}$, with all quantities associated to the ingoing PT structure of ${}^{(int)}\mathcal{M}$ being primed.

In view of the above notations, and the initialization of the ingoing PT structure of ${}^{(top)}\mathcal{M}$ from the outgoing PG structure of ${}^{(ext)}\mathcal{M}$ on $\{u = u'_*\}$, see Section 9.1.3, we have

$$\underline{u} = u + 2\int_{r_0}^{r}\frac{\tilde{r}^2 + a^2}{\tilde{r}^2 - 2m\tilde{r} + a^2}d\tilde{r}, \quad r' = r, \quad \theta' = \theta, \quad \mathfrak{J}' = \mathfrak{J} \quad \text{on} \quad \{u = u'_*\},$$

and

$$e'_4 = \lambda e_4, \qquad e'_3 = \lambda^{-1}e_3, \qquad e'_a = e_a, \ a = 1, 2, \quad \text{on} \quad \{u = u'_*\},$$

where λ is given by

$$\lambda = \frac{\Delta}{|q|^2}.$$

Based on this initialization we derive in particular, for any tangent vector X on $\{u = u'_*\}$,

$$\begin{aligned}
\mathbf{g}(\mathbf{D}_X e'_4, e'_a) &= \mathbf{g}\left(\mathbf{D}_X\left(\lambda e_4\right), e_a\right) = \lambda \mathbf{g}\left(\mathbf{D}_X e_4, e_a\right),\\
\mathbf{g}(\mathbf{D}_X e'_3, e'_a) &= \mathbf{g}\left(\mathbf{D}_X\left(\lambda^{-1} e_3\right), e_a\right) = \lambda^{-1}\mathbf{g}\left(\mathbf{D}_X e_3, e_a\right),\\
\mathbf{g}(\mathbf{D}_X e'_4, e'_3) &= \mathbf{g}\left(\mathbf{D}_X\left(\lambda e_4\right), \lambda^{-1} e_3\right) = -2X(\log\lambda) + \mathbf{g}\left(\mathbf{D}_X e_4, e_3\right).
\end{aligned}$$

Note that any such tangent vector X is a linear combination of the following three tangent directions to $\{u = u'_*\}$,

$$X_1 = e_1 - \frac{e_1(u)}{e_3(u)} e_3, \qquad X_2 = e_2 - \frac{e_2(u)}{e_3(u)} e_3, \qquad X_4 = e_4.$$

Also, note that we have

$$\nabla(u) = a\Re(\mathfrak{J}) + \widetilde{\Gamma}_b, \qquad e_3(u) = \frac{2(r^2+a^2)}{|q|^2} + r\widetilde{\Gamma}_b,$$

where we have introduced the following notation[224]

$$\widetilde{\Gamma}_b := \Gamma_b \setminus \{\underline{\Xi}\}, \qquad \widetilde{\Gamma}_g := \Gamma_g \setminus \{\widetilde{\mathrm{tr}\underline{X}}\}.$$

We infer

$$X_b = e_b + \left(-\frac{a|q|^2}{2(r^2+a^2)}(\Re(\mathfrak{J}))_b + \Gamma_b\right) e_3, \;\; b = 1,2, \qquad X_4 = e_4,$$

as well as

$$X_b = e'_b + \left(-\frac{a|q|^2}{2(r^2+a^2)}(\Re(\mathfrak{J}))_b + \widetilde{\Gamma}_b\right) \lambda e'_3, \;\; b = 1,2, \qquad X_4 = \lambda^{-1} e'_4.$$

We infer

$$\begin{aligned}
&\chi'_{ba} + 2\left(-\frac{a|q|^2}{(r^2+a^2)}(\Re(\mathfrak{J}))_b + \widetilde{\Gamma}_b\right)\lambda\eta'_a\\
&= \lambda\chi_{ba} + 2\left(-\frac{a|q|^2}{2(r^2+a^2)}(\Re(\mathfrak{J}))_b + \Gamma_b\right)\lambda\eta_a,
\end{aligned}$$

[224]Recall that $\underline{\Xi}$ and $\widetilde{\mathrm{tr}\underline{X}}$ exhibit a $r^{-\frac{\delta_B}{2}}$ loss compared to the other components.

$$
\begin{aligned}
& \underline{\chi}'_{ba} + 2\left(-\frac{a|q|^2}{2(r^2+a^2)}(\Re(\mathfrak{J}))_b + \widetilde{\Gamma}_b\right)\lambda\underline{\xi}'_a \\
= \;& \lambda^{-1}\underline{\chi}_{ba} + 2\left(-\frac{a|q|^2}{2(r^2+a^2)}(\Re(\mathfrak{J}))_b + \widetilde{\Gamma}_b\right)\lambda^{-1}\underline{\xi}_a, \\
& \zeta'_b - 2\left(-\frac{a|q|^2}{(r^2+a^2)}(\Re(\mathfrak{J}))_b + \widetilde{\Gamma}_b\right)\lambda\underline{\omega}' \\
= \;& -\left(e_b + \left(-\frac{a|q|^2}{2(r^2+a^2)}(\Re(\mathfrak{J}))_b + \Gamma_b\right)e_3\right)\log\lambda \\
& + \zeta_b - 2\left(-\frac{a|q|^2}{(r^2+a^2)}(\Re(\mathfrak{J}))_b + \widetilde{\Gamma}_b\right)\lambda\underline{\omega},
\end{aligned}
$$

and

$$
\begin{aligned}
\xi' &= \lambda^2\xi, \\
\underline{\eta}' &= \underline{\eta}, \\
\omega' &= -\frac{1}{2}e_4(\log\lambda) + \omega.
\end{aligned}
$$

Together with

- the fact that, by the PT gauge choices, $\underline{\xi}' = 0$, $\underline{\omega}' = 0$, $H' = \frac{aq'}{|q'|^2}$, $\xi = 0$, $\omega = 0$, $\underline{H} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J}$,
- the form of λ,
- the fact that $r' = r$, $\theta' = \theta$, and $\mathfrak{J}' = \mathfrak{J}$ on $\{u = u'_*\}$,
- the definition of the linearized quantities for the PT frame in ${}^{(top)}\mathcal{M}$ and ${}^{(ext)}\mathcal{M}$,

we infer on $\{u = u'_*\}$

$$
\widecheck{\mathrm{tr}X}', \, \widehat{X}', \, \check{Z}' = \widetilde{\Gamma}_g, \qquad \widecheck{\mathrm{tr}\underline{X}}' = \lambda\widecheck{\mathrm{tr}\underline{X}} + O(r^{-1})\underline{\Xi}, \qquad \widehat{\underline{X}}' = \widetilde{\Gamma}_b,
$$

and

$$
\Xi' = 0, \qquad \widecheck{\underline{H}}' = 0, \qquad \check{\omega}' = 0.
$$

We deduce

$$
\begin{aligned}
\sup_{S\subset\{u=u'_*\}} & \Big(r^2\|\widetilde{\mathfrak{d}}^{\le k}(\Xi', \omega', \widecheck{\mathrm{tr}X}', \widehat{X}', \check{Z}', \widecheck{\underline{H}}')\|^2_{L^2(S)} \\
& + r^{2-\delta_B}\|\widetilde{\mathfrak{d}}^{\le k}\widecheck{\mathrm{tr}\underline{X}}'\|^2_{L^2(S)}\Big) + \sup_{S\subset\{u=u'_*\}}\|\widetilde{\mathfrak{d}}^{\le k}\widehat{\underline{X}}'\|^2_{L^2(S)}
\end{aligned}
$$

$$\lesssim \sup_{S\subset\{u=u'_*\}}\Big(r^2\|\widetilde{\mathfrak{d}}^{\leq k}\widetilde{\Gamma}_g\|^2_{L^2(S)}+r^{2-\delta_B}\|\widetilde{\mathfrak{d}}^{\leq k}\widetilde{\mathrm{tr}\underline{X}}\|^2_{L^2(S)}$$
$$+\|\widetilde{\mathfrak{d}}^{\leq k}\widetilde{\Gamma}_b\|^+_{L^2(S)}r^{-\delta_B}\|\widetilde{\mathfrak{d}}^{\leq k}\underline{\Xi}\|^2_{L^2(S)}\Big).$$

Together with Proposition 9.105 on the control of the PT structure of ${}^{(ext)}\mathcal{M}$ on $\{u=u'_*\}$, we infer, for $k\leq k_{large}+7$,

$$\sup_{S\subset\{u=u'_*\}}\Big(r^2\|\widetilde{\mathfrak{d}}^{\leq k}(\Xi',\,\omega',\,\widetilde{\mathrm{tr}X}',\,\widehat{X}',\,\check{Z}',\,\widetilde{\underline{H}}')\|^2_{L^2(S)}$$
$$+r^{2-\delta_B}\|\widetilde{\mathfrak{d}}^{\leq k}\widetilde{\mathrm{tr}\underline{X}}'\|^2_{L^2(S)}\Big)+\sup_{S\subset\{u=u'_*\}}\|\widetilde{\mathfrak{d}}^{\leq k}\widehat{\underline{X}}'\|^2_{L^2(S)} \;\lesssim\; \epsilon_0^2+L_*^2(k)$$

where the constant in the definition of $\lesssim$ is independent of r_0.

Concerning the linearized metric coefficients, we consider

$$\underline{u}=u+2\int_{r_0}^r\frac{\tilde{r}^2+a^2}{\tilde{r}^2-2m\tilde{r}+a^2}d\tilde{r},\quad r'=r,\quad \theta'=\theta,\quad \mathfrak{J}'=\mathfrak{J}\quad\text{on}\quad\{u=u'_*\},$$

and apply to both sides of these identities the tangential vectorfields X to $\{u=u'_*\}$ as above. Then, proceeding similarly to the linearized Ricci coefficients[225], we obtain on $\{u=u'_*\}$ relations between the linearized metric coefficients on ${}^{(top)}\mathcal{M}$ and ${}^{(ext)}\mathcal{M}$, which then, together with the above control for linearized Ricci coefficients, yields

$$\sup_{S\subset\{u=u'_*\}}\Big(r^2\|\widetilde{\mathfrak{d}}^{\leq k}(\Gamma'_g\setminus\{\widetilde{\mathrm{tr}\underline{X}}'\})\|^2_{L^2(S)}+r^{2-\delta_B}\|\widetilde{\mathfrak{d}}^{\leq k}\widetilde{\mathrm{tr}\underline{X}}'\|^2_{L^2(S)}\Big)$$
$$+\sup_{S\subset\{u=u'_*\}}\|\widetilde{\mathfrak{d}}^{\leq k}\Gamma'_b\|^2_{L^2(S)} \;\lesssim\; \epsilon_0^2+L_*^2(k)$$

where the constant in the definition of $\lesssim$ is independent of r_0.

Then, relying on the transport equations along e_3 of the ingoing PT structure of ${}^{(top)}\mathcal{M}$, and on the fact that e_3 is transversal to $\{u=u'_*\}$ so that $\mathfrak{d}$ is spanned by ∇_3 and $\widetilde{\mathfrak{d}}$, we infer, for $k\leq k_{large}+7$,

$$\sup_{S\subset\{u=u'_*\}}\Big(r^2\|\mathfrak{d}^{\leq k}(\Gamma_g\setminus\{\widetilde{\mathrm{tr}\underline{X}}\})\|^2_{L^2(S)}+r^{2-\delta_B}\|\mathfrak{d}^{\leq k}\widetilde{\mathrm{tr}\underline{X}}\|^2_{L^2(S)}\Big)$$
$$+\sup_{S\subset\{u=u'_*\}}\|\mathfrak{d}^{\leq k}\Gamma_b\|^2_{L^2(S)} \;\lesssim\; \epsilon_0^2+L_*^2(k)$$

where the constant in the definition of $\lesssim$ is independent of r_0. This concludes the proof of Proposition 9.106. ◻

[225] See Lemma 7.13 for the analog situation for PG structures.

9.10.3. Proof of Proposition 9.103 We are now in position to prove Proposition 9.103 on the control of the ingoing PT structure of ${}^{(top)}\mathcal{M}$.

9.10.3.1. Propagation lemmas Note that ${}^{(top)}\mathcal{M}$ has by construction the following boundaries

$$\partial\,{}^{(top)}\mathcal{M} = \quad \{u = u'_*\} \cup \{\underline{u} = u'_*\} \cup {}^{(top)}\Sigma \cup (\mathcal{A} \cap \{\underline{u} \geq u'_*\}). \tag{9.184}$$

Also, recall from the construction of the scalar function τ in Proposition 9.29 that the following properties hold on ${}^{(top)}\mathcal{M}$:

1. The future boundary ${}^{(top)}\Sigma$ of $\mathcal{M}$ is given by

   $$ {}^{(top)}\Sigma = \{\tau = u_*\} \tag{9.185}$$

   and $\tau \leq u_*$ on $\mathcal{M}$.
2. Denoting, on each level set of $\underline{u}$ in ${}^{(top)}\mathcal{M}(r \geq r_0)$, by $r_+(\underline{u})$ the maximal value of r and by $r_-(\underline{u})$ the minimal value of r, we have[226]

   $$0 \leq r_+(\underline{u}) - r_-(\underline{u}) \leq 2(2m+1). \tag{9.186}$$

3. In ${}^{(top)}\mathcal{M}(r \leq r_0)$, τ satisfies

   $$u_* - 2(m+2) \leq \tau \leq u_*. \tag{9.187}$$

Remark 9.107. *In view of (9.186) and (9.187), ${}^{(top)}\mathcal{M}$ is in fact a local existence type region.*

We introduce the following norms on ${}^{(top)}\mathcal{M}$

$$\|f\|_2(\underline{u}, r) := \|f\|_{L^2\left(S(\underline{u},r)\right)}, \qquad \|f\|_{2,k}(\underline{u}, r) := \sum_{i=0}^{k} \|\mathfrak{d}^i f\|_2(\underline{u}, r),$$

which allow us to state a first propagation lemma.

Proposition 9.108. *Let U and F anti-selfdual k-tensors. Assume that U verifies one of the following equations, for a real constant c,*

$$\nabla_3 U - \frac{c}{\overline{q}} U = F \tag{9.188}$$

or

[226] Note that (9.186) depends on the choice of ${}^{(top)}\Sigma$ and hence on the choice of τ.

$$\nabla_3 U - \Re\left(\frac{c}{q}\right) U = F. \tag{9.189}$$

In both cases we derive, for any c', and for any $r_-(\underline{u}) \leq r \leq r_1 \leq r_+(\underline{u})$ at fixed $\underline{u}$, with $\underline{u} \geq u'_*$, in ${}^{(top)}\mathcal{M}$

$$r^{c'}\|U\|_{2,k}(\underline{u}, r) \lesssim r_1^{c'}\|U\|_{2,k}(\underline{u}, r_1) + \int_r^{r_1} \lambda^{c'}\|F\|_{2,k}(\underline{u}, \lambda)d\lambda, \tag{9.190}$$

where the constant in the definition of $\lesssim$ is independent of r_0.

Proof. Note first that we have for any $r_-(\underline{u}) \leq r \leq r_1 \leq r_+(\underline{u})$

$$r^{c-1}\|U\|_{2,k}(\underline{u}, r) \lesssim r_1^{c-1}\|U\|_{2,k}(\underline{u}, r_1) + \int_r^{r_1} \lambda^{c-1}\|F\|_{2,k}(\underline{u}, \lambda)d\lambda,$$

whose proof is completely analogous of the one of Proposition 9.83. The stated estimate follows then by multiplying by $r^{c'-c+1}$ and by noticing that $r \leq r_1 \leq r + 2(2m+1)$ in view of (9.186). □

We infer the following corollary.

Corollary 9.109. *Let U and F anti-selfdual k-tensors. Assume that U verifies (9.188) or (9.189). In both cases we derive the following:*

1. *In ${}^{(top)}\mathcal{M}(r \geq r_0)$, we have, for any c',*

$$\begin{aligned}\sup_{\underline{u}_1 \geq u'_*} \int_{{}^{(top)}\mathcal{M}_{r_0,\underline{u}_1}} r^{2c'}|\mathfrak{d}^{\leq k}U|^2 &\lesssim \sup_{S\subset\{u=u'_*\}} r^{2c'}\|\mathfrak{d}^{\leq k}U\|^2_{L^2(S)} \\ &\quad + \sup_{\underline{u}_1 \geq u'_*} \int_{{}^{(top)}\mathcal{M}_{r_0,\underline{u}_1}} r^{2c'}|\mathfrak{d}^{\leq k}F|^2,\end{aligned} \tag{9.191}$$

where the constant in the definition of $\lesssim$ is independent of r_0, and where we recall the notation

$$ {}^{(top)}\mathcal{M}_{r_0,\underline{u}_1} = {}^{(top)}\mathcal{M}(r \geq r_0) \cap \{\underline{u}_1 \leq \underline{u} \leq \underline{u}_1 + 1\}.$$

2. *In ${}^{(top)}\mathcal{M}(r \leq r_0)$, we have, for any c',*

$$\begin{aligned}&\int_{{}^{(top)}\mathcal{M}(r\leq r_0)} |\mathfrak{d}^{\leq k}U|^2 \\ &\lesssim \sup_{S\subset\{u=u'_*\}} r^{2c'}\|\mathfrak{d}^{\leq k}U\|^2_{L^2(S)} + \sup_{\tau}\int_{{}^{(top)}\mathcal{M}(r\leq r_0)\cap\Sigma(\tau)} |\mathfrak{d}^{\leq k}F|^2 \\ &\quad + \sup_{\underline{u}_1 \geq u'_*} \int_{{}^{(top)}\mathcal{M}_{r_0,\underline{u}_1}} r^{2c'}|\mathfrak{d}^{\leq k}F|^2.\end{aligned} \tag{9.192}$$

Proof. We start with the estimate for U in ${}^{(top)}\mathcal{M}(r \geq r_0)$. In view of Proposition 9.108 with the choice $r_1 = r_+(\underline{u})$, we have, for any $r_-(\underline{u}) \leq r \leq r_+(\underline{u})$,

$$r^{c'}\|U\|_{2,k}(\underline{u}, r) \lesssim (r_+(\underline{u}))^{c'}\|U\|_{2,k}(\underline{u}, r_+(\underline{u})) + \int_r^{r_+(\underline{u})} \lambda^{c'}\|F\|_{2,k}(\underline{u}, \lambda)d\lambda.$$

Squaring, using Cauchy-Schwarz and the bound (9.186), as well as $r \geq \max(r_0, r_-(\underline{u}))$, we infer

$$\begin{aligned} r^{2c'}\|U\|^2_{2,k}(\underline{u}, r) \lesssim{}& (r_+(\underline{u}))^{2c'}\|U\|^2_{2,k}(\underline{u}, r_+(\underline{u})) \\ &+ \int_{\max(r_0, r_-(\underline{u}))}^{r_+(\underline{u})} \lambda^{2c'}\|F\|^2_{2,k}(\underline{u}, \lambda)d\lambda. \end{aligned}$$

Since $S(\underline{u}, r_+(\underline{u})) \subset \{u = u'_*\}$ by the definition of $r_+(\underline{u})$, we deduce

$$\begin{aligned} r^{2c'}\|U\|^2_{2,k}(\underline{u}, r) \lesssim{}& \sup_{S \subset \{u = u'_*\}} r^{2c'}\|\mathfrak{d}^{\leq k}U\|^2_{L^2(S)} \\ &+ \int_{\max(r_0, r_-(\underline{u}))}^{r_+(\underline{u})} \lambda^{2c'}\|F\|^2_{2,k}(\underline{u}, \lambda)d\lambda. \end{aligned}$$

Integrating in r on $(\max(r_0, r_-(\underline{u})), r_+(\underline{u}))$, and using again the bound (9.186), this yields

$$\begin{aligned} \int_{\max(r_0, r_-(\underline{u}))}^{r_+(\underline{u})} r^{2c'}\|U\|^2_{2,k}(\underline{u}, r)dr \lesssim{}& \sup_{S \subset \{u = u'_*\}} r^{2c'}\|\mathfrak{d}^{\leq k}U\|^2_{L^2(S)} \\ &+ \int_{\max(r_0, r_-(\underline{u}))}^{r_+(\underline{u})} r^{2c'}\|F\|^2_{2,k}(\underline{u}, r)dr. \end{aligned}$$

Integrating in $\underline{u}$ on $(\underline{u}_1, \underline{u}_1 + 1)$, we infer, in view of the definition of ${}^{(top)}\mathcal{M}_{r_0, \underline{u}_1}$,

$$\int_{{}^{(top)}\mathcal{M}_{r_0, \underline{u}_1}} r^{2c'}|\mathfrak{d}^{\leq k}U|^2 \lesssim \sup_{S \subset \{u = u'_*\}} r^{2c'}\|\mathfrak{d}^{\leq k}U\|^2_{L^2(S)} + \int_{{}^{(top)}\mathcal{M}_{r_0, \underline{u}_1}} r^{2c'}|\mathfrak{d}^{\leq k}F|^2.$$

Taking the supremum in $\underline{u}_1 \geq u'_*$, we deduce

$$\begin{aligned} \sup_{\underline{u}_1 \geq u'_*} \int_{{}^{(top)}\mathcal{M}_{r_0, \underline{u}_1}} r^{2c'}|\mathfrak{d}^{\leq k}U|^2 \lesssim{}& \sup_{S \subset \{u = u'_*\}} r^{2c'}\|\mathfrak{d}^{\leq k}U\|^2_{L^2(S)} \\ &+ \sup_{\underline{u}_1 \geq u'_*} \int_{{}^{(top)}\mathcal{M}_{r_0, \underline{u}_1}} r^{2c'}|\mathfrak{d}^{\leq k}F|^2, \end{aligned}$$

as stated. Note that the constant in the definition of $\lesssim$ is independent of r_0 since it is the case in Proposition 9.108 and in the bound (9.186).

Next, we consider the second case, i.e. the estimate for U in ${}^{(top)}\mathcal{M}(r \leq r_0)$. We proceed as above, but instead of integrating in r on $(\max(r_0, r_-(\underline{u})), r_+(\underline{u}))$, we integrate in r on $(r_-(\underline{u}), r_+(\underline{u}))$ and obtain

$$\int_{\{\underline{u}_1\leq\underline{u}\leq\underline{u}_1+1\}} r^{2c'}|\mathfrak{d}^{\leq k}U|^2 \lesssim \sup_{S\subset\{u=u'_*\}} r^{2c'}\|\mathfrak{d}^{\leq k}U\|^2_{L^2(S)} + \int_{\{\underline{u}_1\leq\underline{u}\leq\underline{u}_1+1\}} r^{2c'}|\mathfrak{d}^{\leq k}F|^2.$$

Now, recall that $\underline{u} \geq u'_*$ on ${}^{(top)}\mathcal{M}$. Also, $\tau \leq u_*$ on ${}^{(top)}\mathcal{M}$ and $\tau = u_* + f(r)$ for an explicit smooth function f, see the proof of Proposition 9.29. Thus, we have $\underline{u} \leq u_* - f(r)$ and hence

$$\max_{{}^{(top)}\mathcal{M}(r\leq r_0)} \underline{u} \leq u_* + \max_{r_+-\delta_{\mathcal{H}}\leq r\leq r_0} f(r).$$

Since $u'_* \geq u_* - 5$ by definition, we infer

$$\begin{aligned}\max_{{}^{(top)}\mathcal{M}(r\leq r_0)} \underline{u} - u'_* &\leq u_* - u'_* + \max_{r_+-\delta_{\mathcal{H}}\leq r\leq r_0} f(r)\\ &\leq 5 + \max_{r_+-\delta_{\mathcal{H}}\leq r\leq r_0} f(r).\end{aligned}$$

Thus, introducing the notation

$$\underline{u}_{max} := \max_{{}^{(top)}\mathcal{M}(r\leq r_0)} \underline{u},$$

we have

$$\underline{u}_{max} - u'_* \leq 5 + \max_{r_+-\delta_{\mathcal{H}}\leq r\leq r_0} f(r) \lesssim 1.$$

In particular, we deduce

$$\int_{{}^{(top)}\mathcal{M}(r\leq r_0)} r^{2c'}|\mathfrak{d}^{\leq k}U|^2 \lesssim \sup_{u'_*\leq\underline{u}\leq\underline{u}_{max}} \int_{\{\underline{u}_1\leq\underline{u}\leq\underline{u}_1+1\}} r^{2c'}|\mathfrak{d}^{\leq k}U|^2$$

and

$$\sup_{u'_*\leq\underline{u}\leq\underline{u}_{max}} \int_{\{\underline{u}_1\leq\underline{u}\leq\underline{u}_1+1\}} r^{2c'}|\mathfrak{d}^{\leq k}F|^2 \lesssim \int_{{}^{(top)}\mathcal{M}(r\leq r_0)} |\mathfrak{d}^{\leq k}F|^2 + \sup_{\underline{u}_1\geq u'_*} \int_{{}^{(top)}\mathcal{M}_{r_0,\underline{u}_1}} r^{2c'}|\mathfrak{d}^{\leq k}F|^2.$$

Thus, coming back to the above estimate

$$\int_{\{\underline{u}_1\leq\underline{u}\leq\underline{u}_1+1\}} r^{2c'}|\mathfrak{d}^{\leq k}U|^2 \lesssim \sup_{S\subset\{u=u'_*\}} r^{2c'}\|\mathfrak{d}^{\leq k}U\|^2_{L^2(S)} + \int_{\{\underline{u}_1\leq\underline{u}\leq\underline{u}_1+1\}} r^{2c'}|\mathfrak{d}^{\leq k}F|^2,$$

and taking the supremum on $u'_*\leq\underline{u}\leq\underline{u}_{max}$, we infer

$$\int_{{}^{(top)}\mathcal{M}(r\leq r_0)} |\mathfrak{d}^{\leq k}U|^2 \lesssim \sup_{S\subset\{u=u'_*\}} r^{2c'}\|\mathfrak{d}^{\leq k}U\|^2_{L^2(S)} + \int_{{}^{(top)}\mathcal{M}(r\leq r_0)} |\mathfrak{d}^{\leq k}F|^2 + \sup_{\underline{u}_1\geq u'_*}\int_{{}^{(top)}\mathcal{M}_{r_0,\underline{u}_1}} r^{2c'}|\mathfrak{d}^{\leq k}F|^2.$$

Finally, since we have $u_*-2(m+2)\leq\tau\leq u_*$ in ${}^{(top)}\mathcal{M}(r\leq r_0)$, see (9.187), we have

$$\int_{{}^{(top)}\mathcal{M}(r\leq r_0)} |\mathfrak{d}^{\leq k}F|^2 \leq 2(m+2)\sup_\tau\int_{{}^{(top)}\mathcal{M}(r\leq r_0)\cap\Sigma(\tau)} |\mathfrak{d}^{\leq k}F|^2$$

and hence

$$\int_{{}^{(top)}\mathcal{M}(r\leq r_0)} |\mathfrak{d}^{\leq k}U|^2 \lesssim \sup_{S\subset\{u=u'_*\}} r^{2c'}\|\mathfrak{d}^{\leq k}U\|^2_{L^2(S)} + \sup_\tau\int_{{}^{(top)}\mathcal{M}(r\leq r_0)\cap\Sigma(\tau)} |\mathfrak{d}^{\leq k}F|^2 + \sup_{\underline{u}_1\geq u'_*}\int_{{}^{(top)}\mathcal{M}_{r_0,\underline{u}_1}} r^{2c'}|\mathfrak{d}^{\leq k}F|^2$$

as stated. This concludes the proof of Corollary 9.109. □

9.10.3.2. Control of ${}^{(top)}\mathcal{M}(r\geq r_0)$ In the region ${}^{(top)}\mathcal{M}(r\geq r_0)$, we proceed as follows:

1. We rely on the transport equation in e_3 of the ingoing PT structure of ${}^{(top)}\mathcal{M}$ for the linearized Ricci and metric coefficients, see Proposition 9.27.
2. We control the transport equations using the first estimate of Corollary 9.109 according to which, for U satisfying a transport equation with RHS F, there holds, for any c',

$$\sup_{\underline{u}_1\geq u'_*}\int_{{}^{(top)}\mathcal{M}_{r_0,\underline{u}_1}} r^{2c'}|\mathfrak{d}^{\leq k}U|^2 \lesssim \sup_{S\subset\{u=u'_*\}} r^{2c'}\|\mathfrak{d}^{\leq k}U\|^2_{L^2(S)}$$

$$+\sup_{\underline{u}_1\geq u'_*}\int_{{}^{(top)}\mathcal{M}_{r_0,\underline{u}_1}} r^{2c'}|\mathfrak{d}^{\leq k}F|^2, \tag{9.193}$$

where the constant in the definition of $\lesssim$ is independent of r_0.

3. We control the first term of the RHS of (9.193), i.e. the term involving the control of U on $\{u=u'_*\}$, thanks to Proposition 9.106 on the control of the linearized Ricci and metric coefficients of the ingoing PT frame of ${}^{(top)}\mathcal{M}$ on $\{u=u'_*\}$ by $L_*(k)$.
4. We control the second term on the RHS of (9.193), i.e. the term involving the control of F on ${}^{(top)}\mathcal{M}_{r_0,\underline{u}_1}$, using:
   (a) The bootstrap assumption (9.178) to control the quadratic terms.
   (b) The definition of the norm ${}^{(top)}\mathfrak{R}_k$ to control the curvature terms.
   (c) The triangular structure to control the linear terms involving the Ricci and metric coefficients. More precisely, we need the linear terms appearing on the RHS of the estimate of any given Ricci or metric coefficient to be already under control. As in Section 9.9.6, this is possible by estimating the quantities in the following order

$$\widecheck{\mathrm{tr}\underline{X}},\ \widehat{\underline{X}},\ \widecheck{\mathcal{D}\cos\theta},\ \widecheck{Z},\ \widecheck{\underline{H}},\ \mathcal{D}r,\ e_4(\cos\theta),\ \widecheck{\omega},\ \mathcal{D}\widehat{\otimes}\mathfrak{J},\ \widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}},$$
$$\widecheck{\mathrm{tr}X},\ \widehat{X},\ \widecheck{e_4(r)},\ \widecheck{\nabla_4\mathfrak{J}},\ \Xi.$$

The above scheme yields the desired estimate in ${}^{(top)}\mathcal{M}(r\geq r_0)$, i.e.

$$ {}^{(top)}\mathfrak{G}_k^{\geq r_0} \lesssim \epsilon_0+L_*(k)+{}^{(top)}\mathfrak{R}_k, \quad k\leq k_{large}+7, \tag{9.194}$$

where the constant in $\lesssim$ is independent of r_0. Since the proof is reminiscent[227] of the strategy used both in ${}^{(ext)}\mathcal{M}$, see Section 9.8.2, and ${}^{(int)}\mathcal{M}$, see Section 9.9.6, we leave the details to the reader.

9.10.3.3. Control of ${}^{(top)}\mathcal{M}(r\leq r_0)$ In the region ${}^{(top)}\mathcal{M}(r\leq r_0)$, the control on solutions of transport equations given by (9.193) is replaced by the second estimate of Corollary 9.109 according to which, for U satisfying a transport equation with RHS F, there holds, for any c',

$$\int_{{}^{(top)}\mathcal{M}(r\leq r_0)}|\mathfrak{d}^{\leq k}U|^2 \lesssim \sup_{S\subset\{u=u'_*\}} r^{2c'}\|\mathfrak{d}^{\leq k}U\|^2_{L^2(S)}$$
$$+\sup_{\tau}\int_{{}^{(top)}\mathcal{M}(r\leq r_0)\cap\Sigma(\tau)}|\mathfrak{d}^{\leq k}F|^2$$

[227] And in fact significantly simpler since ${}^{(top)}\mathcal{M}$ is a local existence type region, see Remark 9.107.

$$+ \sup_{\underline{u}_1 \geq u'_*} \int_{{}^{(top)}\mathcal{M}_{r_0, \underline{u}_1}} r^{2c'} |\mathfrak{d}^{\leq k} F|^2.$$

Then, the proof in the region ${}^{(top)}\mathcal{M}(r \leq r_0)$ follows the same steps as the one in the region ${}^{(top)}\mathcal{M}(r \geq r_0)$ and we finally obtain

$$(9.195) \qquad {}^{(top)}\mathfrak{G}_k \quad \lesssim \quad \epsilon_0 + L_*(k) + {}^{(top)}\mathfrak{R}_k, \quad k \leq k_{large} + 7.$$

Again, since the proof is reminiscent[228] of the strategy used in ${}^{(int)}\mathcal{M}$, see Section 9.9.6, we leave the details to the reader. This concludes the proof of Proposition 9.103.

[228] And in fact significantly simpler since ${}^{(top)}\mathcal{M}$ is a local existence type region, see Remark 9.107.

Appendix A. PROOF OF RESULTS IN CHAPTER 2

A.1. Proof of Corollary 2.13

We start with the equation for f. Assuming that $\xi'=0$, we have

$$\begin{aligned}
0 &= 2\xi'_a = \mathbf{g}\left(\mathbf{D}_{e'_4}e'_4, e'_a\right) = \lambda\mathbf{g}\left(\mathbf{D}_{e'_4}(\lambda^{-1}e'_4), e'_a\right)\\
&= \lambda\mathbf{g}\left(\mathbf{D}_{e'_4}(\lambda^{-1}e'_4), e_a+\frac{1}{2}\underline{f}_a\lambda^{-1}e'_4+\frac{1}{2}f_ae_3\right)\\
&= \lambda\mathbf{g}\left(\mathbf{D}_{e'_4}(\lambda^{-1}e'_4), e_a+\frac{1}{2}f_ae_3\right)\\
&= \lambda^2\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e'_4}\left(e_4+f^be_b+\frac{1}{4}|f|^2e_3\right), e_a+\frac{1}{2}f_ae_3\right)
\end{aligned}$$

and hence

$$\begin{aligned}
0 &= \mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e'_4}e_4, e_a+\frac{1}{2}f_ae_3\right)+\lambda^{-1}e'_4(f_a)+f^b\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e'_4}e_b, e_a+\frac{1}{2}f_ae_3\right)\\
&\quad+\frac{1}{4}|f|^2\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e'_4}e_3, e_a+\frac{1}{2}f_ae_3\right).
\end{aligned}$$

Next, we compute

$$\begin{aligned}
\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e'_4}e_4, e_a+\frac{1}{2}f_ae_3\right) &= \mathbf{g}\left(\mathbf{D}_{e_4+f^be_b+\frac{1}{4}|f|^2e_3}e_4, e_a+\frac{1}{2}f_ae_3\right)\\
&= 2\xi_a+2\omega f_a+f^b\chi_{ba}+(f\cdot\zeta)f_a+\frac{1}{2}|f|^2\eta_a\\
&\quad+O(f^3\Gamma),
\end{aligned}$$

$$\begin{aligned}
&\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e'_4}e_b, e_a+\frac{1}{2}f_ae_3\right)\\
&= -\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e'_4}\left(e_a+\frac{1}{2}f_ae_3\right), e_b\right)\\
&= -\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e'_4}\left(e_a+\frac{1}{2}f_ae_3\right), e_b+\frac{1}{2}f_be_3\right)\\
&\quad+\frac{1}{2}f_a\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e'_4}\left(e_a+\frac{1}{2}f_ae_3\right), e_3\right)\\
&= -\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e'_4}\left(e'_a-\frac{1}{2}\underline{f}_a\lambda^{-1}e'_4\right), e'_b-\frac{1}{2}\underline{f}_b\lambda^{-1}e'_4\right)\\
&\quad-\frac{1}{2}f_b\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e'_4}e_3, e_a+\frac{1}{2}f_ae_3\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_a', e_b'\right) + \underline{f}_a\xi_b' - \underline{f}_b\xi_a' - \frac{1}{2}f_b\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_3, e_a\right)\\
&= -\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_a', e_b'\right) - \underline{\eta}_a f_b - \frac{1}{2}f_b f_c\underline{\chi}_{ca} - \frac{1}{4}|f|^2 f_b\underline{\xi}_a\\
&= -\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_a', e_b'\right) - \underline{\eta}_a f_b + O(f^2\Gamma),
\end{aligned}
$$

and

$$
\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_3, e_a + \frac{1}{2}f_a e_3\right) = 2\underline{\eta}_a + O(f\Gamma).
$$

Plugging in the above, we infer

$$
\begin{aligned}
0 &= 2\xi + 2\omega f + \frac{1}{2}(\mathrm{tr}\,\chi f - {}^{(a)}\mathrm{tr}\chi\,{}^*f) + \widehat{\chi}\cdot f + (f\cdot\zeta)f + \frac{1}{2}|f|^2\eta\\
&\quad +\nabla_{\lambda^{-1}e_4'}f - \frac{1}{2}|f|^2\underline{\eta} + O(f^3\Gamma)
\end{aligned}
$$

and hence

$$
\nabla_{\lambda^{-1}e_4'}f + \frac{1}{2}(\mathrm{tr}\,\chi f - {}^{(a)}\mathrm{tr}\chi\,{}^*f) + 2\omega f = -2\xi - f\cdot\widehat{\chi} + E_1(f,\Gamma)
$$

where

$$
E_1(f,\Gamma) = -(f\cdot\zeta)f - \frac{1}{2}|f|^2\eta + \frac{1}{2}|f|^2\underline{\eta} + O(f^3\Gamma)
$$

as stated.

Next, we derive the equation for λ. Assuming that $\omega'=0$ and $\xi'=0$, we have

$$
\begin{aligned}
0 &= 4\omega' = \mathbf{g}\left(\mathbf{D}_{e_4'}e_4', e_3'\right) = -2e_4'(\log\lambda) + \lambda\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}(\lambda^{-1}e_4'), \lambda e_3'\right)\\
&= -2e_4'(\log\lambda) + \lambda\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}(\lambda^{-1}e_4'), e_3 + \underline{f}^a e_a' - \frac{1}{4}|\underline{f}|^2\lambda^{-1}e_4'\right)\\
&= -2e_4'(\log\lambda) + \lambda\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}(\lambda^{-1}e_4'), e_3\right) + 2\lambda^{-1}\underline{f}^a\xi_a'\\
&= -2e_4'(\log\lambda) + \lambda\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}(\lambda^{-1}e_4'), e_3\right)\\
&= -2e_4'(\log\lambda) + \lambda\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_4, e_3\right) + \lambda f^a\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_a, e_3\right).
\end{aligned}
$$

Next, we compute

$$
\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_4, e_3\right) = 4\omega + 2f\cdot\zeta - |f|^2\underline{\omega}
$$

and

$$\mathbf{g}\left(\mathbf{D}_{\lambda^{-1}e_4'}e_a, e_3\right) = -2\underline{\eta}_a - f^b\underline{\chi}_{ba} + O(f^2\Gamma).$$

Plugging in the above, we infer

$$\begin{aligned} 0 &= -2\lambda^{-1}e_4'(\log\lambda) + 4\omega + 2f\cdot\zeta - |f|^2\underline{\omega} - 2f\cdot\underline{\eta} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}|f|^2 \\ &\quad +O(f^3\Gamma + f^2\widehat{\underline{\chi}}) \end{aligned}$$

and hence

$$\lambda^{-1}e_4'(\log\lambda) = 2\omega + f\cdot(\zeta - \underline{\eta}) + E_2(f,\Gamma)$$

where

$$E_2(f,\Gamma) = -\frac{1}{2}|f|^2\underline{\omega} - \frac{1}{4}\mathrm{tr}\,\underline{\chi}|f|^2 + O(f^3\Gamma + f^2\widehat{\underline{\chi}})$$

as stated.

Finally, we derive the equation for $\underline{f}$. Summing the transformation formula for ζ' and $\underline{\eta}'$, and using $\underline{\eta}' + \zeta' = 0$, we have

$$\begin{aligned} \nabla_{\lambda^{-1}e_4'}\underline{f} + \frac{1}{2}(\mathrm{tr}\,\chi\underline{f} + {}^{(a)}\mathrm{tr}\chi\ {}^*\underline{f}) &= -2(\underline{\eta}+\zeta) + 2\nabla'(\log\lambda) + 2\underline{\omega}f \\ &\quad +2\mathrm{Err}(\underline{\eta},\underline{\eta}') + 2\mathrm{Err}(\zeta,\zeta'). \end{aligned}$$

Now, in view of the form of $\mathrm{Err}(\underline{\eta},\underline{\eta}')$ and $\mathrm{Err}(\zeta,\zeta')$, and since $\xi' = 0$, we have

$$\begin{aligned} &\mathrm{Err}(\underline{\eta},\underline{\eta}') + \mathrm{Err}(\zeta,\zeta') \\ = \;& \frac{1}{2}f\cdot\widehat{\underline{\chi}} + \frac{1}{2}(f\cdot\underline{\eta})\underline{f} - \frac{1}{4}(f\cdot\zeta)\underline{f} - \frac{1}{4}|\underline{f}|^2\lambda^{-2}\xi' \\ & -\frac{1}{2}\widehat{\underline{\chi}}\cdot f + \frac{1}{2}(f\cdot\zeta)\underline{f} - \frac{1}{2}(f\cdot\underline{\eta})\underline{f} + \frac{1}{4}\underline{f}(f\cdot\eta) + \frac{1}{4}\underline{f}(f\cdot\zeta) \\ & +\frac{1}{4}\ {}^*\underline{f}(f\wedge\eta) + \frac{1}{4}\ {}^*\underline{f}(f\wedge\zeta) + \frac{1}{4}\underline{f}\mathrm{div}'f + \frac{1}{4}\ {}^*\underline{f}\mathrm{curl}'f + \frac{1}{2}\lambda^{-1}\underline{f}\cdot\widehat{\chi}' \\ & -\frac{1}{16}(f\cdot\underline{f})\underline{f}\lambda^{-1}\mathrm{tr}\,\chi' + \frac{1}{16}(\underline{f}\wedge f)\underline{f}\lambda^{-1}\,{}^{(a)}\mathrm{tr}\chi' - \frac{1}{16}\ {}^*\underline{f}(f\cdot\underline{f})\lambda^{-1}\,{}^{(a)}\mathrm{tr}\chi' \\ & +\frac{1}{16}\ {}^*\underline{f}\lambda^{-1}(f\wedge\underline{f})\mathrm{tr}\,\chi' + \text{ l.o.t.} \\ = \;& \frac{1}{4}(f\cdot\eta)\underline{f} + \frac{1}{2}(f\cdot\zeta)\underline{f} + \frac{1}{4}\ {}^*\underline{f}(f\wedge\eta) + \frac{1}{4}\ {}^*\underline{f}(f\wedge\zeta) + \frac{1}{4}\underline{f}\mathrm{div}'f \end{aligned}$$

$$+\frac{1}{4}\,{}^*\underline{f}\mathrm{curl}'f+\frac{1}{2}\lambda^{-1}\underline{f}\cdot\widehat{\chi}'$$
$$+O\Big((\lambda^{-1}\mathrm{tr}\,\chi',\lambda^{-1}\,{}^{(a)}\mathrm{tr}\chi')(f,\underline{f})^3+(f,\underline{f})^3\Gamma\Big)$$

and hence

$$\begin{aligned}\nabla_{\lambda^{-1}e_4'}\underline{f}+\frac{1}{2}(\mathrm{tr}\,\chi\underline{f}+{}^{(a)}\mathrm{tr}\chi\,{}^*\underline{f}) &= -2(\underline{\eta}+\zeta)+2\nabla'(\log\lambda)+2\underline{\omega}f\\ &\quad +E_3(\nabla'^{\leq 1}f,\underline{f},\Gamma,\lambda^{-1}\chi'),\end{aligned}$$

where

$$\begin{aligned}&E_3(\nabla'^{\leq 1}f,\underline{f},\Gamma,\lambda^{-1}\chi')\\ =\;&\frac{1}{4}(f\cdot\eta)\underline{f}+\frac{1}{2}(f\cdot\zeta)\underline{f}+\frac{1}{4}\,{}^*\underline{f}(f\wedge\eta)+\frac{1}{4}\,{}^*\underline{f}(f\wedge\zeta)\\ &+\frac{1}{4}\underline{f}\mathrm{div}'f+\frac{1}{4}\,{}^*\underline{f}\mathrm{curl}'f+\frac{1}{2}\lambda^{-1}\underline{f}\cdot\widehat{\chi}'\\ &+O\Big((\lambda^{-1}\mathrm{tr}\,\chi',\lambda^{-1}\,{}^{(a)}\mathrm{tr}\chi')(f,\underline{f})^3+(f,\underline{f})^3\Gamma\Big)\end{aligned}$$

as stated. This concludes the proof of Corollary 2.13.

A.2. Proof of Corollary 2.14

The first three identities, i.e.

$$\begin{aligned}\nabla_{\lambda^{-1}e_4'}F+\frac{1}{2}\overline{\mathrm{tr}X}F+2\omega F &= -2\Xi-\widehat{\chi}\cdot F+E_1(f,\Gamma),\\ \lambda^{-1}\nabla_4'(\log\lambda) &= 2\omega+f\cdot(\zeta-\underline{\eta})+E_2(f,\Gamma),\\ \nabla_{\lambda^{-1}e_4'}\underline{F}+\frac{1}{2}\mathrm{tr}X\underline{F} &= -2(\,\underline{H}+Z)+2\mathcal{D}'(\log\lambda)+2\underline{\omega}F\\ &\quad +E_3(\nabla'^{\leq 1}f,\underline{f},\Gamma,\lambda^{-1}\chi'),\end{aligned}$$

are an immediate consequence of Corollary 2.13 and the notation for F and $\underline{F}$.

Next, we focus on proving the fourth identity. Since we assume $e_4(q)=1$, which also yields $e_4(\overline{q})=1$, we have

$$\begin{aligned}\lambda^{-1}e_4'(q) &= \left(e_4+f\cdot\nabla+\frac{1}{4}|f|^2e_3\right)q=1+\left(f\cdot\nabla+\frac{1}{4}|f|^2e_3\right)q,\\ \lambda^{-1}e_4'(\overline{q}) &= \left(e_4+f\cdot\nabla+\frac{1}{4}|f|^2e_3\right)\overline{q}=1+\left(f\cdot\nabla+\frac{1}{4}|f|^2e_3\right)\overline{q}.\end{aligned}$$

We infer

$$\begin{aligned}\nabla_{\lambda^{-1}e_4'}(\overline{q}F) &= \overline{q}\nabla_{\lambda^{-1}e_4'}F+\lambda^{-1}e_4'(\overline{q})F\\ &= \overline{q}\left(-\frac{1}{2}\overline{\mathrm{tr}X}F-2\omega F-2\Xi-\widehat{\chi}\cdot F+E_1(f,\Gamma)\right)+F\\ &\quad +\left(f\cdot\nabla+\frac{1}{4}|f|^2e_3\right)F\\ &= -2\overline{q}\omega F-2\overline{q}\Xi-\frac{1}{2}\overline{q}\left(\mathrm{tr}X-\frac{2}{\overline{q}}\right)F-\overline{q}\widehat{\chi}\cdot F+\overline{q}E_1(f,\Gamma)\\ &\quad +f\cdot\nabla(\overline{q})F+\frac{1}{4}|f|^2e_3(\overline{q})F\end{aligned}$$

and hence

$$\nabla_{\lambda^{-1}e_4'}(\overline{q}F) = -2\overline{q}\omega F-2\overline{q}\Xi+E_4(f,\Gamma),$$

where $E_4(f,\Gamma)$ is given by

$$\begin{aligned}E_4(f,\Gamma) &= -\frac{1}{2}\overline{q}\left(\mathrm{tr}X-\frac{2}{\overline{q}}\right)F-\overline{q}\widehat{\chi}\cdot F+\overline{q}E_1(f,\Gamma)+f\cdot\nabla(\overline{q})F\\ &\quad +\frac{1}{4}|f|^2e_3(\overline{q})F\end{aligned}$$

as stated.

Finally, we consider the fifth and last identity. Using the first four identities and the above identities for $\lambda^{-1}e_4'(q)$ and $\lambda^{-1}e_4'(\overline{q})$, we have, using also $e_4(e_3(r))=-2\underline{\omega}$,

$$\begin{aligned}&\nabla_{\lambda^{-1}e_4'}\left[q\Big(\underline{F}-2q\mathcal{D}'(\log\lambda)\Big)+\overline{q}e_3(r)F\right]\\ =&q\nabla_{\lambda^{-1}e_4'}\underline{F}+q\Big(-2q\mathcal{D}'\nabla_{\lambda^{-1}e_4'}(\log\lambda)-2[\nabla_{\lambda^{-1}e_4'},q\mathcal{D}']\log\lambda\Big)\\ &+\lambda^{-1}e_4'(q)\Big(\underline{F}-2q\mathcal{D}'(\log\lambda)\Big)+e_3(r)\nabla_{\lambda^{-1}e_4'}(\overline{q}F)+\lambda^{-1}e_4'(e_3(r))\overline{q}F\\ =&q\left(-\frac{1}{2}\mathrm{tr}X\underline{F}-2(\underline{H}+Z)+2\mathcal{D}'(\log\lambda)+2\underline{\omega}F+E_3(\nabla'^{\leq 1}f,\underline{f},\Gamma,\lambda^{-1}\chi')\right)\\ &-2q^2\mathcal{D}'\Big(2\omega+f\cdot(\zeta-\underline{\eta})+E_2(f,\Gamma)\Big)-2q[\nabla_{\lambda^{-1}e_4'},q\mathcal{D}']\log\lambda\\ &+\underline{F}-2q\mathcal{D}'(\log\lambda)+\left(f\cdot\nabla(q)+\frac{1}{4}|f|^2e_3(q)\right)\Big(\underline{F}-2q\mathcal{D}'(\log\lambda)\Big)\\ &+e_3(r)\Big(-2\overline{q}\omega F-2\overline{q}\Xi+E_4(f,\Gamma)\Big)-2\underline{\omega}\overline{q}F.\end{aligned}$$

We infer

$$\begin{aligned}
&\nabla_{\lambda^{-1}e_4'}\Big[q\Big(\underline{F}-2q\mathcal{D}'(\log\lambda)\Big)+\overline{q}e_3(r)F\Big]\\
=\;& -2q(\,\underline{H}+Z)-2q^2\mathcal{D}'\Big(2\omega+f\cdot(\zeta-\underline{\eta})\Big)+e_3(r)\left(-2\overline{q}\omega F-2\overline{q}\Xi\right)\\
&+2\underline{\omega}(q-\overline{q})F+E_5(\nabla'^{\leq 1}f,\underline{f},\nabla'^{\leq 1}\lambda,\mathbf{D}^{\leq 1}\Gamma)
\end{aligned}$$

where $E_5(\nabla'^{\leq 1}f,\underline{f},\nabla'^{\leq 1}\lambda,\mathbf{D}^{\leq 1}\Gamma)$ is given by

$$\begin{aligned}
&E_5(\nabla'^{\leq 1}f,\underline{f},\nabla'^{\leq 1}\lambda,\mathbf{D}^{\leq 1}\Gamma)\\
=\;& -\frac{q}{2}\left(\mathrm{tr}X-\frac{2}{q}\right)\underline{F}+qE_3(\nabla'^{\leq 1}f,\underline{f},\Gamma,\lambda^{-1}\chi')-2q^2\mathcal{D}'\left(E_2(f,\Gamma)\right)\\
&-2q[\nabla_{\lambda^{-1}e_4'},q\mathcal{D}']\log\lambda+\left(f\cdot\nabla(q)+\frac{1}{4}|f|^2e_3(q)\right)\Big(\underline{F}-2q\mathcal{D}'(\log\lambda)\Big)\\
&+e_3(r)E_4(f,\Gamma)
\end{aligned}$$

as stated. This concludes the proof of Corollary 2.14.

A.3. Proof of Proposition 2.54

In the following lemma, we prove refined asymptotic for $\mathrm{tr}\,\chi'$, and $\mathrm{tr}\,\underline{\chi}'$ which will be used in the proof of Proposition 2.54 below.

