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Generalizing the Mukai Conjecture to the symplectic category and the Kostant game

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Abstract: In this paper we pose the question of whether the (generalized) Mukai inequalities hold for compact, positive monotone symplectic manifolds. We first provide a method that enables one to check whether the (generalized) Mukai inequalities hold true. This only makes use of the almost complex structure of the manifold and the analysis of the zeros of the so-called generalized Hilbert polynomial, which takes into account the Atiyah-Singer indices of all possible line bundles.

We apply this method to generalized flag varieties. In order to find the zeros of the corresponding generalized Hilbert polynomial we introduce a modified version of the Kostant game and study its combinatorial properties.

Keywords: Symplectic geometry, combinatorics.

1. Introduction

In 1988 Mukai [19] conjectured that a Fano variety M of complex dimension n with index k_0 and Picard number b should satisfy the following inequality:

$$n \geq b(k_0 - 1),$$

with equality if and only if M is $(\mathbb{C}P^{k_0-1})^b$. This conjecture was generalized by Bonavero, Casagrande, Debarre and Druel in [4], where the index above is replaced by the pseudoindex ρ_M, defined as the minimum of the evaluation of the anticanonical divisor $-K_M$ on rational curves on M. There has been extensive work towards proving these two inequalities in a variety of cases, but general proofs of these conjectures are still missing.

In this paper we start investigating similar questions in a different category. Namely, suppose that (M, ω) is a compact symplectic manifold with

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first Chern class of the tangent bundle given by c_1. The *index* of (M, ω) can be defined as the largest integer k_0 satisfying $c_1 = k_0\eta$ for some primitive element $\eta \in H^2(M; \mathbb{Z})$, modulo torsion elements. The symplectic analogues of Fano varieties are known in the literature as *positive monotone symplectic manifolds*, namely symplectic manifolds satisfying $c_1 = [\omega]$. As for Fano varieties the Picard number is exactly the second Betti number, the following question arises naturally:

Question 1. *Let (M, ω) be a compact, positive monotone symplectic manifold of dimension $2n$ with second Betti number b_2 and index k_0. Does the following inequality hold?*

$$n \geq b_2(k_0 - 1) \qquad \textbf{\textit{Symplectic Mukai inequality}} \tag{1.1}$$

For positive monotone symplectic manifolds the concept of pseudoindex can also be generalized, namely: Consider an embedded symplectic sphere S^2 and observe that, under the positive monotonicity assumption, $c_1[S^2]$ is a positive integer. Therefore one can define the *pseudoindex* ρ_0 of (M, ω) as

$$\rho_0 := \min\{c_1[S^2] \mid S^2 \text{ symplectic sphere embedded in } (M, \omega)\}.$$

In analogy with the generalized Mukai conjecture for Fano varieties we pose the following:

Question 2. *Let (M, ω) be a compact, positive monotone symplectic manifold of dimension $2n$ with second Betti number b_2 and pseudoindex ρ_0. Does the following inequality hold?*

$$n \geq b_2(\rho_0 - 1) \qquad \textbf{\textit{Generalized symplectic Mukai inequality}} \tag{1.2}$$

One of the reasons to believe that such inequalities should hold true for positive monotone symplectic manifolds is that recent work has shown that, at least under some mild symmetry assumptions, this category behaves very similarly to that of Fano varieties, see for instance [7, 18, 24].

As Questions 1 and 2 have been so far investigated only for Fano varieties, many of the tools used to answer them are, not surprisingly, algebraic geometric. Therefore, the first step to tackle them is to generalize the methods with which they can be proved to a category of spaces that includes that of positive monotone symplectic manifolds.

In Section 2 we introduce the so-called *generalized Hilbert polynomial* H for a compact almost complex manifold (M, J). This polynomial takes into account the Atiyah-Singer indices of all the possible line bundles on (M, J).

One of the main results of the section is Theorem 2.4, in which it is proved that the presence of a so-called "pointed box of zeros" of H (see Definition 2.6) implies an inequality that is related to the Mukai inequality and its generalization. This relation is investigated in Corollary 2.8 and Corollary 2.11. In particular the latter gives conditions under which a positive monotone Hamiltonian GKM space satisfies an inequality that is indeed stronger than the generalized Mukai inequality (see (2.13)).

One of the goals of Section 3 is to verify the hypotheses of Corollary 2.11 for generalized flag varieties, thus proving for them the existence of a "pointed box of zeros" of the generalized Hilbert polynomial H, see Theorem 3.3. We point out that the corresponding inequality (3.2) in Corollary 3.4 has already been proved by Pasquier [21], where he checked it through explicit calculations that depend on the type of the Dynkin diagram of the compact Lie group and the parabolic subgroup. The proof that we present in this paper relies on the combinatorics of the *(modified) Kostant game*, see Subsections 3.3 and 3.4. One of the main purposes of the already known Kostant game (see for instance [22, 6]) is to enumerate the positive roots of a compact Lie group from its Dynkin diagram. In Subsection 3.4 we explain a modified version of the Kostant game, which allows us to index the linear factors of H, thus finding its zeros.[1] It turns out that the moves of the modified Kostant game are in one to one correspondence with the reduced expressions of the minimal length representatives in the posets of the quotient of a Weyl group with a parabolic subgroup. This fact is purely combinatorial and can be generalized to any Weyl group and any of its parabolic subgroups.

2. Zeros of the generalized Hilbert polynomial and Mukai inequalities

Let (M, J) be a compact, connected, almost complex manifold of dimension $2n$. Consider the cohomology group $H^2(M;\mathbb{Z})$ and the following map

$$\begin{array}{cccc} \widetilde{H}\colon & H^2(M;\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ & \eta & \mapsto & \mathrm{Ind}(e^{2\pi \mathrm{i}\eta}), \end{array}$$

where the first Chern class of the line bundle $e^{2\pi\mathrm{i}\eta}$ is exactly η. We refer to it as the *index map*, as it computes the indices of all the possible line bundles on (M, J). Thanks to a simple observation (see for instance [23, Lemma 4.1]), the map $\widetilde{H}$ is well-defined on the lattice $\mathcal{L}$ given by the quotient $H^2(M;\mathbb{Z})/\operatorname{Tor}(H^2(M;\mathbb{Z}))$, where $\operatorname{Tor}(H^2(M;\mathbb{Z}))$ denotes the torsion

[1] The referee let us know that the modified version of the Kostant game already appeared in [20].

subgroup of $H^2(M;\mathbb{Z})$. By abuse of notation, let $\widetilde{H}\colon \mathcal{L} \to \mathbb{Z}$ be the induced index map.

Let b_2 denote the second Betti number of M, which is therefore exactly the rank of $\mathcal{L}$. Once a $\mathbb{Z}$-basis $\tau_1, \dots, \tau_{b_2}$ of $\mathcal{L}$ is chosen, we can identify $\mathcal{L}$ with $\mathbb{Z}^{b_2}$

$$\begin{array}{ccc} \mathcal{L} & \longrightarrow & \mathbb{Z}^{b_2} \\ \eta = k_1\tau_1 + \cdots + k_{b_2}\tau_{b_2} & \mapsto & (k_1, \dots, k_{b_2}) \end{array}$$

and obtain a map $\mathrm{H}\colon \mathbb{Z}^{b_2} \to \mathbb{Z}$ given by

$$\mathrm{H}(k_1, \dots, k_{b_2}) = \mathrm{Ind}(e^{2\pi\mathrm{i}\eta}), \quad \text{where} \quad \eta = k_1\tau_1 + \cdots + k_{b_2}\tau_{b_2}. \tag{2.1}$$

Lemma 2.1. *The map $\mathrm{H}\colon \mathbb{Z}^{b_2} \to \mathbb{Z}$ defined in (2.1) is a polynomial in $k_1, \dots, k_{b_2}$ of degree at most n.*

Proof. This lemma is a direct consequence of the Atiyah-Singer cohomological formula for the index [2], as for $\eta = k_1\tau_1 + \cdots k_{b_2}\tau_{b_2}$, the index of $e^{2\pi\mathrm{i}\eta}$ is given by

$$\begin{aligned} \mathrm{Ind}(e^{2\pi\mathrm{i}\eta}) &= \mathrm{Ch}(e^{2\pi\mathrm{i}\eta})\mathcal{T}[M] = \mathrm{Ch}(e^{2\pi\mathrm{i}(k_1\tau_1+\cdots+k_{b_2}\tau_{b_2})})\mathcal{T}[M] \\ &= \left(\sum_{l\geq 0} \frac{(k_1\tau_1 + \cdots + k_{b_2}\tau_{b_2})^l}{l!}\right)\mathcal{T}[M] \\ &= \sum_I a_I k_1^{l_1} \cdot \cdots \cdot k_{b_2}^{l_{b_2}}, \end{aligned}$$

where the sum runs over all the multi-indices $I = (l_1, \dots, l_{b_2})$ satisfying $l_j \in \mathbb{Z}_{\geq 0}$ for all j and $l_1 + \cdots + l_{b_2} \leq n$, $\mathcal{T}$ is the total Todd class of M and $[M]$ is the orientation class of (M, J) in homology. $\square$

Observe that the coefficients a_I above are rational numbers given by rational combinations of the evaluation on the homology class $[M]$ of products of Chern classes of (M, J) and of the classes $\tau_1, \dots, \tau_{b_2}$.

Lemma 2.1 allows us to define the following polynomial on $\mathbb{C}^{b_2}$, which is exactly the (unique) polynomial extension of H from $\mathbb{Z}^{b_2}$ to $\mathbb{C}^{b_2}$ and, by abuse of notation, is still denoted by H.

Definition 2.2. The **generalized Hilbert polynomial** is the polynomial map

$$\begin{array}{cccc} \mathrm{H}\colon & \mathbb{C}^{b_2} & \longrightarrow & \mathbb{C} \\ & (z_1, \dots, z_{b_2}) & \mapsto & \sum_I a_I z_1^{l_1} \cdot \cdots \cdot z_{b_2}^{l_{b_2}}, \end{array}$$

where the sum runs over all the multi-indices $I = (l_1, \ldots, l_{b_2})$ satisfying $l_j \in \mathbb{Z}_{\geq 0}$ for all j and $l_1 + \cdots + l_{b_2} \leq n$, and the a_I's are defined in the proof of Lemma 2.1.

Example 2.3. Let (M, J) be a compact almost complex manifold of dimension 4 with Chern classes of (TM, J) given by c_1 and c_2. The total Todd class is given in this case by $\mathcal{T} = 1 + \frac{c_1}{2} + \frac{c_1^2+c_2}{12}$. Assume that $\mathcal{L}$ has dimension 2 and let τ_1, τ_2 be one of its bases. Let $c_1 = \alpha_1\tau_1 + \alpha_2\tau_2$, where $\alpha_1, \alpha_2 \in \mathbb{Z}$. Then

$$\begin{aligned}
\mathrm{H}(k_1, k_2) &= e^{2\pi \mathrm{i}(k_1\tau_1 + k_2\tau_2)}\mathcal{T}[M] \\
&= \left(1 + k_1\tau_1 + k_2\tau_2 + \frac{(k_1\tau_1 + k_2\tau_2)^2}{2}\right)(1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12})[M] \\
&= k_1^2\frac{\tau_1^2}{2}[M] + k_2^2\frac{\tau_2^2}{2} + k_1k_2\tau_1\tau_2[M] + k_1\frac{\alpha_1\tau_1^2 + \alpha_2\tau_1\tau_2}{2}[M] \\
&+ k_2\frac{\alpha_1\tau_1\tau_2 + \alpha_2\tau_2^2}{2}[M] + \mathrm{Todd}(M),
\end{aligned}$$

where $\mathrm{Todd}(M)$ denotes the Todd genus of M and is given by $\frac{c_1^2+c_2}{12}[M]$. In the notation introduced in Lemma 2.1 we obtain $a_{(2,0)} = \frac{\tau_1^2}{2}[M]$, $a_{(0,2)} = \frac{\tau_2^2}{2}$, $a_{(1,1)} = \tau_1\tau_2[M]$ and so on.

