Pure and Applied Mathematics Quarterly
Volume 19, Number 4, 1943–1973, 2023

Quantum Witten localization*

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Abstract: We prove a quantum version of the localization formula of Witten [31], see also [28, 22, 35], that relates invariants of a GIT quotient with the equivariant invariants of the action.

Keywords: Quantum cohomology, GIT quotients.

1. Introduction

The main result of this paper is a formula relating the equivariant Gromov-Witten graph invariants of a smooth projective variety with group action and the graph invariants of the geometric invariant theory quotient. To state the main result we introduce the following notation. Let G be a connected complex reductive group acting on a smooth polarized projective variety X. Let

$$X /\!\!/ G = X^{\mathrm{ss}}/G$$

denote the *GIT quotient* of X by G, which here means the stack-theoretic quotient of the semistable locus X^{ss} by the group action. We assume that G acts with only finite stabilizers on the semistable locus X^{ss}. In this case the GIT quotient $X /\!\!/ G$ is a smooth proper Deligne-Mumford stack with projective coarse moduli space by Mumford et al. [17]. Let $H(X/\!\!/G)$ resp. $H_G(X)$ denote the rational resp. equivariant rational cohomology of $X/\!\!/G$ resp. X. Kirwan's thesis [14] studies the natural map

$$\kappa_{X,G} : H_G(X) \to H(X/\!\!/G)$$

given by restriction to the semistable locus and descent. Equivariant integration resp. integration over X resp. $X/\!\!/G$ define *trace maps*

$$\tau_{X \times \mathfrak{g}_{\mathbb{R}}} : H_G(X) \to \mathbb{Q}, \quad \tau_{X/\!\!/G} : H(X/\!\!/G) \to \mathbb{Q}$$

arXiv: 0811.3358
Received July 11, 2021.
2010 Mathematics Subject Classification: Primary 14N35, 53D45; secondary 14L24.
*Partially supported by grants DMS1104670 and DMS1207194. A previous version was titled *Area-dependence in gauged Gromov-Witten theory*.

where the first can be defined using equivariant cohomology with distributional coefficients as in [35]. Witten [31] introduced a formula, which he termed *non-abelian localization*, which expresses the difference between the composition $\tau_{X/\!\!/G} \circ \kappa_{X,G}$ and $\tau_{X \times \mathfrak{g}_\mathbb{R}}$ as a sum over critical points of the norm-square of the moment map. Mathematical versions can be found in Paradan [21, 22], Teleman in the case of sheaf cohomology [28], and Woodward [35]. Witten's localization principle quantifies the failure of the following diagram to commute:

$$\begin{array}{ccc} H_G(X) & \xrightarrow{\kappa_{X,G}} & H(X/\!\!/G) \\ & {\scriptstyle \tau_{X\times\mathfrak{g}_\mathbb{R}}} \searrow \quad \swarrow {\scriptstyle \tau_{X/\!\!/G}} & \\ & \mathbb{Q} & \end{array} \quad . \tag{1}$$

A different formula computing the composition is given in Jeffrey-Kirwan [13]. A virtual Witten localization formula has recently appeared in Halpern-Leistner [12, (5)].

Naturally one wishes for a quantum version of Witten's localization formula which computes the Gromov-Witten invariants of the moduli spaces of stable maps to a GIT quotient. For conceptual reasons we explain below one expects such an invariant only for *parametrized* stable maps; in good cases these compactify the space of maps $\mathrm{Hom}(\mathbb{P}^1, X)$ from the projective line $\mathbb{P}^1$. The action of G on X induces an action on $\mathrm{Hom}(\mathbb{P}^1, X)$ and our main result is a formula for the difference in integrals over "compactifications" of

$$\mathrm{Hom}(\mathbb{P}^1, X/\!\!/G), \quad \mathrm{Hom}(\mathbb{P}^1, X)/\!\!/G$$

as a sum of integrals over maps to the Kirwan-Ness strata of X; here the second quotient is defined as the quotient of a suitable semistable locus. As a sample potential application, since the integral over $\mathrm{Hom}(\mathbb{P}^1, X)/\!\!/G$ may be related to the abelian quotient $\mathrm{Hom}(\mathbb{P}^1, X)/\!\!/T$ by Martin's formula [16], this provides an inductive approach to abelianization questions, where the induction is over the rank of the group G.

To state the result in more detail let $\omega \in H^2_G(X)$ be the first Chern class of the linearization (that is, the symplectic class) and let

$$\Lambda_X^G = \left\{ \sum_{i=0}^{\infty} c_i q^{d_i}, c_i \in \mathbb{Q}, d_i \in H_2^G(X, \mathbb{Q}), \ \lim_{i\to\infty} \langle d_i, \omega \rangle = \infty \right\}$$

denote the equivariant Novikov field for X. Let

$$QH_G(X) = H_G(X) \otimes \Lambda_X^G$$

denote the equivariant quantum cohomology of X. Virtual integration over the moduli stack of n-marked genus 0 stable maps $\overline{\mathcal{M}}_{0,n}(X)$ for $n \geq 3$ defines a family of formal quantum products

$$\star_\alpha : T_\alpha QH_G(X)^2 \to T_\alpha QH_G(X), \quad \alpha \in QH_G(X).$$

Formal in this setting means that only the Taylor coefficients of the maps are convergent. Define a quantum version of Witten's trace as follows. Let $\mathbb{P}^1 = (\mathbb{C}^2 - \{0\})/\mathbb{C}^\times$ denote the projective line. For $d \in H_2(X, \mathbb{Z})$ let

$$\overline{\mathcal{M}}_n(\mathbb{P}^1, X, d) := \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1 \times X, (1, d))$$

denote the moduli stack of parametrized stable maps from $\mathbb{P}^1$ to X of class $d \in H_2^G(X, \mathbb{Z})$. The action of G on X induces a natural action on $\overline{\mathcal{M}}_n(\mathbb{P}^1, X, d)$. A natural stability condition for the action is given by requiring that the stable map has generically semistable value for all one-parameter subgroups [5]. Denote by $\overline{\mathcal{M}}_n(\mathbb{P}^1, X, d)/\!\!/G$ the stack-theoretic quotient of the semistable locus by the group action. By, for example, [7, Lemma 2.6], $\overline{\mathcal{M}}_n(\mathbb{P}^1, X, d)/\!\!/G$ is a proper Deligne-Mumford stack with a perfect relative obstruction theory. Via equivariant formality we may consider $H_2(X, \mathbb{Z})/\operatorname{torsion}$ as a subgroup of $H_2^G(X, \mathbb{Q})$. Denote by $\tau_{X,G}$ the formal trace map given by virtual integration over the moduli stacks $\overline{\mathcal{M}}_n(\mathbb{P}^1, X, d)/\!\!/G$:

$$\begin{aligned} \tau_{X,G} : QH_G(X) &\to \Lambda_X^G \\ \alpha \mapsto \sum_{n \geq 0, d \in H_2(X,\mathbb{Z})/\operatorname{torsion}} &\frac{q^d}{n!} \int_{[\overline{\mathcal{M}}_n(\mathbb{P}^1, X, d)/\!\!/G]} \operatorname{ev}^*(\alpha \otimes \ldots \otimes \alpha) \end{aligned}$$

for $\alpha \in H_G(X)$ and by $\tau_{X,G}^n$ its n-th Taylor coefficient. More generally, define a trace "with insertions" for a sequence of classes $\beta := \{\beta_n\}_{n \geq 0}$ such that

$$\beta_n \in \overline{\mathcal{M}}_n := \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1), \quad n \geq 0$$

by

$$\tau_{X,G}(\alpha, \beta) = \sum_{n \geq 0, d \in H_2(X,\mathbb{Z})/\operatorname{torsion}} \frac{q^d}{n!} \int_{[\overline{\mathcal{M}}_n(\mathbb{P}^1, X, d)/\!\!/G]} \operatorname{ev}^*(\alpha \otimes \ldots \otimes \alpha) \cup f_n^* \beta_n$$

where $f_n : \overline{\mathcal{M}}_n(\mathbb{P}^1, X, d)/\!\!/G \to \overline{\mathcal{M}}_n$ is the forgetful map. The map $\tau_{X,G}$ is a quantum version of Witten's trace in the sense that if one sets $q = 0$ and takes β_n to be point classes, which fixes the positions of the markings then one obtains the classical Witten trace, that is, the integral of $\exp(\alpha)$ over $X/\!\!/G$.

A quantum version of Kirwan's map counting maps to the quotient stack with semistability enforced at a marked point was introduced in [32, 33, 34]. The quantum Kirwan map is a non-linear map, still denoted $\kappa_{X,G}$,

$$\kappa_{X,G} : QH_G(X) \to QH(X/\!\!/G) \tag{2}$$

with the property that any linearization

$$D_\alpha \kappa_{X,G} : T_\alpha QH_G(X) \to T_{\kappa_{X,G}(\alpha)} QH(X/\!\!/G)$$

is a homomorphism with respect to the quantum products. In particular, if $\kappa_{X,G}(0) = 0$ (which happens only in very special cases) then $D_0\kappa_{X,G}$ is a homomorphism from the small equivariant quantum cohomology $T_0 QH_G(X)$ of X to the quantum cohomology $T_0 QH(X/\!\!/G)$ of $X/\!\!/G$.

A quantum version of the integration over the geometric invariant theory quotient is defined by a count of stable maps to the graph space. Recall that $\overline{\mathcal{M}}_n(\mathbb{P}^1, X/\!\!/G, d)$ denotes stable maps to $\mathbb{P}^1 \times (X/\!\!/G)$ of class $(1, d)$. Using the Behrend-Fantechi virtual fundamental classes [2] define

$$\tau_{X/\!\!/G} : QH(X/\!\!/G) \to \Lambda_X^G$$
$$\alpha \mapsto \sum_{n \geq 0, d \in H_2(X/\!\!/G, \mathbb{Q})} \frac{q^d}{n!} \int_{[\overline{\mathcal{M}}_n(\mathbb{P}^1, X/\!\!/G, d)]} \mathrm{ev}^*(\alpha \otimes \ldots \otimes \alpha) \cup f^*\beta_n$$

for $\alpha \in H(X/\!\!/G)$. The quantum Witten localization formula gives a precise description of the difference between the traces $\tau_{X,G}$ and $\tau_{X/\!\!/G} \circ \kappa_{X,G}$. That is, it measures the failure of the "quantum integration" to commute with reduction, i.e. the failure of commutativity of the diagram

$$\begin{array}{ccc} QH_G(X) & \xrightarrow{\kappa_{X,G}} & QH(X/\!\!/G) \\ & {\scriptstyle \tau_{X,G}} \searrow \quad \swarrow {\scriptstyle \tau_{X/G}} & \\ & \Lambda_X^G & \end{array} . \tag{3}$$

As in the classical Witten localization formula [31], the failure to commute is given by a sum of fixed point contributions. Each term is a *gauged Gromov-Witten invariant* $\tau_{X,G,\zeta,\rho}$ associated to the action of centralizers on components of the fixed point variety of some one-parameter subgroup $\exp(\mathbb{C}\zeta) \subset$

$G, \zeta \in \mathfrak{g}$, stable with respect to the linearization $\mathfrak{L}_\rho$ for some $\rho \in (0,\infty)$. We denote by $\overline{\mathcal{M}}_n^G(C, X, \mathfrak{L}_\rho, \zeta)$ the moduli of such fixed gauged maps.

