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Stability and bifurcations of symmetric tops

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Abstract: We study the stability and bifurcation of relative equilibria of a particle on the Lie group $SO(3)$ whose motion is governed by an $SO(3) \times SO(2)$ invariant metric and an $SO(2) \times SO(2)$ invariant potential. Our method is to reduce the number of degrees of freedom at *singular* values of the $SO(2) \times SO(2)$ momentum map and study the stability of the equilibria of the reduced systems as a function of spin. The result is an elementary analysis of the fast/slow transition in the Lagrange and Kirchhoff tops.

More generally, since an $SO(2) \times SO(2)$ invariant potential on $SO(3)$ can be thought of as $\mathbb{Z}_2$ invariant function on a circle, we analyze the stability and bifurcation of relative equilibria of the system in terms of the second and fourth derivative of the function.

Keywords: Bifurcation, stability, finite-dimensional Hamiltonian systems.

1. Introduction

Recall the geometric mechanics approach to classical systems which was developed in the early 1960s: A "simple" classical mechanical system consists

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of a manifold Q, the configuration space of the system, a Riemannian metric g in Q, the kinetic energy, and a function $V : Q \to \mathbb{R}$, the potential. These data define a Hamiltonian $H : T^*Q \to \mathbb{R}$ on the cotangent bundle of Q which is given by

$$H(q, p) = \frac{1}{2} g_q^*(p, p) + V(q)$$

where $q \in Q$, $p \in T_q^*Q$ and $g^* \in \mathrm{Sym}^2(T^*Q)$ is the dual metric. The Hamiltonian H together with the canonical symplectic form ω_{T^*Q} on the cotangent bundle give rise to a vector field $\Xi_H : T^*Q \to T(T^*Q)$ which is uniquely defined by the equation

$$\omega_{T^*Q}(\Xi_H, \cdot) = dH.$$

An action of a Lie group G on Q that preserves the metric g and the potential V lifts to an action of G on the cotangent bundle T^*Q that preserves the Hamiltonian H and the symplectic form ω_{T^*Q}. Noether's theorem in this setting translates into the existence of an equivariant moment map $\mu : T^*Q \to \mathfrak{g}^*$ ($\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$ of the Lie group G) with the property that μ is constant along the intergral curves of the vector field Ξ_H. Since the Hamiltonian H and the symplectic form are G-invariant the Hamiltonian vector field Ξ_H is G-invariant as well. Hence its flow Φ_t^H is G-equivariant. Noether's theorem combined with equivariance of the flow implies that we have continuous time dynamical systems on the topological spaces $\{\mu^{-1}(\alpha)/G_\alpha\}_{\alpha \in \mathfrak{g}^*}$, where G_α denotes the stabilizer of $\alpha \in \mathfrak{g}^*$ under the coadjoint action. When α is a regular value of μ and the action of G is proper then, thanks to a theorem of Meyer [15] and of Marsden and Weinstein [14], the topological space $M//_\alpha G := \mu^{-1}(\alpha)/G_\alpha$ is naturally a symplectic orbifold and the flow induced by Φ_t^H is a flow of a Hamiltonian vector field. The theorem of Meyer and Marsden-Weinstein is known as regular symplectic reduction and as (Meyer-)Marsden-Weinstein reduction (a number of people seem to be unaware of Meyer's paper).

A top is a classical mechanical system of the form $(T^*SO(3), H(q,p) = \frac{1}{2}g_q^*(p,p) + V(q))$ where $SO(3)$ is the special orthogonal group and g is a left-invariant metric on the group. A top is symmetric if two of its principal moments of inertia are equal and the potential is invariant under the additional symmetry. This amounts to the metric g being invariant under the multiplication on the right by $SO(2)$ (here $SO(2)$ is the subgroup of $SO(3)$ fixing the third standard basis vector $e_3 = (0, 0, 1)$), and the potential $V \in C^\infty(SO(3))$ being $SO(2) \times SO(2)$-invariant. Here and elsewhere $SO(2) \times SO(2)$ acts on $SO(3)$ by multiplication on the left and right, respectively.

In a symmetric top one can view the rate of spin about its axis of symmetry as a bifurcation parameter. Geometrically this amounts to viewing a top as a family of Hamiltonian systems on the unit 2-sphere S^2:

$$\left\{(T^*S^2, \omega_{T^*S^2} + r\omega_{S^2}, h(q,p) = \frac{1}{2}g^*_q(p,p) + V(q))\right\}_{r\in\mathbb{R}},$$

where $\omega_{T^*S^2}$ is the canonical symplectic form on the cotangent bundle, ω_{S^2} is the standard area form on the sphere, g is the standard round metric, and we have identified the potential of the top with an $SO(2)$-invariant function on S^2. See Corollary 2.3 below. Note that there is no loss of generality in assuming that $r \geq 0$ since $r < 0$ corresponds to spin in the opposite direction.