The following theorem is one of the main results of this section.

Theorem 2.4. *Let (M, J) be a compact, connected, almost complex manifold of dimension $2n$ with second Betti number b_2 and first Chern class c_1. Let $\tilde{\mathcal{L}}$ be a full rank sublattice of $\mathcal{L} = H^2(M;\mathbb{Z})/\operatorname{Tor}(H^2(M;\mathbb{Z}))$ and $\{\eta_1, \ldots, \eta_{b_2}\}$ be a $\mathbb{Z}$-basis of $\tilde{\mathcal{L}}$ such that $c_1 \in \tilde{\mathcal{L}}$ and*

$$c_1 = \sum_{i=1}^{b_2} m_i\eta_i$$

for some positive integers $m_i \in \mathbb{Z}_{>0}$.

Let $\widetilde{H}\colon \mathcal{L} \to \mathbb{Z}$ be the index map and assume that

$$\widetilde{H}(-c_1) \neq 0,$$

and

$$\widetilde{H}(-k_1\eta_1 - \cdots - k_{b_2}\eta_{b_2}) = 0, \tag{2.2}$$

for all $k_1, \dots, k_{b_2} \in \mathbb{Z}$ *such that* $0 < k_i \leq m_i$ *for all* $i = 1, \dots, b_2$ *and* $\sum_{i=1}^{b_2} k_i < \sum_{i=1}^{b_2} m_i$. *Let* $\mathrm{H} \in \mathbb{Q}[z_1, \dots, z_{b_2}]$ *be the generalised Hilbert polynomial. Then*

$$n \geq \deg(\mathrm{H}) \geq \sum_{i=1}^{b_2} (m_i - 1). \tag{2.3}$$

Remark 2.5. It is easy to see that the first inequality in (2.3) is an equality whenever $c_1^n[M] \neq 0$. This holds for instance when M is a complex Fano variety.

In order to have a better understanding of this theorem we introduce the following terminology.

Definition 2.6. For $m = (m_1, \dots, m_l) \in \mathbb{Z}_{\geq 0}^l$, we call the set

$$\{(k_1, \dots, k_l) \in \mathbb{Z}_{\geq 0}^l \mid (k_1, \dots, k_l) \leq (m_1, \dots, m_l)\} \setminus \{(m_1, \dots, m_l)\}$$

a *pointed box at* m. An affine transformation of a pointed box is also called *pointed box*.

With this definition at hand, Theorem 2.4 means that the line bundles corresponding to the integral points of $\tilde{\mathcal{L}}$ inside the "pointed box" sketched in Figure 2.1, namely the white circles, have index zero.

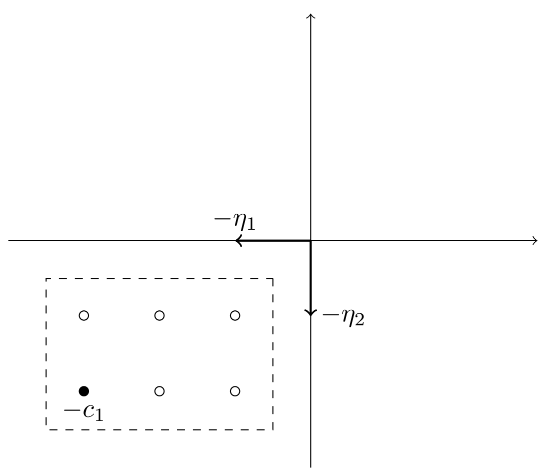


Figure 2.1: pointed box.

The proof of Theorem 2.4 relies on the following algebraic fact. First of all we set the following definitions.

- Let $\leq$ be the partial order on $\mathbb{Z}^l$, defined as follows:

$$(s_1, \ldots, s_l) \leq (k_1, \ldots, k_l) \text{ if and only if } s_1 \leq k_1, \ldots, s_l \leq k_l.$$

- Given the standard order on $\mathbb{Z}$, we define the following order preserving function

$$\begin{aligned} \mathrm{ht} : \mathbb{Z}^l &\to \mathbb{Z} \\ (k_1, \ldots, k_l) &\mapsto \mathrm{ht}(k_1, \ldots, k_l) := k_1 + \ldots + k_l \end{aligned}$$

  and we call it *height.*

Lemma 2.7. *Let $P \in \mathbb{Q}[z_1, \ldots, z_l]$ and assume that there exists $(m_1, \ldots, m_l) \in \mathbb{Z}_{\geq 0}$ such that $P(m_1, \ldots, m_l) \neq 0$ and $P(k_1, \ldots, k_l) = 0$ for every $(k_1, \ldots, k_l) \in \mathbb{Z}^l_{\geq 0}$ such that $(k_1, \ldots, k_l) \leq (m_1, \ldots, m_l)$ and $(k_1, \ldots, k_l) \neq (m_1, \ldots, m_l)$. Then*

$$\deg P \geq \mathrm{ht}(m_1, \ldots, m_l) = m_1 + \ldots + m_l. \tag{2.4}$$

Proof. The idea of the proof is to write the polynomial P in terms of an appropriate basis, show that one of the coefficients of P with respect to this basis is non zero and that the degree of the corresponding basis element equals $\mathrm{ht}(m_1, \ldots, m_l)$.

For $s \in \mathbb{Z}_{\geq 0}$, let

$$M_{s+1}(z) := 1 \cdot z \cdot (z-1) \cdot \ldots \cdot (z-s) = \prod_{j=0}^{s} (z-j)$$

and $M_0(z) := 1$. The set $\{M_s(z)\}_{s \in \mathbb{Z}_{\geq 0}}$ defines a basis of $\mathbb{Q}[z]$. Note that for an integer $k \in \mathbb{Z}_{\geq 0}$

$$M_s(k) = \begin{cases} k! & \text{if } k = s \\ 0 & \text{if } k < s. \end{cases} \tag{2.5}$$

It can be easily checked that a basis of $\mathbb{Q}[z_1, \ldots, z_l]$ is the set

$$\{M_{(s_1,\ldots,s_l)}(z_1, \ldots, z_l) := M_{s_1}(z_1) \cdot \ldots \cdot M_{s_l}(z_l)\}_{(s_1,\ldots,s_l) \in \mathbb{Z}^l_{\geq 0}}.$$

Therefore we can write any polynomial $P \in \mathbb{Q}[z_1, \ldots, z_l]$ as a linear combination of the elements of this basis

$$P(z_1, \ldots, z_l) = \sum_{(s_1,\ldots,s_l) \in \mathbb{Z}^l_{\geq 0}} h_{(s_1,\ldots,s_l)} \cdot M_{(s_1,\ldots,s_l)}(z_1, \ldots, z_l) \tag{2.6}$$

for some rational numbers $h_{(s_1,\dots,s_l)}$, such that all except a finite number of them are equal to zero.

We show the following

Claim. For a polynomial P satisfying the assumptions of the Lemma, the coefficient $h_{(k_1,\dots,k_l)}$ equals zero for every

$$(0,\dots,0) \leq (k_1,\dots,k_l) \leq (m_1,\dots,m_l) \text{ and } (k_1,\dots,k_l) \neq (m_1,\dots,m_l).$$

Moreover the coefficient $h_{(m_1,\dots,m_l)}$ is nonzero.

The claim implies that

$$\deg(P) \geq \deg(M_{(m_1,\dots,m_l)}) = m_1 + \cdots + m_l,$$

which is the desired inequality (2.4).

From (2.5), the definition of $M_{(s_1,\dots,s_l)}$ and (2.6) it follows that

$$P(k_1,\dots,k_l) = \sum_{(0,\dots,0)\leq(s_1,\dots,s_l)\leq(k_1,\dots,k_l)} h_{(s_1,\dots,s_l)} \cdot M_{(s_1,\dots,s_l)}(k_1,\dots,k_l) \tag{2.7}$$

for any $(k_1,\dots,k_l) \in \mathbb{Z}^l_{\geq 0}$. In particular $P(0) = 0$, unless $(m_1,\dots,m_l) = (0,\dots,0)$, in which case we are done. We prove the claim using finite induction on the height function: assume that $h_{(k_1,\dots,k_l)} = 0$ for all $(k_1,\dots,k_l) \in \mathbb{Z}^l_{\geq 0}$ with $(k_1,\dots,k_l) \leq (m_1,\dots,m_l)$ and $\operatorname{ht}(k_1,\dots,k_l) \leq k < \operatorname{ht}(m_1,\dots,m_l)$. We show that for any $(k_1,\dots,k_l) \in \mathbb{Z}^l_{\geq 0}$ with $(k_1,\dots,k_l) \leq (m_1,\dots,m_l)$ and $\operatorname{ht}(k_1,\dots,k_l) = k+1$, either $(k_1,\dots,k_l) \neq (m_1,\dots,m_l)$ and $h_{(k_1,\dots,k_l)} = 0$ or $(k_1,\dots,k_l) = (m_1,\dots,m_l)$ and $h_{(m_1,\dots,m_l)} \neq 0$. The induction on the height stops when we reach $(k_1,\dots,k_l) = (m_1,\dots,m_l)$, which is the only maximal element in the set $\{(s_1,\dots,s_l) \in \mathbb{Z}^l_{\geq 0} : (s_1,\dots,s_l) : (s_1,\dots,s_l) \leq (m_1,\dots,m_l)\}$ with respect to the order that we defined above.

Let $(k_1,\dots,k_l) \in \mathbb{Z}_{\geq 0}$ be such that $\operatorname{ht}(k_1,\dots,k_l) = k+1$. Then from (2.7) and the induction hypothesis we obtain

$$P(k_1,\dots,k_l) = h_{(k_1,\dots,k_l)} M_{(k_1,\dots,k_l)}(k_1,\dots,k_l),$$

as any $(s_1,\dots,s_l)$ satisfying

$$(0,\dots,0) \leq (s_1,\dots,s_l) \leq (k_1,\dots,k_l) \text{ and } (s_1,\dots,s_l) \neq (k_1,\dots,k_l)$$

has height less than $k+1$, and therefore by induction satisfies $h_{(s_1,\dots,s_l)} = 0$.