In order to state the result, we must indicate how the insertions of cohomology classes from the moduli spaces of curves are distributed in the composition. For a fixed curve C let $\overline{\mathcal{M}}_{n,1}(C)$ be the moduli stack of scaled n-marked maps from [32]; a generic element is an n-marked map $\pi : \hat{C} \to C$ with a relative differential $\lambda \in H^0(\hat{C}, \omega_\pi)$. The variety $\overline{\mathcal{M}}_{n,1}(C)$ contains a prime divisor $\overline{\mathcal{M}}_n(C)$ corresponding to maps with zero differential $\lambda = 0$, and for any partition $I_1 \cup \ldots \cup I_r = \{1, \ldots, n\}$ with $|I_j| > 1$, a prime divisor $D_{I_1,\ldots,I_r}$ isomorphic to

$$D_{I_1,\ldots,I_r} \cong \overline{\mathcal{M}}_r(C) \times \prod_{j=1}^r \overline{\mathcal{M}}_{|I_j|}(\mathbb{C})$$

whose generic element is a curve $\hat{C}$ with infinite differential $\lambda = \infty$ on the one unmarked component $C_0 \cong C$ and finite differentials on the remaining components $C_1, \ldots, C_r$.

The main result is the following:

Theorem 1.1. (Quantum Witten localization) *Let C be a smooth connected projective curve of genus 0, X a smooth projective G-variety, and $\mathfrak{L} \to X$ a linearization. Suppose that for every $\zeta \in \mathfrak{g}$ and $\rho \in (0,\infty)$, stable=semistable for $\overline{\mathcal{M}}_n^G(C, X, \mathfrak{L}_\rho, \zeta)$, and stable=semistable for the G-action on X. Then the following equality holds for formal maps from $QH_G(X)$ to Λ_X^G:*

$$\tau_{X,G} - \tau_{X/\!\!/G} \circ \kappa_{X,G} = \sum_{[\zeta]\neq 0, \rho \in (0,\infty)} \tau_{X,G,\zeta,\rho} \tag{4}$$

in the following sense: For any class $\beta \in \overline{\mathcal{M}}_{n,1}(C)$ let

$$\sum_{k=1}^{l} \prod_{I_1 \cup \ldots \cup I_r = \{1,\ldots,n\}} \beta_\infty^k \otimes \beta_1^k \otimes \ldots \otimes \beta_r^k, \quad \textit{resp. } \beta_0$$

be its restrictions to

$$H\left(\overline{\mathcal{M}}_r(C) \times \prod_{j=1}^r \overline{\mathcal{M}}_{|I_j|}(\mathbb{C})\right), \quad \textit{resp. } H(\overline{\mathcal{M}}_n(C))$$

respectively. Then

$$\tau_{X,G}(\alpha,\beta_0) - \sum_{k=1}^{l} \tau^r_{X/\!\!/G}(\alpha,\beta^k_\infty) \circ \kappa^{|I_j|}_{X,G}(\alpha,\beta^k_j) = \sum_{[\zeta]\neq 0, \rho\in(0,\infty)} \tau_{X,G,\zeta,\rho}(\alpha,\beta_0)$$

in the sense that the (well-defined) Taylor coefficients on both sides agree.

The proof is concluded at the end of Section 5. The Theorem arose out of an attempt to compare Givental's results [4] to those of Witten [31].

Example 1.2. Suppose that G is semi-simple and that d is not in the image of $H_2(X) \to H_2^G(X)$. The degree d contribution to $\tau_{X,G}$ vanishes, since the degree d part of $\mathcal{M}_n^G(\mathbb{P}^1, X)$ is empty. We obtain a formula for the degree d contribution of Gromov-Witten invariants in the quotient as a sum of contributions over reducible gauged maps:

$$\mathrm{Coeff}(q^d, \tau_{X/\!\!/G} \circ \kappa_{X,G}) = - \sum_{[\zeta]\neq 0, \rho\in(0,\infty)} \mathrm{Coeff}(q^d, \tau_{X,G,\zeta,\rho}).$$

The particular case of the action of scalars on affine space (adapted to the quasiprojective case) is described in Example 7.6.

2. Mundet stability

Mundet stability combines the slope conditions from Ramanathan stability for bundles and Hilbert-Mumford stability for points in the target. First we recall Mumford-Seshadri stability. Let C be a smooth projective curve and $E \to C$ a vector bundle of vanishing degree $\deg(E) = (c_1(E), [C])$. The bundle

$$E \textit{ is semistable} \text{ resp. } \textit{stable} \iff (\deg(F) \le 0 \quad \text{resp.} < 0, \quad \forall F \subset E)$$

for all holomorphic sub-bundles $F \subset E$ [20]. Ramanathan's stability [25] generalizes the Mumford-Seshadri condition to principal bundles as a condition on parabolic reductions. Let G be a connected reductive group with Lie algebra $\mathfrak{g}$. Let $T \subset G$ be a maximal torus, with Lie algebra $\mathfrak{t}$. Denote the integral resp. rational weights resp. coweights

$$\mathfrak{t}_\mathbb{Z} = \exp^{-1}(e), \quad \mathfrak{t}^\vee_\mathbb{Z} \subset \mathfrak{t}^\vee := \mathrm{Hom}(\mathfrak{t},\mathbb{C}), \quad \mathfrak{t}_\mathbb{Q} = \mathfrak{t}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}, \quad \mathfrak{t}^\vee_\mathbb{Q} = \mathfrak{t}^\vee_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$$

Let $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_- \subset \mathfrak{t}^\vee_\mathbb{Z}$ denote a set of positive and negative roots so that

$$\mathfrak{g} \cong \mathfrak{t} \oplus \bigoplus_{\alpha\in\mathcal{R}_-} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha\in\mathcal{R}_+} \mathfrak{g}_\alpha.$$

A *parabolic subgroup* of G is a subgroup Q such that G/Q is complete. Up to conjugacy this means that the Lie algebra $\mathfrak{q}$ of Q is given by

$$\mathfrak{q} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}_-} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \mathcal{R}_Q} \mathfrak{g}_\alpha$$

for some subset of the roots $\mathcal{R}_Q \subset \mathcal{R}_+$ such that $\mathfrak{q}$ is a Lie subalgebra of $\mathfrak{g}$. A *Levi subgroup* of Q is a maximal reductive subgroup $L(Q)$; again up to conjugacy the Lie algebra $\mathfrak{l}(\mathfrak{q})$ of $L(Q)$ resp. $\mathfrak{u}(\mathfrak{q})$ of a maximal unipotent $U(Q)$ is

$$\mathfrak{l}(\mathfrak{q}) = \mathfrak{t} \oplus \bigoplus_{\alpha \in -\mathcal{R}_Q} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \mathcal{R}_Q} \mathfrak{g}_\alpha, \quad \mathfrak{u}(\mathfrak{q}) = \bigoplus_{\alpha \in -\mathcal{R}_Q} \mathfrak{g}_\alpha.$$

The parabolic subgroup and its Lie algebra admit decompositions into reductive and unipotent parts

$$\mathfrak{q} = \mathfrak{l}(\mathfrak{q}) \oplus \mathfrak{u}(\mathfrak{q}), \quad Q = L(Q)U(Q).$$

Taking the quotient by the maximal unipotent gives a projection

$$\pi_Q : Q \to Q/U(Q) \cong L(Q).$$

This projection has the following alternative description. A *dominant coweight* for Q is a coweight $\lambda \in \mathfrak{t}$ such that

$$(\alpha(\lambda) \geq 0, \quad \forall \alpha \in \mathcal{R}_+) \quad \text{and} \quad (\alpha(\lambda) = 0, \quad \forall \alpha \in \mathcal{R}(Q)).$$

Any rational coweight for Q determines a one-parameter subgroup

$$\phi_\lambda : \mathbb{C}^\times \to Q, \quad z \mapsto \phi_\lambda(z).$$

If $\lambda \in \mathfrak{q}$ is a dominant rational coweight then

$$\pi_Q(q) = \lim_{z \to 0} \operatorname{Ad}(\phi_{-\lambda}(z))q.$$

Choose an equivariant identification $\mathfrak{g} \to \mathfrak{g}^\vee$ that identifies the subspaces of rational weights and coweights $\mathfrak{t}_\mathbb{Q} \to \mathfrak{t}_\mathbb{Q}^\vee$. The identification and $\lambda \in \mathfrak{t}$ determine a rational weight $\lambda^\vee \in \mathfrak{t}^\vee$. After finite cover λ defines a one-dimensional representation

$$\chi_{\lambda^\vee} : Q \to \mathbb{C}^\times, q \mapsto \chi_{\lambda^\vee}(q)$$

which factors through $L(Q)$.