Lemma A.1. *We have, for large r,*

$$\begin{aligned}
&tr\chi'=\frac{2}{r}-\frac{2a^2(\cos\theta)^2}{r^3}-\frac{a^2(\sin\theta)^2}{2r^3}+O\left(\frac{1}{r^4}\right),\\
&tr\underline{\chi}'=-\frac{2\left(1-\frac{2m}{r}\right)}{r}-\frac{2a^2}{r^3}\Big(1-2(\cos\theta)^2\Big)+\frac{3a^2(\sin\theta)^2}{2r^3}+O\left(\frac{1}{r^4}\right).
\end{aligned}\tag{A.1}$$

Proof. We write, making use of the transformation formulas in Proposition 2.12,

$$\begin{aligned}
\lambda^{-1}\mathrm{tr}\,\chi' &= \mathrm{tr}\,\chi+\mathrm{div}'f+f\cdot\eta+f\cdot\zeta+\mathrm{Err}(\mathrm{tr}\,\chi,\mathrm{tr}\,\chi'),\\
\mathrm{Err}(\mathrm{tr}\,\chi,\mathrm{tr}\,\chi') &= \underline{f}\cdot\xi+\frac{1}{4}\underline{f}\cdot\Big(f\mathrm{tr}\,\chi-\;{}^*f\,{}^{(a)}\mathrm{tr}\chi\Big)+\omega(f\cdot\underline{f})-\underline{\omega}|f|^2\\
&\quad-\frac{1}{4}|f|^2\mathrm{tr}\,\underline{\chi}-\frac{1}{4}(f\cdot\underline{f})\lambda^{-1}\mathrm{tr}\,\chi'+\frac{1}{4}(\underline{f}\wedge f)\lambda^{-1}\,{}^{(a)}\mathrm{tr}\chi'+\text{ l.o.t.},\\
\lambda\mathrm{tr}\,\underline{\chi}' &= \mathrm{tr}\,\underline{\chi}+\mathrm{div}'\underline{f}+\underline{f}\cdot\underline{\eta}-\underline{f}\cdot\zeta+\mathrm{Err}(\mathrm{tr}\,\underline{\chi},\mathrm{tr}\,\underline{\chi}'),\\
\mathrm{Err}(\mathrm{tr}\,\underline{\chi},\mathrm{tr}\,\underline{\chi}') &= \frac{1}{2}(f\cdot\underline{f})\mathrm{tr}\,\underline{\chi}+f\cdot\underline{\xi}-|\underline{f}|^2\omega+(f\cdot\underline{f})\underline{\omega}-\frac{1}{4}|\underline{f}|^2\lambda^{-1}\mathrm{tr}\,\chi'\\
&\quad+\text{ l.o.t.}
\end{aligned}$$

Also, recall from Lemma 2.52 the following asymptotic:

- We have $f_1 = \underline{f}_1 = 0$, and

$$f_2 = -\frac{a\sin\theta}{r} + O\big(\sin\theta r^{-3}\big), \qquad \underline{f}_2 = -\frac{a\sin\theta\Upsilon}{r} + O\big(\sin\theta r^{-3}\big).$$

- We have,

$$\begin{aligned} \operatorname{div}' f &= O(\sin^2\theta r^{-5}), \qquad \nabla'\widehat{\otimes} f = O(\sin\theta r^{-5}),\\ \operatorname{div}' \underline{f} &= O(\sin^2\theta r^{-5}), \qquad \nabla'\widehat{\otimes} \underline{f} = O(\sin\theta r^{-5}). \end{aligned}$$

Together with the asymptotic for the outgoing principal frame of Kerr, Corollary 2.51, we deduce

$$\begin{aligned} \lambda^{-1}\operatorname{tr}\chi' &= \operatorname{tr}\chi + f_2\eta_2 + f_2\zeta_2 + \frac{1}{4}\underline{f}_2 f_2 \operatorname{tr}\chi - \frac{1}{4}(f_2)^2\operatorname{tr}\underline{\chi} - \frac{1}{4}f_2\underline{f}_2\operatorname{tr}\chi'\\ &\quad + O\left(\frac{1}{r^4}\right)\\ &= \operatorname{tr}\chi - \frac{3a^2(\sin\theta)^2}{2r^3} + O\left(\frac{1}{r^4}\right),\\ \lambda\operatorname{tr}\underline{\chi}' &= \operatorname{tr}\underline{\chi} + \underline{f}_2\underline{\eta}_2 - \underline{f}_2\zeta_2 + \frac{1}{2}f_2\underline{f}_2\operatorname{tr}\underline{\chi} - \frac{1}{4}|\underline{f}|^2\lambda^{-1}\operatorname{tr}\chi' + O\left(\frac{1}{r^4}\right)\\ &= \operatorname{tr}\underline{\chi} + \frac{a^2(\sin\theta)^2}{2r^3} + O\left(\frac{1}{r^4}\right). \end{aligned}$$

Recall that, relative to the principal frame of Kerr

$$\begin{aligned} \operatorname{tr}\chi &= \frac{2r}{|q|^2} = \frac{2r}{r^2+a^2(\cos\theta)^2} = \frac{2}{r} - \frac{2a^2(\cos\theta)^2}{r^3} + O\left(\frac{1}{r^5}\right),\\ \operatorname{tr}\underline{\chi} &= -\frac{2r\Delta}{|q|^4} = -\frac{2r\left(r^2-2mr+a^2\right)}{(r^2+a^2(\cos\theta)^2)^2}\\ &= -\frac{2\Upsilon}{r} - \frac{2a^2}{r^3}\left(1-2(\cos\theta)^2\right) + O\left(\frac{1}{r^4}\right). \end{aligned}$$

We infer that

$$\begin{aligned} \lambda^{-1}\operatorname{tr}\chi' &= \frac{2}{r} - \frac{2a^2(\cos\theta)^2}{r^3} - \frac{3a^2(\sin\theta)^2}{2r^3} + O\left(\frac{1}{r^4}\right),\\ \lambda\operatorname{tr}\underline{\chi}' &= -\frac{2\left(1-\frac{2m}{r}\right)}{r} - \frac{2a^2}{r^3}\left(1-2(\cos\theta)^2\right) + \frac{a^2(\sin\theta)^2}{2r^3} + O\left(\frac{1}{r^4}\right). \end{aligned}$$

To conclude, we need to derive an improved asymptotic for λ compared to the one in Lemma 2.52. Recall from (2.12) that

$$\lambda = 1+\frac{1}{2}f\cdot\underline{f}+\frac{1}{16}|f|^2|\underline{f}|^2.$$

Together with the above asymptotic for f and $\underline{f}$, this yields

$$\lambda = 1+\frac{a^2(\sin\theta)^2}{2r^2}+O(r^{-3})$$

and hence

$$\begin{aligned}
\mathrm{tr}\,\chi' &= \frac{2}{r}-\frac{2a^2(\cos\theta)^2}{r^3}-\frac{a^2(\sin\theta)^2}{2r^3}+O\left(\frac{1}{r^4}\right),\\
\mathrm{tr}\,\underline{\chi}' &= -\frac{2\left(1-\frac{2m}{r}\right)}{r}-\frac{2a^2}{r^3}\left(1-2(\cos\theta)^2\right)+\frac{3a^2(\sin\theta)^2}{2r^3}+O\left(\frac{1}{r^4}\right),
\end{aligned}$$

as stated. □

In the following lemma, we derive the asymptotic of the area radius of $S(u,r)$ that will be used in the proof of Proposition 2.54 below.

Lemma A.2. *Let r' denote the area radius of $S(u,r)$. r' verifies*

$$r' = r\left(1+\frac{a^2}{3r^2}+O\left(\frac{1}{r^3}\right)\right).$$

Proof. Let g denote the induced metric on $S(u,r)$. Then, (θ,φ) forms a coordinates system on $S(u,r)$, and we have

$$g_{\theta\theta}=\mathbf{g}_{\theta\theta}=|q|^2, \qquad g_{\theta\varphi}=\mathbf{g}_{\theta\varphi}=0, \qquad g_{\varphi\varphi}=\mathbf{g}_{\varphi\varphi}=\frac{\Sigma^2(\sin\theta)^2}{|q|^2}.$$

We deduce

$$\sqrt{g_{\theta\theta}g_{\varphi\varphi}-(g_{\theta\varphi})^2} = \Sigma\sin\theta.$$

Now, in view of the definition of Σ, we have

$$\begin{aligned}
\Sigma &= \sqrt{(r^2+a^2)^2-a^2(\sin\theta)^2(r^2-2mr+a^2)}\\
&= r^2\sqrt{1+\frac{a^2(2-(\sin\theta)^2)}{r^2}+O\left(\frac{1}{r^3}\right)}
\end{aligned}$$

$$= \quad r^2\left(1+\frac{a^2(1+(\cos\theta)^2)}{2r^2}+O\left(\frac{1}{r^3}\right)\right)$$

and hence

$$\sqrt{g_{\theta\theta}g_{\varphi\varphi}-(g_{\theta\varphi})^2} \quad = \quad r^2\sin\theta\left(1+\frac{a^2(1+(\cos\theta)^2)}{2r^2}+O\left(\frac{1}{r^3}\right)\right).$$

We deduce

$$\begin{aligned}
4\pi(r')^2 &= |S(u,r)| = \int_0^{2\pi}\int_0^{\pi}\sqrt{g_{\theta\theta}g_{\varphi\varphi}-(g_{\theta\varphi})^2}d\theta d\varphi\\
&= 2\pi r^2\int_0^{\pi}\sin\theta\left(1+\frac{a^2(1+(\cos\theta)^2)}{2r^2}+O\left(\frac{1}{r^3}\right)\right)d\theta\\
&= 4\pi r^2\left(1+\frac{2a^2}{3r^2}+O\left(\frac{1}{r^3}\right)\right)
\end{aligned}$$

and hence

$$r' \quad = \quad r\left(1+\frac{a^2}{3r^2}+O\left(\frac{1}{r^3}\right)\right)$$

as stated. □

Proof of Proposition 2.54. We make use of the refined asymptotic calculation $\mathrm{tr}\,\chi', \mathrm{tr}\,\underline{\chi}'$ in (A.1), i.e.

$$\begin{aligned}
\mathrm{tr}\,\chi' &= \frac{2}{r}-\frac{2a^2(\cos\theta)^2}{r^3}-\frac{a^2(\sin\theta)^2}{2r^3}+O\left(\frac{1}{r^4}\right),\\
\mathrm{tr}\,\underline{\chi}' &= -\frac{2\left(1-\frac{2m}{r}\right)}{r}-\frac{2a^2}{r^3}\left(1-2(\cos\theta)^2\right)+\frac{3a^2(\sin\theta)^2}{2r^3}+O\left(\frac{1}{r^4}\right).
\end{aligned}$$

Since the area radius r' of $S(u,r)$ verifies, in view of Lemma A.2,

$$r' \quad = \quad r\left(1+\frac{a^2}{3r^2}+O\left(\frac{1}{r^3}\right)\right),$$

and since we have by definition

$$\widetilde{\mathrm{tr}\,\chi}' = \mathrm{tr}\,\chi' - \frac{2}{r'}, \qquad \widetilde{\mathrm{tr}\,\underline{\chi}}' = \mathrm{tr}\,\underline{\chi}' + \frac{2\left(1-\frac{2m}{r'}\right)}{r'},$$

we deduce

$$\widetilde{\mathrm{tr}\,\chi}' = \frac{2a^2}{3r^3} - \frac{2a^2(\cos\theta)^2}{r^3} - \frac{a^2(\sin\theta)^2}{2r^3} + O\left(\frac{1}{r^4}\right),$$
$$\widetilde{\mathrm{tr}\,\underline{\chi}}' = -\frac{2a^2}{3r^3} - \frac{2a^2}{r^3}\left(1-2(\cos\theta)^2\right) + \frac{3a^2(\sin\theta)^2}{2r^3} + O\left(\frac{1}{r^4}\right). \tag{A.2}$$

Since the $O(r^{-3})$ terms in the expansion of $\widetilde{\mathrm{tr}\,\chi}'$ and $\widetilde{\mathrm{tr}\,\underline{\chi}}'$ do not depend on φ, and in view of the form of $J^{(+)}$ and $J^{(-)}$, we immediately deduce that

$$\begin{aligned}\int_{S(u,r)} \widetilde{\mathrm{tr}\,\chi}' J^{(p)} &= O\left(\frac{1}{r^2}\right), \quad p=+,-,\\ \int_{S(u,r)} \widetilde{\mathrm{tr}\,\underline{\chi}}' J^{(p)} &= O\left(\frac{1}{r^2}\right), \quad p=+,-.\end{aligned}$$

It remains to consider the case $p=0$. In view of (A.2) and the form of $J^{(0)}$, this case follows from the calculation

$$\int_0^\pi \left(1, (\cos\theta)^2, (\sin\theta)^2\right)\cos\theta\sin\theta d\theta = (0,0,0).$$

This concludes the proof of Proposition 2.54. □

A.4. Proof of Proposition 2.70

We decompose the proof in the following steps.

Calculation of π_{33}. We have, using the fact that $\underline{\omega} = \underline{\check{\omega}} + \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right)$,

$$\begin{aligned}2\mathbf{g}(\mathbf{D}_3\mathbf{T}, e_3) &= \mathbf{g}\left(\mathbf{D}_3\left(e_3 + \frac{\Delta}{|q|^2}e_4 - 2a\Re(\mathfrak{J})^b e_b\right), e_3\right)\\ &= -2e_3\left(\frac{\Delta}{|q|^2}\right) - \frac{4\Delta}{|q|^2}\underline{\omega} + 4a\Re(\mathfrak{J})\cdot\underline{\xi}\\ &= -2\partial_r\left(\frac{\Delta}{|q|^2}\right)e_3(r) - 2\partial_{\cos\theta}\left(\frac{\Delta}{|q|^2}\right)e_3(\cos\theta) - \frac{4\Delta}{|q|^2}\underline{\omega}\\ &\quad +4a\Re(\mathfrak{J})\cdot\underline{\xi}\\ &= -2\partial_r\left(\frac{\Delta}{|q|^2}\right)\widetilde{e_3(r)} - 2\partial_{\cos\theta}\left(\frac{\Delta}{|q|^2}\right)e_3(\cos\theta) - \frac{4\Delta}{|q|^2}\underline{\check{\omega}}\\ &\quad +4a\Re(\mathfrak{J})\cdot\underline{\xi}\\ &= \Gamma_b.\end{aligned}$$

Calculation of π_{44}.

$$\begin{aligned}2\mathbf{g}(\mathbf{D}_4\mathbf{T}, e_4) &= \mathbf{g}\left(\mathbf{D}_4\left(e_3+\frac{\Delta}{|q|^2}e_4-2a\Re(\mathfrak{J})^b e_b, e_4\right)\right) = -4\omega+4a\Re(\mathfrak{J})\cdot\xi\\ &= 0.\end{aligned}$$

Calculation of π_{34}. Making use of

$$\begin{aligned}&\omega=0, \qquad \underline{\omega}=\frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right)+\Gamma_b,\\ &H+\underline{H}=\frac{a(q-\overline{q})}{|q|^2}\mathfrak{J}+\Re(\check{H}-\check{Z}), \qquad {}^*\mathfrak{J}=-i\mathfrak{J},\end{aligned}$$

we deduce

$$\begin{aligned}2\mathbf{g}(\mathbf{D}_3\mathbf{T}, e_4) &= \mathbf{g}\left(\mathbf{D}_3\left(e_3+\frac{\Delta}{|q|^2}e_4-2a\Re(\mathfrak{J})^b e_b\right), e_4\right) = 4\underline{\omega}+4a\Re(\mathfrak{J})\cdot\eta,\\ 2\mathbf{g}(\mathbf{D}_4\mathbf{T}, e_3) &= \mathbf{g}\left(\mathbf{D}_4\left(e_3+\frac{\Delta}{|q|^2}e_4-2a\Re(\mathfrak{J})^b e_b\right), e_3\right)\\ &= -2e_4\left(\frac{\Delta}{|q|^2}\right)+4a\Re(\mathfrak{J})\cdot\underline{\eta},\end{aligned}$$

and hence

$$\begin{aligned}&2\mathbf{g}(\mathbf{D}_4\mathbf{T}, e_3)+2\mathbf{g}(\mathbf{D}_3\mathbf{T}, e_4)\\ =\ & 4\check{\underline{\omega}}+4a\Re(\mathfrak{J})\cdot(\eta+\underline{\eta})\\ =\ & 4\check{\underline{\omega}}+4a\Re(\mathfrak{J})\cdot\left(\Re\Big(\frac{a(q-\overline{q})}{|q|^2}\mathfrak{J}\Big)+\Re(\check{H}-\check{Z})\right)\\ =\ & 4a^2\frac{2a\cos\theta}{|q|^2}\Re(\mathfrak{J})\cdot\Re\Big(i\mathfrak{J}\Big)+\Gamma_b\\ =\ & -4a^2\frac{2a\cos\theta}{|q|^2}\Re(\mathfrak{J})\cdot\ {}^*\Re\Big(\mathfrak{J}\Big)+\Gamma_b=\Gamma_b.\end{aligned}$$

Calculation of π_{4a}. Next, using

$$\xi=0, \quad \zeta=\Re\left(\frac{a\overline{q}}{|q|^2}\right)+\Gamma_g, \quad \mathrm{tr}X=\frac{2}{q}+\Gamma_g, \quad \nabla_4\mathfrak{J}=-q^{-1}\mathfrak{J}, \quad {}^*\mathfrak{J}=-i\mathfrak{J},$$

we deduce

$$2\mathbf{g}(\mathbf{D}_4\mathbf{T}, e_c) = \mathbf{g}\left(\mathbf{D}_4\left(e_3+\frac{\Delta}{|q|^2}e_4-2a\Re(\mathfrak{J})^b e_b\right), e_c\right)$$

$$\begin{aligned}
&= 2\underline{\eta}_c + \frac{2\Delta}{|q|^2}\xi_c - 2a\nabla_4\Re(\mathfrak{J})_c\\
&= -2\Re\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)_c + 2a\Re(q^{-1}\mathfrak{J})_c + \Gamma_g = \Gamma_g,
\end{aligned}$$

$$\begin{aligned}
2\mathbf{g}(\mathbf{D}_c\mathbf{T}, e_4) &= \mathbf{g}\left(\mathbf{D}_c\left(e_3 + \frac{\Delta}{|q|^2}e_4 - 2a\Re(\mathfrak{J})^b e_b\right), e_4\right)\\
&= -2\zeta_c + 2a\Re(\mathfrak{J})_b\chi_{cb} = -2\zeta_c + a\Re(\mathfrak{J})_b\big(\mathrm{tr}\,\chi\delta_{bc} + {}^{(a)}\mathrm{tr}\chi \in_{bc}\big)\\
&\quad +r^{-1}\Gamma_g\\
&= -2\Re\left(\frac{2a\overline{q}}{|q|^2}\mathfrak{J}\right)_c + a\Big(\mathrm{tr}\,\chi\Re(\mathfrak{J})_c + {}^{(a)}\mathrm{tr}\chi\ {}^*\Re(\mathfrak{J})_c\Big) + \Gamma_g\\
&= -2\Re\left(\frac{2a\overline{q}}{|q|^2}\mathfrak{J}\right)_c + a\left(\frac{2r}{|q|^2}\Re(\mathfrak{J})_c + \frac{2a\cos\theta}{|q|^2}\ {}^*\Re(\mathfrak{J})_c\right) + \Gamma_g\\
&= \frac{2a^2\cos\theta}{|q|^2}\Re(i\mathfrak{J} + {}^*\mathfrak{J})_c + \Gamma_g = \Gamma_g.
\end{aligned}$$

Thus both $\mathbf{g}(\mathbf{D}_4\mathbf{T}, e_c)$ and $\mathbf{g}(\mathbf{D}_c\mathbf{T}, e_4)$ are Γ_g and so is ${}^{(\mathbf{T})}\pi_{c4}$.

Calculation of π_{3a}. Since $\nabla_3\mathfrak{J} = \frac{\Delta q}{|q|^4}\mathfrak{J} + \widetilde{\nabla_3\mathfrak{J}} = \frac{\Delta q}{|q|^4}\mathfrak{J} + r^{-1}\Gamma_b$ and $\eta = \Re\Big(\frac{aq}{|q|^2}\mathfrak{J}\Big) + \Gamma_b$,

$$\begin{aligned}
2\mathbf{g}(\mathbf{D}_3\mathbf{T}, e_c) &= \mathbf{g}\left(\mathbf{D}_3\left(e_3 + \frac{\Delta}{|q|^2}e_4 - 2a\Re(\mathfrak{J})^b e_b\right), e_c\right)\\
&= \frac{2\Delta}{|q|^2}\eta_c - 2a\nabla_3\Re(\mathfrak{J})_c + \Gamma_b = \frac{2\Delta}{|q|^2}\eta_c - 2a\Re\left(\frac{\Delta q}{|q|^4}\mathfrak{J}\right)_c\\
&\quad +r^{-1}\Gamma_b\\
&= \frac{2\Delta}{|q|^2}\Re\left(\frac{aq}{|q|^2}\mathfrak{J}\right)_c - 2a\Re\left(\frac{\Delta q}{|q|^4}\mathfrak{J}\right)_c + \Gamma_b = \Gamma_b.
\end{aligned}$$

Also, since ${}^*\mathfrak{J} = -i\mathfrak{J}$,

$$\begin{aligned}
2\mathbf{g}(\mathbf{D}_c\mathbf{T}, e_3) &= \mathbf{g}\left(\mathbf{D}_c\left(e_3 + \frac{\Delta}{|q|^2}e_4 - 2a\Re(\mathfrak{J})^b e_b\right), e_3\right)\\
&= \frac{2\Delta}{|q|^2}\zeta_c + 2a\Re(\mathfrak{J})_b\underline{\chi}_{cb}\\
&= \frac{2\Delta}{|q|^2}\zeta_c + 2a\Re(\mathfrak{J})_b\left(\frac{1}{2}\delta_{cb}\mathrm{tr}\,\underline{\chi} + \frac{1}{2}\in_{cb}\ {}^{(a)}\mathrm{tr}\underline{\chi}\right) + r^{-1}\Gamma_b\\
&= \frac{2\Delta}{|q|^2}\zeta_c + a\big(\mathrm{tr}\,\underline{\chi}\Re(\mathfrak{J})_c + {}^{(a)}\mathrm{tr}\chi\ {}^*\Re(\mathfrak{J})_c\big) + r^{-1}\Gamma_b
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2\Delta}{|q|^2}\Re\left(\frac{a\overline{q}\mathfrak{J}}{|q|^2}\right)_c + a\left(-\frac{2r\Delta}{|q|^4}\Re(\mathfrak{J})_c - \frac{2a\Delta\cos\theta}{|q|^4}\,{}^*\Re(\mathfrak{J})_c\right)\\
&\quad +r^{-1}\Gamma_b\\
&= \frac{2\Delta}{|q|^2}\Re\left(\frac{-a^2 i\cos\theta\mathfrak{J}}{|q|^2}\right)_c - \frac{2a^2\Delta\cos\theta}{|q|^4}\,{}^*\Re(\mathfrak{J})_c + r^{-1}\Gamma_b\\
&= -\frac{2a^2\Delta\cos\theta}{|q|^4}\Re(i\mathfrak{J}+\,{}^*\mathfrak{J})_c + r^{-1}\Gamma_b = r^{-1}\Gamma_b.
\end{aligned}
$$

Thus both $\mathbf{g}(\mathbf{D}_3\mathbf{T}, e_c)$ and $\mathbf{g}(\mathbf{D}_c\mathbf{T}, e_3)$ are Γ_b and so is $\widetilde{{}^{(\mathbf{T})}\pi_{3c}}$.

Calculation of π_{ab}. We make use of the assumptions $r\,\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}} \in r^{-1}\Gamma_b$ and $r\,\mathcal{D}\widehat{\otimes}\mathfrak{J} \in r^{-1}\Gamma_b$ to deduce

$$
\begin{aligned}
& 2\mathbf{g}(\mathbf{D}_c\mathbf{T}, e_d)\\
&= \mathbf{g}\left(\mathbf{D}_c\left(e_3 + \frac{\Delta}{|q|^2}e_4 - 2a\Re(\mathfrak{J})^b e_b\right), e_d\right) = \underline{\chi}_{cd} + \frac{\Delta}{|q|^2}\chi_{cd} - 2a\nabla_c\Re(\mathfrak{J})_d\\
&= \frac{1}{2}\delta_{cd}\Big(\operatorname{tr}\underline{\chi} + \frac{\Delta}{|q|^2}\operatorname{tr}\chi\Big) + \frac{1}{2}\in_{cd}\left({}^{(a)}\operatorname{tr}\underline{\chi} + \frac{\Delta}{|q|^2}\,{}^{(a)}\operatorname{tr}\chi\right) - 2a\nabla_c\Re(\mathfrak{J})_d + \Gamma_b\\
&= \frac{1}{2}\Big(\frac{-2r\Delta}{|q|^4} + \frac{\Delta}{|q|^2}\big(\frac{2r}{|q|^2}\big)\Big)\delta_{cd} + \frac{1}{2}\in_{cd}\Big(\frac{2a\Delta\cos\theta}{|q|^4} + \frac{\Delta}{|q|^2}\frac{2a\cos\theta}{|q|^2}\Big)\\
&\quad -2a\nabla_c\Re(\mathfrak{J}_d)\\
&= \frac{1}{2}\in_{cd}\Big(\frac{2a\Delta\cos\theta}{|q|^4} + \frac{\Delta}{|q|^2}\frac{2a\cos\theta}{|q|^2}\Big) - 2a\nabla_c\Re(\mathfrak{J}_d) + \Gamma_b\\
&= -\frac{4amr\cos\theta}{|q|^4}\in_{ab} + \Gamma_b.
\end{aligned}
$$

Hence

$$
\begin{aligned}
2\mathbf{g}(\mathbf{D}_c\mathbf{T}, e_d) + 2\mathbf{g}(\mathbf{D}_d\mathbf{T}, e_c) &= -2a\big(\nabla_c\Re(\mathfrak{J}_d) + \nabla_d\Re(\mathfrak{J}_c)\big) + \Gamma_b\\
&= -2a\Re(\nabla\widehat{\otimes}\mathfrak{J})_{cd} - 2a\Re(\widetilde{\operatorname{div}\mathfrak{J}})\delta_{cd} + \Gamma_b = \Gamma_b.
\end{aligned}
$$

Moreover, using the fact that $\widehat{\underline{\chi}}$, $\widehat{\chi}$ and $\nabla\widehat{\otimes}\mathfrak{J}$, are traceless,

$$
\begin{aligned}
2g^{cd}\,{}^{(\mathbf{T})}\pi_{cd} &= \widetilde{\operatorname{tr}}\chi + \frac{\Delta}{|q|^2}\widetilde{\operatorname{tr}}\underline{\chi} - 2a\Re(\widetilde{\operatorname{div}\mathfrak{J}})\\
&= \Gamma_g.
\end{aligned}
$$

This concludes the proof of Proposition 2.70.

Appendix B. PROOF OF RESULTS IN CHAPTER 5

The proofs in this section rely on the the linearized null structure and null Bianchi identities of Proposition 5.18.

B.1. Proof of Proposition 5.22

We proceed as follows.

Step 1. Recall from Proposition 5.18 the linearized the null structure equation for $\nabla_3\zeta$

$$\nabla_3\zeta - \frac{\Upsilon}{r}\zeta \;\; = \;\; -\underline{\beta} - 2\nabla\underline{\check{\omega}} + \frac{\Upsilon}{r}(\eta+\zeta) + \frac{1}{r}\underline{\xi} + \frac{2m}{r^2}(\zeta-\eta) + \Gamma_b\cdot\Gamma_b.$$

It can be rewritten in the form

$$2\nabla\underline{\check{\omega}} - \frac{1}{r}\underline{\xi} \;\; = \;\; -\nabla_3\zeta - \underline{\beta} + \frac{1}{r}\eta + r^{-1}\Gamma_g + \Gamma_b\cdot\Gamma_b$$

which is the first stated identity of the proposition.

Step 2. We make use of the equations for κ and $\operatorname{curl}\eta$ in Proposition 5.18

$$\begin{aligned}
\nabla_3\check{\kappa} &= 2\operatorname{div}\eta + 2\check{\rho} - \frac{1}{r}\underline{\check{\kappa}} + \frac{4}{r}\underline{\check{\omega}} + \frac{2}{r^2}\check{y} + \Gamma_b\cdot\Gamma_b,\\
\operatorname{curl}\eta &= {}^*\rho + \Gamma_b\cdot\Gamma_g,
\end{aligned}$$

which we rewrite in the form

$$\begin{aligned}
2\operatorname{div}\eta &= \nabla_3\check{\kappa} - \frac{4}{r}\underline{\check{\omega}} - \frac{2}{r^2}\check{y} + r^{-1}\Gamma_g + \Gamma_b\cdot\Gamma_b,\\
2\operatorname{curl}\eta &= r^{-1}\Gamma_g + \Gamma_b\cdot\Gamma_g.
\end{aligned}$$

Recalling that $\not{d}_1 = (\operatorname{div}, \operatorname{curl})$, we rewrite in the form

$$2\,\not{d}_1\eta \;\; = \;\; \left(\nabla_3\check{\kappa} - \frac{4}{r}\underline{\check{\omega}} - \frac{2}{r^2}\check{y}, 0\right) + r^{-1}\Gamma_g + \Gamma_b\cdot\Gamma_b.$$

Next, recall that for a pair of scalar functions (f,h), $\not{d}_1^*(f,h) = -\nabla f + {}^*\nabla h$. Hence

$$\begin{aligned}
2\,\not{d}_1^*\not{d}_1\eta &= -\nabla\left(\nabla_3\check{\kappa} - \frac{4}{r}\underline{\check{\omega}} - \frac{2}{r^2}\check{y}\right) + r^{-2}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_g + r^{-1}\,\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b)\\
&= -\nabla_3\nabla\check{\kappa} - [\nabla,\nabla_3]\check{\kappa} + \frac{4}{r}\nabla\underline{\check{\omega}} + \frac{2}{r^2}\nabla\check{y} + r^{-2}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_g
\end{aligned}$$

$$+r^{-1}\not{\mathfrak{d}}^{\le 1}(\Gamma_b\cdot\Gamma_b).$$

Now, in view of Lemma 5.12, we have

$$\begin{aligned}\nabla\check{y} &= \nabla y = -\underline{\xi}+(\zeta-\eta)y = -\underline{\xi}-\Upsilon(\zeta-\eta)+r\Gamma_b\cdot\Gamma_b\\ &= -\underline{\xi}+\eta+\Gamma_g+r\Gamma_b\cdot\Gamma_b,\end{aligned}$$

where we have also used $y=-\Upsilon+r\Gamma_b$. We infer

$$\begin{aligned}2\not{d}_1^*\not{d}_1\eta &= -\nabla_3\nabla\check{\kappa}-[\nabla,\nabla_3]\check{\kappa}+\frac{4}{r}\nabla\underline{\check{\omega}}-\frac{2}{r^2}\underline{\xi}+\frac{2}{r^2}\eta+r^{-2}\not{\mathfrak{d}}^{\le 1}\Gamma_g\\ &\quad +r^{-1}\not{\mathfrak{d}}^{\le 1}(\Gamma_b\cdot\Gamma_b).\end{aligned}$$

We make use the commutation formula of Lemma 5.19 in the particular case of a scalar

$$-[\nabla,\nabla_3]\check{\kappa}=[\nabla_3,\nabla]\check{\kappa}=(\eta-\zeta)\nabla_3\check{\kappa}+\underline{\xi}\nabla_4\check{\kappa}+r^{-2}\not{\mathfrak{d}}^{\le 1}\Gamma_g+r^{-1}\not{\mathfrak{d}}^{\le 1}(\Gamma_b\cdot\Gamma_b).$$

Making use of the equations

$$\begin{aligned}\nabla_3\check{\kappa} &= 2\mathrm{div}\,\eta+2\check{\rho}-\frac{1}{r}\underline{\check{\kappa}}+\frac{4}{r}\underline{\check{\omega}}+\frac{2}{r^2}\check{y}+\Gamma_b\cdot\Gamma_b,\\ \nabla_4\check{\kappa} &= \Gamma_g\cdot\Gamma_g,\end{aligned}$$

we deduce,

$$(\eta-\zeta)\nabla_3\check{\kappa}+\underline{\xi}\nabla_4\check{\kappa} = r^{-1}\not{\mathfrak{d}}^{\le 1}(\Gamma_b\cdot\Gamma_b).$$

Hence,

$$-[\nabla,\nabla_3]\check{\kappa} = r^{-2}\not{\mathfrak{d}}^{\le 1}\Gamma_g+r^{-1}\not{\mathfrak{d}}^{\le 1}(\Gamma_b\cdot\Gamma_b)$$

and therefore,

$$2\not{d}_1^*\not{d}_1\eta = -\nabla_3\nabla\check{\kappa}+\frac{4}{r}\nabla\underline{\check{\omega}}-\frac{2}{r^2}\underline{\xi}+\frac{2}{r^2}\eta+r^{-2}\not{\mathfrak{d}}^{\le 1}\Gamma_g+r^{-1}\not{\mathfrak{d}}^{\le 1}(\Gamma_b\cdot\Gamma_b).$$

We now make use of Step 1 to substitute $\nabla\underline{\omega}$ and deduce

$$\begin{aligned}2\not{d}_1^*\not{d}_1\eta &= -\nabla_3\nabla\check{\kappa}-\frac{2}{r^2}\underline{\xi}+\frac{2}{r^2}\eta+\frac{2}{r}\left(\frac{1}{r}\underline{\xi}-\nabla_3\zeta-\underline{\beta}+\frac{1}{r}\eta\right)\\ &\quad +r^{-2}\not{\mathfrak{d}}^{\le 1}\Gamma_g+r^{-1}\not{\mathfrak{d}}^{\le 1}(\Gamma_b\cdot\Gamma_b)\end{aligned}$$

$$= \quad -\nabla_3\nabla\check{\kappa} + \frac{4}{r^2}\eta - \frac{2}{r}\nabla_3\zeta - \frac{2}{r}\underline{\beta} + r^{-2}\,\mathfrak{d}^{\leq 1}\Gamma_g + r^{-1}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

Recalling $\not{d}_1^*\not{d}_1 = \not{d}_2\not{d}_2^* + 2K$ and $\check{K} = r^{-1}\Gamma_g$, we obtain

$$2\not{d}_2\not{d}_2^*\eta \quad = \quad -\nabla_3\nabla\check{\kappa} - \frac{2}{r}\nabla_3\zeta - \frac{2}{r}\underline{\beta} + r^{-2}\,\mathfrak{d}^{\leq 1}\Gamma_g + r^{-1}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b)$$

as stated.

Step 3. We start with the equation

$$\nabla_3\underline{\check{\kappa}} - \frac{2\Upsilon}{r}\underline{\check{\kappa}} \quad = \quad 2\mathrm{div}\,\underline{\xi} + \frac{4\Upsilon}{r}\underline{\check{\omega}} - \frac{2m}{r^2}\underline{\check{\kappa}} - \left(\frac{2}{r^2} - \frac{8m}{r^3}\right)\check{y} + \Gamma_b\cdot\Gamma_b,$$

which we write in the form

$$2\mathrm{div}\,\underline{\xi} \quad = \quad \nabla_3\underline{\check{\kappa}} - \frac{4}{r}\underline{\check{\omega}} + \frac{2}{r^2}\check{y} + r^{-1}\Gamma_g + \Gamma_b\cdot\Gamma_b.$$

We also have

$$\mathrm{curl}\,\underline{\xi} \quad = \quad \Gamma_b\cdot\Gamma_b.$$

Hence,

$$2\not{d}_1\underline{\xi} \quad = \quad \left(\nabla_3\underline{\check{\kappa}} - \frac{4}{r}\underline{\check{\omega}} + \frac{2}{r^2}\check{y}, 0\right) + r^{-1}\Gamma_g + \Gamma_b\cdot\Gamma_b$$

and thus

$$\begin{aligned} 2\not{d}_1^*\not{d}_1\underline{\xi} \quad &= \quad -\nabla\left(\nabla_3\underline{\check{\kappa}} - \frac{4}{r}\underline{\check{\omega}} + \frac{2}{r^2}\check{y}\right) + r^{-2}\,\mathfrak{d}^{\leq 1}\Gamma_g + r^{-1}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b) \\ &= \quad -\nabla_3\nabla\underline{\check{\kappa}} - [\nabla,\nabla_3]\underline{\check{\kappa}} + \frac{4}{r}\nabla\underline{\check{\omega}} - \frac{2}{r^2}\nabla\check{y} + r^{-2}\,\mathfrak{d}^{\leq 1}\Gamma_g \\ &\quad +r^{-1}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b). \end{aligned}$$

Plugging the identity $\nabla\check{y} = -\underline{\xi} + \eta + \Gamma_g + r\Gamma_b\cdot\Gamma_b$ derived in Step 2, we infer

$$\begin{aligned} 2\not{d}_1^*\not{d}_1\underline{\xi} \quad &= \quad -\nabla_3\nabla\underline{\check{\kappa}} - [\nabla,\nabla_3]\underline{\check{\kappa}} + \frac{4}{r}\nabla\underline{\check{\omega}} + \frac{2}{r^2}\underline{\xi} - \frac{2}{r^2}\eta + r^{-2}\,\mathfrak{d}^{\leq 1}\Gamma_g \\ &\quad +r^{-1}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b). \end{aligned}$$

Next, as in Step 2, we have

$$-[\nabla,\nabla_3]\underline{\check{\kappa}} = [\nabla_3,\nabla]\underline{\check{\kappa}} = (\eta-\zeta)\nabla_3\underline{\check{\kappa}} + \underline{\xi}\nabla_4\underline{\check{\kappa}} + r^{-2}\,\mathfrak{d}^{\leq 1}\Gamma_g + r^{-1}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

Using the equations for $\nabla_3\underline{\check{\kappa}}$ and $\nabla_4\underline{\check{\kappa}}$, we infer

$$-[\nabla, \nabla_3]\underline{\check{\kappa}} = r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

Therefore

$$2\,\not{d}_1^*\not{d}_1\underline{\xi} = -\nabla_3\nabla\underline{\check{\kappa}} + \frac{4}{r}\nabla\underline{\check{\omega}} + \frac{2}{r^2}\underline{\xi} - \frac{2}{r^2}\eta + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

Making use of Step 1 to substitute $\nabla\underline{\omega}$ we deduce

$$\begin{aligned}
2\,\not{d}_1^*\not{d}_1\underline{\xi} &= -\nabla_3\nabla\underline{\check{\kappa}} + \frac{2}{r}\left(\frac{1}{r}\underline{\xi} - \nabla_3\zeta - \underline{\beta} + \frac{1}{r}\eta\right) + \frac{2}{r^2}\underline{\xi} - \frac{2}{r^2}\eta \\
&\quad + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b) \\
&= -\nabla_3\nabla\underline{\check{\kappa}} - \frac{2}{r}\nabla_3\zeta - \frac{2}{r}\underline{\beta} + \frac{4}{r^2}\underline{\xi} + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).
\end{aligned}$$

Using as in Step 2 that $\not{d}_1^*\not{d}_1 = \not{d}_2\not{d}_2^* + 2K$ and $\check{K} = r^{-1}\Gamma_g$, we obtain

$$2\,\not{d}_2\not{d}_2^*\underline{\xi} = -\nabla_3\nabla\underline{\check{\kappa}} - \frac{2}{r}\nabla_3\zeta - \frac{2}{r}\underline{\beta} + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

This ends the proof of Proposition 5.22.

B.2. Proof of Proposition 5.23

We start with the following identity of Proposition 5.22

$$2\,\not{d}_2\not{d}_2^*\eta = -\nabla_3\nabla\check{\kappa} - \frac{2}{r}\nabla_3\zeta - \frac{2}{r}\underline{\beta} + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b)$$

and apply $\not{d}_2^*\not{d}_1^*\not{d}_1$ to derive

$$\begin{aligned}
2\,\not{d}_2^*\not{d}_1^*\not{d}_1\not{d}_2\not{d}_2^*\eta &= -\not{d}_2^*\not{d}_1^*\not{d}_1\nabla_3\nabla\check{\kappa} - \frac{2}{r}\not{d}_2^*\not{d}_1^*\not{d}_1\nabla_3\zeta - \frac{2}{r}\not{d}_2^*\not{d}_1^*\not{d}_1\underline{\beta} \\
&\quad + r^{-5}\mathfrak{d}^{\leq 4}\Gamma_g + r^{-4}\mathfrak{d}^{\leq 4}(\Gamma_b\cdot\Gamma_b) \\
&= -\not{d}_2^*\not{d}_1^*\not{d}_1\nabla_3\nabla\check{\kappa} - \frac{2}{r}\nabla_3\not{d}_2^*\not{d}_1^*\not{d}_1\zeta - \frac{2}{r}\not{d}_2^*\not{d}_1^*\not{d}_1\underline{\beta} \\
&\quad - \frac{2}{r}[\not{d}_2^*\not{d}_1^*\not{d}_1, \nabla_3]\zeta + r^{-5}\mathfrak{d}^{\leq 4}\Gamma_g + r^{-4}\mathfrak{d}^{\leq 4}(\Gamma_b\cdot\Gamma_b).
\end{aligned}$$

Making use of the commutation formulas of Lemma 5.20, we have

$$[\not{d}_2^*\not{d}_1^*\not{d}_1, \nabla_3]\zeta = \not{d}_2^*\not{d}_1^*[\not{d}_1, \nabla_3]\zeta + \not{d}_2^*[\not{d}_1^*, \nabla_3]\not{d}_1\zeta + [\not{d}_2^*, \nabla_3]\not{d}_1^*\not{d}_1\zeta$$

$$
\begin{aligned}
&= \frac{3\Upsilon}{r} \,\not{d}_2^*\not{d}_1^*\not{d}_1\zeta + r^{-2}\,\not{\mathfrak{d}}^{\leq 2}(\Gamma_b\nabla_3\zeta) + r^{-3}\,\not{\mathfrak{d}}^{\leq 2}(\Gamma_b\mathfrak{d}^{\leq 1}\zeta)\\
&= r^{-2}\,\not{\mathfrak{d}}^{\leq 2}(\Gamma_b\nabla_3\zeta) + r^{-2}\,\not{\mathfrak{d}}^{\leq 2}(\Gamma_b\nabla_4\zeta) + r^{-4}\,\not{\mathfrak{d}}^{\leq 3}\Gamma_g.
\end{aligned}
$$

Since the null structure equations for $\nabla_3\zeta$ and $\nabla_4\zeta$ on Σ_* imply that $\nabla_4\zeta = r^{-1}\Gamma_g$ and $\nabla_3\zeta = r^{-1}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_b$, we infer

$$
[\not{d}_2^*\not{d}_1^*\not{d}_1, \nabla_3]\zeta = r^{-4}\,\not{\mathfrak{d}}^{\leq 3}\Gamma_g + r^{-3}\,\not{\mathfrak{d}}^{\leq 3}(\Gamma_b\cdot\Gamma_b),
$$

and hence

$$
\begin{aligned}
2\,\not{d}_2^*\not{d}_1^*\not{d}_1\,\not{d}_2\,\not{d}_2^*\eta &= -\not{d}_2^*\not{d}_1^*\not{d}_1\nabla_3\nabla\check{\kappa} - \frac{2}{r}\nabla_3\,\not{d}_2^*\not{d}_1^*\not{d}_1\zeta - \frac{2}{r}\,\not{d}_2^*\not{d}_1^*\not{d}_1\underline{\beta}\\
&\quad + r^{-5}\,\not{\mathfrak{d}}^{\leq 4}\Gamma_g + r^{-4}\,\not{\mathfrak{d}}^{\leq 4}(\Gamma_b\cdot\Gamma_b).
\end{aligned}
$$

Next, recalling that

$$
\mu = -\mathrm{div}\,\zeta - \rho + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}}, \qquad \mathrm{curl}\,\zeta = {}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}},
$$

we have

$$
\not{d}_1\zeta = \left(-\mu - \rho + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}},\ {}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}}\right)
$$

and hence

$$
\begin{aligned}
\nabla_3\,\not{d}_2^*\not{d}_1^*\not{d}_1\zeta &= \nabla_3\,\not{d}_2^*\not{d}_1^*\left(-\mu - \rho + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}},\ {}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}}\right)\\
&= -\nabla_3\,\not{d}_2^*\not{d}_1^*\check{\mu} + \not{d}_2^*\not{d}_1^*\nabla_3\left(-\check{\rho} + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}},\ {}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}}\right)\\
&\quad + [\nabla_3, \not{d}_2^*\not{d}_1^*]\left(-\check{\rho} + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}},\ {}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}}\right).
\end{aligned}
$$

From Proposition 5.18, we have on Σ_*

$$
\begin{aligned}
&\nabla_3\widehat{\chi} \in r^{-1}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_b, &&\nabla_3\widehat{\underline{\chi}} \in \not{\mathfrak{d}}^{\leq 1}\Gamma_b,\\
&\nabla_3\check{\rho} \in r^{-2}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_b, &&\nabla_3\ {}^*\rho \in r^{-2}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_b,\\
&\nabla_4\widehat{\chi} \in r^{-1}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_g, &&\nabla_4\widehat{\underline{\chi}} \in r^{-1}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_b,\\
&\nabla_4\check{\rho} \in r^{-2}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_g, &&\nabla_4\ {}^*\rho \in r^{-2}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_g,
\end{aligned}
$$

which, together with the commutation formulas of Lemma 5.20, implies

$$\nabla_3 \,\not{d}_2^*\not{d}_1^*\not{d}_1\zeta = -\nabla_3\,\not{d}_2^*\not{d}_1^*\check{\mu} + \not{d}_2^*\not{d}_1^*\Big(-\nabla_3\check{\rho} + \frac{1}{2}\widehat{\chi}\cdot\nabla_3\underline{\widehat{\chi}}, \nabla_3\ {}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\nabla_3\underline{\widehat{\chi}}\Big)$$
$$+ r^{-4}\,\not{\mathfrak{d}}^{\leq 2}\Gamma_g + r^{-3}\,\not{\mathfrak{d}}^{\leq 3}(\Gamma_b\cdot\Gamma_b).$$

Also, we have in view of Proposition 5.18

$$\begin{aligned}
-\nabla_3\check{\rho} + \frac{1}{2}\widehat{\chi}\cdot\nabla_3\underline{\widehat{\chi}} &= -\Big(-\mathrm{div}\,\underline{\beta} - \frac{1}{2}\widehat{\chi}\cdot\underline{\alpha}\Big) + \frac{1}{2}\widehat{\chi}\cdot(-\underline{\alpha}) + r^{-2}\Gamma_g \\
&\quad + r^{-1}\,\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b) \\
&= \mathrm{div}\,\underline{\beta} + r^{-2}\Gamma_g + r^{-1}\,\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b), \\
\nabla_3\ {}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\nabla_3\underline{\widehat{\chi}} &= -\mathrm{curl}\,\underline{\beta} - \frac{1}{2}\widehat{\chi}\wedge\underline{\alpha} - \frac{1}{2}\widehat{\chi}\wedge(-\underline{\alpha}) + r^{-2}\Gamma_g \\
&\quad + r^{-1}\,\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b) \\
&= -\mathrm{curl}\,\underline{\beta} + r^{-2}\Gamma_g + r^{-1}\,\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b)
\end{aligned}$$

so that

$$\begin{aligned}
\nabla_3\,\not{d}_2^*\not{d}_1^*\not{d}_1\zeta &= -\nabla_3\,\not{d}_2^*\not{d}_1^*\check{\mu} + \not{d}_2^*\not{d}_1^*\Big(\mathrm{div}\,\underline{\beta}, -\mathrm{curl}\,\underline{\beta}\Big) + r^{-4}\,\not{\mathfrak{d}}^{\leq 2}\Gamma_g \\
&\quad + r^{-3}\,\not{\mathfrak{d}}^{\leq 3}(\Gamma_b\cdot\Gamma_b).
\end{aligned}$$

We deduce

$$\begin{aligned}
2\,\not{d}_2^*\not{d}_1^*\not{d}_1\,\not{d}_2\,\not{d}_2^*\eta &= -\not{d}_2^*\not{d}_1^*\not{d}_1\nabla_3\nabla\check{\kappa} + \frac{2}{r}\nabla_3\,\not{d}_2^*\not{d}_1^*\check{\mu} - \frac{2}{r}\,\not{d}_2^*\not{d}_1^*\Big(\mathrm{div}\,\underline{\beta}, -\mathrm{curl}\,\underline{\beta}\Big) \\
&\quad - \frac{2}{r}\,\not{d}_2^*\not{d}_1^*\not{d}_1\underline{\beta} + r^{-5}\,\not{\mathfrak{d}}^{\leq 4}\Gamma_g + r^{-4}\,\not{\mathfrak{d}}^{\leq 4}(\Gamma_b\cdot\Gamma_b)
\end{aligned}$$

and hence

$$\begin{aligned}
2\,\not{d}_2^*\not{d}_1^*\not{d}_1\,\not{d}_2\,\not{d}_2^*\eta &= -\not{d}_2^*\not{d}_1^*\not{d}_1\nabla_3\nabla\check{\kappa} + \frac{2}{r}\nabla_3\,\not{d}_2^*\not{d}_1^*\check{\mu} - \frac{4}{r}\,\not{d}_2^*\not{d}_1^*\mathrm{div}\,\underline{\beta} \\
&\quad + r^{-5}\,\not{\mathfrak{d}}^{\leq 4}\Gamma_g + r^{-4}\,\not{\mathfrak{d}}^{\leq 4}(\Gamma_b\cdot\Gamma_b)
\end{aligned}$$

as desired.