If

$$(k_1,\dots,k_l) \le (m_1,\dots,m_l) \text{ and } (k_1,\dots,k_l) \neq (m_1,\dots,m_l),$$

then $h_{(k_1,\dots,k_l)} = 0$ as $P(k_1,\dots,k_l) = 0$ by the assumption of the Lemma and $M_{(k_1,\dots,k_l)}(k_1,\dots,k_l) = k_1! \cdot \ldots \cdot k_l! \neq 0$. Likewise, if $(k_1,\dots,k_l) = (m_1,\dots,m_l)$, then $P(m_1,\dots,m_l) \neq 0$ implies $h_{(m_1,\dots,m_l)} \neq 0$. This finishes the proof of the claim and hence of the Lemma. □

Proof of Theorem 2.4. The first inequality, $n \ge \deg \mathrm{H}$, comes from Lemma 2.1 and the definition of generalized Hilbert polynomial. In order to prove the second inequality we introduce the following: In analogy with the definition of the generalised Hilbert polynomial $\mathrm{H}(z_1,\dots,z_{b_2})$, we define the polynomial $\mathrm{H}'(z_1,\dots,z_{b_2})$ that at integral values $(k_1,\dots,k_{b_2}) \in \mathbb{Z}^{b_2}$ satisfies

$$\mathrm{H}'(k_1,\dots,k_{b_2}) := \widetilde{H}(k_1\eta_1 + \cdots + k_{b_2}\eta_{b_2}).$$

We observe that, as $\tilde{\mathcal{L}}$ is a full rank sublattice of $\mathcal{L}$, $\mathrm{H}(z_1,\dots,z_{b_2})$ and $\mathrm{H}'(z_1,\dots,z_{b_2})$ differ only by a linear, invertible transformation of the coordinates $(z_1,\dots,z_{b_2})$, namely there exists a linear, invertible transformation $A\colon \mathbb{C}^{b_2} \simeq H^2(M;\mathbb{C}) \to \mathbb{C}^{b_2} \simeq H^2(M;\mathbb{C})$ satisfying $A(\tau_j) = \eta_j$ for all $j = 1,\dots,b_2$, implying $\mathrm{H}'(z_1,\dots,z_{b_2}) = \mathrm{H}(A(z_1,\dots,z_{b_2}))$. From the fact that A is linear and invertible we have $\deg(\mathrm{H}') = \deg(\mathrm{H})$.

Consider now the polynomial

$$Q(z_1,\dots,z_{b_2}) := \mathrm{H}'(-z_1-1,\dots,-z_{b_2}-1),$$

whose degree is clearly the degree of H', and hence the degree of H.

We recall that the first Chern class c_1 satisfies $c_1 = \sum_{j=1}^{b_2} m_i\eta_i$ for some positive integers $m_1,\dots,m_{b_2}$ and observe that the polynomial Q satisfies

$$Q(m_1-1,\dots,m_{b_2}-1) = \mathrm{H}'(-m_1,\dots,-m_{b_2}) = \widetilde{H}(-m_1\eta_1 - \cdots - m_{b_2}\eta_{b_2}),$$

the latter being nonzero by assumption. Moreover by (2.2) we have

$$Q(k_1,\dots,k_{b_2}) = 0$$

for all $(k_1,\dots,k_{b_2}) \in \mathbb{Z}^{b_2}_{\ge 0}$ such that

$$(k_1,\dots,k_{b_2}) \le (m_1-1,\dots,m_{b_2}-1) \text{ and } (k_1,\dots,k_{b_2}) \neq (m_1-1,\dots,m_{b_2}-1).$$

Therefore Lemma 2.1 implies that

$$\deg(\mathrm{H}) = \deg(Q) \geq \sum_{i=1}^{b_2}(m_i - 1),$$

which concludes the proof. $\square$

We are now ready to derive the corollaries of this section that concern the Mukai inequality. Suppose that $\{\tau_1, \ldots, \tau_{b_2}\}$ is a basis of $\mathcal{L}$ such that $c_1 = \sum_{i=1}^{b_2} n_i\tau_i$ for some positive integers n_i, for all $i = 1, \ldots, b_2$. Observe that the index k_0 of (M, J) satisfies $k_0 = \gcd(n_1, \ldots, n_{b_2})$.

Corollary 2.8. *Let (M, J) be a compact, connected, almost complex manifold of dimension $2n$ with second Betti number b_2, first Chern class c_1 and index k_0.*

Let $\{\tau_1, \ldots, \tau_{b_2}\}$ be a basis of $\mathcal{L} = H^2(M;\mathbb{Z})/\operatorname{Tor}(H^2(M;\mathbb{Z}))$ such that

$$c_1 = \sum_{i=1}^{b_2} n_i\tau_i$$

for some positive integers $n_i \in \mathbb{Z}_{>0}$. For $i = 1, \ldots, b_2$ define $\eta_i := \frac{n_i}{k_0}\tau_i \in \mathcal{L}$ and assume that the index map $\widetilde{H}\colon \mathcal{L} \to \mathbb{Z}$ satisfies

$$\widetilde{H}(-c_1) \neq 0,$$

and

$$\widetilde{H}(-k_1\eta_1 - \cdots - k_{b_2}\eta_{b_2}) = 0, \tag{2.8}$$

for all $k_1, \ldots, k_{b_2} \in \mathbb{Z}$ such that $0 < k_i \leq k_0$ for all $i = 1, \ldots, b_2$ and $\sum_{i=1}^{b_2} k_i < k_0 b_2$.

Let $\mathrm{H} \in \mathbb{Q}[z_1, \ldots, z_{b_2}]$ be the generalised Hilbert polynomial. Then (M, J) satisfies the Mukai inequality, more precisely

$$n \geq \deg(\mathrm{H}) \geq b_2(k_0 - 1). \tag{2.9}$$

Proof. The first inequality is a consequence of Lemma 2.1 and the definition of Hilbert polynomial. The second inequality follows easily from Theorem 2.4, as the set $\{\eta_1, \ldots, \eta_{b_2}\}$ is $\mathbb{Z}$-independent and the lattice $\mathcal{L}'$ given by $\mathbb{Z}\langle\eta_1, \ldots, \eta_{b_2}\rangle$ is a full rank sublattice of $\mathcal{L}$. Moreover $c_1 \in \mathcal{L}'$ and it is given by $c_1 = \sum_{i=1}^{b_2} k_0\eta_i$. $\square$

The second corollary of Theorem 2.4 concerns the generalized Mukai inequality for a positive monotone symplectic manifold with pseudoindex ρ_0. Henceforth we focus on the following category of spaces, which is also that appearing in the next section, and which enables us to find a special basis of $H^2(M;\mathbb{Z})$ such that the coefficients of the first Chern class in this basis are symplectic volumes of some special embedded spheres.

Suppose that (M,ω) is a compact symplectic manifold endowed with a Hamiltonian action of a compact torus T: We recall that the action of a compact torus T on a symplectic manifold (M,ω) is called Hamiltonian if there exists a T-invariant map $\psi\colon M \to \mathrm{Lie}(T)^*$, called **moment map**, satisfying

$$d\langle\psi(\cdot),\xi\rangle = -\iota_{\xi^\#}\omega,$$

where $\langle\cdot,\cdot\rangle$ denotes the natural pairing between $Lie(T)^*$ and $Lie(T)$, $\xi^\#$ is the vector field generated by the action and $\iota_{\xi^\#}\cdot$ denotes the contraction operator. The function $\psi^\xi\colon M \to \mathbb{R}$ defined as $\psi^\xi(\cdot) = \langle\psi(\cdot),\xi\rangle$ is called the ξ-component of the moment map. The triple (M,ω,ψ) is called a **(compact) Hamiltonian T-space**. In this paper we consider Hamiltonian T-spaces whose fixed point set – denoted by M^T – is discrete and hence, by compactness of M, finite. Moreover we assume that (M,ω,ψ) is a **GKM (Goresky-Kottwitz-MacPherson) space**, namely, for every codimension one subtorus H, the connected components of the set of points fixed by H is of dimension at most 2. This condition can also be rephrased as follows. Consider the isotropy action of T on the tangent space at a fixed point $p \in M^T$ and its corresponding weights, called **isotropy weights** of p. Then the action is GKM if and only if for every $p \in M^T$ the weights of the isotropy action are pairwise linearly independent. (See [10] for the original reference or for instance [12, Chapter 11].) Indeed the two-dimensional components mentioned above are spheres corresponding exactly to the fixed points of the codimension one subtorus $\exp\{\xi \in \mathrm{Lie}(T) \mid \langle\alpha,\xi\rangle = 0\}$. Note that, as they are fixed components of a subgroup of T, they are symplectic submanifolds of M. The restriction of the T-action to any of those has exactly two fixed points $p,q \in M^T$, and the intersection properties of the set of these isotropy spheres is encoded in a graph (V,E_{GKM}) called the **GKM graph**: the vertex set is exactly the fixed point set M^T, and every edge represents an isotropy sphere. Henceforth we restrict to Hamiltonian T-spaces whose action is GKM.

The basis of $H^2(M;\mathbb{Z})$ that allows us to translate Theorem 2.4 in terms of the pseudoindex comes from a well-known basis in the equivariant cohomology ring of (M,ω,ψ), called the **canonical basis**. In order to introduce it we need the following terminology: Consider a generic component ψ^ξ of the moment map, where generic means that $\langle \alpha, \xi \rangle \neq 0$ for every isotropy weight α occurring at a fixed point p. Define λ_p to be the number of negative weights at p, namely the number of isotropy weights α of p such that $\langle \alpha, \xi \rangle < 0$, and let Λ_p^- be the product of these weights. We say that a generic component $\psi^\#$ is **index increasing** if $\lambda_p < \lambda_q$ for every edge connecting p to q in E_{GKM} such that $\psi^\xi(p) < \psi^\psi(q)$.

Since the manifold is acted on upon a torus T, which we assume to have dimension d, it is useful to consider the equivariant cohomology ring with $\mathbb{Z}$-coefficients $H_T^*(M;\mathbb{Z})$, which has the structure of a $H_T^*(\{p\};\mathbb{Z})$-module, where $\{p\}$ is simply a point. Once a basis $\{x_1,\ldots,x_d\}$ of the dual lattice of $\mathrm{Lie}(T)^*$ is chosen, the latter can be identified with the polynomial ring $\mathbb{Z}[x_1,\ldots,x_d]$. Notice that $\Lambda_p^- \in \mathbb{Z}[x_1,\ldots,x_d]$. More generally the restriction $\tau(p)$ to a fixed point p of a class $\tau \in H_T^*(M;\mathbb{Z})$ lives in this polynomial ring.

The theorem below was proved by Guillemin and Zara [13] over the rationals and then extended to the integers by Goldin and Tolman [9].

Theorem 2.9. *Let (M,ω,ψ) be a compact Hamiltonian T-space such that the T-action is GKM. Suppose that there exists a $\xi \in \mathrm{Lie}(T)$ such that ψ^ξ is index increasing. Then for every $p \in M^T$ there exists a unique element $\tilde{\tau}_p \in H_T^{2\lambda_p}(M;\mathbb{Z})$ satisfying the following properties:*

(i) $\tilde{\tau}_p(p) = \Lambda_p^-$;
(ii) $\tilde{\tau}_p(q) = 0$ *for all* $q \in M^T \setminus \{p\}$ *such that* $\lambda_q \leq \lambda_p$.

*Moreover the set $\{\tilde{\tau}_p\}_{p\in M^T}$ of these elements is a basis of $H_T^*(M;\mathbb{Z})$ as a module over $\mathbb{Z}[x_1,\ldots,x_d]$ and is called the **canonical basis** of (M,ω,ψ) w.r.t. the component ψ^ξ.*

Consider the natural restriction $r\colon H_T^*(M;\mathbb{Z}) \to H^*(M;\mathbb{Z})$. Another virtue of this basis is that it restricts to a basis of $H^*(M;\mathbb{Z})$ (regarded as a $\mathbb{Z}$-module). Therefore the elements $\{\tau_p := r(\tilde{\tau}_p), \lambda_p = 1\}$ form a basis of $H^2(M;\mathbb{Z})$. By abuse of notation we refer to these elements as the canonical basis of $H^2(M;\mathbb{Z})$.