The analog for principal bundles of the stability condition for sub-bundles is a condition for parabolic reductions together with dominant coweights. Let $P \to C$ be a principal G bundle on a curve C over a scheme S; bundles are by assumption locally trivial in the étale topology. A *parabolic reduction* is a section $\sigma : C \to P/Q$. Any parabolic reduction induces a reduction of structure group given by a sub-bundle $\sigma^*(P) \subset P$ with structure group Q, given by pull-back of the Q-bundle $P \to P/Q$. We denote by

$$\mathrm{Gr}(P) := (\pi_Q)_*(\sigma^* P) \to C$$

the corresponding $L(Q)$-bundle, called the *associated graded* bundle of P associated to the parabolic reduction σ. In case $G = GL(\mathbb{C}^n)$, a parabolic reduction is equivalent to a partial flag of sub-bundles

$$E^{i_1} \subset E^{i_2} \subset \ldots \subset E^{i_l} = E$$

in the associated vector bundle $E = P(\mathbb{C}^n)$; the corresponding parabolic reduction $\sigma^* P$ is the bundle of frames whose first i_k elements belong to E^{i_k} for $k = 1, \ldots, l$. The associated graded principal bundle is the principal bundle of frames of the associated graded vector bundle

$$\mathrm{Gr}(E) = \bigoplus_j (E^{i_{j+1}}/E^{i_j}), \quad \mathrm{Gr}(P) = \mathrm{Fr}(\mathrm{Gr}(E)).$$

The construction of the associated graded bundle also has an interpretation via degeneration. The family of elements $\phi_\lambda(z)$ defines a family of automorphisms $\mathrm{Ad}(\phi_\lambda(z)) : G \to G$. Consider the family of bundles $P^\lambda \to C \times \mathbb{C}^\times$ defined by conjugating the transition maps of $\sigma^* P$ by $\phi_\lambda(z)^{-1}$. Then P^λ extends over the central fiber $C \times \{0\}$ as the bundle $\mathrm{Gr}(P)$. The *Ramanathan weight* of a principal bundle with respect to a parabolic reduction and dominant weight is the degree of the line bundle corresponding to the given dominant coweight:

$$\mu(\sigma, \lambda) = \deg(\mathrm{Gr}(P) \times_{L(Q)} \mathbb{C}_\lambda) = ([C], c_1(\mathrm{Gr}(P) \times_{L(Q)} \mathbb{C}_\lambda)).$$

Define

$$P \text{ is } \textit{semistable} \text{ resp. } \textit{stable} \iff \mu(\sigma, \lambda) \leq 0 \quad \text{resp.} < 0, \forall (\sigma, \lambda).$$

Remark 2.1. This version of stability omits a constant on the right hand side corresponding to the first Chern class of the bundle, and stability forces the existence of a flat, rather than merely central curvature, connection. Indeed the first Chern class of the bundle must vanish, by taking trivial parabolic reductions. For rational curves, Birkhoff-Grothendieck [10] implies that a principal bundle with vanishing degree is semistable if and only if it is trivial, since it must admit a reduction to the maximal torus with all sub-bundles of zero weight.

As for vector bundles, it suffices to check the stability condition for reduction to *maximal* parabolic subgroups Q. Ramanathan [25] shows the existence of a projective coarse moduli space for semistable principal bundles with reductive structure group and fixed numerical invariants.

Mundet semistability [18, 26] generalizes Ramanathan stability to the case of maps to a quotient stack. Let G be a connected reductive group acting on a smooth projective variety X. By a *gauged map* with domain a curve C we mean a map from C to the quotient stack X/G, given by a pair (P, u) of a G-bundle and section of the associated X-fiber bundle:

$$P \to C, \quad u : C \to P \times_G X.$$

Given a pair $(P \to C, u : C \to P(X))$, the section u defines a section u^λ of $P^\lambda(X)$ as follows: In any local trivialization $P(X)|_U \cong U \times X$ the section u is given by a map $u|_U : U \to X$, and the sections $\phi_\lambda(z)u$ patch together to a section of $P^\lambda(X)$. By Gromov compactness, u^λ extends over the central fiber $C \times \{0\}$ as a stable map denoted $\mathrm{Gr}(u) : \hat{C} \to \mathrm{Gr}(P)(X)$. Associated to this limit there is an associated *Hilbert-Mumford weight* defined as follows. The principal component C_0 of $\hat{C}$ is the irreducible component such that the restriction u_0 of u to C_0 maps isomorphically to C. The principal component $\mathrm{Gr}(u)_0$ of the associated graded section $\mathrm{Gr}(u)$ takes values in the fixed point set $(\mathrm{Gr}(P)(X))^\lambda = \mathrm{Gr}(P)(X^\lambda)$ of the infinitesimal automorphism of $\mathrm{Gr}(P)(X)$ induced by λ. The *Hilbert-Mumford weight*

$$\mu_H(\sigma, \lambda) \in \mathbb{Z} \tag{5}$$

determined by the linearization $\mathfrak{L}$, is the weight of the $\mathbb{C}^\times$-action generated by $-\lambda$ on the fiber of the bundle $(\mathrm{Gr}(P))(\mathfrak{L}) \to (\mathrm{Gr}(P))(X)$ over a generic value of $\mathrm{Gr}(u)_0$:

$$\phi_\lambda(z)\tilde{x} = z^{\mu_H(\sigma,\lambda)}\tilde{x}, \quad z \in \mathbb{C}^\times.$$

The *Mundet weight* is the sum of the Hilbert-Mumford and Ramanathan weights:

$$\mu_M(\sigma, \lambda) := \mu_H(\sigma, \lambda) + \mu_R(\sigma, \lambda).$$

Then

$$(P, u) \quad \textit{semistable} \text{ resp. } \textit{stable} \iff \mu(\sigma, \lambda) \leq 0 \quad \text{resp.} < 0, \forall(\sigma, \lambda).$$

Mundet's original definition allowed possibly irrational λ, but this is unnecessary in the case that the symplectic class is rational by [33, Remark 5.8]. Mundet semistability is realized as a GIT stability condition in Schmitt [26, 27].

The moduli stack of Mundet-semistable morphisms admits a natural Kontsevich-style compactification that allows formation of bubbles in the fibers of the associated bundle: An *n-marked gauged map* from C to X over a scheme S is a datum $(\hat{C}, P, u, \underline{z})$ where $\hat{C} \to S$ is a proper flat morphism with reduced nodal curves as fibers, $P \to C \times S$ is a principal G-bundle; and

$$u : \hat{C} \to P(X) := (P \times X)/G$$

is a family of stable maps with base class $[C]$, that is, for a fixed fiber over $s \in S$, the composition of u with the projection $P(X) \to C$ has class $[C]$. A *morphism* between gauged maps $(S, \hat{C}, P, u)$ and $(S', \hat{C}', P', u')$ consists of a morphism $\beta : S \to S'$, a morphism $\phi : P \to (\beta \times 1)^* P'$, and a morphism $\psi : \hat{C} \to \hat{C}'$ such that the first diagram below is Cartesian and the second and third commute:

$$\begin{array}{ccc} \hat{C} & \longrightarrow & S \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\beta} \\ \hat{C}' & \longrightarrow & S' \end{array} \qquad \begin{array}{ccc} P & \longrightarrow & S \times C \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\beta\times\mathrm{id}} \\ (\beta \times 1)^* P' & \longrightarrow & S' \times C \end{array} \qquad \begin{array}{ccc} \hat{C} & \xrightarrow{u} & P(X) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle[\phi\times\mathrm{id}_X]} \\ \hat{C}' & \xrightarrow[u']{} & P'(X). \end{array}$$

An *n-marked* nodal gauged map is equipped with an n-tuple $(z_1, \ldots, z_n) \in \hat{C}^n$ of distinct smooth S-points on $\hat{C}$. An n-marked nodal gauged map $(\hat{C}, P, \underline{z}, u)$ is *Mundet semistable* resp. *stable* if the principal component is Mundet semistable resp. stable and the section $u : \hat{C} \to P(X)$ is a stable section, in the sense that any component on which u is constant has at least three special (nodal or marked) points.

3. Gauged maps

We introduce the following notations for moduli stacks. Denote by $\overline{\mathfrak{M}}_n^G(C, X, d)$ resp. $\overline{\mathcal{M}}_n^G(C, X, d)$ the category of gauged maps resp. Mundet semistable gauged maps from C to X/G of homology class d and n markings.

Theorem 3.1. *For any d, n, if stable=semistable then the stack $\overline{\mathcal{M}}_n^G(C,X,d)$ is a proper Deligne-Mumford stack equipped with evaluation morphisms*

$$\mathrm{ev} : \overline{\mathcal{M}}_n^G(C,X,d) \to (X/G)^n, \quad (\hat{C}, P, u) \mapsto (\underline{z}^* P, \underline{z}^* u)$$

and virtual fundamental class $[\overline{\mathcal{M}}_n^G(C,X,d)] \in H_n(\overline{\mathcal{M}}_n^G(C,X,d))$.

The properties of the moduli stacks in the above theorem were proved elsewhere. Properness is covered in [9, Theorem 1.1]. Virtual fundamental classes are [33, Example 6.6]. We sketch the construction for completeness. The proof of properness uses a simpler Grothendieck-style compactification obtained by allowing the maps to acquire base points, studied by Schmitt [26], [27, Section 2.7]. Suppose that $X \subset \mathbb{P}(V)$ is embedded in the projectivization $\mathbb{P}(V)$ of a G-representation V. A map $C \to P(\mathbb{P}(V))$ gives rise to a line sub-bundle $L \subset C \times P(V)$. By dualization such a sub-bundle gives rise to a quotient map $q : C \times P(V)^\vee \to L^\vee$. A *gauged quotient* is a datum $(P, L, q, \underline{z})$, called by Schmitt [26] a *bundle with map*. Denote by $\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C,X,d)$ the compactification of the space of gauged quotients whose section takes values in $P(X) \subset P(V)$. The moduli stacks $\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C,X,d)$ only admit evaluation morphisms to the quotient stacks for the ambient vector spaces,

$$\mathrm{ev} : \overline{\mathcal{M}}_n^G(C,X,d) \to (V/(G \times \mathbb{C}^\times))^n, \quad (\hat{C}, P, u) \mapsto (\underline{z}^* P, \underline{z}^* L, \underline{z}^* q).$$

The moduli stack of stable gauged quotients admits a construction as a geometric invariant theory quotient by Schmitt [26, 27]. Choose a faithful representation $G \to GL(V)$, so that $X \subset \mathbb{P}(V)$. A *k-level structure* for a stable gauged quotient is a collection of sections $s_1, \ldots, s_k : C \to P(V)$ generating $P(V)$. Equivalently, a level structure is a surjective morphism $\mathcal{O}_C^{\oplus k} \to P(V)^\vee$. The action of $GL(k)$ on $C \times \mathbb{C}^k$ induces an action on the set of level structures by composition. The stack $\overline{\mathcal{M}}_n^{G,\mathrm{lev,quot}}(C,X,d)$ of gauged quotients with level structure is naturally an Artin stack with an action of $GL(k)$ on the sections. Schmitt [26, Section 2.7] constructs a linearization $D(\mathfrak{L}) \to \overline{\mathcal{M}}_n^{G,\mathrm{lev,quot}}(C,X,d)$ giving rise to a projective embedding of the coarse moduli space, so that the GIT quotient is the stack of gauged quotients:

$$\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C,X,d) = \overline{\mathcal{M}}_n^{G,\mathrm{lev,quot}}(C,X,d) /\!\!/ GL(k).$$

