Pure and Applied Mathematics Quarterly
Volume 19, Number 4, 1839–1852, 2023

Classical and Quantum mechanics on 3D contact manifolds

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Dedicated to Victor, teacher and friend

Abstract: In this survey paper, I describe some aspects of the dynamics and the spectral theory of sub-Riemannian 3D contact manifolds. We use Toeplitz quantization of the characteristic cone as introduced by Louis Boutet de Monvel and Victor Guillemin. We also discuss trace formulae following our work as well as the Duistermaat-Guillemin trace formula.

1. Introduction

The goal of our work with Luc Hillairet and Emmanuel Trélat, in particular in [C-H-T-18], was to see if we can extend what is known for spectral asymptotics of the Laplace operator on a Riemannian manifold to sub-Riemannian ("sR" in what follows) manifolds, in particular concerning

- Trace formulae relating the spectrum of the Laplace operator to the lengths of periodic geodesics (see [CdV-73, D-G-75] and the survey [CdV-07]).
- Quantum limits and quantum ergodicity (Schnirelman theorem, see [CdV-85] and the excellent review [Dy-21]).
- Approximation of eigenfunctions by the construction of quasi-modes, i.e. approximate solutions of the eigenvalue equation, supported by invariant sets of the geodesic flow (see [CdV-77, Ze-17]).

What I find nice with sR geometry is that we have to take into account a topological set of data, namely the distribution (a sub-bundle of the tangent bundle), and also the metric data which allow to define the distance, the geodesics and the Laplace operator. In this review, we will only speak of the case of 3D contact manifold going a little beyond the paper [C-H-T-18]. While starting our project, we discovered new things about the sR geodesic

Received September 9, 2021.

flow, namely the important role played by the Reeb vector field as a way to "compactify" the geodesic flow on the unit cotangent bundle whose fibers are not compact (in contrast with the Riemannian case). We tried also to study interesting examples like magnetic Laplacians and the so-called Liouville Laplacians where the sR manifold M is the unit cotangent bundle of a Riemannian surface and the distribution is the kernel of the Liouville form restricted to M. Special cases of these examples linked to surfaces of constant curvature turn out to be "integrable" in a weak sense.

2. The setup

Let us consider a 3D smooth closed manifold M equipped with a smooth contact distribution $D \subset TM$ (assumed to be oriented), a smooth metric g on D and a smooth density $|dx|$. Recall that D is contact if there exists a non vanishing real 1-form α with $D = \ker \alpha$ so that the 3-form $\alpha \wedge d\alpha$ is a volume form. Such a set of data $(M, D, g, |dx|)$ defines an sR manifold with a volume form. To such an sR manifold, we associate the following objects:

1. The cometric $g^\star : T^\star M \to \mathbb{R}^+$ is defined in local coordinates by
$$g^\star(x, \xi) = \|\xi_{|D_x}\|^2_{g(x)}$$
2. The geodesic flow which is the flow of the Hamiltonian vector field X_g of $\frac{1}{2}g^\star$. When restricted to the unit cotangent bundle $U^\star M := \{(x, \xi) \in T^\star M \mid g^\star(x, \xi) = 1\}$ the integral curves project onto M as geodesics with unit speed and, conversely, any geodesic is the projection of such an integral curve.
3. The Laplacian Δ_{sR} which is the Friedrichs extension on $L^2(M, |dx|)$ of the quadratic form $q(f) = \int_M g^\star(df)|dx|$. The self-adjoint second order differential operator Δ_{sR} can be written locally as $\Delta = X^\star X + Y^\star Y = -X^2 - Y^2 + \text{l.o.t}$ where (X, Y) is a local smooth orthonormal frame of D and $X^\star$ and $Y^\star$ are the adjoints of X and Y with respect to $|dx|$. The operator Δ_{sR} is sub-elliptic (a well known result due to Lars Hörmander) and hence has a discrete spectrum $(\lambda_j),\ j \in \mathbb{N}$, with an o.n.b. $(\phi_j),\ j \in \mathbb{N}$, of eigenfunctions $(\phi_j),\ j \in \mathbb{N}$ of L^2. The principal symbol of this Laplacian is the co-metric.
4. The canonical contact 1-form α_g which is defined by $\ker \alpha_g = D$ and $(d\alpha_g)_{|D} = v_g$ where v_g is the volume form on D induced by the metric and the orientation.
5. The characteristic manifold $\Sigma := D^\perp = (g^\star)^{-1}(0)$ which is a 4D symplectic subcone of $T^\star M \setminus 0$.