Before proving the main property of this canonical basis, which allows us to link Theorem 2.4 to the generalized Mukai conjecture, we observe the following:

- Since (M,ω) is compact and symplectic, $H^2(M;\mathbb{Z}) \neq 0$, therefore there must be fixed points p with $\lambda_p = 1$;
- For every fixed point p with $\lambda_p = 1$ there exists a unique symplectic, invariant sphere containing the (unique) minimum p_0 of ψ^ξ and p as respectively "south" and "north pole". Let $S_1^2,\ldots,S_{b_2}^2$ be the collection of these spheres.

We are ready for the following

Proposition 2.10. *Let (M, ω, ψ) be a compact Hamiltonian T-space such that the T-action is GKM. Suppose that there exists a $\xi \in \mathrm{Lie}(T)$ such that ψ^ξ is index increasing and consider the canonical basis $\{\tau_1, \ldots, \tau_{b_2}\}$ described above.*

Let $\alpha \in H^2(M;\mathbb{Z})$ and write it as $\alpha = \sum_{i=1}^{b_2} m_i \tau_i$, where $m_i \in \mathbb{Z}$ for all $i = 1, \ldots, b_2$. Let $S_1^2, \ldots, S_{b_2}^2$ be the collection of symplectic spheres connecting p_0 to the fixed points p with $\lambda_p = 1$. Then, modulo reordering the elements τ_i,

$$m_i = \int_{S_i^2} \alpha, \quad \text{for all } i = 1, \ldots, b_2.$$

Proof. By the Kirwan surjectivity theorem [16], every cohomology class $\tau \in H^*(M;\mathbb{Z})$ admits an equivariant extension $\tilde{\tau} \in H_T^*(M;\mathbb{Z})$, namely there exists $\tilde{\tau}$ such that $r(\tilde{\tau}) = \tau$. This extension is not unique, but for degree 2 elements any two extensions differ by a degree one polynomial in $\mathbb{Z}[x_1, \ldots, x_d]$ without constant term. Therefore, for the given $\alpha \in H^2(M;\mathbb{Z})$ there exists $\tilde{\alpha} \in H_T^2(M;\mathbb{Z})$ such that

$$\tilde{\alpha} = \sum_{i=1}^{b_2} m_i \tilde{\tau}_i + P, \tag{2.10}$$

for some degree one polynomial $P \in \mathbb{Z}[x_1, \ldots, x_d]$ without constant term. By evaluating (2.10) at the minimum p_0 of ψ^ξ and using property (ii) in Theorem 2.9 we obtain that $\tilde{\alpha}(p_0) = P$. Let p_j be a point with $\lambda_{p_j} = 1$. Then, evaluating (2.10) at p_j, using property (ii) in Theorem 2.9 and the previous computation, we have

$$\tilde{\alpha}(p_j) = m_j \tilde{\tau}_j(p_j) + \tilde{\alpha}(p_0) = m_j \alpha_j + \tilde{\alpha}(p_0),$$

where α_j is the unique negative weight at p_j. Therefore m_j, thought as a rational function in $x_1, \ldots, x_d$, is exactly given by

$$m_j = \frac{\tilde{\alpha}(p_j) - \tilde{\alpha}(p_0)}{\alpha_j}. \tag{2.11}$$

Let S_j^2 be the unique invariant symplectic sphere stabilized pointwise by $\exp\{\xi \in \mathrm{Lie}(T) \mid \langle \alpha_j, \xi \rangle = 0\}$ and containing p_0 and p_j. Then the weight of the isotropy T-action on the tangent space $T_{p_0} S_{p_j}^2$ is exactly $-\alpha_j$. Therefore, by the Atiyah-Bott [1] and Berline-Vergne [3] localization formula in equivariant cohomology, m_j is exactly $\int_{S_j^2} \tilde{\alpha}$ which, by degree reasons, is equal to $\int_{S_j^2} \alpha$. □

Corollary 2.11. *Let (M, ω) be a positive monotone compact symplectic manifold, therefore $c_1 = [\omega]$, with pseudoindex ρ_0. Assume that (M, ω) can be endowed with a Hamiltonian T-action which is also GKM. Suppose that there exists a $\xi \in \mathrm{Lie}(T)$ such that ψ^ξ is index increasing and consider the canonical basis $\{\tau_1, \dots, \tau_{b_2}\} \subset H^2(M;\mathbb{Z})$ described above. Let*

$$c_1 = \sum_{i=1}^{b_2} m_i \tau_i$$

for some integers $m_i \in \mathbb{Z}$. Then $m_i > 0$ for all $i = 1, \dots, b_2$.

Moreover, if $\widetilde{H} \colon \mathcal{L} \to \mathbb{Z}$ denotes the index map and

$$\widetilde{H}(-k_1\tau_1 - \cdots - k_{b_2}\tau_{b_2}) = 0, \tag{2.12}$$

for all $k_1, \dots, k_{b_2} \in \mathbb{Z}$ such that $0 < k_i \leq m_i$ for all $i = 1, \dots, b_2$ and $\sum_{i=1}^{b_2} k_i < \sum_{i=1}^{b_2} m_i$, then

$$n \geq \sum_{i=1}^{b_2}(m_i - 1) \geq b_2(\rho_0 - 1), \tag{2.13}$$

and therefore (M, ω) satisfies the generalized Mukai inequality.

Proof. Let $S_1^2, \dots, S_{b_2}^2$ be the collection of symplectic spheres connecting p_0, the minimum of ψ^ξ, to the fixed points p_j with $\lambda_{p_j} = 1$, for $j = 1, \dots, b_2$. Then by Proposition 2.10 and the monotonicity condition $c_1 = [\omega]$ we have that

$$m_j = \int_{S_j^2} c_1 = \int_{S_j^2} \omega > 0,$$

thus proving the first claim.

Now we observe that for the class of spaces described in the hypotheses, $\widetilde{H}(-c_1) \neq 0$ (this holds indeed for all compact Hamiltonian T-spaces with discrete fixed point set). Indeed, by [23, Proposition 41], $\widetilde{H}(-c_1) = (-1)^n N_0$, where N_0 is the number of fixed points with zero negative weights which, for connected, compact Hamiltonian T-spaces is exactly 1. Therefore, the inequality

$$n \geq \sum_{i=1}^{b_2}(m_i - 1)$$

in (2.13) is a direct consequence of Theorem 2.4, where we can take $\eta_i = \tau_i$ for all $i = 1, \ldots, b_2$.

The last inequality in (2.13) follows from Proposition 2.10, because $m_j = \int_{S_j^2} c_1 \geq \rho_0$, where the last inequality follows directly by the definition of pseudoindex. $\square$

In the next section we apply these results, in particular Corollary 2.11, to generalized flag varieties. Indeed these are Hamiltonian T-spaces whose action is GKM (see [11]), they admit an index increasing component of the moment map (see [25, Lemma 6.4]) and a symplectic form with respect to which they are positive monotone (see [8, Proposition 5.24]). In order to obtain the inequalities in (2.13) it is then sufficient to find the "pointed box" of zeros of the generalized Hilbert polynomial, which is the content of Theorem 3.3.

3. Mukai Conjecture for coadjoint orbits and the Kostant game

3.1. Set up and reformulation of our main claim

Let G be a compact Lie group. We denote by $G_{\mathbb{C}}$ the complexification of G. Let $P \subset G_{\mathbb{C}}$ be a parabolic subgroup of $G_{\mathbb{C}}$.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. We denote by $(\cdot, \cdot)$ an Ad-invariant inner product defined on $\mathfrak{g}$ and identify the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ via this inner product.

Let $T \subset G$ be a maximal torus and let $B \subset G_{\mathbb{C}}$ be a Borel subgroup with $T_{\mathbb{C}} \subset B \subset P$, where $T_{\mathbb{C}}$ denotes the complexification of T. Let $R \subset \mathfrak{t}^*$ denote the set of roots and R^+ be the system of positive roots compatible with the choice of the Borel subgroup $B \subset G_{\mathbb{C}}$ with simple roots $S \subset R^+$. Let $W := N_G(T)/T$ be the Weyl group of G. For every root $\alpha \in R$, let $s_\alpha \in W$ be the reflection associated to it. For the parabolic subgroup $P \subset G_{\mathbb{C}}$, let $W_P := N_P(T)/T$ be the Weyl group of P, $S_P \subset S$ be the subset of simple roots whose corresponding reflections are in W_P and R_P^+ be the set of positive roots generated by the simple roots S_P.

We say that a simple root α is *adjacent* to P if $\alpha \in S \setminus S_P$ and if in the Dynkin diagram of G, the simple root α is adjacent to the Dynkin diagram of S_P.

The set of fundamental weights of G will be denoted by $\{\varpi_\alpha \mid \alpha \in S\}$. Recall that they are defined as the dual basis to the basis of coroots $\check{S} := \{\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} \mid \alpha \in S\}$. Just as before, we define $\check{S}_P := \{\check{\alpha} \in \check{S} \mid \alpha \in S_P\}$ and let $\check{R}_P^+$ be the set of positive coroots generated by the simple coroots $\check{S}_P$.

If $\varpi \in \mathbb{Z}\{\varpi_\alpha \mid \alpha \in S\}$ is a weight that vanishes on all β in S_P, it determines a character on P, and so a line bundle $\mathbb{L}_\varpi = G_\mathbb{C} \times^P \mathbb{C}(\varpi)$ on $G_\mathbb{C}/P$. We identify the Chern class $c_1(\mathbb{L}_\varpi) \in H^2(G_\mathbb{C}/P, \mathbb{Z})$ with the weight ϖ and we obtain an isomorphism

$$\begin{aligned} \mathbb{Z}\{\lambda_\alpha \mid \alpha \in S \setminus S_P\} &\to H^2(G_\mathbb{C}/P, \mathbb{Z}) \\ \varpi &\mapsto c_1(\mathbb{L}_\varpi) \end{aligned} \tag{3.1}$$

(see for instance [26]).

We denote by H_P the index map of $G_\mathbb{C}/P$. More precisely, for $\varpi \in \mathbb{Z}\{\varpi_\alpha \mid \alpha \in S \setminus S_P\} \cong \mathrm{H}^2(G_\mathbb{C}/P;\mathbb{Z})$, the value of $\mathrm{H}_P(\varpi)$ is the index $\mathrm{Ind}(\mathbb{L}_\varpi)$ of the line bundle $\mathbb{L}_\varpi$. If we write $\varpi = \sum_{\alpha \in S \setminus S_P} k_\alpha \varpi_\alpha$, Lemma 2.1 allows us to view $\mathrm{H}_P(\varpi)$ as a complex polynomial of degree at most the complex dimension of $G_\mathbb{C}/P$ in k_α's over $\alpha \in S \setminus S_P$. We will call this polynomial the *Hilbert polynomial of* $G_\mathbb{C}/P$. The Bott-Borel-Weil Theorem gives us an explicit formula for the Hilbert polynomial H_P. We provide this formula and a short explanation of from where it follows in the next statement.

Theorem 3.1. *For*

$$\varpi = \sum_{\alpha \in S \setminus S_P} k_\alpha \varpi_\alpha \in \mathbb{Z}\{\varpi_\alpha \mid \alpha \in S \setminus S_P\} \cong \mathrm{H}^2(G_\mathbb{C}/P;\mathbb{Z}),$$
$$\mathrm{H}_P(\varpi) = \mathrm{Ind}(\mathbb{L}_\varpi) = \prod_{\alpha \in R_+ \setminus R_P^+} \frac{\langle \varpi + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle},$$

where $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha = \sum_{\alpha \in S} \varpi_\alpha$ is half of the sum of positive roots which is also equal to the sum of all fundamental weights.

