Pure and Applied Mathematics Quarterly
Volume 19, Number 4, 1873–1898, 2023

Lower bounds for Steklov eigenfunctions

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Abstract: Let (Ω, g) be a compact, real analytic Riemannian manifold with real analytic boundary $\partial\Omega = M$. We give L^2-lower bounds for Steklov eigenfunctions and their restrictions to interior hypersurfaces $H \subset \Omega^\circ$ in a geometrically defined neighborhood of M. Our results are optimal in the entire geometric neighborhood and complement the results on eigenfunction upper bounds in [GT19].

Keywords: Steklov, high energy asymptotics, lower bounds.

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Received January 12, 2022.
2010 Mathematics Subject Classification: Primary 35P20, 58J50.
*J.G. is grateful to the EPSRC for support under Early Career Fellowship EP/V001760/1 and Standard Grant EP/V051636/1.
†J.T. was partially supported by NSERC Discovery Grant # OGP0170280 and by the French National Research Agency project Gerasic-ANR-13-BS01-0007-0.

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1. Introduction

Let (Ω, g) be an $n+1$-dimensional, compact C^∞ Riemannian manifold with boundary M and corresponding unit exterior normal ν. By some abuse of notation, we also let ν denote a smooth vector field extension and $\gamma_M : C^0(\Omega) \to C^0(M)$ be the boundary restriction map. Let $\mathcal{D} : C^\infty(M) \to C^\infty(M)$ be the associated Dirichlet-to-Neumann (DtN) operator defined by

$$\mathcal{D}f := \gamma_M \partial_\nu u \tag{1.1}$$

where u solves the Dirichlet problem

$$\Delta_g u(x) = 0, \;\; x \in \Omega, \qquad\qquad u(q) = f(q), \;\; q \in M. \tag{1.2}$$

The operator $\mathcal{D}$ is an ellptic, first order, self-adjoint pseudodifferential operator (see for example [Tay11, Section 7.11]) with an L^2-normalized basis of eigenfunctions φ_j; $j = 1, 2, \dots$. It is useful to work in the semiclasscial setting from the outset. Choosing $h^{-1} \in \operatorname{Spec} \mathcal{D}$, the corresponding eigenfunction φ_h then satisfies the semiclassical eigenfunction equation

$$h\mathcal{D}\varphi_h = \varphi_h.$$

The harmonic extension, $u_h \in C^\infty(\Omega)$, of a DtN eigenfunction φ_h is called a *Steklov eigenfunction.* Throughout the article, we will use the notation u_h for these Steklov eigenfunctions which have L^2 normalized boundary restriction.

There has been a substantial amount of recent work devoted to the study of the asymptotic behaviour of the DtN eigenvalues and both DtN and Steklov eigenfunctions, including the asymptotics of eigenfunction nodal sets (see for example [BL15, GP17, GPPS14, HL01, PST15, Sha71, SWZ16, Zel15, Zhu15, Zhu16] and references therein).

For large eigenvalues, Steklov eigenfunctions possess both high oscillation inherited from the boundary DtN eigenfunctions and very sharp decay into the interior of Ω. As a consequence, even though Steklov eigenfunctions decay rapidly, the oscillation implies that the nodal sets have intricate structure. It has been conjectured [GP17] that the analogue of Yau's conjecture [Yau82, Yau93] for nodal volumes holds in the Steklov case. This was recently proved for real-analytic Riemann surfaces in [PST15]. In the case of smooth manifolds, the recent work [Dec21b, Dec21a] gives the best available polynomial upper and lower bounds on the nodal volume. Despite these bounds on the nodal volume, it is likely that a typical high energy Steklov eigenfunction exhibits regions of fixed sign with inner radius uniformly bounded from below [BG20].

The question of decay of Steklov eigenfunctions into the interior of M when (M, g) is real analytic was first raised by Hislop–Lutzer [HL01] where they conjecture that the Steklov eigenfunctions decay into the interior as $e^{-d(x,\partial\Omega)/h}$. In the special case where $\dim \Omega = 2$ and Ω is analytic, exponential decay with respect to $d(x, \partial\Omega)$ was proved in [PST15]. The case of general dimension and analytic Ω was handled in [GT19]. There, the authors prove that

$$\sup_{x\in\Omega_\partial(\varepsilon_0)} |\partial_x^\alpha u_h(x)| e^{[d(x,\partial\Omega)-C_{sup}(\Omega)d^2(x,\partial\Omega)]/h} \leq C_\alpha, \tag{1.3}$$

where $\Omega_\partial(\varepsilon_0)$ is a tubular neighbourhood of the boundary of width $\varepsilon_0 = \varepsilon_0(\Omega, M, g) > 0$ and $C_{\sup}$ is a constant depending on the second fundamental form of the boundary. Here, $\varepsilon_0 > 0$ is an h-independent positive constant that depends on the analyticity properties of the boundary and is difficult to quantify explicitly.

In this article, we consider the complementary question of lower bounds on Steklov eigenfunctions. As in the case of [GT19], we restrict our attention to the case of analytic Ω and M. Our first result is a *lower* bound for L^2 restrictions of eigenfunctions in a small ε_0-neighbourhood of the boundary. In analogy with (1.3) we prove

Theorem 1. *Let Ω be a real analytic manifold with real analytic boundary, $M = \partial\Omega$. There exist a neighbourhood, $\Omega_\partial(\varepsilon_0)$ of M and constants $C_j(\varepsilon_0) >$*

$0; j = 1, 2$ *such that for any connected component,* N*, of the boundary and any* $\varepsilon > 0$ *there are* $C > 0$ *and* $h_0 > 0$ *such that for* $h \in (0, h_0(\varepsilon)]$ *and* $0 \le t \le \varepsilon_0$*, and* $H_t := \{x : d(x, N) = t\}$*,*

$$e^{[t+C_1(\varepsilon_0)t^2]/h}\|u_h\|_{L^2(H_t)} \ge Ce^{-C_2(\varepsilon_0)\,\varepsilon/h}\|u_h\|_{L^2(N)} - C_1 e^{-1/(hC_1)}\|u_h\|_{L^2(M)}.$$

Here $\varepsilon_0 = \varepsilon_0(M, \Omega, g) > 0$ is a possibly small constant (but independent of h) that is the same in both the upper bounds (1.3) and lower bounds in Theorem 1 and is difficult to quantify precisely.

Our second result extends eigenfunction lower bounds to an explicit *geometric* neighbourhood of the boundary. Specifically, we use Carleman estimates to "bootstrap" the local result in Theorem 1 to the full geometric neighbourhood of the boundary.

To define the geometric tubular neighbourhood more precisely, let N be a connected component of $\partial\Omega$. We consider the map $\varphi_N : N \times [0, r) \to \overline{\Omega}$ given by

$$\varphi_N(x, r) = \exp_x(-r\nu), \quad r \in [0, r_0),\ x \in N, \tag{1.4}$$

where exp is the Riemannian exponential map induced by the metric g and $-\nu$ is the unit *interior* normal to $\partial\Omega$. By the collar neighbourhood theorem, for sufficiently small $r_0 > 0$. the map φ_N is a diffeomorphism onto its image $\varphi_N([0, r_0))$. We let $r_{\max,N}$ be the maximal choice of r_0 with this property and set

$$\Omega_N(r_{\max,N}) := \varphi([0, r_{\max,N})). \tag{1.5}$$

We refer to $\Omega_N(r_{\max,N})$ as the *geometric* neighbourhood of the boundary component N. In the following, we sometimes abuse notation and just write Ω_N for $\Omega_N(r_{\max,N})$. See Figure 2 for a description of these domains for the annulus.