In particular this construction implies that $\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C,X,d)$ has proper coarse moduli space. If stable=semistable then all stabilizers are finite, and since we are in characteristic zero, this implies that $\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C,X,d)$ is Deligne-Mumford and proper. Now the Kontsevich-style compactification $\overline{\mathcal{M}}_n^{G}(C,X,d)$ admits a Givental style proper relative morphism given by Popa-Roth [24, Theorem 7.1]

$$\overline{\mathcal{M}}_n^{G}(C,X,d) \to \overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C,X,d)$$

and so is also proper. Denote by $\overline{\mathcal{M}}_n^{G,\mathrm{lev}}(C,X,d)$ the moduli stack of gauged maps with level structure on the associated vector bundle $P(V)$. The Givental construction on the moduli stack of maps with level structure gives a morphism $\pi : \overline{\mathcal{M}}_n^{G,\mathrm{lev}}(C,X,d) \to \overline{\mathcal{M}}_n^{G,\mathrm{lev},\mathrm{quot}}(C,X,d)$. Then the moduli stack of gauged maps is also a stack-theoretic quotient

$$\overline{\mathcal{M}}_n^{G}(C,X,d) = \pi^{-1}(\overline{\mathcal{M}}_n^{G,\mathrm{lev},\mathrm{quot}}(C,X,d)^{\mathrm{ss}})/G.$$

However, the pull-back of $D(\mathfrak{L})$ is not ample on $\overline{\mathcal{M}}_n^{G,\mathrm{lev}}(C,X,d)$. Thus this quotient cannot be considered a GIT quotient without further perturbation of the linearization.

Virtual fundamental classes are obtained from the construction of Behrend-Fantechi [2]. The universal curve $\overline{\mathcal{C}}_n^{G}(C,X)$ is the stack whose objects are tuples $(\hat{C},P,u,\underline{z},z')$ where $(\hat{C},P,u,\underline{z})$ is a gauged map and $z' \in \hat{C}$ is a (possibly singular) point. Forgetting z' defines a projection

$$p : \overline{\mathcal{C}}_n^{G}(C,X) \to \overline{\mathcal{M}}_n^{G}(C,X)$$

while evaluating at z' defines a universal gauged map

$$e : \overline{\mathcal{C}}_n^{G}(C,X) \to X/G.$$

The relative obstruction theory is the complex $R\pi_* e^* T(X/G)^\vee$ equipped with its canonical morphism to the cotangent complex of $\overline{\mathcal{M}}_n^{G}(C,X)$. If stable=semistable then the obstruction theory is perfect and $\overline{\mathcal{M}}_n^{G}(C,X,d)$ is a proper smooth Deligne-Mumford stack with perfect relative obstruction theory over the stack of *semistable* n-marked maps to C, see [33]. Denote by $[\overline{\mathcal{M}}_n^{G}(C,X,d)] \in H(\overline{\mathcal{M}}_n^{G}(C,X,d))$ the virtual fundamental classes constructed via Behrend-Fantechi machinery.

Using the virtual fundamental classes, gauged Gromov-Witten potentials are defined as follows. Suppose that stable=semistable for all gauged maps. The gauged potential $\tau_{X,G}$ is the formal map defined by

$$\tau_{X,G} : QH_G(X) \to \Lambda_X^G$$

$$\alpha \mapsto \sum_{n\geq 0, d \in H_2^G(X,\mathbb{Z})/\text{torsion}} \frac{q^d}{n!} \int_{[\overline{\mathcal{M}}_n^G(C,X,d)]} \operatorname{ev}^*(\alpha \otimes \ldots \otimes \alpha) \cup f^*\beta_n$$

for $\alpha \in H_G(X)$. We also write $\tau_{X,G,\mathfrak{L}}$ to emphasize the dependence on the linearization.

4. The master space

The strategy for the localization formula is the same as that outlined in the finite-dimensional case described by Thaddeus [29] for the case of variation of linearization in geometric invariant theory. The wall-crossing formula is obtained from a *master space* construction as follows. Suppose $\mathfrak{L}_\pm \to X$ are polarizations. Given integers $\underline{r}_-, \underline{r}_+ > 0$, a class $d_G \in H^2(BG)$, integers $d_{\underline{L}} = (d_{L_-}, d_{L_+})$ and G-modules V_-, V_+ of ranks $\underline{r}_-, \underline{r}_+$, a *bundle with pair* is a tuple

$$\left\{ (P, L_-, L_+, \varphi_-, \varphi_+) \,\middle|\, \begin{pmatrix} \varphi_- : P(V_-^\vee) =: E_- \to L_- \\ \varphi_+ : P(V_+^\vee) =: E_+ \to L_+ \end{pmatrix} \right\}$$

consisting of a G-bundle P with first Chern class d_G, line bundles L_-, L_+ of degrees $d_{\underline{L}}$ and non-zero maps φ_-, φ_+ from the associated vector bundles $E_\pm := P(V_\pm^\vee)$. For weights $\rho_-, \rho_+ > 0$ a parabolic reduction σ and Lie algebra element λ generating a one-parameter subgroup define

$$\mu_{\rho_-,\rho_+}(\sigma,\lambda) = \mu_R(\sigma,\lambda) + \rho_- \mu_{H,-}(\sigma,\lambda) + \rho_+ \mu_{H,+}(\sigma,\lambda)$$

where $\mu_{H,\pm}(\sigma,\lambda)$ is the weight of the one-parameter subgroup on the associated graded for the map $\varphi_\pm$. A datum $(P, L_-, L_+, \varphi_-, \varphi_+)$ is *semistable* if and only if

$$\mu_{\rho_-,\rho_+}(\sigma,\lambda) \leq 0 \quad \forall (\sigma,\lambda).$$

Let $\overline{\mathcal{M}}^{G,\text{quot}}(C, V_-, V_+, d_G, d_{\underline{L}})$ denote the moduli stack of semistable tuples $(P, L_-, L_+, \varphi_-, \varphi_+)$. For sufficiently large $d_{L_\pm}$, there exists a projective scheme $\overline{\mathcal{M}}^{G,\text{quot,lev}}(C, V_-, V_+, d_G, d_{\underline{L}})$ with $GL(\underline{r}_-) \times GL(\underline{r}_+)$ action such that for any ρ_+, ρ_-, the stack $\overline{\mathcal{M}}^{G,\text{quot}}(C, V_-, V_+, d_G, d_{\underline{L}})$ has coarse moduli space that is the good quotient of an open subset of semistable points in $\overline{\mathcal{M}}^{G,\text{quot,lev}}(C, V_-, V_+, d_G, d_{\underline{L}})$

$$\overline{\mathcal{M}}^{G,\text{quot}}(C, V_-, V_+, d_G, d_{\underline{L}}) = \overline{\mathcal{M}}^{G,\text{quot,lev}}(C, V_-, V_+, d_G, d_{\underline{L}})^{\text{ss}} / GL(\underline{r}_-) \times GL(\underline{r}_+).$$

Furthermore, for any choice of $\rho_\pm > 0$ the semistable locus is GIT semistable in the sense that there exists a finite injective equivariant morphism

$$\overline{\mathcal{M}}^{G,\mathrm{quot},\mathrm{lev}}(C, V_-, V_+, d_G, d_{\underline{L}}) \to \overline{\mathfrak{Q}}^{G,\mathrm{lev}}(C, V_-, V_+, d_G, d_{\underline{L}})$$

to a $GL(\underline{r}_-) \times GL(\underline{r}_+)$-scheme $\overline{\mathfrak{Q}}^{G,\mathrm{lev}}(C, V_-, V_+, d_G, d_{\underline{L}})$ and a line bundle

$$D(\mathfrak{L}_-, \mathfrak{L}_+) \to \overline{\mathfrak{Q}}^{G,\mathrm{lev}}(C, V_-, V_+, d_G, d_{\underline{L}})$$

so that the following holds: A bundle with pair $(P, L_-, L_+, \varphi_-, \varphi_+)$ is semistable if and only if its image in $\overline{\mathfrak{Q}}^{G,\mathrm{lev}}(C, V_-, V_+, d_G, d_{\underline{L}})$ is semistable, that is, there exists a non-trivial invariant section of $D(\mathfrak{L}_-, \mathfrak{L}_+)$ non-vanishing at $(P, L_-, L_+, \varphi_-, \varphi_+)$. This follows from [26, Theorem 2.3.5.11] and its generalization to G-bundles in [26, Section 2.7]. More generally, given the projective G-variety X and G-equivariant embeddings

$$\iota_\pm : X \to \mathbb{P}(V_\pm)$$

let $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C, X, V_-, V_+, d_G, \mathrm{d}_{\underline{L}}) \to \overline{\mathcal{M}}^{G,\mathrm{quot}}(C, V_-, V_+, d_G, d_{\underline{L}})$ be the substack consisting of data

$$(P, \varphi_- : E_- \to L_-, \varphi_+ : E_+ \to L_+)$$

so that

$$([\varphi_-(z)], [\varphi_+(z)]) \in (\iota_- \times \iota_+)(X) \subset \mathbb{P}(V_-) \times \mathbb{P}(V_+)$$

for the generic point $z \in C$. Including level structures for the bundles E_-, E_+ into the data gives a $GL(\underline{r}_-) \times GL(\underline{r}_+)$-stack

$$\overline{\mathcal{M}}^{G,\mathrm{quot},\mathrm{lev}}(C, X, V_-, V_+, d_G, \mathrm{d}_{\underline{L}}) \quad \subset \quad \overline{\mathcal{M}}^{G,\mathrm{quot},\mathrm{lev}}(C, V_-, V_+, d_G, d_{\underline{L}})$$

Let $\overline{\mathfrak{Q}}^{G,\mathrm{lev}}(C, X, d_G, d_{\underline{L}})$ denote its image in $\overline{\mathfrak{Q}}^{G,\mathrm{lev}}(C, V_-, V_+, d_G, d_{\underline{L}})$. Denote the quotient stack

$$\overline{\mathfrak{M}}^{G,\mathrm{quot}}(C, V_-, V_+, d_G, d_{\underline{L}}) = \overline{\mathcal{M}}^{G,\mathrm{quot},\mathrm{lev}}(C, V_-, V_+, d_G, d_{\underline{L}})/(GL(\underline{r}_-) \times GL(\underline{r}_+)).$$

Denote by $D(\mathfrak{L}_\pm)$ the line bundle obtained by pull-back from the case $V_\mp$ is trivial. Consider the rank two bundle obtained from the direct sum:

$$D(\mathfrak{L}_-) \oplus D(\mathfrak{L}_+) \to \overline{\mathcal{M}}_n^{G,\mathrm{lev},\mathrm{quot}}(C, X, d).$$