6. The Reeb vector field R of the form α_g (ie $\alpha_g(R) = 1,\ \iota(R)d\alpha_g = 0$) which is the projection onto M of the Hamiltonian vector field of $\rho : \Sigma \to \mathbb{R}$ defined by $\rho(s\alpha_g) = s$.

The main object of this review is to describe asymptotic properties of the geodesic flow and of the spectral data of the sR Laplacian. The geodesics of large momenta spiral around the Reeb flow. This leads to the existence of infinitely many periodic geodesics spiraling around a generic closed Reeb orbit. Concerning the Laplace operator, most eigenfunctions concentrate microlocally on the characteristic manifold. Here a natural object is the quantization of the Reeb Hamiltonian as a Toeplitz operator "à la Boutet de Monvel/Guillemin" ([B-G-81]). We recover band spectra which we call "Landau bands": they are indeed Landau levels in some magnetic examples. The starting point of this presentation is works in collaboration with Luc Hillairet (Orléans) and Emmanuel Trélat (Paris). There are no fundamentally new results in the present paper, but some new definitions, examples and conjectures. The conjecture that I propose (see Section 14) is the following one

Conjecture 2.1. *The periods of the Reeb orbits are spectral invariants of the sR Laplacian.*

3. Example 0: Heisenberg quotients

This is the most basic example (see [C-H-T-18], sec. 3.1). We consider the presentation of H^3 as $\mathbb{R}^3$ equipped with the group law $(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' - xy')$. We choose the subgroup $\Gamma := \{(x, y, z) \mid (x, y) \in (\sqrt{2\pi}\mathbb{Z})^2,\ z \in 2\pi\mathbb{Z}\}$. Our sR manifold is then $\mathbb{R}^3/\Gamma$ with the orthonormal basis for D given by

$$X = \partial_x,\ Y = \partial_y - x\partial_z$$

The spectrum of $\Delta = -(X^2 + Y^2)$ is then explicitly computable: one gets the union of the eigenvalues of the flat torus $\mathbb{R}^2/\sqrt{2\pi}\mathbb{Z}^2$ and the set of integers $m(2l + 1)$, $m = 1, \ldots, l = 0, \ldots$ with multiplicities $2m$. Note that the multiplicities are very high.

The lengths spectrum (the set of lengths of closed geodesics) is the set of $2\pi\sqrt{2n},\ n \in \mathbb{N}$.

Note that $\alpha_g = dz + xdy$ and the Reeb vector field is ∂_z which is a Killing vector field.

4. Example 1: magnetic fields over a Riemannian surface

Let $\pi : M \to X$ be a principal S^1_θ-bundle on an oriented Riemannian surface (X, h). We assume that this bundle is equipped with an Hermitian connection ∇ whose horizontal distribution is our D. If the curvature of the connection does not vanish, the distribution D is contact. We take for g the pull-back on D of the metric h by π. The curvature of ∇ is a 2-form B (the magnetic field) and one introduces the magnetic scalar $b = B/dx_h$ where dx_h is the Riemannian volume form of X. The sR metric is invariant by the S^1 action, this gives an invariant momentum $e : T^\star M \to \mathbb{R}$ which is the principal symbol of $-i\partial_\theta$. The geodesics of (M, D, g) with momentum e project onto the trajectories on X with the magnetic fields b and electric charge e.

The Reeb flow is

$$R = b\partial_\theta - \vec{b}$$

where $\vec{b}$ is the horizontal lift of the Hamiltonian vector field of b w.r. to the symplectic form B on X. We define the Laplacian using the volume form $dx_g = |d\theta \wedge \pi^\star dx_h|$. Then Δ commutes with the S^1-action and $L^2(M, dx_g)$ splits into a direct sum $\oplus_{n\in\mathbb{Z}} H_n$ where H_n is unitarily equivalent to the Schrödinger operator on X with magnetic field nB.