Next, we focus on the second identity. We consider the following identity of Proposition 5.22

$$2\,\not{d}_2\,\not{d}_2^*\underline{\xi} = -\nabla_3\nabla\underline{\check{\kappa}} - \frac{2}{r}\nabla_3\zeta - \frac{2}{r}\underline{\beta} + r^{-2}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_g + r^{-1}\,\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b)$$

and apply $\not{d}_2^*\not{d}_1^*\not{d}_1$ to derive

$$2\,\not{d}_2^*\not{d}_1^*\not{d}_1\,\not{d}_2\,\not{d}_2^*\underline{\xi}$$

$$
\begin{aligned}
&= -\,\not{d}_2^*\not{d}_1^*\not{d}_1\nabla_3\nabla\underline{\check{\kappa}} - \frac{2}{r}\not{d}_2^*\not{d}_1^*\not{d}_1\nabla_3\zeta - \frac{2}{r}\not{d}_2^*\not{d}_1^*\not{d}_1\underline{\beta} + r^{-5}\not{\mathfrak{d}}^{\leq 4}\Gamma_g\\
&\quad + r^{-4}\not{\mathfrak{d}}^{\leq 4}(\Gamma_b\cdot\Gamma_b)\\
&= -\nabla_3\not{d}_2^*\not{d}_1^*\not{d}_1\nabla\underline{\check{\kappa}} - \frac{2}{r}\nabla_3\not{d}_2^*\not{d}_1^*\not{d}_1\zeta - \frac{2}{r}\not{d}_2^*\not{d}_1^*\not{d}_1\underline{\beta}\\
&\quad -[\not{d}_2^*\not{d}_1^*\not{d}_1, \nabla_3]\nabla\underline{\check{\kappa}} - \frac{2}{r}[\not{d}_2^*\not{d}_1^*\not{d}_1, \nabla_3\zeta] + r^{-5}\not{\mathfrak{d}}^{\leq 4}\Gamma_g + r^{-4}\not{\mathfrak{d}}^{\leq 4}(\Gamma_b\cdot\Gamma_b).
\end{aligned}
$$

Recall from above that we have

$$
[\not{d}_2^*\not{d}_1^*\not{d}_1, \nabla_3]\zeta = r^{-4}\not{\mathfrak{d}}^{\leq 3}\Gamma_g + r^{-3}\not{\mathfrak{d}}^{\leq 3}(\Gamma_b\cdot\Gamma_b),
$$

so that

$$
\begin{aligned}
2\not{d}_2^*\not{d}_1^*\not{d}_1\not{d}_2\not{d}_2^*\underline{\xi} &= -\nabla_3\not{d}_2^*\not{d}_1^*\not{d}_1\nabla\underline{\check{\kappa}} - \frac{2}{r}\nabla_3\not{d}_2^*\not{d}_1^*\not{d}_1\zeta - \frac{2}{r}\not{d}_2^*\not{d}_1^*\not{d}_1\underline{\beta}\\
&\quad -[\not{d}_2^*\not{d}_1^*\not{d}_1, \nabla_3]\nabla\underline{\check{\kappa}} + r^{-5}\not{\mathfrak{d}}^{\leq 4}\Gamma_g + r^{-4}\not{\mathfrak{d}}^{\leq 4}(\Gamma_b\cdot\Gamma_b).
\end{aligned}
$$

Also, since the null structure equations for $\nabla_3\underline{\check{\kappa}}$ and $\nabla_4\underline{\check{\kappa}}$ on Σ_* imply $\nabla_4\underline{\check{\kappa}} = r^{-1}\not{\mathfrak{d}}^{\leq 1}\Gamma_g$ and $\nabla_3\underline{\check{\kappa}} = r^{-1}\not{\mathfrak{d}}^{\leq 1}\Gamma_b$, and making use of the commutation formulas of Lemma 5.20, we infer, arguing as for the commutator $[\not{d}_2^*\not{d}_1^*\not{d}_1, \nabla_3]\zeta$,

$$
[\not{d}_2^*\not{d}_1^*\not{d}_1, \nabla_3]\underline{\check{\kappa}} = r^{-4}\not{\mathfrak{d}}^{\leq 3}\Gamma_g + r^{-3}\not{\mathfrak{d}}^{\leq 3}(\Gamma_b\cdot\Gamma_b),
$$

and hence

$$
\begin{aligned}
2\not{d}_2^*\not{d}_1^*\not{d}_1\not{d}_2\not{d}_2^*\underline{\xi} &= -\nabla_3\not{d}_2^*\not{d}_1^*\not{d}_1\nabla\underline{\check{\kappa}} - \frac{2}{r}\nabla_3\not{d}_2^*\not{d}_1^*\not{d}_1\zeta - \frac{2}{r}\not{d}_2^*\not{d}_1^*\not{d}_1\underline{\beta}\\
&\quad + r^{-5}\not{\mathfrak{d}}^{\leq 4}\Gamma_g + r^{-4}\not{\mathfrak{d}}^{\leq 4}(\Gamma_b\cdot\Gamma_b).
\end{aligned}
$$

Now, recall from above that we have

$$
\begin{aligned}
\nabla_3\not{d}_2^*\not{d}_1^*\not{d}_1\zeta &= -\nabla_3\not{d}_2^*\not{d}_1^*\check{\underline{\mu}} + \not{d}_2^*\not{d}_1^*\Big(\mathrm{div}\,\underline{\beta}, -\mathrm{curl}\,\underline{\beta}\Big) + r^{-4}\not{\mathfrak{d}}^{\leq 2}\Gamma_g\\
&\quad + r^{-3}\not{\mathfrak{d}}^{\leq 3}(\Gamma_b\cdot\Gamma_b).
\end{aligned}
$$

We deduce

$$
\begin{aligned}
2\not{d}_2^*\not{d}_1^*\not{d}_1\not{d}_2\not{d}_2^*\underline{\xi} &= -\nabla_3\not{d}_2^*\not{d}_1^*\not{d}_1\nabla\underline{\check{\kappa}} + \frac{2}{r}\nabla_3\not{d}_2^*\not{d}_1^*\check{\underline{\mu}} - \frac{4}{r}\not{d}_2^*\not{d}_1^*\mathrm{div}\,\underline{\beta}\\
&\quad + r^{-5}\not{\mathfrak{d}}^{\leq 4}\Gamma_g + r^{-4}\not{\mathfrak{d}}^{\leq 4}(\Gamma_b\cdot\Gamma_b).
\end{aligned}
$$

Also, we have

$$
-\not{d}_2^*\not{d}_1^*\not{d}_1\nabla\underline{\check{\kappa}} = \not{d}_2^*\not{d}_1^*\not{d}_1\not{d}_1^*(\underline{\check{\kappa}}, 0) = \not{d}_2^*(\not{d}_2\not{d}_2^* + 2K)\not{d}_1^*(\underline{\check{\kappa}}, 0)
$$

$$
\begin{aligned}
&= \ \not{d}_2^\star\Big(\not{d}_2\not{d}_2^\star + \frac{2}{r^2}\Big)\not{d}_1^\star(\underline{\check{\kappa}}, 0) + 2\not{d}_2^\star\Big(\check{K}\not{d}_1^\star(\underline{\check{\kappa}}, 0)\Big)\\
&= \ \Big(\not{d}_2\not{d}_2^\star + \frac{2}{r^2}\Big)\not{d}_2^\star\not{d}_1^\star(\underline{\check{\kappa}}, 0) + 2\not{d}_2^\star\Big(\check{K}\not{d}_1^\star(\underline{\check{\kappa}}, 0)\Big)
\end{aligned}
$$

and hence

$$
\begin{aligned}
2\not{d}_2^\star\not{d}_1^\star\not{d}_1\not{d}_2\not{d}_2^\star\underline{\xi} \ &= \ \nabla_3\Big(\not{d}_2\not{d}_2^\star + \frac{2}{r^2}\Big)\not{d}_2^\star\not{d}_1^\star\underline{\check{\kappa}} + \frac{2}{r}\nabla_3\not{d}_2^\star\not{d}_1^\star\check{\underline{\mu}} - \frac{4}{r}\not{d}_2^\star\not{d}_1^\star\mathrm{div}\,\underline{\beta}\\
&+\nabla_3\Big(2\not{d}_2^\star\Big(\check{K}\not{d}_1^\star(\underline{\check{\kappa}}, 0)\Big)\Big) + r^{-5}\,\not{\mathfrak{d}}^{\leq 4}\Gamma_g + r^{-4}\,\not{\mathfrak{d}}^{\leq 4}(\Gamma_b\cdot\Gamma_b).
\end{aligned}
$$

Now, from Proposition 5.18, we have $\nabla_4\underline{\check{\kappa}} = r^{-1}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_g$, $\nabla_3\underline{\check{\kappa}} = r^{-1}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_b$, $\nabla_4\check{K} = r^{-2}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_g$ and $\nabla_3\check{K} = r^{-2}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_b$ on Σ_*, which, together with the commutation formulas of Lemma 5.20, implies

$$
\nabla_3\Big(2\not{d}_2^\star\Big(\check{K}\not{d}_1^\star(\underline{\check{\kappa}}, 0)\Big)\Big) \ = \ r^{-4}\,\not{\mathfrak{d}}^{\leq 3}(\Gamma_b\cdot\Gamma_b)
$$

and hence

$$
\begin{aligned}
2\not{d}_2^\star\not{d}_1^\star\not{d}_1\not{d}_2\not{d}_2^\star\underline{\xi} \ &= \ \nabla_3\Big(\not{d}_2\not{d}_2^\star + \frac{2}{r^2}\Big)\not{d}_2^\star\not{d}_1^\star\underline{\check{\kappa}} + \frac{2}{r}\nabla_3\not{d}_2^\star\not{d}_1^\star\check{\underline{\mu}} - \frac{4}{r}\not{d}_2^\star\not{d}_1^\star\mathrm{div}\,\underline{\beta}\\
&+r^{-5}\,\not{\mathfrak{d}}^{\leq 4}\Gamma_g + r^{-4}\,\not{\mathfrak{d}}^{\leq 4}(\Gamma_b\cdot\Gamma_b)
\end{aligned}
$$

as stated. This concludes the proof of Proposition 5.23.

B.3. Proof of Lemma 5.25

Below, recall that the notation $O(r^a)$, for $a\in\mathbb{R}$, denotes an explicit function of r which is bounded by r^a as $r\to+\infty$.

First identity. We start with the following equation, see Proposition 5.18,

$$
\nabla_3\underline{\check{\kappa}} - \frac{2\Upsilon}{r}\underline{\check{\kappa}} \ = \ 2\mathrm{div}\,\underline{\xi} + \frac{4\Upsilon}{r}\underline{\check{\omega}} - \frac{2m}{r^2}\underline{\check{\kappa}} - \left(\frac{2}{r^2} - \frac{8m}{r^3}\right)\check{y} + \Gamma_b\cdot\Gamma_b.
$$

Commuting the first equation with the Laplacian we deduce

$$
\begin{aligned}
\nabla_3\Delta\underline{\check{\kappa}} - \frac{2\Upsilon}{r}\Delta\underline{\check{\kappa}} \ &= \ [\nabla_3, \Delta]\underline{\check{\kappa}} + 2\Delta\mathrm{div}\,\underline{\xi} + \frac{4\Upsilon}{r}\Delta\underline{\check{\omega}} - \frac{2m}{r^2}\Delta\underline{\check{\kappa}}\\
&-\left(\frac{2}{r^2} - \frac{8m}{r^3}\right)\Delta\check{y} + r^{-2}\,\not{\mathfrak{d}}^{\leq 2}(\Gamma_b\cdot\Gamma_b).
\end{aligned}
$$

According to Lemma 5.20, we have

$$[\nabla_3, \Delta]\underline{\check{\kappa}} = \frac{2\Upsilon}{r}\Delta\underline{\check{\kappa}} + r^{-1}\,\mathfrak{d}\!\!/^{\leq 1}\Big(\Gamma_b \cdot \nabla_3\underline{\check{\kappa}} + r^{-1}\Gamma_b \cdot \mathfrak{d}\underline{\check{\kappa}}\Big)$$

and, using again the equation for $\nabla_3\underline{\kappa}$,

$$[\nabla_3, \Delta]\underline{\check{\kappa}} = \frac{2\Upsilon}{r}\Delta\underline{\check{\kappa}} + r^{-2}\,\mathfrak{d}\!\!/^{\leq 2}(\Gamma_b \cdot \Gamma_b).$$

Thus

$$\begin{aligned}\nabla_3\Delta\underline{\check{\kappa}} - \frac{4\Upsilon}{r}\Delta\underline{\check{\kappa}} &= 2\Delta\mathrm{div}\,\underline{\xi} + \frac{4\Upsilon}{r}\Delta\underline{\check{\omega}} - \frac{2m}{r^2}\Delta\underline{\check{\kappa}} - \left(\frac{2}{r^2} - \frac{8m}{r^3}\right)\Delta\underline{\check{y}} \\ &\quad + r^{-2}\,\mathfrak{d}\!\!/^{\leq 2}(\Gamma_b \cdot \Gamma_b). \end{aligned} \tag{B.1}$$

To remove the term involving $\Delta\underline{\omega}$, we consider the null structure equation for ζ in the form

$$\nabla_3\zeta - \frac{\Upsilon}{r}\zeta = -\underline{\beta} - 2\nabla\underline{\check{\omega}} + \frac{\Upsilon}{r}(\eta + \zeta) + \frac{1}{r}\underline{\xi} + \frac{2m}{r^2}(\zeta - \eta) + \Gamma_b \cdot \Gamma_b.$$

Differentiating w.r.t. div , and using the commutation

$$[\nabla_3, \mathrm{div}\,]\zeta = \frac{\Upsilon}{r}\mathrm{div}\,\zeta + \Gamma_b \cdot \nabla_3\zeta + r^{-1}\Gamma_b \cdot \mathfrak{d}\zeta = \frac{\Upsilon}{r}\mathrm{div}\,\zeta + r^{-1}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_b \cdot \Gamma_b),$$

we obtain

$$\begin{aligned}\nabla_3\mathrm{div}\,\zeta - \frac{2\Upsilon}{r}\mathrm{div}\,\zeta &= -\mathrm{div}\,\underline{\beta} - 2\Delta\underline{\check{\omega}} + \frac{\Upsilon}{r}\mathrm{div}\,(\eta + \zeta) + \frac{1}{r}\mathrm{div}\,\underline{\xi} \\ &\quad + \frac{2m}{r^2}\mathrm{div}\,(\zeta - \eta) + r^{-1}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_b \cdot \Gamma_b)\end{aligned}$$

from which we easily deduce

$$\begin{aligned}\nabla_3\left(\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right) &= \frac{2\Upsilon}{r}\nabla_3\mathrm{div}\,\zeta + \left(-\frac{2e_3(r)}{r^2} + \frac{8me_3(r)}{r^3}\right)\mathrm{div}\,\zeta \\ &= \frac{4\Upsilon^2}{r^2}\mathrm{div}\,\zeta - \frac{2\Upsilon}{r}\mathrm{div}\,\underline{\beta} - \frac{4\Upsilon}{r}\Delta\underline{\check{\omega}} + \frac{2\Upsilon^2}{r^2}\mathrm{div}\,(\eta + \zeta) \\ &\quad + \frac{2\Upsilon}{r^2}\mathrm{div}\,\underline{\xi} + \frac{4m\Upsilon}{r^3}\mathrm{div}\,(\zeta - \eta) + \left(\frac{2\Upsilon}{r^2} - \frac{8m\Upsilon}{r^3}\right)\mathrm{div}\,\zeta \\ &\quad + r^{-2}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_b \cdot \Gamma_b)\end{aligned}$$

and hence

$$\begin{aligned}\frac{4\Upsilon}{r}\Delta\underline{\check{\omega}} &= -\nabla_3\left(\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right)+\frac{4\Upsilon^2}{r^2}\mathrm{div}\,\zeta-\frac{2\Upsilon}{r}\mathrm{div}\,\underline{\beta}+\frac{2\Upsilon^2}{r^2}\mathrm{div}\,(\eta+\zeta)\\ &\quad+\frac{2\Upsilon}{r^2}\mathrm{div}\,\underline{\xi}+\frac{4m\Upsilon}{r^3}\mathrm{div}\,(\zeta-\eta)+\left(\frac{2\Upsilon}{r^2}-\frac{8m\Upsilon}{r^3}\right)\mathrm{div}\,\zeta\\ &\quad+r^{-2}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).\end{aligned}$$

Combining this with the previous identity (B.1), we deduce

$$\begin{aligned}\nabla_3\left(\Delta\underline{\check{\kappa}}+\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right) &= O(r^{-1})\Delta\underline{\check{\kappa}}+2\Delta\mathrm{div}\,\underline{\xi}+O(r^{-2})\Delta\check{y}+O(r^{-2})\mathrm{div}\,\zeta\\ &\quad+O(r^{-1})\mathrm{div}\,\underline{\beta}+O(r^{-2})\mathrm{div}\,\eta+O(r^{-2})\mathrm{div}\,\underline{\xi}\\ &\quad+r^{-2}\,\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_b).\end{aligned}$$

Similarly, proceeding as above, starting with the following equations, see Proposition 5.18,

$$\nabla_4\underline{\check{\kappa}}+\frac{1}{r}\underline{\check{\kappa}} \;=\; -2\mathrm{div}\,\zeta+2\check{\rho}+\Gamma_b\cdot\Gamma_g,$$

we deduce

$$\nabla_4\Delta\underline{\check{\kappa}}+\frac{3}{r}\Delta\underline{\check{\kappa}} \;=\; -2\Delta\mathrm{div}\,\zeta+2\Delta\check{\rho}+r^{-2}\,\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g).$$

Also, using

$$\nabla_4\zeta+\frac{2}{r}\zeta \;=-\beta+\Gamma_g\cdot\Gamma_g,$$

and differentiating w.r.t. div , we have

$$\nabla_4\mathrm{div}\,\zeta+\frac{3}{r}\mathrm{div}\,\zeta \;=-\mathrm{div}\,\beta+r^{-1}\,\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g)$$

and hence

$$\begin{aligned}\nabla_4\left(\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right) &= \frac{2\Upsilon}{r}\nabla_4\mathrm{div}\,\zeta+\left(-\frac{2}{r^2}+\frac{8m}{r^3}\right)\mathrm{div}\,\zeta\\ &= -\frac{2\Upsilon}{r}\mathrm{div}\,\beta-\frac{4}{r^2}\left(2-\frac{5m}{r}\right)\mathrm{div}\,\zeta+r^{-2}\,\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g).\end{aligned}$$

Combining, we obtain

$$\nabla_4\left(\Delta\underline{\check{\kappa}}+\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right) = O(r^{-1})\Delta\underline{\check{\kappa}}-2\Delta\mathrm{div}\,\zeta+2\Delta\check{\rho}+O(r^{-1})\mathrm{div}\,\beta +O(r^{-2})\mathrm{div}\,\zeta+r^{-2}\,\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g)$$

and, since $b_*=-1-\frac{2m}{r}+r\Gamma_b$,

$$\begin{aligned}b_*\nabla_4\left(\Delta\underline{\check{\kappa}}+\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right) &= O(r^{-1})\Delta\underline{\check{\kappa}}+2\left(1+O(r^{-1})\right)\Delta\mathrm{div}\,\zeta\\ &\quad -2\left(1+O(r^{-1})\right)\Delta\check{\rho}+O(r^{-1})\mathrm{div}\,\beta\\ &\quad +O(r^{-2})\mathrm{div}\,\zeta+r^{-2}\,\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g).\end{aligned}$$

Since

$$\nabla_\nu\left(\Delta\underline{\check{\kappa}}+\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right) = \nabla_3\left(\Delta\underline{\check{\kappa}}+\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right)+b_*\nabla_4\left(\Delta\underline{\check{\kappa}}+\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right),$$

we infer from the above

$$\begin{aligned}&\nabla_\nu\left(\Delta\underline{\check{\kappa}}+\frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right)\\ =\ & O(r^{-1})\Delta\underline{\check{\kappa}}+2\Delta\mathrm{div}\,\underline{\xi}+O(r^{-2})\Delta\check{\underline{y}}+O(r^{-2})\mathrm{div}\,\zeta+O(r^{-1})\mathrm{div}\,\underline{\beta}\\ &+O(r^{-2})\mathrm{div}\,\eta+O(r^{-2})\mathrm{div}\,\underline{\xi}+2\left(1+O(r^{-1})\right)\Delta\mathrm{div}\,\zeta\\ &-2\left(1+O(r^{-1})\right)\Delta\check{\rho}+O(r^{-1})\mathrm{div}\,\beta+r^{-2}\,\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_b)\end{aligned}$$

as stated.

Identities 2 and 3. We have, see Proposition 5.18,

$$\nabla_3\beta-\frac{2\Upsilon}{r}\beta = (\nabla\rho+{}^*\nabla{}^*\rho)+\frac{2m}{r^2}\beta-\frac{6m}{r^3}\eta+r^{-1}\Gamma_b\cdot\Gamma_g.$$

Taking the divergence and the curl, we infer

$$\begin{aligned}\nabla_3\mathrm{div}\,\beta-\frac{2\Upsilon}{r}\mathrm{div}\,\beta &= [\nabla_3,\mathrm{div}\,]\beta+\Delta\rho+\frac{2m}{r^2}\mathrm{div}\,\beta-\frac{6m}{r^3}\mathrm{div}\,\eta\\ &\quad +r^{-2}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g),\\ \nabla_3\mathrm{curl}\,\beta-\frac{2\Upsilon}{r}\mathrm{curl}\,\beta &= [\nabla_3,\mathrm{curl}\,]\beta-\Delta{}^*\rho+\frac{2m}{r^2}\mathrm{curl}\,\beta-\frac{6m}{r^3}\mathrm{curl}\,\eta\\ &\quad +r^{-2}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g).\end{aligned}$$

According to Lemma 5.20, using the equation for $\nabla_3\beta$ and $\nabla_4\beta$,

$$
\begin{aligned}
[\nabla_3, \mathrm{div}\,]\beta &= \frac{\Upsilon}{r}\mathrm{div}\,\beta + \Gamma_b\cdot\nabla_3\beta + r^{-2}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_g\cdot\Gamma_b)\\
&= \frac{\Upsilon}{r}\mathrm{div}\,\beta + r^{-2}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_g\cdot\Gamma_b),\\
[\nabla_3, \mathrm{curl}\,]\beta &= \frac{\Upsilon}{r}\mathrm{curl}\,\beta + \Gamma_b\cdot\nabla_3\beta + r^{-2}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_g\cdot\Gamma_b)\\
&= \frac{\Upsilon}{r}\mathrm{curl}\,\beta + r^{-2}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\nabla_3\mathrm{div}\,\beta &= \left(\frac{3}{r}-\frac{4m}{r^2}\right)\mathrm{div}\,\beta + \Delta\rho - \frac{6m}{r^3}\mathrm{div}\,\eta + r^{-2}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_b\cdot\Gamma_g),\\
\nabla_3\mathrm{curl}\,\beta &= \left(\frac{3}{r}-\frac{4m}{r^2}\right)\mathrm{curl}\,\beta - \Delta\ {}^*\rho - \frac{6m}{r^3}\mathrm{curl}\,\eta + r^{-2}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_b\cdot\Gamma_g).
\end{aligned}
$$

Since $\mathrm{curl}\,\eta = {}^*\rho + \Gamma_b\cdot\Gamma_g$, we infer

$$
\begin{aligned}
\nabla_3\mathrm{div}\,\beta &= \left(\frac{3}{r}-\frac{4m}{r^2}\right)\mathrm{div}\,\beta + \Delta\rho - \frac{6m}{r^3}\mathrm{div}\,\eta + r^{-2}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_b\cdot\Gamma_g),\\
\nabla_3\mathrm{curl}\,\beta &= \left(\frac{3}{r}-\frac{4m}{r^2}\right)\mathrm{curl}\,\beta - \Delta\ {}^*\rho - \frac{6m}{r^3}\ {}^*\rho + r^{-2}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_b\cdot\Gamma_g).
\end{aligned}
$$

Also, we have

$$
\nabla_4\beta + \frac{4}{r}\beta = -\mathrm{div}\,\alpha + r^{-1}\Gamma_g\cdot\Gamma_g.
$$

Taking the divergence and the curl, we infer

$$
\begin{aligned}
\nabla_4\mathrm{div}\,\beta + \frac{4}{r}\mathrm{div}\,\beta &= [\nabla_4, \mathrm{div}\,]\beta - \mathrm{div}\,\mathrm{div}\,\alpha + r^{-2}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_g\cdot\Gamma_g),\\
\nabla_4\mathrm{curl}\,\beta + \frac{4}{r}\mathrm{curl}\,\beta &= [\nabla_4, \mathrm{curl}\,]\beta - \mathrm{curl}\,\mathrm{div}\,\alpha + r^{-2}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_g\cdot\Gamma_g).
\end{aligned}
$$

According to Lemma 5.20, using again the equation for $\nabla_3\beta$ and $\nabla_4\beta$,

$$
\begin{aligned}
[\nabla_4, \mathrm{div}\,]\beta &= -\frac{1}{r}\mathrm{div}\,\beta + r^{-2}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_g\cdot\Gamma_g),\\
[\nabla_4, \mathrm{curl}\,]\beta &= -\frac{1}{r}\mathrm{curl}\,\beta + r^{-2}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_g\cdot\Gamma_g).
\end{aligned}
$$

Hence

$$
\nabla_4\mathrm{div}\,\beta = -\frac{5}{r}\mathrm{div}\,\beta - \mathrm{div}\,\mathrm{div}\,\alpha + r^{-2}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_g\cdot\Gamma_g),
$$

$$\nabla_4 \text{curl}\,\beta = -\frac{5}{r}\text{curl}\,\beta - \text{curl}\,\text{div}\,\alpha + r^{-2}\,\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g).$$

Since $\nu = e_3 + b_* e_4$ and $b_* = -1 - \frac{2m}{r} + r\Gamma_b$, we infer

$$\begin{aligned}\nabla_\nu \text{div}\,\beta &= O(r^{-1})\text{div}\,\beta + \Delta\rho + (1+O(r^{-1}))\text{div}\,\text{div}\,\alpha + O(r^{-3})\text{div}\,\eta\\ &\quad + r^{-2}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g),\\ \nabla_\nu \text{curl}\,\beta &= \frac{8}{r}(1+O(r^{-1}))\text{curl}\,\beta - \Delta\ {}^*\rho + (1+O(r^{-1}))\text{curl}\,\text{div}\,\alpha\\ &\quad + O(r^{-3})\ {}^*\rho + r^{-2}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g).\end{aligned}$$

Identity 4. We have, see Proposition 5.18,

$$\nabla_3\check{\rho} - \frac{3\Upsilon}{r}\check{\rho} = -\text{div}\,\underline{\beta} + \frac{3m}{r^3}\underline{\check{\kappa}} - \frac{6m}{r^4}\check{y} - \frac{1}{2}\widehat{\chi}\cdot\underline{\alpha} + r^{-1}\Gamma_b\cdot\Gamma_b.$$

Also, using the equation for $\widehat{\chi}$ and $\underline{\widehat{\chi}}$, we have

$$\begin{aligned}\nabla_3(\widehat{\chi}\cdot\underline{\widehat{\chi}}) &= \widehat{\chi}\cdot\left(\frac{2\Upsilon}{r}\underline{\widehat{\chi}} - \underline{\alpha} - \frac{2m}{r^2}\underline{\widehat{\chi}} + \nabla\widehat{\otimes}\underline{\xi} + \Gamma_b\cdot\Gamma_b\right)\\ &\quad + \left(\frac{\Upsilon}{r}\widehat{\chi} + \nabla\widehat{\otimes}\eta - \frac{1}{r}\underline{\widehat{\chi}} + \frac{2m}{r^2}\widehat{\chi} + \Gamma_b\cdot\Gamma_b\right)\cdot\underline{\widehat{\chi}}\\ &= -\widehat{\chi}\cdot\underline{\alpha} + r^{-1}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).\end{aligned}$$

Hence, we deduce

$$\nabla_3\left(\check{\rho} - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right) = -\text{div}\,\underline{\beta} + \frac{3\Upsilon}{r}\check{\rho} + \frac{3m}{r^3}\underline{\check{\kappa}} - \frac{6m}{r^4}\check{y} + r^{-1}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

Also, we have

$$\nabla_4\check{\rho} + \frac{3}{r}\check{\rho} = \text{div}\,\beta + r^{-1}\Gamma_b\cdot\Gamma_g$$

and

$$\nabla_4(\widehat{\chi}\cdot\underline{\widehat{\chi}}) = r^{-1}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g),$$

and hence

$$\nabla_4\left(\check{\rho} - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right) = \text{div}\,\beta - \frac{3}{r}\check{\rho} + r^{-1}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g).$$

Since $\nu = e_3 + b_* e_4$ and $b_* = -1 - \frac{2m}{r} + r\Gamma_b$, we infer

$$\begin{aligned}\nabla_\nu\left(\check{\rho} - \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}}\right) &= -\mathrm{div}\,\underline{\beta} - (1+O(r^{-1}))\mathrm{div}\,\beta + O(r^{-1})\check{\rho} + O(r^{-3})\check{\underline{\kappa}}\\ &\quad +O(r^{-4})\check{y} + r^{-1}\,\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b)\end{aligned}$$

as stated. This concludes the proof of Lemma 5.25.

B.4. Proof of Corollary 5.41

Since $\nu(J^{(p)}) = 0$, we have in view of Corollary 5.32 for any scalar function h on Σ_* and any $S \subset \Sigma_*$

$$\nu\left(\int_S hJ^{(p)}\right) = \int_S \nu(h)J^{(p)} - \frac{4}{r}\int_S hJ^{(p)} + r^3\Gamma_b\nu(h) + r^2\Gamma_b h \tag{B.2}$$

where we also used $J^{(p)} = O(1)$, and where we recall that the notation $O(r^a)$, for $a \in \mathbb{R}$, denotes an explicit function of r which is bounded by r^a as $r \to +\infty$.

Next, recall from Lemma 5.25 that we have along Σ_*

$$\begin{aligned}\nabla_\nu\left(\Delta\check{\underline{\kappa}} + \frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right) &= O(r^{-1})\Delta\check{\underline{\kappa}} + 2\Delta\mathrm{div}\,\underline{\xi} + O(r^{-2})\Delta\check{y} + O(r^{-2})\mathrm{div}\,\zeta\\ &\quad +O(r^{-1})\mathrm{div}\,\underline{\beta} + O(r^{-2})\mathrm{div}\,\eta + O(r^{-2})\mathrm{div}\,\underline{\xi}\\ &\quad +2\left(1+O(r^{-1})\right)\Delta\mathrm{div}\,\zeta - 2\left(1+O(r^{-1})\right)\Delta\check{\rho}\\ &\quad +O(r^{-1})\mathrm{div}\,\beta + r^{-2}\,\not{\mathfrak{d}}^{\leq 2}(\Gamma_b\cdot\Gamma_b).\end{aligned}$$

Together with (B.2), and noticing that the terms $O(r^a)$ only depend on r and are thus constant on S, we infer

$$\begin{aligned}&\nu\left(\int_S\left(\Delta\check{\underline{\kappa}} + \frac{2\Upsilon}{r}\mathrm{div}\,\zeta\right)J^{(p)}\right)\\ &= O(r^{-1})\int_S\Delta\check{\underline{\kappa}}J^{(p)} + 2\int_S\Delta\mathrm{div}\,\underline{\xi}J^{(p)} + O(r^{-2})\int_S\Delta\check{y}J^{(p)}\\ &\quad +O(r^{-2})\int_S\mathrm{div}\,\zeta J^{(p)} + O(r^{-1})\int_S\mathrm{div}\,\underline{\beta}J^{(p)} + O(r^{-2})\int_S\mathrm{div}\,\eta J^{(p)}\\ &\quad +O(r^{-2})\int_S\mathrm{div}\,\underline{\xi}J^{(p)} + 2\left(1+O(r^{-1})\right)\int_S\Delta\mathrm{div}\,\zeta J^{(p)}\\ &\quad -2\left(1+O(r^{-1})\right)\int_S\Delta\check{\rho}J^{(p)} + O(r^{-1})\int_S\mathrm{div}\,\beta J^{(p)} + \not{\mathfrak{d}}^{\leq 2}(\Gamma_b\cdot\Gamma_b).\end{aligned}$$

Integrating by parts, we infer

$$
\begin{aligned}
& \nu\left(\int_S \left(\Delta\check{\underline{\kappa}} + \frac{2\Upsilon}{r}\operatorname{div}\zeta\right) J^{(p)}\right) \\
= \;& O(r^{-3})\int_S \check{\underline{\kappa}} J^{(p)} + O(r^{-2})\int_S \operatorname{div}\underline{\xi} J^{(p)} + O(r^{-2})\int_S \Delta\check{\underline{y}} J^{(p)} \\
& + O(r^{-2})\int_S \operatorname{div}\zeta J^{(p)} + O(r^{-1})\int_S \operatorname{div}\underline{\beta} J^{(p)} + O(r^{-2})\int_S \operatorname{div}\eta J^{(p)} \\
& + O(r^{-2})\int_S \check{\rho} J^{(p)} + O(r^{-1})\int_S \operatorname{div}\beta J^{(p)} + r\left|\left(\Delta + \frac{2}{r^2}\right) J^{(p)}\right| \not{\mathfrak{d}}^{\leq 1}\Gamma_b \\
& + \not{\mathfrak{d}}^{\leq 2}(\Gamma_b\cdot\Gamma_b).
\end{aligned}
$$

Also, in view of Lemma 5.12, we have

$$
\Delta\check{\underline{y}} \;=\; \operatorname{div}(-\underline{\xi} + (\zeta-\eta)\underline{y}) = -\operatorname{div}\underline{\xi} + \underline{y}(\operatorname{div}\zeta - \operatorname{div}\eta) + (\zeta-\eta)\cdot\nabla\underline{y},
$$

and hence, since $\underline{y} = -\Upsilon + r\Gamma_b$,

$$
\Delta\check{\underline{y}} \;=\; -\operatorname{div}\underline{\xi} - \Upsilon(\operatorname{div}\zeta - \operatorname{div}\eta) + \not{\mathfrak{d}}(\Gamma_b\cdot\Gamma_b)
$$

so that

$$
\begin{aligned}
& \nu\left(\int_S \left(\Delta\check{\underline{\kappa}} + \frac{2\Upsilon}{r}\operatorname{div}\zeta\right) J^{(p)}\right) \\
= \;& O(r^{-3})\int_S \check{\underline{\kappa}} J^{(p)} + O(r^{-2})\int_S \operatorname{div}\underline{\xi} J^{(p)} + O(r^{-2})\int_S \operatorname{div}\zeta J^{(p)} \\
& + O(r^{-1})\int_S \operatorname{div}\underline{\beta} J^{(p)} + O(r^{-2})\int_S \operatorname{div}\eta J^{(p)} + O(r^{-2})\int_S \check{\rho} J^{(p)} \\
& + O(r^{-1})\int_S \operatorname{div}\beta J^{(p)} + r\left|\left(\Delta + \frac{2}{r^2}\right) J^{(p)}\right| \not{\mathfrak{d}}^{\leq 1}\Gamma_b + \not{\mathfrak{d}}^{\leq 2}(\Gamma_b\cdot\Gamma_b).
\end{aligned}
$$

Together with the GCM conditions $(\operatorname{div}\eta)_{\ell=1} = 0$ and $(\operatorname{div}\underline{\xi})_{\ell=1} = 0$, we infer

$$
\begin{aligned}
& \nu\left(\int_S \left(\Delta\check{\underline{\kappa}} + \frac{2\Upsilon}{r}\operatorname{div}\zeta\right) J^{(p)}\right) \\
= \;& O(r^{-3})\int_S \check{\underline{\kappa}} J^{(p)} + O(r^{-2})\int_S \operatorname{div}\zeta J^{(p)} + O(r^{-1})\int_S \operatorname{div}\underline{\beta} J^{(p)} \\
& + O(r^{-2})\int_S \check{\rho} J^{(p)} + O(r^{-1})\int_S \operatorname{div}\beta J^{(p)} + r\left|\left(\Delta + \frac{2}{r^2}\right) J^{(p)}\right| \not{\mathfrak{d}}^{\leq 1}\Gamma_b \\
& + \not{\mathfrak{d}}^{\leq 2}(\Gamma_b\cdot\Gamma_b)
\end{aligned}
$$

as stated.

Next, recall from Lemma 5.25 that we have along Σ_*

$$\begin{aligned}\nabla_\nu \mathrm{div}\,\beta &= O(r^{-1})\mathrm{div}\,\beta + \Delta\rho + (1+O(r^{-1}))\mathrm{div}\,\mathrm{div}\,\alpha\\ &\quad +O(r^{-3})\mathrm{div}\,\eta + r^{-2}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g).\end{aligned}$$

Together with (B.2), and noticing that the terms $O(r^a)$ only depend on r and are thus constant on S, we infer

$$\begin{aligned}\nu\left(\int_S \mathrm{div}\,\beta J^{(p)}\right) &= O(r^{-1})\int_S \mathrm{div}\,\beta J^{(p)} + \int_S \Delta\check{\rho} J^{(p)}\\ &\quad +(1+O(r^{-1}))\int_S \mathrm{div}\,\mathrm{div}\,\alpha J^{(p)} + O(r^{-3})\int_S \mathrm{div}\,\eta J^{(p)}\\ &\quad + \mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g).\end{aligned}$$

Together with the GCM condition $(\mathrm{div}\,\eta)_{\ell=1}=0$, we infer

$$\begin{aligned}\nu\left(\int_S \mathrm{div}\,\beta J^{(p)}\right) &= O(r^{-1})\int_S \mathrm{div}\,\beta J^{(p)} + \int_S \Delta\check{\rho} J^{(p)}\\ &\quad +(1+O(r^{-1}))\int_S \mathrm{div}\,\mathrm{div}\,\alpha J^{(p)} + \mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g).\end{aligned}$$

Integrating by parts, we deduce the desired identity for $\mathrm{div}\,\beta$

$$\begin{aligned}\nu\left(\int_S \mathrm{div}\,\beta J^{(p)}\right) &= O(r^{-1})\int_S \mathrm{div}\,\beta J^{(p)} + O(r^{-2})\int_S \check{\rho} J^{(p)}\\ &\quad +r\left(\left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right| + \left|{\not{d}}_2^*{\not{d}}_1^* J^{(p)}\right|\right)\Gamma_g + \mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g),\end{aligned}$$

where by ${\not{d}}_1^* J^{(p)}$, we mean ${\not{d}}_1^*(J^{(p)},0)$. Similarly, starting from

$$\begin{aligned}\nabla_\nu \mathrm{curl}\,\beta &= \frac{8}{r}(1+O(r^{-1}))\mathrm{curl}\,\beta - \Delta\ {}^*\rho + (1+O(r^{-1}))\mathrm{curl}\,\mathrm{div}\,\alpha\\ &\quad +O(r^{-3})\ {}^*\rho + r^{-2}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g),\end{aligned}$$

we deduce the desired identity for $\mathrm{curl}\,\beta$

$$\begin{aligned}&\nu\left(\int_S \mathrm{curl}\,\beta J^{(p)}\right)\\ =\ & \frac{4}{r}(1+O(r^{-1}))\int_S \mathrm{curl}\,\beta J^{(p)} + \frac{2}{r^2}(1+O(r^{-1}))\int_S {}^*\rho J^{(p)}\\ & +r\left(\left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right| + \left|{\not{d}}_2^*{\not{d}}_1^* J^{(p)}\right|\right)\Gamma_g + \mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g),\end{aligned}$$

where by ${\not{d}}_1^* J^{(p)}$, we mean ${\not{d}}_1^*(0,J^{(p)})$.

Finally, recall from Lemma 5.25 that we have along Σ_*

$$\nabla_\nu\left(\check{\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right) = -\text{div}\,\underline{\beta}-(1+O(r^{-1}))\text{div}\,\beta+O(r^{-1})\check{\rho}+O(r^{-3})\underline{\check{\kappa}} +O(r^{-4})\check{y}+r^{-1}\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

Together with (B.2), and noticing that the terms $O(r^a)$ only depend on r and are thus constant on S, we infer

$$\begin{aligned}
&\nu\left(\int_S\left(\check{\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)J^{(p)}\right)\\
=\;& -\int_S\text{div}\,\underline{\beta}J^{(p)}-(1+O(r^{-1}))\int_S\text{div}\,\beta J^{(p)}\\
&+O(r^{-1})\int_S\left(\check{\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)J^{(p)}+O(r^{-3})\int_S\underline{\check{\kappa}}J^{(p)}\\
&+O(r^{-4})\int_S\check{y}J^{(p)}+r\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).
\end{aligned}$$

Next, recalling from the above that

$$\Delta\check{y} = -\text{div}\,\underline{\xi}-\Upsilon(\text{div}\,\zeta-\text{div}\,\eta)+\mathfrak{d}(\Gamma_b\cdot\Gamma_b)$$

we write, using also integration by parts,

$$\begin{aligned}
O(r^{-4})\int_S\check{y}J^{(p)} &= O(r^{-2})\int_S\Delta\check{y}J^{(p)}+r\left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right|\Gamma_b\\
&= O(r^{-2})\int_S\text{div}\,\underline{\xi}J^{(p)}+O(r^{-2})\int_S\text{div}\,\zeta J^{(p)}\\
&\quad+O(r^{-2})\int_S\text{div}\,\eta J^{(p)}+r\left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right|\Gamma_b\\
&\quad+\mathfrak{d}(\Gamma_b\cdot\Gamma_b).
\end{aligned}$$

Together with the GCM conditions $(\text{div}\,\eta)_{\ell=1}=0$ and $(\text{div}\,\underline{\xi})_{\ell=1}=0$, we infer

$$O(r^{-4})\int_S\check{y}J^{(p)} = O(r^{-2})\int_S\text{div}\,\zeta J^{(p)}+r\left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right|\Gamma_b+\mathfrak{d}(\Gamma_b\cdot\Gamma_b)$$

and hence

$$\begin{aligned}
&\nu\left(\int_S\left(\check{\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)J^{(p)}\right)\\
=\;& -\int_S\text{div}\,\underline{\beta}J^{(p)}-(1+O(r^{-1}))\int_S\text{div}\,\beta J^{(p)}
\end{aligned}$$

$$+O(r^{-1})\int_S\left(\check{\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\right)J^{(p)}+O(r^{-3})\int_S\underline{\check{\kappa}}J^{(p)}$$
$$+O(r^{-2})\int_S \mathrm{div}\,\zeta J^{(p)}+r\left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right|\Gamma_b+r\,\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b)$$

as stated. This concludes the proof of Corollary 5.41.

B.5. Proof of Proposition 5.26

In the proof, we denote by

- (e_1, e_2, e_3, e_4) the null frame of Σ_*,
- (e'_1, e'_2, e'_3, e'_4) the null frame of ${}^{(ext)}\mathcal{M}$,
- $(e''_1, e''_2, e''_3, e''_4)$ the second null frame of ${}^{(ext)}\mathcal{M}$ of Proposition 3.26,
- $(e'''_1, e'''_2, e'''_3, e'''_4)$ the global null frame of $\mathcal{M}$ of Proposition 3.33.

Also, we denote

- by $(f, \underline{f}, \lambda)$ the change coefficients from (e_1, e_2, e_3, e_4) to (e'_1, e'_2, e'_3, e'_4),
- by $(f', \underline{f}', \lambda')$ the change coefficients from (e'_1, e'_2, e'_3, e'_4) to $(e''_1, e''_2, e''_3, e''_4)$.

Since Proposition 5.26 involves identities on Σ_*, it suffices to consider a neighborhood of Σ_* where we have in view of Proposition 3.33

$$e'''_4=\lambda e''_4, \qquad e'''_3=\lambda^{-1}e''_3, \qquad e'''_a=e''_a, \;\; a=1,2, \qquad \lambda=\frac{\Delta}{|q|^2}. \tag{B.3}$$

Also, in view of the construction of $(e''_1, e''_2, e''_3, e''_4)$ in Proposition 3.26, and in view of the definition of $(f', \underline{f}', \lambda')$, we have $\underline{f}'=0$, $\lambda'=1$, and

$$e''_4=e_4+f'^a e'_a+\frac{1}{4}|f'|^2e'_3, \qquad e''_a=e'_a+\frac{1}{2}f'_a e'_3, \;\; a=1,2, \qquad e''_3=e'_3, \tag{B.4}$$

where f' satisfies on ${}^{(ext)}\mathcal{M}$, and hence in a neighborhood of Σ_*, for any $k\leq k_*$,

$$|\mathfrak{d}^k f'|\lesssim \frac{\epsilon}{ru^{\frac{1}{2}+\frac{\delta_{dec}}{2}}}, \qquad |\mathfrak{d}^{k-1}\nabla_3 f'|\lesssim\frac{\epsilon}{ru^{1+\frac{\delta_{dec}}{2}}}. \tag{B.5}$$

Moreover, in view of the initialization of the PG frame of ${}^{(ext)}\mathcal{M}$ on Σ_*, see Section 3.2.5, and in view of the definition of $(f, \underline{f}, \lambda)$, we have $\lambda=1$, and

(B.6)
$$
\begin{aligned}
e_4' &= e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3,\\
e_a' &= \left(\delta_a^b + \frac{1}{2}\underline{f}_a f^b\right) e_b + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2} f_a + \frac{1}{8}|f|^2 \underline{f}_a\right) e_3, \;\; a = 1, 2,\\
e_3' &= \left(1 + \frac{1}{2} f \cdot \underline{f} + \frac{1}{16}|f|^2 |\underline{f}|^2\right) e_3 + \left(\underline{f}^b + \frac{1}{4}|\underline{f}|^2 f^b\right) e_b\\
&\quad + \frac{1}{4}|\underline{f}|^2 e_4,
\end{aligned}
$$

where f and $\underline{f}$ are given respectively by

(B.7)
$$
f_1 = 0, \qquad f_2 = \frac{a\sin\theta}{r}, \quad \text{on} \quad S_*, \qquad \nabla_\nu(rf) = 0 \quad \text{on} \quad \Sigma_*,
$$

and

(B.8)
$$
\underline{f} = -\frac{(\nu(r) - b_*)}{1 - \frac{1}{4} b_* |f|^2} f \quad \text{on} \quad \Sigma_*.
$$

Finally, recall that the quantity $\mathfrak{q}$ is defined with respect to the null frame $(e_1''', e_2''', e_3''', e_4''')$ as follows, see Section 3.6.1,

$$
\mathfrak{q} \;\; = \;\; q\overline{q}^3\Big((\nabla_{e_3'''} - 2\underline{\omega}''')(\nabla_{e_3'''} - 4\underline{\omega}''')A''' + C_1(\nabla_{e_3'''} - 4\underline{\omega}''')A''' + C_2 A'''\Big)
$$

where the scalar functions C_1 and C_2 are given by[229]

(B.9)
$$
\begin{aligned}
C_1 &= 2\mathrm{tr}\,\underline{\chi}''' - 2\frac{{}^{(a)}\mathrm{tr}\underline{\chi}'''^2}{\mathrm{tr}\,\underline{\chi}'''} - 4i\,{}^{(a)}\mathrm{tr}\underline{\chi}''',\\
C_2 &= \frac{1}{2}\mathrm{tr}\,\underline{\chi}'''^2 - 4\,{}^{(a)}\mathrm{tr}\underline{\chi}'''^2 + \frac{3}{2}\frac{{}^{(a)}\mathrm{tr}\underline{\chi}'''^4}{\mathrm{tr}\,\underline{\chi}'''^2} + i\left(-2\mathrm{tr}\,\underline{\chi}'''\,{}^{(a)}\mathrm{tr}\underline{\chi}''' + 4\frac{{}^{(a)}\mathrm{tr}\underline{\chi}'''^3}{\mathrm{tr}\,\underline{\chi}'''}\right).
\end{aligned}
$$

We now ready to prove the identities (5.57) and (5.58) starting with the first one.

B.5.1. Proof of (5.57) In view of Proposition 2.9 applied to the frame $(e_1''', e_2''', e_3''', e_4''')$, we have in particular

$$
(\nabla_{e_3'''} - 4\underline{\omega}''')A''' = \frac{1}{2}\mathcal{D}'''\widehat{\otimes}B''' - \frac{1}{2}\mathrm{tr}\underline{X}'''A''' + \frac{1}{2}(Z''' + 4H''')\widehat{\otimes}B''' - 3\overline{P'''}\widehat{X}'''.
$$

[229] See Definition 5.2.2 in [28].