Theorem 2. *Let* Ω *be a real analytic manifold with real analytic boundary,* $M = \partial\Omega$*, and* $\Omega_N \subset \overline{\Omega}$ *(as in (1.5)) be the* g *Fermi neighbourhood of the connected component,* N*, of* M *and* $0 < t < r_{\max,N}$*. Then, for any tubular neighbourhood,* U_{H_t}*, of* $H_t := \varphi_N(N, t)$ *and* $\varepsilon > 0$*, there are* $h_0 > 0$ *and* $C > 0$ *such that for* $h \in (0, h_0]$*,*

$$\|e^{\psi_N(t)/h}\, u_h\|_{L^2(U_{H_t})} \ge Ce^{-\varepsilon/h}(\|\, u_h\|_{L^2(N)} - e^{-1/Ch}\|u_h\|_{L^2(M)}),$$

where

$$\psi_N(x_{n+1}) = \int_0^{x_{n+1}} e^{\int_0^s Q_{\sup,N}(t)dt} ds,$$

$$Q_{\sup,N}(t) := \sup\{Q(t,x',\xi') \,:\, x' \in N,\, |\xi'|_{g_t(x')} = 1\},$$

g_t is the metric induced on H_t, and $Q(t,x',\xi')$ is the second fundamental form on H_t induced by the inward pointing normal.

The examples of the disk, cylinder, and annulus in Sections 2.1–2.3 show that Theorem 2 is optimal.

Remark 1. Notice that, although the right hand sides of the estimates in Theorems 1 and 2 have an error $e^{-1/Ch}$ with a constant C depending on the analyticity properties of Ω and M, these do not cause losses in the estimates when $\|u_h\|_{L^2(N)} \gg e^{-1/Ch}\|u_h\|_{L^2(M)}$. Since there are finitely many boundary components, there are *always* boundary components where this is the case. Furthermore, we may replace N in Theorem 2 by a union of boundary components, $\tilde{N} := \cup_{j=1}^{L} N_j$ by applying Theorem 2 for each N_j if we replace ψ_N by $\psi_{\tilde{N}}$.

By Taylor expansion at the boundary $M = \{x_{n+1} = 0\}$,

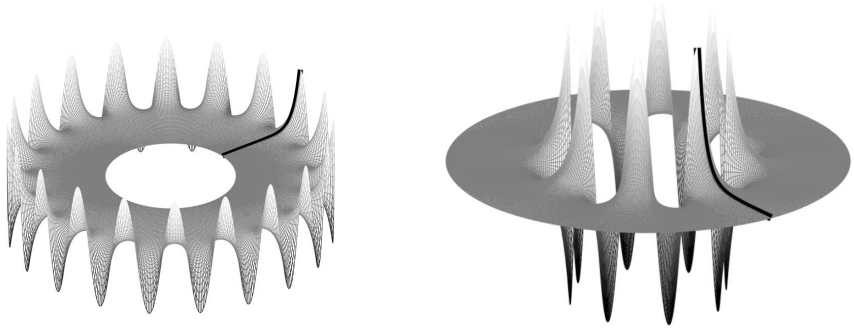
$$\psi_N(x_{n+1}) = x_{n+1} + \frac{Q_{\sup,N}(0)}{2} x_{n+1}^2 + O(x_{n+1}^3), \tag{1.6}$$

where $Q_{\sup,N}(0)$ is the maximum of the second fundamental form along $N \subset \partial\Omega$. Thus, near the boundary, eigenfunction decay is given to first order by $x_{n+1} = d(x, M)$. However, when the boundary is strictly convex, the quadratic correction in (1.6) is actually *negative* and so the rate of decay in our estimate may be faster than $e^{-d(x,M)/h}$. The simple example of the disc (see Section 2.1) shows that this extra decay *does* occur. Likewise, when a boundary component is strictly concave, the quadratic correction is positive, producing a sub-linear rate of decay. This behavior can be seen in the example of the annulus (see Section 2.3).

Our final results concerns lower bounds of the L^2-restriction of eigenfunciton Cauchy data along $H \subset \Omega_\partial(r_{\max,\partial\Omega})$. We recall that given a hypersurface $H \subset \Omega$, the *Cauchy data* along H is the pair

$$CD_H(h) := (u_h|_H, h\partial_\nu u_h|_H).$$

The lower bound in Theorem 2 combined with a potential layer formula in the tube $U_H(\varepsilon)$ allows us prove goodness for $CD_H(h)$ for hypersurfaces H potentially far inside Ω.



(a) $u_{\sigma_{20,1}}$ (b) $u_{\sigma_{8,2}}$

Figure 1: Steklov eigenfunctions with $\sigma \sim 20$ on the annulus with $r_0 = .4$. The black line shows the lower bound predicted by Theorem 2. The labels $\sigma_{k,j}$ are as in (2.3).

Theorem 3. *Let Ω be a real analytic manifold with real analytic boundary, $M = \partial\Omega$ and $H_t := \varphi_{\partial\Omega}(\partial\Omega, t)$. Then for $0 < t < r_{\max,\partial\Omega}$ and $\varepsilon > 0$ there are $C(\varepsilon) > 0$ and $h_0(\varepsilon) > 0$ such that for $h \in (0, h_0(\varepsilon)]$,*

$$e^{\psi_{\partial\Omega}(t)/h}\left(\|u_h\|_{L^2(H_t)} + \|h\partial_\nu u_h\|_{L^2(H_t)}\right) \geq C(\varepsilon)e^{-\varepsilon/h}\|u_h\|_{L^2(M)}.$$

Theorems 2 and 3 are optimal in a sense made precise in the next section.

1.1. Organization of the paper

The proof of Theorem 1 follows first from the existence of a parametrix for the Poisson kernel modulo analytic errors, and second, from the construction of an approximate inverse for this operator valid at frequencies $\lesssim h^{-1}$. However, the parametrix construction is only valid in a collar of radius $\varepsilon_0 = \varepsilon_0(M, \Omega, g)$ that while h-independent, is possibly smaller than $r_{\max,N}$. The proof of Theorem 1 is taken up in Section 3.

The proof of Theorem 2 is given in Section 4. Here, we use the local result in Theorem 1 as a control estimate and use Carleman estimates to extend the lower bound in $\Omega_\partial(\varepsilon_0)$ to the full geometric neighbourhood $\Omega_N(r_{\max,N})$ of the boundary. Finally Section 5 applies a layer potential formula together with Theorem 2 to prove Theorem 3.

2. Examples

2.1. The disk

Let $\Omega = B(0,R) \subset \mathbb{R}^2$. Then the Steklov eigenvalues are precisely $\sigma = 0, \frac{1}{R}, \frac{2}{R} \dots$ with corresponding Steklov eigenfunctions given by

$$u_k^{\pm} = \frac{1}{\sqrt{2\pi R} R^k} r^k e^{\pm ik\theta}, \quad \sigma = \frac{k}{R}. \tag{2.1}$$

In particular, letting $h = \sigma^{-1} = k^{-1}R$,

$$u_h^{\pm} = \frac{1}{\sqrt{2\pi R}} e^{[R\log(1-(R-r)/R))]/h} e^{\pm i\theta R/h}.$$

Now, we recall that the metric in Fermi normal coordinates relative to $\partial B(0,R)$ (i.e. with $x_2 = R - r$) is given by

$$\xi_2^2 + \frac{1}{(R-x_2)^2}\xi_\theta^2,$$

and hence, the metric induced on H_t and second fundamental form on H_t are given by

$$|\xi_\theta|_{g_t}^2 = \frac{\xi_\theta^2}{(R-t)^2}, \qquad Q(t,\theta,\xi_\theta) = \frac{\xi_\theta^2}{(R-t)^3}.$$

In particular,

$$Q_{\sup}(t) = \frac{1}{R-t},$$

and therefore,

$$\psi(x_2) = \int_0^{x_2} e^{\int_0^s Q_{\sup}(t)dt} ds = R\log(1 - \tfrac{x_2}{R}).$$

Undoing, the change of variables and applying Theorem 3 to obtain a lower bound, we have that

$$C(\varepsilon)e^{-\varepsilon/h} \le e^{R\log(1-\frac{R-r_0}{R})/h}[\|u_h^{\pm}\|_{L^2(r=r_0))} + \|h\partial_r u_h^{\pm}\|_{L^2(r=r_0)}] \le C.$$

In particular, the exponential weight in Theorems 2 and 3 is optimal.

The case of spheres in higher dimensions is nearly identical if we replace $e^{\pm ik\theta}$ by a spherical harmonic.

