Pure and Applied Mathematics Quarterly
Volume 19, Number 4, 1975–1983, 2023

Almost invariant subspaces and operators

David Kazhdan and Alexander Polishchuk*

Abstract: We prove an efficient version of the Wagner's theorem on almost invariant subspaces (see [5]) and deduce some consequences in the context of Galois extensions.

1. Introduction

The goal of this note is to present an elementary proof of the following linear algebra result and to consider its consequences in the context of Galois extensions (see Theorem B). In [3] this result is applied to the study of the Schmidt rank of quartic polynomials.

Theorem A. *Let V be a vector space over a field $\mathbf{k}$, X a collection of subspaces of V, G a group acting linearly on V and preserving X setwise. For $x \in X$ we denote by $A_x \subset V$ the corresponding subspace. Assume that for some $r \geq 1$ the following condition holds: for any $x, y \in X$ one has $\dim A_x/(A_x \cap A_y) \leq r$. Then there exists a G-invariant subspace $W \subset V$, which is a finite sum of some finite intersections of subspaces from X, such that*

$$\dim W/(W \cap A_x) \leq r, \quad \dim A_x/(W \cap A_x) \leq r \cdot (r+1)^{r+1}.$$

Originally, this theorem was proved by Wagner [5], using model theory and without an explicit bound on $\dim A_x/(W \cap A_x)$. To get an explicit bound we combine Wagner's proof with the idea of the proof of Neumann's explicit bound in Bergman-Lenstra's theorem on almost normal subgroups (see [1]). We conjecture that it should be possible to improve the bound $r \cdot (r+1)^{r+1}$ to a polynomial (possibly linear) function of r, at least in the case when G is finite.

In Section 3 we show that in the special case when $X = G$, G is a finite group interchanging vectors of a basis, subspaces are spanned by subsets of this basis, one can replace $r \cdot (r+1)^{r+1}$ by $2r$. This reduces to the known

Received July 23, 2021.
*A.P. is partially supported by the NSF grant DMS-2001224 and within the framework of the HSE University Basic Research Program.

set-theoretic result (see [4, 2]), for which we give a new short proof assuming that G is finite (see Theorem 3.1). Our version of this result suggests a formula for the approximating G-invariant subspace, for which a better bound (polynomial or even linear in r) might hold, see Section 3 for a discussion.

As a corollary of Theorem A, we obtain the following result about linear subspaces and linear operators that are almost invariant under the action of the Galois group. This result is used in our study of the Schmidt rank of quartics over non-closed fields (see [3]).

Theorem B. *Let $E/\mathbf{k}$ be a finite Galois extension with Galois group G, and let V_0 be a finite dimensional $\mathbf{k}$-vector space. Let us set $V = V_0 \otimes_{\mathbf{k}} E$. We consider the natural action of G on V and the induced action of G on the set of E-linear subspaces of V and on $\mathrm{End}_E(V)$.*

(i) *Suppose $A \subset V$ is an E-linear subspace such that for each $\sigma \in G$, one has*

$$\dim_E\big(A/(A \cap \sigma A)\big) \le r,$$

for some $r \ge 0$. Then there exists a $\mathbf{k}$-linear subspace $W_0 \subset V_0$ such that for $W = W_0 \otimes_{\mathbf{k}} E \subset V$ one has

$$\dim_E\big(W/(W \cap A)\big) \le r, \quad \dim_E\big(A/(W \cap A)\big) \le r \cdot (r+1)^{r+1}.$$

(ii) *Suppose V_0' is another finite dimensional $\mathbf{k}$-vector space, $V' = V_0' \otimes_{\mathbf{k}} E$, and $T : V \to V'$ is an E-linear operator such that for any $\sigma \in G$, one has*

$$\mathrm{rk}_E\big(\sigma(T) - T\big) \le r,$$

for some $r \ge 0$. Then there exists a $\mathbf{k}$-linear operator $T_0 : V_0 \to V_0'$, such that

$$\mathrm{rk}_E\big(T - (T_0)_E\big) \le 2r + r \cdot (r+1)^{r+1},$$

where $(T_0)_E : V \to V'$ is obtained from T_0 by the extension of scalars.

2. Linear algebra results

2.1. Proof of Theorem A

Notation For a subset $S \subset X$ we set $A_S := \cap_{x \in S} A_x$.