Also, recall that in the frame $(e_1''', e_2''', e_3''', e_4''')$, we have $\check{H}''' \in \Gamma_g$ and the normalization in ingoing, so that

$$\mathrm{tr}\underline{X}''' = -\frac{2}{q} + \Gamma_g''', \qquad Z''' = \frac{aq}{|q|^2}\mathfrak{J} + \Gamma_g''', \qquad H''' = \frac{aq}{|q|^2}\mathfrak{J} + \Gamma_g''',$$
$$P''' = -\frac{2m}{q^3} + r^{-1}\Gamma_g'''.$$

Together with the fact that $A''' \in r^{-1}\Gamma_g'''$ and $B''' \in r^{-1}\Gamma_g'''$, we infer

$$(\nabla_{e_3'''} - 4\underline{\omega}''')A''' = \frac{1}{2}\mathcal{D}'''\widehat{\otimes}B''' + \frac{1}{q}A''' + \frac{5}{2}\frac{aq}{|q|^2}\mathfrak{J}\widehat{\otimes}B''' + \frac{6m}{\overline{q}^3}\widehat{X}''' + r^{-1}\Gamma_g''' \cdot \Gamma_g'''.$$

Plugging in the definition of $\mathfrak{q}$, and using additionally

$$C_1 = -\frac{4}{r} + O(r^{-2}) + \Gamma_g''', \quad C_2 = \frac{2}{r^2} + O(r^{-3}) + r^{-1}\Gamma_g''', \quad q = r + O(1),$$

we infer

$$\begin{aligned}\mathfrak{q} \;\; = \;\; & q\overline{q}^3\Bigg[(\nabla_{e_3'''} - 2\underline{\omega}''')\left(\frac{1}{2}\mathcal{D}'''\widehat{\otimes}B''' + \frac{1}{q}A''' + \frac{5}{2}\frac{aq}{|q|^2}\mathfrak{J}\widehat{\otimes}B''' + \frac{6m}{\overline{q}^3}\widehat{X}'''\right) \\ & -\frac{4}{r}\left(\frac{1}{2}\mathcal{D}'''\widehat{\otimes}B''' + \frac{1}{q}A'''\right) + \frac{2}{r^2}A'''\Bigg] + \mathfrak{d}^{\leq 1}\Gamma_g''' + r^2\mathfrak{d}^{\leq 1}(\Gamma_b''' \cdot \Gamma_g''').\end{aligned}$$

where we have used in particular the fact that $\nabla_{e_3'''}(\Gamma_g''') = r^{-1}\mathfrak{d}\Gamma_b'''$ and $\nabla_{3'''}(r) = -1 + r\Gamma_b'''$. Also, using again $q = r + O(1)$, $\nabla_{3'''}(r) = -1 + r\Gamma_b'''$, as well as

$$\underline{\omega}''' = O(r^{-2}) + \Gamma_b''', \qquad \nabla_{e_3'''}\mathfrak{J} = O(r^{-1})\mathfrak{J} + r^{-1}\Gamma_b,$$
$$[\nabla_{3'''}, \mathcal{D}'''\widehat{\otimes}]B''' = \frac{1}{r}\mathcal{D}'''\widehat{\otimes}B''' + r^{-4}\mathfrak{d}^{\leq 1}\Gamma_g''' + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b''' \cdot \Gamma_g'''),$$

we obtain

$$\begin{aligned}\mathfrak{q} \;\; = \;\; & q\overline{q}^3\left[\frac{1}{2}\mathcal{D}'''\widehat{\otimes}\nabla_{e_3'''}B''' + \frac{1}{q}\nabla_{e_3'''}A''' + \frac{5}{2}\frac{aq}{|q|^2}\mathfrak{J}\widehat{\otimes}\nabla_{e_3'''}B''' + \frac{6m}{\overline{q}^3}\nabla_{e_3'''}\widehat{X}'''\right] \\ & + r^4\left[-\frac{3}{2r}\mathcal{D}'''\widehat{\otimes}B''' - \frac{1}{r^2}A'''\right] + \mathfrak{d}^{\leq 1}\Gamma_g''' + r^2\mathfrak{d}^{\leq 1}(\Gamma_b''' \cdot \Gamma_g''').\end{aligned}$$

Next, we use the null structure equation for $\nabla_{e_3'''}\widehat{X}'''$ of Proposition 2.8 and the Bianchi identities for $\nabla_{e_3'''}B'''$ and $\nabla_{e_3'''}A'''$ of Proposition 2.9 according to which we have

$$
\begin{aligned}
\nabla_{e_3'''}A''' &= \frac{1}{2}\mathcal{D}'''\widehat{\otimes}B''' + \frac{1}{r}A''' + r^{-3}\Gamma_g''' + r^{-1}\Gamma_b'''\cdot\Gamma_g''',\\
\nabla_{e_3'''}B''' &= \mathcal{D}'''\overline{P'''} + \frac{2}{r}B''' + 3\overline{P'''}H''' + r^{-3}\Gamma_g''' + r^{-1}\Gamma_b'''\cdot\Gamma_g'''\\
&= \mathcal{D}'''\overline{P'''} + \frac{2}{r}B''' + O(r^{-5}) + r^{-3}\Gamma_g''' + r^{-1}\Gamma_b'''\cdot\Gamma_g''',\\
\nabla_{e_3'''}\widehat{X}''' &= -\frac{1}{r}\widehat{\underline{X}}''' + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g''' + \Gamma_b'''\cdot\Gamma_g'''.
\end{aligned}
$$

where we recall that $O(r^a)$ denotes, for $a\in\mathbb{R}$, a function of $(r,\cos\theta)$ bounded by r^a as $r\to+\infty$. We infer

$$
\begin{aligned}
\mathfrak{q} &= r^4\left[\frac{1}{2}\mathcal{D}'''\widehat{\otimes}\left(\mathcal{D}'''\overline{P'''} + \frac{2}{r}B'''\right) + \frac{1}{r}\left(\frac{1}{2}\mathcal{D}'''\widehat{\otimes}B''' + \frac{1}{r}A'''\right) - \frac{6m}{r^4}\widehat{\underline{X}}'''\right]\\
&\quad + r^4\left[-\frac{3}{2r}\mathcal{D}'''\widehat{\otimes}B''' - \frac{1}{r^2}A'''\right] + O(r^{-2}) + \mathfrak{d}^{\leq 2}\Gamma_g''' + r^2\mathfrak{d}^{\leq 2}(\Gamma_b'''\cdot\Gamma_g''')
\end{aligned}
$$

and hence

$$
\begin{aligned}
\mathfrak{q} &= r^4\left[\frac{1}{2}\mathcal{D}'''\widehat{\otimes}\mathcal{D}'''\overline{P'''} - \frac{6m}{r^4}\widehat{\underline{X}}'''\right] + O(r^{-2}) + \mathfrak{d}^{\leq 2}\Gamma_g'''\\
&\quad + r^2\mathfrak{d}^{\leq 2}(\Gamma_b'''\cdot\Gamma_g''').
\end{aligned}
\tag{B.10}
$$

This yields in particular

$$
\mathfrak{q} = \frac{1}{2}r^4\mathcal{D}'''\widehat{\otimes}\mathcal{D}'''\overline{P'''} + O(r^{-2}) + \mathfrak{d}^{\leq 2}\Gamma_b''' + r^2\mathfrak{d}^{\leq 2}(\Gamma_b'''\cdot\Gamma_g''').
$$

Next, we deduce an identity in the second frame of ${}^{(ext)}\mathcal{M}$, i.e. in the frame $(e_1'', e_2'', e_3'', e_4'')$. In view of (B.3), the change of frame coefficients $(f'', \underline{f}'', \lambda'')$ from $(e_1'', e_2'', e_3'', e_4'')$ to $(e_1''', e_2''', e_3''', e_4''')$ satisfy

$$
f'' = 0, \qquad \underline{f}'' = 0, \qquad \lambda'' = 1 + O(r^{-1}).
$$

Together with the transformation formulas of Proposition 2.12, we obtain

$$
P''' = P'', \qquad \mathcal{D}'''\widehat{\otimes}\mathcal{D}''' = \mathcal{D}''\widehat{\otimes}\mathcal{D}'',
$$

and hence

$$
\mathfrak{q} = \frac{1}{2}r^4\mathcal{D}''\widehat{\otimes}\mathcal{D}''\overline{P''} + O(r^{-2}) + \mathfrak{d}^{\leq 2}\Gamma_b''' + r^2\mathfrak{d}^{\leq 2}(\Gamma_b'''\cdot\Gamma_g''').
$$

Next, we deduce an identity in the frame of ${}^{(ext)}\mathcal{M}$, i.e. in the null frame (e_1', e_2', e_3', e_4'). Recall that the change of frame coefficients $(f', \underline{f}', \lambda')$ from (e_1', e_2', e_3', e_4') to $(e_1'', e_2'', e_3'', e_4'')$ satisfy in particular $\underline{f}' = 0$ and $\lambda' = 1$, see (B.4). Together with the transformation formulas of Proposition 2.12, we obtain

$$P'' = P' + r^{-1} f' \cdot \Gamma_b' + \text{ l.o.t.}, \qquad \mathcal{D}'' = \mathcal{D}' + \frac{1}{2} f' \nabla_3'.$$

We deduce

$$\mathfrak{q} = \frac{1}{2} r^4 \mathcal{D}' \widehat{\otimes} \mathcal{D}' \overline{P'} + O(r^{-2}) + \mathfrak{d}^{\leq 2} \Gamma_b''' + r^2 \mathfrak{d}^{\leq 2} (\Gamma_b''' \cdot \Gamma_g''') + r^2 \mathfrak{d}^{\leq 2} (r^{-1} f' \cdot \Gamma_b).$$

Finally, we derive an identity in the frame of Σ_*, i.e. in the frame (e_1, e_2, e_3, e_4). Recall that the change of frame coefficients $(f, \underline{f}, \lambda)$ from (e_1, e_2, e_3, e_4) to (e_1', e_2', e_3', e_4') satisfy in particular $\lambda = 1$, as well as $f = O(r^{-1})$ and $\underline{f} = O(r^{-1}) + \Gamma_b$, see (B.6). Together with the transformation formulas of Proposition 2.12, we obtain

$$P' = P + O(r^{-5}) + r^{-2} \Gamma_b, \qquad \mathcal{D}' = \mathcal{D} + \frac{1}{2} f \nabla_3 + \frac{1}{2} \underline{f} \nabla_4.$$

We deduce

$$\begin{aligned} \mathfrak{q} \;&=\; \frac{1}{2} r^4 \mathcal{D} \widehat{\otimes} \mathcal{D} \overline{P} + O(r^{-2}) + \not{\mathfrak{d}}^{\leq 2} \Gamma_b + \mathfrak{d}^{\leq 2} \Gamma_b''' + r^2 \mathfrak{d}^{\leq 2} (\Gamma_b''' \cdot \Gamma_g''') \\ &\quad + r^2 \mathfrak{d}^{\leq 2} (r^{-1} f' \cdot \Gamma_b). \end{aligned}$$

As Γ_b''' and Γ_g''', $r^{-1} f'$, satisfy for $k \leq k_*$ the same estimates as Γ_b, respectively Γ_g, we write, by a slight abuse of notations

$$\mathfrak{q} \;=\; \frac{1}{2} r^4 \mathcal{D} \widehat{\otimes} \mathcal{D} \overline{P} + O(r^{-2}) + \not{\mathfrak{d}}^{\leq 2} \Gamma_b + r^2 \not{\mathfrak{d}}^{\leq 2} (\Gamma_b \cdot \Gamma_g).$$

Since we have

$$\Re(\mathcal{D} \widehat{\otimes} \mathcal{D} \overline{P}) \;=\; 2 \nabla \widehat{\otimes} \nabla \rho + 2 \nabla \widehat{\otimes} {}^* \nabla \; {}^* \rho = 2 \not{d}_2^* \not{d}_1^* (-\rho, \; {}^* \rho),$$

we infer

$$\Re(\mathfrak{q}) \;=\; r^4 \not{d}_2^* \not{d}_1^* (-\rho, \; {}^* \rho) + O(r^{-2}) + \not{\mathfrak{d}}^{\leq 2} \Gamma_b + r^2 \not{\mathfrak{d}}^{\leq 2} (\Gamma_b \cdot \Gamma_g)$$

which is the desired identity (5.57).

B.5.2. Proof of (5.58) Recall (B.10)

$$\mathfrak{q} = r^4\left[\frac{1}{2}\mathcal{D}'''\widehat{\otimes}\mathcal{D}'''\overline{P'''} - \frac{6m}{r^4}\widehat{\underline{X}}'''\right] + O(r^{-2}) + \mathfrak{d}^{\leq 2}\Gamma_g''' + r^2\mathfrak{d}^{\leq 2}(\Gamma_b'''\cdot\Gamma_g''').$$

We multiply by r and differentiate w.r.t. $\nabla_{e_3'''}$. Using $e_3'''(r) = -1 + r\Gamma_b'''$ and

$$[\nabla_{e_3'''}, \mathcal{D}'''\widehat{\otimes}\mathcal{D}''']\overline{P'''} = r^{-4}\mathfrak{d}^{\leq 2}\Gamma_g''' + r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b'''\cdot\Gamma_g'''),$$

we infer

$$\begin{aligned}\nabla_{e_3'''}(r\mathfrak{q}) &= r^5\left[\frac{1}{2}\mathcal{D}'''\widehat{\otimes}\mathcal{D}'''\nabla_{e_3'''}\overline{P'''} - \frac{6m}{r^4}\nabla_{e_3'''}\widehat{\underline{X}}'''\right] + O(r^{-2}) + r\mathfrak{d}^{\leq 3}\Gamma_g''' \\ &\quad + r^3\mathfrak{d}^{\leq 3}(\Gamma_b'''\cdot\Gamma_g''').\end{aligned}$$

Next, we use the null structure equation for $\nabla_{e_3'''}\widehat{\underline{X}}'''$ of Proposition 2.8 and the Bianchi identity for $\nabla_{e_3'''}P'''$ of Proposition 2.9 according to which we have

$$\begin{aligned}\nabla_{e_3'''}\widehat{\underline{X}}''' &= -\underline{A}''' + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b''', \\ \nabla_{e_3'''}P''' &= -\frac{1}{2}\overline{\mathcal{D}'''}\cdot\underline{B}''' - \frac{6m}{\overline{q}q^3} + r^{-2}\Gamma_g''' + \Gamma_b'''\cdot\Gamma_g''',\end{aligned}$$

and hence

$$\begin{aligned}\nabla_{e_3'''}(r\mathfrak{q}) &= r^5\left[-\frac{1}{4}\mathcal{D}'''\widehat{\otimes}\mathcal{D}'''\mathcal{D}'''\cdot\overline{\underline{B}'''} - 3m\mathcal{D}'''\widehat{\otimes}\mathcal{D}'''\left(\frac{1}{q\overline{q}^3}\right) + \frac{6m}{r^4}\underline{A}'''\right] \\ &\quad + O(r^{-2}) + r\mathfrak{d}^{\leq 3}\Gamma_g''' + r^3\mathfrak{d}^{\leq 3}(\Gamma_b'''\cdot\Gamma_g''').\end{aligned}$$

Since $\mathcal{D}'''(q) = O(r^{-1}) + r\Gamma_g'''$, we have

$$\mathcal{D}'''\widehat{\otimes}\mathcal{D}'''\left(\frac{1}{q\overline{q}^3}\right) = O(r^{-7}) + r^{-5}\mathfrak{d}^{\leq 1}\Gamma_g'''$$

and thus

$$\begin{aligned}\nabla_{e_3'''}(r\mathfrak{q}) &= -\frac{1}{4}r^5\mathcal{D}'''\widehat{\otimes}\mathcal{D}'''\mathcal{D}'''\cdot\overline{\underline{B}'''} + O(r)\underline{A}''' + O(r^{-2}) + r\mathfrak{d}^{\leq 3}\Gamma_g''' \\ &\quad + r^3\mathfrak{d}^{\leq 3}(\Gamma_b'''\cdot\Gamma_g''').\end{aligned}$$

As in the above proof of (5.57), we now come back to the frame of Σ_*. First, since $f'' = 0$, $\underline{f}'' = 0$, and $\lambda'' = 1 + O(r^{-1})$, we have, together with the transformation formulas of Proposition 2.12,

$$\underline{B}''' = (1+O(r^{-1}))\underline{B}'', \qquad \underline{A}''' = (1+O(r^{-1}))\underline{A}'', \qquad \mathcal{D}''' = \mathcal{D}'',$$
$$e_3''' = (1+O(r^{-1}))e_3'',$$

and hence

$$\nabla_{e_3''}(r\mathfrak{q}) = -\frac{1}{4}r^5\mathcal{D}''\widehat{\otimes}\mathcal{D}''\mathcal{D}''\cdot\overline{\underline{B}''} + O(r)\underline{A}'' + O(r^{-2}) + r\mathfrak{d}^{\leq 3}\Gamma_g''' + r^3\mathfrak{d}^{\leq 3}(\Gamma_b'''\cdot\Gamma_g''').$$

Next, since $\underline{f}' = 0$ and $\lambda' = 1$, we have, together with the transformation formulas of Proposition 2.12,

$$\underline{B}'' = \underline{B}' + f'\cdot\Gamma_b' + O(r^{-3})f', \qquad \underline{A}'' = \underline{A}' + r^{-1}f'\cdot\Gamma_b',$$
$$\mathcal{D}'' = \mathcal{D}' + \frac{1}{2}f'\nabla_3', \qquad e_3'' = e_3'.$$

We deduce

$$\nabla_{e_3'}(r\mathfrak{q}) = -\frac{1}{4}r^5\mathcal{D}'\widehat{\otimes}\mathcal{D}'\mathcal{D}'\cdot\overline{\underline{B}'} + O(r)\underline{A}' + O(r^{-2}) + r\mathfrak{d}^{\leq 3}\Gamma_g''' + \mathfrak{d}^{\leq 3}f' + r^3\mathfrak{d}^{\leq 3}(\Gamma_b'''\cdot\Gamma_g''') + r^3\mathfrak{d}^{\leq 2}\big(r^{-1}f'\cdot\Gamma_b\big).$$

Finally, since $\lambda = 1$, as well as $f = O(r^{-1})$ and $\underline{f} = O(r^{-1}) + \Gamma_b$, together with the transformation formulas of Proposition 2.12, we have

$$\underline{B}' = \underline{B} + O(r^{-1})\underline{A} + O(r^{-4}) + r^{-3}\Gamma_b, \qquad \underline{A}' = \underline{A} + r^{-2}\Gamma_b,$$
$$\mathcal{D}' = \mathcal{D} + O(r^{-1})\nabla_3 + O(r^{-1})\nabla_4,$$
$$e_3' = (1+O(r^{-2}))e_3 + O(r^{-1})\nabla + O(r^{-2})\nabla_4.$$

We infer

$$\nabla_3(r\mathfrak{q}) = -\frac{1}{4}r^5\mathcal{D}\widehat{\otimes}\mathcal{D}\mathcal{D}\cdot\overline{\underline{B}} + O(r)\,\not{\mathfrak{d}}^{\leq 3}\underline{A} + O(r^{-2}) + r\mathfrak{d}^{\leq 3}\Gamma_g''' + \mathfrak{d}^{\leq 3}f' + r\,\not{\mathfrak{d}}^{\leq 3}\Gamma_g + r^3\mathfrak{d}^{\leq 3}(\Gamma_b'''\cdot\Gamma_g''') + r^3\mathfrak{d}^{\leq 2}\big(r^{-1}f'\cdot\Gamma_b\big) + r^3\,\not{\mathfrak{d}}^{\leq 3}(\Gamma_b\cdot\Gamma_g) + O(r^{-1})\mathfrak{d}^{\leq 1}\mathfrak{q}.$$

As Γ_b''' and Γ_g''', $r^{-1}f'$, $r^{-1}\mathfrak{q}$, satisfy for $k \leq k_*$ the same estimates as Γ_b, respectively Γ_g, we write, by a slight abuse of notations

$$\nabla_3(r\mathfrak{q}) = -\frac{1}{4}r^5\mathcal{D}\widehat{\otimes}\mathcal{D}\mathcal{D}\cdot\overline{\underline{B}} + O(r)\,\not{\mathfrak{d}}^{\leq 3}\underline{A} + O(r^{-2}) + r\,\not{\mathfrak{d}}^{\leq 3}\Gamma_g + r^3\,\not{\mathfrak{d}}^{\leq 3}(\Gamma_b\cdot\Gamma_g).$$

Since we have

$$\Re(\mathcal{D}\widehat{\otimes}\mathcal{D}\mathcal{D}\cdot\overline{\underline{B}}) = 4\nabla\widehat{\otimes}\nabla\text{div}\,\underline{\beta}+4\nabla\widehat{\otimes}\,{}^*\nabla\text{curl}\,\underline{\beta} = 4\,\not{d}_2^*\not{d}_1^*(-\text{div}\,\underline{\beta},\text{curl}\,\underline{\beta}),$$

we infer

$$\begin{aligned}\Re(\nabla_3(r\mathfrak{q})) &= r^5\,\not{d}_2^*\not{d}_1^*(\text{div}\,\underline{\beta},-\text{curl}\,\underline{\beta})+O(r)\,\not{\mathfrak{d}}^{\leq 3}\underline{\alpha}+O(r^{-2})+r\,\not{\mathfrak{d}}^{\leq 3}\Gamma_g\\ &\quad +r^3\,\not{\mathfrak{d}}^{\leq 3}(\Gamma_b\cdot\Gamma_g).\end{aligned}$$

which is the desired identity (5.58). This concludes the proof of Proposition 5.26.

B.6. Proof of Lemma 5.62

Let the orthonormal basis (e_1,e_2) of S_* given by (5.129), i.e.

$$e_1=\frac{1}{re^\phi}\partial_\theta,\quad e_2=\frac{1}{r\sin\theta e^\phi}\partial_\varphi.$$

In order to prove Lemma 5.62, we will need the following simple lemma.

Lemma B.1. *On S_*, for (e_1,e_2) given by (5.129), we have*

$$\begin{aligned}(\Lambda_1)_{12} &:= \mathbf{g}(\mathbf{D}_1e_2,e_1)=\frac{1}{r\sin\theta e^\phi}\partial_\varphi(\phi),\\ (\Lambda_2)_{12} &:= \mathbf{g}(\mathbf{D}_2e_2,e_1)=-\frac{1}{re^\phi}\Big(\cot\theta+\partial_\theta(\phi)\Big).\end{aligned}\tag{B.11}$$

Proof. Recall that we have on S_*

$$g_{S_*}=r^2e^{2\phi}\Big((d\theta)^2+(\sin\theta)^2(d\varphi)^2\Big),\qquad e_1=\frac{1}{re^\phi}\partial_\theta,\qquad e_2=\frac{1}{r\sin\theta e^\phi}\partial_\varphi.$$

This allows us to compute

$$\begin{aligned}(\Lambda_1)_{12} &= \mathbf{g}(\mathbf{D}_1e_2,e_1)=\frac{1}{r^3\sin\theta e^{3\phi}}\mathbf{g}(\mathbf{D}_{\partial_\theta}\partial_\varphi,\partial_\theta)=\frac{1}{r^3\sin\theta e^{3\phi}}\mathbf{g}(\mathbf{D}_{\partial_\varphi}\partial_\theta,\partial_\theta)\\ &= \frac{1}{r^3\sin\theta e^{3\phi}}\frac{1}{2}\partial_\varphi(\mathbf{g}_{\theta\theta})=\frac{1}{r^3\sin\theta e^{3\phi}}\frac{1}{2}\partial_\varphi(r^2e^{2\phi})\\ &= \frac{1}{r\sin\theta e^\phi}\partial_\varphi(\phi)\end{aligned}$$

and

$$\begin{aligned}(\Lambda_2)_{12} &= \mathbf{g}(\mathbf{D}_2e_2,e_1)=\frac{1}{r^3(\sin\theta)^2e^{3\phi}}\mathbf{g}(\mathbf{D}_{\partial_\varphi}\partial_\varphi,\partial_\theta)\\ &= -\frac{1}{r^3(\sin\theta)^2e^{3\phi}}\mathbf{g}(\partial_\varphi,\mathbf{D}_{\partial_\varphi}\partial_\theta)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{r^3(\sin\theta)^2e^{3\phi}}\mathbf{g}(\partial_\varphi, \mathbf{D}_{\partial_\theta}\partial_\varphi) = -\frac{1}{r^3(\sin\theta)^2e^{3\phi}}\frac{1}{2}\partial_\theta(\mathbf{g}_{\varphi\varphi})\\
&= -\frac{1}{r^3(\sin\theta)^2e^{3\phi}}\frac{1}{2}\partial_\theta(r^2(\sin\theta)^2e^{2\phi})\\
&= -\frac{1}{r(\sin\theta)^2e^{\phi}}\Big(\sin\theta\cos\theta + (\sin\theta)^2\partial_\theta(\phi)\Big)\\
&= -\frac{1}{re^{\phi}}\Big(\cot\theta + \partial_\theta(\phi)\Big)
\end{aligned}
$$

as stated. □

We are now ready to prove Lemma 5.62. We start with the identities for f_0. We have

$$
\begin{aligned}
\mathrm{div}\,(f_0) &= \nabla_1(f_0)_1 + \nabla_2(f_0)_2\\
&= e_1((f_0)_1) - \mathbf{g}(\mathbf{D}_1e_1, e_2)(f_0)_2 + e_2((f_0)_2) - \mathbf{g}(\mathbf{D}_2e_2, e_1)(f_0)_1\\
&= -\mathbf{g}(\mathbf{D}_1e_1, e_2)\sin\theta + e_2(\sin\theta) = (\Lambda_1)_{12}\sin\theta + \frac{1}{r\sin\theta e^{\phi}}\partial_\varphi(\sin\theta)\\
&= \frac{1}{re^{\phi}}\partial_\varphi(\phi) = f_0\cdot\nabla\phi
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{curl}\,(f_0) &= \nabla_1(f_0)_2 - \nabla_2(f_0)_1\\
&= e_1((f_0)_2) - \mathbf{g}(\mathbf{D}_1e_2, e_1)(f_0)_1 - e_2((f_0)_1) + \mathbf{g}(\mathbf{D}_2e_1, e_2)(f_0)_2\\
&= \frac{1}{re^{\phi}}\partial_\theta(\sin\theta) - (\Lambda_2)_{12}\sin\theta\\
&= \frac{1}{re^{\phi}}\cos\theta + \frac{1}{re^{\phi}}\Big(\cos\theta + \sin\theta\partial_\theta(\phi)\Big)\\
&= \frac{2}{re^{\phi}}\cos\theta + \frac{1}{re^{\phi}}\sin\theta\partial_\theta(\phi) = \frac{2}{re^{\phi}}\cos\theta - f_0\wedge\nabla\phi
\end{aligned}
$$

as stated. Also, we have

$$
\begin{aligned}
(\nabla\widehat{\otimes}f_0)_{11} &= \nabla_1(f_0)_1 - \nabla_2(f_0)_2\\
&= e_1((f_0)_1) - \mathbf{g}(\mathbf{D}_1e_1, e_2)(f_0)_2 - e_2((f_0)_2) + \mathbf{g}(\mathbf{D}_2e_2, e_1)(f_0)_1\\
&= -\mathbf{g}(\mathbf{D}_1e_1, e_2)\sin\theta - e_2(\sin\theta)\\
&= (\Lambda_1)_{12}\sin\theta - \frac{1}{r\sin\theta e^{\phi}}\partial_\varphi(\sin\theta) = \frac{1}{re^{\phi}}\partial_\varphi(\phi)\\
&= f_0\cdot\nabla\phi,
\end{aligned}
$$

and

$$
\begin{aligned}
(\nabla\widehat{\otimes} f_0)_{12} &= \nabla_1(f_0)_2+\nabla_2(f_0)_1\\
&= e_1((f_0)_2)-\mathbf{g}(\mathbf{D}_1e_2,e_1)(f_0)_1+e_2((f_0)_1)-\mathbf{g}(\mathbf{D}_2e_1,e_2)(f_0)_2\\
&= \frac{1}{re^{\phi}}\partial_\theta(\sin\theta)+(\Lambda_2)_{12}\sin\theta\\
&= \frac{1}{re^{\phi}}\cos\theta-\frac{1}{re^{\phi}}\Big(\cos\theta+\sin\theta\partial_\theta(\phi)\Big)\\
&= -\frac{1}{re^{\phi}}\sin\theta\partial_\theta(\phi)=f_0\wedge\nabla\phi
\end{aligned}
$$

so that

$$
\nabla\widehat{\otimes} f_0 = \begin{pmatrix} f_0\cdot\nabla\phi & f_0\wedge\nabla\phi\\ f_0\wedge\nabla\phi & -f_0\cdot\nabla\phi\end{pmatrix}
$$

as stated.

Next, we consider the identities for f_+. We have

$$
\begin{aligned}
\mathrm{div}\,(f_+) &= \nabla_1(f_+)_1+\nabla_2(f_+)_2\\
&= e_1((f_+)_1)-\mathbf{g}(\mathbf{D}_1e_1,e_2)(f_+)_2+e_2((f_+)_2)-\mathbf{g}(\mathbf{D}_2e_2,e_1)(f_+)_1\\
&= \frac{1}{re^{\phi}}\Bigg(-\sin\theta\cos\varphi+\frac{1}{\sin\theta}\partial_\varphi(\phi)(-\sin\varphi)-\frac{\cos\varphi}{\sin\theta}\\
&\qquad +\Big(\cot\theta+\partial_\theta(\phi)\Big)\cos\theta\cos\varphi\Bigg)\\
&= -\frac{2}{re^{\phi}}J^{(+)}+f_+\cdot\nabla\phi
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{curl}\,(f_+) &= \nabla_1(f_+)_2-\nabla_2(f_+)_1\\
&= e_1((f_+)_2)-\mathbf{g}(\mathbf{D}_1e_2,e_1)(f_+)_1-e_2((f_+)_1)+\mathbf{g}(\mathbf{D}_2e_1,e_2)(f_+)_2\\
&= \frac{1}{re^{\phi}}\Bigg(-\frac{1}{\sin\theta}\partial_\varphi(\phi)\cos\theta\cos\varphi+\cot\theta\sin\varphi\\
&\qquad +\Big(\cot\theta+\partial_\theta(\phi)\Big)(-\sin\varphi)\Bigg)\\
&= f_+\wedge\nabla\phi,
\end{aligned}
$$

as stated. Also, we have

$$\begin{aligned}
(\nabla\widehat{\otimes} f_+)_{11} &= \nabla_1(f_+)_1 - \nabla_2(f_+)_2\\
&= e_1((f_+)_1) - \mathbf{g}(\mathbf{D}_1 e_1, e_2)(f_+)_2 - e_2((f_+)_2) + \mathbf{g}(\mathbf{D}_2 e_2, e_1)(f_+)_1\\
&= \frac{1}{re^\phi}\Bigg(-\sin\theta\cos\varphi + \frac{1}{\sin\theta}\partial_\varphi(\phi)(-\sin\varphi) + \frac{\cos\varphi}{\sin\theta}\\
&\qquad -\Big(\cot\theta + \partial_\theta(\phi)\Big)\cos\theta\cos\varphi\Bigg)\\
&= (f_+)_2\nabla_2\phi - (f_+)_1\nabla_1\phi,
\end{aligned}$$

and

$$\begin{aligned}
(\nabla\widehat{\otimes} f_+)_{12} &= \nabla_1(f_+)_2 + \nabla_2(f_+)_1\\
&= e_1((f_+)_2) - \mathbf{g}(\mathbf{D}_1 e_2, e_1)(f_+)_1 + e_2((f_+)_1) - \mathbf{g}(\mathbf{D}_2 e_1, e_2)(f_+)_2\\
&= \frac{1}{re^\phi}\Bigg(-\frac{1}{\sin\theta}\partial_\varphi(\phi)\cos\theta\cos\varphi - \cot\theta\sin\varphi\\
&\qquad -\Big(\cot\theta + \partial_\theta(\phi)\Big)(-\sin\varphi)\Bigg)\\
&= -(f_+)_1\nabla_2\phi - (f_+)_2\nabla_1\phi
\end{aligned}$$

so that

$$\nabla\widehat{\otimes} f_+ = \begin{pmatrix} (f_+)_2\nabla_2\phi - (f_+)_1\nabla_1\phi & -(f_+)_1\nabla_2\phi - (f_+)_2\nabla_1\phi\\ -(f_+)_1\nabla_2\phi - (f_+)_2\nabla_1\phi & -(f_+)_2\nabla_2\phi + (f_+)_1\nabla_1\phi \end{pmatrix}$$

as stated.

Finally, we consider the identities for f_-. We have

$$\begin{aligned}
\mathrm{div}\,(f_-) &= \nabla_1(f_-)_1 + \nabla_2(f_-)_2\\
&= e_1((f_-)_1) - \mathbf{g}(\mathbf{D}_1 e_1, e_2)(f_-)_2 + e_2((f_-)_2) - \mathbf{g}(\mathbf{D}_2 e_2, e_1)(f_+)_1\\
&= \frac{1}{re^\phi}\Bigg(-\sin\theta\sin\varphi + \frac{1}{\sin\theta}\partial_\varphi(\phi)\cos\varphi - \frac{\sin\varphi}{\sin\theta}\\
&\qquad +\Big(\cot\theta + \partial_\theta(\phi)\Big)\cos\theta\sin\varphi\Bigg)\\
&= -\frac{2}{re^\phi}J^{(-)} + f_-\cdot\nabla\phi
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{curl}\,(f_-) &= \nabla_1(f_-)_2 - \nabla_2(f_-)_1 \\
&= e_1((f_-)_2) - \mathbf{g}(\mathbf{D}_1 e_2, e_1)(f_-)_1 - e_2((f_-)_1) + \mathbf{g}(\mathbf{D}_2 e_1, e_2)(f_-)_2 \\
&= \frac{1}{re^{\phi}}\Bigg(- \frac{1}{\sin\theta}\partial_\varphi(\phi)\cos\theta\sin\varphi - \cot\theta\cos\varphi \\
&\qquad\qquad + \Big(\cot\theta + \partial_\theta(\phi)\Big)\cos\varphi\Bigg) \\
&= -f_- \wedge \nabla\phi
\end{aligned}$$

as stated. Also, we have

$$\begin{aligned}
(\nabla\widehat{\otimes} f_-)_{11} &= \nabla_1(f_-)_1 - \nabla_2(f_-)_2 \\
&= e_1((f_-)_1) - \mathbf{g}(\mathbf{D}_1 e_1, e_2)(f_-)_2 - e_2((f_-)_2) + \mathbf{g}(\mathbf{D}_2 e_2, e_1)(f_-)_1 \\
&= \frac{1}{re^{\phi}}\Bigg(- \sin\theta\sin\varphi + \frac{1}{\sin\theta}\partial_\varphi(\phi)\cos\varphi + \frac{\sin\varphi}{\sin\theta} \\
&\qquad\qquad - \Big(\cot\theta + \partial_\theta(\phi)\Big)\cos\theta\sin\varphi\Bigg) \\
&= (f_-)_2\nabla_2\phi - (f_-)_1\nabla_1\phi,
\end{aligned}$$

and

$$\begin{aligned}
(\nabla\widehat{\otimes} f_-)_{12} &= \nabla_1(f_-)_2 + \nabla_2(f_-)_1 \\
&= e_1((f_-)_2) - \mathbf{g}(\mathbf{D}_1 e_2, e_1)(f_-)_1 + e_2((f_-)_1) - \mathbf{g}(\mathbf{D}_2 e_1, e_2)(f_-)_2 \\
&= \frac{1}{re^{\phi}}\Bigg(- \frac{1}{\sin\theta}\partial_\varphi(\phi)\cos\theta\sin\varphi + \cot\theta\cos\varphi \\
&\qquad\qquad - \Big(\cot\theta + \partial_\theta(\phi)\Big)\cos\varphi\Bigg) \\
&= -(f_-)_1\nabla_2\phi - (f_-)_2\nabla_1\phi
\end{aligned}$$

so that

$$\nabla\widehat{\otimes} f_- = \begin{pmatrix} (f_-)_2\nabla_2\phi - (f_-)_1\nabla_1\phi & -(f_-)_1\nabla_2\phi - (f_-)_2\nabla_1\phi \\ -(f_-)_1\nabla_2\phi - (f_-)_2\nabla_1\phi & -(f_-)_2\nabla_2\phi + (f_-)_1\nabla_1\phi \end{pmatrix}$$

as stated. This concludes the proof of Lemma 5.62.

B.7. Proof of Lemma 5.65

Let f a 1-form and $F = f + i\,{}^*f$. Then, we have

$$\overline{\mathcal{D}} \cdot F = 2\text{div}\,(f) + 2i\text{curl}\,(f).$$

Also, since $|q|^2 = r^2 + a^2(\cos\theta)^2$, $J^{(0)} = \cos\theta$, and $\nabla(r) = 0$, we have

$$\begin{aligned}
\text{div}\left(\frac{1}{|q|}f\right) &= \frac{1}{|q|}\text{div}\,(f) - \frac{\nabla(|q|)}{|q|^2}\cdot f \\
&= \frac{1}{|q|}\text{div}\,(f) - \frac{\nabla(r^2 + a^2(\cos\theta)^2)}{2|q|^3}\cdot f \\
&= \frac{1}{|q|}\text{div}\,(f) - \frac{a^2\cos\theta\nabla(J^{(0)})}{|q|^3}\cdot f \\
&= \frac{1}{|q|}\text{div}\,(f) + \frac{a^2\cos\theta}{r|q|^3}\,{}^*f_0\cdot f - \frac{a^2\cos\theta}{|q|^3}\widetilde{\nabla J^{(0)}}\cdot f
\end{aligned}$$

and

$$\begin{aligned}
\text{curl}\left(\frac{1}{|q|}f\right) &= \frac{1}{|q|}\text{curl}\,(f) + \frac{{}^*\nabla(|q|)}{|q|^2}\cdot f \\
&= \frac{1}{|q|}\text{curl}\,(f) - \frac{{}^*\nabla(r^2 + a^2(\cos\theta)^2)}{2|q|^3}\cdot f \\
&= \frac{1}{|q|}\text{curl}\,(f) - \frac{a^2\cos\theta\,{}^*\nabla(J^{(0)})}{|q|^3}\cdot f \\
&= \frac{1}{|q|}\text{curl}\,(f) - \frac{a^2\cos\theta}{r|q|^3}f_0\cdot f - \frac{a^2\cos\theta}{|q|^3}\,{}^*\widetilde{\nabla J^{(0)}}\cdot f.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\overline{\mathcal{D}}\cdot\left(\frac{1}{|q|}F\right) &= \frac{2}{|q|}\text{div}\,(f) + \frac{2a^2\cos\theta}{r|q|^3}\,{}^*f_0\cdot f - \frac{2a^2\cos\theta}{|q|^3}\widetilde{\nabla J^{(0)}}\cdot f \\
&\quad + i\left(\frac{2}{|q|}\text{curl}\,(f) - \frac{2a^2\cos\theta}{r|q|^3}f_0\cdot f - \frac{2a^2\cos\theta}{|q|^3}\,{}^*\widetilde{\nabla J^{(0)}}\cdot f\right).
\end{aligned}$$

Since we have by definition on Σ_*

$$\mathfrak{J} = \frac{1}{|q|}\left(f_0 + i\,{}^*f_0\right), \qquad \mathfrak{J}_\pm = \frac{1}{|q|}\left(f_\pm + i\,{}^*f_\pm\right),$$

we infer

$$\overline{\mathcal{D}}\cdot\mathfrak{J} = \frac{2}{|q|}\mathrm{div}\,(f_0) + \frac{2a^2\cos\theta}{r|q|^3}\,{}^*f_0\cdot f_0 - \frac{2a^2\cos\theta}{|q|^3}f_0\cdot\widetilde{\nabla J^{(0)}}$$
$$+i\left(\frac{2}{|q|}\mathrm{curl}\,(f_0) - \frac{2a^2\cos\theta}{r|q|^3}f_0\cdot f_0 - \frac{2a^2\cos\theta}{|q|^3}f_0\cdot\,{}^*\widetilde{\nabla J^{(0)}}\right)$$

and

$$\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm = \frac{2}{|q|}\mathrm{div}\,(f_\pm) + \frac{2a^2\cos\theta}{r|q|^3}\,{}^*f_0\cdot f_\pm - \frac{2a^2\cos\theta}{|q|^3}f_\pm\cdot\widetilde{\nabla J^{(0)}}$$
$$+i\left(\frac{2}{|q|}\mathrm{curl}\,(f_\pm) - \frac{2a^2\cos\theta}{r|q|^3}f_0\cdot f_\pm - \frac{2a^2\cos\theta}{|q|^3}f_\pm\cdot\,{}^*\widetilde{\nabla J^{(0)}}\right).$$

Since we have on Σ_*

$$f_0\cdot f_0 = (\sin\theta)^2,\quad f_+\cdot f_0 = -J^{(-)},\quad f_-\cdot f_0 = J^{(+)},$$
$$f_0\cdot\,{}^*f_0 = 0,\quad f_+\cdot\,{}^*f_0 = \sin\theta\cos\theta\cos\varphi,\quad f_-\cdot\,{}^*f_0 = \sin\theta\cos\theta\sin\varphi,$$

we deduce

$$\overline{\mathcal{D}}\cdot\mathfrak{J} = \frac{2}{|q|}\mathrm{div}\,(f_0) - \frac{2a^2\cos\theta}{|q|^3}f_0\cdot\widetilde{\nabla J^{(0)}}$$
$$+i\left(\frac{2}{|q|}\mathrm{curl}\,(f_0) - \frac{2a^2\cos\theta(\sin\theta)^2}{r|q|^3} - \frac{2a^2\cos\theta}{|q|^3}f_0\cdot\,{}^*\widetilde{\nabla J^{(0)}}\right),$$

$$\overline{\mathcal{D}}\cdot\mathfrak{J}_+ = \frac{2}{|q|}\mathrm{div}\,(f_+) + \frac{2a^2\cos\theta}{r|q|^3}\sin\theta\cos\theta\cos\varphi - \frac{2a^2\cos\theta}{|q|^3}f_+\cdot\widetilde{\nabla J^{(0)}}$$
$$+i\left(\frac{2}{|q|}\mathrm{curl}\,(f_+) + \frac{2a^2\cos\theta\sin\theta\sin\varphi}{r|q|^3} - \frac{2a^2\cos\theta}{|q|^3}f_+\cdot\,{}^*\widetilde{\nabla J^{(0)}}\right),$$

and

$$\overline{\mathcal{D}}\cdot\mathfrak{J}_- = \frac{2}{|q|}\mathrm{div}\,(f_-) + \frac{2a^2\cos\theta}{r|q|^3}\sin\theta\cos\theta\sin\varphi - \frac{2a^2\cos\theta}{|q|^3}f_-\cdot\widetilde{\nabla J^{(0)}}$$
$$+i\left(\frac{2}{|q|}\mathrm{curl}\,(f_-) - \frac{2a^2\cos\theta\sin\theta\cos\varphi}{r|q|^3} - \frac{2a^2\cos\theta}{|q|^3}f_-\cdot\,{}^*\widetilde{\nabla J^{(0)}}\right).$$

In view of the definition of $\widetilde{\mathrm{curl}\,(f_0)}$, $\widetilde{\mathrm{div}\,(f_\pm)}$, $\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}$, and $\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm}$ we obtain

$$\begin{aligned}\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}} &= O(r^{-4})+\frac{2}{|q|}\mathrm{div}\,(f_0)-\frac{2a^2\cos\theta}{|q|^3}f_0\cdot\widetilde{\nabla J^{(0)}}\\ &\quad +i\left(\frac{2}{|q|}\widetilde{\mathrm{curl}\,(f_0)}-\frac{2a^2\cos\theta}{|q|^3}f_0\cdot\ {}^*\widetilde{\nabla J^{(0)}}\right),\end{aligned}$$

$$\begin{aligned}\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}_\pm} &= O(r^{-4})+\frac{2}{|q|}\widetilde{\mathrm{div}\,(f_\pm)}-\frac{2a^2\cos\theta}{|q|^3}f_\pm\cdot\widetilde{\nabla J^{(0)}}\\ &\quad +i\left(\frac{2}{|q|}\mathrm{curl}\,(f_\pm)-\frac{2a^2\cos\theta}{|q|^3}f_\pm\cdot\ {}^*\widetilde{\nabla J^{(0)}}\right),\end{aligned}$$

where $O(r^a)$ denotes, for $a\in\mathbb{R}$, a function of $(r,\cos\theta)$ bounded by r^a as $r\to+\infty$. This concludes the proof of Lemma 5.65.

Appendix C. PROOF OF RESULTS IN CHAPTER 6

C.1. Proof of Lemma 6.15

The proof relies on the null structure equations and Bianchi identities of Proposition 6.9, the definition of the linearized quantities and of Γ_g and Γ_b in Section 6.1.2, the notation $O(r^{-p})$ made in Definition 6.3, the fact that a and m are constants, and the following identities

$$e_4(r) = 1, \quad e_4(\theta) = 0, \quad \nabla_4\mathfrak{J} = -\frac{1}{q}\mathfrak{J}, \quad e_4(q) = 1, \quad e_4(\overline{q}) = 1, \quad \nabla(r) = 0,$$

where we used in particular the fact that $q = r + ai\cos\theta$ and $\overline{q} = r - ai\sin\theta$.

We start with the equation for $\widetilde{\mathrm{tr}X}$. Recall

$$\nabla_4\mathrm{tr}X + \frac{1}{2}(\mathrm{tr}X)^2 = -\frac{1}{2}\widehat{X}\cdot\overline{\widehat{X}}.$$

Since $e_4(q) = 1$, we infer

$$\begin{aligned}\nabla_4(\widetilde{\mathrm{tr}X}) &= -\frac{1}{2}(\mathrm{tr}X)^2 - \frac{1}{2}\widehat{X}\cdot\overline{\widehat{X}} - \partial_r\left(\frac{2}{q}\right)\\ &= -\frac{1}{2}(\mathrm{tr}X)^2 + \frac{1}{2}\left(\frac{2}{q}\right)^2 - \frac{1}{2}\widehat{X}\cdot\overline{\widehat{X}}\end{aligned}$$

and hence

$$\nabla_4(\widetilde{\mathrm{tr}X}) + \frac{2}{q}\widetilde{\mathrm{tr}X} = \Gamma_g\cdot\Gamma_g.$$

Next, recall

$$\nabla_4\widehat{X} + \Re(\mathrm{tr}X)\widehat{X} = -A$$

and hence

$$\nabla_4\widehat{X} + \frac{2r}{|q|^2}\widehat{X} = -A + \Gamma_g\cdot\Gamma_g.$$

Next, recall

$$\nabla_4 Z + \mathrm{tr}XZ = -\widehat{X}\cdot\overline{Z} - B.$$

We infer

$$\nabla_4\check{Z} = -\mathrm{tr}XZ - \widehat{X}\cdot\overline{Z} - B - \nabla_4\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right).$$

Now, since $e_4(q) = 1$ and $\nabla_4\mathfrak{J} = -q^{-1}\mathfrak{J}$, we have

$$\begin{aligned}\nabla_4\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right) &= \left(\frac{a}{|q|^2} - \frac{2a\overline{q}e_4(|q|)}{|q|^3}\right)\mathfrak{J} - \frac{a\overline{q}}{q|q|^2}\mathfrak{J}\\ &= \frac{a}{|q|^4}\Big(|q|^2 - \overline{q}(q+\overline{q}) - \overline{q}^2\Big)\mathfrak{J} = -\frac{2a\overline{q}^2}{|q|^4}\mathfrak{J}\end{aligned}$$

and thus

$$\nabla_4\check{Z} + \frac{2}{q}\check{Z} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J}\widecheck{\mathrm{tr}X} - \frac{aq}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X} - B + \Gamma_g\cdot\Gamma_g.$$

We infer

$$\nabla_4\check{Z} + \frac{2}{q}\check{Z} = -\frac{aq}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X} - B + O(r^{-2})\widecheck{\mathrm{tr}X} + \Gamma_g\cdot\Gamma_g.$$

Next, recall

$$\nabla_4 H + \frac{1}{2}\overline{\mathrm{tr}X}(H+Z) = -\frac{1}{2}\widehat{X}\cdot(\overline{H}+\overline{Z}) - B,$$

we infer

$$\nabla_4\check{H} = -\frac{1}{2}\overline{\mathrm{tr}X}(H+Z) - \frac{1}{2}\widehat{X}\cdot(\overline{H}+\overline{Z}) - B - \nabla_4\left(\frac{aq}{|q|^2}\mathfrak{J}\right).$$

Now, since $e_4(q) = 1$ and $\nabla_4\mathfrak{J} = -q^{-1}\mathfrak{J}$, we have

$$\nabla_4\left(\frac{aq}{|q|^2}\mathfrak{J}\right) = \left(\frac{a}{|q|^2} - \frac{2aqe_4(|q|)}{|q|^3}\right)\mathfrak{J} - \frac{a}{|q|^2}\mathfrak{J} = -\frac{aq(q+\overline{q})}{|q|^4}\mathfrak{J}$$

and thus

$$\nabla_4\check{H} + \frac{1}{\overline{q}}\check{H} = -\frac{1}{\overline{q}}\check{Z} - \frac{ar}{|q|^2}\overline{\widecheck{\mathrm{tr}X}}\mathfrak{J} - \frac{ar}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X} - B + \Gamma_b\cdot\Gamma_g.$$

We infer

$$\nabla_4\check{H} + \frac{1}{\overline{q}}\check{H} = -\frac{1}{\overline{q}}\check{Z} - \frac{ar}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X} - B + O(r^{-2})\widecheck{\mathrm{tr}X} + \Gamma_b\cdot\Gamma_g.$$

Next, recall

$$\nabla_4 \mathrm{tr}\underline{X} + \frac{1}{2}\mathrm{tr}X\mathrm{tr}\underline{X} = -\mathcal{D}\cdot\overline{\underline{Z}} + Z\cdot\overline{\underline{Z}} + 2\overline{\underline{P}} - \frac{1}{2}\widehat{X}\cdot\overline{\widehat{\underline{X}}}.$$

We infer

$$\begin{aligned}
\nabla_4\widecheck{\mathrm{tr}\underline{X}} &= \nabla_4\left(\frac{2q\Delta}{|q|^4}\right) - \frac{1}{2}\mathrm{tr}X\mathrm{tr}\underline{X} - \mathcal{D}\cdot\overline{\underline{Z}} + Z\cdot\overline{\underline{Z}} + 2\overline{\underline{P}}\\
&\quad -\frac{1}{2}\widehat{X}\cdot\overline{\widehat{\underline{X}}}\\
&= \partial_r\left(\frac{2q\Delta}{|q|^4}\right) - \frac{1}{2}\left(\frac{2}{q} + \widecheck{\mathrm{tr}X}\right)\left(-\frac{2q\Delta}{|q|^4} + \widecheck{\mathrm{tr}\underline{X}}\right)\\
&\quad -\mathcal{D}\cdot\overline{\underline{Z}} + Z\cdot\overline{\underline{Z}} + 2\left(-\frac{2m}{(\overline{q})^3} + \overline{\widecheck{\underline{P}}}\right) - \frac{1}{2}\widehat{X}\cdot\overline{\widehat{\underline{X}}},
\end{aligned}$$

and hence

$$\begin{aligned}
\nabla_4\widecheck{\mathrm{tr}\underline{X}} + \frac{1}{q}\widecheck{\mathrm{tr}\underline{X}} &= -\mathcal{D}\cdot\overline{\underline{Z}} + Z\cdot\overline{\underline{Z}} + 2\overline{\widecheck{\underline{P}}} + \partial_r\left(\frac{2q\Delta}{|q|^4}\right) + \frac{2\Delta}{|q|^4} - \frac{4m}{(\overline{q})^3}\\
&\quad +O(r^{-1})\widecheck{\mathrm{tr}X} + \Gamma_b\cdot\Gamma_g.
\end{aligned}$$

Next, we compute

$$\begin{aligned}
& -\mathcal{D}\cdot\overline{\underline{Z}} + Z\cdot\overline{\underline{Z}}\\
={}& -\mathcal{D}\cdot\overline{\frac{a\overline{q}}{|q|^2}\mathfrak{J} + \widecheck{\underline{Z}}} + \left(\frac{a\overline{q}}{|q|^2}\mathfrak{J} + \widecheck{Z}\right)\cdot\left(\overline{\frac{a\overline{q}}{|q|^2}\mathfrak{J} + \widecheck{\underline{Z}}}\right)\\
={}& -\frac{aq}{|q|^2}\mathcal{D}\cdot\overline{\mathfrak{J}} - \mathcal{D}\left(\frac{aq}{|q|^2}\right)\cdot\overline{\mathfrak{J}} - \mathcal{D}\cdot\overline{\widecheck{\underline{Z}}} + \frac{a^2}{|q|^2}\mathfrak{J}\cdot\overline{\mathfrak{J}} + \frac{a\overline{q}}{|q|^2}\mathfrak{J}\cdot\overline{\widecheck{\underline{Z}}} + \frac{aq}{|q|^2}\overline{\mathfrak{J}}\cdot\widecheck{Z}\\
& +\Gamma_g\cdot\Gamma_g.
\end{aligned}$$

Since $\mathfrak{J}\cdot\overline{\mathfrak{J}} = \frac{2(\sin\theta)^2}{|q|^2}$, we infer

$$\begin{aligned}
& -\mathcal{D}\cdot\overline{\underline{Z}} + Z\cdot\overline{\underline{Z}}\\
={}& -\frac{aq}{|q|^2}\left(-\frac{4i(r^2+a^2)\cos\theta}{|q|^4} + \widecheck{\mathcal{D}\cdot\overline{\mathfrak{J}}}\right) + \frac{aq^2}{|q|^4}\mathcal{D}(\overline{q})\cdot\overline{\mathfrak{J}} - \mathcal{D}\cdot\overline{\widecheck{\underline{Z}}}\\
& +\frac{2a^2(\sin\theta)^2}{|q|^4} + \frac{a\overline{q}}{|q|^2}\mathfrak{J}\cdot\overline{\widecheck{\underline{Z}}} + \frac{aq}{|q|^2}\overline{\mathfrak{J}}\cdot\widecheck{Z} + \Gamma_g\cdot\Gamma_g\\
={}& \frac{4aiq(r^2+a^2)\cos\theta}{|q|^6} + \frac{2a^2(\sin\theta)^2}{|q|^4} - \frac{aq}{|q|^2}\widecheck{\mathcal{D}\cdot\overline{\mathfrak{J}}}
\end{aligned}$$

$$
\begin{aligned}
& -\frac{ia^2q^2}{|q|^4}\Big(i\mathfrak{J}+\widetilde{\mathcal{D}(\cos\theta)}\Big)\cdot\overline{\mathfrak{J}}-\mathcal{D}\cdot\overline{\check{Z}}+\frac{a\overline{q}}{|q|^2}\mathfrak{J}\cdot\overline{\check{Z}}+\frac{aq}{|q|^2}\overline{\mathfrak{J}}\cdot\check{Z}+\Gamma_g\cdot\Gamma_g\\
= \quad & \frac{4aiq(r^2+a^2)\cos\theta}{|q|^6}+\frac{2a^2(\sin\theta)^2}{|q|^4}+\frac{a^2q^2}{|q|^4}\frac{2(\sin\theta)^2}{|q|^2}\\
& -\frac{aq}{|q|^2}\widetilde{\mathcal{D}\cdot\overline{\mathfrak{J}}}-\frac{ia^2q^2}{|q|^4}\widetilde{\mathcal{D}(\cos\theta)}\cdot\overline{\mathfrak{J}}-\mathcal{D}\cdot\overline{\check{Z}}+\frac{a\overline{q}}{|q|^2}\mathfrak{J}\cdot\overline{\check{Z}}+\frac{aq}{|q|^2}\overline{\mathfrak{J}}\cdot\check{Z}\\
& +\Gamma_g\cdot\Gamma_g.
\end{aligned}
$$

This yields

$$
\begin{aligned}
\nabla_4\widetilde{\mathrm{tr}\underline{X}}+\frac{1}{q}\widetilde{\mathrm{tr}\underline{X}} \quad = \quad & -\frac{aq}{|q|^2}\widetilde{\mathcal{D}\cdot\overline{\mathfrak{J}}}-\frac{ia^2q^2}{|q|^4}\widetilde{\mathcal{D}(\cos\theta)}\cdot\overline{\mathfrak{J}}-\mathcal{D}\cdot\overline{\check{Z}}+\frac{a\overline{q}}{|q|^2}\mathfrak{J}\cdot\overline{\check{Z}}\\
& +\frac{aq}{|q|^2}\overline{\mathfrak{J}}\cdot\check{Z}+2\overline{\check{P}}+O(r^{-1})\widetilde{\mathrm{tr}X}+\Gamma_b\cdot\Gamma_g
\end{aligned}
$$

and hence

$$
\begin{aligned}
\nabla_4\widetilde{\mathrm{tr}\underline{X}}+\frac{1}{q}\widetilde{\mathrm{tr}\underline{X}} \quad = \quad & -\mathcal{D}\cdot\overline{\check{Z}}+2\overline{\check{P}}+O(r^{-2})\check{Z}+O(r^{-1})\widetilde{\mathrm{tr}X}\\
& +O(r^{-1})\widetilde{\mathcal{D}\cdot\overline{\mathfrak{J}}}+O(r^{-3})\widetilde{\mathcal{D}(\cos\theta)}+\Gamma_b\cdot\Gamma_g.
\end{aligned}
$$