2.2. Cylinders

Let (M, g) be a real analytic manifold of dimension n without boundary and $\Omega = (-1, 1)_s \times M_x$ with metric $ds^2 + g(x)$. Then

$$\Delta_\Omega = \partial_s^2 + \Delta_M.$$

Let φ_k be an orthonormal basis for $L^2(M)$ with

$$(-\Delta_M - \lambda_k^2)\varphi_k = 0.$$

Then the Steklov eigenfunctions are given by

$$u_h(s, x) = \frac{\cosh(\lambda_k s)}{\cosh(\lambda_k)}\varphi_k(x), \qquad v_h(x, s) = \frac{\sinh(\lambda_k t)}{\sinh(\lambda_k)}\varphi_k(x)$$

with Steklov eigenvalues $\sigma_k = \lambda_k \tanh(\lambda_k)$ and $\sigma_k' = \lambda_k \coth(\lambda_k)$ respectively. Notice that

$$\cosh(x) = \frac{1}{2}e^{|x|} + O(e^{-|x|}), \qquad \sinh(x) = \frac{\operatorname{sgn}(x)}{2}e^{|x|} + O(e^{-|x|}),$$

and

$$\sigma_k = \lambda_k(1 + O(e^{-\lambda_k})).$$

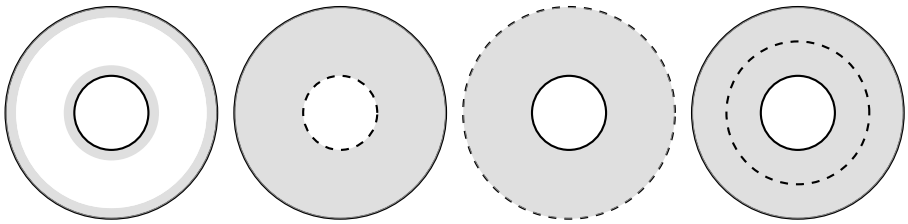
It is easy to see that $Q_{\sup}(s) \equiv 0$ and hence, taking $N_L = \{s = -1\}$, we have $\psi_{N_L}(s) = 1 + s$, and taking $N_R = \{s = 1\}$, we have $\psi_{N_R} = 1 - s$. Combining the lower bounds from Theorem 2 applied with N_L and N_R, we obtain optimal lower bounds on M. Similarly, we obtain optimal lower bounds with an application of Theorem 3, but this time the hypersurface is given by $H_t = \{s = -1 + t\} \sqcup \{s = 1 - t\}$ and the Theorem is valid for $0 < t < 1$.

Remark 2. Notice that a cylinder has the unusual feature that there are Steklov eigenfunctions with non-negligible mass on multiple boundary components. This is why one must apply Theorem 2 twice (once from the left hypersurface and once from the right) to obtain the correct lower bounds.

2.3. The annulus

Now, consider $B(0, 1) \setminus B(0, r_0) \subset \mathbb{R}^2$. Then a simple computation shows that the Steklov eigenvalues are the roots of

$$p_k(\sigma) = \sigma^2 - \sigma k\left(\frac{1 + r_0}{r_0}\right)\left(\frac{1 + r_0^{2k}}{1 - r_0^{2k}}\right) + \frac{k^2}{r_0}, \quad k = 0, 1, \ldots$$



(a) $\Omega_\partial(\varepsilon)$. (b) $\Omega_{\partial B(0,1)}$. (c) $\Omega_{\partial B(0,r_0)}$. (d) $\Omega_{\partial\Omega}$.

Figure 2: Regions of applicability for Theorems 1, 2, and 3 for $\Omega = B(0,1) \setminus \overline{B(0,r_0)}$. The relevant regions are shaded in gray with dashed lines not included in the region.

with corresponding eigenfunctions

$$u_\sigma^\pm(r,\theta) = C_{k,\sigma} e^{\pm ik\theta}\left(r^k + \frac{k-\sigma}{k+\sigma} r^{-k}\right). \tag{2.2}$$

See Figure 1 for graphs of two such eigenfunctions.

It is easy to show that the roots of $p_n(\sigma)$ have

$$\sigma_{k,1} = k + O(kr_0^{2k}), \quad \sigma_{k,2} = \frac{k}{r_0} + O(kr_0^{2k}). \tag{2.3}$$

Then,

$$u^\pm_{\sigma_{k,1}} = \frac{1}{\sqrt{2\pi}} e^{\pm ik\theta}(r^k + O(r_0^{2k})r^{-k}), \quad u^\pm_{\sigma_{k,2}} = \frac{1}{\sqrt{2\pi r_0}} r_0^k e^{\pm ik\theta}(r^{-k} + O(1)r^k).$$

The case of $u_{\sigma_{k,1}}$ is identical to that for the disk when $r > r_0 + \varepsilon$, so we focus on $u_{\sigma_{k,2}}$. Let $h = \sigma_{k,2}^{-1} = r_0 k^{-1} + O(e^{-ck})$. Then, for $r < 1 - \varepsilon$,

$$|u^\pm_{\sigma_{k,2}}(r,\theta)| \geq \frac{1}{\sqrt{2\pi r_0}} e^{-r_0 \log[1+(r-r_0)/r_0]/h}(1 + O(e^{-c/h})).$$

Using exactly the same computation as for the interior of the disk, one again sees that the lower bound in Theorem 2 is optimal for $u_{\sigma_{k,2}}$. Indeed, $\|u_{\sigma_{k,2}}\|_{L^2(\partial B(0,r_0))} = 1 + e^{-c/h}$ and $\psi(r) = r_0 \log(1 + (r - r_0)/r_0)$.

3. Lower bounds sufficiently close to the boundary

The main goal of this section is the proof of Theorem 1. As we already indicated in the introduction, here $\varepsilon_0 = \varepsilon_0(M, \Omega, g) > 0$ is a possibly small

constant that depends on the analyticity properties of M. As such, it is difficult to quantify.

3.1. Notation for pseudodifferential operators

Below, we will need notation for semiclassical pseudodifferential operators. We say that $a \in C^\infty(T^*M)$ *is a symbol of order* k, and write $a_h = a_h(x,\xi;h) \in S^k(T^*M)$, if

$$|\partial_x^\alpha \partial_\xi^\beta a_h(x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{k-|\beta|}, \qquad \langle\xi\rangle = (1+|\xi|^2)^{1/2}.$$

Note that, a_h depends on the small parameter h. We also define the set of semiclassical pseudodifferential operators of order k, $\Psi^k(M)$ as in [Zwo12, Chapter 14] or [DZ19, Appendix E]. Note, in particular, that semiclassical pseudodifferential operators of order $-\infty$ are smoothing, but their norms do not vanish as $h \to 0$. We also define the elliptic set of a pseudodifferential operator as in [DZ19, Definition E.31].

3.2. Analytic symbols

In this section, we will need the notion of a classical analytic symbol, which we recall from [Sjö82] (see also [SU16]). We say that a is *classical analytic of order* k and write $a = a(x,\xi) \in S^k_{cl,a}$ if there exist $C_0 > 0$ and functions a_j analytic on a fixed neighbourhood of $T^*M \setminus \{0\}$, homogeneous degree j, satisfying

$$\Big|a_j\Big(x, \frac{\xi}{|\xi|}\Big)\Big| \leq C_0^{j+1}(j+1)!,$$

and for every $C_1 > 0$ large enough, there is $C_2 > 0$ such that

$$\Big|a(x,\xi) - \sum_{0\leq j\leq |\xi|/C_1} a_{k-j}(x,\xi)\Big| \leq C_2 e^{-|\xi|/C_2}, \qquad |\xi| \geq 1.$$

The key fact that we will use about such symbols is that, after rescaling $\xi \to \xi/h$, it is possible to deform contours away from $|\xi| = 0$ modulo errors of order $e^{-1/Ch}$.

We also recall the notion of a semiclassical, classical analytic symbol from [Sjö96]. We say that $a_h = a_h(x,\xi;h) \in S^k_{h,cl,a}$ provided there are $a_j \in S^{k-j}(T^*M)$, h independent and analytic in a conic neighbourhood of T^*M,

and C_0, $C > 0$ such that

$$\partial_x^{l_1}\partial_\xi^{l_2}\bar{\partial}_{(x,\xi)}a_h(x,\xi;h) = O_{l_1,l_2}(e^{-\langle\xi\rangle/Ch}),$$
$$\Big|a_h(x,\xi;h) - \sum_{0\le j\le\langle\xi\rangle/Ch} h^j a_j(x,\xi)\Big| \le Ce^{-\langle\xi\rangle/Ch}, \qquad |a_j| \le C_0 C^j j!\langle\xi\rangle^{k-j}.$$

Contours can, again modulo errors of the form $e^{-1/Ch}$, be readily deformed when this type of symbol is involved.