For each m, $0 \le m \le r$, let $\mathcal{S}_m$ denote the set of all nonempty subsets $S \subset X$ such that

$$\dim(A_S + A_x)/A_x \le m \quad \text{for any } x \in X.$$

Note that $S = X$ is contained in $\mathcal{S}_0$ and $\mathcal{S}_{m-1} \subset \mathcal{S}_m$. Set

$$h(m) = \min_{S \in \mathcal{S}_m} |S|$$

which is either a natural number or ∞. Note that by assumption $h(r) = 1$. We also have $h(m-1) \geq h(m)$.

We consider separately two cases.

Case 1. There exists m, $1 \leq m \leq r$, such that $h(m)$ is finite and $h(m-1) > (r+1)h(m) + 1$. Let us take the maximal m with this property and set

$$W := \sum_{S \in \mathcal{S}_m : |S| = h(m)} A_S.$$

Note that W is clearly G-invariant. Since for all $m' > m$ we have $h(m'-1) \leq (r+1)h(m') + 1$, we get

$$h(m) \leq (r+1)^{r-m} + \cdots + (r+1)^2 + (r+1) + 1 \leq (r+1)^r.$$

Note that since $A_S \subset W$ for some $S \in \mathcal{S}_m$ with $|S| = h(m)$, we have for any $x \in X$,

$$\dim A_x / A_x \cap W \leq \dim A_x / A_x \cap A_S \leq r \cdot h(m) \leq r \cdot (r+1)^r.$$

Next, we claim that for any $x \in X$ one has $\dim W/W \cap A_x \leq r$. Indeed, suppose there exists $x \in X$ such that $\dim(W + A_x)/A_x \geq r+1$. Then there exist $r+1$ subsets $S_1, \ldots, S_{r+1} \in \mathcal{S}_m$ with $|S_i| = h(m)$, such that

$$\dim\left(\sum_{i=1}^{r+1} A_{S_i} + A_x\right)/A_x \geq r+1.$$

Let us consider the subset

$$T := \{x\} \cup S_1 \cup \cdots \cup S_{r+1} \subset X.$$

Then we have $|T| \leq (r+1)h(m) + 1 < h(m-1)$. Hence, $T \notin \mathcal{S}_{m-1}$, so there exists an element $y \in X$ such that

$$\dim(A_T + A_y)/A_y \geq m.$$

But for any $i = 1 \ldots, r+1$, we have

$$(A_T + A_y)/A_y \subset (A_{S_i} + A_y)/A_y,$$

and the latter space has dimension $\leq m$ by the definition of $\mathcal{S}_m$. Hence, we have $A_T + A_y = A_{S_i} + A_y$ for any i. Therefore,

$$\sum_{i=1}^{r+1} A_{S_i} \subset A_T + A_y \subset A_x + A_y,$$

so

$$\dim\left(\sum_{i=1}^{r+1} A_{S_i} + A_x\right)/A_x \leq \dim(A_x + A_y)/A_x \leq r,$$

which is a contradiction.

Finally, we note that since $W/(W \cap A_S)$ is finite dimensional, for any $S \in \mathcal{S}_m$, W can be written as a finite sum of some intersections A_S.

Case 2. For each $m = 1, \ldots, r$ one has $h(m-1) \leq (r+1)h(m) + 1$. This implies that

$$h(0) \leq (r+1)^r + \cdots + (r+1)^2 + (r+1) + 1 \leq (r+1)^{r+1}.$$

Note that $\mathcal{S}_0$ consists of S such that $A_S = \cap_{x \in X} A_x$. In this case we set

$$W := \cap_{x \in X} A_x.$$

Then W is G-invariant, and since there exists a subset $S \in \mathcal{S}_0$ with $|S| = h(0)$, we get

$$\dim A_x/A_x \cap W = \dim A_x/A_x \cap A_S \leq r \cdot h(0) \leq r(r+1)^{r+1}.$$

This ends the proof.

2.2. Almost invariant operators

In this section we use Theorem A to approximate almost G-invariant operators by G-invariant ones by considering their graphs.

Lemma 2.1. *Let V and V' be linear representations of a group G, such that any G-invariant subspace $W \subset V$ (resp., $W' \subset V'$) of finite codimension (resp., dimension) admits a G-invariant complement. Assume that $T : V \to V'$ is a linear operator such that for any $g \in G$, one has* $\mathrm{rk}(gTg^{-1} - T) \leq r$, *for some $r \geq 0$. Then there exists a G-invariant operator $T_0 : V \to V'$ such that*

$$\mathrm{rk}(T - T_0) \leq 2r + r \cdot (r+1)^{r+1}.$$

Proof. Let $A = A_T \subset V \oplus V'$ denote the graph of T, i.e., $A = \{(v, Tv)\}$. Note that if A and A' are graphs of T and T' then $A \cap A' = \{(v, Tv) \mid v \in \ker(T - T')\}$, so