5. Example 2: Liouville form on the unit cotangent bundle of a Riemann surface

Again (X, h) is a Riemannian surface and M is the unit cotangent bundle of X. The distribution D is $D = \ker\lambda$ where λ is the restriction to M of the Liouville 1-form $\xi dx + \eta dy$. We take on D any metric so that $\alpha_g = \lambda$. Then the Reeb vector field is the geodesic flow of h.

In particular, the case of hyperbolic surfaces is of special interest. This example is studied in [C-H-W-?]. Using the representations of $\mathrm{SL}_2(\mathbb{R})$, we reduce the computation of the spectrum and periodic geodesics to 1D-problems.

6. Example 3: Jacobi metric for the sR Kepler problem

Following [Sh-21], we take the metric $g = D^{-\frac{1}{2}} g_0$ on $\mathbb{R}^3 \setminus 0$ where g_0 is the Heisenberg metric and $D = (x^2 + y^2)^2 + 16z^2$. This metric is the Jacobi metric for a sR Kepler problem at energy 0. The metric g is invariant by the dilations $\delta_\lambda : (x, y, z) \to (\lambda x, \lambda y, \lambda^2 z)$. It admits compact quotients by the groups generated by δ_{λ_0} for some $\lambda_0 > 1$. Hence the geodesic flow is complete and the Laplacian is essentially self-adjoint.

7. Example 4: boundary of complex domains

If Ω is a smooth domain in $\mathbb{C}^2$, we consider, on $M := \partial\Omega$, the distribution $D = TM \cap iTM$. If Ω is strictly pseudo-convex, (M, D) is a contact manifold. We can take the metric induced by the Euclidean metric on $\mathbb{C}^2$.

Another example after Louis Boutet de Monvel [BdM-80]: let $Z \subset \mathbb{C}^N$ be a complex subcone of complex dimension 2 (for example defined as the zero set of complex valued homogeneous polynomials), smooth outside 0, and B the unit ball of $\mathbb{C}^N$. The 3D manifold $M = \partial(Z \cap B)$ is an S^1-bundle over the projective complex curve $Z \setminus 0/\mathbb{C} \setminus 0$. The form $\alpha = \sum_j \Im(z_j d\bar{z}_j)$ is contact on M. The Reeb flow R of α is 2π periodic. A convenient choice of g has the Reeb flow R. We call such a manifold a Zoll-Reeb sR manifold.

8. Classical Birkhoff normal forms

We will assume for simplicity that the fiber bundle $D \to M$ is topologically trivial. This holds for the magnetic sR if X is a torus. This holds for the Liouville sR if the surface X is orientable. This holds also if M is a neighbourhood of a periodic Reeb orbit. Then

Theorem 8.1. *There exists an homogeneous canonical transformation $\chi : C \to C'$ with C a conic neighbourhood of Σ in $T^\star M \setminus 0$ and C' a conic neighbourhood of $\Sigma \times 0$ in $\Sigma \times \mathbb{R}^2$, with $\mathbb{R}^2_{u,v}$ equipped with the symplectic form $dv \wedge du$ and the cone structure $\lambda.(u, v) = (\sqrt{\lambda}u, \sqrt{\lambda}v)$, so that $\chi_{|\Sigma} = \mathrm{Id} \times 0$ and*

$$g^\star \circ \chi^{-1} = \sum_{j=1}^{\infty} \rho_j(\sigma) I^j + O\left((I/\rho)^\infty\right)$$

with $\rho_j : \Sigma \setminus 0 \to \mathbb{R}$ homogeneous of degree $2 - j$, $\rho_1 = |\rho|$ with ρ the Reeb Hamiltonian and $I = u^2 + v^2$. The function ρ_2 is uniquely defined modulo Lie derivatives w.r. to Reeb.

This is proved in Section 5 of [C-H-T-18].