3.3. A geometric FBI transform

We also recall a particular Fourier-Bros-Iagolnltzer (FBI) transform on M. Define the operator $T : C^\infty(M) \to C^\infty(T^*M)$ by

$$Tu(x,\xi) = \frac{1}{(2\pi h)^{3n/4}}\int e^{i(\langle \exp_y^{-1}(x),\xi\rangle + i\frac{\langle\xi\rangle}{2}d(x,y)^2)/h}a_h(x,\xi,y)u(y)dy, \tag{3.1}$$

where $a_h \in S^{n/4}_{h,cla}$, which is uniformly bounded from $L^2(M) \to L^2(T^*M)$, and has a left parametrix $S : L^2(T^*M) \to L^2(M)$ given by

$$Sv(x) = \frac{1}{(2\pi h)^{3n/4}}\int e^{-i(\langle \exp_x^{-1}(y),\xi\rangle - i\frac{\langle\xi\rangle}{2}d(x,y)^2)/h}b_h(x,\xi,y)v(x,\xi)dxd\xi, \tag{3.2}$$

for some $b_h \in S^{n/4}_{h,cla}$, which is also uniformly bounded with the property that

$$ST = I + R, \quad |\partial_x^\alpha\partial_y^\beta R(x,y,h)| = O_{\alpha,\beta}(e^{-C/h}),\ C > 0.$$

As in [GT19, Theorem 2], the FBI transform and its left parametrix play a key role in localization of symbols and associated operators modulo exponential errors.

3.4. Preliminaries on the Poisson operator

Let $P : C^\infty(\partial\Omega) \to C^\omega(\Omega)$ be a parametrix for the Poisson operator modulo analytic errors of the form

$$[Pf](x) = \frac{1}{(2\pi)^n}\int e^{i(\psi(x_{n+1},x',\xi')-\langle y',\xi'\rangle)}p(x_{n+1},x',\xi')f(y')dy'd\xi'. \tag{3.3}$$

That is, there is an operator $K : C^\infty(\partial\Omega) \to C^\omega(\Omega)$ with an analytic kernel such that

$$\Delta_g(P+K) = 0 \text{ in } \Omega \qquad (P+K)|_{\partial\Omega} = f.$$

Such an operator exists by [SU16] (see also [Leb18, Ste14, Zel12, Gui97]). In addition, $p\in S^0_{cl,a}$ is classical analytic of order 0 and $\psi \in C^\infty([0,\varepsilon)\times T^*M\setminus\{0\})$ satisfies,

$$\partial_{x_{n+1}}\psi - i\sqrt{r(x,\partial_{x'}\psi)} = 0, \qquad \psi(0,x',\xi') = \langle x',\xi'\rangle, \tag{3.4}$$

where, in Fermi normal coordinates, the symbol of $-\Delta_g$ is $\xi_n^2 + r(x,\xi')$.

Let $t>0$ and define the smooth hypersurface

$$H_t := \{(x',t) \mid x'\in\partial\Omega\}$$

In the following, we identify $C^\infty(\partial\Omega)$ with $C^\infty(H_t)$ under the diffeomorphism $\partial\Omega \ni x' \mapsto (x',t)\in H_t$.

Let $\varphi\in C_c^\infty(0,\infty)$ with $\varphi(x)\equiv 1$ near $\{x\equiv 1\}$. We define the family of operators $E_h : L^2(\partial\Omega)\to L^2(\partial\Omega)$ by

$$[E_h u](x') := \frac{1}{(2\pi h)^n}\int e^{i(\langle x',\xi'\rangle-\psi(t,y',\xi'))/h}\varphi(|\xi'|_{g(x')})u(y')dy'd\xi'. \tag{3.5}$$

Let also γ_H denote the restriction operator from Ω to $\{H_t\}$.

Remark 3. Note that, because we include the compactly supported cutoff φ in the amplitude of the integral defining E_h, E_h is well-defined as an operator on $L^2(\partial\Omega)$. It would, however, be possible to define a left inverse for P acting on sufficiently analytic functions as in e.g. [Gui97], but this is not necessary here.

Lemma 3.1. *Let T and S be the FBI transform and its left parametrix from (3.1) and (3.2). There is $\varepsilon_0>0$ such that for $0<t<\varepsilon_0$, $A_H := E_h\gamma_H PS\varphi(|\xi'|_g)T\in\Psi^0(\partial\Omega)$. Moreover, A is elliptic on*

$$S^*\partial\Omega = \{(x',\xi') \,:\, |\xi'|_{g(0,x')} = 1\}.$$

Proof. The kernel of $E_h\gamma_H PS\varphi(|\eta'|_g)$ is:

$$\begin{aligned}&[E_h\gamma_H PS\varphi(|\eta'|_g)](x',y',\eta')\\ &= \frac{1}{(2\pi h)^{2n+3n/4}}\int e^{i(\langle x',\xi'\rangle-\psi(t,z',\xi')+\psi(t,z',\omega')-\langle s',\omega'\rangle-\langle \exp_{s'}^{-1}(y'),\eta'\rangle+\frac{i\langle\eta'\rangle}{2}d(y',s')^2)/h}\\ &\qquad p(t,z',\omega'/h)\varphi(|\xi'|_{g(x')})b_h(y',\eta',s')\varphi(|\eta'|_{g(y')})ds'd\omega'dz'd\xi'.\end{aligned}$$

We start by formally computing the critical points of the phase in z', ω', s'. Let

$$\Phi = \langle x', \xi' \rangle - \psi(t, z', \xi') + \psi(t, z', \omega') - \langle s', \omega' \rangle - \langle \exp_{s'}^{-1}(y'), \eta' \rangle + \frac{i\langle \eta' \rangle}{2} d(y', s')^2.$$

Then, using Fermi normal coordinates centered around y' in the s'-variables to compute

$$\partial_{z'}\Phi = \partial_{z'}(\psi(t, z', \omega') - \psi(t, z', \xi')), \qquad \partial_{\omega'}\Phi = \partial_{\omega'}\psi(t, z', \omega') - s',$$
$$\partial_{s'}\Phi = \eta' - \omega' + i\langle \eta' \rangle (s' - y').$$

Now, observe that (3.4) implies that

$$\partial^2_{(z', \omega', s')}\Phi = \begin{pmatrix} 0 & I & 0 \\ I & 0 & -I \\ 0 & -I & i\langle \eta' \rangle \end{pmatrix} + \begin{pmatrix} O(t) & O(t) & 0 \\ O(t) & O(t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.6}$$

and hence that the phase is non-degenerate for t small. Moreover, by Taylor expansion,

$$\partial_{z'}\Phi = (I + t\, A(t, z', \omega', \xi'))(\omega' - \xi'),$$

for some $A \in C^\infty$. In particular, denoting the critical points of Φ by (ω'_c, z'_c, s'_c), for t small, we have

$$\omega'_c = \xi', \;\; s'_c = y' + i\langle \eta' \rangle^{-1}(\eta' - \xi'), \;\; z'_c = s'_c + O(t).$$

Thus,

$$\begin{aligned} \Phi(x, \xi', z'_c, \omega'_c, s'_c, y', \eta') &= \langle x', \xi' \rangle + \langle s'_c, \eta' - \xi' \rangle - \langle y', \eta' \rangle + \frac{i\langle \eta' \rangle}{2} d(y', s'_c)^2 \\ &= \langle x' - y', \xi' \rangle + \frac{i}{2\langle \eta' \rangle}(\xi' - \eta')^2. \end{aligned}$$

We will need to deform the contour to a good contour in order to perform complex stationary phase [Sjö82, Theorem 2.8]. However, before doing this, we must show that the region near $\omega' = 0$ can be neglected. Since the integrand is supported on $|\eta'| > c > 0$, this can be done by deforming the contour in s' alone by $s' \mapsto s' + i\delta\langle \eta' \rangle^{-1}(\omega' - \eta')$.