$$\dim A/A \cap A' = \mathrm{rk}(T - T').$$

Thus, the assumption on T implies that $\dim(A/(A \cap gA)) \leq r$ for all $g \in G$. Applying Theorem A, we obtain a G-invariant subspace $A_0 \subset V \oplus V'$ such that

$$\dim A_0/(A_0 \cap A) \leq r, \quad \dim A/(A_0 \cap A) \leq r(r+1)^{r+1}.$$

Let $p_1 : V \oplus V' \to V$ and $p_2 : V \oplus V' \to V'$ be the projections. Set $K := \{v' \in V' \mid (0, v') \in A_0\} \subset V'$ and $I := p_1(A_0) \subset V$. Note that the subspaces K and I are G-invariant, and $p_2 : A_0 \to V'$ induces a well defined G-invariant linear map

$$\overline{T}_0 : I \to V'/K,$$

such that A_0 is the pull-back of the graph $\overline{A}_0 \subset I \oplus V'/K$ of $\overline{T}_0$ under the projection $\pi : I \oplus V' \to I \oplus V'/K$. Furthermore, we have

$$\dim K \leq r, \quad \dim V/I \leq r(r+1)^{r+1}.$$

Hence, we can find a G-linear projector $p_I : V \to I$ and a G-linear projector $p_C : V' \to C \subset V'$, where C is a G-invariant complement to K, such that $p_C(K) = 0$. Now we set

$$T_0 = p_C \circ \overline{T}_0 \circ p_I.$$

Note that $\overline{T}_0$ is obtained as the composition of $T_0|_I$ with the projection $V' \to V'/K$. Let $\overline{T} : I \to V'/K$ denote the composition of $T|_I$ with the projection $V' \to V'/K$, and let $\overline{A} \subset I \oplus V'/K$ denote the graph of $\overline{T}$. It is easy to see that π induces a surjection

$$A_0/(A_0 \cap A) \to \overline{A}_0/(\overline{A}_0 \cap \overline{A}),$$

so we get

$$\mathrm{rk}(\overline{T} - \overline{T}_0) = \dim \overline{A}_0/(\overline{A}_0 \cap \overline{A}) \leq r.$$

But $\overline{T} - \overline{T}_0$ is obtained from $T - T_0$ by restricting to the subspace $I \subset V$ and composing with the projection $V' \to V'/K$, so we get

$$\mathrm{rk}(T - T_0) \leq \dim K + \dim V/I + \mathrm{rk}(\overline{T} - \overline{T}_0) \leq 2r + r(r+1)^{r+1}. \qquad \square$$

2.3. Proof of Theorem B

(i) Let us apply Theorem A to the collection (σA) of all Galois conjugates of A. Note that the action of G is only $\mathbf{k}$-linear, so we should view this is a collection of $\mathbf{k}$-linear subspaces, and use the dimension function $\dim_{\mathbf{k}} = [E : \mathbf{k}] \cdot \dim_E$. However, the resulting G-invariant $\mathbf{k}$-subspace W is in the lattice generated by (σA), so it is actually an E-linear subspace. But G-invariant E-linear subspaces of V are precisely subspaces obtained by extension of scalars from $\mathbf{k}$-linear subspaces of V_0. This gives the statement.

(ii) We apply the same strategy as in the proof of Lemma 2.1. First, we find a G-invariant E-subspace W approximating the graph A of T. Then we use the fact that W is an extension of scalars from $W_0 \subset V_0 \oplus V_0'$ and construct $\mathbf{k}$-linear projectors $p_C : V_0' \to C \subset V_0'$ and $p_I : V_0 \to I$, as in the proof of Lemma 2.1, where $I = p_1(W_0)$ and C is a complement in V_0' to $(0 \oplus V_0') \cap W_0$. This gives the required operator defined over $\mathbf{k}$.

3. Almost invariant subsets

The following result is a more concrete version of a theorem of Neumann [4] (see also [2] for related results), stating that if a subset A in a G-set X satisfies $|A \setminus gA| \le r$ for every $g \in G$ and some $r > 0$, then there exists a G-invariant subset $A_0 \subset X$ such that

$$|A \setminus A_0| + |A_0 \setminus A| < 2r.$$

We make an extra assumption that G is finite but as a bonus we get an explicit construction of A_0.