9. Spiraling of the sR geodesics around Reeb orbits

The goal of this section is to explain the following fact: given $x_0 \in M$ and $v_0 \in D$ of length 1 for the metric $g(x_0)$, there exists a 1-parameter family of geodesics with these Cauchy data at time 0. They are associated to initial momenta whose component vanishing on D is not fixed. When this transverse momentum tends to ∞, these geodesics will spiral more and more around a Reeb orbit like helices with small radii. See [C-H-T-21] for more details.

9.1. A simple Hamiltonian

Let us assume that our Hamiltonian is $H_0 = \frac{1}{2}\rho I$ on $\Sigma \times \mathbb{R}^2$ with ρ the Reeb Hamiltonian. Then the Hamiltonian vector field is

$$\vec{H}_0 = \frac{1}{2} I \vec{\rho} + \rho \partial_\theta$$

The Poisson bracket $\{\rho, I\}$ vanishes, hence ρ and I are first integrals of the motion. The dynamics can be integrated as follows:

$$\Phi_t(\sigma_0, u + iv) = (\phi_{It/2}(\sigma_0), (u + iv)e^{i\rho(\sigma_0)t})$$

where Φ_t is the flow of H_0, ϕ_t the flow of ρ (the Reeb flow). If we fix the energy $H_0 = 1$, we have

$$\Phi_t(\sigma_0, u + iv) = (\phi_{It/2}(\sigma_0), (u + iv)e^{it/I})$$

As I is small, ρ is large and we get a spiraling flow around the Reeb orbits. Note that this Hamiltonian is exactly the Heisenberg one. In particular, there exists closed geodesics $\gamma_k,\ k \in \mathbb{N}$, spiraling around any periodic Reeb orbit of period T_0 of lengths $l_k = 2\sqrt{\pi k T_0}$.

9.2. Spiraling

Let us choose an orthonormal frame (X, Y) of D in some tubular neighbourhood of a Reeb orbit Γ defined on some interval $t \in [0, T]$. Denote by $Z = [X, Y]$. Moreover, we can deduce from the Birkhoff normal form the existence of a well defined parallel transport of vectors in D along the Reeb flow.

Theorem 9.1. *Let $q_0 \in M$ be arbitrary, and let $(q_0, p_0) \in T^\star_{q_0}M$ be the Cauchy data of a geodesic $t \mapsto \gamma(t)$ starting at q_0 with unit speed $\dot\gamma(0) = X_0 \in D(q_0)$. We assume that $p_0 \to \infty$ (large initial momentum) and denote by $h_0 = p_0(Z) \to \infty$.*

Then, there exists a point $Q_0 = Q_0(q_0, p_0) \in M$ close to q_0, and a vector $Y_0 \in D(Q_0)$ close to X_0, such that, denoting by $\Gamma(\tau) = \mathcal{R}_\tau(Q_0)$ the Reeb orbit of Q_0, and by $Y(t)$ the parallel transport of Y_0 along Γ, we have, using the complex structure on D, for $t = \mathrm{O}(h_0)$,

$$\gamma(t) = \Gamma(J_0 t/2) - iJ_0 e^{it/J_0} Y(J_0 t/2)) + \mathrm{O}(J_0^2)$$

with $J_0 = h_0^{-1} + \mathrm{O}(h_0^{-3})$.

In words, the sR geodesic γ spirals along a Reeb orbit with a slow speed $\sim 1/2h_0$ along that orbit and a fast angular speed $\sim h_0$ transversally.

9.3. Periodic geodesics around a generic periodic Reeb orbit

In the paper [CdV-22a], I proved the following

Theorem 9.2. *If Γ is a non degenerate periodic orbit of the Reeb flow of period $T_0 > 0$, there exist infinitely many periodic sR geodesics (γ_k), $k \geq k_0$, accumulating on Γ as $k \to +\infty$, whose lengths admit a full asymptotic expansion*

$$L_k = 2\sqrt{\pi T_0}k^{\frac{1}{2}} + \sum_{j=1}^{\infty} a_j k^{-j/2} + O\left(k^{-\infty}\right)$$

as $k \to +\infty$.