Next, we need to find a good contour for the phase. That is, we want to find a contour, Γ, such that the critical point, $\rho_c = (\omega'_c, z'_c, s'_c) \in \Gamma$ and

$\operatorname{Im}\Phi|_\Gamma(\rho) \geq c|\rho-\rho_c|^2$. To do this, we use the Hessian (3.6) to choose a contour such that

$$\operatorname{Im}\langle \partial^2_{(z',\omega',s')}\Phi|_{\rho_c}\, v, v\rangle \geq c|v|^2, \qquad v = \rho - \rho_c, \qquad \rho \in \Gamma. \tag{3.7}$$

Using formula (3.6), we observe that if the tangent space to Γ at ρ_c is spanned by

$$\left\{ \begin{pmatrix} z' + i\delta\omega' \\ \omega' + i\delta z' \\ s' \end{pmatrix} \,\middle|\, (z',\omega',s') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \right\}$$

then (3.7) holds when t is small enough. In addition to (3.7), we need to make sure that the contour remains in the domain of analyticity for the integrand. Thus, imaginary parts of the deformation must remain small even away from the critical point, ρ_c. Motivated by this, we let $\chi \in C_c^\infty(\mathbb{R};[0,1])$ with $\chi \equiv 1$ near 0 and consider the one-parameter family of contours Γ_r defined for $r \in [0,1]$ by

$$\begin{aligned} &\Gamma_r : (z',\omega',s') \mapsto \\ &\Big(z' + r\Big(z_c + \frac{i\delta\omega'}{\langle\omega'\rangle}\Big),\ \omega' + \xi' + ir\delta z',\ s' + y' + ir\langle\eta'\rangle^{-1}(\eta'-\xi')\chi(\delta^{-1}\langle\eta'\rangle^{-1}|\eta'-\xi'|) \\ &\qquad\qquad + ir\delta\Big(\frac{\eta'-\xi'}{|\eta'-\xi'|}(1-\chi(\delta^{-1}\langle\eta'\rangle^{-1}|\eta'-\xi'|))\Big)\Big). \end{aligned}$$

First, note that for $|\omega'| \gg 1$, $r \in [0,1]$, $\operatorname{Im}\Phi|_{\Gamma_r} \geq c|\omega'|$. Thus, the terms coming from infinity in ω' vanish and the contour deformation from $\Gamma_0 = \mathbb{R}^{3n}$ to Γ_1 is justified by Stokes' theorem.

Moreover, for $|\xi'-\eta'| \leq \delta$, by Taylor expansion of $\Phi|_{\Gamma_1}$ around (z_c', ω_c', s_c'), the phase satisfies

$$\begin{aligned} \Phi|_{\Gamma_1} = \langle x'-y', \xi'\rangle &+ \frac{i}{2\langle\eta'\rangle}(\xi'-\eta')^2 \\ &+ \Big\langle z' + \frac{i\delta\omega'}{\langle\omega\rangle}, \omega' + i\delta z'\Big\rangle + \langle s', \omega' + i\delta z\rangle + i\frac{\langle\eta'\rangle|s'|^2}{2} \\ &+ O\Big(t\Big(|\omega' + i\delta z'|^2 + \Big|z' + \frac{i\delta\omega'}{\langle\omega'\rangle}\Big|^2\Big)\Big), \end{aligned}$$

and for $|\xi'-\eta'| \geq \delta$, $\operatorname{Im}\Phi|_{\Gamma_1} \geq c\delta$. In particular, for $t \ll \delta$, Γ_1 is a good contour for Φ and we may apply the method of analytic stationary phase in

(z', ω', s') to obtain

$$\begin{aligned}
&[E_h \gamma_H P S \varphi(|\eta'|_g)](x', y', \eta') \\
&= \frac{1}{(2\pi h)^{n+\frac{n}{4}}} \int e^{i[\langle x'-y', \xi'\rangle + \frac{i}{2\langle \eta'\rangle}(\xi'-\eta')^2]/h} \\
&\qquad\qquad (p(t, z_c', \xi'/h)\varphi(|\xi'|_{g(x')})\varphi(|\eta'|_{g(x')})\bar{b}_h(y', \eta', s_c') + O(h)_{C_c^\infty})d\xi' \\
&\quad + O(e^{-C/h}).
\end{aligned}$$

Here, we crucially use that, since a is polyhomogeneous of order 0, $a(t, z', \omega'/h)$ is uniformly bounded with all derivatives when $|\omega'| > c > 0$.

Finally, we precompose with T to obtain the phase

$$\tilde{\Phi} = \langle x' - w', \xi'\rangle + \frac{i}{2\langle \eta'\rangle}(\xi' - \eta')^2 + \frac{i\langle \eta'\rangle}{2}(w' - y')^2 + \langle w' - y', \eta'\rangle$$

and, using that $\mathbb{R}^{2n}_{(w',\eta')}$ is a good contour, we may perform (analytic) stationary phase in w', η' to obtain

$$[E_h \gamma_H P S \varphi(|\xi|_g) T](x', y') = \frac{1}{(2\pi h)^n} \int e^{i[\langle x'-y', \xi'\rangle]/h}\, \tilde{a}_h(x', y', \xi) d\xi' + O(h^\infty)_{C^\infty},$$

where $\tilde{a}_h \in C_c^\infty$, $|\tilde{a}_h(x', x', \xi')| > c > 0$ on $|\xi'|_g = 1$, $|\partial_{x'}^{\alpha_1}\partial_{y'}^{\alpha_2}\partial_\xi^\beta \tilde{a}_h(x', y', \xi')| \leq C_{\alpha_1\alpha_2\beta}$ and. In particular, $E_h \gamma_H P S \varphi(|\xi'|_g) T \in \Psi^0(M)$, a semiclassical pseudodifferential operator of order zero which is elliptic as claimed. □

3.5. Lower bounds in a fixed non-geometric neighborhood: Proof of Theorem 1

Proof. Recall from [GT19, Theorem 2] that, with T and S as in (3.1) and (3.2) respectively, for $\varphi \in C_c^\infty((0, \infty))$ with $\varphi \equiv 1$ near 1,

$$u|_{\partial\Omega} = S\varphi(|\xi'|_g) T u|_{\partial\Omega} + O(e^{-C/h})_{L^2}.$$

Next,

$$u|_H = \gamma_H (P + K) u|_{\partial\Omega} = \gamma_H (P + K) S \varphi(|\xi'|_g) T u|_{\partial\Omega} + O(e^{-C/h})_{L^2}.$$

Now, we have

$$K S \varphi(|\xi'|_g) T$$

$$= \frac{1}{(2\pi h)^{3n/2}} \int K(x', w') e^{\frac{i}{h}(-\langle \exp_{w'}^{-1}(z'), \zeta'\rangle + \langle \exp_{y'}^{-1}(z'), \zeta'\rangle + \frac{i\langle \zeta'\rangle}{2} d(w', z')^2 + \frac{i\langle \zeta'\rangle}{2} d(y', z'))}$$
$$b_h(w', y'\zeta') a_h(z', \zeta', y') \varphi(|\zeta'|_{g(z')}) dz' d\zeta' dw',$$

Since K is analytic, by Stokes' theorem we make the contour deformation $w' \mapsto w' + i\delta\zeta'$ with $\delta > 0$ sufficiently small, and use that $|\zeta'| > c > 0$ to obtain

$$KS\varphi(|\xi'|_g)T = O(e^{-C/h})_{L^2 \to L^2},$$

and hence

$$u|_H = \gamma_H PS\varphi(|\xi'|_g)Tu|_{\partial\Omega} + O(e^{-C/h})_{L^2}.$$

Now, note that E_h naturally decomposes into a sum of operators acting on from $L^2(N_j) \to L^2(N_j)$ where $M = \sqcup N_j$ and N_j are the connected components of M. We assume that $H_t = \varphi_{N_0}(N_0, t)$ with φ_{N_0} as in (1.4) and write the component of E_h acting on N_0 as E_h^0. Therefore, with E_h as in (3.5) and $\varphi \in C_c^\infty(0, 1+\varepsilon)$ with $\varphi \equiv 1$ near 1, we have by Lemma 3.1

$$E_h^0(u|_{H_t}) = E_h^0 \gamma_H PS\varphi(|\xi'|_g)Tu|_{\partial\Omega} = Au|_{N_0} + E_h^0 O(e^{-C/h})_{L^2}$$

where $A \in \Psi^0(\partial\Omega)$ is elliptic on

$$S^*\partial\Omega = \{(x', \xi') \,:\, |\xi'|_{g(0,x')} = 1\}.$$

In particular,

$$\begin{aligned} \|Au|_{N_0}\| &\leq \|E_h^0\|_{L^2 \to L^2}(\|u|_{H_t}\|_{L^2(H_t)} + O(e^{-C/h})) \\ &\leq C_\varepsilon \sup_{x \in H_t, |\eta|_{g_{H_t}} \leq 1+\varepsilon} e^{\operatorname{Im}\psi(x,\eta)/h}(\|u|_{H_t}\|_{L^2(H_t)} + O(e^{-C/h})_{L^2}) \end{aligned}$$