Proof. It follows from the Hirzebruch-Riemann-Roch Thereom that the index of the bundle $\mathbb{L}_\varpi$ is the Euler characteristic of $\mathbb{L}_\varpi$ in sheaf cohomology, namely the alternating sum

$$\mathrm{Ind}(\mathbb{L}_\varpi) = \sum_j (-1)^j \dim \mathrm{H}^j(G_\mathbb{C}/P; \mathbb{L}_\varpi).$$

The Bott-Borel-Weil Theorem implies that for ϖ a dominant weight, i.e. $\varpi = \sum_{\alpha \in S \setminus S_P} k_\alpha \varpi_\alpha$ with $k_\alpha \in \mathbb{Z}_{>0}$, higher cohomology vanishes and thus

$$\mathrm{H}_P(\varpi) = \mathrm{Ind}(\mathbb{L}_\varpi) = \dim \mathrm{H}^0(G_\mathbb{C}/P; \mathbb{L}_\varpi)$$

holds for a dominant weight (see for instance [5, 17]). The Borel-Weil Theorem states that for ϖ a dominant weight, the action of G on $\mathrm{H}^0(G_\mathbb{C}/B; \mathbb{L}_\varpi)$ is the irreducible representation of $G_{\mathbb{C}}$ with highest weight ϖ (see e.g. [26]). The dimension of $\mathrm{H}^0(G_{\mathbb{C}}/P; \mathbb{L}_{\varpi})$ for a dominant weight ϖ follows from the Weyl character formula and equals

$$\prod_{\alpha \in R_+} \frac{\langle \varpi + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} = \prod_{\alpha \in R_+ \setminus R_P^+} \frac{\langle \varpi + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}$$

where ρ is half of the sum of positive roots and we are done. □

The first Chern class $c_1(T(G_{\mathbb{C}}/P))$ corresponds to

$$\sum_{\alpha \in R^+ \smallsetminus R_P^+} \alpha = \sum_{\alpha \in R^+} \alpha - \sum_{\alpha \in R_P^+} \alpha = \sum_{\alpha \in S} 2\varpi_\alpha - \sum_{\alpha \in R_P^+} \alpha$$

via the identification in (3.1) and can be written as a linear combination of fundamental weights as

$$\sum_{\beta \in S} n_\beta \varpi_\beta,$$

where

$$n_\beta := \sum_{\alpha \in R \smallsetminus R_P^+} \langle \alpha, \beta^\vee \rangle = 2 - \Big\langle \sum_{\alpha \in R_P^+} \alpha, \beta^\vee \Big\rangle.$$

Note for instance that in type A, $n_\beta = 0$ if $\beta \in S_P$ and $n_\beta = 2 + l$ if $\beta \in S \setminus S_P$, where l denotes the sum of the sizes of the connected components of the Dynkin diagram of S_P adjacent to β. In particular, if $\beta \in S \setminus S_P$ is not adjacent to P, then $n_\beta = 2$. We generalise some of these remarks for any type in the following lemma.

Lemma 3.2. *If $\beta \in S_P$, then $n_\beta = 0$. If $\beta \in S \setminus S_P$ is a simple root not adjacent to P, then $n_\beta = 2$.*

Proof. If $\beta \in S_P$, then $\Big\langle \sum_{\alpha \in R_P^+} \alpha, \beta^\vee \Big\rangle = 2$ and $n_\beta = 0$. Indeed, if we look at the Dynkin diagram for G, and let Γ_P be the subgraph corresponding to P, i.e. the subgraph containing all the simple roots that are in P and all the edges between them. We call by Γ_{P_β} the connected component of Γ_P containing β and by P_β the parabolic subgroup associated to the simple roots with Dynkin diagram Γ_{P_β}. The connected component Γ_P^β is either a point or a Dynkin diagram of some classical group. Only the roots α generated by the simple roots contained in Γ_{P_β} contribute to $\langle \sum_{\alpha \in R_P^+} \alpha, \beta^\vee \rangle$. For other roots

$\alpha \in R_P^+$ we have that $\langle \alpha, \beta^\vee \rangle = 0$. Therefore, for such β, one has that

$$\Big\langle \sum_{\alpha\in R_P^+} \alpha, \beta^\vee \Big\rangle = \Big\langle \sum_{\alpha\in R_{P_\beta}^+} \alpha, \beta^\vee \Big\rangle = 2\Big\langle \sum_{\alpha\in S_{P_\beta}} \varpi_\alpha, \beta^\vee \Big\rangle = 2.$$

If $\beta \notin S_P$ and β is not adjacent to any simple root in Γ_P, then $\langle \alpha, \beta^\vee \rangle = 0$ for all $\alpha \in R_P^+$ and $n_\beta = 2$. $\square$

Theorem 3.3. *Let $\{\varpi_\alpha \mid \alpha \in S\}$ be the set of fundamental weights. If we write*

$$c_1(T(G_{\mathbb{C}}/P)) = \sum_{\alpha\in R^+\smallsetminus R_P^+} \alpha = \sum_{\beta\in S\smallsetminus S_P} n_\beta \varpi_\beta$$

where n_β is the positive integer $\sum_{\alpha\in R\smallsetminus R_P^+}\langle \alpha, \beta^\vee \rangle$, then for each

$$\varpi = -\sum_{\beta\in S\setminus S_P} \tilde{n}_\beta \varpi_\beta$$

with

$$0 < \tilde{n}_\beta \le n_\beta \ \ \textit{but} \ \sum_{\beta\in S\setminus S_P} \tilde{n}_\beta < \sum_{\beta\in S\setminus S_P} n_\beta,$$

we have that

$$\mathrm{H}_P(\varpi) = \mathrm{Ind}(\mathbb{L}_\varpi) = 0.$$

Note that the last Theorem together with Theorem 2.4 imply the Mukai conjecture for coadjoint orbits and indeed an stronger inequality.

Corollary 3.4. *The following inequality*

$$\sum_{\beta\in S\smallsetminus S_P} \Big(\sum_{\alpha\in R\smallsetminus R_P^+} n_\beta - 1 \Big) = \sum_{\beta\in S\smallsetminus S_P} \Big(\sum_{\alpha\in R\smallsetminus R_P^+} \langle \alpha, \beta^\vee \rangle - 1 \Big) \le \sharp(R^+\smallsetminus R_P^+) \tag{3.2}$$

holds for a parabolic subgroup P of $G_{\mathbb{C}}$ and it implies the Mukai inequality for $G_{\mathbb{C}}/P$

$$\sharp(S\smallsetminus S_P)\cdot\big(\gcd_{\beta\in S\smallsetminus S_P}(n_\beta) - 1\big) \le \sharp(R^+\smallsetminus R_P^+)$$

as the complex dimension, the second Betti number and the index of $G_{\mathbb{C}}/P$ equal $\sharp(R^+\smallsetminus R_P^+), \sharp(S\smallsetminus S_P)$ and $\gcd_{\beta\in S\smallsetminus S_P} n_\beta$, respectively.

Remark 3.5. The inequality in 3.2 was already proved by B. Pasquier in [21, Lemma 4.8]. In Pasquier's proof the inequality is proven by first reducing its verification to a list of cases that depend on the type of the Dynkin diagram of the parabolic subgroup and the position of the simple roots adjacent to the subgroup and then by checking directly the resulting inequalities through explicit calculations. The proof that we present in this paper of the inequality will rely on the combinatorics of the *Kostant game* and we hope that it will contribute to its understanding.

If we view the Hilbert polynomial $\mathrm{H}_P(\varpi) = \mathrm{H}_P\big(\sum_{\beta\in S\smallsetminus S_P} k_\beta\varpi_\beta\big)$ as a polynomial in variables k_β over $\beta \in S \smallsetminus S_P$, Theorem 3.3 will follow from the following.

Theorem 3.6. *For each $\beta \in S \smallsetminus S_P$ and each $j = 1, \dots, n_\beta - 1$ the Hilbert polynomial $\mathrm{H}_P\big(\sum_{\beta\in S\smallsetminus S_P} k_\beta\varpi_\beta\big)$ vanishes at $k_\beta = -j$.*

This is the theorem we are going to prove. As part of this theorem, it is not hard to prove that the statement holds when $k_\beta = -1$. We present this proof already here.

Lemma 3.7. *For each $\beta \in S\smallsetminus S_P$ the Hilbert polynomial $\mathrm{H}_P\big(\sum_{\beta\in S\smallsetminus S_P} k_\beta\varpi_\beta\big)$ vanishes at $k_\beta = -1$.*

Proof. By Theorem 3.1, $\langle \varpi + \rho, \beta^\vee\rangle = k_\beta + 1$ is a factor of H_P. □

Corollary 3.8. *Theorem 3.6 holds for a simple root β not adjacent to P.*

3.2. String of coroots

In order to prove Theorem 3.6, we want to show that $k_\beta + j$ is a linear factor of the Hilbert polynomial

$$\mathrm{H}_P\Big(\sum_{\beta\in S\smallsetminus S_P} k_\beta\varpi_\beta\Big) = \mathrm{Ind}(\mathbb{L}_\varpi) = \prod_{\alpha\in R_+\backslash R_P^+} \frac{\langle \varpi+\rho, \alpha^\vee\rangle}{\langle \rho, \alpha^\vee\rangle}$$

for $\beta \in S\smallsetminus S_P$ and $j \in \{1, \dots, n_\beta - 1\}$. That means that for every $\beta \in S\smallsetminus S_P$ we want to find a string of roots $\alpha_1, \dots, \alpha_{n_\beta-1} \in R^+ \smallsetminus R_P^+$ such that

$$\langle \varpi + \rho, \check{\alpha}_j\rangle = \Big\langle \sum_{\beta\in S\smallsetminus S_P} k_\beta\varpi_\beta + \rho, \check{\alpha}_j\Big\rangle = k_\beta + j$$

for every $j = 1, \dots, n_\beta - 1$. Note that it follows from Corollary 3.8 that this task is already accomplished when β is a simple root not adjacent to P.

The previous analysis motivates the definitions of strings of coroots for a parabolic subgroup and a simple root adjacent to it that we define below in this section. Before defining them formally we set up some notation. Let P be a parabolic subgroup of $G_{\mathbb{C}}$ and $\beta \in S \setminus S_P$ be a simple root adjacent to P. We denote by $P \cup \{\beta\}$ the parabolic subgroup corresponding to $S_P \cup \{\beta\}$. Let Γ be the Dynkin diagram of G. Let Γ_P and $\Gamma_P \cup \beta$ be the subgraphs of Γ corresponding to the sets of simple roots S_P and $S_P \cup \{\beta\}$, respectively.

Definition 3.9. A **string of coroots** (for P and β) is a string of positive coroots of the form

$$\check{\beta}, \check{\beta} + \gamma_1, \ldots, \check{\beta} + \gamma_l \in \check{R}^+_{P \cup \beta},$$

such that every γ_j can be written as a positive integer linear combination of coroots in $\check{R}^+_P$ and

$$\mathrm{ht}(\check{\beta} + \gamma_j) < \mathrm{ht}(\check{\beta} + \gamma_{j+1})$$

for every $1 \leq j < l$, where ht stands for the height function defined on the set of positive coroots. The integer $\mathrm{ht}(\check{\beta} + \gamma_l)$ will be called the **length of the string**. A string of coroots is called **maximal** if its length is exactly equal to

$$n_\beta - 1 = \sum_{\alpha \in R^+ \setminus R^+_P} \langle \alpha, \check{\beta} \rangle - 1 = 1 - \sum_{\alpha \in R^+_P} \langle \alpha, \check{\beta} \rangle.$$

A string of coroots is called **good** if

$$\mathrm{ht}(\check{\beta} + \gamma_j) = j + 1$$

for every $1 \leq j \leq l$. A string of coroots that is not good is called a **string with a jump**. A **gap of a string with a jump** is an integer j such that $\mathrm{ht}(\check{\beta} + \gamma_j) + 1 < \mathrm{ht}(\check{\beta} + \gamma_{j+1})$.