Theorem 2 of [20] implies that the algebra $C^\infty(S^2)^{SO(2)}$ of invariant functions on the sphere is isomorphic to $C^\infty(S^1)^{\mathbb{Z}/2}$, where $\mathbb{Z}/2 = \{\pm 1\}$ acts on $S^1 = \{(x,z) \in \mathbb{R}^2 \mid x^2 + z^2 = 1\}$ by $(-1)\cdot(x,z) = (-x,z)$. Parameterize the upper half of S^1 by

$$f : (-1,1) \to S^1, \qquad f(u) = (u, \sqrt{1-u^2}).$$

Then f pulls back $\mathbb{Z}/2$-invariant functions on S^1 to $\mathbb{Z}/2$ invariant functions on $(-1,1)$ (where $-1 \in \mathbb{Z}/2$ acts on $(-1,1)$ by $(-1)\cdot u = -u$). It follows from a theorem of Whitney that $C^\infty((-1,1))^{\mathbb{Z}/2}$ is isomorphic to $C^\infty([0,1))$. That is, for any $k \in C^\infty((-1,1))^{\mathbb{Z}/2}$ there is a unique $\ell \in C^\infty([0,1))$ with $k(u) = \ell(u^2)$. (Recall that $\ell \in C^\infty([0,1))$ iff there is $\varepsilon > 0$ and $\tilde{\ell} \in C^\infty((-\varepsilon,1))$ with $\ell = \tilde{\ell}|_{[0,1)}$.) Therefore given a function $V \in C^\infty(S^2)^{SO(2)}$ there is a unique function $W \in C^\infty([0,1))$ so that

$$W(u^2) = V(u, 0, \sqrt{1-u^2}) \qquad \text{for all } u \in [0,1).$$

We are now in position to formulate our stability and bifurcation result. Note that for us "stability" means Lyapunov stability.

Theorem 1.1. Consider a 1-parameter family of $SO(2)$-invariant Hamiltonian systems

$$\left\{(T^*S^2, \omega_{T^*S^2} + r\omega_{S^2}, h(q,p) = \frac{1}{2}g^*_q(p,p) + V(q))\right\}_{r\geq 0}, \tag{1.2}$$

where $SO(2)$ acts by the lift of rotations about $e_3 = (0,0,1)$, $\omega_{T^*S^2}$ is the canonical symplectic form on the cotangent bundle, ω_{S^2} is the standard area form on the sphere, g is the standard round round metric and

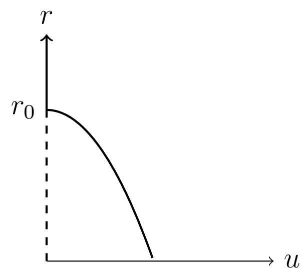
$V \in C^\infty(S^2)^{SO(2)}$. Let $W \in C^\infty([0,1))$ be the function with

$$W(u^2) = V(u, 0, \sqrt{1-u^2})$$

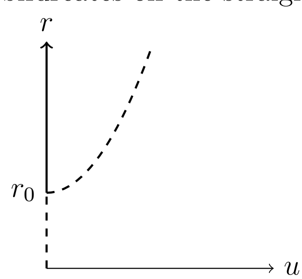
for all $u \in [0,1)$. Then

(i) If $W'(0) > 0$ then the straight up top (i.e. the point $(e_3, 0) \in T^*S^2$ where $e_3 = (0,0,1)$) is stable for all values of r.

(ii) Suppose $W'(0) < 0$ and $W''(0) > W'(0)$. Then for $r > r_0 := \sqrt{-8W'(0)}$ the straight up top is stable. As r decreases below r_0 the top loses stability and we get a branch of stable relative equilibria bifurcating off the straight up position. That is, a "Hamiltonian Hopf" bifurcation takes place.



(iii) Suppose $W'(0) < 0$ and $W''(0) < W'(0)$. Then for $0 \le r \le r_0$ the straight up top is unstable. As r increases past $r_0 = \sqrt{-W'(0)}$ the straight up top gains stability. Additionally a branch of unstable relative equilibria bifurcates off the straight up position.



Example 1.3 (Lagrange Top)**.** For the Lagrange top the potential $V \in C^\infty(SO(3))^{SO(2)\times SO(2)}$ is given by $V(A) = \langle Ae_3, e_3\rangle$ for all $A \in SO(3)$. Hence the corresponding function $V \in C^\infty(S^2)^{SO(2)}$ is

$$V(q) = \langle q, e_3\rangle = q_3.$$

Since $V(u, 0, \sqrt{1-u^2}) = \sqrt{1-u^2}$, the function $W \in C^\infty([0,1))$ is given by

$$W(t) = \sqrt{1-t}.$$

Since $W'(0) = -\frac{1}{2} < 0$ and since $W'(0) < -\frac{1}{4} = W''(0)$ we are in the case (ii) of Theorem 1.1: as the rate of spin decreases the straight up top loses stability at $r_0 = 2$ and a stable relative periodic orbit appears nearby, i.e., the tip of the top will trace out a circle around the vertical axis.