Now, since

$$\mathrm{WF_h}(u|_{\partial\Omega}) \subset S^*\partial\Omega,$$

and A is elliptic on $S^*\partial\Omega$, we have by e.g. [DZ19, Theorem E.33]

$$\|u\|_{L^2(N_0)} \leq C_\varepsilon \|Au\|_{L^2(N_0)} + O(h^\infty)\|u\|_{L^2(N_0)}.$$

In particular, for h small enough

$$\begin{aligned} \|u\|_{L^2(N_0)} &\leq C_\varepsilon \|Au\|_{L^2(N_0)} \\ &\leq C_\varepsilon \sup_{x \in H_t, |\eta'|_{g_{H_t}} \leq 1+\varepsilon} e^{\operatorname{Im}\psi(x,\eta)/h}(\|u|_{H_t}\|_{L^2(H_t)} + O(e^{-C/h})_{L^2}). \end{aligned}$$

Therefore, for t small enough, the proof is complete. □

4. Carleman estimates under control assumptions: Proof of Theorem 2

Although the collar neighbourhood U in which the Poisson representation in (3.3) is valid is fixed independent of h, the size of U is difficult to make precise and could be quite small since it depends in a complicated fashion on the analyticity properies of (Ω, M, g). Our aim in the next section is to "bootstrap" the lower bounds in Theorem 1 further into the interior of Ω. To set notation, we let N be a connected component of $\partial\Omega$ and $H_t = \varphi_N(N, t)$ with $0 < t < r_{\max,N}$ and φ_N as in (1.4).

Proof. Let $(x', x_{n+1}) : \Omega_N \to \mathbb{R}^n \times \mathbb{R}$ be the Fermi coordinates above adapted to the boundary component $N = \{x_{n+1} = 0\}$ and let $\Omega_N := \{0 \leq x_{n+1} < r_{\max,N}\}$ be the maximal Fermi tube containing the hypersurface $H_t = \{x_{n+1} = t\}$ with $t < r_{\max,N}$. Fix U_{H_t} a neighbourhood of H_t and $\varepsilon > 0$. Let $\delta_{U_t} > 0$ such that $\{(x', x_{n+1}) : |t - x_{n+1}| < 3\delta_{H_t}\} \subset \Omega_N \cap U_t$ and $\varepsilon > 0$.

Remark 4. We note here that Fermi neighbourhood Ω_N of the boundary depends only on the geometry of geodesic flow inside Ω and *not* on the analytic modulus of the data $(\partial\Omega, H, g)$. As such, it is often comparatively easy to determine the maximal tube width $t > 0$ in Theorem 2.

4.1. Definition of cutoff functions

Let $\varepsilon_1 > 0$ to be chosen small later such that

$$\varepsilon_1 < \min\left(\frac{\varepsilon_0}{4}, t, r_{\max,N} - t, \delta_{U_{H_t}}\right), \tag{4.1}$$

where we recall that $H_t = \{x_{n+1} = t\} \subset \Omega_N$ and $\varepsilon_0 = \varepsilon_0(M, \Omega, g)$ is the radius for which Theorem 1 holds.

Let $\chi^+_{\varepsilon_1} \in C^\infty(\mathbb{R}; [0, 1])$ with

$$\operatorname{supp} \partial\chi^+_{\varepsilon_1}, \subset \{\tfrac{\varepsilon_1}{2} < x_{n+1} < \varepsilon_1\}$$

and

$$\chi^+_{\varepsilon_1}(x_{n+1}) = 1; \quad x_{n+1} > 2\varepsilon_1, \qquad \chi^+_{\varepsilon_1}(x_{n+1}) = 0; \quad x_{n+1} < \tfrac{\varepsilon_1}{2}.$$

We let $\chi^-_{\varepsilon_1} \in C^\infty(\mathbb{R}; [0, 1])$ be another cutoff with derivative localized around the hypersurface $\{x_{n+1} = t\}$, satisfying

$$\operatorname{supp} \partial\chi^-_{\varepsilon_1}, \subset \{t - \varepsilon_1 < x_{n+1} < t + \varepsilon_1\}$$

and

$$\chi^-_{\varepsilon_1}(x_{n+1}) = 1; \quad x_{n+1} < t - 2\varepsilon_1, \qquad \chi^-_{\varepsilon_1}(x_{n+1}) = 0; \quad x_{n+1} > t + 2\varepsilon_1.$$

Finally, we set

$$\chi_{\varepsilon_1}(x', x_{n+1}) := \chi^+_{\varepsilon_1}(x_{n+1}) \cdot \chi^-_{\varepsilon_1}(x_{n+1}) \in C_0^\infty(\Omega_N(r_{\max,N}); [0,1]), \tag{4.2}$$

where by Leibniz rule it follows that

$$\operatorname{supp} \partial\chi_{\varepsilon_1} \subset \{(x', x_{n+1}) \in \Omega_N \,:\, x_{n+1} \in \operatorname{supp} \partial\chi^+_{\varepsilon_1} \cup \operatorname{supp} \partial\chi^-_{\varepsilon_1}\}. \tag{4.3}$$

Next, we set

$$v_h := \chi_{\varepsilon_1} e^{\psi/h} u_h \in C_0^\infty(\Omega_N),$$

where $\psi \in C^\infty(\Omega_N)$ is a weight function that is defined below. As usual, one then considers the conjugated operator $P_\psi(h) := e^{\psi/h} P(h) e^{-\psi/h} : C_0^\infty(\Omega_N) \to C_0^\infty(\Omega_N)$ with principal symbol

$$p_\psi(x, \xi) = p(x, \xi + i\partial_x \psi),$$

where $p(x, \xi) = |\xi|_g^2$. In Fermi coordinates $(x', x_{n+1}) : \Omega_N \to \mathbb{R}^{n+1}$,

$$p(x, \xi) = \xi_n^2 + r(x, \xi'), \quad r(x, \xi') = r_\partial(x', \xi') + 2x_{n+1}\kappa_\partial(x, \xi') \tag{4.4}$$

where r is a quadratic form in ξ'. In the Taylor expansion (4.4), r_∂ is the dual metric form on $T^*\partial\Omega$ induced from the interior and $\kappa_\partial(x', x_{n+1} = 0, \xi')$ is the second fundamental form of the boundary.

4.2. Carleman weight

Fix $\delta > 0$. We define the putative weight function to be

$$\psi_N(x_{n+1}) = \int_0^{x_{n+1}} e^{\frac{1}{2}\int_0^s f_\delta(t)dt} ds \quad x_{n+1} \in [0, r_{\max,N}], \tag{4.5}$$

where $f_\delta \in C^\infty([0, r_{\max,N}])$ and satisfies

$$\delta \le f_\delta(x_{n+1}) - \sup_{\{(x',\xi');r(x_{n+1},x',\xi')=1\}} \partial_{x_{n+1}} r(x_{n+1}, x', \xi') \le 2\delta, \tag{4.6}$$

so that

(4.7)
$$
\begin{aligned}
&(|\psi_N'(x_{n+1})|^2)' \\
&= f_\delta(x_{n+1})\,(\psi_N'(x_{n+1}))^2 \\
&\geq \Big(\sup_{\{(x',\xi');\, r(x_{n+1},x',\xi')=1\}} \partial_{x_{n+1}} r(x_{n+1},x',\xi')\Big)\,(\psi_N'(x_{n+1}))^2 + \delta\,(\psi_N'(x_{n+1}))^2, \\
&= \Big(\sup_{\{(x',\xi');\, r(x_{n+1},x',\xi')=|\psi_N'(x_{n+1})|^2\}} \partial_{x_{n+1}} r(x_{n+1},x',\xi')\Big) \;+\; \delta\,(\psi_N'(x_{n+1}))^2,
\end{aligned}
$$

and

$$
\partial_{x_{n+1}}\psi_N|_{x_{n+1}=0} = 1, \qquad \partial^2_{x_n+1}\psi_N|_{x_{n+1}=0} = \tfrac{1}{2} f_\delta(0).
$$

The last line in (4.7) follows since $r(x,\xi')$ is quadratic in the fiber ξ'-variables.