9.4. Periodic geodesics in the weakly integrable case

We say that the geodesic flow is *weakly integrable* if the BNF converges, ie we can write $g^\star = F(\sigma, I)$ in some conic neighbourhood of Σ with F admitting an expansion in powers of I as given in Theorem 8.1. Let us sketch a proof of Theorem 9.2 in this case.

First, if I is small enough, there exists a closed orbit Γ_I, of the Hamiltonian $\rho + \rho_2 I + \cdots$ of period $T(I) = T_0 - AI + \cdots$ contained in $H(\sigma, I) = 1$. One then consider at the return map of the angles. This gives

$$\int_0^{T(I)} (\rho + 2I\rho_2 + \cdots)\, dt = 2k\pi$$

One then use the fact that $H = 1$ and concludes by eliminating I.

Note that this asymptotic formula is exact in the examples of Heisenberg ($T_0 = 2\pi$) and S^3 ($T_0 = \pi$) (see [K-V-19]).

10. Weyl measures, QL and QE for the sR Laplacian

10.1. Weyl

The Weyl formula is given by:

Theorem 10.1. *If $N(\lambda) = \#\{j \mid \lambda_j \leq \lambda\}$, we have, as $\lambda \to +\infty$,*

$$N(\lambda) \sim \frac{\int_M |\alpha_g \wedge d\alpha_g|}{32}\lambda^2$$

Note that the exponent 2 of λ is larger than the exponent 3/2 of the Riemannian case. The smooth measure $|\alpha_g \wedge d\alpha_g|$ is called the Popp measure. Note that the measure μ which is used in order to define the Laplacian does not need to be the Popp measure. The measure μ plays a very minor role in the spectral asymptotics. This is because, for any pair $\mu,\ \mu_0$ of densities, $\Delta_{g,\mu}$ is unitary equivalent to $\Delta_{g,\mu_0} + V$ for some smooth potential V. Hence, one gets, for example, by using the minimax principle, that

$$\exists C > 0 \text{ so that}, \forall j \in \mathbb{N},\ |\lambda_j(\Delta_{g,\mu}) - \lambda_j(\Delta_{g,\mu_0})| \leq C$$

The volume $\int_M |\alpha_g \wedge d\alpha_g|$ which is a spectral invariant corresponds to the inverse of Arnold's asymptotic linking number for the Reeb flow if $M = S^3$ [Ar-86].

There is a microlocal version of the Weyl law, namely

Theorem 10.2. *If A is self-adjoint pseudo-differential operator of degree 0 whose principal symbol $a : T^\star M \setminus 0 \to \mathbb{R}$ is homogeneous of degree 0 and is identified with a function on the sphere bundle $S(T^\star M)$, we have*

$$\lim_{\lambda \to +\infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle A\phi_j | \phi_j \rangle = \int_{S(\Sigma)} a dL$$

where dL is the unique probability measure on $S(\Sigma)$ which is invariant by antipody and whose direct image by the projection onto M is the probability measure $|\alpha_g \wedge d\alpha_g| / \int_M |\alpha_g \wedge d\alpha_g|$.

Both theorems are proved in [C-H-T-18]. The first one is classical, but we provided a new proof in that paper.

10.2. QL and QE

Let us recall what are *quantum limits* (in short QL's): a QL is a probability measure dm on $S^\star M := S(T^\star M)$ such that there exists a sequence of eigenfunctions $\phi_{j_k},\ k \in \mathbb{N}$, of our Laplacian such that for any self-adjoint pseudo-differential operator A of degree 0 with homogeneous principal symbol $a \in C^\infty(S^\star M)$, on has

$$\lim_{k \to +\infty} \langle A\phi_{j_k} | \phi_{j_k} \rangle = \int_{S^\star M} a dm$$

The eigenbasis $\phi_j, j \in \mathbb{N}$, is said to satisfy *Quantum Ergodicity* (in short QE) with a probability measure dE on $S^\star M$ if there exists a subsequence $(j_k), k \in \mathbb{N}$, of density one w.r. to the Weyl law, admitting dL as QL.