Example 1.4 (Kirchhoff top)**.** A Kirchhoff top is a family of symmetric tops with the potential $V(A) = \langle Ae_1, e_1\rangle^2 + \langle Ae_2, e_2\rangle^2 + c\langle Ae_3, e_3\rangle^2$ where $c > 0$ is a constant. It corresponds to the $SO(2)$-invariant function $V : S^2 \to \mathbb{R}$ given by

$$V(q) = q_1^2 + q_2^2 + cq_3^2.$$

Since $V(u, 0, \sqrt{1-u^2}) = u^2 + c(1-u^2) = c + (1-c)u^2$ the function $W \in C^\infty([0, 1))$ is given by $W(t) = c+(1-c)t$. Then $W'(0) = 1-c$ and $W''(0) = 0$. Therefore if $c < 1$ the derivative $W'(0) > 0$ and there is no bifurcation: the upright top is stable for all values of r. If $c > 1$

$$W'(0) = 1 - c < 0 = W''(0),$$

so a "Hamiltonian Hopf" bifurcation takes place.

It has been argued that Lagrange and Kirchhoff tops undergo Hamiltonian Hopf bifurcations as the rate of spin decreases past a critical value: the sleeping top loses stability and a stable periodic orbit appears nearby [4, 8, 3]. Since in symmetric tops these periodic orbits are in fact relative equilibria it may be better to view the bifurcation as figure 8 bifurcations. These figure 8 bifurcations are typical for one degree of freedom Hamiltonian systems with $\mathbb{Z}/2$ symmetries [7].

Organization of the paper

In Section 2 we review a result of Satzer and Kummer on the symplectic quotients of T^*P where $S^1 \to P \to B$ is a principal S^1 bundle. We use it reduce the study of symmetric tops to the study of families of Hamiltonian systems on a magnetic 2-sphere. This is an old idea and I include it to keep the paper self-contained. Section 3 recalls the theory of singular symplectic reduction. We start with the developments in late 1980s – early 1990s and continue with more recent work. We then recall the notion of a C^∞-ring and formulated the differential spaces of Sikorski in terms of C^∞-rings. We describe Śniatycki's view of singular symplectic quotients as differential spaces. In Section 4 we reduce the study of families of S^1-symmetric Hamiltonian systems on a magnetic 2-sphere to one variable calculus. Section 5 proves the main result of the paper.

2. Regular S^1 reduction

It is well known that symplectic quotients of the cotangent bundle T^*P of a principal bundle $G \to P \to B$ by the lifted action of G are symplectic fiber bundles over T^*B with coadjoint orbits of G as fibers. If the group in question is abelian the proof is simpler since the fibers are points. The result is due to several people. I believe that the version below, Theorem 2.1, is mostly due to Satzer [21] and Kummer [10]. Kummer, in turn, relies on the work of Sternberg [24] on minimal coupling as reformulated by Weinstein in [26]. I learned the formulation and the proof of the theorem from Victor Guillemin in the mid 1980s.

Theorem 2.1. Let P be a manifold with a free S^1 action, $h(q,p)=\frac{1}{2}g_q^*(p,p)+V(q)$ an S^1-invariant Hamiltonian on the cotangent bundle T^*P (so g is an S^1-invariant metric on P and $V \in C^\infty(P)^{S^1}$ an invariant potential) and $\mu : T^*P \to \mathbb{R} = Lie(S^1)^*$ the associated invariant moment map. View P as a principal S^1-bundle over $B := P/S^1$ with the projection $\pi : P \to B$.

Then for any $r \in \mathbb{R}$ metric g induces a diffeomorphism

$$\varphi_r : T^*B \to T^*P/\!/_r S^1 := \mu^{-1}(r)/S^1$$

between the cotangent bundle of the base B and the symplectic quotient. Moreover

$${\varphi_r}^*\omega_r = \omega_{T^*B} + r\tau^*F$$

where ω_r is the reduced symplectic form on the quotient $T^*P/\!/_r S^1$, $F \in \Omega^2(B)$ is the curvature of the connection 1-form A on P defined by the metric g and $\tau : T^*B \to B$ the canonical projection. Finally the pullback by φ_r of the induced Hamiltonian h_r is

$$({\varphi_r}^* h_r)(b,\eta) = \frac{1}{2}\bar{g}_b^*(\eta,\eta) + \frac{1}{2}g_q^*(A_q,A_q) + V(b) \tag{2.2}$$

for all $b \in B, \eta \in T_b^*B$, where $q \in \pi^{-1}(b)$ is any point in the fiber of π above b and $\bar{g}$ is the metric induced on B by g. Note that we are identifying $V \in C^\infty(P)^{S^1}$ with the corresponding function on B.