Next, recall that

$$
\begin{aligned}
\nabla_3Z+\frac{1}{2}\mathrm{tr}\underline{X}(Z+H)-2\underline{\omega}(Z-H) \quad = \quad & -2\mathcal{D}\underline{\omega}-\frac{1}{2}\widehat{\underline{X}}\cdot(\overline{Z}+\overline{H})+\frac{1}{2}\mathrm{tr}X\underline{\Xi}\\
& -\underline{B}+\frac{1}{2}\underline{\Xi}\cdot\widehat{X},\\
\nabla_3Z+\nabla_4\underline{\Xi} \quad = \quad & -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}(Z+H)-\frac{1}{2}\widehat{\underline{X}}\cdot(\overline{Z}+\overline{H})\\
& -\underline{B}.
\end{aligned}
$$

We infer

$$
\nabla_4\underline{\Xi}+\frac{1}{2}\mathrm{tr}X\underline{\Xi} \quad = \quad 2\mathcal{D}\underline{\omega}+i\Im(\mathrm{tr}\underline{X})(Z+H)-2\underline{\omega}(Z-H)+\Gamma_b\cdot\Gamma_g.
$$

This yields

$$
\begin{aligned}
& \nabla_4\underline{\Xi}+\frac{1}{q}\underline{\Xi}\\
= \quad & \mathcal{D}\left(\partial_r\left(\frac{\Delta}{|q|^2}\right)\right)+i\Im\left(-\frac{2q\Delta}{|q|^4}\right)\frac{a(q+\overline{q})}{|q|^2}\mathfrak{J}-\partial_r\left(\frac{\Delta}{|q|^2}\right)\frac{a(\overline{q}-q)}{|q|^2}\mathfrak{J}
\end{aligned}
$$

$$+O(r^{-1})\,\mathfrak{d}^{\leq 1}(\underline{\check{\omega}}) + O(r^{-2})\check{Z} + O(r^{-2})\check{H} + O(r^{-2})\widetilde{\mathrm{tr}\underline{X}} + \Gamma_b\cdot\Big(\underline{\check{\omega}},\Gamma_g\Big).$$

Now, using

$$\mathcal{D}(r)=0, \qquad \Im(q)=a\cos\theta, \qquad \mathcal{D}(|q|^2)=2a^2\cos\theta\mathcal{D}(\cos\theta),$$
$$\mathcal{D}(\cos\theta)=-i\mathfrak{J}+\widetilde{\mathcal{D}(\cos\theta)},$$

we have by direct check

$$\begin{aligned}
&\mathcal{D}\left(\partial_r\left(\frac{\Delta}{|q|^2}\right)\right)+i\Im\left(-\frac{2q\Delta}{|q|^4}\right)\frac{a(q+\overline{q})}{|q|^2}\mathfrak{J}-\partial_r\left(\frac{\Delta}{|q|^2}\right)\frac{a(\overline{q}-q)}{|q|^2}\mathfrak{J}\\
=\;& O(r^{-3})\widetilde{\mathcal{D}(\cos\theta)}
\end{aligned}$$

and hence

$$\begin{aligned}
\nabla_4\underline{\Xi}+\frac{1}{q}\underline{\Xi} \;=\;& O(r^{-1})\,\mathfrak{d}^{\leq 1}(\underline{\check{\omega}}) + O(r^{-2})\check{Z} + O(r^{-2})\check{H} + O(r^{-2})\widetilde{\mathrm{tr}\underline{X}}\\
&+O(r^{-3})\widetilde{\mathcal{D}(\cos\theta)}+\Gamma_b\cdot\Big(\underline{\check{\omega}},\Gamma_g\Big).
\end{aligned}$$

Next, recall that

$$\nabla_4\widehat{\underline{X}}+\frac{1}{2}\mathrm{tr}X\widehat{\underline{X}} \;=\; -\frac{1}{2}\mathcal{D}\widehat{\otimes}Z+\frac{1}{2}Z\widehat{\otimes}Z-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\widehat{X}$$

and hence

$$\nabla_4\widehat{\underline{X}}+\frac{1}{q}\widehat{\underline{X}} \;=\; -\frac{1}{2}\mathcal{D}\widehat{\otimes}Z+\frac{1}{2}Z\widehat{\otimes}Z+O(r^{-1})\widehat{X}+\Gamma_b\cdot\Gamma_g.$$

Now, we have

$$\begin{aligned}
&-\mathcal{D}\widehat{\otimes}Z+Z\widehat{\otimes}Z\\
=\;& -\mathcal{D}\widehat{\otimes}\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}+\check{Z}\right)+\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}+\check{Z}\right)\widehat{\otimes}\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}+\check{Z}\right)\\
=\;& -\frac{a\overline{q}}{|q|^2}\mathcal{D}\widehat{\otimes}\mathfrak{J}-\mathcal{D}\left(\frac{a\overline{q}}{|q|^2}\right)\widehat{\otimes}\mathfrak{J}-\mathcal{D}\widehat{\otimes}\check{Z}+\frac{a^2\overline{q}^2}{|q|^4}\mathfrak{J}\widehat{\otimes}\mathfrak{J}+\frac{2a\overline{q}}{|q|^2}\mathfrak{J}\widehat{\otimes}\check{Z}+\Gamma_g\cdot\Gamma_g\\
=\;& \frac{a^2\overline{q}^2}{|q|^4}\mathfrak{J}\widehat{\otimes}\mathfrak{J}+\frac{a}{q^2}\mathcal{D}(q)\widehat{\otimes}\mathfrak{J}-\mathcal{D}\widehat{\otimes}\check{Z}+O(r^{-2})\check{Z}+O(r^{-1})\mathcal{D}\widehat{\otimes}\mathfrak{J}+\Gamma_g\cdot\Gamma_g\\
=\;& \frac{a^2i\overline{q}^2}{|q|^4}\Big(\mathcal{D}(\cos\theta)-i\mathfrak{J}\Big)\widehat{\otimes}\mathfrak{J}-\mathcal{D}\widehat{\otimes}\check{Z}+O(r^{-2})\check{Z}+O(r^{-1})\mathcal{D}\widehat{\otimes}\mathfrak{J}+\Gamma_g\cdot\Gamma_g
\end{aligned}$$

$$= \quad -\mathcal{D}\widehat{\otimes}\check{Z} + O(r^{-2})\check{Z} + O(r^{-1})\mathcal{D}\widehat{\otimes}\mathfrak{J} + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} + \Gamma_g\cdot\Gamma_g.$$

We infer

$$\begin{aligned}\nabla_4\widehat{\underline{X}} + \frac{1}{q}\widehat{\underline{X}} &= -\frac{1}{2}\mathcal{D}\widehat{\otimes}\check{Z} + O(r^{-2})\check{Z} + O(r^{-1})\widehat{X} + O(r^{-1})\mathcal{D}\widehat{\otimes}\mathfrak{J}\\ &\quad + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} + \Gamma_b\cdot\Gamma_g.\end{aligned}$$

Next, recall that

$$\nabla_4\underline{\omega} - (2\eta+\zeta)\cdot\zeta = \rho,$$

which we rewrite

$$\nabla_4\underline{\omega} - \frac{1}{2}\Re((2H+Z)\cdot\overline{Z}) = \Re(P).$$

We infer

$$\begin{aligned}\nabla_4(\underline{\check{\omega}}) &= \frac{1}{2}\Re((2H+Z)\cdot\overline{Z}) + \Re(P) - \frac{1}{2}\nabla_4\left(\partial_r\left(\frac{\Delta}{|q|^2}\right)\right)\\ &= \frac{1}{2}\Re\left(\frac{a(2q+\overline{q})}{|q|^2}\mathfrak{J}\cdot\frac{aq}{|q|^2}\overline{\mathfrak{J}}\right) + \Re\left(-\frac{2m}{q^3}\right) - \frac{1}{2}\partial_r^2\left(\frac{\Delta}{|q|^2}\right)\\ &\quad + \Re(\check{P}) + O(r^{-2})\check{Z} + O(r^{-2})\check{H} + \Gamma_b\cdot\Gamma_g.\end{aligned}$$

Now, using $\mathfrak{J}\cdot\overline{\mathfrak{J}} = \frac{2(\sin\theta)^2}{|q|^2}$, we have

$$\frac{1}{2}\Re\left(\frac{a^2q(2q+\overline{q})}{|q|^4}\mathfrak{J}\cdot\overline{\mathfrak{J}}\right) + \Re\left(-\frac{2m}{q^3}\right) - \frac{1}{2}\partial_r^2\left(\frac{\Delta}{|q|^2}\right) = 0$$

and hence

$$\nabla_4(\underline{\check{\omega}}) = \Re(\check{P}) + O(r^{-2})\check{Z} + O(r^{-2})\check{H} + \Gamma_b\cdot\Gamma_g.$$

Next, recall

$$\frac{1}{2}\overline{\mathcal{D}}\cdot\widehat{X} + \frac{1}{2}\widehat{X}\cdot\overline{Z} = \frac{1}{2}\mathcal{D}\overline{\mathrm{tr}X} + \frac{1}{2}\overline{\mathrm{tr}X}Z - i\Im(\mathrm{tr}X)H - B.$$

We infer

$$\frac{1}{2}\overline{\mathcal{D}}\cdot\widehat{X} + O(r^{-2})\widehat{X} = \mathcal{D}\left(\frac{1}{\overline{q}}\right) + \frac{1}{\overline{q}}\frac{a\overline{q}}{|q|^2}\mathfrak{J} + \frac{1}{\overline{q}}\check{Z} - i\Im\left(\frac{2}{q}\right)\frac{aq}{|q|^2}\mathfrak{J} + O(r^{-2})\check{H}$$

$$+O(r^{-1})\,\not{\mathfrak{d}}^{\leq 1}\widecheck{\mathrm{tr}X} - B + \Gamma_b\cdot\Gamma_g.$$

Since we have

$$\begin{aligned}\mathcal{D}\left(\frac{1}{\overline{q}}\right)+\frac{1}{\overline{q}}\frac{a\overline{q}}{|q|^2}\mathfrak{J}-i\Im\left(\frac{2}{q}\right)\frac{aq}{|q|^2}\mathfrak{J} &= -\frac{1}{\overline{q}^2}\mathcal{D}(\overline{q})+\frac{a}{|q|^2}\mathfrak{J}+\frac{2ia^2\cos\theta q}{|q|^4}\mathfrak{J}\\ &= \frac{iaq^2}{|q|^4}\left(\mathcal{D}(\cos\theta)-i\mathfrak{J}\right)=\frac{iaq^2}{|q|^4}\widecheck{\mathcal{D}(\cos\theta)},\end{aligned}$$

we deduce

$$\begin{aligned}\frac{1}{2}\overline{\mathcal{D}}\cdot\widehat{X} &= \frac{1}{q}\check{Z}-B+O(r^{-2})\widehat{X}+O(r^{-2})\check{H}+O(r^{-1})\,\not{\mathfrak{d}}^{\leq 1}\widecheck{\mathrm{tr}X}\\ &\quad +O(r^{-2})\widecheck{\mathcal{D}(\cos\theta)}+\Gamma_b\cdot\Gamma_g.\end{aligned}$$

Next, recall

$$\nabla_3 B-\mathcal{D}\overline{P} = -\mathrm{tr}\underline{X}B+2\underline{\omega}B+\underline{\overline{B}}\cdot\widehat{X}+3\overline{P}H+\frac{1}{2}A\cdot\underline{\Xi}.$$

We infer

$$\begin{aligned}\nabla_3 B-\mathcal{D}\overline{\check{P}} &= \frac{2}{r}B+O(r^{-2})B+O(r^{-2})\check{P}+O(r^{-3})\check{H}-\mathcal{D}\left(\frac{2m}{(\overline{q})^3}\right)\\ &\quad -\frac{6m}{(\overline{q})^3}\frac{aq}{|q|^2}\mathfrak{J}+r^{-1}\Gamma_b\cdot\Gamma_g.\end{aligned}$$

Now, we have

$$\begin{aligned}-\mathcal{D}\left(\frac{2m}{(\overline{q})^3}\right)-\frac{6m}{(\overline{q})^3}\frac{aq}{|q|^2}\mathfrak{J} &= \frac{6m}{\overline{q}^4}\mathcal{D}(\overline{q})-\frac{6am}{(\overline{q})^4}\mathfrak{J}=-\frac{6iam}{\overline{q}^4}\mathcal{D}(\cos\theta)-\frac{6am}{(\overline{q})^4}\mathfrak{J}\\ &= -\frac{6iam}{\overline{q}^4}\Big(\mathcal{D}(\cos\theta)-i\mathfrak{J}\Big)=-\frac{6iam}{\overline{q}^4}\widecheck{\mathcal{D}(\cos\theta)}\end{aligned}$$

and hence

$$\begin{aligned}\nabla_3 B-\mathcal{D}\overline{\check{P}} &= \frac{2}{r}B+O(r^{-2})B+O(r^{-2})\check{P}+O(r^{-3})\check{H}+O(r^{-4})\widecheck{\mathcal{D}(\cos\theta)}\\ &\quad +r^{-1}\Gamma_b\cdot\Gamma_g.\end{aligned}$$

Next, recall

$$\nabla_4 B-\frac{1}{2}\overline{\mathcal{D}}\cdot A = -2\overline{\mathrm{tr}X}B+\frac{1}{2}A\cdot\overline{Z},$$

and hence

$$\nabla_4 B + \frac{4}{\overline{q}} B \quad = \quad \frac{1}{2}\overline{\mathcal{D}}\cdot A + \frac{aq}{2|q|^2}\overline{\mathfrak{J}}\cdot A + \Gamma_g\cdot(B, A).$$

Next, recall

$$\nabla_4 P - \frac{1}{2}\mathcal{D}\cdot\overline{B} \quad = \quad -\frac{3}{2}\mathrm{tr}XP - \frac{1}{2}Z\cdot\overline{B} - \frac{1}{4}\underline{\widehat{X}}\cdot\overline{A}.$$

We infer

$$\begin{aligned}\nabla_4\left(-\frac{2m}{q^3} + \check{P}\right) - \frac{1}{2}\mathcal{D}\cdot\overline{B} \quad &= \quad \frac{6m\overline{q}}{|q|^2q^3} - \frac{3}{q}\check{P} - \frac{a\overline{q}}{2|q|^2}\mathfrak{J}\cdot\overline{B} + O(r^{-3})\widecheck{\mathrm{tr}X}\\ &\quad +r^{-1}\Gamma_g\cdot\Gamma_g + \Gamma_b\cdot A.\end{aligned}$$

Now, we have

$$\nabla_4\left(-\frac{2m}{q^3}\right) = \partial_r\left(-\frac{2m}{(r+ia\cos\theta)^4}\right) = \frac{6m}{(r+ia\cos\theta)^4} = \frac{6m\overline{q}}{|q|^2q^3}$$

and hence

$$\nabla_4\left(\check{P}\right) - \frac{1}{2}\mathcal{D}\cdot\overline{B} = -\frac{3}{q}\check{P} - \frac{a\overline{q}}{2|q|^2}\mathfrak{J}\cdot\overline{B} + O(r^{-3})\widecheck{\mathrm{tr}X} + r^{-1}\Gamma_g\cdot\Gamma_g + \Gamma_b\cdot A.$$

Finally, recall

$$\nabla_4\underline{B} + \mathcal{D}P \quad = \quad -\mathrm{tr}X\underline{B} + \overline{B}\cdot\underline{\widehat{X}} + 3PZ.$$

We infer

$$\begin{aligned}\nabla_4\underline{B} + \mathcal{D}\left(\check{P}\right) \quad &= \quad -\frac{2}{q}\underline{B} + O(r^{-2})\check{P} + O(r^{-3})\check{Z} + \mathcal{D}\left(\frac{2m}{q^3}\right) - \frac{6m}{q^3}\frac{a\overline{q}}{|q|^2}\mathfrak{J}\\ &\quad +r^{-1}\Gamma_b\cdot\Gamma_g.\end{aligned}$$

Now, we have

$$\begin{aligned}\mathcal{D}\left(\frac{2m}{q^3}\right) - \frac{6m}{q^3}\frac{a\overline{q}}{|q|^2}\mathfrak{J} \quad &= \quad -\frac{6m}{q^4}\mathcal{D}(q) - \frac{6am}{q^4}\mathfrak{J} = -\frac{6iam}{q^4}\mathcal{D}(\cos\theta) - \frac{6am}{q^4}\mathfrak{J}\\ &= \quad -\frac{6iam}{q^4}\Big(\mathcal{D}(\cos\theta) - i\mathfrak{J}\Big) = -\frac{6iam}{q^4}\widecheck{\mathcal{D}(\cos\theta)}\end{aligned}$$

and hence

$$\nabla_4\underline{B}+\mathcal{D}\left(\check{P}\right) = -\frac{2}{q}\underline{B}+O(r^{-2})\check{P}+O(r^{-3})\check{Z}+O(r^{-4})\widecheck{\mathcal{D}(\cos\theta)} \\ +r^{-1}\Gamma_b\cdot\Gamma_g.$$

This concludes the proof of Lemma 6.15.

C.2. Proof of Lemma 6.16

We have from Lemma 6.10.

$$e_4(e_3(r)) = -2\underline{\omega},$$

and hence, since $e_4(r)=1$, $e_4(\theta)=0$,

$$e_4\left(\widecheck{e_3(r)}\right) = e_4\left(e_3(r)+\frac{\Delta}{|q|^2}\right) = -2\underline{\omega}+\partial_r\left(\frac{\Delta}{|q|^2}\right) \\ = -2\underline{\check{\omega}}.$$

Also, we have from Proposition 6.10.

$$\nabla_4\mathcal{D}u+\frac{1}{2}\mathrm{tr}X\mathcal{D}u = -\frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}u$$

and hence

$$\nabla_4\mathcal{D}u+\frac{1}{q}\mathcal{D}u = O(r^{-1})\widecheck{\mathrm{tr}X}+O(r^{-1})\widehat{X}+\Gamma_b\cdot\Gamma_g.$$

Since

$$\nabla_4\mathcal{D}u+\frac{1}{q}\mathcal{D}u = \nabla_4\left(a\mathfrak{J}+\widecheck{\mathcal{D}u}\right)+\frac{1}{q}\left(a\mathfrak{J}+\widecheck{\mathcal{D}u}\right)=\nabla_4\widecheck{\mathcal{D}u}+\frac{1}{q}\widecheck{\mathcal{D}u},$$

where we have used $\nabla_4\mathfrak{J}=-q^{-1}\mathfrak{J}$, we infer

$$\nabla_4\widecheck{\mathcal{D}u}+\frac{1}{q}\widecheck{\mathcal{D}u} = O(r^{-1})\widecheck{\mathrm{tr}X}+O(r^{-1})\widehat{X}+\Gamma_b\cdot\Gamma_g.$$

Also, recall that we have

$$e_4(e_3(u)) = -\Re\Big((Z+H)\cdot\overline{\mathcal{D}}u\Big).$$

We infer

$$e_4\left(\frac{2(r^2+a^2)}{|q|^2}+\widetilde{e_3(u)}\right) = -\Re\left(\frac{a(q+\overline{q})}{|q|^2}\mathfrak{J}\cdot a\overline{\mathfrak{J}}\right)+O(r^{-1})\check{H}+O(r^{-1})\check{Z}$$
$$+O(r^{-2})\widetilde{\mathcal{D}u}+\Gamma_b\cdot\Gamma_b.$$

Now, using $e_4(r)=1$, $e_4(\theta)=0$ and $\mathfrak{J}\cdot\overline{\mathfrak{J}}=\frac{2(\sin\theta)^2}{|q|^2}$, we have

$$\begin{aligned}
&-e_4\left(\frac{2(r^2+a^2)}{|q|^2}\right)-\Re\left(\frac{a(q+\overline{q})}{|q|^2}\mathfrak{J}\cdot a\overline{\mathfrak{J}}\right)\\
=\ & -\partial_r\left(\frac{2(r^2+a^2)}{|q|^2}\right)-\Re\left(\frac{2ra^2}{|q|^2}\mathfrak{J}\cdot\overline{\mathfrak{J}}\right)\\
=\ & -\frac{4r}{|q|^2}+\frac{4r(r^2+a^2)}{|q|^4}-\frac{4ra^2(\sin\theta)^2}{|q|^4}=0
\end{aligned}$$

and hence

$$e_4\left(\widetilde{e_3(u)}\right) = O(r^{-1})\check{H}+O(r^{-1})\check{Z}+O(r^{-2})\widetilde{\mathcal{D}u}+\Gamma_b\cdot\Gamma_b.$$

Also, recall that we have

$$\nabla_4\mathcal{D}\cos\theta+\frac{1}{2}\mathrm{tr}X\mathcal{D}\cos\theta = -\frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}\cos\theta.$$

and hence

$$\nabla_4\mathcal{D}\cos\theta+\frac{1}{q}\mathcal{D}\cos\theta = \frac{i}{2}\overline{\mathfrak{J}}\cdot\widehat{X}+O(r^{-1})\widetilde{\mathrm{tr}X}+\Gamma_b\cdot\Gamma_g.$$

Since

$$\begin{aligned}
\nabla_4\mathcal{D}\cos\theta+\frac{1}{q}\mathcal{D}\cos\theta &= \nabla_4\left(i\mathfrak{J}+\widetilde{\mathcal{D}\cos\theta}\right)+\frac{1}{q}\left(i\mathfrak{J}+\widetilde{\mathcal{D}\cos\theta}\right)\\
&= \nabla_4\widetilde{\mathcal{D}\cos\theta}+\frac{1}{q}\widetilde{\mathcal{D}\cos\theta},
\end{aligned}$$

where we have used $\nabla_4\mathfrak{J}=-q^{-1}\mathfrak{J}$, we infer

$$\nabla_4\widetilde{\mathcal{D}\cos\theta}+\frac{1}{q}\widetilde{\mathcal{D}\cos\theta} = \frac{i}{2}\overline{\mathfrak{J}}\cdot\widehat{X}+O(r^{-1})\widetilde{\mathrm{tr}X}+\Gamma_b\cdot\Gamma_g.$$

Also, recall that we have

$$e_4(e_3(\cos\theta)) = -\Re\Big((Z+H)\cdot\overline{\mathcal{D}}\cos\theta\Big)$$

and hence

$$\begin{aligned}
& e_4(e_3(\cos\theta)) \\
= & \Re\left(\frac{a(q+\overline{q})}{|q|^2}\mathfrak{J}\cdot i\overline{\mathfrak{J}}\right)+O(r^{-1})\check{H}+O(r^{-1})\check{Z}+O(r^{-2})\widetilde{\mathcal{D}\cos\theta}+\Gamma_b\cdot\Gamma_b \\
= & \Re\left(i\frac{2ra}{|q|^2}\frac{2(\sin\theta)^2}{|q|^2}\right)+O(r^{-1})\check{H}+O(r^{-1})\check{Z}+O(r^{-2})\widetilde{\mathcal{D}\cos\theta}+\Gamma_b\cdot\Gamma_b \\
= & O(r^{-1})\check{H}+O(r^{-1})\check{Z}+O(r^{-2})\widetilde{\mathcal{D}\cos\theta}+\Gamma_b\cdot\Gamma_b.
\end{aligned}$$

This concludes the proof of Lemma 6.16.

C.3. Proof of Lemma 6.28

In view of Lemma 2.10, we have

$$e'_a = e_a+\frac{1}{2}\underline{f}_a\lambda^{-1}e'_4+\frac{1}{2}f_ae_3.$$

We compute

$$\begin{aligned}
& \mathbf{g}(\mathbf{D}_{e'_a}e'_b,e'_c) = \mathbf{g}\left(\mathbf{D}_{e'_a}\left(e_b+\frac{1}{2}\underline{f}_b\lambda^{-1}e'_4+\frac{1}{2}f_be_3\right),e'_c\right) \\
= & \frac{1}{2}e'_a(f_b)\mathbf{g}(e_3,e'_c)+\mathbf{g}(\mathbf{D}_{e'_a}e_b,e'_c)+\frac{1}{2}\underline{f}_b\lambda^{-1}\mathbf{g}(\mathbf{D}_{e'_a}e'_4,e'_c) \\
& +\frac{1}{2}f_b\mathbf{g}(\mathbf{D}_{e'_a}e_3,e'_c) \\
= & -\frac{1}{2}e'_a(f_b)\underline{f}_c+\mathbf{g}(\mathbf{D}_{e'_a}e_b,e_c)+\frac{1}{2}\underline{f}_c\lambda^{-1}\mathbf{g}(\mathbf{D}_{e'_a}e_b,e'_4)+\frac{1}{2}f_c\mathbf{g}(\mathbf{D}_{e'_a}e_b,e_3) \\
& +\frac{1}{2}\underline{f}_b\lambda^{-1}\chi'_{ac}+\frac{1}{2}f_b\mathbf{g}\left(\mathbf{D}_{e'_a}e_3,\left(\delta_c^d+\frac{1}{2}\underline{f}_cf^d\right)e_d+\frac{1}{2}\underline{f}_ce_4\right) \\
= & -\frac{1}{2}e'_a(f_b)\underline{f}_c+\mathbf{g}\left(\mathbf{D}_{\left(\delta_a^d+\frac{1}{2}\underline{f}_af^d\right)e_d+\frac{1}{2}\underline{f}_ae_4+\left(\frac{1}{2}f_a+\frac{1}{8}|f|^2\underline{f}_a\right)e_3}e_b,e_c\right) \\
& +\frac{1}{2}\underline{f}_c\lambda^{-1}e'_a(\mathbf{g}(e_b,e'_4))-\frac{1}{2}\underline{f}_c\lambda^{-1}\mathbf{g}(e_b,\mathbf{D}_{e'_a}e'_4) \\
& +\frac{1}{2}f_c\mathbf{g}\left(\mathbf{D}_{\left(\delta_a^d+\frac{1}{2}\underline{f}_af^d\right)e_d+\frac{1}{2}\underline{f}_ae_4+\left(\frac{1}{2}f_a+\frac{1}{8}|f|^2\underline{f}_a\right)e_3}e_b,e_3\right)+\frac{1}{2}\underline{f}_b\lambda^{-1}\chi'_{ac}
\end{aligned}$$

$$+\frac{1}{2}f_b\mathbf{g}\Bigg(\mathbf{D}_{\left(\delta_a^p+\frac{1}{2}\underline{f}_af^p\right)e_p+\frac{1}{2}\underline{f}_ae_4+\left(\frac{1}{2}f_a+\frac{1}{8}|f|^2\underline{f}_a\right)e_3}e_3, \left(\delta_c^d+\frac{1}{2}\underline{f}_cf^d\right)e_d$$
$$+\frac{1}{2}\underline{f}_ce_4\Bigg).$$

We further deduce

$$\begin{aligned}
&\mathbf{g}(\mathbf{D}_{e'_a}e'_b, e'_c)\\
=\;& \left(\delta_a^d+\frac{1}{2}\underline{f}_af^d\right)\mathbf{g}\left(\mathbf{D}_{e_d}e_b, e_c\right)+\frac{1}{2}\underline{f}_a\mathbf{g}\left(\mathbf{D}_{e_4}e_b, e_c\right)\\
&+\left(\frac{1}{2}f_a+\frac{1}{8}|f|^2\underline{f}_a\right)\mathbf{g}\left(\mathbf{D}_{e_3}e_b, e_c\right)+\frac{1}{2}\underline{f}_cf_be'_a(\log\lambda)\\
&-\frac{1}{2}\underline{f}_c\lambda^{-1}\mathbf{g}\Big(e_b, \chi'_{ad}e'_d-\zeta'_ae'_4\Big)\\
&-\frac{1}{2}f_c\mathbf{g}\left(\mathbf{D}_{\left(\delta_a^d+\frac{1}{2}\underline{f}_af^d\right)e_d+\frac{1}{2}\underline{f}_ae_4+\left(\frac{1}{2}f_a+\frac{1}{8}|f|^2\underline{f}_a\right)e_3}e_3, e_b\right)+\frac{1}{2}\underline{f}_b\lambda^{-1}\chi'_{ac}\\
&+\frac{1}{2}f_b\left(\delta_c^d+\frac{1}{2}\underline{f}_cf^d\right)\left(\left(\delta_a^p+\frac{1}{2}\underline{f}_af^p\right)\underline{\chi}_{pd}+\underline{f}_a\underline{\eta}_d+\left(f_a+\frac{1}{4}|f|^2\underline{f}_a\right)\underline{\xi}_d\right)\\
&+\frac{1}{4}f_b\underline{f}_c\left(-2\left(\delta_a^p+\frac{1}{2}\underline{f}_af^p\right)\zeta_p-2\omega\underline{f}_a+\underline{\omega}\left(2f_a+\frac{1}{2}|f|^2\underline{f}_a\right)\right)
\end{aligned}$$

and hence

$$\begin{aligned}
&\mathbf{g}(\mathbf{D}_{e'_a}e'_b, e'_c)\\
=\;& \left(\delta_a^d+\frac{1}{2}\underline{f}_af^d\right)\mathbf{g}\left(\mathbf{D}_{e_d}e_b, e_c\right)+\frac{1}{2}\underline{f}_a\mathbf{g}\left(\mathbf{D}_{e_4}e_b, e_c\right)\\
&+\left(\frac{1}{2}f_a+\frac{1}{8}|f|^2\underline{f}_a\right)\mathbf{g}\left(\mathbf{D}_{e_3}e_b, e_c\right)+\frac{1}{2}\underline{f}_cf_be'_a(\log\lambda)-\frac{1}{2}\underline{f}_c\lambda^{-1}\chi'_{ab}\\
&+\frac{1}{2}\underline{f}_b\lambda^{-1}\chi'_{ac}+\frac{1}{2}\lambda^{-1}\zeta'_a\underline{f}_cf_b-\frac{1}{4}\underline{f}_c\lambda^{-1}\chi'_{ad}\underline{f}_bf_d\\
&-\frac{1}{2}f_c\left(\delta_a^d+\frac{1}{2}\underline{f}_af^d\right)\underline{\chi}_{db}-\frac{1}{2}f_c\underline{f}_a\underline{\eta}_b-\frac{1}{2}f_c\left(f_a+\frac{1}{4}|f|^2\underline{f}_a\right)\underline{\xi}_b\\
&+\frac{1}{2}f_b\left(\delta_c^d+\frac{1}{2}\underline{f}_cf^d\right)\left(\left(\delta_a^p+\frac{1}{2}\underline{f}_af^p\right)\underline{\chi}_{pd}+\underline{f}_a\underline{\eta}_d+\left(f_a+\frac{1}{4}|f|^2\underline{f}_a\right)\underline{\xi}_d\right)\\
&+\frac{1}{4}f_b\underline{f}_c\left(-2\left(\delta_a^p+\frac{1}{2}\underline{f}_af^p\right)\zeta_p-2\omega\underline{f}_a+\underline{\omega}\left(2f_a+\frac{1}{2}|f|^2\underline{f}_a\right)\right).
\end{aligned}$$

We infer

$$\mathbf{g}(\mathbf{D}_{e'_a}e'_b, e'_c)$$

$$
\begin{aligned}
&= \left(\delta_a^d + \frac{1}{2}\underline{f}_a f^d\right) \mathbf{g}\left(\mathbf{D}_{e_d} e_b, e_c\right) + \frac{1}{2}\underline{f}_a \mathbf{g}\left(\mathbf{D}_{e_4} e_b, e_c\right) \\
&\quad + \left(\frac{1}{2} f_a + \frac{1}{8}|f|^2 \underline{f}_a\right) \mathbf{g}\left(\mathbf{D}_{e_3} e_b, e_c\right) + \frac{1}{2}\underline{f}_c f_b e'_a(\log\lambda) - \frac{1}{2}\underline{f}_c \lambda^{-1}\chi'_{ab} \\
&\quad + \frac{1}{2}\underline{f}_b \lambda^{-1}\chi'_{ac} + \frac{1}{2}\zeta'_a \underline{f}_c f_b - \frac{1}{4}\underline{f}_c \lambda^{-1}\chi'_{ad}\underline{f}_b f_d - \frac{1}{2} f_c \underline{\chi}_{ab} + \frac{1}{2} f_b \underline{\chi}_{ac} \\
&\quad + \mathrm{Err}[\mathbf{g}(\mathbf{D}_{e'_a} e'_b, e'_c)],
\end{aligned}
$$

with

$$
\begin{aligned}
&\mathrm{Err}[\mathbf{g}(\mathbf{D}_{e'_a} e'_b, e'_c)] \\
&= -\frac{1}{2} f_c \left(\delta_a^d + \frac{1}{2}\underline{f}_a f^d\right) \underline{\chi}_{db} - \frac{1}{2} f_c \underline{f}_a \underline{\eta}_b - \frac{1}{2} f_c \left(f_a + \frac{1}{4}|f|^2 \underline{f}_a\right) \underline{\xi}_b \\
&\quad + \frac{1}{2} f_b \left(\delta_c^d + \frac{1}{2}\underline{f}_c f^d\right) \left(\left(\delta_a^p + \frac{1}{2}\underline{f}_a f^p\right) \underline{\chi}_{pd} + \underline{f}_a \underline{\eta}_d + \left(f_a + \frac{1}{4}|f|^2 \underline{f}_a\right) \underline{\xi}_d\right) \\
&\quad + \frac{1}{4} f_b \underline{f}_c \left(-2\left(\delta_a^p + \frac{1}{2}\underline{f}_a f^p\right)\zeta_p - 2\omega \underline{f}_a + \underline{\omega}\left(2 f_a + \frac{1}{2}|f|^2 \underline{f}_a\right)\right),
\end{aligned}
$$

where $\mathrm{Err}[\mathbf{g}(\mathbf{D}_{e'_a} e'_b, e'_c)]$ contains all the terms depending on $(f, \underline{f}, \Gamma)$, without derivative, and at least quadratic in $(f, \underline{f})$. This concludes the proof of Lemma 6.28.

C.4. Proof of Proposition 6.35

In order to prove Proposition 6.35, we start with the following lemma.

Lemma C.1. *For all $J = J^{(p)}$, we have*

$$
\nabla_4(\mathcal{D}\widehat{\otimes}\mathcal{D}J) + \frac{2}{q}\mathcal{D}\widehat{\otimes}\mathcal{D}J = r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g
$$

and

$$
\nabla_4(\overline{\mathcal{D}}\cdot\mathcal{D}J) + \frac{2}{r}\overline{\mathcal{D}}\cdot\mathcal{D}J = O(r^{-5}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g.
$$

Proof. We start with the first identity. We use $e_4(J) = 0$ and the commutation Lemma 6.12 to deduce

$$
\begin{aligned}
\nabla_4(\mathcal{D}\widehat{\otimes}\mathcal{D}J) &= [\nabla_4, \mathcal{D}\widehat{\otimes}]\mathcal{D}J + \mathcal{D}\widehat{\otimes}[\nabla_4, \mathcal{D}]J \\
&= -\frac{1}{2}\mathrm{tr}X\left(\mathcal{D}\widehat{\otimes}\mathcal{D}J - Z\widehat{\otimes}\mathcal{D}J\right) + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}\mathcal{D}J
\end{aligned}
$$

$$
\begin{aligned}
& +\mathcal{D}\widehat{\otimes}\left(-\frac{1}{2}\mathrm{tr}X\mathcal{D}J + r^{-1}\Gamma_g\cdot\not{\mathfrak{d}}J\right)\\
= & -\mathrm{tr}X\mathcal{D}\widehat{\otimes}\mathcal{D}J - \frac{1}{2}\Big(\mathcal{D}\mathrm{tr}X - \mathrm{tr}XZ\Big)\widehat{\otimes}\mathcal{D}J + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g\\
= & -\frac{2}{q}\mathcal{D}\widehat{\otimes}\mathcal{D}J - \frac{1}{2}\left(\mathcal{D}\left(\frac{2}{q}\right) - \frac{2}{q}\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)\widehat{\otimes}\mathcal{D}J + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g\\
= & -\frac{2}{q}\mathcal{D}\widehat{\otimes}\mathcal{D}J - \frac{1}{2}\left(-\frac{2ia}{q^2}\mathcal{D}(\cos\theta) - \frac{2}{q}\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)\widehat{\otimes}\mathcal{D}J\\
& + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g\\
= & -\frac{2}{q}\mathcal{D}\widehat{\otimes}\mathcal{D}J + \frac{ia}{q^2}\Big(\mathcal{D}(\cos\theta) - i\mathfrak{J}\Big)\widehat{\otimes}\mathcal{D}J + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g\\
= & -\frac{2}{q}\mathcal{D}\widehat{\otimes}\mathcal{D}J + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g\\
= & -\frac{2}{q}\mathcal{D}\widehat{\otimes}\mathcal{D}J + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g
\end{aligned}
$$

and hence

$$
\nabla_4(\mathcal{D}\widehat{\otimes}\mathcal{D}J) + \frac{2}{q}\mathcal{D}\widehat{\otimes}\mathcal{D}J \quad = \quad r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g
$$

which is the first identity.

Similarly, to prove the second identity, we use $e_4(J) = 0$ and the commutation Lemma 6.12 to deduce

$$
\begin{aligned}
& \nabla_4(\overline{\mathcal{D}}\cdot\mathcal{D}J)\\
= & [\nabla_4, \overline{\mathcal{D}}\cdot]\mathcal{D}J + \overline{\mathcal{D}}\cdot[\nabla_4, \mathcal{D}]J\\
= & -\frac{1}{2}\overline{\mathrm{tr}X}\Big(\overline{\mathcal{D}}\cdot\mathcal{D}J + \overline{Z}\cdot\mathcal{D}J\Big) + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}\mathcal{D}J\\
& + \overline{\mathcal{D}}\cdot\left(-\frac{1}{2}\mathrm{tr}X\mathcal{D}J + r^{-1}\Gamma_g\cdot\not{\mathfrak{d}}J\right)\\
= & -\frac{1}{2}(\mathrm{tr}X + \overline{\mathrm{tr}X})\overline{\mathcal{D}}\cdot\mathcal{D}J - \frac{1}{2}\Big(\overline{\mathcal{D}}\mathrm{tr}X + \overline{\mathrm{tr}X}\,\overline{Z}\Big)\cdot\mathcal{D}J + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g\\
= & -\frac{2r}{|q|^2}\overline{\mathcal{D}}\cdot\mathcal{D}J - \frac{1}{2}\left(\overline{\mathcal{D}}\left(\frac{2}{q}\right) - \frac{2}{\overline{q}}\frac{aq}{|q|^2}\overline{\mathfrak{J}}\right)\cdot\mathcal{D}J + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g\\
= & -\frac{2r}{|q|^2}\overline{\mathcal{D}}\cdot\mathcal{D}J - \frac{1}{2}\left(-\frac{2ia}{q^2}\overline{\mathcal{D}}(\cos\theta) - \frac{2}{\overline{q}}\frac{aq}{|q|^2}\overline{\mathfrak{J}}\right)\cdot\mathcal{D}J + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g\\
= & -\frac{2r}{|q|^2}\overline{\mathcal{D}}\cdot\mathcal{D}J + a\left(\frac{i}{q^2}\overline{\mathcal{D}(\cos\theta) - i\mathfrak{J}} + \left(\frac{1}{(\overline{q})^2} - \frac{1}{q^2}\right)\overline{\mathfrak{J}}\right)\cdot\mathcal{D}J + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g\\
= & -\frac{2}{r}\overline{\mathcal{D}}\cdot\mathcal{D}J + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} + O(r^{-5}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g
\end{aligned}
$$

$$= -\frac{2}{r}\overline{\mathcal{D}}\cdot\mathcal{D}J + O(r^{-5}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g$$

and hence

$$\nabla_4(\overline{\mathcal{D}}\cdot\mathcal{D}J) + \frac{2}{r}\overline{\mathcal{D}}\cdot\mathcal{D}J = O(r^{-5}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g$$

as desired. □

Corollary C.2. *For all $J = J^{(p)}$, we have*

$$\sup_{^{(ext)}\mathcal{M}} r^3 u^{\frac{1}{2}+\delta_{dec}}|\mathcal{D}\widehat{\otimes}\mathcal{D}J| \lesssim \epsilon$$

and

$$\left|\left(\Delta + \frac{2}{r^2}\right)J\right| \lesssim \frac{1}{r^4} + \frac{\epsilon}{r^3 u^{\frac{1}{2}+\delta_{dec}}} \quad on \quad {}^{(ext)}\mathcal{M}.$$

Proof. Recall that we have

$$\nabla_4(\mathcal{D}\widehat{\otimes}\mathcal{D}J) + \frac{2}{q}\mathcal{D}\widehat{\otimes}\mathcal{D}J = r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g$$

so that

$$\nabla_4(q^2\mathcal{D}\widehat{\otimes}\mathcal{D}J) = \mathfrak{d}^{\leq 1}\Gamma_g.$$

Integrating from Σ_*, and together with the control on Σ_* of Lemma 5.68, we infer

$$\sup_{^{(ext)}\mathcal{M}} r^3 u^{\frac{1}{2}+\delta_{dec}}|\mathcal{D}\widehat{\otimes}\mathcal{D}J| \lesssim \epsilon$$

as desired.

Also, recall that we have

$$\nabla_4(\overline{\mathcal{D}}\cdot\mathcal{D}J) + \frac{2}{r}\overline{\mathcal{D}}\cdot\mathcal{D}J = O(r^{-5}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g$$

so that

$$\nabla_4(r^2\overline{\mathcal{D}}\cdot\mathcal{D}J + 4) = O(r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_g.$$

Integrating from Σ_*, we infer on ${}^{(ext)}\mathcal{M}$

$$\left|r^2\overline{\mathcal{D}}\cdot\mathcal{D}J+4\right| \lesssim \sup_{\Sigma_*}\left|r^2\overline{\mathcal{D}}\cdot\mathcal{D}J+4\right|+\frac{1}{r^2}+\frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}}}.$$

Now, we have, for a scalar function h

$$\begin{aligned}\overline{\mathcal{D}}\cdot\mathcal{D}h &= 2\Delta h+2i\in_{ab}\nabla_a\nabla_b h\\ &= 2\Delta h+i\in_{ab}\Big(\mathbf{D}_a\mathbf{D}_b+\chi_{ab}e_3+\underline{\chi}_{ab}e_4\Big)h\\ &= 2\Delta h+i\Big({}^{(a)}\mathrm{tr}\chi e_3(h)+{}^{(a)}\mathrm{tr}\underline{\chi}e_4(h)\Big).\end{aligned}$$

Using this formula with $h=J^{(p)}$, and since $e_4(J^{(p)})=0$, $e_3(J^{(p)})=O(r^{-2})+\Gamma_b$ and ${}^{(a)}\mathrm{tr}\chi=O(r^{-2})+\Gamma_g$, we infer

$$\begin{aligned}\overline{\mathcal{D}}\cdot\mathcal{D}J &= 2\Delta J+\Big(O(r^{-2})+\Gamma_g\Big)\Big(O(r^{-2})+\Gamma_b\Big)\\ &= 2\Delta J+O(r^{-4})+r^{-2}\Gamma_b.\end{aligned}$$

Plugging in the above, and using the control of Γ_b, we infer

$$\left|r^2\Delta J+2\right| \lesssim \sup_{\Sigma_*}\left|r^2\Delta J+2\right|+\frac{1}{r^2}+\frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}}}.$$

Together with the control on Σ_* of Lemma 5.68, we deduce

$$\left|r^2\Delta J+2\right| \lesssim \frac{1}{r^2}+\frac{\epsilon}{ru^{\frac{1}{2}+\delta_{dec}}}$$

as desired. □

We are now ready to prove Proposition 6.35.

Proof of Proposition 6.35. Recall from Corollary 6.30 that we have

$$\nabla' = \Big(1+O(r^{-2})\Big)\nabla+O(r^{-1})\not{\mathcal{L}}_{\mathbf{T}}+O(r^{-1})\nabla_4+O(r^{-3})+r^{-1}\Gamma_b.$$

We infer

$$\begin{aligned}\nabla'\widehat{\otimes}\nabla' &= \nabla\widehat{\otimes}\nabla+O(r^{-4})\,\not{\mathfrak{d}}^{\leq 2}+O(r^{-2})\mathfrak{d}^{\leq 1}\not{\mathcal{L}}_{\mathbf{T}}+O(r^{-2})\mathfrak{d}^{\leq 1}\nabla_4+O(r^{-4})\\ &\quad +r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b,\\ \Delta' &= \Delta+O(r^{-4})\,\not{\mathfrak{d}}^{\leq 2}+O(r^{-2})\mathfrak{d}^{\leq 1}\not{\mathcal{L}}_{\mathbf{T}}+O(r^{-2})\mathfrak{d}^{\leq 1}\nabla_4+O(r^{-4})\\ &\quad +r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b.\end{aligned}$$

Since $\nabla_4 J = 0$ and $\mathbf{T}(J) = \Gamma_b$, see Lemma 6.19, we deduce

$$\begin{aligned}\nabla'\widehat{\otimes}\nabla' J &= \nabla\widehat{\otimes}\nabla J + O(r^{-4}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b,\\ \Delta' J &= \Delta J + O(r^{-4}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b.\end{aligned}$$

Together with the above estimate for $\nabla\widehat{\otimes}\nabla J$ and ΔJ, and in view of the control of Γ_b, we infer

$$|\mathcal{D}'\widehat{\otimes}\mathcal{D}' J| + |r^2\Delta' J + 2| \lesssim \frac{\epsilon}{r^3 u^{\frac{1}{2}+\delta_{dec}}} + \frac{1}{r^4}$$

as desired. □

C.5. Proof of Proposition 6.41

Recall the definition of our renormalized quantities:

$$\begin{aligned}[\check{H}]_{ren} &= \frac{1}{\overline{q}}\left(\overline{q}\check{H} - q\check{Z} + \frac{1}{3}\left(-\overline{q}^2 + |q|^2\right)B + \frac{a}{2}(q-\overline{q})\overline{\mathfrak{J}}\cdot\widehat{X}\right),\\ [\widecheck{\mathcal{D}\cos\theta}]_{ren} &= \frac{1}{q}\left(q\widecheck{\mathcal{D}\cos\theta} + \frac{i}{2}|q|^2\overline{\mathfrak{J}}\cdot\widehat{X}\right),\\ [\widecheck{M}]_{ren} &= \frac{1}{\overline{q}q^2}\Bigg[\overline{q}\,\overline{\mathcal{D}}\cdot\left(q^2\check{Z} + \left(-\frac{a}{2}q^2 - \frac{a}{2}|q|^2\right)\mathfrak{J}\cdot\widehat{X}\right) + 2\overline{q}^3\,\overline{\check{P}} - 2aq^2\overline{\mathfrak{J}}\cdot\check{Z}\\ &\quad + \left(-\frac{1}{3}q^2\overline{q}^2 - \frac{1}{3}q\overline{q}^3 + \frac{2}{3}\overline{q}^4\right)\overline{\mathcal{D}}\cdot B + a\left(q^2\overline{q} + \frac{2}{3}q\overline{q}^2 - \frac{13}{6}\overline{q}^3\right)\overline{\mathfrak{J}}\cdot B\\ &\quad + a^2(q^2+|q|^2)\overline{\mathfrak{J}}\cdot\widehat{X}\cdot\overline{\mathfrak{J}}\Bigg].\end{aligned}$$

Our goal is to prove the following three identities

$$\begin{aligned}\nabla_4\left(\overline{q}[\check{H}]_{ren}\right) &= O(r^{-1})\widecheck{\mathrm{tr}X} + O(1)\,\mathfrak{d}^{\leq 1}A + r\Gamma_b\cdot\Gamma_g,\\ \nabla_4\left(q[\widecheck{\mathcal{D}\cos\theta}]_{ren}\right) &= O(1)\widecheck{\mathrm{tr}X} + O(r)A + r\Gamma_b\cdot\Gamma_g,\\ \nabla_4\left(\overline{q}q^2[\widecheck{M}]_{ren}\right) &= O(1)\,\mathfrak{d}^{\leq 1}\widecheck{\mathrm{tr}X} + O(r)\,\mathfrak{d}^{\leq 2}A + r^2\,\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g) + r^3\Gamma_b\cdot A.\end{aligned}$$

First identity. To check

$$\nabla_4\left(\overline{q}[\check{H}]_{ren}\right) = O(r^{-1})\widecheck{\mathrm{tr}X} + O(1)\,\mathfrak{d}^{\leq 1}A + r\Gamma_b\cdot\Gamma_g,$$

we start with, see Lemma 6.15,

$$\begin{aligned}
\nabla_4 \widecheck{H} + \frac{1}{\overline{q}}\widecheck{H} &= -\frac{1}{\overline{q}}\widecheck{Z} - \frac{ar}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X} - B + O(r^{-2})\widecheck{\mathrm{tr}X} + \Gamma_b\cdot\Gamma_g,\\
\nabla_4 \widecheck{Z} + \frac{2}{q}\widecheck{Z} &= -\frac{aq}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X} - B + O(r^{-2})\widecheck{\mathrm{tr}X} + \Gamma_g\cdot\Gamma_g.
\end{aligned}$$

The goal is to eliminate the presence of B and $\widehat{X}$ on the RHS. We first combine the two as follows

$$\begin{aligned}
\nabla_4(\overline{q}\widecheck{H} - q\widecheck{Z}) &= \overline{q}\nabla_4\widecheck{H} + \widecheck{H} - q\nabla_4\widecheck{Z} - \widecheck{Z}\\
&\quad -q\left(-\frac{2}{q}\widecheck{Z} - \frac{aq}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X} - B\right) - \widecheck{Z} + O(r^{-1})\widecheck{\mathrm{tr}X} + r\Gamma_b\cdot\Gamma_g\\
&= (q-\overline{q})B + \frac{a\big(q^2 - r\overline{q}\big)}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X} + O(r^{-1})\widecheck{\mathrm{tr}X} + r\Gamma_b\cdot\Gamma_g.
\end{aligned}$$

To eliminate $(q-\overline{q})B$ we make use of the equation

$$\nabla_4 B + \frac{4}{\overline{q}}B = O(r^{-1})\not{\mathfrak{d}}^{\leq 1}A + \Gamma_g\cdot(B, A).$$

Since $-\overline{q}^2 + |q|^2 = O(r)$, we obtain

$$\begin{aligned}
&\nabla_4\left(\left(-\frac{1}{3}\overline{q}^2 + \frac{1}{3}|q|^2\right)B\right)\\
&= \nabla_4\left(-\frac{1}{3}\overline{q}^2 + \frac{1}{3}|q|^2\right)B + \left(-\frac{1}{3}\overline{q}^2 + \frac{1}{3}|q|^2\right)\nabla_4 B\\
&= \left(-\frac{2}{3}\overline{q} + \frac{1}{3}(q+\overline{q})\right)B\\
&\quad + \left(-\frac{1}{3}\overline{q}^2 + \frac{1}{3}|q|^2\right)\left(-\frac{4}{\overline{q}}B + O(r^{-1})\not{\mathfrak{d}}^{\leq 1}A + \Gamma_g\cdot(B,A)\right)\\
&= (\overline{q} - q)B + O(1)\not{\mathfrak{d}}^{\leq 1}A + r\Gamma_g\cdot(B,A).
\end{aligned}$$

To eliminate $\frac{a\left(q^2 - r\overline{q}\right)}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X}$ we make use of

$$\nabla_4\widehat{X} + \frac{2r}{|q|^2}\widehat{X} = -A + \Gamma_g\cdot\Gamma_g.$$

We infer, using also $\nabla_4\mathfrak{J} = -q^{-1}\mathfrak{J}$, $\mathfrak{J} = O(r^{-1})$ and $q - \overline{q} = O(1)$,

$$\nabla_4\left(a\left(\frac{1}{2}q - \frac{1}{2}\overline{q}\right)\overline{\mathfrak{J}}\cdot\widehat{X}\right)$$

$$\begin{aligned}
&= a\nabla_4\left(\frac{1}{2}q-\frac{1}{2}\overline{q}\right)\overline{\mathfrak{J}}\cdot\widehat{X}+a\left(\frac{1}{2}q-\frac{1}{2}\overline{q}\right)\nabla_4\overline{\mathfrak{J}}\cdot\widehat{X}\\
&\quad +a\left(\frac{1}{2}q-\frac{1}{2}\overline{q}\right)\overline{\mathfrak{J}}\cdot\nabla_4\widehat{X}\\
&= -\frac{a}{\overline{q}}\left(\frac{1}{2}q-\frac{1}{2}\overline{q}\right)\overline{\mathfrak{J}}\cdot\widehat{X}+a\left(\frac{1}{2}q-\frac{1}{2}\overline{q}\right)\overline{\mathfrak{J}}\cdot\left(-\frac{2r}{|q|^2}\widehat{X}-A+\Gamma_g\cdot\Gamma_g\right)\\
&= a\left(-\frac{1}{\overline{q}}-\frac{2r}{|q|^2}\right)\left(\frac{1}{2}q-\frac{1}{2}\overline{q}\right)\overline{\mathfrak{J}}\cdot\widehat{X}+O(r^{-1})A+r^{-1}\Gamma_g\cdot\Gamma_g\\
&= -\frac{a(q^2-r\overline{q})}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X}+O(r^{-1})A+r^{-1}\Gamma_g\cdot\Gamma_g.
\end{aligned}$$

Combining the above three identities we deduce

$$\begin{aligned}
\nabla_4\left([\overline{q}\check{H}]_{red}\right) &= \nabla_4\left(\overline{q}\check{H}-q\check{Z}+\left(-\frac{1}{3}\overline{q}^2+\frac{1}{3}|q|^2\right)B+a\left(\frac{1}{2}q-\frac{1}{2}\overline{q}\right)\overline{\mathfrak{J}}\cdot\widehat{X}\right)\\
&= O(r^{-1})\widecheck{\mathrm{tr}X}+O(1)\,\mathfrak{d}^{\leq 1}A+r\Gamma_b\cdot\Gamma_g
\end{aligned}$$

as desired.