To show that the function ψ_N in (4.5) is a legitmate Carleman weight, we compute that in Fermi coordinates $(x', x_{n+1}) : \Omega_N(r_{\max,N}) \to \mathbb{R}^n \times \mathbb{R}$,

$$
\begin{aligned}
p_{\psi_N}(x,\xi) &= (\xi_{n+1} + i\partial_{x_{n+1}}\psi_N)^2 + r(x,\xi') \\
&= \xi_{n+1}^2 + r(x,\xi') - (\partial_{x_{n+1}}\psi_N)^2 + 2i\xi_{n+1}\partial_{x_{n+1}}\psi_N.
\end{aligned}
$$

Since $\partial_{x_{n+1}}\psi_N \geq c > 0$ it follows that

(4.8)
$$
\mathrm{Char}(p_{\psi_N})(x,\xi) = \{(x,\xi) \in T^*\Omega_N \,:\, r(x,\xi') = (\partial_{x_{n+1}}\psi_N)^2,\;\; \xi_{n+1} = 0\}
$$

Then, since

$$
\mathrm{Re}\, p_{\psi_N} = \xi_{n+1}^2 + r(x,\xi') - (\partial_{x_{n+1}}\psi_N)^2 \qquad \text{and} \qquad \mathrm{Im}\, p_{\psi_N} = 2\partial_{x_{n+1}}\psi_N\xi_{n+1},
$$

a direct computation gives

(4.9)
$$
\begin{aligned}
&\{\mathrm{Re}\, p_{\psi_N}, \mathrm{Im}\, p_{\psi_N}\}(x,\xi) \\
&= \{\xi_{n+1}^2 + r(x,\xi') - (\partial_{x_{n+1}}\psi_N)^2,\; 2\partial_{x_{n+1}}\psi_N\xi_{n+1}\} \qquad (x,\xi) \in \mathrm{Char}\, p_{\psi_N}. \\
&= 2\partial_{x_{n+1}}\psi_N \cdot \Big(\partial_{x_{n+1}}[(\partial_{x_{n+1}}\psi_N)^2] - \partial_{x_{n+1}} r(x,\xi')\Big)
\end{aligned}
$$

Then, since $\partial_{x_{n+1}}\psi_N \geq c \geq 0$, it follows from (4.7) and (4.9) that

(4.10)
$$
\{\mathrm{Re}\, p_{\psi_N}, \mathrm{Im}\, p_{\psi_N}\}(x,\xi) \geq C_1(\delta)|\partial_{x_{n+1}}\psi_N|^2 \geq C_2(\delta) > 0, \quad (x,\xi) \in \mathrm{Char}(p_{\psi_N}).
$$

Consequently ψ_N is a legitimate Carleman weight in $\Omega_N(r_{\max,N})$ and so, by the subelliptic Carleman estimates (see e.g. [Zwo12, Theorem 7.5])

$$\|P_{\psi_N}(h)v_h\|^2_{L^2(\Omega_N)} \geq Ch\,\|v_h\|^2_{H^1_h(\Omega_N)}. \tag{4.11}$$

4.3. Lower bounds: completion of the proof

Since $P(h)u_h = 0$, it follows that $P_{\psi_N}(h)v_h = e^{\psi_N/h}[P(h),\chi_{\varepsilon_1}]u_h$. Since $[P(h),\chi_{\varepsilon_1}]$ is an h-differential operator of order one supported in supp $\partial\chi_{\varepsilon_1}$, it follows from (4.11) and (4.3), that with Carleman weight $\psi_N(x_{n+1})$ in (4.5),

$$\begin{aligned}\|e^{\psi_N/h}[P(h),\chi_{\varepsilon_1}]u_h\|^2_{\operatorname{supp}\partial\chi^+_{\varepsilon_1}} + \|e^{\psi_N/h}[P(h),\chi_{\varepsilon_1}]u_h\|^2_{\operatorname{supp}\partial\chi^-_{\varepsilon_1}} \\ \geq Ch\,\|e^{\psi_N/h}\chi_{\varepsilon_1}u_h\|^2_{H^1_h(\Omega_N)}.\end{aligned} \tag{4.12}$$

Since $\chi_{\varepsilon_1} = 1$ on the set $\Gamma(t,\varepsilon_1) := \{2\varepsilon_1 < x_{n+1} < t - 2\varepsilon_1\}$, from (4.12),

$$\begin{aligned}\|e^{\psi_N/h}[P(h),\chi_{\varepsilon_1}]u_h\|^2_{\operatorname{supp}\partial\chi^+_{\varepsilon_1}} + \|e^{\psi_N/h}[P(h),\chi_{\varepsilon_1}]u_h\|^2_{\operatorname{supp}\partial\chi^-_{\varepsilon_1}} \\ \geq Ch\,\|e^{\psi_N/h}u_h\|^2_{H^1_h(\Gamma(t,\varepsilon_1))},\end{aligned} \tag{4.13}$$

and so,

$$\begin{aligned}h^2\Big(\|e^{\psi_N/h}u_h\|^2_{H^1_h(\operatorname{supp}\partial\chi^+_{\varepsilon_1})} + \|e^{\psi_N/h}u_h\|^2_{H^1_h(\operatorname{supp}\partial\chi^-_{\varepsilon_1})}\Big) \\ \geq Ch\|e^{\psi_N/h}u_h\|^2_{H^1_h(\Gamma(t,\varepsilon_1))}.\end{aligned} \tag{4.14}$$

To bound the first term on the LHS of (4.14) from above, we recall the upper bound from [GT19]: for $x \in \Omega_N(\varepsilon_0)$

$$|\partial^\alpha_x u_h(x)| \leq C_{\Omega,\alpha}\exp\left([-x_{n+1} + C_{sup}(\Omega)x^2_{n+1})]/h\right)\|u\|_{L^2(N)} + Ce^{-1/Ch}.$$

Since supp $\partial\chi^+_{\varepsilon_1} \subset \{\frac{\varepsilon_1}{2} < x_{n+1} < \varepsilon_1\}$ and $\varepsilon_1 < \varepsilon_0$, it follows that

$$\begin{aligned}&h^2\|e^{\psi_N/h}u_h\|^2_{H^1_h(\operatorname{supp}\partial\chi^+_{\varepsilon_1})} \\ &\leq h^2C^2_\Omega\|e^{\psi_N/h}e^{[-x_{n+1}+C_{sup}(\Omega)x^2_{n+1})]/h}\|^2_{L^2(\{0<x_{n+1}<\varepsilon_1\})}\|u_h\|^2_{L^2(N)} + Ce^{-1/Ch}.\end{aligned}$$

Using that $\psi_N(x_{n+1}) = x_{n+1} + \frac{1}{2}\psi'_N(0)x^2_{n+1} + O(x^3_{n+1})$,

$$\|e^{\psi_N/h}e^{[-x_{n+1}+C_{sup}(\Omega)x^2_{n+1})]/h}\|^2_{L^2(\{0<x_{n+1}<\varepsilon_1\})} \leq Ce^{(f_\delta(0)+C_{sup})\varepsilon_1{}^2+C\varepsilon_1{}^3/h},$$

and consequently, for the first term on the LHS of (4.11),

(4.15)
$$h^2\|e^{\psi_N/h}u_h\|^2_{H^1_h(\operatorname{supp}\partial\chi^+_{\varepsilon_1})} \le C_\Omega^2 h^2 e^{(f_\delta(0)+C_{sup})\varepsilon_1{}^2+C\varepsilon_1{}^3/h}\|u_h\|^2_{L^2(N)} + Ce^{-1/Ch}.$$