We have the following results which shows the prominent role of the Reeb vector field in the spectral asymptotics ([C-H-T-18], Theorems A and B):

Theorem 10.3. *Let us decompose the sphere bundle $S(T^\star M \setminus 0)$ as the disjoint union of the unit bundle $U^\star M := \{g^\star = 1\}$ and the sphere bundle of the characteristic manifold $S\Sigma$.*

1. *Any QL μ (a probability measure on $S(T^\star M \setminus 0)$), can be uniquely written as the sum $\mu = \mu_0 + \mu_\infty$ where μ_∞ is supported by $S\Sigma$ and is invariant under the Reeb flow, while $\mu_0(S\Sigma) = 0$ and μ_0 is invariant under the geodesic flow.*
2. *If (ϕ_j) is an ONB of eigenfunctions, there exists a subsequence (ϕ_{j_k}) of density 1, so that all corresponding QL's are supported on $S\Sigma$ (and hence invariant by Reeb).*
3. *If the Reeb flow is ergodic, then we have QE for any real eigenbasis with the limit measure the measure dL on $S\Sigma$.*

11. Toeplitz quantization of the Reeb Hamiltonian and Landau levels

Recall that, if Σ is a symplectic sub-cone of $T^\star M$, one can associate to it an Hilbert space $\mathcal{H} \subset L^2$ of functions whose wavefront set is included in Σ and an algebra of operators which obey to the usual rules of the pseudo-differential calculus where the symbols are functions on Σ (see [B-G-81]). In particular in the case of boundaries of complex domains, one recovers the original definition of Toeplitz operators.

We have the following normal form:

Theorem 11.1. *Assuming that D is a trivial bundle, we can use a FIO associated to the canonical transform χ defined in Section 8 to transform Δ into*

$$\Delta_0 = \sum_{j=1}^{\infty} R_j \otimes \Omega^j + R_\infty$$

where the R_j are Toeplitz operators on Σ of degree $1-j$, R_0 is elliptic with symbol $|\rho|$, Ω is an harmonic oscillator on $\mathbb{R}^2$ and R_∞ is smoothing along Σ.

The proof is a standard extension of the classical Birkhoff normal form using Fourier integral operators.

It follows that we have, for each value of $l \in \mathbb{N}$ a sequence of eigenvalues of Δ which are the eigenvalues of the Toeplitz operator

$$\Delta_l := \sum_{j=1}^{\infty} (2l+1)^j R_j$$

modulo a fast decaying sequence.

We call the spectrum of Δ_l a "Landau band" because in the case of constant magnetic field on surfaces (see Section 4) it is the union of the l-th Landau clusters for all magnetic field $kB,\ k \in \mathbb{Z} \setminus 0$.

Note that the approximate eigenvalues given by the normal form are "almost all" eigenvalues: we have

$$N_l(\lambda) \sim \frac{\lambda^2}{(2\pi(2l+1))^2} \int_{|\rho|\leq 1} dL_\Sigma$$

with dL_Σ the Liouville measure on Σ. This gives

$$\sum_l N_l(\lambda) \sim \frac{\lambda^2}{32} \int_M |\alpha_g \wedge d\alpha_g|$$

which fits with the Weyl formula.

12. $\Gamma \backslash PSL_2(\mathbb{R})$

Here Γ is a lattice in $G = PSL_2(\mathbb{R})$. We will look at operators invariant under left translations. Their symbols are functions on the dual of the Lie algebra. The Lie algebra is the 3D space of trace free 2×2 matrices

$$M(x,y,z) := \begin{pmatrix} z & x \\ y & -z \end{pmatrix}$$

We write $M = zA + xX + yY$. The Casimir operator is $\Box = -A^2 - 2(XY + YX)$. The Liouville Laplacian is

$$\Delta_L = -(X^2 + Y^2)$$

The magnetic Laplacian is

$$\Delta_B = -A^2 - (X+Y)^2$$

Their principal symbols are $c = \zeta^2 + 4\xi\eta$, $l = \xi^2 + \eta^2$ and $b = \zeta^2 + (\xi+\eta)^2$ respectively. The characteristic cones are $\Sigma_l = \{\xi = \eta = 0\} \subset \{c > 0\}$ and $\Sigma_b = \{\zeta = 0, \xi + \eta = 0\} \subset \{c < 0\}$. The co-adjoint orbits lying in $c > 0$ support the principal series of irreducible representations (H1 hyperboloïds), while the orbits lying in $c < 0$ support the discrete series of irreps ($H2$ hyperboloïds).