Taking the projectivization of the total space gives a $\mathbb{P}^1$-fibration

$$\mathbb{P}(D(\mathfrak{L}_-) \oplus D(\mathfrak{L}_+)) \to \overline{\mathcal{M}}_n^{G,\mathrm{lev},\mathrm{quot}}(C, X, d).$$

The action of $GL(\underline{r}_-) \times GL(\underline{r}_+)$ lifts to the fibration, since the bundles $D(\mathfrak{L}_\pm)$ are $GL(\underline{r}_-) \times GL(\underline{r}_+)$-equivariant. The bundle

$$\mathcal{O}_{\mathbb{P}(D(\mathfrak{L}_-)\oplus D(\mathfrak{L}_+))}(1) \to \mathbb{P}(D(\mathfrak{L}_-) \oplus D(\mathfrak{L}_+))$$

is automatically ample on the coarse moduli space of $\mathbb{P}(D(\mathfrak{L}_-)\oplus D(\mathfrak{L}_+))$. Let the quotient of the pull-back of the semistable locus on $\overline{\mathfrak{Q}}^{G,\mathrm{lev}}(C, X, d_G, d_{\underline{L}})$ be

$$\overline{\mathcal{M}}_n^{G,\mathrm{lev},\mathrm{quot}}(C, X, \mathfrak{L}_-, \mathfrak{L}_+, d) = \mathbb{P}(D(\mathfrak{L}_-) \oplus D(\mathfrak{L}_+)) /\!\!/ GL(\underline{r}_-) \times GL(\underline{r}_+).$$

Similarly, let

$$\pi^*\mathbb{P}(D(\mathfrak{L}_-) \oplus D(\mathfrak{L}_+)) \to \overline{\mathcal{M}}_n^{G,\mathrm{lev}}(C, X, d)$$

denote the pull-back to the stack of stable gauged maps with level structure, where as before $\pi : \overline{\mathcal{M}}_n^{G,\mathrm{lev}}(C, X, d) \to \overline{\mathcal{M}}_n^{G,\mathrm{lev},\mathrm{quot}}(C, X, d)$. Let

$$\overline{\mathcal{M}}_n^G(C, X, \mathfrak{L}_-, \mathfrak{L}_+, d) = \pi^{-1}(\mathbb{P}(D(\mathfrak{L}_-) \oplus D(\mathfrak{L}_+)))^{\mathrm{ss}}/GL(\underline{r}_-) \times GL(\underline{r}_+)$$

denote the quotient of the pull-back of the semistable locus. The action of $\mathbb{C}^\times$ on $\mathbb{P}(D(\mathfrak{L}_-)\oplus D(\mathfrak{L}_+))$ induces an action of $\mathbb{C}^\times$ on $\overline{\mathcal{M}}_n^G(C, X, \mathfrak{L}_-, \mathfrak{L}_+, d)$.

The fixed point components for the natural circle action are of two types. First, there are inclusions

$$\mathbb{P}(D(\mathfrak{L}_\pm) \oplus 0) \to \mathbb{P}(D(\mathfrak{L}_-) \oplus D(\mathfrak{L}_+))$$

and isomorphisms

$$\mathbb{P}(D(\mathfrak{L}_\pm) \oplus 0) \cong \overline{\mathcal{M}}_n^G(C, X, \mathfrak{L}_\pm).$$

These induce embeddings

$$\overline{\mathcal{M}}_n^G(C, X, \mathfrak{L}_\pm) \to \overline{\mathcal{M}}_n^G(C, X, \mathfrak{L}_-, \mathfrak{L}_+)^{\mathbb{C}^\times}$$

in the locus of fixed points of the $\mathbb{C}^\times$-action. On the other hand, there are fixed point components correspond to *reducible* gauged maps for some stability condition $\mathfrak{L}_t := \mathfrak{L}_-^{(1-t)/2} \otimes \mathfrak{L}_+^{(1+t)/2}$, $t \in (-1, 1)$ interpolating between those defined by $\mathfrak{L}_\pm$. Reducibility means that the fixed point components consist of maps $v = (P, u) : \hat{C} \to X/G$ that admit a one-parameter family of automorphisms $\phi : \mathbb{C}^\times \to \operatorname{Aut}(P)$; via evaluation at a point $\operatorname{Aut}(P) \to \operatorname{Aut}(P_z)$, any such one-parameter family may be identified with a one-parameter family of automorphisms of G generated by some element $\zeta \in \mathfrak{g}$. Euler-twisted integration over the fixed point components gives rise to fixed point contributions

$$\tau_{X,G,\zeta,t} : QH_G(X) \to \Lambda_X^G.$$

The fixed point contributions are curves with bubble trees consisting of maps to the quotient stack with one-parameter automorphisms and stable maps fixed up to isomorphism by one-parameter subgroups. Suppose that a gauged map $(P \to C, u : \hat{C} \to P(X))$ is reducible, that is, has a one-parameter family of automorphisms $\phi : \mathbb{C}^\times \to \operatorname{Aut}(P)$ covering the identity on the principal component so that the associated automorphism satisfies

$$\phi(X) : P(X) \to P(X), \quad \phi(X)^* u = u.$$

Evaluation at any fiber defines a homomorphism

$$\phi_z : \mathbb{C}^\times \to \operatorname{Aut}(P_z) \cong G$$

and so identifies ϕ_z with a one-parameter subgroup of G. Let $\zeta \in \mathfrak{g}$ be a generator of ϕ_z and $G_\zeta \subset G$ the centralizer. The structure group of P reduces to the centralizer G_ζ of ζ. Furthermore, the restriction $u|\hat{C}_0$ of u to the principal component $\hat{C}_0$ takes values in $P(X^\zeta)$ where $X^\zeta = \{x \in X | \zeta_X(x) = 0\}$. Any bubble tree attached at $z \in \hat{C}_0$ must be fixed, up to isomorphism, by the action of $\phi_z \in \operatorname{Aut}(P_z(X))$. That is, there exists a one-parameter family of automorphisms $\psi : \mathbb{C}^\times \to \operatorname{Aut}(\hat{C})$ so that $\psi^* u = \phi(X) \circ u$.

We introduce notation for these fixed point stacks and their normal complexes as follows. For each $\zeta \in \mathfrak{g}$, let $\overline{\mathcal{M}}_n^G(C, X, \mathfrak{L}_t, \zeta)$ denote the stack of $\mathfrak{L}_t$-Mundet-semistable morphisms from C to X/G_ζ that are $\mathbb{C}_\zeta^\times$-fixed and take values in X^ζ on the principal component. Via the inclusion $G_\zeta \to G$ the universal curve over $\overline{\mathcal{M}}_n^G(C, X, \mathfrak{L}_t, \zeta)$ admits a morphism to X/G. Denote by ν_ζ the virtual normal complex for the morphism $\overline{\mathcal{M}}_n^G(C, X, \mathfrak{L}_t, \zeta) \to \overline{\mathcal{M}}_n^G(C, X, \mathfrak{L}_-, \mathfrak{L}_+, d)$.

The virtual fundamental classes on these fixed point stacks lead to fixed point contributions appearing in the wall-crossing formula. Let $QH_{G,\mathrm{fin}}(X)$

denote the tensor product of $H_G(X)$ with the sub-ring

$$\Lambda_X^{G,\mathrm{fin}} = \left\{ \sum_{i=1}^n c_i q^{d_i}, d_i \in H_2^G(X), c_i \in \mathbb{Q} \right\} \subset \Lambda_X^G$$

of finite sums. Let ξ denote the equivariant parameter for the action of the one-parameter subgroup generated by ζ and

$$\mathrm{Resid}_\xi : \mathbb{C}[\xi^{-1}, \xi] \to \mathbb{C}, \quad f \mapsto \frac{1}{2\pi i} \oint f(\xi) \mathrm{d}\xi$$

the residue of ξ at 0, that is, the map taking the coefficient of ξ^{-1}. Virtual integration over $\overline{\mathcal{M}}_n^G(C, X, \mathfrak{L}, \zeta)$ defines a "fixed point contribution"

$$\begin{aligned} (6) \quad & \tau_{X,G,\zeta,\mathfrak{L}} : QH_{G,\mathrm{fin}}(X) \to \tilde{\Lambda}_X^G \otimes H(B\mathbb{C}^\times), \\ & \alpha \mapsto \sum_{d \in H_G^2(X,\mathbb{Z})} \sum_{n \geq 0} \mathrm{Resid}_\xi \int_{[\overline{\mathcal{M}}_n^G(C,X,\mathfrak{L},\zeta,d)]} \frac{q^d}{n!} \mathrm{ev}^*(\alpha, \ldots, \alpha) \cup \mathrm{Eul}(\nu_\zeta)^{-1} \end{aligned}$$

for $\alpha \in H_G(X)$. Here we omit the restriction map $H_{G,\mathrm{fin}}(X) \to H_{G_\zeta}(X)$ to simplify notation. In case $\mathfrak{L}_t$ is a family of linearizations we write $\tau_{X,G,\zeta,t} := \tau_{X,G,\zeta,\mathfrak{L}_t}$. The following is [7, Theorem 3.14].

Theorem 4.1 (Wall-crossing for gauged Gromov-Witten potentials)**.** *Let X be a smooth projective G-variety. Suppose that $\mathfrak{L}_\pm \to X$ are linearizations such that semistable=stable for the stack of polarized gauged maps in [7]. Then the gauged Gromov-Witten potentials are related by*

$$(7) \qquad \tau_{X,G,\mathfrak{L}_+} - \tau_{X,G,\mathfrak{L}_-} = \sum_{[\zeta], t \in (-1,1)} \tau_{X,G,\zeta,t}$$

where the sum is over equivalence classes $[\zeta]$ of unparametrized one-parameter subgroups generated by $\zeta \in \mathfrak{g}$.