Proposition 3.10. *If $\check{\beta}, \check{\beta} + \gamma_1, \ldots, \check{\beta} + \gamma_l$ is a good string of coroots for P and β, then the Hilbert polynomial $\mathrm{H}_P(\varpi) = \mathrm{H}_P\Big(\sum_{\beta \in S \setminus S_P} k_\beta \varpi_\beta\Big)$ vanishes at $k_\beta = -1, \ldots, -(l+1)$.*

Proof. For each $j = 1, \ldots, l$ there is a factor $\langle \varpi + \rho, \check{\beta} + \gamma_j \rangle$ in H_P which, by the conditions satisfied by good strings, is equal to

$$\langle \varpi + \rho, \check{\beta} + \gamma_j \rangle = k_\beta + (j + 1)$$

thus H_P vanishes at $k_\beta = -(j + 1)$. ∎

In order to prove Theorem 3.6, our aim now is to find for every simple root adjacent to a parabolic subgroup a maximal good string of coroots for the parabolic subgroup and the simple root. Maximal good strings of coroots can be obtained via Kostant games which we define in the following section.

3.3. The Kostant game

Let Γ be the Dynkin diagram of G and $S = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots.

A configuration is a vector $c = \sum_{i=1}^n c_i\alpha_i$ where $c_i \in \mathbb{Z}_{\geq 0}$. The **height** $\mathrm{ht}(c)$ of c is the sum $\sum_{i=1}^n c_i$. We can think of a configuration as placing c_i chips on the i-vertex of the Dynkin diagram. The height of the configuration is the total number of chips placed on the diagram.

For $i \in V$, let $N(i)$ denote the neighbors of i. For $j \in N(i)$, we denote by $n_{i,j}$ the number of arrows coming to the i-vertex from the j-vertex. In our convention, when the simple roots α_i and α_j have the same length, then $n_{i,j} = 1 = n_{j,i}$. Also in our convention, for instance in Figure 3.1 $n_{i,j} = 2$ and $n_{j,i} = 1$. Note that $n_{i,j} = -\langle \alpha_j, \check{\alpha}_i \rangle$.

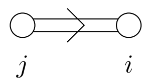


Figure 3.1

Let $c = \sum_{i=1}^n c_i\alpha_i$ be a configuration. We say that the i-th entry of the configuration is:

- Happy if $c_i = \frac{1}{2}\sum_{j\in N(i)} n_{i,j}c_j$.
- Unhappy if $c_i < \frac{1}{2}\sum_{j\in N(i)} n_{i,j}c_j$.
- Excited if $c_i > \frac{1}{2}\sum_{j\in N(i)} n_{i,j}c_j$.

We play the Kostant game by first placing a chip at a vertex of the Dynkin diagram. The goal of the Kostant game is to make every vertex of the Dynkin diagram either happy or excited. The game is played as follows. Given a configuration c, we pick any unhappy vertex i, and replace the number of chips c_i at i by

$$c_i \mapsto -c_i + \sum_{j\in N(i)} n_{i,j}c_j.$$

Equivalently, we replace c by

$$\begin{aligned} s_{\alpha_i}(c) := c - \langle c, \check{\alpha}_i \rangle \alpha_i &= \sum_{j=1}^n c_j\alpha_j - \left\langle \sum_{j=1}^n c_j\alpha_j, \check{\alpha}_i \right\rangle \alpha_i \\ &= \sum_{j\neq i} c_j\alpha_j + \Big(-c_i + \sum_{j\in N(i)} n_{i,j}c_j \Big)\alpha_i. \end{aligned}$$

Note that if the vertex i is unhappy, then $\mathrm{ht}(s_{\alpha_i}(c)) > \mathrm{ht}(c)$. It follows from this remark that we play the Kostant game by consecutively replacing one positive root by another of greatest height. Indeed, the set of all the possible configurations of the Kostant game played by initially placing a chip on a vertex of the Dynkin diagram corresponds to the set of positive roots R^+. When we start playing the game by placing one chip at one of the vertices, the game will reach the same terminating configuration regardless of the sequence of moves. If the Dynkin diagram is simply laced, the game always leads to a unique terminating configuration regardless of the vertex where we place the first chip. This unique terminating configuration corresponds to the highest positive root. If the Dynkin diagram is not simply-laced, any sequence of moves of the Kostant game will lead to two possible terminating configurations, and one of them corresponds to the highest positive root. For these and other facts concerning the combinatorics of the Kostant game we suggest the reader to check the class notes written by E. Chen of the course *"Topics on Combinatorics"* taught by A. Postnikov in 2017 at MIT [22] and B. Elek's notes on Reflection Groups [6, Section 5.4].

Example 3.11. In Figure 3.2 we play the Kostant game on the Dynkin diagram A_4 starting from its simple roots until we reach the configuration of the highest positive root.

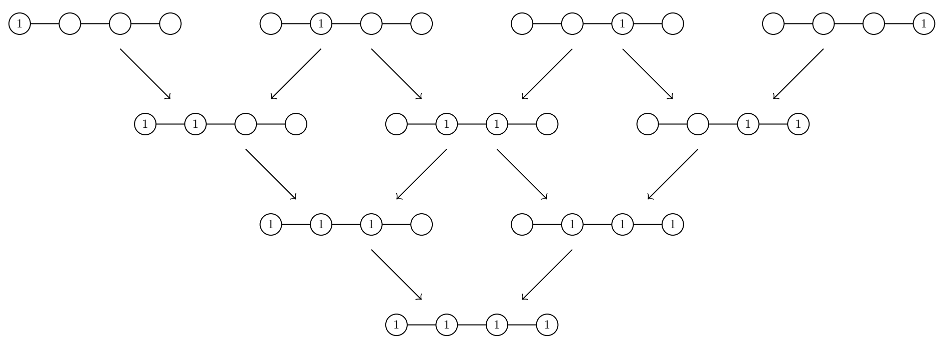


Figure 3.2: Kostant game on A_4.

3.4. The modified Kostant game

We modify the Kostant game on Dynkin diagrams of compact Lie groups in order to construct maximal good strings. We modify the graph of Γ by adding an extra vertex adjacent only to a vertex j of the graph. We denote the new vertex by $\tilde{j}$. We also draw k-arrows pointing from the new vertex $\tilde{j}$ to j. We denote the new graph by Γ_j^k. When $k = 1$, we just denote it by Γ_j. We play a modified version of the Kostant game on Γ_j^k by placing one chip at the vertex $\tilde{j}$, with the same rules as before but with a new rule in which the vertex $\tilde{j}$ is *always* happy. We call the resulting game on Γ_j^k the **modified Kostant game** at the vertex j with k-arrows. When $k = 1$, we just call the resulting game the modified Kostant game at the vertex j. We define configurations and their heights as we did for the standard Kostant game.

The proofs of the two theorems below will be given in the later section.

Theorem 3.12. *The modified Kostant game at a vertex of a Dynkin diagram of a compact Lie groups leads to a unique terminating configuration.*

Remark 3.13. Note that given a Dynkin diagram, if we denote by $h+1$ the height of the final configuration of the modified Kostant game on the Dynkin diagram at a fixed vertex, then the height of the terminating configuration of the modified Kostant game at the same vertex but with k-arrows equals $kh + 1$.

Theorem 3.14. *Given a Dynkin diagram Γ of a compact Lie group G, if we denote by $h_j + 1$ the height of the unique terminating configuration of the modified Kostant game on the Dynkin diagram of simple coroots $\check{\Gamma}$ at the vertex j, then*

$$\sum_{\alpha \in R^+} \alpha = \sum_{\alpha_j \in S} h_j \alpha_j.$$

Example 3.15. We illustrate Theorem 3.14 with one example. Let $\Gamma = A_4$. We enumerate the vertices of A_4 from the left to the right in increasing order. We denote by $h_i + 1$, $i = 1, \ldots, 4$, the height of the unique terminating configuration of the modified Kostant game on $\Gamma = \check{\Gamma}$ at the vertex i. By symmetry, $h_1 = h_4$ and $h_2 = h_3$. We play the modified Kostant game at the first and second vertex and illustrate all its possible configurations in the Figure 3.3.

From Figure 3.3 we conclude that $h_1 = h_4 = 4, h_2 = h_3 = 6$ and Theorem 3.14 states that the sum of the positive roots of A_4 equals

$$\sum_{\alpha \in R^+} \alpha = 4\alpha_1 + 6\alpha_2 + 6\alpha_3 + 4\alpha_4.$$

We would like to prove the existence of maximal good strings for a parabolic group and a root adjacent to it from Theorem 3.12 and Theorem 3.14. Before doing so we review the following two lemmas on roots whose proofs can be found for instance in [14, Section 9.4].

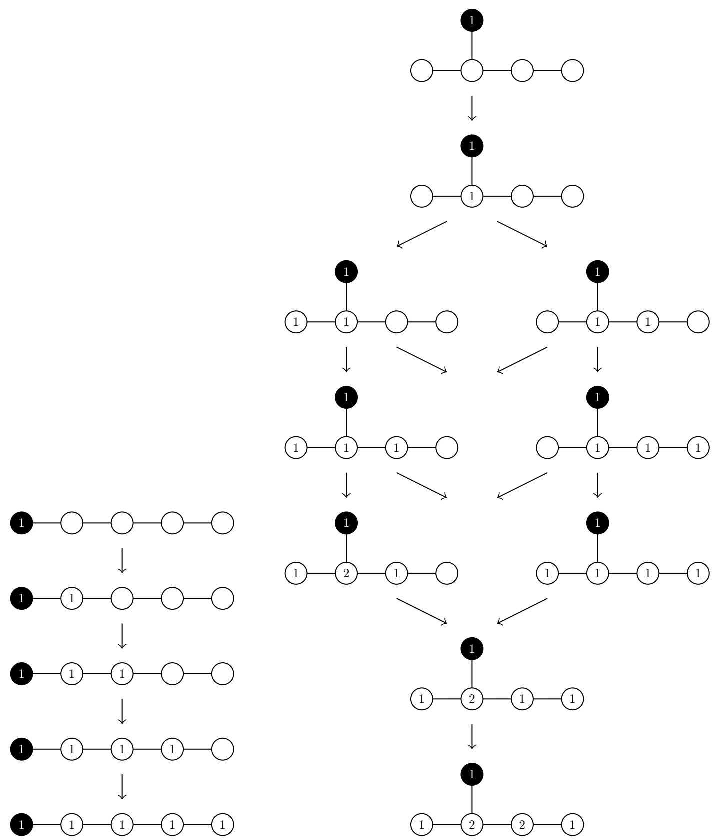


Figure 3.3: Modified Kostant game on A_4.