Sketch of proof. The zero level set of $\mu : T^*P \to \mathbb{R}$ is the annihilator $\mathscr{V}^\circ$ of the vertical bundle of $\pi : P \to B$. It is not hard to show that the map

$$\psi_r : \mathscr{V}^\circ \to \mu^{-1}(r), \qquad \psi_r(q,p) = (q, p + rA_q)$$

is an S^1-equivariant diffeomorphism. Here as above $q \in P$ is a point, $p \in \mathscr{V}_q^\circ$ a covector. Hence ψ_r descends to a diffeomorphism

$$\varphi_r : \mathscr{V}^\circ/S^1 \simeq T^*B \to \mu^{-1}(r)/S^1.$$

The pullback by ψ_r of the tautological 1-form θ_{T^*P} on the cotangent bundle of P turns out to be the sum of $pr^*\theta_{T^*B}$ and $r\,A$ (where $pr : \mathscr{V}^\circ \to T^*B$ is the canonical submersion and we identified $A \in \Omega^1(P)$ with its pullback to $\mathscr{V}^\circ$):

$$\psi_r^*\theta_{T^*P} = pr^*\theta_{T^*B} + r\,A,$$

see [10]. Consequently

$$\varphi_r^*\omega_r = d\theta_{T^*B} + rdA = \omega_{T^*B} + r\tau^*F.$$

Finally equation (2.2) follows from the definition of the diffeomorphism ψ_r and the fact that the connection 1-form A is induced by the metric g. $\square$

Corollary 2.3. The symplectic quotient at $r \in \mathbb{R} = (Lie(SO(2))^*$ of a symmetric top

$$(T^*SO(3), \omega_{T^*SO(3)}, h(q,p) = \frac{1}{2}g_q^*(p,p) + V(q))$$

with respect to the action of $SO(2)$ by multiplication on the right is the classical mechanical system

$$(T^*S^2, \omega_{T^*S^2} + r\omega_{S^2}, h(q,p) = \frac{1}{2}\bar{g}_q^*(p,p) + V(q)) \tag{2.4}$$

with $SO(2)$ symmetry, where $\bar{g}$ is an $SO(3)$ invariant metric on the sphere S^2 induced by g, $\bar{g}^*$ the dual metric and $V \in C^\infty(SO(3))^{SO(2)\times SO(2)}$ is identified with an $SO(2)$ invariant function on S^2 which we again call V.

Proof. We apply Theorem 2.1 to the action of $S^1 = SO(2)$ on $P = SO(3)$ by multiplication on the right. Then $B = P/S^1$ is the standard 2-sphere S^2, the metric g is left $SO(3)$ and right $SO(2)$-invariant and $V \in C^\infty(SO(3))^{SO(2)\times SO(2)} = C^\infty(SO(3)/SO(2))^{SO(2)}$. Then the induced metric $\bar{g}$ on S^2 is $SO(3)$-invariant hence (up to a scalar multiple that we ignore) is the standard round metric on S^2. The connection 1-form A and its curvature $F \in \Omega^2(S^2)$ are both $SO(3)$-invariant. Hence F is the standard area form on S^2 (possibly up to a factor that depends on normalization that we again ignore). Finally the function

$$q \mapsto \frac{1}{2} g^*(A_q, A_q)$$

on $SO(3)$ is $SO(3)$-invariant, hence constant. There is no harm in dropping it. We conclude that under the diffeomorphism $\varphi_r : T^*S^2 \to T^*SO(3)//_r S^1$ the reduced classical mechanical system is

$$(T^*S^2, \omega_{T^*S^2} + r\omega_{S^2}, h(q,p) = \frac{1}{2}\bar{g}_q^*(p,p) + V(q)). \qquad \square$$

Remark 2.5. It is convenient to identify the cotangent bundle T^*S^2 of the 2-sphere with a submanifold of $\mathbb{R}^6$:

$$T^*S^2 = \{(q,p) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle q, q\rangle = 1, \langle q, p\rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^3$. With this identification the action of $SO(2)$ on T^*S^2 becomes identified with the restriction of the diagonal action on $\mathbb{R}^3 \times \mathbb{R}^3$ by rotations about $e_3 = (0,0,1)$. The corresponding moment map $\mu_r : (T^*S^2, \omega_{T^*S^2} + r\omega_{S^2}) \to \mathbb{R}$ is given by

$$\mu_r(q,p) = \langle q \times p, e_3\rangle + rq_3 \tag{2.6}$$

where $\times$ here is the cross product.

3. Digression: C^∞-rings and differential spaces

In the next section we will use symplectic reduction of the $SO(2)$-symmetric Hamiltonian system

$$(T^*S^2, \omega_{T^*S^2} + r\omega_{S^2}, \mu_r : T^*S^2 \to \mathbb{R}, h(q,p) = \frac{1}{2} g_q^*(p,p) + V(q))$$

at $r = \mu_r(e_3, 0)$ to analyze stability and bifurcation of the fixed point $(e_3, 0)$. Since r is a singular value of the $SO(2)$ moment map μ_r (μ_r is given by (2.6)) the reduced space $T^*S^2//_r SO(2) = \mu_r^{-1}(r)/SO(2)$ is singular. We will argue that a neighborhood of the image of $(e_3, 0)$ in $T^*S^2//_r SO(2)$ is isomorphic to $T^*((-1,1))/\mathbb{Z}_2$. To carry this out we need to explain what we mean by "isomorphic." In order to do this we will need to recall the notion of a C^∞-ring and of a differential space.