Remark 4.2. The fixed point contributions can be re-written as contributions from gauged Gromov-Witten invariants with structure group of smaller rank as follows. For $\zeta \in \mathfrak{g}$ let $\mathbb{C}_\zeta^\times \subset G_\zeta$ denote the one-parameter subgroup generated by ζ, and $G_\zeta / \mathbb{C}_\zeta^\times$ the quotient. Let $X^\zeta \subset X$ denote the fixed point set of $\mathbb{C}_\zeta^\times$. Let

$$\overline{\mathcal{M}}_{0,n}(X)^{\mathbb{C}_\zeta^\times} \subset \overline{\mathcal{M}}_{0,n}(X)$$

denote the $\mathbb{C}_\zeta^\times$-fixed point stack of stable maps to X. The evaluation map restricted to $\overline{\mathcal{M}}_{0,n}(X)^{\mathbb{C}_\zeta^\times}$ automatically takes values in the fixed point locus $X^\zeta \subset X$, that is, $\mathrm{ev} : \overline{\mathcal{M}}_{0,n}(X)^{\mathbb{C}_\zeta^\times} \to (X^\zeta)^n$. Push-pull over the moduli stack $\overline{\mathcal{M}}_{0,n+1}(X)^{\mathbb{C}_\zeta^\times}$ defines a *quantum restriction map*

$$\iota_\zeta : QH(X) \to QH_{G_\zeta}(X^\zeta), \quad \alpha \mapsto \alpha|_{X^\zeta} + \sum_{n,d} \frac{q^d}{n!} \mathrm{ev}_{n+1,*} \mathrm{ev}_1^* \alpha \cup \ldots \cup \mathrm{ev}_n^* \alpha$$

combined with the restriction map $QH_G(X) \to QH_{G_\zeta}(X)$.

Let $\pi_{G_\zeta}^G : \Lambda_X^{G_\zeta} \to \Lambda_X^G$ be the canonical map of Novikov rings induced by the natural map $H_2^{G_\zeta}(X) \to H_2^G(X)$.

Lemma 4.3. *Suppose that stable=semistable for ζ-fixed gauged maps. Then*

$$\tau_{X,\mathfrak{L}^t,G,\zeta}\alpha = \pi_{G_\zeta}^G \circ \tau_{X^\zeta,\mathfrak{L}^t|X^\zeta,G_\zeta/\mathbb{C}_\zeta^\times} \circ \iota_\zeta. \tag{8}$$

Proof. Decomposing the fixed point locus according to the number of markings on each bubble tree gives an isomorphism

$$\begin{aligned} \overline{\mathcal{M}}_n^G(C,X,\mathfrak{L},\zeta) &\cong \bigcup_{I_1\cup\ldots\cup I_r=\{1,\ldots,n\}} \prod_{j=1}^r (\{\mathrm{pt}\} \cup \overline{\mathcal{M}}_{0,i_j+1}(X)^{\mathbb{C}_\zeta^\times}) \\ &\qquad \times_{(X^\zeta)^r} \overline{\mathcal{M}}_r^{G_\zeta,\mathrm{fr}}(C,X^\zeta)/G_\zeta^r \\ &\cong \bigcup_{I_1\cup\ldots\cup I_r=\{1,\ldots,n\}} \prod_{j=1}^r (\{\mathrm{pt}\}/G_\zeta \cup \overline{\mathcal{M}}_{0,|i_j|+1}(X)^{\mathbb{C}_\zeta^\times}/G_\zeta) \\ &\qquad \times_{(X^\zeta/G_\zeta)^r} \mathcal{M}_r^{G_\zeta/\mathbb{C}_\zeta^\times}(C,X^\zeta) \end{aligned}$$

where $\{\mathrm{pt}\}$ represents a trivial bubble tree attached at the j-th node on the principal component. Since these isomorphisms are induced by the natural decomposition of markings on bubble trees, they are compatible with the obstruction theories and thus, it follows that integration over $\overline{\mathcal{M}}_n^G(C,X,\mathfrak{L},\zeta)$ is given by push-forward of $\mathrm{ev}_1^* \alpha \cup \ldots \mathrm{ev}_{i_j}^* \alpha$ over each

$$\mathrm{ev}_{i_j+1} : \{\mathrm{pt}\} \cup \overline{\mathcal{M}}_{0,i_j+1}(X)^{\mathbb{C}_\zeta^\times}/G_\zeta \to X^\zeta/G_\zeta$$

followed by integration over $\overline{\mathcal{M}}_r^{G_\zeta}(C,X^\zeta)$, or more precisely, $\mathbb{C}_\zeta^\times$-equivariant integration over the Deligne-Mumford stack $\overline{\mathcal{M}}_r^{G_\zeta/\mathbb{C}_\zeta^\times}(C,X^\zeta)$ (for which stable=semistable). □

The rank of the structure group for the fixed point contributions is less than the rank of the original group. More precisely, there exists a canonical isomorphism

$$\mathcal{M}^{G_\zeta}(C, X^\zeta) \to \mathcal{M}^{G_\zeta/\mathbb{C}^\times}(C, X^\zeta).$$

Indeed via the projection map $G_\zeta \to G_\zeta/\mathbb{C}^\times_\zeta$ any gauged map to X^ζ/G defines a gauged map to $X^\zeta/(G_\zeta/\mathbb{C}^\times_\zeta)$ and we obtain a map

$$\mathcal{M}^{G_\zeta}(C, X^\zeta) \to \mathcal{M}^{G_\zeta/\mathbb{C}^\times}(C, X^\zeta). \tag{9}$$

Up to finite cover the exact sequence

$$1 \to \mathbb{C}^\times_\zeta \to G_\zeta \to G_\zeta/\mathbb{C}^\times_\zeta \to 1$$

splits. Given a gauged map to $X^\zeta/(G_\zeta/\mathbb{C}^\times))$, let c denote the weight of the $C^\times_\zeta$-action on $\mathfrak{L}|X^\zeta$. Taking the bundle $\mathbb{C}^\times_\zeta$-bundle with first Chern class $-c$ defines the inverse map to (9). This ends the Remark.

5. Quantum Kirwan map

In this section we recall the quantum Kirwan map $\kappa_{X,G}$ of (2). The map $\kappa_{X,G}$ is defined by virtual integration over a moduli stack *scaled affine gauged maps* to X.

Definition 5.1 (Affine gauged maps)**.** Let $n \geq 0$ be an integer. An n-marked affine gauged map is a tuple

$$(P \to C, u : C \to P(X), \lambda : C \to \mathbb{P}(\omega_C \oplus \mathbb{C}), \underline{z} = (z_0, \ldots, z_n))$$

where C is a twisted balanced curve as in orbifold Gromov-Witten theory [1], $P \to C$ is a principal G-bundle, ω_C is the dualizing sheaf on C, and λ is a section of its projectivization $\mathbb{P}(\omega_C \oplus \mathbb{C})$ which satisfies the

(Monotonicity Condition): On any maximal non-self-crossing path of components $C_0, C_1, \ldots, C_l$ of C starting with the component C_0 containing z_0, $\lambda|C_i$ is non-zero and finite on exactly one component C_i, on which $\lambda|C_i$ has a single double pole.

A tuple $(P, u, \lambda, \underline{z})$ is *semistable* if u takes values in $X/\!\!/G$ on the infinity locus $\lambda^{-1}(\infty) \subset C$, the bundle P is trivial on the locus $\lambda^{-1}(0)$, $z_0 \in \lambda^{-1}(\infty)$ while $z_1, \ldots, z_n \in \lambda^{-1}(< \infty)$ and the datum admits no non-trivial automorphisms. The last condition means the following: Each component on which (i) the scaling λ is finite and non-zero resp. zero or infinite and (ii) on which (P, u) is trivializable has at least two resp. three special points (nodes and markings.)

We introduce notation for moduli stacks and evaluation maps. Denote by $\overline{\mathcal{M}}^G_{n,1}(\mathbb{C}, X)$ the moduli stack and $\overline{\mathcal{M}}_{n,1}(\mathbb{C})$ the case that G and X are trivial. Each component $\overline{\mathcal{M}}^G_{n,1}(\mathbb{C}, X, d)$ of homology class $d \in H^G_2(X, \mathbb{Z})/$ torsion and n markings has evaluation maps

$$\begin{aligned} \mathrm{ev}_\infty \times \mathrm{ev} : \overline{\mathcal{M}}^G_{n,1}(\mathbb{C}, X, d) &\to (X/\!\!/G) \times (X/G)^n \\ (P, u, \lambda, \underline{z}) &\mapsto (u(z_0), \dots, u(z_n)). \end{aligned}$$

The formula for the quantum Kirwan map $\kappa_{X,G}$ is

$$\kappa_{X,G}(\alpha) = \sum_{n \geq 0, d} \frac{q^d}{n!} \mathrm{ev}_{\infty,*} \mathrm{ev}^*(\alpha, \dots, \alpha)$$

where again, the map is formal in the sense that only each Taylor coefficient is convergent.

Remark 5.2. If the target satisfies an equivariant Fano condition then the derivative of the quantum Kirwan map at zero is a homomorphism of small quantum cohomologies. Namely suppose that the first Chern class $c^G_1(TX)$ has pairing at least 2 with any non-zero curve class d for which there is a generically semistable map to the quotient stack. In this case, by [34, Remark 8.7] the moduli stacks $\overline{\mathcal{M}}^G_n(\mathbb{C}, X, d)$ have dimension

$$\mathrm{vdim}\, \overline{\mathcal{M}}^G_n(\mathbb{C}, X, d) \geq \dim(X/\!\!/G) + 1.$$

Hence

$$\kappa_{X,G}(0) = 0, \quad D_0\kappa_{X,G} : T_0QH_G(X) \to T_0QH(X/\!\!/G). \tag{10}$$

This ends the remark.

Example 5.3. (Quantum Kirwan map for the scalar multiplication on affine space) Let $G = \mathbb{C}^\times$ act on $X = \mathbb{C}^k$ with $k \in \mathbb{Z}, k \geq 2$ by scalar multiplication, so that $X/\!\!/G = \mathbb{P}^{k-1}$. We have

$$T_0QH_G(X) = \Lambda^G_X[\xi],$$

with ξ the equivariant parameter, while

$$T_0QH(X/\!\!/G) = \Lambda^G_X[\omega]/(\omega^k - q),$$

with $\omega \in H(\mathbb{P}^{k-1})$ the standard hyperplane class. By the previous remark $\kappa_{X,G}(0) = 0$ and

$$D_0\kappa_{X,G}(\xi^l) = \omega^l, \quad l < k.$$

A special case of the main result of [8] (quantum Stanley-Reisner relations) implies that

$$D_0\kappa_{X,G}(\xi^k) = q.$$

Hence $D_0\kappa_{X,G}$ is surjective and

$$T_0QH(X/\!\!/G) = T_0QH_G(X)/\ker D_0\kappa_{X,G} = \Lambda_X^G[\xi]/(\xi^k - q)$$

as expected.

5.1. Adiabatic limit theorem

The following theorem describes the relationship between the gauged potential and the graph potential of the quotient. Let ρ be a positive integer and consider the family of linearizations $\mathfrak{L}^\rho$ with $\rho \to \infty$.