Theorem 3.1. *Let X be a set with action of a finite group G. Let $A \subset X$ be a subset such that for any $g \in G$ one has $|A \setminus gA| \le r$ for some $r > 0$. For every subset $S \subset G$ we denote $A_S := \cap_{g \in S} gA$. Now consider the G-invariant subset*

$$A_0 := \bigcup_{|S| > |G|/2} A_S.$$

Then

$$|A \setminus A_0| + |A_0 \setminus A| \le 2r.$$

Proof. We have

$$A \setminus A_0 = \bigcap_{|S| > |G|/2} A \setminus A_S = \{a \in A \mid |\{g \in G \mid ga \in A\}| \le |G|/2\}.$$

Now let us consider the set

$$P := \{(a \in A, g \in G) \mid ga \notin A\}.$$

Considering the fibers of the projection of P to G and using the assumption, we see that

$$|P| \leq r \cdot |G|.$$

Now, let us consider the projection $p_A : P \to A$. For every $a \in A \setminus A_0$, we have

$$|p_A^{-1}(a)| = |\{g \in G \mid ga \notin A\}| \geq \frac{|G|}{2}.$$

Hence, setting

$$P_1 := p_A^{-1}(A \setminus A_0),$$

we get

$$\frac{|G|}{2} \cdot |A \setminus A_0| \leq |P_1|. \tag{1}$$

Next, setting $B = X \setminus A$, we observe that

$$A_0 \setminus A = A_0 \cap B = \bigcup_{|S|>|G|/2} A_S \cap B = \{b \in B \mid |\{g \in G \mid gb \in A\}| > |G|/2\}.$$

Let us consider the projection

$$p_B : P \to B : (a, g) \mapsto ga$$

and set

$$P_2 := p_B^{-1}(A_0 \cap B).$$

Since for every $b \in A_0 \cap B$, we have

$$|p_B^{-1}(b)| = |\{g \in G \mid g^{-1}b \in A\}| > \frac{|G|}{2},$$

we deduce that

$$\frac{|G|}{2} \cdot |A_0 \cap B| \leq |P_2|. \tag{2}$$

Finally, we claim that P_1 and P_2 do not intersect. Indeed, suppose $(a, g_0) \in P_1 \cap P_2$. Then

$$|\{g \in G \mid ga \notin A\}| \geq \frac{|G|}{2}$$

and

$$|\{g \in G \mid gg_0a \in A\}| > \frac{|G|}{2},$$

which is impossible since

$$|\{g \in G \mid gg_0a \in A\}| = |\{g \in G \mid ga \in A\}| = |G| - |\{g \in G \mid gg_0a \in A\}|.$$

Thus, combining (1) and (2), we get

$$\frac{|G|}{2} \cdot (|A \setminus A_0| + |A_0 \setminus A|) \leq |P_1| + |P_2| \leq |P| \leq r \cdot |G|,$$

which gives

$$|A \setminus A_0| + |A_0 \setminus A| \leq 2r. \qquad \square$$

Remark 3.2. One can see from the proof that the only case when we possibly do not get a strict inequality

$$|A \setminus A_0| + |A_0 \setminus A| < 2r$$

is when $A_0 \subset A$ and $P_1 = P$, which is equivalent to $A_S = \cap_{g \in G} gA$ whenever $|S| > |G|/2$. In this case we can replace A_0 with

$$A_0' := \bigcup_{|S| \geq |G|/2} A_S.$$

Since for any $S \subset G \setminus \{1\}$ with $|S| \geq |G|/2$, we have $A \cap A_S = \cap_{g \in G} gA$, assuming that A is not G-invariant, we get $A \setminus A_0' \neq \emptyset$. In this case, running the similar argument to the above proof, we get that

$$|A \setminus A_0'| + |A_0' \setminus A| < 2r.$$

Theorem 3.1 suggests that in the linear algebra setup with A a linear subspace of a G-representation V, such that $\dim(A/A \cap gA) \leq r$, for finite G, one can try to define the approximating G-invariant subspace as

$$A_0 := \sum_{|S| > |G|/2} A_S, \tag{3}$$

where $A_S = \cap_{g \in S} gA$.

Question. Does there exist a polynomial (or even a linear) function $c(r)$ such that for (V, G, A) as above and A_0 given by (3) we have

$$\dim A/(A_0 \cap A) \leq c(r) \quad \text{and} \quad \dim A_0/(A_0 \cap A) \leq c(r)?$$

Acknowledgment

We are grateful to Ehud Hrushovski for pointing out the references and for useful discussions.

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David Kazhdan
The Hebrew University of Jerusalem
Jerusalem 91904
Israel
E-mail: kazhdan@math.huji.ac.il

Alexander Polishchuk
University of Oregon
Eugene, OR 97403
USA
National Research University Higher School of Economics
Russia
E-mail: apolish@uoregon.edu