To set the stage we briefly recall the theory of singular reduction as it was developed in late 1980s – early 1990s. It has been known since 1970s that if α is a regular value of a G-equivariant moment map $\mu : M \to \mathfrak{g}^*$ that arises from a proper action of a Lie group G on a symplectic manifold (M, ω) and if the action of the stabilizer G_α of α on the level set $\mu^{-1}(\alpha)$ is free then the α-level set is a manifold and the restriction of the symplectic form $\omega|_{\mu^{-1}(\alpha)}$ descends to a symplectic form ω_α on the symplectic quotient $M//_\alpha G := \mu^{-1}(\alpha)/G_\alpha$. If additionally $h \in C^\infty(M)^G$ is an invariant Hamiltonian then its (local) flow on M is G-equivariant and preserves $\mu^{-1}(\alpha)$ hence induces a flow on the quotient $M//_\alpha G$. On the other hand $h|_{\mu^{-1}(\alpha)}$ descends to a smooth function $h_\alpha \in C^\infty(M//_\alpha G)$ and the flow of h_α on $(M//_\alpha G, \omega_\alpha)$ agrees with the flow induced by h. If the action of G_α on $\mu^{-1}(\alpha)$ is only locally free then the level set $\mu^{-1}(\alpha)$ is still a manifold, the quotient $M//_\alpha G$ is naturally a symplectic orbifold and again the flow on $M//_\alpha G$ induced by an invariant Hamiltonian h is the flow of the function h_α.

If the action of G_α on $\mu^{-1}(\alpha)$ is not locally free then the flow of an invariant Hamiltonian $h \in C^\infty(M)^G$ still preserves the level set $\mu^{-1}(\alpha)$ and induces a flow on the space $M//_\alpha G = \mu^{-1}(\alpha)/G_\alpha$. The set $M//_\alpha G$ is naturally a topological space (take the subspace topology on $\mu^{-1}(\alpha)$ and quotient topology on $\mu^{-1}(\alpha)/G_\alpha$). One refers to $M//_\alpha G$ as the reduced space at $\alpha \in \mathfrak{g}^*$ and as a (singular) symplectic quotient.

The singular symplectic quotients are highly structured:

3.0.i. The space $M//_\alpha G$ is a symplectic stratified space [22, 2, 12]. This means that the topological space $M//_\alpha G$ naturally decomposes into a collection of symplectic manifolds (these manifolds are called symplectic strata) and that the singularities of $M//_\alpha G$ are tame — see [22, 12]. More precisely for any Lie subgroup H of G the intersection $M_{(H)} \cap \mu^{-1}(\alpha)$ is a manifold ($M_{(H)}$ denotes the subset of points of M whose G-stabilizer is conjugate to H). The quotient

$$(M//_\alpha G)_{(H)} := (M_{(H)} \cap \mu^{-1}(\alpha))/G_\alpha$$

is a manifold as well and the restriction of the symplectic form ω on M to $M_{(H)} \cap \mu^{-1}(\alpha)$ descends to a symplectic form on $(M//_\alpha G)_{(H)}$. The manifolds $(M//_\alpha G)_{(H)}$ are the symplectic strata of $M//_\alpha G$.

3.0.ii. The quotient map $\surd : \mu^{-1}(\alpha) \to M//_\alpha G$ induces an isomorphism $\surd^* : C^0(M//_\alpha G) \to C^0(\mu^{-1}(\alpha))^{G_\alpha}$. Consequently the preimage of the restriction $C^\infty(M)^G|_{\mu^{-1}(\alpha)}$ under $\surd^*$ is a subalgebra of $C^0(M//_\alpha G)$

which is denoted by $C^\infty(M/\!/_\alpha G)$. Since

$$C^\infty(M/\!/_\alpha G) \simeq C^\infty(M)^G/\mathscr{I}_\alpha$$

where $\mathscr{I}_\alpha = \{f \in C^\infty(M)^G \mid f|_{\mu^{-1}(\alpha)} = 0\}$ and since $\mathscr{I}_\alpha$ is a *Poisson* ideal [1] the algebra $C^\infty(M/\!/_\alpha G)$ is naturally a Poisson algebra.

3.0.iii. The Poisson bracket on $C^\infty(M/\!/_\alpha G)$ is compatible with the symplectic forms on the strata of the reduced space $M/\!/_\alpha G$: the restriction map

$$C^\infty(M/\!/_\alpha G) \to C^\infty((M/\!/_\alpha G)_{(H)})$$

is Poisson for every subgroup $H < G$ (see [22]).

3.0.iv. The compatibility of the Poisson brackets and of the symplectic forms on the strata of $M/\!/_\alpha G$ gives two complementary ways to view the flow on the symplectic quotient $M/\!/_\alpha G$ induced by an invariant Hamiltonian $h \in C^\infty(M)^G$. One can view it as a collection of Hamiltonian flows on the strata. Alternatively the Hamiltonian $h_\alpha \in C^\infty(M/\!/G)$ induced by h defines a derivation

$$\{h_\alpha, -\} : C^\infty(M/\!/G) \to C^\infty(M/\!/G),$$

and the flowlines $\gamma(t)$ of the induced flow are integral curves of this derivation:

$$\left.\frac{d}{dt}\right|_t f \circ \gamma = \{h_\alpha, f\}(\gamma(t))$$

for all functions $f \in C^\infty(M/\!/G)$.