Theorem 5.4 (Adiabatic limit theorem [34])**.** *If stable=semistable for the action of G on X then stable=semistable for gauged maps for ρ sufficiently large (more precisely, for any class $d \in H_2^G(X,\mathbb{Z})$ there exists an $r > 0$ such that $\rho > r$ implies stable=semistable) and*

$$\tau_{X/\!\!/G} \circ \kappa_{X,G} = \lim_{\rho\to\infty} \tau_{X,G} : QH_G(X) \to \Lambda_X^G$$

in the following sense of Taylor coefficients: For any class $\beta \in \overline{\mathcal{M}}_{n,1}(C)$ let

$$\sum_{k=1}^{l} \prod_{I_1\cup\ldots\cup I_r=\{1,\ldots,n\}} \beta_\infty^k \otimes \beta_1^k \otimes \ldots \beta_r^k, \quad \textit{resp. } \beta_0$$

be its restrictions to

$$H\left(\overline{\mathcal{M}}_r(C) \times \prod_{j=1}^{r} \overline{\mathcal{M}}_{|I_j|}(\mathbb{C})\right), \quad \textit{resp. } H(\overline{\mathcal{M}}_n(C))$$

respectively. Then

$$\lim_{\rho\to\infty} \tau_{X,G} = \sum_{k=1}^{l} \tau_{X/\!\!/G}^r(\alpha, \beta_\infty^k) \circ \kappa_X^{G,|I_j|}(\alpha, \beta_j^k)$$

In other words, the diagram

$$\begin{array}{ccc} QH_G(X) & \xrightarrow{\kappa_{X,G}} & QH(X/\!\!/G) \\ & \searrow^{\tau_{X,G}} \quad \swarrow^{\tau_{X/G}} & \\ & \Lambda_X^G & \end{array} \tag{11}$$

commutes in the limit $\rho \to \infty$.

The limit of the Mundet semistability condition in which the linearization goes to zero is studied in the paper [5]. In this limit, the bundle must be semistable since the Ramanathan weight dominates the Hilbert-Mumford weight. By Remark 2.1, the bundle must be trivial. Furthermore, after trivialization the value of the resulting map $u : C \to X$ satisfies the Hilbert-Mumford condition for any one-parameter subgroups at a generic point in C. Hence the moduli stack of gauged maps is a quotient of the moduli space of parametrized stable maps to X:

$$\exists \rho(d), \quad (\rho > \rho(d)) \implies \overline{\mathcal{M}}_n^G(C, X, \mathfrak{L}^\rho, d) = \overline{\mathcal{M}}_n(C, X, d)/\!\!/G.$$

where $\overline{\mathcal{M}}_n(C, X, d)/\!\!/G$ is the quotient of a “semistable locus” so defined. Theorem 1.1 follows from Theorems 4.1 and Theorem 5.4.

6. Quantum abelianization

In this section we sketch an application of quantum Witten localization to a version of the *quantum Martin conjecture* of Bertram et al. [3] that compares Gromov-Witten invariants of a GIT quotient $X/\!\!/G$ and the quotient $X/\!\!/T = X^{\mathrm{ss},T}/T$ by a maximal torus $T \subset G$. Recall the classical version of abelianization due to Martin [16]. Let $\nu_{\mathfrak{g}/\mathfrak{t}}$ denote the bundle over $X/\!\!/T$ induced from the trivial bundle with fiber $\mathfrak{g}/\mathfrak{t}$ over X and $\tau_{X/\!\!/T}^{\mathfrak{g}/\mathfrak{t}}$ the Euler-twisted integration map

$$\nu_{\mathfrak{g}/\mathfrak{t}} = X^{\mathrm{ss},T} \times_T (\mathfrak{g}/\mathfrak{t}), \quad \tau_{X/\!\!/T}^{\mathfrak{g}/\mathfrak{t}} : H(X/\!\!/T) \to \mathbb{Q}, \quad \alpha \mapsto \int_{[X/\!\!/T]} \alpha \cup \mathrm{Eul}(\nu_{\mathfrak{g}/\mathfrak{t}}).$$

Let $W = N(T)/T$ denote the Weyl group of $T \subset G$ and r_T^G the isomorphism with Weyl-invariants

$$\mathrm{r}_T^G : H_G(X) \cong H_T(X)^W.$$

Suppose that stable=semistable for the actions of T and G on X. According to Martin [16] the integrations over $X/\!\!/G$ and $X/\!\!/T$ are related by

$$\tau_{X/\!\!/G} \circ \kappa_{X,G} = |W|^{-1} \tau_{X/\!\!/T}^{\mathfrak{g}/\mathfrak{t}} \circ \kappa_{X,T} \circ \mathrm{r}_T^G .$$

Let $Q\mathbb{H}_G(X) \subset QH_G(X)$ denote the subspace generated by Chern characters of algebraic vector bundles,

$$\mathbb{H}_G(X) := \{ \mathrm{Ch}_G(E) \mid E \to X \ \text{ vector bundle } \}, \quad Q\mathbb{H}_G(X) := \mathbb{H}_G(X) \otimes \Lambda_X^G .$$

The restriction to Chern characters is necessary because our arguments use sheaf cohomology.

Theorem 6.1 (Quantum Martin formula)**.** *Let C be a smooth connected projective genus 0 curve and X a smooth linearized projective G-variety. Suppose that stable=semistable for T and G actions on X. The following equality holds on $Q\mathbb{H}_G(X)$:*

$$\begin{aligned} \tau_{X/\!\!/G} \circ \kappa_{X,G} &= |W|^{-1} \pi_T^G \circ \tau_{X/\!\!/T}^{\mathfrak{g}/\mathfrak{t}} \circ \kappa_{X,T}^{\mathfrak{g}/\mathfrak{t}} \circ \mathrm{r}_T^G \\ &= |W|^{-1} \pi_T^G \circ \tau_{X,T}^{\mathfrak{g}/\mathfrak{t}} \circ \mathrm{r}_T^G : \quad Q\mathbb{H}_G(X) \to \Lambda_X^G . \end{aligned}$$

That is, there is a commutative diagram

$$\begin{array}{ccc} Q\mathbb{H}_G(X) & \longrightarrow & QH_T(X) \\ \Big\downarrow{\scriptstyle \kappa_{X,G}} & & \Big\downarrow{\scriptstyle \kappa_{X,T}^{\mathfrak{g}/\mathfrak{t}}} \\ QH(X/\!\!/G) & & QH(X/\!\!/T) \\ \Big\downarrow{\scriptstyle \tau_{X/G}} & & \Big\downarrow{\scriptstyle |W|^{-1}\tau_{X/T}^{\mathfrak{g}/\mathfrak{t}}} \\ \Lambda_X^G & \longleftarrow & \Lambda_X^T \end{array}$$

We sketch a proof the Theorem in the case stable=semistable for linearized gauged maps. We take as the inductive hypothesis that Theorem 6.1 holds for any group of rank less than $\dim(G)$. We wish to compare the fixed point contributions in the quantum Witten localization formulas

$$\tau_{X,G} - \tau_{X/\!\!/G} \circ \kappa_{X,G} = \sum_{[\lambda] \neq 0, \rho} \tau_{X, \tilde{X}^\rho, G, \lambda} \tag{12}$$

and

$$\tau_X^{T,\mathfrak{g}/\mathfrak{t}} - \tau_{X/\!\!/T}^{\mathfrak{g}/\mathfrak{t}} \circ \kappa_{X,T}^{\mathfrak{g}/\mathfrak{t}} = \sum_{[\lambda]\neq 0,\rho} \tau_{X,\tilde{X}^\rho,T,\lambda}^{\mathfrak{g}/\mathfrak{t}}. \tag{13}$$

In the version for T, both the traces and quantum Kirwan maps have been twisted by the Euler class of the index of $\mathfrak{g}/\mathfrak{t}$. Now $\tau_X^{T,\mathfrak{g}/\mathfrak{t}}$ resp. $\tau_{X,G}$ is defined by integration over $\overline{\mathcal{M}}_n(C,X)/\!\!/T$ resp. $\overline{\mathcal{M}}_n(C,X)/\!\!/G$. This is essentially the setting considered by Martin [16]. In González-Woodward [5, Chapter 5] we show

$$\tau_{X,G} = |W|^{-1}\pi_T^G \circ \tau_X^{T,\mathfrak{g}/\mathfrak{t}}$$

This follows by Martin's argument in [16], if the moduli spaces of stable maps are smooth and the virtual fundamental classes are the usual ones, or by a virtual version of Martin's argument if the moduli spaces of stable maps are only virtually smooth. We note that the virtual non-abelian localization formula used in [5] had a gap in the proof, which was fixed by Halpern-Leistner [12, Formula (2)].

Using abelianization in the small-area limit to prove abelianization it suffices to show abelianization for the right-hand-sides in (12), (13). Each contribution X^λ corresponds to $|W/W_\lambda|$ contributions $X^{w\lambda}, w \in W/W_\lambda$ for the T-action. (Note that $W_{\mathbb{C}\lambda}$ is not necessarily equal to W_λ, but the action of $W_{\mathbb{C}\lambda}/W_\lambda$ gives a double cover of the corresponding fixed point set in the master space.) The identity we wish to show is

$$\tau_{X,\tilde{X}^\rho,G,\lambda} = |W_\lambda|^{-1}\pi_T^{G_\lambda} \circ \tau_{X,\tilde{X}^\rho,T,\lambda}^{\mathfrak{g}/\mathfrak{t}}. \tag{14}$$

In the case G_λ is abelian the group W is trivial and the equality holds automatically. More generally the equation (8) gives

$$\tau_{X,\tilde{X}^t,G,\lambda} = \tau_{X^\lambda,G_\lambda/\mathbb{C}_\lambda^\times} \circ \iota_\lambda$$

and

$$\tau_{X,\tilde{X}^t,T,\lambda} = \tau_{X^\lambda,T/\mathbb{C}_\lambda^\times} \circ \iota_\lambda.$$

By the inductive hypothesis,

$$\tau_{X^\lambda,G_\lambda/\mathbb{C}_\lambda^\times} = |W_\lambda|^{-1}\pi_T^{G_\lambda} \circ \tau_{X^\lambda,T/\mathbb{C}_\lambda^\times}^{\mathfrak{g}_\lambda/\mathfrak{t}}.$$

Equation (14) follows. See Guillemin-Kalkman [11, Section 4] for similar arguments involving recursive applications of fixed point formulae. The equality with $\tau_{X/\!\!/T}^{\mathfrak{g}/\mathfrak{t}} \circ \kappa_{X,T}^{\mathfrak{g}/\mathfrak{t}} \circ \mathrm{r}_T^G$ follows from by combining wall-crossing with Theorem 5.4.