Second identity. To check

$$\nabla_4\left(q[\widecheck{\mathcal{D}\cos\theta}]_{ren}\right) = O(1)\widecheck{\mathrm{tr}X}+O(r)A+r\Gamma_b\cdot\Gamma_g,$$

we start with the equation, see Lemma 6.16,

$$\nabla_4\widecheck{\mathcal{D}\cos\theta}+\frac{1}{q}\widecheck{\mathcal{D}\cos\theta} = \frac{i}{2}\overline{\mathfrak{J}}\cdot\widehat{X}+O(r^{-1})\widecheck{\mathrm{tr}X}+\Gamma_b\cdot\Gamma_g.$$

We infer

$$\nabla_4\Big(q\widecheck{\mathcal{D}\cos\theta}\Big) = \frac{i}{2}q\overline{\mathfrak{J}}\cdot\widehat{X}+O(1)\widecheck{\mathrm{tr}X}+r\Gamma_b\cdot\Gamma_g.$$

To eliminate the term in $\widehat{X}$ we make use of

$$\nabla_4\widehat{X}+\frac{2r}{|q|^2}\widehat{X} = -A+\Gamma_g\cdot\Gamma_g.$$

We infer, using also $\nabla_4\mathfrak{J}=-q^{-1}\mathfrak{J}$ and $\mathfrak{J}=O(r^{-1})$,

$$\nabla_4\left(\frac{i}{2}|q|^2\overline{\mathfrak{J}}\cdot\widehat{X}\right) = \nabla_4\left(\frac{i}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}+\left(\frac{i}{2}|q|^2\right)\nabla_4\overline{\mathfrak{J}}\cdot\widehat{X}$$

$$
\begin{aligned}
& + \left(\frac{i}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\nabla_4\widehat{X}\\
= \quad & \frac{i}{2}(q+\overline{q})\overline{\mathfrak{J}}\cdot\widehat{X} + \left(\frac{i}{2}|q|^2\right)\left(-\frac{1}{\overline{q}}\right)\overline{\mathfrak{J}}\cdot\widehat{X}\\
& + \left(\frac{i}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\left(-\frac{2r}{|q|^2}\widehat{X} - A + \Gamma_g\cdot\Gamma_g\right)\\
= \quad & -\frac{i}{2}q\overline{\mathfrak{J}}\cdot\widehat{X} + O(r)A + r\Gamma_g\cdot\Gamma_g.
\end{aligned}
$$

Summing the two identities above, we infer

$$
\begin{aligned}
\nabla_4\left(q[\widetilde{\mathcal{D}\cos\theta}]_{ren}\right) &= \nabla_4\left(q\widetilde{\mathcal{D}\cos\theta} + \frac{i}{2}|q|^2\overline{\mathfrak{J}}\cdot\widehat{X}\right)\\
&= O(1)\widetilde{\mathrm{tr}X} + O(r)A + r\Gamma_b\cdot\Gamma_g
\end{aligned}
$$

as desired.

Third identity. To check

$$
\nabla_4\left(\overline{q}q^2[\widecheck{M}]_{ren}\right) = O(1)\,\not{\mathfrak{d}}^{\leq 1}\widetilde{\mathrm{tr}X} + O(r)\,\not{\mathfrak{d}}^{\leq 2}A + r^2\,\not{\mathfrak{d}}^{\leq 1}(\Gamma_g\cdot\Gamma_g) + r^3\Gamma_b\cdot A,
$$

we start with the equation

$$
\nabla_4\widecheck{Z} + \frac{2}{q}\widecheck{Z} = -\frac{aq}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X} - B + O(r^{-2})\widetilde{\mathrm{tr}X} + \Gamma_g\cdot\Gamma_g
$$

which we rewrite in the form

$$
\nabla_4(q^2\widecheck{Z}) = -\frac{aq^3}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X} - q^2B + O(1)\widetilde{\mathrm{tr}X} + r^2\Gamma_g\cdot\Gamma_g.
$$

To eliminate the term in $\widehat{X}$ on the right we make use of the equation

$$
\nabla_4\widehat{X} + \frac{2r}{|q|^2}\widehat{X} = -A + \Gamma_g\cdot\Gamma_g.
$$

Using also $\nabla_4\mathfrak{J} = -q^{-1}\mathfrak{J}$ and $\mathfrak{J} = O(r^{-1})$ we infer

$$
\begin{aligned}
& \nabla_4\left(\left(-\frac{a}{2}q^2 - \frac{a}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\right)\\
= \quad & \nabla_4\left(-\frac{a}{2}q^2 - \frac{a}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X} + \left(-\frac{a}{2}q^2 - \frac{a}{2}|q|^2\right)\nabla_4\overline{\mathfrak{J}}\cdot\widehat{X}
\end{aligned}
$$

$$
\begin{aligned}
& + \left(-\frac{a}{2}q^2 - \frac{a}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\nabla_4\widehat{X}\\
= & \left(-aq - \frac{a}{2}(q+\overline{q})\right)\overline{\mathfrak{J}}\cdot\widehat{X} + \left(-\frac{a}{2}q^2 - \frac{a}{2}|q|^2\right)\left(-\frac{1}{q}\right)\overline{\mathfrak{J}}\cdot\widehat{X}\\
& + \left(-\frac{a}{2}q^2 - \frac{a}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\left(-\frac{2r}{|q|^2}\widehat{X} - A + \Gamma_g\cdot\Gamma_g\right)\\
= & \frac{aq^3}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X} + O(r)A + r\Gamma_g\cdot\Gamma_g.
\end{aligned}
$$

Summing the two identities above, we infer

$$
\nabla_4\left(q^2\check{Z} + \left(-\frac{a}{2}q^2 - \frac{a}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\right) = -q^2B + O(1)\widetilde{\mathrm{tr}X} + O(r)A + r^2\Gamma_g\cdot\Gamma_g.
$$

Commuting with $\overline{q}\,\overline{\mathcal{D}}\cdot$ we deduce

$$
\begin{aligned}
& \nabla_4\left(\overline{q}\,\overline{\mathcal{D}}\cdot\left(q^2\check{Z} + \left(-\frac{a}{2}q^2 - \frac{a}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\right)\right)\\
= & [\nabla_4, \overline{q}\,\overline{\mathcal{D}}\cdot]\left(q^2\check{Z} + \left(-\frac{a}{2}q^2 - \frac{a}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\right)\\
& -\overline{q}\,\overline{\mathcal{D}}\cdot(q^2B) + O(1)\,\mathfrak{d}\!\!/^{\leq 1}\widetilde{\mathrm{tr}X} + O(r)\,\mathfrak{d}\!\!/^{\leq 1}A + r^2\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_g\cdot\Gamma_g)\\
= & \left(-\overline{q}\,\overline{\mathrm{tr}X}\,\overline{Z} + \Gamma_g\cdot\mathfrak{d}^{\leq 1}\right)\cdot\left(q^2\check{Z} + \left(-\frac{a}{2}q^2 - \frac{a}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\right)\\
& -\overline{q}\,\overline{\mathcal{D}}\cdot(q^2B) + O(1)\,\mathfrak{d}\!\!/^{\leq 1}\widetilde{\mathrm{tr}X} + O(r)\,\mathfrak{d}\!\!/^{\leq 1}A + r^2\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_g\cdot\Gamma_g)
\end{aligned}
$$

and hence

$$
\begin{aligned}
& \nabla_4\left(\overline{q}\,\overline{\mathcal{D}}\cdot\left(q^2\check{Z} + \left(-\frac{a}{2}q^2 - \frac{a}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\right)\right)\\
= & -\frac{2aq^3}{|q|^2}\overline{\mathfrak{J}}\cdot\check{Z} + \frac{a^2q}{|q|^2}\left(q^2 + |q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\cdot\overline{\mathfrak{J}} - \overline{q}\,\overline{\mathcal{D}}\cdot(q^2B)\\
& +O(1)\,\mathfrak{d}\!\!/^{\leq 1}\widetilde{\mathrm{tr}X} + O(r)\,\mathfrak{d}\!\!/^{\leq 1}A + r^2\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_g\cdot\Gamma_g).
\end{aligned}
$$

Next we make use of

$$
\nabla_4\left(\check{P}\right) - \frac{1}{2}\mathcal{D}\cdot\overline{B} = -\frac{3}{q}\check{P} - \frac{a\overline{q}}{2|q|^2}\mathfrak{J}\cdot\overline{B} + O(r^{-3})\widetilde{\mathrm{tr}X} + r^{-1}\Gamma_g\cdot\Gamma_g + \Gamma_b\cdot A
$$

from which we obtain

$$
\nabla_4(2q^3\check{P}) = q^3\mathcal{D}\cdot\overline{B} - aq^2\mathfrak{J}\cdot\overline{B} + O(1)\widetilde{\mathrm{tr}X} + r^2\Gamma_g\cdot\Gamma_g + r^3\Gamma_b\cdot A.
$$

Taking the complex conjugate, we infer

$$\nabla_4\Big(2\overline{q}^3\,\overline{\check{P}}\Big) = \overline{q}^3\overline{\mathcal{D}}\cdot B - a\overline{q}^2\overline{\mathfrak{J}}\cdot B + O(1)\widecheck{\operatorname{tr}X} + r^2\Gamma_g\cdot\Gamma_g + r^3\Gamma_b\cdot A.$$

Adding to the previous identity, we deduce

$$\begin{aligned}
&\nabla_4\left(\overline{q}\,\overline{\mathcal{D}}\cdot\left(q^2\check{Z} + \left(-\frac{a}{2}q^2 - \frac{a}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\right) + 2\overline{q}^3\,\overline{\check{P}}\right)\\
= & -\frac{2aq^3}{|q|^2}\overline{\mathfrak{J}}\cdot\check{Z} + \frac{a^2q}{|q|^2}\left(q^2+|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\cdot\overline{\mathfrak{J}} - \overline{q}\,\overline{\mathcal{D}}\cdot(q^2B) + \overline{q}^3\overline{\mathcal{D}}\cdot B\\
& - a\overline{q}^2\overline{\mathfrak{J}}\cdot B + O(1)\,\mathfrak{d}^{\leq 1}\widecheck{\operatorname{tr}X} + O(r)\,\mathfrak{d}^{\leq 1}A + r^2\,\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g) + r^3\Gamma_b\cdot A.
\end{aligned}$$

Since we have

$$\begin{aligned}
-\overline{q}\,\overline{\mathcal{D}}\cdot(q^2B) + \overline{q}^3\overline{\mathcal{D}}\cdot B &= \overline{q}(\overline{q}^2 - q^2)\overline{\mathcal{D}}\cdot B - 2|q|^2\overline{\mathcal{D}}q\cdot B\\
&= \overline{q}(\overline{q}^2 - q^2)\overline{\mathcal{D}}\cdot B - 2ai|q|^2\overline{\mathcal{D}}(\cos\theta)\cdot B\\
&= \overline{q}(\overline{q}^2 - q^2)\overline{\mathcal{D}}\cdot B - 2a|q|^2\overline{\mathfrak{J}}\cdot B + r\Gamma_b\cdot\Gamma_g,
\end{aligned}$$

we infer

$$\begin{aligned}
&\nabla_4\left(\overline{q}\,\overline{\mathcal{D}}\cdot\left(q^2\check{Z} + \left(-\frac{a}{2}q^2 - \frac{a}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\right) + 2\overline{q}^3\,\overline{\check{P}}\right)\\
= & -\frac{2aq^3}{|q|^2}\overline{\mathfrak{J}}\cdot\check{Z} + \overline{q}(\overline{q}^2 - q^2)\overline{\mathcal{D}}\cdot B - a\Big(2|q|^2 + \overline{q}^2\Big)\overline{\mathfrak{J}}\cdot B\\
& + \frac{a^2q}{|q|^2}\left(q^2+|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\cdot\overline{\mathfrak{J}} + O(1)\,\mathfrak{d}^{\leq 1}\widecheck{\operatorname{tr}X} + O(r)\,\mathfrak{d}^{\leq 1}A\\
& + r^2\,\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g) + r^3\Gamma_b\cdot A.
\end{aligned}$$

To eliminate the term in $\check{Z}$ on the right we write

$$\begin{aligned}
&\nabla_4\Big(-2aq^2\overline{\mathfrak{J}}\cdot\check{Z}\Big)\\
= & \; a\nabla_4(-2q^2)\overline{\mathfrak{J}}\cdot\check{Z} - 2aq^2\nabla_4\overline{\mathfrak{J}}\cdot\check{Z} - 2aq^2\overline{\mathfrak{J}}\cdot\nabla_4\check{Z}\\
= & -4aq\overline{\mathfrak{J}}\cdot\check{Z} + \frac{2aq^2}{\overline{q}}\overline{\mathfrak{J}}\cdot\check{Z}\\
& -2aq^2\overline{\mathfrak{J}}\cdot\left(-\frac{2}{q}\check{Z} - \frac{aq}{|q|^2}\mathfrak{J}\cdot\widehat{X} - B + O(r^{-2})\widecheck{\operatorname{tr}X} + \Gamma_g\cdot\Gamma_g\right)\\
= & \; \frac{2aq^3}{|q|^2}\overline{\mathfrak{J}}\cdot\check{Z} + \frac{2a^2q^3}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X}\cdot\overline{\mathfrak{J}} + 2aq^2\overline{\mathfrak{J}}\cdot B + O(r^{-1})\widecheck{\operatorname{tr}X} + r\Gamma_g\cdot\Gamma_g.
\end{aligned}$$

Summing with the previous identity, we infer

$$
\begin{aligned}
&\nabla_4\left(\overline{q}\,\overline{\mathcal{D}}\cdot\left(q^2\check{Z}+\left(-\frac{a}{2}q^2-\frac{a}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\right)+2\overline{q}^3\,\overline{\check{P}}-2aq^2\overline{\mathfrak{J}}\cdot\check{Z}\right)\\
&=\overline{q}(\overline{q}^2-q^2)\overline{\mathcal{D}}\cdot B+a\Big(-2|q|^2-\overline{q}^2+2q^2\Big)\overline{\mathfrak{J}}\cdot B\\
&+\frac{a^2q}{|q|^2}\left(3q^2+|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\cdot\overline{\mathfrak{J}}+O(1)\,\mathfrak{d}\mkern-9mu/^{\leq 1}\widetilde{\mathrm{tr}X}+O(r)\,\mathfrak{d}\mkern-9mu/^{\leq 1}A\\
&+r^2\,\mathfrak{d}\mkern-9mu/^{\leq 1}(\Gamma_g\cdot\Gamma_g)+r^3\Gamma_b\cdot A.
\end{aligned}
\tag{C.1}
$$

To eliminate the term in $\overline{\mathcal{D}}\cdot B$ we make use of the equation

$$
\nabla_4 B+\frac{4}{\overline{q}}B \;=\; O(r^{-1})\,\mathfrak{d}\mkern-9mu/^{\leq 1}A+\Gamma_g\cdot(B,A).
$$

Differentiating w.r.t.$\overline{\mathcal{D}}\cdot$, we infer

$$
\begin{aligned}
\nabla_4(\overline{\mathcal{D}}\cdot B) &= [\nabla_4,\overline{\mathcal{D}}\cdot]B+\overline{\mathcal{D}}\cdot\left(-\frac{4}{\overline{q}}B+O(r^{-1})\,\mathfrak{d}\mkern-9mu/^{\leq 1}A+\Gamma_g\cdot(B,A)\right)\\
&= -\frac{1}{2}\overline{\mathrm{tr}X}\Big(\overline{\mathcal{D}}\cdot B+2\overline{Z}\cdot B\Big)+r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}B-\frac{4}{\overline{q}}\overline{\mathcal{D}}\cdot B\\
&\quad+\frac{4}{\overline{q}^2}\overline{\mathcal{D}}(\overline{q})\cdot B+O(r^{-2})\,\mathfrak{d}\mkern-9mu/^{\leq 2}A+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g)\\
&= -\frac{5}{\overline{q}}\overline{\mathcal{D}}\cdot B-\frac{2}{\overline{q}}\overline{Z}\cdot B-\frac{4ai}{\overline{q}^2}\overline{\mathcal{D}}(\cos\theta)\cdot B\\
&\quad+O(r^{-2})\,\mathfrak{d}\mkern-9mu/^{\leq 2}A+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g)
\end{aligned}
$$

and hence

$$
\nabla_4(\overline{\mathcal{D}}\cdot B) \;=\; -\frac{5}{\overline{q}}\overline{\mathcal{D}}\cdot B-\frac{6a}{\overline{q}^2}\overline{\mathfrak{J}}\cdot B+O(r^{-2})\,\mathfrak{d}\mkern-9mu/^{\leq 2}A+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g).
$$

We then compute

$$
\begin{aligned}
&\nabla_4\left(\left(-\frac{1}{3}q^2\overline{q}^2-\frac{1}{3}q\overline{q}^3+\frac{2}{3}\overline{q}^4\right)\overline{\mathcal{D}}\cdot B\right)\\
=\;&\nabla_4\left(-\frac{1}{3}q^2\overline{q}^2-\frac{1}{3}q\overline{q}^3+\frac{2}{3}\overline{q}^4\right)\overline{\mathcal{D}}\cdot B\\
&+\left(-\frac{1}{3}q^2\overline{q}^2-\frac{1}{3}q\overline{q}^3+\frac{2}{3}\overline{q}^4\right)\nabla_4\overline{\mathcal{D}}\cdot B\\
=\;&\left(-\frac{2}{3}q\overline{q}^2-\frac{2}{3}q^2\overline{q}-\frac{1}{3}\overline{q}^3-q\overline{q}^2+\frac{8}{3}\overline{q}^3\right)\overline{\mathcal{D}}\cdot B
\end{aligned}
$$

$$
\begin{aligned}
& +\left(-\frac{1}{3}q^2\overline{q}^2-\frac{1}{3}q\overline{q}^3+\frac{2}{3}\overline{q}^4\right)\\
& \times\left(-\frac{5}{\overline{q}}\overline{\mathcal{D}}\cdot B-\frac{6a}{\overline{q}^2}\overline{\mathfrak{J}}\cdot B+O(r^{-2})\,\not{\mathfrak{d}}^{\leq 2}A+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g)\right)\\
= & \ \overline{q}(q^2-\overline{q}^2)\overline{\mathcal{D}}\cdot B-\frac{6a}{\overline{q}^2}\left(-\frac{1}{3}q^2\overline{q}^2-\frac{1}{3}q\overline{q}^3+\frac{2}{3}\overline{q}^4\right)\overline{\mathfrak{J}}\cdot B\\
& +O(r)\,\not{\mathfrak{d}}^{\leq 2}A+r\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g)
\end{aligned}
$$

where we used in particular the fact that $-\frac{1}{3}q^2\overline{q}^2-\frac{1}{3}q\overline{q}^3+\frac{2}{3}\overline{q}^4=O(r^3)$. Together with (C.1), we deduce, for the intermediary quantity $[\widetilde{M'}]_{red}$ defined by

$$
\begin{aligned}
[\widetilde{M'}]_{red}: \ = & \ \overline{q}\,\overline{\mathcal{D}}\cdot\left(q^2\check{Z}+\left(-\frac{a}{2}q^2-\frac{a}{2}|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\right)+2\overline{q}^3\,\check{\overline{P}}-2aq^2\overline{\mathfrak{J}}\cdot\check{Z}\\
& +\left(-\frac{1}{3}q^2\overline{q}^2-\frac{1}{3}q\overline{q}^3+\frac{2}{3}\overline{q}^4\right)\overline{\mathcal{D}}\cdot B
\end{aligned}
$$

the following identity

(C.2)
$$
\begin{aligned}
\nabla_4\left([\widetilde{M'}]_{red}\right) = & \ a\Big(4q^2-5\overline{q}^2\Big)\overline{\mathfrak{J}}\cdot B+\frac{a^2q}{|q|^2}\left(3q^2+|q|^2\right)\overline{\mathfrak{J}}\cdot\widehat{X}\cdot\overline{\mathfrak{J}}\\
& +O(1)\,\not{\mathfrak{d}}^{\leq 1}\widetilde{\mathrm{tr}X}+O(r)\,\not{\mathfrak{d}}^{\leq 2}A+r^2\,\not{\mathfrak{d}}^{\leq 1}(\Gamma_g\cdot\Gamma_g)+r^3\Gamma_b\cdot A.
\end{aligned}
$$

To eliminate the term in $\mathfrak{J}\cdot B$ on the RHS of (C.2), we compute

$$
\begin{aligned}
& \nabla_4\left(-a\left(-q^2\overline{q}-\frac{2}{3}q\overline{q}^2+\frac{13}{6}\overline{q}^3\right)\overline{\mathfrak{J}}\cdot B\right)\\
= & \ -a\nabla_4\left(-q^2\overline{q}-\frac{2}{3}q\overline{q}^2+\frac{13}{6}\overline{q}^3\right)\overline{\mathfrak{J}}\cdot B\\
& -a\left(-q^2\overline{q}-\frac{2}{3}q\overline{q}^2+\frac{13}{6}\overline{q}^3\right)\nabla_4\overline{\mathfrak{J}}\cdot B\\
& -a\left(-q^2\overline{q}-\frac{2}{3}q\overline{q}^2+\frac{13}{6}\overline{q}^3\right)\overline{\mathfrak{J}}\cdot\nabla_4 B\\
= & \ -a\left(-q^2-\frac{10}{3}|q|^2+\frac{35}{6}\overline{q}^2\right)\overline{\mathfrak{J}}\cdot B\\
& -a\left(-q^2\overline{q}-\frac{2}{3}q\overline{q}^2+\frac{13}{6}\overline{q}^3\right)\left(-\frac{1}{\overline{q}}\right)\overline{\mathfrak{J}}\cdot B\\
& -a\left(-q^2\overline{q}-\frac{2}{3}q\overline{q}^2+\frac{13}{6}\overline{q}^3\right)\overline{\mathfrak{J}}\cdot\left(-\frac{4}{\overline{q}}B+O(r^{-1})\,\not{\mathfrak{d}}^{\leq 1}A+\Gamma_g\cdot(B,A)\right)
\end{aligned}
$$

$$= \quad -a(4q^2 - 5\overline{q}^2)\overline{\mathfrak{J}} \cdot B + O(r)\,\slashed{\mathfrak{d}}^{\leq 1} A + r\Gamma_g \cdot \Gamma_g.$$

Summing with (C.2), we infer

$$\begin{aligned}
& \nabla_4\left\{[\widecheck{M'}]_{ren} + a\left(q^2\overline{q} + \frac{2}{3}q\overline{q}^2 - \frac{13}{6}\overline{q}^3\right)\overline{\mathfrak{J}} \cdot B\right\} \\
= \quad & \frac{a^2 q}{|q|^2}\left(3q^2 + |q|^2\right)\overline{\mathfrak{J}} \cdot \widehat{X} \cdot \overline{\mathfrak{J}} + O(1)\,\slashed{\mathfrak{d}}^{\leq 1}\widecheck{\mathrm{tr} X} + O(r)\,\slashed{\mathfrak{d}}^{\leq 2} A + r^2\,\slashed{\mathfrak{d}}^{\leq 1}(\Gamma_g \cdot \Gamma_g) \\
& + r^3\Gamma_b \cdot A.
\end{aligned}$$

It remains to eliminate the term in $\widehat{X}$. To do this we write

$$\begin{aligned}
& \nabla_4(a^2(q^2 + |q|^2)\overline{\mathfrak{J}} \cdot \widehat{X} \cdot \overline{\mathfrak{J}}) \\
= \quad & a^2\nabla_4(q^2 + |q|^2)\overline{\mathfrak{J}} \cdot \widehat{X} \cdot \overline{\mathfrak{J}} + 2a^2(q^2 + |q|^2)\overline{\mathfrak{J}} \cdot \widehat{X} \cdot \nabla_4\overline{\mathfrak{J}} \\
& + a^2(q^2 + |q|^2)\overline{\mathfrak{J}} \cdot \nabla_4\widehat{X} \cdot \overline{\mathfrak{J}} \\
= \quad & a^2(3q + \overline{q})\overline{\mathfrak{J}} \cdot \widehat{X} \cdot \overline{\mathfrak{J}} + 2a^2(q^2 + |q|^2)\overline{\mathfrak{J}} \cdot \widehat{X} \cdot \left(-\frac{1}{\overline{q}}\overline{\mathfrak{J}}\right) \\
& + a^2(q^2 + |q|^2)\overline{\mathfrak{J}} \cdot \left(-\frac{2r}{|q|^2}\widehat{X} - A + \Gamma_g \cdot \Gamma_g\right) \cdot \overline{\mathfrak{J}} \\
= \quad & -\frac{a^2 q}{|q|^2}(3q^2 + |q|^2)\overline{\mathfrak{J}} \cdot \widehat{X} \cdot \overline{\mathfrak{J}} + O(1)A + \Gamma_g \cdot \Gamma_g.
\end{aligned}$$

Summing with the previous identity above, we infer that

$$\begin{aligned}
& \nabla_4\left([\widecheck{M}]_{ren}\right) \\
= \quad & \nabla_4\left\{[\widecheck{M'}]_{ren} + a\left(q^2\overline{q} + \frac{2}{3}q\overline{q}^2 - \frac{13}{6}\overline{q}^3\right)\overline{\mathfrak{J}} \cdot B + a^2(q^2 + |q|^2)\overline{\mathfrak{J}} \cdot \widehat{X} \cdot \overline{\mathfrak{J}}\right\} \\
= \quad & O(1)\,\slashed{\mathfrak{d}}^{\leq 1}\widecheck{\mathrm{tr} X} + O(r)\,\slashed{\mathfrak{d}}^{\leq 2} A + r^2\,\slashed{\mathfrak{d}}^{\leq 1}(\Gamma_g \cdot \Gamma_g) + r^3\Gamma_b \cdot A
\end{aligned}$$

as stated. This concludes the proof of Proposition 6.41.

C.6. Proof of Lemma 6.44

The goal of this section is to prove the following identity

$$\left(\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b\right)\left(r[\overline{\mathcal{D}}\cdot]_{ren}(r^4[B]_{ren})\right) = \frac{r^5}{2}\overline{\mathcal{D}}' \cdot \overline{\mathcal{D}}' \cdot \left(A - a(\mathfrak{J}\widehat{\otimes}B)\right) + \mathrm{Err},$$

where the $\mathcal{D}'$ is taken with respect to the integral frame (e_1', e_2') adapted to $S(u,r)$, see Section 6.2.1, and where the error term has the following structure

$$\begin{aligned}\mathrm{Err} =& O(1)\mathfrak{d}^{\leq 1}\widehat{X} + O(r)\mathfrak{d}^{\leq 2}B + O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 B + O(r)\mathfrak{d}^{\leq 2}\check{P} + O(1)\mathfrak{d}^{\leq 1}\widetilde{\mathrm{tr} X}\\ &+ O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 A + O(r)\mathfrak{d}^{\leq 2}A + r^4\mathfrak{d}^{\leq 1}(\Gamma_g\cdot(B,A)) + r^4\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\nabla_3 A)\\ &+ r^2\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_g).\end{aligned}$$

Step 1. We start with the following lemma.

Lemma C.3. *We have*

$$\begin{aligned}&\Big(\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b\Big)[B]_{ren} + \frac{4}{q}[B]_{ren}\\ =\;& -\frac{3a}{4}(\overline{\mathcal{D}}\cdot B)\mathfrak{J} - a\Re(\mathfrak{J})^b\nabla_b B + \frac{1}{2}\overline{\mathcal{D}}\cdot A - \frac{a}{4}\overline{\mathfrak{J}}\cdot\nabla_4 A - \frac{a}{4\overline{q}}\overline{\mathfrak{J}}\cdot A\\ &+O(r^{-3})B + O(r^{-3})\,\mathfrak{\not{d}}^{\leq 1}\check{P} + O(r^{-3})\,\mathfrak{\not{d}}^{\leq 1}A + O(r^{-4})\widetilde{trX} + \Gamma_g\cdot(B,A)\\ &+r^{-2}\Gamma_g\cdot\Gamma_g.\end{aligned}$$

Proof. We start with the linearized Bianchi equations

$$\begin{aligned}\nabla_4 B + \frac{4}{\overline{q}}B &= \frac{1}{2}\overline{\mathcal{D}}\cdot A + \frac{aq}{2|q|^2}\overline{\mathfrak{J}}\cdot A + \Gamma_g\cdot(B,A),\\ \nabla_4\left(\check{P}\right) - \frac{1}{2}\mathcal{D}\cdot\overline{B} &= -\frac{3}{q}\check{P} - \frac{a\overline{q}}{2|q|^2}\mathfrak{J}\cdot\overline{B} + O(r^{-3})\widetilde{\mathrm{tr} X} + r^{-1}\Gamma_g\cdot\Gamma_g\\ &\quad+\Gamma_b\cdot A.\end{aligned}$$

We infer, using $\nabla_4\mathfrak{J} = -q^{-1}\mathfrak{J}$,

$$\begin{aligned}&\nabla_4([B]_{ren})\\ =\;& \nabla_4 B - \frac{3a}{2}\mathfrak{J}\nabla_4\overline{\check{P}} - \frac{3a}{2}\overline{\check{P}}\nabla_4\mathfrak{J} - \frac{a}{4}\overline{\mathfrak{J}}\cdot\nabla_4 A - \frac{a}{4}A\cdot\nabla_4\overline{\mathfrak{J}}\\ =\;& -\frac{4}{\overline{q}}B + \frac{1}{2}\overline{\mathcal{D}}\cdot A + \frac{aq}{2|q|^2}\overline{\mathfrak{J}}\cdot A + \Gamma_g\cdot(B,A)\\ &-\frac{3a}{2}\mathfrak{J}\left(\frac{1}{2}\overline{\mathcal{D}}\cdot B - \frac{3}{\overline{q}}\overline{\check{P}} - \frac{aq}{2|q|^2}\overline{\mathfrak{J}}\cdot B + O(r^{-3})\widetilde{\mathrm{tr} X} + r^{-1}\Gamma_g\cdot\Gamma_g + \Gamma_b\cdot A\right)\\ &+\frac{3a}{2q}\overline{\check{P}}\mathfrak{J} - \frac{a}{4}\overline{\mathfrak{J}}\cdot\nabla_4 A + \frac{a}{4\overline{q}}\overline{\mathfrak{J}}\cdot A\\ =\;& -\frac{4}{\overline{q}}[B]_{ren} - \frac{3a}{2}\mathfrak{J}\left(\frac{1}{2}\overline{\mathcal{D}}\cdot B - \frac{aq}{2|q|^2}\overline{\mathfrak{J}}\cdot B\right) + \frac{3a}{2}\left(\frac{1}{q} - \frac{1}{\overline{q}}\right)\overline{\check{P}}\mathfrak{J}\end{aligned}$$

$$+\frac{1}{2}\overline{\mathcal{D}}\cdot A-\frac{a}{4}\overline{\mathfrak{J}}\cdot\nabla_4A-\frac{a}{4\overline{q}}\overline{\mathfrak{J}}\cdot A+O(r^{-4})\widecheck{\mathrm{tr}X}+\Gamma_g\cdot(B,A)$$
$$+r^{-2}\Gamma_g\cdot\Gamma_g$$

and hence

$$\begin{aligned}\nabla_4([B]_{ren})+\frac{4}{q}[B]_{ren} &= -\frac{3a}{4}(\overline{\mathcal{D}}\cdot B)\mathfrak{J}+\frac{1}{2}\overline{\mathcal{D}}\cdot A-\frac{a}{4}\overline{\mathfrak{J}}\cdot\nabla_4A-\frac{a}{4\overline{q}}\overline{\mathfrak{J}}\cdot A\\ &+O(r^{-3})B+O(r^{-3})\check{P}+O(r^{-4})\widecheck{\mathrm{tr}X}\\ &+\Gamma_g\cdot(B,A)+r^{-2}\Gamma_g\cdot\Gamma_g.\end{aligned}$$

Also, we have

$$a\Re(\mathfrak{J})^b\nabla_b[B]_{ren} = a\Re(\mathfrak{J})^b\nabla_bB-\frac{3a^2}{2}\Re(\mathfrak{J})^b\nabla_b(\overline{\check{P}}\mathfrak{J})+O(r^{-3})\,\mathfrak{d}\!\!/^{\leq 1}A$$

and hence

$$\begin{aligned}&\Big(\nabla_4-a\Re(\mathfrak{J})^b\nabla_b\Big)[B]_{ren}+\frac{4}{q}[B]_{ren}\\ =\ & -\frac{3a}{4}(\overline{\mathcal{D}}\cdot B)\mathfrak{J}-a\Re(\mathfrak{J})^b\nabla_bB+\frac{1}{2}\overline{\mathcal{D}}\cdot A-\frac{a}{4}\overline{\mathfrak{J}}\cdot\nabla_4A-\frac{a}{4\overline{q}}\overline{\mathfrak{J}}\cdot A\\ &+O(r^{-3})B+O(r^{-3})\,\mathfrak{d}\!\!/^{\leq 1}\check{P}+O(r^{-3})\,\mathfrak{d}\!\!/^{\leq 1}A+O(r^{-4})\widecheck{\mathrm{tr}X}+\Gamma_g\cdot(B,A)\\ &+r^{-2}\Gamma_g\cdot\Gamma_g\end{aligned}$$

as desired. This concludes the proof of Lemma C.3. ◻

Step 2. Next, we prove the following lemma.

Lemma C.4. *The following holds true, with ∇' the covariant derivative on $S(u,r)$ and $\mathcal{D}'$ the corresponding complex operator,*

$$\begin{aligned}&\overline{\mathcal{D}}\cdot A-\frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_4A-\frac{a}{2\overline{q}}\overline{\mathfrak{J}}\cdot A\\ =\ &\overline{\mathcal{D}}'\cdot A+\frac{a}{4}\overline{\mathfrak{J}}\cdot\mathcal{D}\widehat{\otimes}B+O(r^{-3})B+O(r^{-4})\widehat{X}+O(r^{-2})\nabla_3A\\ &+O(r^{-3})\mathfrak{d}^{\leq 1}A+\Gamma_b\cdot\nabla_3A+r^{-1}\Gamma_b\cdot(A,B)+r^{-2}\Gamma_g\cdot\Gamma_g.\end{aligned}$$

Proof. We make use of the formula

$$\begin{aligned}\mathcal{D}'_aV_B=\ &\mathcal{D}_aV_B+\frac{1}{2}\underline{F}_af^c\nabla_cV_B+\frac{1}{2}\underline{F}_a\nabla_4V_B+\left(\frac{1}{2}F_a+\frac{1}{8}|f|^2\underline{F}_a\right)\nabla_3V_B\\ &+\Big\{O(r^{-3})+r^{-1}\Gamma_b\Big\}V\end{aligned}\tag{C.3}$$

which follows from the last identity of Proposition 6.29, i.e.

$$\mathcal{D}'_a V_B = \mathcal{D}_a V_B + \frac{1}{2}\underline{F}_a f^c \nabla_c V_B + \frac{1}{2}\underline{F}_a \nabla_4 V_B + \left(\frac{1}{2}F_a + \frac{1}{8}|f|^2\underline{F}_a\right)\nabla_3 V_B + (E[V])_{aB},$$

and formula (6.33) for $(E[V])_{aB}$ derived in the proof of Corollary 6.30. Here, the horizontal 1-forms f and $\underline{f}$ are given by (6.32) and

$$F = f + i\,{}^*f, \qquad \underline{F} = \underline{f} + i\,{}^*\underline{f}.$$

In view of (C.3) we deduce

$$\overline{\mathcal{D}}' \cdot A = \overline{\mathcal{D}} \cdot A + \frac{1}{2}\overline{\underline{F}} \cdot \nabla_4 A + \frac{1}{2}\overline{F} \cdot \nabla_3 A + O(r^{-3})\mathfrak{d}^{\leq 1}A + r^{-1}\Gamma_b \cdot A.$$

Also, in view of (6.32), the horizontal 1-forms f and $\underline{f}$ satisfy

$$f = \Big(-1 + O(r^{-1}) + r\Gamma_b\Big)\nabla u, \qquad \underline{f} = \Big(-1 + O(r^{-1}) + r\Gamma_b\Big)\nabla u$$

and hence

$$\begin{aligned} F &= \Big(-1 + O(r^{-1}) + r\Gamma_b\Big)\mathcal{D}u = -a\mathfrak{J} + O(r^{-2}) + \Gamma_b, \\ \underline{F} &= \Big(-1 + O(r^{-1}) + r\Gamma_b\Big)\mathcal{D}u = -a\mathfrak{J} + O(r^{-2}) + \Gamma_b. \end{aligned}$$

We infer

$$\begin{aligned} \overline{\mathcal{D}}' \cdot A &= \overline{\mathcal{D}} \cdot A - \frac{a}{2}\overline{\mathfrak{J}} \cdot \nabla_4 A - \frac{a}{2}\overline{\mathfrak{J}} \cdot \nabla_3 A + O(r^{-2})\nabla_3 A + O(r^{-3})\mathfrak{d}^{\leq 1}A \\ &\quad + \Gamma_b \cdot \nabla_3 A + r^{-1}\Gamma_b \cdot A. \end{aligned}$$

Making use of the Bianchi identity

$$\begin{aligned} \nabla_3 A &= \frac{1}{2}\mathcal{D}\widehat{\otimes}B - \frac{1}{2}\mathrm{tr}\underline{X}A + 4\underline{\omega}A + \frac{1}{2}(Z + 4H)\widehat{\otimes}B - 3\overline{P}\widehat{X} \\ &= \frac{1}{2}\mathcal{D}\widehat{\otimes}B + \frac{1}{r}A + \frac{a(\overline{q} + 4q)}{2|q|^2}\mathfrak{J}\widehat{\otimes}B + \frac{6m}{\overline{q}^3}\widehat{X} + O(r^{-2})A + \Gamma_b \cdot (A, B) \\ &\quad + r^{-1}\Gamma_g \cdot \Gamma_g, \end{aligned}$$

we deduce

$$\overline{\mathcal{D}}' \cdot A = \overline{\mathcal{D}} \cdot A - \frac{a}{2}\overline{\mathfrak{J}} \cdot \nabla_4 A - \frac{a}{2r}\overline{\mathfrak{J}} \cdot A - \frac{a}{4}\overline{\mathfrak{J}} \cdot \mathcal{D}\widehat{\otimes}B - \frac{a^2(\overline{q} + 4q)}{4|q|^2}\overline{\mathfrak{J}} \cdot (\mathfrak{J}\widehat{\otimes}B)$$

$$-\frac{3am}{\overline{q}^3}\overline{\mathfrak{J}}\cdot\widehat{X}+O(r^{-2})\nabla_3 A+O(r^{-3})\mathfrak{d}^{\leq 1}A+\Gamma_b\cdot\nabla_3 A$$
$$+r^{-1}\Gamma_b\cdot(A,B)+r^{-2}\Gamma_g\cdot\Gamma_g$$

and hence

$$\begin{aligned}
&\overline{\mathcal{D}}\cdot A-\frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_4 A-\frac{a}{2\overline{q}}\overline{\mathfrak{J}}\cdot A\\
=\;&\overline{\mathcal{D}}'\cdot A+\frac{a}{4}\overline{\mathfrak{J}}\cdot\mathcal{D}\widehat{\otimes}B+O(r^{-3})B+O(r^{-4})\widehat{X}\\
&+O(r^{-2})\nabla_3 A+O(r^{-3})\mathfrak{d}^{\leq 1}A+\Gamma_b\cdot\nabla_3 A+r^{-1}\Gamma_b\cdot(A,B)+r^{-2}\Gamma_g\cdot\Gamma_g
\end{aligned}$$

as stated. This concludes the proof of Lemma C.4. □

Step 3. Next, we derive the following corollary.

Corollary C.5. *We have*

$$\begin{aligned}
&\Big(\nabla_4-a\Re(\mathfrak{J})^b\nabla_b\Big)[B]_{red}+\frac{4}{r}[B]_{red}\\
=\;&\frac{1}{2}\overline{\mathcal{D}}'\cdot A-\frac{3a}{4}(\overline{\mathcal{D}}\cdot B)\mathfrak{J}-a\Re(\mathfrak{J})^b\nabla_b B+\frac{a}{8}\overline{\mathfrak{J}}\cdot\mathcal{D}\widehat{\otimes}B-\frac{4ia\cos\theta}{r^2}B\\
&+O(r^{-4})\widehat{X}+O(r^{-3})B+O(r^{-3})\mathfrak{d}^{\leq 1}\check{P}+O(r^{-4})\widecheck{trX}\\
&+O(r^{-2})\nabla_3 A+O(r^{-3})\mathfrak{d}^{\leq 1}A+\Gamma_g\cdot(B,A)+\Gamma_b\cdot\nabla_3 A+r^{-2}\Gamma_g\cdot\Gamma_g.
\end{aligned}$$

Proof. Plugging the identity of Lemma C.4 in the identity of Lemma C.3, we obtain

$$\begin{aligned}
&\Big(\nabla_4-a\Re(\mathfrak{J})^b\nabla_b\Big)[B]_{red}+\frac{4}{q}[B]_{red}\\
=\;&\frac{1}{2}\overline{\mathcal{D}}'\cdot A-\frac{3a}{4}(\overline{\mathcal{D}}\cdot B)\mathfrak{J}-a\Re(\mathfrak{J})^b\nabla_b B+\frac{a}{8}\overline{\mathfrak{J}}\cdot\mathcal{D}\widehat{\otimes}B\\
&+O(r^{-4})\widehat{X}+O(r^{-3})B+O(r^{-3})\,\not{\mathfrak{d}}^{\leq 1}\check{P}+O(r^{-4})\widecheck{\mathrm{tr}X}\\
&+O(r^{-2})\nabla_3 A+O(r^{-3})\mathfrak{d}^{\leq 1}A+\Gamma_g\cdot(B,A)+\Gamma_b\cdot\nabla_3 A+r^{-2}\Gamma_g\cdot\Gamma_g.
\end{aligned}$$

Since

$$\frac{4}{q}[B]_{red}\;=\;\frac{4}{r}[B]_{red}+\frac{4ia\cos\theta}{r^2}B+O(r^{-3})\check{P}+O(r^{-3})B+O(r^{-3})A,$$

we deduce

$$\Big(\nabla_4-a\Re(\mathfrak{J})^b\nabla_b\Big)[B]_{red}+\frac{4}{r}[B]_{red}$$

$$
\begin{aligned}
= & \ \frac{1}{2}\overline{\mathcal{D}}'\cdot A - \frac{3a}{4}(\overline{\mathcal{D}}\cdot B)\mathfrak{J} - a\Re(\mathfrak{J})^b\nabla_b B + \frac{a}{8}\overline{\mathfrak{J}}\cdot\mathcal{D}\widehat{\otimes}B - \frac{4ia\cos\theta}{r^2}B \\
& +O(r^{-4})\widehat{X} + O(r^{-3})B + O(r^{-3})\mathfrak{d}^{\leq 1}\check{P} + O(r^{-4})\widetilde{\mathrm{tr} X} \\
& +O(r^{-2})\nabla_3 A + O(r^{-3})\mathfrak{d}^{\leq 1}A + \Gamma_g\cdot(B,A) + \Gamma_b\cdot\nabla_3 A + r^{-2}\Gamma_g\cdot\Gamma_g
\end{aligned}
$$

as desired. This concludes the proof of Corollary C.5. □

Step 4. Next, we derive the following lemma.

Lemma C.6. *We have*

$$
\begin{aligned}
\overline{\mathcal{D}}\cdot(\mathfrak{J}\widehat{\otimes}B) \ = & \ -\frac{1}{4}\overline{\mathfrak{J}}\cdot(\mathcal{D}\widehat{\otimes}B) + \frac{3}{2}(\overline{\mathcal{D}}\cdot B)\mathfrak{J} + 2\Re(\mathfrak{J})^b\nabla_b B \\
& + \frac{8i(r^2+a^2)\cos\theta}{|q|^4}B + r^{-1}\Gamma_b\cdot B.
\end{aligned}
$$

Proof. The proof follows immediately by combining the following three identities

$$
\begin{aligned}
\overline{\mathcal{D}}\cdot(\mathfrak{J}\widehat{\otimes}B)_b = & \ (\overline{\mathcal{D}}\cdot\mathfrak{J})B_b + \delta_{cd}\overline{\mathcal{D}}_c\mathfrak{J}_b B_d - \overline{\mathcal{D}}_b\mathfrak{J}\cdot B + \mathfrak{J}_b\overline{\mathcal{D}}\cdot B \\
& + \delta_{cd}\mathfrak{J}_d\overline{\mathcal{D}}_c B_b - \mathfrak{J}\cdot\overline{\mathcal{D}}_b B,
\end{aligned} \tag{C.4}
$$

$$
\begin{aligned}
& (\overline{\mathcal{D}}\cdot\mathfrak{J})B_b + \delta_{cd}\overline{\mathcal{D}}_c\mathfrak{J}_b B_d - \overline{\mathcal{D}}_b\mathfrak{J}\cdot B \\
= & \ \frac{8i(r^2+a^2)\cos\theta}{|q|^4}B_b + r^{-1}\Gamma_b\cdot B,
\end{aligned} \tag{C.5}
$$

and

$$
\begin{aligned}
& \mathfrak{J}_b\overline{\mathcal{D}}\cdot B + \delta_{cd}\mathfrak{J}_d\overline{\mathcal{D}}_c B_b - \mathfrak{J}\cdot\overline{\mathcal{D}}_b B \\
= & \ \left(-\frac{1}{4}\overline{\mathfrak{J}}\cdot(\mathcal{D}\widehat{\otimes}B) + \frac{3}{2}(\overline{\mathcal{D}}\cdot B)\mathfrak{J} + 2\Re(\mathfrak{J})^c\nabla_c B\right)_b.
\end{aligned} \tag{C.6}
$$

Thus, from now on, we focus on the proof of (C.4), (C.5) and (C.6). To prove (C.4), we write

$$
\begin{aligned}
\overline{\mathcal{D}}\cdot(\mathfrak{J}\widehat{\otimes}B)_b = & \ \delta_{cd}\overline{\mathcal{D}}_c(\mathfrak{J}\widehat{\otimes}B)_{db} = \delta_{cd}(\overline{\mathcal{D}}_c\mathfrak{J}\widehat{\otimes}B)_{db} + \delta_{cd}(\mathfrak{J}\widehat{\otimes}\overline{\mathcal{D}}_c B)_{db} \\
= & \ \delta_{cd}\Big(\overline{\mathcal{D}}_c\mathfrak{J}_d B_b + \overline{\mathcal{D}}_c\mathfrak{J}_b B_d - \delta_{bd}\overline{\mathcal{D}}_c\mathfrak{J}\cdot B\Big) \\
& + \delta_{cd}\Big(\mathfrak{J}_b\overline{\mathcal{D}}_c B_d + \mathfrak{J}_d\overline{\mathcal{D}}_c B_b - \delta_{bd}\mathfrak{J}\cdot\overline{\mathcal{D}}_c B\Big) \\
= & \ (\overline{\mathcal{D}}\cdot\mathfrak{J})B_b + \delta_{cd}\overline{\mathcal{D}}_c\mathfrak{J}_b B_d - \overline{\mathcal{D}}_b\mathfrak{J}\cdot B + \mathfrak{J}_b\overline{\mathcal{D}}\cdot B + \delta_{cd}\mathfrak{J}_d\overline{\mathcal{D}}_c B_b \\
& -\mathfrak{J}\cdot\overline{\mathcal{D}}_b B,
\end{aligned}
$$

as stated.