While the theory described above caused a bit of excitement in 1990s, a particularly nagging question remained: what were singular symplectic spaces an example of? Several decades later it seems to me that the best answer so far was found by Jedrzej Śniyaticki [23]: singular symplectic quotients are differential spaces in the sense of Sikorski or, more generally, C^∞-locally ringed spaces [9]. It is not a complete answer and more work remains to be done: see Remark 3.28 below. One can also plausibly argue that derived differential-geometric symplectic stacks would be another promising home for singular symplectic quotients. However at the moment derived differential geometry is not developed enough for the study of dynamics on singular symplectic quotients.

To explain what differential spaces are we start by recalling the notion of a C^∞-ring. The definition below is not standard, but it's equivalent to the standard one [9] and is easier to make sense of in the first pass (unless you have some background in categorical universal algebra which I don't).

Definition 3.1. A C^∞-ring is a *set* $\mathscr{C}$ together with an infinite collection of operations

$$\{g_{\mathscr{C}} : \mathscr{C}^m \to \mathscr{C}\}_{m\geq 0, g\in C^\infty(\mathbb{R}^m)}$$

(where $\mathscr{C}^0 := \{*\}$, a one-point set and $\mathbb{R}^0 := \{0\}$) so that for all $n, m \in \mathbb{N}$, all $g \in C^\infty(\mathbb{R}^m)$ and all $f_1, \ldots, f_m \in C^\infty(\mathbb{R}^n)$

$$(g \circ (f_1, \ldots, f_m))_{\mathscr{C}}(c_1, \ldots, c_n) = g_{\mathscr{C}}((f_1)_{\mathscr{C}}(c_1, \ldots, c_n), \ldots, (f_m)_{\mathscr{C}}(c_1, \ldots, c_n))$$

for all $(c_1, \ldots, c_n) \in \mathscr{C}^n$. Additionally we require that for every coordinate function $x_j : \mathbb{R}^m \to \mathbb{R}$,

$$(x_j)_{\mathscr{C}}(c_1, \ldots, c_m) = c_j.$$

Example 3.2. The algebra of functions $C^\infty(M)$ on a smooth manifold M is a C^∞-ring: for any $n > 0$, any function $f \in C^\infty(\mathbb{R}^n)$ one defines

$$f_{C^\infty(M)} : (C^\infty(M))^n \to C^\infty(M)$$

by

$$f_{C^\infty(M)}(a_1, \ldots, a_n) := f \circ (a_1, \ldots, a_n)$$

for any n-tuple of functions $a_1, \ldots, a_n \in C^\infty(M)$.

Example 3.3. The real line $\mathbb{R}$ is a C^∞ ring since it's the algebra of smooth functions on a one point manifold $*$. Explicitly, given $f \in C^\infty(\mathbb{R}^n)$ the corresponding operation $f_{\mathbb{R}} : \mathbb{R}^n \to \mathbb{R}$ "is" the function f:

$$f_{\mathbb{R}}(a_1, \ldots, a_n) := f(a_1, \ldots, a_n),$$

That is, the operation $f_{\mathbb{R}}(a_1, \ldots, a_n)$ is the evaluation of the function f on the n-tuple $(a_1, \ldots, a_n))$ of real numbers.

Remark 3.4. Any C^∞-ring $\mathscr{A}$ has an underlying $\mathbb{R}$-algebra. This is because addition and multiplication functions $f(x, y) = x+y$, $g(x, y) = xy$ are smooth functions as are multiplications by scalars $m_\lambda(x) = \lambda x$ (for all $\lambda \in \mathbb{R}$) hence define appropriate binary and unary operations on the set $\mathscr{A}$ making it into an $\mathbb{R}$ algebra.

We will not notationally distinguish between C^∞-rings and their underlying $\mathbb{R}$-algebras.

Definition 3.5. A morphism of $\mathscr{A} \to \mathscr{B}$ of C^∞-rings is a map of sets $\varphi : \mathscr{A} \to \mathscr{B}$ which preserves all the operations: for any $n > 0$, any $a_1, \dots, a_n \in \mathscr{A}$ and any $f \in C^\infty(\mathbb{R}^n)$

$$\varphi(f_{\mathscr{A}}(a_1, \dots, a_n)) = f_{\mathscr{B}}(\varphi(a_1), \dots, \varphi(a_n)).$$

Definition 3.6. A point of a C^∞-ring $\mathscr{A}$ is a map of C^∞-rings $p : \mathscr{A} \to \mathbb{R}$.

Example 3.7. Let M be a manifold, $p \in M$ a point and $ev_p : C^\infty(M) \to \mathbb{R}$ the evaluation at p: $ev_p(f) = f(p)$. Then ev_p is a point of the C^∞-ring $C^\infty(M)$.

Definition 3.8. An ideal in a C^∞ ring $\mathscr{A}$ is an ideal in the underlying $\mathbb{R}$-algebra.

The following theorem is useful in defining a C^∞-ring structure on singular symplectic quotients.