Lemma 3.16. *Let α, β be non-proportional roots. If $(\alpha, \beta) > 0$, then $\alpha - \beta$ is a root. If $(\alpha, \beta) < 0$, then $\alpha + \beta$ is a root.*

Lemma 3.17. *Let α and β be non-proportional roots. Let $r, q \in \mathbb{Z}_{\geq 0}$ be the largest integers for which $\beta - r\alpha \in R, \beta + q\alpha \in R$. Then $\beta + i\alpha \in R$ for all $-r \leq i \leq q$. We call the set of roots $\beta + i\alpha$ ($i \in \mathbb{Z}$) the* α***-string through*** *β.*

Theorem 3.18. *Let Γ be a Dynkin diagram and Γ_P be a connected subgraph. Let β be a simple root adjacent to Γ_P and let α_j be the root in Γ_P adjacent to β. Let us assume that there are k-arrows pointing from α_j to β.*

When we play the modified Kostant game on the Dynkin diagram of co-roots of Γ_P at the j-vertex with k-arrows until it reaches its terminating con-

figuration, we obtain a maximal string of coroots

$$\check{\beta}, \check{\beta} + \gamma_1, \ldots, \check{\beta} + \gamma_l.$$

for P and β. This string of coroots can always be completed into a good string of coroots for P and β.

Proof. The string of coroots

$$\check{\beta}, \check{\beta} + \gamma_1, \ldots, \check{\beta} + \gamma_l.$$

is indeed a string of coroots for P and β because the string is obtained by playing the standard Kostant game on the coroots of $\Gamma_P \cup \beta$ by leaving always one chip at the β-vertex.

We write

$$\sum_{\alpha \in R_P^+} \alpha = h_j \alpha_j + \cdots .$$

The coroot $\check{\beta}$ is orthogonal to every root in S_P except α_j, thus

$$\left\langle \sum_{\alpha \in R_P^+} \alpha, \check{\beta} \right\rangle = \langle h_j \alpha_j + \cdots , \check{\beta} \rangle = h_j \langle \alpha_j, \check{\beta} \rangle = -k h_j.$$

The string of coroots is maximal because Theorem 3.14 implies that

$$\mathrm{ht}(\check{\beta} + \gamma_l) = k h_j + 1 = 1 - \left\langle \sum_{\alpha \in \Gamma_P^+} \alpha, \check{\beta} \right\rangle.$$

Let us suppose that

$$\check{\beta}, \check{\beta} + \gamma_1, \ldots, \check{\beta} + \gamma_l$$

is a string with a jump. That means that there exist r such that

$$\mathrm{ht}(\check{\beta} + \gamma_r) + 1 < \mathrm{ht}(\check{\beta} + \gamma_{r+1}).$$

As we have obtained the string by playing the Kostant game, there exist a simple root $\alpha \in S_P$ and a positive integer n such that

$$\check{\beta} + \gamma_{r+1} = \check{\beta} + \gamma_r + n\check{\alpha}.$$

Lemma 3.17 implies that

$$\check{\beta} + \gamma_r + m\check{\alpha}$$

is a coroot for every $0 \leq m \leq n$ (a $\check{\alpha}$-string through $\check{\beta} + \gamma_r$ is unbroken). We can always fill all the gaps of the maximal string in this way to get a good string of coroots for P and β, and we are done. $\square$

Proof of Theorem 3.6. Let Γ be the Dynkin diagram of G and Γ_P be the subgraph corresponding to P (not necessarily connected). Assume that β is a simple root adjacent to P. We show that there exists a maximal good string of coroots for P and β and the Theorem will follow from Proposition 3.10.

We consider the connected components $\Gamma_{P_j} = \Gamma_{P_j}(\beta)$ of Γ_P which are adjacent to β. Note that there are at most three such components. Let

$$\check{\beta}, \check{\beta} + \gamma_1^j, \ldots, \check{\beta} + \gamma_{l_j}^j.$$

be a maximal string of coroots for P_j and β. We know from the previous Theorem that they exist. The mechanics of the standard Kostant game and Lemma 3.17 imply that we can glue the strings to obtain a good string of coroots, for instance the following is a good string of coroots for P and β

$$\check{\beta}, \check{\beta} + \gamma_1^1, \ldots, \check{\beta} + \gamma_{l_1}^1, \check{\beta} + \gamma_{l_1}^1 + \gamma_1^2, \ldots, \check{\beta} + \gamma_{l_1}^1 + \gamma_{l_2}^2, \ldots, \check{\beta} + \sum_j \gamma_{l_j}^j.$$

Now we verify that it is maximal:

$$\begin{aligned} n_\beta - 1 = 1 - \sum_{\alpha \in R_P^+} \langle \alpha, \check{\beta} \rangle &= 1 - \sum_j \sum_{\alpha \in R_{P_j}^+} \langle \alpha, \check{\beta} \rangle \\ &= 1 + \sum_j \Big(\mathrm{ht}(\check{\beta} + \gamma_{l_j}^j) - 1 \Big) = \mathrm{ht}\Big(\check{\beta} + \sum_j \gamma_{l_j}^j \Big). \end{aligned}$$

The Theorem now follows from Proposition 3.10. $\square$

3.5. Proofs of Theorem 3.12 and Theorem 3.14

In this section we give the proofs of Theorem 3.12 and Theorem 3.14. It turns out that the configurations of a modified Kostant game correspond to minimal length representatives of the quotient of the Weyl group of a compact Lie group with a parabolic subgroup. We show Theorem 3.12 and Theorem 3.14 using this fact together with the fact that the set of minimal length representatives of the quotient contains a unique maximal length representative whose set of inversions is the set of positive roots not belonging to the set of positive roots spanned by the simple roots in the parabolic. Before giving formal explanations and proofs of our claims, we recall some notation.

Let G be a compact Lie group and T be a maximal torus. Let R be the root system defined by G and T with a choice of positive roots R^+ and simple roots $S = \{\alpha_i\}_{i=1}^n$. We denote by Γ the corresponding Dynkin diagram. We identify the Lie algebra $\mathfrak{t}$ of T with its dual $\mathfrak{t}^*$ via an invariant inner product $(\cdot,\cdot)$ on the Lie algebra $\mathfrak{g}$ of G. For every $\alpha \in R$, we denote by s_α the reflection defined by α:

$$\begin{aligned} s_\alpha : \mathfrak{t} &\to \mathfrak{t} \\ x &\mapsto x - \langle x, \alpha \rangle \check{\alpha} \\ &= x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha. \end{aligned}$$

We denote by W the Weyl group generated by the set of simple reflections $\{s_\alpha \mid \alpha \in S\}$. When $\alpha = \alpha_i$ is a simple root, we denote the corresponding reflection just by $s_{\alpha_i} = s_i$.

We define the **length** $l(w)$ of an element $w \in W$ as the smallest positive integer such that w can be written as a product of simple reflections

$$w = s_{i_t} \cdot \ldots \cdot s_{i_k} \cdot \ldots \cdot s_{i_1}$$

and call such an expression **reduced**.

The length of an element in the Weyl group can be characterized in another way. Let

$$I(w) = \{\alpha \in R^+ \mid w(\alpha) \in -R^+\}$$

be the **inversion set of** w. Then $l(w) = \mathrm{Car}(I(w))$. A reduced expression $s_{i_t} \ldots \cdot s_{i_k} \cdot \ldots s_{i_1}$ of w allows us to enumerate all the positive roots in $I(w)$ as follows: define

$$\tilde{\alpha}_k = s_{i_1} \ldots s_{i_k}(\alpha_{i_{k+1}}) \text{ with } \tilde{\alpha}_t = \alpha_{i_1},$$

then $I(w) = \{\tilde{\alpha}_k\}_{k=1}^t$, with all $\tilde{\alpha}_k$ different (see for instance [15, Section 1.7]).

For $j \in \{1,\ldots,n\}$, we will denote by W_j the parabolic subgroup of W spanned by the set of simple reflections $\{s_i\}_{i\neq j}$. Likewise, we denote by R_j the set of roots spanned by the set of simple roots $\{\alpha_i\}_{i\neq j}$. We denote the set of minimal length representatives of the quotient W/W_j by

$$W^j := \{w \in W \mid I(w) \subset R^+ \backslash R_j^+\}.$$

Theorem 3.19. *A sequence of moves of the modified Kostant game on the Dynkin diagram of coroots of* Γ *at the vertex* j *encodes the reduced expression* *of some element in W^j. Conversely, any reduced expression of an element in W^j can be obtained in this way.*

Proof. We encode a sequence of moves of the modified Kostant game with an ordered arrangement of integers which indicates the position of the vertices where we place the chips every time that we play the modified Kostant game. So for instance the arrangement $(i_1, i_2, \dots, i_t)$ indicates that in our first move we place chips on the i_1-vertex, then in our second move we place chips on the i_2-vertex, and we end by placing chips on the i_t-vertex. Note that at every step we have not encoded the number of chips that we place on each vertex but we explain now how to obtain this number.

Given an arrangement $(i_1, i_2, \dots, i_t)$ that codifies a sequence of moves of the modified Kostant game, we define a sequence $\{w_1, \dots, w_t\}$ of elements in the Weyl group by

$$w_l := s_{i_l} \dots s_{i_2} s_{i_1}$$

and a set of roots $\{\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_t\}$ by

$$\tilde{\alpha}_l = w_l^{-1}(\alpha_{i_{l+1}}) = s_{i_1} s_{i_2} \dots s_{i_l}(\alpha_{i_{l+1}}) \tag{3.3}$$

for $1 \leq l < t$ and $\tilde{\alpha}_t = \alpha_j$. Note that $w_1 = s_j$.

We write

$$\tilde{\alpha}_l = k^l \alpha_j + \cdots$$

as a linear combination of simple roots. We claim that k^l is the number of chips that we locate at the i_{l+1}-vertex when we play the modified Kostant game.

We construct an auxiliary vector space by adding an element $\tilde{\beta}$ to the basis of simple coroots $\{\check{\alpha}_1, \dots, \check{\alpha}_n\}$ and extend the invariant inner product defined on $\mathfrak{t}$ to a bilinear product on $\mathfrak{t} \oplus \langle \tilde{\beta} \rangle$ by saying that

$$(\tilde{\beta}, \tilde{\beta}) = 2, \ (\alpha_i, \tilde{\beta}) = -\delta_{ij}$$

for $j = 1, \dots, n$. We also extend the action of the Weyl group W defined on $\mathfrak{t}$ to $\mathfrak{t} \oplus \langle \tilde{\beta} \rangle$ by

$$s_{\alpha_i}(\tilde{\beta}) = \tilde{\beta} - (\alpha_i, \tilde{\beta})\check{\alpha}_i = \tilde{\beta} + \delta_{ij}\check{\alpha}_j$$

for every simple root α_i. The bilinear product that we define for $\mathfrak{t} \oplus \langle \tilde{\beta} \rangle$ is invariant with respect to the action of W that we just defined on it.