Next, to prove (C.5), we write, since $\Im(\mathfrak{J}) = \Re({}^*\mathfrak{J})$,

$$\begin{aligned}
\overline{\mathcal{D}}\cdot\mathfrak{J} &= (\nabla - i\,{}^*\nabla)\cdot(\Re(\mathfrak{J}) + i\Im(\mathfrak{J})) = 2\mathrm{div}\,(\Re(\mathfrak{J})) + 2i\mathrm{curl}\,(\Re(\mathfrak{J})),\\
\mathcal{D}\widehat{\otimes}\mathfrak{J} &= (\nabla + i\,{}^*\nabla)\widehat{\otimes}(\Re(\mathfrak{J}) + i\Im(\mathfrak{J})) = 2\nabla\widehat{\otimes}\Re(\mathfrak{J}) + 2i\,{}^*\nabla\Re(\mathfrak{J}),
\end{aligned}$$

so that

$$\begin{aligned}
&\nabla\widehat{\otimes}\Re(\mathfrak{J}) \in r^{-1}\Gamma_b, \qquad \mathrm{div}\,(\Re(\mathfrak{J})) \in r^{-1}\Gamma_b,\\
&\mathrm{curl}\,(\Re(\mathfrak{J})) = \frac{2(r^2+a^2)\cos\theta}{|q|^4} + r^{-1}\Gamma_b.
\end{aligned}$$

This yields $\nabla_a\Re(\mathfrak{J})_b = \frac{(r^2+a^2)\cos\theta}{|q|^4}\in_{ab} + r^{-1}\Gamma_b$ and hence

$$\begin{aligned}
\overline{\mathcal{D}}_a\mathfrak{J}_b &= (\nabla_a - i\in_{ac}\nabla_c)(\Re(\mathfrak{J})_b + i\in_{bd}\Re(\mathfrak{J})_d)\\
&= \nabla_a\Re(\mathfrak{J})_b + \in_{ac}\in_{bd}\nabla_c\Re(\mathfrak{J})_d + i\Big(\in_{bd}\nabla_a\Re(\mathfrak{J})_d - \in_{ac}\nabla_c\Re(\mathfrak{J})_b\Big)\\
&= \frac{(r^2+a^2)\cos\theta}{|q|^4}\Big[\in_{ab} + \in_{ac}\in_{bd}\in_{cd} + i(\in_{bd}\in_{ad} - \in_{ac}\in_{cb})\Big] + r^{-1}\Gamma_b\\
&= \frac{2(r^2+a^2)\cos\theta}{|q|^4}\Big[\in_{ab} + i\delta_{ab}\Big] + r^{-1}\Gamma_b.
\end{aligned}$$

We infer

$$\begin{aligned}
&(\overline{\mathcal{D}}\cdot\mathfrak{J})B_b + \delta_{cd}\overline{\mathcal{D}}_c\mathfrak{J}_bB_d - \overline{\mathcal{D}}_b\mathfrak{J}\cdot B_b\\
=\;& \frac{4i(r^2+a^2)\cos\theta}{|q|^4}B_b + \delta_{cd}\frac{2(r^2+a^2)\cos\theta}{|q|^4}\Big[\in_{cb} + i\delta_{cb}\Big]B_d\\
&-\frac{2(r^2+a^2)\cos\theta}{|q|^4}\Big[\in_{bc} + i\delta_{bc}\Big]B_c + r^{-1}\Gamma_b\cdot B\\
=\;& \frac{4i(r^2+a^2)\cos\theta}{|q|^4}B_b + \frac{2(r^2+a^2)\cos\theta}{|q|^4}\Big[\in_{db} + i\delta_{db}\Big]B_d\\
&-\frac{2(r^2+a^2)\cos\theta}{|q|^4}\Big[\in_{bc} + i\delta_{bc}\Big]B_c + r^{-1}\Gamma_b\cdot B\\
=\;& \frac{4i(r^2+a^2)\cos\theta}{|q|^4}B_b - \frac{4(r^2+a^2)\cos\theta}{|q|^4}\,{}^*B_b + r^{-1}\Gamma_b\cdot B
\end{aligned}$$

and hence, since ${}^*B = -iB$, we obtain

$$(\overline{\mathcal{D}}\cdot\mathfrak{J})B_b + \delta_{cd}\overline{\mathcal{D}}_c\mathfrak{J}_bB_d - \overline{\mathcal{D}}_b\mathfrak{J}\cdot B_b = \frac{8i(r^2+a^2)\cos\theta}{|q|^4}B_b + r^{-1}\Gamma_b\cdot B$$

as stated in (C.5).

Next, to prove (C.6), we write

$$\begin{aligned}
&\mathfrak{J}_b\overline{\mathcal{D}}\cdot B+\delta_{cd}\mathfrak{J}_d\overline{\mathcal{D}}_cB_b-\mathfrak{J}\cdot\overline{\mathcal{D}}_bB-\left(-\frac{1}{4}\overline{\mathfrak{J}}\cdot(\mathcal{D}\widehat{\otimes}B)+\frac{3}{2}(\overline{\mathcal{D}}\cdot B)\mathfrak{J}\right)_b\\
&=-\frac{1}{2}(\overline{\mathcal{D}}\cdot B)\mathfrak{J}_b+\frac{1}{4}(\overline{\mathfrak{J}}\cdot(\mathcal{D}\widehat{\otimes}B))_b+\delta_{cd}\mathfrak{J}_d\overline{\mathcal{D}}_cB_b-\mathfrak{J}\cdot\overline{\mathcal{D}}_bB.
\end{aligned} \tag{C.7}$$

Next we make use of the following identity

$$\begin{aligned}
&-\frac{1}{2}(\overline{\mathcal{D}}\cdot B)\mathfrak{J}_b+\frac{1}{4}(\overline{\mathfrak{J}}\cdot(\mathcal{D}\widehat{\otimes}B))_b+\delta_{cd}\mathfrak{J}_d\overline{\mathcal{D}}_cB_b-\mathfrak{J}\cdot\overline{\mathcal{D}}_bB\\
&=\ 2\Re(\mathfrak{J})^c\nabla_cB_b.
\end{aligned} \tag{C.8}$$

Using (C.8), the identity (C.7) becomes

$$\begin{aligned}
&\mathfrak{J}_b\overline{\mathcal{D}}\cdot B+\delta_{cd}\mathfrak{J}_d\overline{\mathcal{D}}_cB_b-\mathfrak{J}\cdot\overline{\mathcal{D}}_bB-\left(-\frac{1}{4}\overline{\mathfrak{J}}\cdot(\mathcal{D}\widehat{\otimes}B)+\frac{3}{2}(\overline{\mathcal{D}}\cdot B)\mathfrak{J}\right)_b\\
=\ &2\Re(\mathfrak{J})^c\nabla_cB_b
\end{aligned}$$

which we rewrite in the form

$$\begin{aligned}
&\mathfrak{J}_b\overline{\mathcal{D}}\cdot B+\delta_{cd}\mathfrak{J}_d\overline{\mathcal{D}}_cB_b-\mathfrak{J}\cdot\overline{\mathcal{D}}_bB\\
=\ &\left(-\frac{1}{4}\overline{\mathfrak{J}}\cdot(\mathcal{D}\widehat{\otimes}B)+\frac{3}{2}(\overline{\mathcal{D}}\cdot B)\mathfrak{J}+2\Re(\mathfrak{J})^c\nabla_cB\right)_b
\end{aligned}$$

as stated in (C.6).

It only remains to check the identity (C.8). Since $B=\beta+i\,{}^*\beta$ and $\mathfrak{J}_2=-i\mathfrak{J}_1$, we have

$$\begin{aligned}
&\left(-\frac{1}{2}(\overline{\mathcal{D}}\cdot B)\mathfrak{J}_b+\frac{1}{4}(\overline{\mathfrak{J}}\cdot(\mathcal{D}\widehat{\otimes}B))_b+\delta_{cd}\mathfrak{J}_d\overline{\mathcal{D}}_cB_b-\mathfrak{J}\cdot\overline{\mathcal{D}}_bB\right)_{b=1}\\
=\ &-\mathfrak{J}_1(\mathrm{div}\,\beta+i\mathrm{curl}\,\beta)\\
&+\frac{1}{4}\overline{\mathfrak{J}_1}\Big((\nabla_1+i\nabla_2)(\beta_1+i\beta_2)-(\nabla_2-i\nabla_1)(\beta_2-i\beta_1)\Big)\\
&+\frac{1}{4}\overline{\mathfrak{J}_2}\Big((\nabla_1+i\nabla_2)(\beta_2-i\beta_1)+(\nabla_2-i\nabla_1)(\beta_1+i\beta_2)\Big)\\
&+\mathfrak{J}_1(\nabla_1-i\nabla_2)(\beta_1+i\beta_2)+\mathfrak{J}_2(\nabla_2+i\nabla_1)(\beta_1+i\beta_2)\\
&-\mathfrak{J}_1(\nabla_1-i\nabla_2)(\beta_1+i\beta_2)-\mathfrak{J}_2(\nabla_1-i\nabla_2)(\beta_2-i\beta_1)\\
=\ &-\mathfrak{J}_1(\mathrm{div}\,\beta+i\mathrm{curl}\,\beta)\\
&+\frac{1}{2}\overline{\mathfrak{J}_1}\Big(\nabla_1\beta_1-\nabla_2\beta_2+i(\nabla_1\beta_2+\nabla_2\beta_1)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{i}{2}\overline{\mathfrak{J}_1}\Big(\nabla_1\beta_2+\nabla_2\beta_1+i(\nabla_2\beta_2-\nabla_1\beta_1)\Big)\\
&+\mathfrak{J}_1\Big(\operatorname{div}\beta+i\operatorname{curl}\beta\Big)-i\mathfrak{J}_1\Big(-\operatorname{curl}\beta+i\operatorname{div}\beta\Big)\\
&-\mathfrak{J}_1\Big(\operatorname{div}\beta+i\operatorname{curl}\beta\Big)+i\mathfrak{J}_1\Big(\operatorname{curl}\beta-i\operatorname{div}\beta\Big)\\
=\;&\overline{\mathfrak{J}_1}\Big(\nabla_1\beta_1-\nabla_2\beta_2+i(\nabla_1\beta_2+\nabla_2\beta_1)\Big)+\mathfrak{J}_1\Big(\operatorname{div}\beta+i\operatorname{curl}\beta\Big).
\end{aligned}
$$

Since $\Im(\mathfrak{J})={}^*\Re(\mathfrak{J})$, we obtain

$$
\begin{aligned}
&\left(-\frac{1}{2}(\overline{\mathcal{D}}\cdot B)\mathfrak{J}_b+\frac{1}{4}(\overline{\mathfrak{J}}\cdot(\mathcal{D}\widehat{\otimes}B))_b+\delta_{cd}\mathfrak{J}_d\overline{\mathcal{D}}_cB_b-\mathfrak{J}\cdot\overline{\mathcal{D}}_bB\right)_{b=1}\\
=\;&\Big(\Re(\mathfrak{J})_1-i\Re(\mathfrak{J})_2\Big)\Big(\nabla_1\beta_1-\nabla_2\beta_2+i(\nabla_1\beta_2+\nabla_2\beta_1)\Big)\\
&+\Big(\Re(\mathfrak{J})_1+i\Re(\mathfrak{J})_2\Big)\Big(\operatorname{div}\beta+i\operatorname{curl}\beta\Big)\\
=\;&2\Re(\mathfrak{J})_1\Big(\nabla_1\beta_1+i\nabla_1\beta_2\Big)+2\Re(\mathfrak{J})_2\Big(\nabla_2\beta_1+i\nabla_2\beta_2\Big)\\
=\;&\Big(2\Re(\mathfrak{J})^c\nabla_cB_b\Big)_{b=1}.
\end{aligned}
$$

Since the complex tensors on the LHS and RHS verify $V_2=-iV_1$, the equality holds also for $b=2$ and hence we have obtained

$$
-\frac{1}{2}(\overline{\mathcal{D}}\cdot B)\mathfrak{J}_b+\frac{1}{4}(\overline{\mathfrak{J}}\cdot(\mathcal{D}\widehat{\otimes}B))_b+\delta_{cd}\mathfrak{J}_d\overline{\mathcal{D}}_cB_b-\mathfrak{J}\cdot\overline{\mathcal{D}}_bB \;=\; 2\Re(\mathfrak{J})^c\nabla_cB_b
$$

as stated in (C.8). This concludes the proof of Lemma C.6. □

Step 5. Next, we prove the following corollary.

Corollary C.7. *We have*

$$
\begin{aligned}
&\Big(\nabla_4-a\Re(\mathfrak{J})^b\nabla_b\Big)\Big(r^4[B]_{red}\Big)\\
=\;&\frac{r^4}{2}\overline{\mathcal{D}}'\cdot\Big(A-a(\mathfrak{J}\widehat{\otimes}B)\Big)+O(1)\widehat{X}+O(r)\mathfrak{d}^{\leq1}B+O(r^2)\nabla_3B\\
&+O(r)\,\not{\mathfrak{d}}^{\leq1}\check{P}+O(1)\widetilde{trX}+O(r^2)\nabla_3A+O(r)\mathfrak{d}^{\leq1}A+r^4\Gamma_g\cdot(B,A)\\
&+r^4\Gamma_b\cdot\nabla_3A+r^2\mathfrak{d}^{\leq1}(\Gamma_g\cdot\Gamma_g).
\end{aligned}
$$

Proof. We combine Corollary C.5 with Lemma C.6 to derive

$$
\Big(\nabla_4-a\Re(\mathfrak{J})^b\nabla_b\Big)[B]_{ren}+\frac{4}{r}[B]_{ren}
$$

$$
\begin{aligned}
= \quad & \frac{1}{2}\overline{\mathcal{D}}'\cdot A-\frac{a}{2}\overline{\mathcal{D}}\cdot(\mathfrak{J}\widehat{\otimes}B)+\frac{3a^2}{4r}(\overline{\mathfrak{J}}\cdot B)\mathfrak{J}+O(r^{-4})\widehat{X}+O(r^{-3})B\\
&+O(r^{-3})\,\mathfrak{d}\!\!/^{\leq 1}\check{P}+O(r^{-4})\widetilde{\mathrm{tr}X}+O(r^{-2})\nabla_3A+O(r^{-3})\mathfrak{d}^{\leq 1}A\\
&+\Gamma_g\cdot(B,A)+\Gamma_b\cdot\nabla_3A+r^{-2}\Gamma_g\cdot\Gamma_g.
\end{aligned}
$$

Also, recalling (C.3)

$$
\begin{aligned}
\mathcal{D}'_aV_B \quad = \quad & \mathcal{D}_aV_B+\frac{1}{2}\underline{F}_af^c\nabla_cV_B+\frac{1}{2}\underline{F}_a\nabla_4V_B+\left(\frac{1}{2}F_a+\frac{1}{8}|f|^2\underline{F}_a\right)\nabla_3V_B\\
&+O(r^{-3})V+r^{-1}\Gamma_bV,
\end{aligned}
$$

and applying it to $V=\mathfrak{J}\widehat{\otimes}B$ we obtain

$$
\begin{aligned}
\overline{\mathcal{D}}'\cdot(\mathfrak{J}\widehat{\otimes}B) \quad = \quad & \overline{\mathcal{D}}(\mathfrak{J}\widehat{\otimes}B)+\frac{1}{2}\overline{\underline{F}}^cf^c\nabla_c(\mathfrak{J}\widehat{\otimes}B)_a+\frac{1}{2}\overline{\underline{F}}\cdot\nabla_4(\mathfrak{J}\widehat{\otimes}B)\\
&+\left(\frac{1}{2}\overline{F}+\frac{1}{8}|f|^2\overline{\underline{F}}\right)\cdot\nabla_3(\mathfrak{J}\widehat{\otimes}B)+\Big(O(r^{-4})+r^{-2}\Gamma_b\Big)B.
\end{aligned}
$$

We deduce

$$
\begin{aligned}
&\Big(\nabla_4-a\Re(\mathfrak{J})^b\nabla_b\Big)[B]_{ren}+\frac{4}{r}[B]_{ren}\\
= \quad & \frac{1}{2}\overline{\mathcal{D}}'\cdot\Big(A-a(\mathfrak{J}\widehat{\otimes}B)\Big)+\frac{a}{4}\overline{\underline{F}}^cf^c\nabla_c(\mathfrak{J}\widehat{\otimes}B)_a+\frac{a}{4}\overline{\underline{F}}\cdot\nabla_4(\mathfrak{J}\widehat{\otimes}B)\\
&+\frac{a}{4}\left(\overline{F}+\frac{1}{4}|f|^2\overline{\underline{F}}\right)\cdot\nabla_3(\mathfrak{J}\widehat{\otimes}B)\\
&+O(r^{-4})\widehat{X}+O(r^{-3})B+O(r^{-3})\,\mathfrak{d}\!\!/^{\leq 1}\check{P}+O(r^{-4})\widetilde{\mathrm{tr}X}\\
&+O(r^{-2})\nabla_3A+O(r^{-3})\mathfrak{d}^{\leq 1}A+\Gamma_g\cdot(B,A)+\Gamma_b\cdot\nabla_3A+r^{-2}\Gamma_g\cdot\Gamma_g.
\end{aligned}
$$

Since $e_4(r)=1$ and $e_1(r)=e_2(r)=0$, we infer

$$
\begin{aligned}
&\Big(\nabla_4-a\Re(\mathfrak{J})^b\nabla_b\Big)\Big(r^4[B]_{ren}\Big)\\
= \quad & \frac{r^4}{2}\overline{\mathcal{D}}'\cdot\Big(A-a(\mathfrak{J}\widehat{\otimes}B)\Big)+\frac{ar^4}{4}\overline{\underline{F}}^cf^c\nabla_c(\mathfrak{J}\widehat{\otimes}B)_a+\frac{ar^4}{4}\overline{\underline{F}}\cdot\nabla_4(\mathfrak{J}\widehat{\otimes}B)\\
&+\frac{ar^4}{4}\left(\overline{F}+\frac{1}{4}|f|^2\overline{\underline{F}}\right)\cdot\nabla_3(\mathfrak{J}\widehat{\otimes}B)\\
&+O(1)\widehat{X}+O(r)B+O(r)\,\mathfrak{d}\!\!/^{\leq 1}\check{P}+O(1)\widetilde{\mathrm{tr}X}\\
&+O(r^2)\nabla_3A+O(r)\mathfrak{d}^{\leq 1}A+r^4\Gamma_g\cdot(B,A)+r^4\Gamma_b\cdot\nabla_3A+r^2\Gamma_g\cdot\Gamma_g.
\end{aligned}
$$

Also, we have

$$f = \Big(-1 + O(r^{-1}) + r\Gamma_b\Big)\nabla u, \qquad \underline{f} = \Big(-1 + O(r^{-1}) + r\Gamma_b\Big)\nabla u,$$

and hence

$$f = -a\Re(\mathfrak{J}) + O(r^{-1})\Re(\mathfrak{J}) + \Gamma_b, \quad \underline{f} = -a\Re(\mathfrak{J}) + O(r^{-1})\Re(\mathfrak{J}) + \Gamma_b, \tag{C.9}$$

which yields

$$\begin{aligned}
&\frac{ar^4}{4}\overline{\underline{F}}^c f^c\nabla_c(\mathfrak{J}\widehat{\otimes}B)_a + \frac{ar^4}{4}\overline{\underline{F}}\cdot\nabla_4(\mathfrak{J}\widehat{\otimes}B)\\
&+\frac{ar^4}{4}\left(\overline{F} + \frac{1}{4}|f|^2\overline{\underline{F}}\right)\cdot\nabla_3(\mathfrak{J}\widehat{\otimes}B)\\
= \;& O(r)\mathfrak{d}^{\leq 1}B + O(r^2)\nabla_3 B + r^2\Gamma_b\cdot\mathfrak{d}^{\leq 1}B + r^3\Gamma_b\nabla_3 B
\end{aligned}$$

and hence

$$\begin{aligned}
&\Big(\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b\Big)\Big(r^4[B]_{ren}\Big)\\
=\;& \frac{r^4}{2}\overline{\mathcal{D}}'\cdot\Big(A - a(\mathfrak{J}\widehat{\otimes}B)\Big) + O(1)\widehat{X} + O(r)\mathfrak{d}^{\leq 1}B + O(r^2)\nabla_3 B\\
&+O(r)\,\not{\mathfrak{d}}^{\leq 1}\check{P} + O(1)\widecheck{\mathrm{tr}X} + O(r^2)\nabla_3 A + O(r)\mathfrak{d}^{\leq 1}A + r^4\Gamma_g\cdot(B,A)\\
&+r^3\Gamma_b\nabla_3 B + r^4\Gamma_b\cdot\nabla_3 A + r^2\Gamma_g\cdot\Gamma_g.
\end{aligned}$$

Using the Bianchi identity for $\nabla_3 B$, i.e.

$$\begin{aligned}
\nabla_3 B - \mathcal{D}\overline{\check{P}} \;=\;& \frac{2}{r}B + O(r^{-2})B + O(r^{-2})\check{P} + O(r^{-3})\check{H} + O(r^{-4})\mathcal{D}\widecheck{(\cos\theta)}\\
&+r^{-1}\Gamma_b\cdot\Gamma_g,
\end{aligned}$$

we infer $\nabla_3 B = r^{-2}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_g$ and hence $r^3\Gamma_b\nabla_3 B = r\Gamma_b\,\not{\mathfrak{d}}^{\leq 1}\Gamma_g$ which is included in error terms of the type $r^2\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g)$. We deduce

$$\begin{aligned}
&\Big(\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b\Big)\Big(r^4[B]_{ren}\Big)\\
=\;& \frac{r^4}{2}\overline{\mathcal{D}}'\cdot\Big(A - a(\mathfrak{J}\widehat{\otimes}B)\Big) + O(1)\widehat{X} + O(r)\mathfrak{d}^{\leq 1}B + O(r^2)\nabla_3 B\\
&+O(r)\,\not{\mathfrak{d}}^{\leq 1}\check{P} + O(1)\widecheck{\mathrm{tr}X} + O(r^2)\nabla_3 A + O(r)\mathfrak{d}^{\leq 1}A + r^4\Gamma_g\cdot(B,A)\\
&+r^4\Gamma_b\cdot\nabla_3 A + r^2\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g)
\end{aligned}$$

as desired. This concludes the proof of Corollary C.7. □

Step 6. Next, we derive the following lemma.

Lemma C.8. *For a complex horizontal 1-form U, we have*

$$\Big[\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b, r[\mathcal{D}]_{ren}\Big]U = O(r^{-3})\mathfrak{d}^{\leq 1}U + O(r^{-2})\nabla_3 U + \Gamma_g\cdot\mathfrak{d}^{\leq 1}U,$$

where we recall that $[\mathcal{D}]_{ren} = \Big(\overline{\mathcal{D}}\cdot - \frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_4 - \frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_3\Big)$.

Proof. We compute, using $e_1(r) = e_2(r) = 0$,

$$\begin{aligned}
&\Big[\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b, r[\mathcal{D}]_{red}\Big]\\
= \; & [\nabla_4, r\overline{\mathcal{D}}\cdot] - ar[\Re(\mathfrak{J})^b\nabla_b, \overline{\mathcal{D}}\cdot] - \frac{a}{2}\nabla_4(r\overline{\mathfrak{J}})\cdot\nabla_4 - \frac{a}{2}\nabla_4(r\overline{\mathfrak{J}})\cdot\nabla_3\\
&-\frac{a}{2}r\overline{\mathfrak{J}}\cdot[\nabla_4,\nabla_3] + \frac{a^2}{2}r\Big[\Re(\mathfrak{J})^b\nabla_b, \overline{\mathfrak{J}}\cdot(\nabla_4+\nabla_3)\Big].
\end{aligned}$$

Since $\nabla_4(\overline{q}\overline{\mathfrak{J}}) = 0$, and, for a complex 1-form U,

$$\begin{aligned}
[\nabla_4,\nabla_3]U &= 2\underline{\omega}\nabla_4 U + 2(\eta+\zeta)\cdot\nabla U + 4\eta\widehat{\otimes}(\zeta\cdot U) - 4\zeta\widehat{\otimes}(\eta\cdot U)\\
&\quad +4i\,{}^*\rho U,\\
[\nabla_4, \overline{\mathcal{D}}\cdot]U &= -\frac{1}{\overline{q}}\overline{\mathcal{D}}\cdot - \frac{1}{\overline{q}}\overline{Z}\cdot U + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}U,
\end{aligned}$$

we infer, for a complex 1-form U,

$$\begin{aligned}
&\Big[\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b, r[\mathcal{D}]_{red}\Big]U\\
= \; & \left(1 - \frac{r}{\overline{q}}\right)\overline{\mathcal{D}}\cdot U - \frac{r}{\overline{q}}\overline{Z}\cdot U - ar[\Re(\mathfrak{J})^b\nabla_b, \overline{\mathcal{D}}\cdot]U\\
&+\frac{a^2}{2}r\Big[\Re(\mathfrak{J})^b\nabla_b, \overline{\mathfrak{J}}\cdot(\nabla_4+\nabla_3)\Big]U\\
&-\frac{a}{2}r\overline{\mathfrak{J}}\cdot\Big(2\underline{\omega}\nabla_4 U + 2(\eta+\zeta)\cdot\nabla U + 4\eta\widehat{\otimes}(\zeta\cdot U) - 4\zeta\widehat{\otimes}(\eta\cdot U) + 4i\,{}^*\rho U\Big)\\
&+\Gamma_g\cdot\mathfrak{d}^{\leq 1}U - \frac{a}{2}\left(1-\frac{r}{\overline{q}}\right)\overline{\mathfrak{J}}\cdot\nabla_4 - \frac{a}{2}\left(1-\frac{r}{\overline{q}}\right)\overline{\mathfrak{J}}\cdot\nabla_3.
\end{aligned}$$

We infer

$$\begin{aligned}
&\Big[\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b, r[\mathcal{D}]_{red}\Big]U\\
= \; & -\frac{ia\cos\theta}{r}\overline{\mathcal{D}}\cdot U - \frac{a}{r}\overline{\mathfrak{J}}\cdot U - ar[\Re(\mathfrak{J})^b\nabla_b, \overline{\mathcal{D}}\cdot]U + O(r^{-3})\mathfrak{d}^{\leq 1}U
\end{aligned}$$

$$+O(r^{-2})\nabla_3 U + \Gamma_g \cdot \mathfrak{d}^{\leq 1} U.$$

Next, we compute

$$[\Re(\mathfrak{J})^b \nabla_b, \overline{\mathcal{D}}\cdot]U \quad = \quad -\overline{\mathcal{D}}_a \Re(\mathfrak{J})_b \nabla_b U_a + \Re(\mathfrak{J})^b [\nabla_b, \overline{\mathcal{D}}_a] U_a.$$

Recall that we have

$$\nabla_a \Re(\mathfrak{J})_b = \frac{(r^2+a^2)\cos\theta}{|q|^4} \in_{ab} + r^{-1}\Gamma_b$$

and hence

$$\begin{aligned}
\overline{\mathcal{D}}_a \Re(\mathfrak{J})_b &= (\nabla_a - i \in_{ac} \nabla_c)\Re(\mathfrak{J})_b \\
&= \frac{(r^2+a^2)\cos\theta}{|q|^4}\Big[\in_{ab} - i \in_{ac}\in_{cb}\Big] + r^{-1}\Gamma_b \\
&= \frac{(r^2+a^2)\cos\theta}{|q|^4}\Big[\in_{ab} + i\delta_{ab}\Big] + r^{-1}\Gamma_b \\
&= \frac{\cos\theta}{r^2}\Big[\in_{ab} + i\delta_{ab}\Big] + O(r^{-4}) + r^{-1}\Gamma_b.
\end{aligned}$$

We infer

$$\begin{aligned}
-\overline{\mathcal{D}}_a \Re(\mathfrak{J})_b \nabla_b U_a &= -\frac{\cos\theta}{r^2}\Big[\in_{ab} + i\delta_{ab}\Big]\nabla_b U_a + O(r^{-5})\, \mathfrak{d}\!\!/ U + r^{-2}\Gamma_b\, \mathfrak{d}\!\!/ U \\
&= -\frac{\cos\theta}{r^2}\Big[-\,\mathrm{curl}\,(U) + i\mathrm{div}\,(U)\Big] + O(r^{-5})\, \mathfrak{d}\!\!/ U \\
&\quad + r^{-2}\Gamma_b\, \mathfrak{d}\!\!/ U \\
&= -\frac{i\cos\theta}{r^2}\overline{\mathcal{D}}\cdot U + O(r^{-5})\, \mathfrak{d}\!\!/ U + r^{-2}\Gamma_b\, \mathfrak{d}\!\!/ U.
\end{aligned}$$

Also, we have, see Proposition 2.1.41 in [28],

$$[\nabla_b, \nabla_a]U_c \quad = \quad \frac{1}{2}\in_{ba}\left({}^{(a)}\mathrm{tr}\chi \nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\right)U_c - \frac{1}{2}E_{cdba}U_d - \in_{cd}\in_{ba}\rho U_d$$

where

$$E_{abcd} \quad := \quad \chi_{ac}\underline{\chi}_{bd} + \underline{\chi}_{ac}\chi_{bd} - \chi_{bc}\underline{\chi}_{ad} - \underline{\chi}_{bc}\chi_{ad}.$$

Since

$$\chi_{ab} = \frac{1}{r}\delta_{ab} + O(r^{-2}) + \Gamma_g, \qquad \underline{\chi}_{ab} = -\frac{1}{r}\delta_{ab} + O(r^{-2}) + \widehat{\underline{\chi}}_{ab},$$

we have[230]

$$E_{abcd} = -\frac{2}{r^2}\Big(\delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad}\Big) + O(r^{-3}) + r^{-1}\Gamma_g$$

and hence

$$\begin{aligned}
[\nabla_b, \nabla_a]U_c &= \frac{1}{r^2}\Big(\delta_{cb}\delta_{da} - \delta_{db}\delta_{ca}\Big)U_d \\
&+O(r^{-3})\mathfrak{d}^{\leq 1}U + O(r^{-2})\nabla_3 U + r^{-1}\Gamma_g\mathfrak{d}^{\leq 1}U + \Gamma_g\nabla_3 U.
\end{aligned}$$

We infer

$$\begin{aligned}
[\nabla_b, \overline{\mathcal{D}}_a]U_a &= \delta_{ac}[\nabla_b, \nabla_a]U_c - i\in_{ca}[\nabla_b, \nabla_a]U_c \\
&= \frac{1}{r^2}(\delta_{ac} - i\in_{ca})\Big(\delta_{cb}\delta_{da} - \delta_{db}\delta_{ca}\Big)U_d \\
&+O(r^{-3})\mathfrak{d}^{\leq 1}U + O(r^{-2})\nabla_3 U + r^{-1}\Gamma_g\mathfrak{d}^{\leq 1}U + \Gamma_g\nabla_3 U \\
&= \frac{1}{r^2}\Big(-\delta_{db} - i\in_{bd}\Big)U_d \\
&+O(r^{-3})\mathfrak{d}^{\leq 1}U + O(r^{-2})\nabla_3 U + r^{-1}\Gamma_g\mathfrak{d}^{\leq 1}U + \Gamma_g\nabla_3 U \\
&= -\frac{1}{r^2}\Big(U_b + i\,{}^*U_b\Big) + O(r^{-3})\mathfrak{d}^{\leq 1}U + O(r^{-2})\nabla_3 U \\
&+r^{-1}\Gamma_g\mathfrak{d}^{\leq 1}U + \Gamma_g\nabla_3 U
\end{aligned}$$

and hence, using also $\Im(\mathfrak{J}) = {}^*\Re(\mathfrak{J})$, we obtain

$$\begin{aligned}
& \Re(\mathfrak{J})^b[\nabla_b, \overline{\mathcal{D}}_a]U_a \\
=\ & -\Re(\mathfrak{J})^b\frac{1}{r^2}\Big(U_b + i\,{}^*U_b\Big) \\
& +O(r^{-4})\mathfrak{d}^{\leq 1}U + O(r^{-3})\nabla_3 U + r^{-2}\Gamma_g\mathfrak{d}^{\leq 1}U + r^{-1}\Gamma_g\nabla_3 U \\
=\ & -\frac{1}{r^2}\Big(\Re(\mathfrak{J})\cdot U + i\Re(\mathfrak{J})\cdot {}^*U\Big) \\
& +O(r^{-4})\mathfrak{d}^{\leq 1}U + O(r^{-3})\nabla_3 U + r^{-2}\Gamma_g\mathfrak{d}^{\leq 1}U + r^{-1}\Gamma_g\nabla_3 U \\
=\ & -\frac{1}{r^2}\Big(\Re(\mathfrak{J})\cdot U - i\,{}^*\Re(\mathfrak{J})\cdot U\Big) \\
& +O(r^{-4})\mathfrak{d}^{\leq 1}U + O(r^{-3})\nabla_3 U + r^{-2}\Gamma_g\mathfrak{d}^{\leq 1}U + r^{-1}\Gamma_g\nabla_3 U
\end{aligned}$$

[230] A priori, one also expects a contribution $r^{-1}\Gamma_b$ coming from

$$\delta_{ac}\widehat{\underline{\chi}}_{bd} + \widehat{\underline{\chi}}_{ac}\delta_{bd} - \delta_{bc}\widehat{\underline{\chi}}_{ad} - \widehat{\underline{\chi}}_{bc}\delta_{ad},$$

but this tensor vanishes in fact identically.

$$= -\frac{1}{r^2}\overline{\mathfrak{J}}\cdot U + O(r^{-4})\mathfrak{d}^{\leq 1}U + O(r^{-3})\nabla_3 U + r^{-2}\Gamma_g\mathfrak{d}^{\leq 1}U + r^{-1}\Gamma_g\nabla_3 U.$$

In view of the above, we deduce

$$\begin{aligned}
[\Re(\mathfrak{J})^b\nabla_b, \overline{\mathcal{D}}\cdot]U &= -\overline{\mathcal{D}}_a\Re(\mathfrak{J})_b\nabla_b U_a + \Re(\mathfrak{J})^b[\nabla_b, \overline{\mathcal{D}}_a]U_a \\
&= -\frac{i\cos\theta}{r^2}\overline{\mathcal{D}}\cdot U - \frac{1}{r^2}\overline{\mathfrak{J}}\cdot U + O(r^{-4})\mathfrak{d}^{\leq 1}U + O(r^{-3})\nabla_3 U \\
&\quad + r^{-2}\Gamma_b\mathfrak{d}^{\leq 1}U + r^{-1}\Gamma_g\nabla_3 U.
\end{aligned}$$

This yields

$$\begin{aligned}
& -\frac{ia\cos\theta}{r}\overline{\mathcal{D}}\cdot U - \frac{a}{r}\overline{\mathfrak{J}}\cdot U - ar[\Re(\mathfrak{J})^b\nabla_b, \overline{\mathcal{D}}\cdot]U \\
= & -\frac{ia\cos\theta}{r}\overline{\mathcal{D}}\cdot U - \frac{a}{r}\overline{\mathfrak{J}}\cdot U - ar\left(-\frac{i\cos\theta}{r^2}\overline{\mathcal{D}}\cdot U - \frac{1}{r^2}\overline{\mathfrak{J}}\cdot U\right) \\
& + O(r^{-3})\mathfrak{d}^{\leq 1}U + O(r^{-2})\nabla_3 U + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1}U + \Gamma_g\nabla_3 U \\
= & \ O(r^{-3})\mathfrak{d}^{\leq 1}U + O(r^{-2})\nabla_3 U + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1}U + \Gamma_g\nabla_3 U.
\end{aligned}$$

Coming back to

$$\begin{aligned}
& \Big[\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b, r[\mathcal{D}]_{red}\Big]U \\
= & -\frac{ia\cos\theta}{r}\overline{\mathcal{D}}\cdot U - \frac{a}{r}\overline{\mathfrak{J}}\cdot U - ar[\Re(\mathfrak{J})^b\nabla_b, \overline{\mathcal{D}}\cdot]U \\
& + O(r^{-3})\mathfrak{d}^{\leq 1}U + O(r^{-2})\nabla_3 U + \Gamma_g\cdot\mathfrak{d}^{\leq 1}U,
\end{aligned}$$

we infer

$$\Big[\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b, r[\mathcal{D}]_{red}\Big]U = O(r^{-3})\mathfrak{d}^{\leq 1}U + O(r^{-2})\nabla_3 U + \Gamma_g\cdot\mathfrak{d}^{\leq 1}U$$

as desired. This concludes the proof of Lemma C.8. □

Step 7. We are finally ready to prove Lemma 6.44. We start with the identity of Corollary C.7

$$\begin{aligned}
& \Big(\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b\Big)\Big[r^4[B]_{red}\Big] \\
= & \ \frac{r^4}{2}\overline{\mathcal{D}}'\cdot\Big(A - a(\mathfrak{J}\widehat{\otimes}B)\Big) + O(1)\widehat{X} + O(r)\mathfrak{d}^{\leq 1}B + O(r^2)\nabla_3 B \\
& + O(r)\,\not{\mathfrak{d}}^{\leq 1}\check{P} + O(1)\widecheck{\mathrm{tr}X} + O(r^2)\nabla_3 A + O(r)\mathfrak{d}^{\leq 1}A + r^4\Gamma_g\cdot(B, A) \\
& + r^4\Gamma_b\cdot\nabla_3 A + r^2\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g).
\end{aligned}$$

According to Lemma C.8 we have, with $U = A - a(\mathfrak{J}\widehat{\otimes}B)$,

$$\Big[\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b, r[\mathcal{D}]_{ren}\Big]U = O(r^{-3})\mathfrak{d}^{\leq 1}U + O(r^{-2})\nabla_3 U + \Gamma_g\cdot\mathfrak{d}^{\leq 1}U.$$

We deduce

(C.10)

$$\begin{aligned}
&\Big(\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b\Big)\Big(r[\overline{\mathcal{D}}\cdot]_{ren}(r^4[B]_{ren})\Big)\\
&= r[\mathcal{D}]_{ren}\left[\frac{r^4}{2}\overline{\mathcal{D}}'\cdot\Big(A - a(\mathfrak{J}\widehat{\otimes}B)\Big)\right]\\
&\quad + O(1)\mathfrak{d}^{\leq 1}\widehat{X} + O(r)\mathfrak{d}^{\leq 2}B + O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 B + O(r)\mathfrak{d}^{\leq 2}\check{P} + O(1)\mathfrak{d}^{\leq 1}\widetilde{\mathrm{tr}X}\\
&\quad + O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 A + O(r)\mathfrak{d}^{\leq 2}A + r^4\mathfrak{d}^{\leq 1}(\Gamma_g\cdot(B,A)) + r^3\mathfrak{d}^{\leq 1}(\Gamma_b\nabla_3 B)\\
&\quad + r^4\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\nabla_3 A) + r^2\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g).
\end{aligned}$$

Next we note that

$$r\overline{\mathcal{D}'}\cdot = r[\mathcal{D}]_{ren} + O(r^{-2})\mathfrak{d}^{\leq 1} + O(r^{-1})\nabla_3 + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1} + \Gamma_b\nabla_3. \tag{C.11}$$

This follows in view of the formula (C.3)

$$\begin{aligned}
\mathcal{D}'_a V_B &= \mathcal{D}_a V_B + \frac{1}{2}\underline{F}_a f^c\nabla_c V_B + \frac{1}{2}\underline{F}_a\nabla_4 V_B + \left(\frac{1}{2}F_a + \frac{1}{8}|f|^2\underline{F}_a\right)\nabla_3 V_B\\
&\quad +\Big\{O(r^{-3}) + r^{-1}\Gamma_b\Big\}V
\end{aligned}$$

from which we infer, using also (C.9),

$$\begin{aligned}
r\overline{\mathcal{D}'}\cdot &= r\left(\overline{\mathcal{D}}\cdot - \frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_4 - \frac{a}{2}\overline{\mathfrak{J}}\cdot\nabla_3\right) + O(r^{-2})\mathfrak{d}^{\leq 1} + O(r^{-1})\nabla_3\\
&\quad + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1} + \Gamma_b\nabla_3\\
&= r[\mathcal{D}]_{ren} + O(r^{-2})\mathfrak{d}^{\leq 1} + O(r^{-1})\nabla_3 + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1} + \Gamma_b\nabla_3
\end{aligned}$$

as stated above.

We deduce

$$\begin{aligned}
& r[\mathcal{D}]_{ren}\left[\frac{r^4}{2}\overline{\mathcal{D}}'\cdot\Big(A - a(\mathfrak{J}\widehat{\otimes}B)\Big)\right]\\
&= r\overline{\mathcal{D}'}\cdot\left[\frac{r^4}{2}\overline{\mathcal{D}}'\cdot\Big(A - a(\mathfrak{J}\widehat{\otimes}B)\Big)\right] + O(r)\mathfrak{d}^{\leq 2}A + O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 A
\end{aligned}$$

$$+r^2\Gamma_b\mathfrak{d}^{\leq 1}A + r^3\Gamma_b\mathfrak{d}^{\leq 1}\nabla_3 A + O(1)\mathfrak{d}^{\leq 2}B + O(r)\mathfrak{d}^{\leq 1}\nabla_3 B + r\Gamma_b\mathfrak{d}^{\leq 1}B$$
$$+r^2\Gamma_b\mathfrak{d}^{\leq 1}\nabla_3 B.$$

Inserting this on the right hand side of (C.10), since $e_1'(r) = e_2'(r) = 0$, we deduce

$$\begin{aligned}
&\Big(\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b\Big)\Big(r[\overline{\mathcal{D}}\cdot]_{ren}(r^4[B]_{ren})\Big)\\
=\;& \frac{r^5}{2}\overline{\mathcal{D}}'\cdot\overline{\mathcal{D}}'\cdot\Big(A - a(\mathfrak{J}\widehat{\otimes}B)\Big) + O(1)\mathfrak{d}^{\leq 1}\widehat{X} + O(r)\mathfrak{d}^{\leq 2}B + O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 B\\
&+O(r)\mathfrak{d}^{\leq 2}\check{P} + O(1)\mathfrak{d}^{\leq 1}\widetilde{\mathrm{tr} X} + O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 A + O(r)\mathfrak{d}^{\leq 2}A\\
&+r^4\mathfrak{d}^{\leq 1}(\Gamma_g\cdot(B,A)) + r^3\mathfrak{d}^{\leq 1}(\Gamma_b\nabla_3 B) + r^4\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\nabla_3 A)\\
&+r^2\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_g).
\end{aligned}$$

Using, as before, that $\Gamma_b\nabla_3 B$ in included in $r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_g)$, we deduce

$$\begin{aligned}
&\Big(\nabla_4 - a\Re(\mathfrak{J})^b\nabla_b\Big)\Big(r[\overline{\mathcal{D}}\cdot]_{ren}(r^4[B]_{ren})\Big)\\
=\;& \frac{r^5}{2}\overline{\mathcal{D}}'\cdot\overline{\mathcal{D}}'\cdot\Big(A - a(\mathfrak{J}\widehat{\otimes}B)\Big) + O(1)\mathfrak{d}^{\leq 1}\widehat{X} + O(r)\mathfrak{d}^{\leq 2}B + O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 B\\
&+O(r)\mathfrak{d}^{\leq 2}\check{P} + O(1)\mathfrak{d}^{\leq 1}\widetilde{\mathrm{tr} X} + O(r^2)\mathfrak{d}^{\leq 1}\nabla_3 A + O(r)\mathfrak{d}^{\leq 2}A\\
&+r^4\mathfrak{d}^{\leq 1}(\Gamma_g\cdot(B,A)) + r^4\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\nabla_3 A) + r^2\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_g)
\end{aligned}$$

as desired. This concludes the proof of the Lemma 6.44.

Appendix D. PROOF OF RESULTS IN CHAPTER 9

D.1. Proof of Proposition 9.19

Recall that we have

$$\nabla_4 \mathrm{tr} X + \frac{1}{2}(\mathrm{tr} X)^2 \;=\; -\frac{1}{2}\widehat{X}\cdot\overline{\widehat{X}} = \Gamma_g\cdot\Gamma_g.$$

We infer

$$\nabla_4\left(\frac{2}{q} + \widetilde{\mathrm{tr} X}\right) + \frac{1}{2}\left(\frac{2}{q} + \widetilde{\mathrm{tr} X}\right)^2 \;=\; \Gamma_g\cdot\Gamma_g$$

and since $e_4(q) = 1$, we deduce

$$\nabla_4\widetilde{\mathrm{tr} X} + \frac{2}{q}\widetilde{\mathrm{tr} X} \;=\; \Gamma_g\cdot\Gamma_g$$

as desired.

Next, recall

$$\nabla_4 Z + \frac{1}{2}\mathrm{tr} X Z \;=\; \frac{1}{2}\mathrm{tr} X\,\underline{H} + \frac{1}{2}\widehat{X}\cdot(-\overline{Z} + \overline{\underline{H}}) - B.$$

We infer

$$\begin{aligned}
&\nabla_4\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J} + \check{Z}\right) + \frac{1}{2}\left(\frac{2}{q} + \widetilde{\mathrm{tr} X}\right)\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J} + \check{Z}\right)\\
&= \frac{1}{2}\left(\frac{2}{q} + \widetilde{\mathrm{tr} X}\right)\left(-\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right) + \frac{1}{2}\widehat{X}\cdot\left(-\overline{\frac{a\overline{q}}{|q|^2}\mathfrak{J}} + \overline{-\frac{a\overline{q}}{|q|^2}\mathfrak{J}}\right) - B + \Gamma_g\cdot\Gamma_g.
\end{aligned}$$

We have

$$\nabla_4\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right) \;=\; \partial_r\left(\frac{a\overline{q}}{|q|^2}\right)\mathfrak{J} + \frac{a\overline{q}}{|q|^2}\left(-\frac{1}{q}\mathfrak{J}\right) = -\frac{2}{q}\frac{a\overline{q}}{|q|^2}\mathfrak{J}$$

and hence

$$\nabla_4\check{Z} + \frac{1}{q}\check{Z} \;=\; -\frac{a\overline{q}}{|q|^2}\widetilde{\mathrm{tr} X}\mathfrak{J} - \frac{aq}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X} - B + \Gamma_g\cdot\Gamma_g.$$

Next, recall

$$\nabla_4 H \;=\; -\frac{1}{2}\overline{\mathrm{tr} X}(H - \underline{H}) - \frac{1}{2}\widehat{X}\cdot(\overline{H} - \overline{\underline{H}}) - B.$$

We infer

$$\nabla_4\left(\frac{aq}{|q|^2}\mathfrak{J}+\check{H}\right) = -\frac{1}{2}\left(\frac{2}{\overline{q}}+\overline{\widecheck{\mathrm{tr}X}}\right)\left(\frac{aq}{|q|^2}\mathfrak{J}+\check{H}+\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right) -\frac{1}{2}\widehat{X}\cdot\left(\overline{\frac{aq}{|q|^2}\mathfrak{J}}+\overline{\frac{a\overline{q}}{|q|^2}\mathfrak{J}}\right)-B+\Gamma_b\cdot\Gamma_g.$$

Now, we have

$$\begin{aligned}\nabla_4\left(\frac{aq}{|q|^2}\mathfrak{J}\right) &= \partial_r\left(\frac{aq}{|q|^2}\right)\mathfrak{J}+\frac{aq}{|q|^2}\left(-\frac{1}{q}\right)\mathfrak{J}\\ &= -\frac{a}{\overline{q}^2}\mathfrak{J}-\frac{a}{|q|^2}\mathfrak{J}\end{aligned}$$

and hence

$$\nabla_4\check{H}+\frac{1}{\overline{q}}\check{H} = -\frac{ar}{|q|^2}\overline{\widecheck{\mathrm{tr}X}}\mathfrak{J}-\frac{ar}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{X}-B+\Gamma_b\cdot\Gamma_g.$$

Next, recall

$$\nabla_4\underline{\omega}-(\eta-\underline{\eta})\cdot\zeta+\eta\cdot\underline{\eta} = \rho$$

which we rewrite

$$\nabla_4\underline{\omega}-\frac{1}{2}\Re\Big((H-\underline{H})\cdot\overline{Z}-H\cdot\overline{\underline{H}}\Big) = \Re(P).$$

We infer

$$\begin{aligned}&\frac{1}{2}\partial_r^2\left(\frac{\Delta}{|q|^2}\right)+\nabla_4\underline{\check{\omega}}\\ &-\frac{1}{2}\Re\left(\left(\frac{aq}{|q|^2}\mathfrak{J}+\check{H}+\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)\cdot\overline{\frac{a\overline{q}}{|q|^2}\mathfrak{J}+\check{Z}}+\left(\frac{aq}{|q|^2}\mathfrak{J}+\check{H}\right)\cdot\overline{\frac{a\overline{q}}{|q|^2}\mathfrak{J}}\right)\\ =\ &\Re\left(-\frac{2m}{q^3}+\check{P}\right)\end{aligned}$$

and hence

$$\begin{aligned}\nabla_4\underline{\check{\omega}} &= -\frac{1}{2}\partial_r^2\left(\frac{\Delta}{|q|^2}\right)+\frac{a^2}{2|q|^2}\Re\left(\mathfrak{J}\cdot\overline{\mathfrak{J}}\right)+\frac{a^2}{|q|^4}\Re\left(q^2\mathfrak{J}\cdot\overline{\mathfrak{J}}\right)+\Re\left(-\frac{2m}{q^3}\right)\\ &+\Re\left(\check{P}\right)+\frac{ar}{|q|^2}\Re\left(\mathfrak{J}\cdot\overline{\check{Z}}\right)+\frac{2a}{|q|^2}\Re\left(q\overline{\mathfrak{J}}\cdot\check{H}\right)+\Gamma_g\cdot\Gamma_g.\end{aligned}$$

Since $\mathfrak{J}\cdot\overline{\mathfrak{J}} = \frac{2(\sin\theta)^2}{|q|^2}$, we infer

$$\nabla_4\underline{\check{\omega}} = \Re\left(\check{P}\right) + \frac{ar}{|q|^2}\Re\left(\mathfrak{J}\cdot\overline{\check{Z}}\right) + \frac{2a}{|q|^2}\Re\left(q\overline{\mathfrak{J}}\cdot\check{H}\right) + \Gamma_g\cdot\Gamma_g.$$

Next, recall

$$\nabla_4\mathrm{tr}\underline{X} + \frac{1}{2}\mathrm{tr}X\mathrm{tr}\underline{X} = \mathcal{D}\cdot\overline{\underline{H}} + \underline{H}\cdot\overline{\underline{H}} + 2\overline{P} - \frac{1}{2}\widehat{X}\cdot\overline{\widehat{\underline{X}}}.$$

We infer

$$\begin{aligned}
&\nabla_4\left(-\frac{2q\Delta}{|q|^4} + \widecheck{\mathrm{tr}\underline{X}}\right) + \frac{1}{2}\left(\frac{2}{q} + \widecheck{\mathrm{tr}X}\right)\left(-\frac{2q\Delta}{|q|^4} + \widecheck{\mathrm{tr}\underline{X}}\right)\\
&= \mathcal{D}\cdot\overline{\underline{H}} + \underline{H}\cdot\overline{\underline{H}} + 2\left(-\frac{2m}{\overline{q}^3} + \overline{\check{P}}\right) + \Gamma_b\cdot\Gamma_g.
\end{aligned}$$

Now, we have, using in particular $\mathfrak{J}\cdot\overline{\mathfrak{J}} = \frac{2(\sin\theta)^2}{|q|^2}$,

$$\begin{aligned}
\mathcal{D}\cdot\overline{\underline{H}} + \underline{H}\cdot\overline{\underline{H}} &= -\mathcal{D}\cdot\left(\overline{\frac{a\overline{q}}{|q|^2}\mathfrak{J}}\right) + \frac{a\overline{q}}{|q|^2}\mathfrak{J}\cdot\overline{\frac{a\overline{q}}{|q|^2}\mathfrak{J}}\\
&= -\frac{aq}{|q|^2}\mathcal{D}\cdot\overline{\mathfrak{J}} + \frac{a}{\overline{q}^2}\mathcal{D}(\overline{q})\cdot\overline{\mathfrak{J}} + \frac{a^2}{|q|^2}\mathfrak{J}\cdot\overline{\mathfrak{J}}\\
&= -\frac{aq}{|q|^2}\left(\frac{4i(r^2+a^2)\cos\theta}{|q|^4} + \widecheck{\mathcal{D}\cdot\overline{\mathfrak{J}}}\right) + \frac{a}{\overline{q}^2}\mathcal{D}(r)\cdot\overline{\mathfrak{J}}\\
&\quad -\frac{ia^2}{\overline{q}^2}\left(i\mathfrak{J} + \widecheck{\mathcal{D}(\cos\theta)}\right)\cdot\overline{\mathfrak{J}} + \frac{a^2}{|q|^2}\mathfrak{J}\cdot\overline{\mathfrak{J}}\\
&= -\frac{4iaq(r^2+a^2)\cos\theta}{|q|^6} + \frac{2(\sin\theta)^2a^2(q^2+|q|^2)}{|q|^6}\\
&\quad -\frac{aq}{|q|^2}\widecheck{\mathcal{D}\cdot\overline{\mathfrak{J}}} + \frac{a}{\overline{q}^2}\mathcal{D}(r)\cdot\overline{\mathfrak{J}} - \frac{ia^2}{\overline{q}^2}\widecheck{\mathcal{D}(\cos\theta)}\cdot\overline{\mathfrak{J}}.
\end{aligned}$$

We deduce

$$\begin{aligned}
\nabla_4\widecheck{\mathrm{tr}\underline{X}} + \frac{1}{q}\widecheck{\mathrm{tr}\underline{X}} &= 2\overline{\check{P}} + \frac{q\Delta}{|q|^4}\widecheck{\mathrm{tr}X} - \frac{aq}{|q|^2}\widecheck{\mathcal{D}\cdot\overline{\mathfrak{J}}} + \frac{a}{\overline{q}^2}\mathcal{D}(r)\cdot\overline{\mathfrak{J}}\\
&\quad -\frac{ia^2}{\overline{q}^2}\widecheck{\mathcal{D}(\cos\theta)}\cdot\overline{\mathfrak{J}} + \Gamma_b\cdot\Gamma_g.
\end{aligned}$$

Next, recall

$$\nabla_4\widehat{\underline{X}}+\frac{1}{2}\mathrm{tr}X\,\widehat{\underline{X}} \;\; = \;\; \mathcal{D}\widehat{\otimes}\,\underline{H}+\,\underline{H}\widehat{\otimes}\,\underline{H}-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\widehat{X}.$$

We have

$$\begin{aligned}
\mathcal{D}\widehat{\otimes}\,\underline{H}+\,\underline{H}\widehat{\otimes}\,\underline{H} &= -\mathcal{D}\widehat{\otimes}\left(\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)+\frac{a\overline{q}}{|q|^2}\mathfrak{J}\widehat{\otimes}\frac{a\overline{q}}{|q|^2}\mathfrak{J}\\
&= -\frac{a\overline{q}}{|q|^2}\mathcal{D}\widehat{\otimes}\mathfrak{J}+\frac{a}{q^2}\mathcal{D}(q)\widehat{\otimes}\mathfrak{J}+\frac{a^2\overline{q}^2}{|q|^4}\mathfrak{J}\widehat{\otimes}\mathfrak{J}\\
&= -\frac{a\overline{q}}{|q|^2}\mathcal{D}\widehat{\otimes}\mathfrak{J}+\frac{a}{q^2}\mathcal{D}(r)\widehat{\otimes}\mathfrak{J}+\frac{ia^2}{q^2}\Big(i\mathfrak{J}+\widetilde{\mathcal{D}(\cos\theta)}\Big)\widehat{\otimes}\mathfrak{J}\\
&\quad +\frac{a^2\overline{q}^2}{|q|^4}\mathfrak{J}\widehat{\otimes}\mathfrak{J}\\
&= -\frac{a\overline{q}}{|q|^2}\mathcal{D}\widehat{\otimes}\mathfrak{J}+\frac{a}{q^2}\mathcal{D}(r)\widehat{\otimes}\mathfrak{J}+\frac{ia^2}{q^2}\widetilde{\mathcal{D}(\cos\theta)}\widehat{\otimes}\mathfrak{J}
\end{aligned}$$

and hence

$$\nabla_4\widehat{\underline{X}}+\frac{1}{q}\widehat{\underline{X}}=-\frac{a\overline{q}}{|q|^2}\mathcal{D}\widehat{\otimes}\mathfrak{J}+\frac{a}{q^2}\mathcal{D}(r)\widehat{\otimes}\mathfrak{J}+\frac{ia^2}{q^2}\widetilde{\mathcal{D}(\cos\theta)}\widehat{\otimes}\mathfrak{J}+\frac{q\Delta}{|q|^4}\widehat{X}+\Gamma_b\cdot\Gamma_g.$$

Next, recall

$$\nabla_4\underline{\Xi} \;\; = \;\; \nabla_3\,\underline{H}+\frac{1}{2}\overline{\mathrm{tr}\underline{X}}(\,\underline{H}-H)+\frac{1}{2}\widehat{\underline{X}}\cdot(\overline{\underline{H}}-\overline{H})-\underline{B}.$$

We infer

$$\nabla_4\underline{\Xi} \;\; = \;\; \nabla_3\,\underline{H}+\frac{1}{2}\overline{-\frac{2q\Delta}{|q|^4}+\widetilde{\mathrm{tr}\underline{X}}}\left(-\frac{2ar}{|q|^2}\mathfrak{J}-\check{H}\right)-\frac{ar}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{\underline{X}}-\underline{B}+\Gamma_b\cdot\Gamma_b.$$

Also, we have

$$\begin{aligned}
\nabla_3\,\underline{H} &= \nabla_3\left(-\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)=-\frac{a\overline{q}}{|q|^2}\nabla_3\mathfrak{J}+\frac{a}{q^2}\nabla_3(q)\mathfrak{J}\\
&= -\frac{a\overline{q}}{|q|^2}\left(\frac{\Delta q}{|q|^4}\mathfrak{J}+\widetilde{\nabla_3\mathfrak{J}}\right)+\frac{a}{q^2}\Big(e_3(r)+iae_3(\cos\theta)\Big)\mathfrak{J}\\
&= -\frac{a\Delta(|q|^2+\overline{q}^2)}{|q|^6}\mathfrak{J}-\frac{a\overline{q}}{|q|^2}\widetilde{\nabla_3\mathfrak{J}}+\frac{a}{q^2}\left(\widetilde{e_3(r)}+iae_3(\cos\theta)\right)\mathfrak{J}.
\end{aligned}$$

We infer

$$\begin{aligned}\nabla_4\underline{\Xi} &= \frac{\overline{q}\Delta}{|q|^4}\check{H} - \frac{ar}{|q|^2}\widetilde{\mathrm{tr}\underline{X}}\mathfrak{J} - \frac{ar}{|q|^2}\overline{\mathfrak{J}}\cdot\widehat{\underline{X}} - \underline{B} - \frac{a\overline{q}}{|q|^2}\widetilde{\nabla_3\mathfrak{J}}\\ &+\frac{a}{q^2}\left(\widetilde{e_3(r)} + iae_3(\cos\theta)\right)\mathfrak{J} + \Gamma_b\cdot\Gamma_b.\end{aligned}$$

This concludes the proof of Proposition 9.19.