To estimate the RHS of (4.14) from below, we use the local L^2-restriction lower bounds in Theorem 1 which gives that for all $\tau \in [0,\varepsilon_0]$ and any $\delta_1 > 0$, then, with $h \in (0, h_0(\delta_1,\varepsilon_0)]$,

$$\|u_h\|_{L^2(\{x_{n+1}=\tau\})} \ge C(\delta_1)e^{-[(\tau+\frac{1}{4}f_0(0)\tau^2+C\tau^3)(1+\delta_1)]/h}\|u_h\|_{L^2(N)} - Ce^{-1/Ch}.$$

Now, let $\varepsilon_1 < \varepsilon_2 \ll \min(\delta,\varepsilon_0)$. since $\{\frac{\varepsilon_2}{2} < x_{n+1} < \varepsilon_2\} \subset \Gamma(t,\varepsilon_1)$, for the RHS in (4.14),

$$\begin{aligned}
&\|e^{\psi_N/h}u_h\|^2_{H^1_h(\Gamma(t,\varepsilon_1))}\\
&\ge \|e^{\psi_N/h}u_h\|^2_{L^2(\{\frac{\varepsilon_2}{2}<x_{n+1}<\varepsilon_2\}}\\
&\ge C(\delta_1)\int_{\tau=\frac{\varepsilon_2}{2}}^{\tau=\varepsilon_2} e^{2\tau+\frac{1}{2}f_\delta(0)\tau^2-C\tau^3/h}e^{-(2\tau+\frac{1}{2}f_0(0)\tau^2+C\tau^3)(1+\delta_1))/h}\,dt\|u_h\|^2_{L^2(N)}\\
&\qquad - Ce^{-1/Ch}\\
&= C(\delta_1)\int_{\tau=\frac{\varepsilon_2}{2}}^{\tau=\varepsilon_2} e^{\delta\tau^2-C\tau^3-\delta_1(2\tau+\frac{1}{2}f_0(0)\tau^2)/h}\,d\tau\|u_h\|^2_{L^2(N)} - Ce^{-1/Ch}\\
&\ge C(\delta_1)\int_{\tau=\frac{\varepsilon_2}{2}}^{\tau=\varepsilon_2} e^{\frac{\delta}{2}\tau^2/h}\,d\tau\|u_h\|^2_{L^2(N)} - Ce^{-1/Ch},
\end{aligned}$$

where the last line follows by choosing $\varepsilon_2 \ll \delta$ and $\delta_1 \ll \varepsilon_2\delta$ and noting that from (4.6), $f_\delta(0) - f_0(0) \ge \delta$.

Consequently, the end result is the following lower bound for the RHS in (4.14):

$$\|e^{\psi_N/h}u_h\|^2_{H^1_h(\Gamma(t,\varepsilon_1))} \ge C(\delta_1)\varepsilon_2 e^{\frac{\delta\varepsilon_2{}^2}{2h}}\|u_h\|^2_{L^2(N)} - e^{-1/Ch}. \tag{4.16}$$

By possibly shrinking $\varepsilon_1 > 0$ further so that $\varepsilon_1 \ll \sqrt{\delta}\varepsilon_2$, it follows from the upper bound in (4.15) and the lower bound in (4.16), that the first term on the LHS of (4.14) can be absorbed in the RHS. Consequently,

$$\|e^{\psi_N/h}u_h\|^2_{H^1_h(\operatorname{supp}\partial\chi^-_{\varepsilon_1})} \ge C(\varepsilon_1))\|u_h\|^2_{L^2(N)} - Ce^{-1/Ch}. \tag{4.17}$$

Now, choose $\varepsilon_1 > 0$ small enough so that

$$|\psi_N(x_{n+1}) - \psi_N(t)| < \varepsilon, \qquad |x_{n+1} - t| < 2\varepsilon_1.$$

Then, we have

$$\|e^{\psi_N/h}u_h\|^2_{H^1_h(\operatorname{supp}\partial\chi^-_{\varepsilon_1})} \leq e^{\psi_N(t)/h+\varepsilon/h}\|u_h\|^2_{H^1_h((t-2\varepsilon_1<x_{n+1}<t+2\varepsilon_1)}. \tag{4.18}$$

Now, since $\varepsilon_1 < \delta_{U_{H_t}}$,

$$\{(x', x_{n+1}) \,:\, t-2\varepsilon_1 < x_{n+1} < t+2\varepsilon_1\} \Subset U_{H_t},$$

by interior elliptic regularity for the Laplacian and the fact that $\Delta_g u_h|_{U_{H_t}} = 0$, it follow that

$$\begin{aligned} \|u_h\|^2_{H^1_h((t-2\varepsilon_1<x_{n+1}<t+2\varepsilon_1))} &\leq C_{\varepsilon_1}(\|-\Delta_g u_h\|^2_{U_{H_t}} + \|u_h\|^2_{L^2(U_{H_t})}) \\ &\leq C_{\varepsilon_1}\|u_h\|^2_{L^2(U_{H_t})}. \end{aligned} \tag{4.19}$$

Combining (4.17), (4.18), and (4.19) implies,

$$C(\varepsilon_1)\|u_h\|^2_{L^2(N)} - Ce^{-1/Ch} \leq C_{\varepsilon_1}e^{\psi_N(t)/h+\varepsilon/h}\|u_h\|^2_{L^2(U_{H_t})},$$

which completes the proof of Theorem 2. □

5. Eigenfunction goodness estimates for Cauchy data: Proof of Theorem 3

Proof. Let $0 \leq t < r_{\max,\partial\Omega}$ and $H_t := \varphi_{\partial\Omega}(\partial\Omega, t)$ so that in Fermi coordinates (x', x_{n+1}) around $\partial\Omega$,

$$H_t = \{(x', t)\}.$$

Let $U_{H_t} \subset \mathring{\Omega}$ be the domain interior to Ω bounded by H_t and for any fixed $\varepsilon > 0$, let $U_{H_t,\varepsilon} \Subset U_{H_t}$ be a compact manifold with boundary, $H_{t,\varepsilon}$, strictly contained in U_{H_t} with $\frac{\varepsilon}{2} \leq d(H_{t,\varepsilon}, H_t) \leq \max_{(x,y)\in H_{t,\varepsilon}\times H_t} d(x,y) \leq \varepsilon$. Then, by e.g. [Aub82, Chapter 4], there exists a Green's function $G \in \mathcal{D}'(U_{H_t} \times U_{H_t})$ satisfying

$$-\Delta_x G(x,y) = \delta_x(y), \quad (x,y) \in U_{H_t} \times U_{H_t},$$

with $G(\cdot,\cdot) \in C^\infty((U_{H_t} \times U_{H_t}) \setminus \{x = y\})$.

Then, for $x \in U_{H_t,\varepsilon}$ an application of Green's formula gives

$$hu_h(x) = \int_{H_t} G(x,s)h\partial_\nu u_h(s)d\sigma(s) - h\int_{H_t} \partial_{\nu(s)}G(x,s)u_h(s)d\sigma(s). \tag{5.1}$$

Since $d(x, H_t) > \varepsilon$ when $x \in U_{H_t,\varepsilon}$ and so, $G \in C^\infty(U_{H_t,\varepsilon}, H_t)$, differentiation of (5.1) in the x-variables gives a similar formula for the derivatives $\partial_{x_k} u_h$. An application of Cauchy-Schwarz then implies that

$$h^2 \|u_h\|^2_{H^1_h(U_{H_t,\varepsilon})} \leq C(\varepsilon)\Big(\|u_h\|^2_{L^2(H_t)} + \|h\partial_\nu u_h\|^2_{L^2(H_t)}\Big). \tag{5.2}$$

Finally, by Theorem 2, applied with $H_{t,\varepsilon} = \{d(x, \partial\Omega) = t + \varepsilon\}$, for any $\varepsilon > 0$,

$$\|u\|_{H^1_h(U_{H_t,\varepsilon})} \geq C_\varepsilon e^{(-\psi_{\partial\Omega}(t)-\varepsilon)/h},$$

which completes the proof. □

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