Theorem 3.9. Let $\mathscr{A}$ be a C^∞-ring and $I \subset \mathscr{A}$ an ideal in the underlying $\mathbb{R}$-algebra. Then the quotient $\mathbb{R}$-algebra $\mathscr{A}/I$ is naturally a C^∞-ring: for any $n > 0$ and any function $f \in C^\infty(\mathbb{R}^n)$ the map

$$f_{\mathscr{C}/I} : (\mathscr{A}/I)^n \to \mathscr{A}/I, \qquad f_{\mathscr{C}/I}(c_1 + I, \dots, c_n + I) := f_{\mathscr{C}}(c_1, \dots, c_n) + I$$

is well-defined for all $(c_1 + I, \dots, c_n + I) \in (\mathscr{A}/I)^n$.

Proof. The result is well-known. See [16] or [9] for a proof. $\square$

Definition 3.10. A C^∞-ring $\mathscr{A}$ is point determined if points separate elements of $\mathscr{A}$. That is, if $a \in \mathscr{A}$ and $a \neq 0$ then there is a point $p : \mathscr{A} \to \mathbb{R}$ so that $p(a) \neq 0$.

Remark 3.11. There are many C^∞-rings that are not point determined. The simplest example is the quotient ring $C^\infty(\mathbb{R})/\langle x^2 \rangle$ where $\langle x^2 \rangle$ is the ideal generated by the function x^2. This ring has only one point p which is given by

$$p(f + \langle x^2 \rangle) = f(0).$$

See [16].

Definition 3.12. A differential space (in the sense of Sikorski) is a pair $(N, C^\infty(N))$ where N is a topological space and $C^\infty(N)$ is a set of real-valued function on N subject to the following three conditions (the notation $C^\infty(N)$ is meant to be suggestive of a smooth structure on the space N):

3.12.i. The topology on N is the smallest topology making every function in $C^\infty(N)$ continuous.

3.12.ii. For any $n > 0$, any smooth function $f \in C^\infty(\mathbb{R}^n)$ and any n-tuple $a_1, \dots, a_n \in C^\infty(N)$, the composite $f \circ (a_1, \dots, a_n)$ is in $C^\infty(N)$.

3.12.iii. For any open cover $\{U_i\}_{i \in I}$ and any function $g : N \to \mathbb{R}$ so that for each $i \in I$ there is $a_i \in C^\infty(N)$ with $g|_{U_i} = a_i|_{U_i}$ the function g is in $C^\infty(N)$.

Remark 3.13. Condition (3.12.i) amounts to requiring that the sets $\{\{f \neq 0\} \mid f \in C^\infty(N)\}$ generate the topology on N. This may be viewed as a C^∞-analogue of the Zariski topology on affine varieties. One can also show that (3.12.i) is equivalent to existence of bump functions: for any open set $U \subset N$ and for any point $x \in U$ there is a function $f \in C^\infty(N, [0, 1])$ with $\operatorname{supp} f \subset U$ and f identically 1 near x.

Condition (3.12.ii) amounts to saying that the $\mathbb{R}$-algebra $C^\infty(N)$ is in fact a C^∞-ring. For some reason most papers that deal with differential spaces never explicitly mention C^∞-rings. Note that the C^∞-ring $C^\infty(N)$ is point determined since it consists of actual functions and for any point $p \in N$ the evaluation map $ev_p : C^\infty(N) \to \mathbb{R}$ is a point of the C^∞-ring $C^\infty(N)$.

The third condition can be interpreted as follows: by restricting the functions in $C^\infty(N)$ to open subsets of N one obtains a presheaf. Denote its sheafication by C^∞_N. Condition (3.12.iii) then amounts to requiring that the C^∞-ring of global section $C^\infty_N(N)$ of this sheaf is the C^∞-ring $C^\infty(N)$. Note that in particular a differential space is implicitly a C^∞-ringed space.

Definition 3.14. A map or a morphism from a differential space $(M, C^\infty(M))$ to a differential space $(N, C^\infty(N))$ is a map of underlying sets $\varphi : M \to N$ so that for any $f \in C^\infty(N)$ the composite $f \circ \varphi$ is in $C^\infty(M)$.

Notation 3.15. Differential spaces and their morphisms form a category which we denote by DiffSp.

The reader unfamiliar with C^∞-schemes should feel free to ignore the following remark. It will play no role in the paper.

Remark 3.16. It is not too hard to show that the category of differential spaces embeds into the larger category of C^∞-ringed spaces. C^∞-schemes [5, 9] also embed into the category of C^∞-ringed spaces. It is not at all clear which differential spaces are C^∞-schemes and conversely.

A possible exception is formed by affine schemes that come from finitely generated and point determined C^∞-rings. These are exactly the differential spaces that are isomorphic to closed subsets of Euclidean spaces [11]. This class includes all second countable Hausdorff manifolds.

Part of the problem of comparing differential spaces and affine C^∞-schemes is that for given a differential space $(N, C^\infty(N))$ it is not clear that a point $p : C^\infty(N) \to \mathbb{R}$ of the C^∞-ring $C^\infty(N)$ has to come from evaluation at a point $x \in N$. If N is a second countable manifold then any point $p : C^\infty(N) \to \mathbb{R}$ of the C^∞-ring $C^\infty(N)$ does come from an evaluation at some point $x \in N$ by the famous "Milnor's exercise."