D.2. Proof of Lemma 9.24

In view of the initialization of the PG and PT structures of ${}^{(ext)}\mathcal{M}$ on Σ_*, we have

$$\lambda = 1, \qquad f = 0, \qquad u' = u, \qquad r' = r, \qquad \theta' = \theta, \qquad \mathfrak{J}' = \mathfrak{J} \quad \text{on} \quad \Sigma_*,$$

see in particular Remark 9.10.

Next, let $F = f + i\,{}^*f$. Since $\xi = \xi' = 0$ and $\omega = \omega' = 0$, we have in view of Corollary 2.81

$$\begin{aligned}\nabla_{\lambda^{-1}e_4'}F + \frac{1}{2}\overline{\mathrm{tr}X}F + 2\omega F &= -2\Xi - \widehat{\chi}\cdot F + E_1(f,\Gamma),\\ \lambda^{-1}\nabla_4'(\log\lambda) &= 2\omega + f\cdot(\zeta - \underline{\eta}) + E_2(f,\Gamma),\end{aligned}$$

where $E_1(f,\Gamma) = O(f^2\Gamma)$ and $E_2(f,\Gamma) = O(f^2\Gamma)$. Since $\lambda = 1$ and $f = 0$ on Σ_*, this propagates immediately to ${}^{(ext)}\mathcal{M}$, and hence

$$\lambda = 1, \qquad f = 0, \qquad e_4' = e_4 \quad \text{on} \quad {}^{(ext)}\mathcal{M}.$$

Next, since $e_4' = e_4$ on ${}^{(ext)}\mathcal{M}$, and since $e_4(r) = e_4'(r') = 1$, $e_4(u) = e_4'(u') = 0$, and $e_4(\theta) = e_4'(\theta') = 0$, we infer $e_4(r' - r) = 0$, $e_4(u' - u) = 0$ and $e_4(\theta' - \theta) = 0$. Hence, since $u' = u$, $r' = r$ and $\theta' = \theta$ on Σ_*, this propagates immediately to ${}^{(ext)}\mathcal{M}$

$$u' = u, \qquad r' = r, \qquad \theta' = \theta, \qquad q' = q \quad \text{on} \quad {}^{(ext)}\mathcal{M}.$$

Also, we compute $\nabla_4(q\mathfrak{J}')$. Since $e_4' = e_4$, $q' = q$, and $\nabla_4'(q'\mathfrak{J}') = 0$, we have

$$\begin{aligned}0 &= \nabla_4'(q'\mathfrak{J}')_a = e_4'(q'\mathfrak{J}_a') - \mathbf{g}(\mathbf{D}_{e_4'}e_a', e_b')\mathfrak{J}_b' = e_4(q\mathfrak{J}_a') - \mathbf{g}(\mathbf{D}_{e_4'}e_a', e_b')\mathfrak{J}_b'\\ &= \nabla_4(q\mathfrak{J}')_a - \Big(\mathbf{g}(\mathbf{D}_{e_4'}e_a', e_b') - \mathbf{g}(\mathbf{D}_{e_4}e_a, e_b)\Big)\mathfrak{J}_b'.\end{aligned}$$

Since, using $\lambda = 1$ and $f = 0$, we have in view of the frame transformation

$$\begin{aligned}\mathbf{g}(\mathbf{D}_{e_4'}e_a', e_b') &= \mathbf{g}\left(\mathbf{D}_{e_4}\left(e_a + \frac{1}{2}\underline{f}_a e_4\right), e_b + \frac{1}{2}\underline{f}_b e_4\right)\\ &= \mathbf{g}(\mathbf{D}_{e_4}e_a, e_b) - \underline{f}_b\xi_a + \underline{f}_a\xi_b = \mathbf{g}(\mathbf{D}_{e_4}e_a, e_b)\end{aligned}$$

where we used the fact that $\xi = 0$, we infer $\nabla_4(q\mathfrak{J}') = 0$ on ${}^{(ext)}\mathcal{M}$. Hence, since $\nabla_4(q\mathfrak{J}) = 0$, we deduce on ${}^{(ext)}\mathcal{M}$

$$\nabla_4(q(\mathfrak{J} - \mathfrak{J}')) = 0.$$

As $\mathfrak{J}' = \mathfrak{J}$ on Σ_*, this propagates immediately to ${}^{(ext)}\mathcal{M}$, and we deduce

$$\mathfrak{J}' = \mathfrak{J} \quad \text{on} \quad {}^{(ext)}\mathcal{M}.$$

Next, we consider the transition coefficients $(f', \underline{f}', \lambda')$ from the PT frame to the PG one, i.e.$(f', \underline{f}', \lambda')$ corresponds to the inverse transformation of $(f, \underline{f}, \lambda)$. Since $\lambda = 1$ and $f = 0$, we infer immediately from (2.8) that

$$\lambda' = 1, \quad f' = 0, \quad \underline{f}' = -\underline{f}.$$

Next, we derive the transport equation for $\underline{f}$. We compute[231]

$$\begin{aligned}2\underline{\eta}_a' &= \mathbf{g}(\mathbf{D}_{e_4'}e_3', e_a') = \mathbf{g}\left(\mathbf{D}_{e_4}\left(e_3 + \underline{f}^b e_b + \frac{1}{4}|\underline{f}|^2 e_4\right), e_a + \frac{1}{2}\underline{f}_a e_4\right)\\ &= \mathbf{g}\left(\mathbf{D}_{e_4}e_3, e_a + \frac{1}{2}\underline{f}_a e_4\right) + e_4(\underline{f}_a) + \underline{f}^b\mathbf{g}\left(\mathbf{D}_{e_4}e_b, e_a + \frac{1}{2}\underline{f}_a e_4\right)\\ &\quad + \frac{1}{4}|\underline{f}|^2\mathbf{g}\left(\mathbf{D}_{e_4}e_4, e_a + \frac{1}{2}\underline{f}_a e_4\right)\\ &= 2\underline{\eta}_a - 2\omega\underline{f}_a + \nabla_4\underline{f}_a - \underline{f}_b\underline{f}_a\xi_b + \frac{1}{2}|\underline{f}|^2\xi_a\end{aligned}$$

and since $\xi = 0$ and $\omega = 0$, we infer

$$\nabla_4\underline{f} = 2(\underline{\eta}' - \underline{\eta}).$$

Since (e_1, e_2, e_3, e_4) is a PG frame and (e_1', e_2', e_3', e_4') is a PT frame, we have $\underline{\eta} = -\zeta$ and $\underline{H}' = -\frac{a\overline{q'}}{|q'|^2}\mathfrak{J}'$. Since $q' = q$ and $\mathfrak{J}' = \mathfrak{J}$, we infer in view of the

[231] We use here a more precise transformation formula for $\underline{\eta}$ than the one derived in Proposition 2.12.

definition of $\check{Z}$ in Definition 2.66,

$$\underline{H}' - \underline{H} \quad = \quad -\frac{a\overline{q'}}{|q'|^2}\mathfrak{J}' + Z = Z - \frac{a\overline{q}}{|q|^2}\mathfrak{J} = \check{Z}$$

so that $\underline{\eta}' - \underline{\eta} = \check{\zeta}$ and hence

$$\nabla_4 \underline{f} \quad = \quad 2\check{\zeta}.$$

Finally, we derive the identity for $\nabla_4' \underline{F}'$ where $\underline{F}' = \underline{f}' + i\,{}^*\underline{f}'$. To this end, we compute[232]

$$\begin{aligned}
2\zeta_a &= \mathbf{g}(\mathbf{D}_{e_a} e_4, e_3) = \mathbf{g}\left(\mathbf{D}_{e'_a + \frac{1}{2}\underline{f}'_a e'_4} e'_4, e'_3 + \underline{f}'^b e'_b + \frac{1}{4}|\underline{f}'|^2 e'_4\right)\\
&= \mathbf{g}\left(\mathbf{D}_{e'_a} e'_4, e'_3 + \underline{f}'^b e'_b\right) + \frac{1}{2}\underline{f}'_a \mathbf{g}\left(\mathbf{D}_{e'_4} e'_4, e'_3 + \underline{f}'^b e'_b\right)\\
&= 2\zeta'_a + \underline{f}'^b \chi'_{ab} + 2\omega' \underline{f}'_a + (\underline{f}' \cdot \xi')\underline{f}'_a
\end{aligned}$$

and since $\xi' = 0$ and $\omega' = 0$, we infer

$$\zeta \quad = \quad \zeta' + \frac{1}{4}\mathrm{tr}\,\chi' \underline{f}' + \frac{1}{4}{}^{(a)}\mathrm{tr}\chi'\;{}^*\underline{f}' + \frac{1}{2}\underline{f}' \cdot \widehat{\chi}'.$$

Since $q' = q$ and $\mathfrak{J}' = \mathfrak{J}$, and in view of the linearization of ζ in Definition 2.66 and the one for ζ' in Definition 9.16, we have

$$\zeta' - \zeta \quad = \quad \Re\left(\frac{a\overline{q'}}{|q'|^2}\mathfrak{J}' - \frac{a\overline{q}}{|q|^2}\mathfrak{J}\right) + \check{\zeta}' - \check{\zeta} = \check{\zeta}' - \check{\zeta}$$

and hence

$$\check{\zeta} \quad = \quad \check{\zeta}' + \frac{1}{4}\mathrm{tr}\,\chi' \underline{f}' + \frac{1}{4}{}^{(a)}\mathrm{tr}\chi'\;{}^*\underline{f}' + \frac{1}{2}\underline{f}' \cdot \widehat{\chi}'.$$

Together with the above equation for $\nabla_4 \underline{f}$, this yields

$$\nabla_4 \underline{f} \quad = \quad 2\check{\zeta} = 2\check{\zeta}' + \frac{1}{2}\mathrm{tr}\,\chi' \underline{f}' + \frac{1}{2}{}^{(a)}\mathrm{tr}\chi'\;{}^*\underline{f}' + \underline{f}' \cdot \widehat{\chi}'.$$

Since we have obtained above

$$e'_4 = e_4, \qquad \underline{f}' = -\underline{f}, \qquad \mathbf{g}(\mathbf{D}_{e'_4} e'_a, e'_b) = \mathbf{g}(\mathbf{D}_{e_4} e_a, e_b),$$

[232] We use here a more precise transformation formula for ζ than the one derived in Proposition 2.12.

we infer

$$\begin{aligned}\nabla'_4 \underline{f}'_a &= e'_4(\underline{f}'_a) - \mathbf{g}(\mathbf{D}_{e'_4} e'_a, e'_b)\underline{f}'_b \\ &= -e_4(\underline{f}_a) + \mathbf{g}(\mathbf{D}_{e_4} e_a, e_b)\underline{f}_b = -\nabla_4 \underline{f}_a\end{aligned}$$

and hence

$$\nabla'_4 \underline{f}' + \frac{1}{2}\mathrm{tr}\,\chi' \underline{f}' + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi'\ {}^*\underline{f}' = -2\check{\zeta}' - \underline{f}' \cdot \widehat{\chi}'.$$

With the notation $\underline{F}' = \underline{f}' + i\ {}^*\underline{f}'$, we infer

$$\nabla'_4 \underline{F}' + \frac{1}{2}\mathrm{tr}X'\underline{F}' = -2\check{Z}' - \underline{F}' \cdot \widehat{\chi}'$$

as desired. This concludes the proof of Lemma 9.24.

D.3. Proof of Proposition 9.29

In this section, we prove Proposition 9.29 in Kerr, see Proposition D.5 below. The general case follows by obvious modifications and is left to the reader.

D.3.1. Setting in Kerr Let $r_0 \gg m$ a fixed large constant. In the Boyer-Lindquist coordinates, we introduce

$$u := t - \int_{r_0}^{r} \frac{r'^2 + a^2}{\Delta(r')} dr', \qquad \underline{u} := t + \int_{r_0}^{r} \frac{r'^2 + a^2}{\Delta(r')} dr', \tag{D.1}$$

where the above normalization is such that we have

$$u = \underline{u} \quad \text{on} \quad r = r_0. \tag{D.2}$$

The following lemma will be useful.

Lemma D.1. *We have for* $r \geq r_0$

$$\frac{1}{2}(\underline{u} - u) = r - r_0 + 2m\log\left(\frac{r}{r_0}\right) - \frac{4m^2}{r} + \frac{4m^2}{r_0} + \int_{r_0}^{r} O\left(\frac{m^3}{r'^3}\right) dr'. \tag{D.3}$$

Proof. We have

$$\frac{r^2 + a^2}{\Delta} = \frac{r^2 + a^2}{r^2 - 2mr + a^2} = \frac{1 + \frac{a^2}{r^2}}{1 - \frac{2m}{r} + \frac{a^2}{r^2}}$$

$$\begin{aligned} &= \left(1+\frac{a^2}{r^2}\right)\left(1+\frac{2m}{r}-\frac{a^2}{r^2}+\frac{4m^2}{r^2}+O\left(\frac{m^3}{r^3}\right)\right)\\ &= 1+\frac{2m}{r}+\frac{4m^2}{r^2}+O\left(\frac{m^3}{r^3}\right) \end{aligned}$$

and hence

$$\begin{aligned} \frac{1}{2}(\underline{u}-u) &= \int_{r_0}^{r}\frac{r'^2+a^2}{\Delta(r')}dr' = \int_{r_0}^{r}\left(1+\frac{2m}{r'}+\frac{4m^2}{r'^2}+O\left(\frac{m^3}{r'^3}\right)\right)dr'\\ &= r-r_0+2m\log\left(\frac{r}{r_0}\right)-\frac{4m^2}{r}+\frac{4m^2}{r_0}+\int_{r_0}^{r}O\left(\frac{m^3}{r'^3}\right)dr' \end{aligned}$$

as desired. □

Also, let $\delta_{\mathcal{H}} > 0$ a small enough constant, and r_+ is given by

$$r_+ := m+\sqrt{m^2-a^2}. \tag{D.4}$$

Moreover, let $r_* \gg r_0$. We define the spacelike hypersurfaces

$$\mathcal{A} := \Big\{r = r_+ - \delta_{\mathcal{H}}\Big\}, \qquad \Sigma_* := \Big\{u + r = u_* + r_*, \ 1 \le u \le u_*\Big\}. \tag{D.5}$$

Also, note that ${}^{(int)}\mathcal{M}' = {}^{(int)}\mathcal{M}$ in Kerr with

$${}^{(int)}\mathcal{M} := \Big\{r_+ - \delta_{\mathcal{H}} \le r \le r_0, \quad 1 \le \underline{u} \le u_*\Big\}, \tag{D.6}$$

and that ${}^{(ext)}\mathcal{M}$ is given by

$${}^{(ext)}\mathcal{M} := \Big\{r \ge r_0, \quad 1 \le u \le u_*, \quad u + r \le u_* + r_*\Big\}. \tag{D.7}$$

Next, we look for the scalar function τ under the form

$$\tau := \underline{u} + f(r), \tag{D.8}$$

where f will be carefully chosen such that:

1. The level sets of τ are spacelike, i.e.

$$\mathbf{g}(\mathbf{D}\tau, \mathbf{D}\tau) < 0 \quad \text{on } r \ge r_+ - \delta_{\mathcal{H}}. \tag{D.9}$$

2. $\tau = \underline{u}$ at $(\underline{u} = u_*, r = r_+ - \delta_{\mathcal{H}})$, i.e.

$$f(r_+ - \delta_{\mathcal{H}}) = 0. \tag{D.10}$$

3. $\tau = u$ at $(u = u_*, r = r_*)$, i.e.

$$f(r_*) = -2\int_{r_0}^{r_*} \frac{r^2+a^2}{\Delta(r)}dr. \tag{D.11}$$

The choice of τ allows us to define ${}^{(top)}\Sigma$.

Definition D.2. *Let $f(r)$ a function satisfying* (D.9) (D.10) (D.11)*, and let τ given by* (D.8)*. Then, we define the hypersurface ${}^{(top)}\Sigma$ by*

$$^{(top)}\Sigma \quad := \quad \Big\{\tau = u_*,\ r_+ - \delta_{\mathcal{H}} \leq r \leq r_*\Big\}. \tag{D.12}$$

Remark D.3. *In view of the definition of τ and the fact that f satisfies properties* (D.9) (D.10) (D.11)*, the hypersurface ${}^{(top)}\Sigma$ given by Definition D.2 satisfies:*

1. *${}^{(top)}\Sigma$ is spacelike,*
2. *${}^{(top)}\Sigma \cap \mathcal{A} = \{\underline{u} = u_*\} \cap \mathcal{A}$,*
3. *${}^{(top)}\Sigma \cap \Sigma_* = \{u = u_*\} \cap \Sigma_*$.*

We then define ${}^{(top)}\mathcal{M}$ as follows.

Definition D.4. *Let $f(r)$ a function satisfying* (D.9) (D.10) (D.11)*, and let τ given by* (D.8)*. Then, we define the spacetime region ${}^{(top)}\mathcal{M}$ by*

$$^{(top)}\mathcal{M} \quad := \quad \{\tau \leq u_*\} \cap \{u \geq u_*\} \cap \{\underline{u} \geq u_*\}. \tag{D.13}$$

Finally, $\mathcal{M}$ is given by

$$\mathcal{M} \quad := \quad {}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M} \cup {}^{(top)}\mathcal{M}, \tag{D.14}$$

and we also introduce the region ${}^{(top)}\mathcal{M}'$ given by

$$^{(top)}\mathcal{M}' \quad := \quad \{\tau \leq u_*\} \cap \{u \geq u'_*\} \cap \{\underline{u} \geq u'_*\}, \tag{D.15}$$

where $u_* - 2 \leq u'_* \leq u_* - 1$.

We are now ready to state the analog of Proposition 9.29 in Kerr.

Proposition D.5. *Let τ a scalar function given by* (D.8)*. There exists a particular choice of $f(r)$ satisfying* (D.9) (D.10) (D.11)*, and such that:*

1. *We have on* $\mathcal{M}$

$$\mathbf{g}(\mathbf{D}\tau, \mathbf{D}\tau) \leq -\frac{m^2}{4|q|^2} < 0, \tag{D.16}$$

so that the level sets of τ *are spacelike and asymptotically null.*

2. *Denoting, on each level set of* $\underline{u}$ *in* ${}^{(top)}\mathcal{M}'(r \geq r_0)$, *by* $r_+(\underline{u})$ *the maximal value of* r *and by* $r_-(\underline{u})$ *the minimal value of* r, *we have*[233]

$$0 \leq r_+(\underline{u}) - r_-(\underline{u}) \leq 2m + 1. \tag{D.17}$$

3. *In* ${}^{(top)}\mathcal{M}'(r \leq r_0)$, τ *satisfies*

$$u_* - (m+2) \leq \tau \leq u_*. \tag{D.18}$$

4. *In* $\mathcal{M}(r \leq r_0)$, τ *satisfies*

$$e_4(\tau) = \frac{2(r^2+a^2) - \frac{m^2}{r^2}\Delta}{|q|^2}, \qquad e_3(\tau) = \frac{m^2}{r^2}, \qquad \nabla(\tau) = a\Re(\mathfrak{J}). \tag{D.19}$$

In the next section, we construct the suitable function $f(r)$. This will allow us to prove Proposition D.5 in Section D.3.3.

D.3.2. Construction of a suitable function $f(r)$ In this section, we exhibit a function $f(r)$ satisfying in particular the properties (D.9) (D.10) (D.11).

We start with the following lemma.

Lemma D.6. *We have*

$$\mathbf{g}(\mathbf{D}\tau, \mathbf{D}\tau) = \frac{1}{|q|^2}\Big(\Delta(f'(r))^2 + 2(r^2+a^2)f'(r) + a^2(\sin\theta)^2\Big). \tag{D.20}$$

Proof. We have, using the fact that $\tau = \underline{u} + f(r)$, $e_3(\underline{u}) = 0$, $e_3(r) = -1$, and $\nabla(r) = 0$,

$$\mathbf{g}(\mathbf{D}\tau, \mathbf{D}\tau) = -e_4(\tau)e_3(\tau) + |\nabla(\tau)|^2 = \Big(e_4(\underline{u}) + e_4(r)f'(r)\Big)f'(r) + |\nabla(\underline{u})|^2.$$

Since

$$e_4(\underline{u}) = \frac{2(r^2+a^2)}{|q|^2}, \qquad e_4(r) = \frac{\Delta}{|q|^2}, \qquad |\nabla(\underline{u})|^2 = \frac{a^2(\sin\theta)^2}{|q|^2},$$

[233] Note that (D.17) depends on the choice of ${}^{(top)}\Sigma$ and hence on the choice of τ.

we infer

$$\mathbf{g}(\mathbf{D}\tau, \mathbf{D}\tau) = \left(\frac{2(r^2+a^2)}{|q|^2} + \frac{\Delta}{|q|^2} f'(r)\right) f'(r) + \frac{a^2(\sin\theta)^2}{|q|^2}$$

as desired. □

Motivated by the above lemma, we consider, for $r > r_+$, the following second order polynomial

$$P(X) := \Delta X^2 + 2(r^2+a^2)X + a^2. \tag{D.21}$$

Lemma D.7. *For $r > r_+$, we have $P(X) < 0$ if and only if*

$$X_-(r) < X < X_+(r),$$

where

$$\begin{aligned} X_-(r) &:= -\frac{r^2+a^2}{\Delta} - \frac{\sqrt{r^4+a^2r^2+2a^2mr}}{\Delta}, \\ X_+(r) &:= -\frac{a^2}{r^2+a^2+\sqrt{r^4+a^2r^2+2a^2mr}}. \end{aligned}$$

Proof. We compute

$$\begin{aligned} D_P(r) &= 4(r^2+a^2)^2 - 4a^2\Delta \\ &= 4\Big(r^4+2a^2r^2+a^4-a^2(r^2-2mr+a^2)\Big) \\ &= 4\Big(r^4+a^2r^2+2a^2mr\Big) \end{aligned}$$

so that $D_P(r) > 0$ for any $r > 0$. In particular, for $r > r_+$, since $\Delta > 0$ in this case, P is a second order polynomial in X, and hence, P has two distinct roots $X_\pm(r)$ given by

$$X_\pm(r) = -\frac{r^2+a^2}{\Delta} \pm \frac{\sqrt{D_P(r)}}{2\Delta}, \qquad X_-(r) < X_+(r) < 0.$$

Note that we may rewrite $X_-(r)$ and $X_+(r)$ as

$$\begin{aligned} X_-(r) &= -\frac{r^2+a^2}{\Delta} - \frac{\sqrt{r^4+a^2r^2+2a^2mr}}{\Delta}, \\ X_+(r) &= -\frac{a^2}{r^2+a^2+\sqrt{r^4+a^2r^2+2a^2mr}}. \end{aligned}$$

Also, for $r > r_+$, since $\Delta > 0$, we have $P(X) < 0$ if and only if $X_-(r) < X < X_+(r)$ as desired. ☐

Lemma D.8. *For r large, we have the following expansions*

$$\begin{aligned} X_-(r) &= -2 - \frac{4m}{r} - \frac{8m^2 - \frac{1}{2}a^2}{r^2} + O\left(\frac{m^3}{r^3}\right), \\ X_+(r) &= -\frac{a^2}{2r^2} + O\left(\frac{m^3}{r^3}\right). \end{aligned}$$

Proof. For r large, we have the expansions

$$\begin{aligned} \sqrt{r^4 + a^2r^2 + 2a^2mr} &= r^2\sqrt{1 + \frac{a^2}{r^2} + O\left(\frac{m^3}{r^3}\right)} \\ &= r^2\left(1 + \frac{a^2}{2r^2} + O\left(\frac{m^3}{r^3}\right)\right), \end{aligned}$$

$$\begin{aligned} & -\frac{r^2 + a^2}{\Delta} - \frac{\sqrt{r^4 + a^2r^2 + 2a^2mr}}{\Delta} \\ = & -\frac{2 + \frac{3a^2}{2r^2} + O\left(\frac{m^3}{r^3}\right)}{1 - \frac{2m}{r} + \frac{a^2}{r^2}} \\ = & -\left(2 + \frac{3a^2}{2r^2} + O\left(\frac{m^3}{r^3}\right)\right)\left(1 + \frac{2m}{r} + \frac{4m^2 - a^2}{r^2} + O\left(\frac{m^3}{r^3}\right)\right) \\ = & -2 - \frac{4m}{r} - \frac{8m^2 - \frac{1}{2}a^2}{r^2} + O\left(\frac{m^3}{r^3}\right), \end{aligned}$$

and

$$-\frac{a^2}{r^2 + a^2 + \sqrt{r^4 + a^2r^2 + 2a^2mr}} = -\frac{a^2}{2r^2} + O\left(\frac{m^3}{r^3}\right),$$

which concludes the proof of the lemma. ☐

Lemma D.9. *Let, for $r \geq r_+ - \delta_{\mathcal{H}}$,*

$$\begin{aligned} f_1(r) &:= \frac{m^2}{r} - \frac{m^2}{r_+ - \delta_{\mathcal{H}}}, \\ f_2(r) &:= -2(r - r_0) - 4m\log\left(\frac{r}{r_0}\right) - \frac{m^2}{r} + c_{0,*}, \end{aligned} \tag{D.22}$$

with the constant $c_{0,*}$ chosen such that f_2 satisfies (D.11). *Then:*

1. *We have, for all* $r \geq r_+ - \delta_{\mathcal{H}}$,
$$f_1'(r) > f_2'(r).$$
2. *There exists a unique solution* $r_1 \geq r_+ - \delta_{\mathcal{H}}$ *of*
$$f_1(r_1) = f_2(r_1).$$
3. *We have* $r_1 \in (r_0, r_0 + m)$.
4. *For* $r_+ - \delta_{\mathcal{H}} < r < r_1$, *we have* $f_1(r) < f_2(r)$, *and for* $r > r_1$, *we have* $f_1(r) > f_2(r)$.
5. *The following holds:*
   - *For all* $r \geq r_+ - \delta_{\mathcal{H}}$, *we have*
   $$P(f_1'(r)) \leq -\frac{m^2}{4}.$$
   - *For all* $r \geq r_0$, *we have*
   $$P(f_2'(r)) \quad \leq \quad -8m^2.$$
   - *For all* $r \geq r_0$, *we have*
   $$P\big(\sigma f_1'(r) + (1-\sigma) f_2'(r)\big) \leq -\frac{m^2}{4}, \qquad 0 \leq \sigma \leq 1.$$

Proof. In view of the definition of $f_1(r)$ and $f_2(r)$, we have

$$\begin{aligned} f_1'(r) &= -\frac{m^2}{r^2}, \\ f_2'(r) &= -2 - \frac{4m}{r} + \frac{m^2}{r^2}. \end{aligned}$$

In particular, note that, on $r > r_+ - \delta_{\mathcal{H}}$,

$$\begin{aligned} f_1'(r) - f_2'(r) &= -\frac{2m^2}{r^2} + \frac{4m}{r} + 2 = \frac{2r^2 + 4mr - 2m^2}{r^2} \\ &= \frac{2}{r^2}(r + m - \sqrt{2}m)(r + m + \sqrt{2}m) \\ &\geq \frac{2}{r^2}\Big(r_+ - (\sqrt{2}-1)m\Big)\Big(r_+ + (1+\sqrt{2})m\Big) \\ &\geq \frac{2}{r^2}(2-\sqrt{2})(2+\sqrt{2})m^2 > 0. \end{aligned}$$

Also, since f_2 satisfies (D.11), we have

$$f_2(r_*) = -2\int_{r_0}^{r_*} \frac{r^2+a^2}{\Delta(r)}dr,$$

and hence, using in particular Lemma D.1,

$$\begin{aligned} c_{0,*} &= \frac{9m^2}{r_*} - \frac{8m^2}{r_0} + \int_{r_0}^{r_*} O\left(\frac{m^3}{r'^3}\right) dr' \\ &= -\frac{8m^2}{r_0}\left(1 + O\left(\frac{r_0}{r_*}\right) + O\left(\frac{m}{r_0}\right)\right). \end{aligned}$$

We infer

$$\begin{aligned} f_1(r_0) &= -\frac{m^2}{r_+ - \delta_{\mathcal{H}}} + \frac{m^2}{r_0}, \\ f_2(r_0) &= O\left(\frac{m^2}{r_0}\right), \\ f_1(r_0+m) &= -\frac{m^2}{r_+ - \delta_{\mathcal{H}}} + \frac{m^2}{r_0+m}, \\ f_2(r_0+m) &= -2m + O\left(\frac{m^2}{r_0}\right), \end{aligned}$$

and hence

$$f_1(r_0) < f_2(r_0), \qquad f_1(r_0+m) > f_2(r_0+m)$$

so that, there exists, by the mean value theorem, $r_1 \in (r_0, r_0+m)$ such that

$$f_1(r_1) = f_2(r_1).$$

Note that, since $f_1'(r) > f_2'(r)$ for all $r > r_+ - \delta_{\mathcal{H}}$, r_1 is the unique 0 of $f_1 - f_2$ on $r > r_+ - \delta_{\mathcal{H}}$, and we have $f_1(r) < f_2(r)$ for $r < r_1$, and $f_1(r) > f_2(r)$ for $r > r_1$.

Next, we compare $f_1'(r)$ to the roots $X_\pm(r)$ of P. We have

$$\begin{aligned} X_+(r) - f_1'(r) &= -\frac{a^2}{r^2+a^2+\sqrt{r^4+a^2r^2+2a^2mr}} + \frac{m^2}{r^2} \\ &= \frac{m^2(r^2+a^2+\sqrt{r^4+a^2r^2+2a^2mr}) - a^2r^2}{r^2(r^2+a^2+\sqrt{r^4+a^2r^2+2a^2mr})} \end{aligned}$$

$$\begin{aligned} &= \frac{(m^2-a^2)r^2+m^2a^2+m^2\sqrt{r^4+a^2r^2+2a^2mr}}{r^2(r^2+a^2+\sqrt{r^4+a^2r^2+2a^2mr})} \\ &\geq \frac{(2m^2-a^2)r^2+m^2a^2}{r^2(r^2+a^2+\sqrt{r^4+a^2r^2+2a^2mr})} \\ &\geq \frac{m^2r^2}{r^2(r^2+m^2+\sqrt{(r^2+m^2)^2})} \\ &\geq \frac{m^2}{4r^2} \end{aligned}$$

and

$$\begin{aligned} & f_1'(r) - X_-(r) \\ =\ & -\frac{m^2}{r^2} + \frac{r^2+a^2}{\Delta} + \frac{\sqrt{r^4+a^2r^2+2a^2mr}}{\Delta} \\ =\ & \frac{r^2(r^2+a^2+\sqrt{r^4+a^2r^2+2a^2mr}) - m^2\Delta}{r^2\Delta} \\ =\ & \frac{r^4-(m^2-a^2)r^2+2m^3r-a^2m^2+r^2\sqrt{r^4+a^2r^2+2a^2mr}}{r^2\Delta}. \end{aligned}$$

Note that we have, for $r > r_+$,

$$\begin{aligned} (r^4-(m^2-a^2)r^2+2m^3r-a^2m^2)' &= 4r^3-2(m^2-a^2)r+2m^3 \\ &= 2r(2r^2-2m^2+2a^2)+2m^3 \\ &\geq 4ma^2+2m^3 > 0 \end{aligned}$$

which implies, for $r > r_+$,

$$\begin{aligned} & f_1'(r) - X_-(r) \\ \geq\ & \frac{m^4-(m^2-a^2)m^2+2m^4-a^2m^2+r^2\sqrt{r^4+a^2r^2+2a^2mr}}{r^2\Delta} \\ =\ & \frac{2m^4+r^2\sqrt{r^4+a^2r^2+2a^2mr}}{r^2\Delta} \\ \geq\ & \frac{r^2}{\Delta}. \end{aligned}$$

For $r > r_+$, we infer

$$\begin{aligned} P(f_1'(r)) &= \Delta(f_1'(r)-X_-(r))(f_1'(r)-X_+(r)) \\ &= -\Delta(f_1'(r)-X_-(r))(X_+(r)-f_1'(r)) \end{aligned}$$

$$\leq \quad -\frac{m^2}{4}.$$

Also, we have for $r_+ - \delta_{\mathcal{H}} \leq r \leq r_+$

$$\begin{aligned} P(f_1'(r)) &= -|\Delta|(f_1'(r))^2 + 2(r^2+a^2)f_1'(r) + a^2 \\ &\leq 2(r^2+a^2)\left(-\frac{m^2}{r^2}\right) + a^2 \\ &\leq -2m^2 + a^2 \\ &\leq -m^2. \end{aligned}$$

Thus, we have for $r \geq r_+ - \delta_{\mathcal{H}}$

$$P(f_1'(r)) \quad \leq \quad -\frac{m^2}{4}.$$

Next, we derive an upper bound for $P(f_2'(r))$ when $r \geq r_0$. Recall that we have for r large the following expansions

$$\begin{aligned} X_-(r) &= -2 - \frac{4m}{r} - \frac{8m^2 - \frac{1}{2}a^2}{r^2} + O\left(\frac{m^3}{r^3}\right), \\ X_+(r) &= -\frac{a^2}{2r^2} + O\left(\frac{m^3}{r^3}\right). \end{aligned}$$

We infer

$$\begin{aligned} X_+(r) - f_2'(r) &= -\frac{a^2}{2r^2} + O\left(\frac{m^3}{r^3}\right) + 2 + \frac{4m}{r} - \frac{m^2}{r^2} \\ &= 2 + O\left(\frac{m}{r}\right) \end{aligned}$$

and

$$\begin{aligned} f_2'(r) - X_-(r) &= -2 - \frac{4m}{r} + \frac{m^2}{r^2} + 2 + \frac{4m}{r} + \frac{8m^2 - \frac{1}{2}a^2}{r^2} + O\left(\frac{m^3}{r^3}\right) \\ &= \frac{9m^2 - \frac{1}{2}a^2}{r^2} + O\left(\frac{m^3}{r^3}\right). \end{aligned}$$

Since

$$\Delta \quad = \quad r^2\left(1 + O\left(\frac{m}{r}\right)\right),$$

we infer, for $r \geq r_0$,

$$\begin{aligned} P(f_2'(r)) &= \Delta(f_2'(r) - X_-(r))(f_2'(r) - X_+(r)) \\ &= -\left(1+O\left(\frac{m}{r}\right)\right)\left(2+O\left(\frac{m}{r}\right)\right)\left(9m^2 - \frac{1}{2}a^2 + O\left(\frac{m^3}{r}\right)\right) \\ &\leq -\frac{17}{2}m^2\left(1+O\left(\frac{m}{r_0}\right)\right) \\ &\leq -8m^2 \end{aligned}$$

for r_0 large enough compared to m.

Finally, for $r > r_+$, we have $P''(X) = 2\Delta > 0$ and hence P is convex in X which implies, for any $0 \leq \sigma \leq 1$, for $r \geq r_0$,

$$\begin{aligned} P\big(\sigma f_1'(r) + (1-\sigma) f_2'(r)\big) &\leq \sigma P(f_1'(r)) + (1-\sigma)P(f_2'(r)) \\ &\leq -\frac{\sigma m^2}{4} - 8m^2(1-\sigma) \\ &\leq -\frac{m^2}{4} \end{aligned}$$

as desired. This concludes the proof of Lemma D.9. □

Corollary D.10. *The exists a smooth function f such that*

1. *$f(r) = f_1(r)$ for $r \leq r_0$.*
2. *$f(r) = f_2(r)$ for $r \geq r_0 + m$.*
3. *For all $r \geq r_+ - \delta_{\mathcal{H}}$, $\tau := \underline{u} + f(r)$ verifies*

$$\mathbf{g}(\mathbf{D}\tau, \mathbf{D}\tau) \leq -\frac{m^2 + 4a^2(\cos\theta)^2}{4|q|^2} < 0$$

so that the level sets of τ are spacelike in $r \geq r_+ - \delta_{\mathcal{H}}$.

Proof. Recall from Lemma D.9 that

1. We have, for all $r \geq r_+ - \delta_{\mathcal{H}}$,

$$f_1'(r) > f_2'(r).$$

2. there exists a unique solution $r_1 \geq r_+ - \delta_{\mathcal{H}}$ of

$$f_1(r_1) = f_2(r_1).$$

3. We have $r_1 \in (r_0, r_0 + m)$.

4. For $r_+ - \delta_{\mathcal{H}} < r < r_1$, we have $f_1(r) < f_2(r)$, and for $r > r_1$, we have $f_1(r) > f_2(r)$.

We deduce the existence of a smooth function f on $r \geq r_+ - \delta_{\mathcal{H}}$ such that

1. $f(r) = f_1(r)$ for $r_+ - \delta_{\mathcal{H}} \leq r \leq r_0$.
2. $f(r) = f_2(r)$ for $r \geq r_0 + m$.
3. For all $r_0 \leq r \leq r_0 + m$, $f'(r)$ verifies

$$f_2'(r) \leq f'(r) \leq f_1'(r).$$

It remains to check the property of the level sets of $\tau = \underline{u} + f(r)$. From the above properties of f, we have, for all $r \geq r_+ - \delta_{\mathcal{H}}$

$$P(f'(r)) \quad \leq \quad -\frac{m^2}{4}$$

and hence

$$\Delta(f'(r))^2 + 2(r^2 + a^2)f'(r) + a^2 \quad \leq \quad -\frac{m^2}{4}.$$

We deduce, for $r \geq r_+ - \delta_{\mathcal{H}}$,

$$\begin{aligned} \mathbf{g}(\mathbf{D}\tau, \mathbf{D}\tau) &= \frac{1}{|q|^2}\Big(\Delta(f'(r))^2 + 2(r^2 + a^2)f'(r) + a^2(\sin\theta)^2\Big) \\ &= \frac{1}{|q|^2}\Big(\Delta(f'(r))^2 + 2(r^2 + a^2)f'(r) + a^2 - a^2(\cos\theta)^2\Big) \\ &\leq -\frac{m^2 + 4a^2(\cos\theta)^2}{4|q|^2} \end{aligned}$$

as desired. □

Corollary D.11. *The function f in Corollary D.10 satisfies* (D.9) (D.10) (D.11).

Proof. Since $\tau = \underline{u} + f(r)$ verifies $\mathbf{g}(\mathbf{D}\tau, \mathbf{D}\tau) < 0$ in view of Corollary D.10, f satisfies (D.9). Also, since $f(r) = f_1(r)$ for $r_+ - \delta_{\mathcal{H}} \leq r \leq r_0$, and since $f_1(r_+ - \delta_{\mathcal{H}}) = 0$, f satisfies (D.10). Finally, since $f(r) = f_2(r)$ for $r \geq r_0 + m$, and since

$$f_2(r_*) = -2\int_{r_0}^{r_*} \frac{r^2 + a^2}{\Delta(r)} dr$$

by the choice of the constant $c_{0,*}$ appearing in the definition of f_2, f satisfies (D.11). This concludes the proof of the corollary. □

D.3.3. Proof of Proposition D.5 Let τ be given by (D.8), i.e.$\tau = \underline{u} + f(r)$. We choose $f(r)$ as in Corollary D.10. In particular, $f(r)$ satisfies (D.9) (D.10) (D.11) in view of Corollary D.11. Also, we have, in view of Corollary D.10, for all $r \geq r_+ - \delta_{\mathcal{H}}$,

$$\mathbf{g}(\mathbf{D}\tau, \mathbf{D}\tau) \leq -\frac{m^2}{4|q|^2} < 0$$

so that the first property of Proposition D.5 is satisfied.

Next, we consider the second property of Proposition D.5, i.e. the upper bound for $r_+(\underline{u}) - r_-(\underline{u})$. First, one easily sees that on each level set of $\underline{u}$ in ${}^{(top)}\mathcal{M}'(r \geq r_0)$, the maximal value $r_+(\underline{u})$ of r corresponds to the value of r on $\{u = u'_*\}$, and that the minimal value $r_-(\underline{u})$ of r corresponds to the value of r on ${}^{(top)}\Sigma \cup \{r = r_0\}$. Since $r_+(\underline{u})$ is the value on $u = u'_*$, and since we have for $r \geq r_0$

$$\frac{1}{2}(\underline{u} - u) = r - r_0 + 2m \log\left(\frac{r}{r_0}\right) - \frac{4m^2}{r} + \frac{4m^2}{r_0} + \int_{r_0}^{r} O\left(\frac{m^3}{r'^3}\right) dr',$$

we infer

$$\frac{1}{2}(\underline{u} - u'_*) = r_+(\underline{u}) - r_0 + 2m \log\left(\frac{r_+(\underline{u})}{r_0}\right) - \frac{4m^2}{r_+(\underline{u})} + \frac{4m^2}{r_0} + \int_{r_0}^{r_+(\underline{u})} O\left(\frac{m^3}{r'^3}\right) dr'.$$

We first focus on the case where $r_-(\underline{u}) \geq r_0 + m$. In that case, $r_-(\underline{u})$ is the value on $\underline{u} + f(r) = u_*$ and hence

$$\underline{u} + f(r_-(\underline{u})) = u_*.$$

We infer

$$\frac{1}{2}\Big(u_* - u'_* - f(r_-(\underline{u}))\Big) = r_+(\underline{u}) - r_0 + 2m \log\left(\frac{r_+(\underline{u})}{r_0}\right) - \frac{4m^2}{r_+(\underline{u})} + \frac{4m^2}{r_0} + \int_{r_0}^{r_+(\underline{u})} O\left(\frac{m^3}{r'^3}\right) dr'.$$

For $r \geq r_0 + m$, we have $f = f_2$ and hence

$$\frac{1}{2}(u_* - u'_*) + r_-(\underline{u}) - r_0 + 2m \log\left(\frac{r_-(\underline{u})}{r_0}\right) + \frac{m^2}{2r_-(\underline{u})} - \frac{1}{2}c_{0,*}$$

$$= r_+(\underline{u}) - r_0 + 2m\log\left(\frac{r_+(\underline{u})}{r_0}\right) - \frac{4m^2}{r_+(\underline{u})} + \frac{4m^2}{r_0} + \int_{r_0}^{r_+(\underline{u})} O\left(\frac{m^3}{r'^3}\right)dr'.$$

This yields

$$\begin{aligned} r_+(\underline{u}) - r_-(\underline{u}) + 2m\log\left(\frac{r_+(\underline{u})}{r_-(\underline{u})}\right) &= \frac{1}{2}(u_* - u'_*) + O\left(\frac{m^2}{r_*}\right) + O\left(\frac{m^3}{r_0^2}\right) \\ &\quad + \frac{m^2}{2r_-(\underline{u})} + \frac{4m^2}{r_+(\underline{u})}, \end{aligned}$$

and thus, for $r_-(\underline{u}) \geq r_0 + m$, and since $u'_* \geq u_* - 2$, we infer

$$0 \leq r_+(\underline{u}) - r_-(\underline{u}) \leq 1 + O\left(\frac{m^2}{r_0}\right) \leq 1 + m.$$

In the other case, i.e.$r_0 \leq r_-(\underline{u}) \leq r_0 + m$, denoting by $\underline{u}_0$ the value of $\underline{u}$ such that $r_-(\underline{u}_0) = r_0 + m$, we have $r_+(\underline{u}) \leq r_+(\underline{u}_0)$ and $r_-(\underline{u}) \geq r_0$ and hence

$$0 \leq r_+(\underline{u}) - r_-(\underline{u}) \leq r_+(\underline{u}_0) - r_0 = r_+(\underline{u}_0) - r_-(\underline{u}_0) + m \leq 1 + 2m$$

in this case. Thus, in both cases, we have obtained on ${}^{(top)}\mathcal{M}'(r \geq r_0)$

$$0 \leq r_+(\underline{u}) - r_-(\underline{u}) \leq 1 + 2m$$

so that the second property of Proposition D.5 is satisfied.

Next, we consider the third property of Proposition D.5, i.e. the lower bound for τ on ${}^{(top)}\mathcal{M}'(r \leq r_0)$. Since $\tau = \underline{u} + f(r)$, and since $\underline{u} \geq u'_* \geq u_* - 2$ on ${}^{(top)}\mathcal{M}'$, we infer on ${}^{(top)}\mathcal{M}'(r \leq r_0)$

$$\tau = \underline{u} + f(r) \geq u'_* + f(r) \geq u_* - 2 + \min_{r_+ - \delta_{\mathcal{H}} \leq r \leq r_0} f(r).$$

In view of the definition of f, f is strictly decreasing so that its minimum is achieved at $r = r_0$. Also, at $r = r_0$, $f = f_1$. We infer on ${}^{(top)}\mathcal{M}'(r \leq r_0)$

$$\tau \geq u_* - 2 + f_1(r_0) = u_* - 2 - \frac{m^2}{r_+ - \delta_{\mathcal{H}}} + \frac{m^2}{r_0}$$

and hence

$$\tau \geq u_* - 2 - m$$

so that the third property of Proposition D.5 is satisfied.

Finally, we consider the last property of Proposition D.5. Since $\tau = \underline{u} + f((r)$, since $f = f_1$ in for $r \leq r_0$, and since $f_1'(r) = -\frac{m^2}{r^2}$, we have in $\mathcal{M}(r \leq r_0)$

$$\begin{aligned} e_4(\tau) &= e_4(\underline{u}) + e_4(r)f'(r) = \frac{2(r^2+a^2)}{|q|^2} - \frac{m^2}{r^2}\frac{\Delta}{|q|^2}, \\ e_3(\tau) &= e_3(\underline{u}) + e_3(r)f'(r) = \frac{m^2}{r^2}, \\ \nabla(\tau) &= \nabla(\underline{u}) + f'(r)\nabla(r) = a\Re(\mathfrak{J}), \end{aligned}$$

as desired. This concludes the proof of Proposition D.5.

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