7. The convex case

A slightly modified version of the quantum Witten localization formula holds in quasiprojective cases under a *convexity* assumption.

Definition 7.1. A finite dimensional complex G-vector space V with weights $\mu \in \mathfrak{g}_{\mathbb{Z}}^{\vee}$ will be called *convex* if there exists a central one-parameter subgroup $\phi_\zeta : \mathbb{C}^\times \to G$ such that V has positive weights (μ, ζ) for the induced action of ϕ,

$$V = \bigoplus_\mu V_\mu, \quad (\mu, \zeta) > 0.$$

Given a convex G-vector space, the *projectivization* of V is the quotient

$$\overline{V} = ((V \times \mathbb{C})^\times - \{(0,0)\})/\mathbb{C}^\times$$

where $\mathbb{C}^\times$ acts on $\mathbb{C}$ with weight one. Thus $\overline{V}$ is a weighted projective space and contains V as an open subset. A quasiprojective G-variety X will be called *convex* if there exists a projective morphism $\pi : X \to V$ to a convex G-vector space V. Denote by $V_\infty = \overline{V} - V$ and $X_\infty = \pi^{-1}(V_\infty)$ the *divisor at infinity.*

The following is a simple application of the technique called *symplectic cutting* in the literature [15]:

Lemma 7.2. *Any convex G-variety X admits a G-equivariant compactification $\overline{X}$ by adding single $\mathbb{C}_\zeta^\times$-fixed divisor.*

Proof. Let $\mathfrak{L} \to X$ denote the given linearization on X and $\mathfrak{L}(k)$ the linearization on $X \times \mathbb{C}$ obtained by twisting by the $\mathbb{C}^\times$-character with weight k. Consider the GIT quotient

$$\overline{X} = (X \times \mathbb{C}) /\!\!/ \mathbb{C}^\times.$$

The inverse image of $(0,0) \in V \times \mathbb{C}$ is unstable, for sufficiently large k. Thus the proper morphism $X \to V$ induces a proper morphism $\overline{X}$ to $\overline{V}$. In particular, the quotient $\overline{X}$ is also proper. The G action on $X \times \mathbb{C}$ given by $g(x,z) = (gx, z)$ descends to a G-action on $\overline{X}$, and restricts to the given action on the open subset $X \subset \overline{X}$. $\square$

Corollary 7.3. *Let $d \in H_G^2(\overline{X})$ be a class that pairs trivially with the divisor class $[X_\infty] = [\overline{X} - X] \in H_2^G(\overline{X})$. Then for $k \gg 0$ the moduli stack $\overline{\mathcal{M}}_n^G(C, \overline{X}, \mathfrak{L}(k), d)$ consists of maps whose images are disjoint from $(\overline{X} -$*

$X)/G$. Similarly, if $\mathfrak{L}_\pm \to X$ are two different linearizations then for $k \gg 0$ the moduli stack $\overline{\mathcal{M}}_n^G(C, \overline{X}, d)$ consists of maps whose images are disjoint from $(\overline{X} - X)/G$.

Proof. The intersection number of any curve $u : \mathbb{P}^1 \to \overline{X}$ contained in $\overline{X} - X$ with $\overline{X} - X$ is non-negative. Indeed $\overline{X} - X$ has ample normal bundle in $\overline{X}$ being the pull-back of the compactifying divisor in a weighted projective space. On the other hand, there are no stable gauged maps $C \to X/G$ with image in $(\overline{X} - X)/G$ for sufficiently large d since the trivial reduction σ together with the generator ζ of the one-parameter subgroup $\mathbb{C}^\times$ has weight $\mu(\sigma, \zeta) \to \infty$ as $k \to \infty$. Combining these observations let $v : \hat{C} \to \overline{X}/G$ be a stable gauged map intersecting $(\overline{X} - X)/G$. The intersection number $\# u^{-1}(P(\overline{X} - X)) > 0$ is positive and equal to the pairing $(d, [\overline{X} - X]) \in \mathbb{Q}$ of $d \in H_2^G(X, \mathbb{Q})$ with $[\overline{X} - X] \in H_G^2(\overline{X}, \mathbb{Q})$. The latter vanishes by assumption, a contradiction. □

The corollary implies that the wall-crossing formula also holds for convex varieties by applying the formula to the compactified variety with compactifying divisor sufficiently far away at infinity. However, the quantum Witten localization formula in Theorem 1.1 does not hold as stated because, eventually, the compactifying divisor $[\overline{X} - X]$ will make a contribution in the localization formula.

The following alternative argument gives a formula similar to that in quantum Witten localization. Let χ be a character of G that is negative on the one-parameter subgroup generated by ξ and $\underline{\mathbb{C}}_\chi$ the corresponding trivial line bundle over X. Consider the path of linearizations $\mathfrak{L}_\rho \to X$ obtained by shifting by multiples of the character χ:

$$\mathfrak{L}_\rho = \begin{cases} \mathfrak{L} \otimes \underline{\mathbb{C}}_\chi^{\rho^{-1}-1} & \rho \leq 1 \\ \mathfrak{L}^\rho & \rho \geq 1 \end{cases} \tag{15}$$

considered as elements in the rational Picard group $\mathrm{Pic}^G(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Lemma 7.4. *For any class $d \in H_2^G(X, \mathbb{Z})$, the stack $\overline{\mathcal{M}}_n^G(C, X, \mathfrak{L}_\rho, d)$ is empty for $\rho \gg 0$.*

Proof. Let $\sigma : C \to P/G$ be the trivial parabolic reduction, and ζ the generator of the one-parameter subgroup in the definition of convexity. Given a gauged map $v : \hat{C} \to X/G$, the associated graded pair $\mathrm{Gr}(P), \mathrm{Gr}(u)$ for (σ, ζ) projects to the origin in V/G. The Mundet weight picks up a term $(\rho^{-1} - 1)(\chi, \zeta)$ which goes to infinity as $\rho \to 0$. Hence there are no Mundet-semistable gauged maps with class d, for ρ sufficiently small. □

Theorem 7.5 (Quantum Witten formula for convex varieties)**.** *Let X be a convex G-variety, C a genus zero curve, and suppose that stable=semistable for the G-action on X, for gauged maps with linearization $\mathfrak{L}$, and for polarized gauged maps for the path $\mathfrak{L}_\rho$. Then*

$$\tau_{X/\!\!/G} \circ \kappa_{X,G} = - \sum_{[\zeta]\neq 0,\rho} \tau_{X,G,\zeta,\rho}. \tag{16}$$

where the sum is over conjugacy classes unparametrized one-parameter subgroups generated by $\zeta \in \mathfrak{g}$.

Proof. This is a combination of the adiabatic limit theorem 5.4, the wall-crossing formula Theorem 4.1, the vanishing of the invariants for large ρ in Lemma 7.4. The application of these results to the non-proper variety X is justified by the relationship between invariants of the compactification $\overline{X}$ with those of X in Corollary 7.3. □

Example 7.6 (Quantum Witten formula for the scalar multiplication on affine space)*.* To explain the notation we use (16) to compute the three-point Gromov-Witten invariants of projective space using quantum Witten localization. Suppose that $G = \mathbb{C}^\times$ acts diagonally on $X = \mathbb{C}^k$ so that

$$X/\!\!/G = \mathbb{C}^k/\!\!/\mathbb{C}^\times = \mathbb{P}^{k-1}.$$

We have

$$H_2^G(X,\mathbb{Z}) \cong H_2(X/\!\!/G) = \mathbb{Z}[\mathbb{P}^1], \quad H^2_G(X,\mathbb{Z}) \cong H^2(X/\!\!/G) = \mathbb{Z}\omega$$

where ω is the hyperplane class, the image of the equivariant generator $\xi \in H^2_G(X,\mathbb{Z})$ under the Kirwan map. We compute the class $d = 1$ three-point invariants using quantum Witten localization. Let $\beta \in H^6(\overline{\mathcal{M}}_3(C))$ be the fundamental class. We identify $QH_G(X) \cong \Lambda_X^G[\xi]$. Consider the three-point invariants with insertions

$$\xi^a, \xi^b, \xi^c \in QH_G(X) \cong S(\mathfrak{g})^\vee.$$

Since $c_1^G(X)$ is at least $2k$ on classes $d > 0$, the derivative $D_0\kappa_{X,G}$ of the quantum Kirwan map has no quantum corrections by (10). The image of ξ^a, ξ^b, ξ^c under $D_0\kappa_{X,G}$ is equal to $\omega^a, \omega^b, \omega^c$ respectively. We consider a path $\mathfrak{L}_\rho$ obtained by shifting by a negative character χ; this means that in the fixed point formula we take the residue with respect to $-\xi$. By the formula (16),

$$\sum_{d\geq 0} q^d \langle \omega^a, \omega^b, \omega^c\rangle_{0,d} = \tau^3_{X/\!\!/G}(\omega^a, \omega^b, \omega^c, \beta)$$

$$= -\sum_{\rho,[\zeta]} \partial_{\xi^a}\partial_{\xi^b}\partial_{\xi^c}\tau^3_{X,G,\zeta,\rho}(0,\beta).$$

There is a unique G-fixed point in X. The G-bundle P with first Chern class $d = 1$ together with the zero section $u \in H^0(C, P \times_G X)$ forms a Mundet semistable map for a unique value of the parameter ρ. For $d = 1$ the index bundle and its Euler class are

$$\mathrm{Ind}(T(X/G)) = H^0(\mathcal{O}(k)^\times \times_{\mathbb{C}^\times} \mathbb{C}^k) \cong \mathbb{C}^{2k}, \quad \mathrm{Eul}(\mathrm{Ind}(T(X/G))) = \xi^{2k}.$$

The unique fixed point contribution

$$\begin{aligned}\partial_{\xi^a}\partial_{\xi^b}\partial_{\xi^c}\tau^3_{X,G,\zeta,\rho}(0,\beta) &= q\,\mathrm{Resid}_{-\xi}\frac{\xi^{a+b+c}}{\xi^{2k}} \\ &= \begin{cases} q & a+b+c = 2k-1 \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

We obtain

$$\langle \omega^a, \omega^b, \omega^c\rangle_{0,1} = \begin{cases} 1 & a+b+c = 2k-1, \\ 0 & \text{otherwise}\end{cases}$$

as expected.

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