The elements

$$w_{l+1}(\tilde{\beta}) := s_{i_{l+1}}(w_l(\tilde{\beta})) = w_l(\tilde{\beta}) - (\alpha_{i_{l+1}}, w_l(\tilde{\beta}))\check{\alpha}_{i_{l+1}}$$

follow the same recursion formula defined by the sequence of moves $(i_1, \ldots, i_t)$ of the modified Kostant game at the vertex j, i.e., if

$$w_l(\tilde{\beta}) = \sum_{i=1}^{n} c_i \check{\alpha}_i + \tilde{\beta},$$

and $N(i_{l+1})$ is the set of vertices in the Dynkin diagram that are adjacent to i_{l+1} and $\check{n}_{i,j} := -(\alpha_i, \check{\alpha}_j)$ is the number of arrows coming to i from j in the Dynkin diagram of simple coroots (that is the same as the number of arrows $n_{j,i}$ from i to j in the Dynkin diagram of simple roots), then

$$\begin{aligned} w_{l+1}(\tilde{\beta}) &= \sum_{i=1}^{n} c_i \check{\alpha}_i + \tilde{\beta} - \Big(\alpha_{i_{l+1}}, \sum_{i=1}^{n} c_i \check{\alpha}_i + \tilde{\beta}\Big) \check{\alpha}_{i_{l+1}} \\ &= \sum_{i \neq i_{l+1}} c_i \check{\alpha}_i + \tilde{\beta} + \Big(-c_{i_{l+1}} - \sum_{k \in N(i_{l+1})} c_k (\alpha_{i_{l+1}}, \check{\alpha}_k) + \delta_{j i_{l+1}}\Big) \check{\alpha}_{i_{l+1}} \\ &= \sum_{i \neq i_{l+1}} c_i \check{\alpha}_i + \tilde{\beta} + \Big(-c_{i_{l+1}} + \sum_{k \in N(i_{l+1})} \check{n}_{i_{l+1},k} c_k + \delta_{j i_{l+1}}\Big) \check{\alpha}_{i_{l+1}}. \end{aligned}$$

As the bilinear form that we define for $\mathfrak{t} \oplus \langle \tilde{\beta} \rangle$ is invariant with respect to the action of W, we get

$$k^l = -(\tilde{\beta}, \tilde{\alpha}_l) = -(\tilde{\beta}, w_l^{-1}(\alpha_{i_{l+1}})) = -(w_l(\tilde{\beta}), \alpha_{i_{l+1}}) = \frac{w_{l+1}(\tilde{\beta}) - w_l(\tilde{\beta})}{\check{\alpha}_{i_{l+1}}}$$

and the claim follows.

The Theorem now follows from the fact that $k^l > 0$ for all $1 \leq l \leq t$ if and only if $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_t\} \subset R^+ \backslash R_j^+$. The latter is equivalent to say that $w_l \in W^j$ and $w_l = s_{i_l} \ldots s_{i_2} s_{i_1}$ is a reduced expression for all $1 \leq l \leq t$. □

Now we are ready to give the proofs of Theorem 3.12 and Theorem 3.14:

Proof of Theorem 3.12 and Theorem 3.14. We keep the notation of the proof of the previous Theorem. We show first that the modified Kostant game at the vertex j terminates. Let w_0 be the longest element in W^j, i.e, the unique element in W^j that sends all positive roots in $R^+ \setminus R_j^+$ to negative roots.

Let $(i_1, i_2, \ldots, i_t)$ be an arrangement that encodes a sequence of moves of the modified Kostant game. The previous Theorem implies that

$$s_{i_t} \ldots s_{i_2} s_{i_1}$$

is a reduced expression of an element in W^j. We can always complete the reduced expression into a reduced expression of w_0, i.e., there exist simple roots $\alpha_{i_{t+1}}, \dots, \alpha_{i_m}$ such that

$$w_0 = s_{i_m} \dots s_{i_{t+1}} s_{i_t} \dots s_{i_2} s_{i_1}$$

is a reduced expression.

The previous statement implies that the arrangement

$$(i_1, i_2, \dots, i_t, i_{t+1}, \dots, i_m)$$

encodes a sequence of moves of the modified Kostant game. In particular, the modified Kostant game terminates. To end the proof of Theorem 3.12, we need to show that the game ends in a unique terminating state. But this follow from the fact the longest element in W^j is unique.

We are done with the proof of Theorem 3.12 and now we continue with the proof of Theorem 3.14. Let us assume now that

$$w_0 = s_{i_m} \cdot \dots \cdot s_{i_2} s_{i_1}$$

is a reduced expression, and let $\{\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_m\}$ be the set of roots defined in Equation 3.3. We know that this sets equals to $R^+ \backslash R_j^+$ and also from the proof of the previous Proposition that if we write

$$\tilde{\alpha}_l = k^l \alpha_j + \cdots,$$

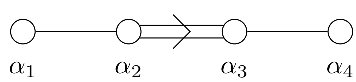
then k^l is the number of chips that we locate at the i_l-vertex when we play the Kostant game. Thus

$$h_j = \sum_l k^l = -\sum_l (\tilde{\alpha}_l, \tilde{\beta}) = -\sum_{\alpha \in R^+ \backslash R_j^+} (\alpha, \tilde{\beta}) = -\sum_{\alpha \in R^+} (\alpha, \tilde{\beta}),$$

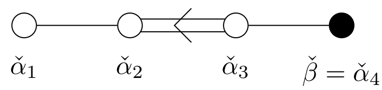
and we are done. $\square$

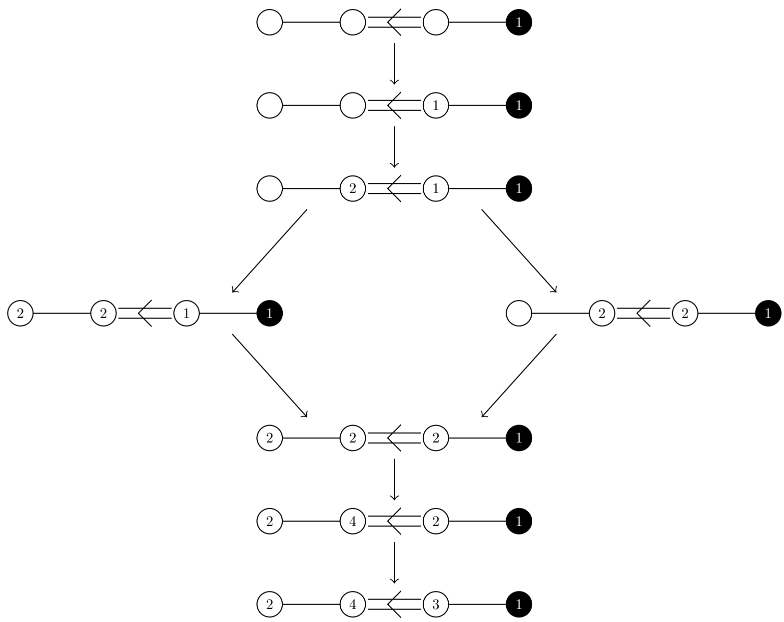
We illustrate the overall idea of the proof of Theorem 3.6 with the following Example.

Example 3.20. Let $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be the set of simple roots of F_4 with Dynkin diagram shown below: Assume that $S_P = \{\alpha_1, \alpha_2, \alpha_3\}$.



1. First we select the simple roots adjacent to S_P in the Dynkin diagram. In this case the only one is $\beta = \alpha_4$ and it is adjacent to α_3 in S_P.
2. We play the modified Kostant game on the Dynkin diagram of coroots $\{\check{\alpha}_1, \check{\alpha}_2, \check{\alpha}_3\}$ at the 3-vertex until it terminates to produce a string of coroots for P and β:





   In this case we obtain two arrangements that encode the moves of the Kostant game until it ends: $(3, 2, 1, 3, 2, 3)$ and $(3, 2, 3, 1, 2, 3)$. The corresponding strings of coroots are $\{\check{\beta}, \check{\alpha}_3 + \check{\beta}, 2\check{\alpha}_2 + \check{\alpha}_3 + \check{\beta}, 2\check{\alpha}_1 + 2\check{\alpha}_2 + \check{\alpha}_3 + \check{\beta}, 2\check{\alpha}_1 + 2\check{\alpha}_2 + 2\check{\alpha}_3 + \check{\beta}, 2\check{\alpha}_1 + 4\check{\alpha}_2 + 2\check{\alpha}_3 + \check{\beta}, 2\check{\alpha}_1 + 4\check{\alpha}_2 + 3\check{\alpha}_3 + \check{\beta}\}$ and $\{\check{\beta}, \check{\alpha}_3 + \check{\beta}, 2\check{\alpha}_2 + \check{\alpha}_3 + \check{\beta}, 2\check{\alpha}_2 + 2\check{\alpha}_3 + \check{\beta}, 2\check{\alpha}_1 + 2\check{\alpha}_2 + 2\check{\alpha}_3 + \check{\beta}, 2\check{\alpha}_1 + 4\check{\alpha}_2 + 2\check{\alpha}_3 + \check{\beta}, 2\check{\alpha}_1 + 4\check{\alpha}_2 + 3\check{\alpha}_3 + \check{\beta}\}$, respectively.
3. It is possible that the strings have gaps, but whenever there is a gap between two coroots, we can always fill the gaps. For instance, we fill the gaps for the string of coroots encoded by $(3, 2, 3, 1, 2, 3)$ and obtain the following good string $\{\check{\beta}, \check{\alpha}_3 + \check{\beta}, \check{\alpha}_2 + \check{\alpha}_3 + \check{\beta}, 2\check{\alpha}_2 + \check{\alpha}_3 + \check{\beta}, 2\check{\alpha}_2 + 2\check{\alpha}_3 + \check{\beta}, \check{\alpha}_1 + 2\check{\alpha}_2 + 2\check{\alpha}_3 + \check{\beta}, 2\check{\alpha}_1 + 2\check{\alpha}_2 + 2\check{\alpha}_3 + \check{\beta}, 2\check{\alpha}_1 + 3\check{\alpha}_2 + 2\check{\alpha}_3 + \check{\beta}, 2\check{\alpha}_1 + 4\check{\alpha}_2 + 2\check{\alpha}_3 + \check{\beta}, 2\check{\alpha}_1 + 4\check{\alpha}_2 + 3\check{\alpha}_3 + \check{\beta}\}$.

4. The sequence of moves of the modified Kostant game at the 3-vertex gives a reduced expression of the longest element of the set of minimal length representatives of the quotient

$$W_{\alpha_1,\alpha_2,\alpha_3}/W_{\alpha_1,\alpha_2}.$$

With this reduced expression, we obtain all the positive roots spanned by $\{\alpha_1, \alpha_2, \alpha_3\}$ that pair non-trivial with β, i.e., the roots of the form

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$$

where k_1, k_2, k_3 are non-negative integers and $k_3 > 0$. For instance, the arrangement of moves $(3, 2, 3, 1, 2, 3)$ gives us the reduced expression

$$s_3s_2s_1s_3s_2s_3$$

and we obtain the following roots

- $\tilde{\alpha}_1 = \alpha_3$
- $\tilde{\alpha}_2 = s_3(\alpha_2) = \alpha_2 + 2\alpha_3$
- $\tilde{\alpha}_3 = s_3s_2(\alpha_3) = \alpha_2 + \alpha_3$
- $\tilde{\alpha}_4 = s_3s_2s_3(\alpha_1) = \alpha_1 + \alpha_2 + 2\alpha_3$
- $\tilde{\alpha}_5 = s_3s_2s_3s_1(\alpha_2) = \alpha_1 + 2\alpha_2 + 2\alpha_3$
- $\tilde{\alpha}_6 = s_3s_2s_3s_1s_2(\alpha_3) = \alpha_1 + \alpha_2 + \alpha_3$

Note that the coefficient k^l of α_3 in $\tilde{\alpha}_l$ equals the number of chips that we place at the Dynkin diagram every time we play the Kostant game. The resulting good string is maximal because

$$\begin{aligned} \mathrm{ht}(2\check{\alpha}_1 + 4\check{\alpha}_2 + 3\check{\alpha}_3 + \check{\beta}) &= \sum_{l=1}^{6} k^l + 1 \\ &= -\sum_{\alpha\in R_P^+} (\alpha, \check{\beta}) + 1 = 10 \end{aligned}$$

and the Hilbert polynomial $H_P(k_4)$ is divided by $\prod_{l=1}^{10}(k_4 + l)$.

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