The calculation of the action of these operators on irreducible representations is the subject of [C-H-W-?]. The spectrum of Δ_B is described in [Ch-20]. Note that the magnetic Hamiltonian is Liouville integrable while the Liouville one is only weakly integrable (see Section 9.4) thanks to the Euler equations.

13. Traces

13.1. Wave traces

Richard Melrose proved in the paper [Me-84] that the Duistermaat-Guillemin trace formula applies for the singularities of $\mathrm{Trace}(\exp(-it\sqrt{\Delta}))$ outside $t = 0$. I gave in [CdV-22b] a simpler proof of this result.

The wave traces of the Δ_l's: the Δ_l's are self-adjoint elliptic Toeplitz operators of degree 1 to which the Theorems 9 and 10 of [B-G-81] apply. The corresponding closed orbits are the Reeb orbits. These theorems say in particular that the singular support of the distributions $Z_l(t) := \mathrm{trace}(\exp(it\sqrt{\Delta_l}))$ is contained in the set of periods of the Reeb flow divided by $(2l+1)$. In fact, under some genericity assumption on the Reeb flow the two sets are the same. Summing with respect to l gives a dense set of singularities. It would be nice to say more on the precise structure of these singularities.

13.2. Schrödinger trace

The Heisenberg case: for $\Re(z) > 0$, one defines $Z(z) := \sum_{j=1}^{\infty} e^{-z\lambda_j}$ For the flat Heisenberg, one gets

$$Z_o(z) = \sum_{m=1}^{\infty} 2m \sum_{n=0}^{\infty} e^{-(2n+1)mz} = \frac{1}{2}\sum_{m=1}^{\infty} \frac{m}{\sinh mz}$$

$$= \frac{1}{2z}\sum_{m\in\mathbb{Z}} \frac{mz}{\sinh mz} - 1$$

We will apply the Poisson summation formula. The Fourier transform of $z/\sinh z$ is $\frac{\pi^2}{1+\cosh \pi z}$. We get

$$Z_o(z) = \frac{\pi^2}{4z^2} - \frac{1}{2z} + \frac{\pi^2}{z^2}\sum_{l=1}^{\infty} \frac{1}{1+\cosh(2\pi^2 l/z)}$$

One recovers the Weyl law, $Z_o(t) \sim \pi^2/8t^2$.

As $z \to 0^+$, one gets exponential corrections as in [CdV-73]

$$\pi^2 \sum_{l=1}^{\infty} e^{-2\pi^2 l/z}$$

Each exponent identifies with $L^2/4$, hence one gets also the lengths of periodic geodesics, namely the set of $2\pi\sqrt{2l}, l \in \mathbb{N}$.

Let us look at the boundary values as $\Re(z) \to 0^+$. There are infinitely many "poles", namely the zeroes of $1 + \cosh 2\pi^2 l/i\tau$. The poles are $z_l = \frac{2\pi i l}{2k+1}$. This corresponds to the periodic orbits of $(2k+1)$-times Reeb as expected. This is a dense set in the boundary, there exist no meromorphic extensions.

14. A conjecture

From what we know, I propose the following conjecture:

"the periods of the Reeb flow are generically spectral invariants of the sR Laplacian".

There are two heuristic arguments for that:

1. Using Theorem 9.2, one recover the lengths of the periodic geodesics. Then the asymptotics of the lengths of closed geodesics accumulating around a Reeb periodic orbit involve the Reeb periods (see Theorem 9.2).
2. The second argument follows from the Boutet de Monvel-Guillemin trace formula appplied to each of the Δ_l's (see Section 13.1) whose righthandsides involve the Reeb periods.
   Of course, it would be even nicer to extend the Schrödinger trace formula from Section 13.2 to a more general case.

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