A vector field v on a manifold M can be defined as a derivation $v : C^\infty(M) \to C^\infty(M)$ of the $\mathbb{R}$-algebra of smooth functions on M with values in $C^\infty(M)$: v is $\mathbb{R}$-linear and for any two functions $f, g \in C^\infty(M)$

$$v(fg) = v(f)g + fv(g).$$

For C^∞-ring $\mathscr{A}$ there is another notion of a derivation of $\mathscr{A}$ with values in $\mathscr{A}$:

Definition 3.17. A C^∞-derivation of a C^∞-ring $\mathscr{A}$ is a map $X : \mathscr{A} \to \mathscr{A}$ so that for any $n > 0$, any $f \in C^\infty(\mathbb{R}^n)$ and any $a_1, \dots, a_n \in \mathscr{A}$

$$X(f_{\mathscr{A}}(a_1, \dots, a_n)) = \sum_{i=1}^{n} (\partial_i f)_{\mathscr{A}}(a_1, \dots, a_n)\, X(a_i).$$

Remark 3.18. One can show that for a large class of C^∞-rings that includes point determined C^∞-rings the two notions of derivations coincide. See [27]. Thus if $(N, C^\infty(N))$ is a differential space then an $\mathbb{R}$-algebra derivation $v : C^\infty(N) \to C^\infty(N)$ is a C^∞-ring derivation.

We are now in position to define integral curves of derivations on differential spaces.

Definition 3.19. Let $v : C^\infty(M) \to C^\infty(M)$ be a derivation on a differential space M. An integral curve γ of v through a point $p \in M$ is either a map $\gamma : \{0\} \to M$ with $\gamma(0) = p$ or a smooth map $\gamma : (J, C^\infty(J)) \to (M, C^\infty(M))$ from an interval $J \subset \mathbb{R}$ containing 0 so that

$$\frac{d}{dx}(f \circ \gamma) = v(f) \circ \gamma \tag{3.20}$$

for all function $f \in C^\infty(M)$. Note that unlike [23] we do *not* require J to be an open interval.

The reader may be puzzled why we allow integral curves to only exist for time $t = 0$ or to have non-open intervals as domains of definition. Example 3.21 below illustrates why this may be useful. Note that for singular symplectic quotients $M/\!/_\alpha G$ and for Hamiltonian $h_\alpha \in C^\infty(M/\!/_\alpha G)$ induced by $h \in C^\infty(M)^G$ the maximal integral curves of the derivation

$$\{h_\alpha, \cdot\} : C^\infty(M/\!/_\alpha G) \to C^\infty(M/\!/_\alpha G)$$

are maximal integral curves of vector fields on manifolds and so have open intervals as domains of definition.

Example 3.21. Let M be the standard closed disk in $\mathbb{R}^2$: $M = \{(x, y) \mid x^2 + y^2 \leq 1\}$. Then M is a manifold with boundary and a differential space. Consider the vector field $v = \frac{\partial}{\partial x}$ on M. The integral curve of v through $(0, 1)$ is $\gamma : \{0\} \to M$, $\gamma(0) = (0, 1)$; it only exists for zero time. Note that v does have a flow. It's a smooth map Φ from $U = \{((x, y), t) \in \mathbb{R}^2 \times \mathbb{R} \mid x^2 + y^2 \leq 1, (x - t)^2 + y^2 \leq 1\} \to M$. It is given by $\Phi((x, y), t) = (x + t, y)$. Note that here we view $U \subset \mathbb{R}^3$ as a differential space. Note also that U is not a manifold with corners as one can see by looking at its singularities.

As in the case of manifolds related derivations have related integral curves:

Lemma 3.22. Let $\varphi : (M, C^\infty(M)) \to (N, C^\infty(N))$ be a map of differential spaces, $X : C^\infty(M) \to C^\infty(M)$, $Y : C^\infty(N) \to C^\infty(N)$ two derivations so that the diagram

$$\begin{tikzcd}
C^\infty(M) \arrow[r, "X"] & C^\infty(M) \\
C^\infty(N) \arrow[r, "Y"] \arrow[u, "\varphi^*"] & C^\infty(N) \arrow[u, "\varphi^*"']
\end{tikzcd}$$

commutes. I.e., for all $f \in C^\infty(N)$

$$Y(f) \circ \varphi = X(f \circ \varphi).$$

Then for any integral curve $\gamma : I \to M$ of X, $\varphi \circ \gamma : I \to N$ is an integral curve of Y.

Proof. The proof is the same as in the case of manifolds:

$$\frac{d}{dt}(f \circ \varphi \circ \gamma) = X(f \circ \varphi) \circ \gamma = Y(f) \circ \varphi \circ \gamma. \qquad \square$$

We now come to the main reason for discussing differential spaces, which is a theorem due to Śniatycki (see [23], for example).

Theorem 3.23 (Śniatycki)**.** Let (M, ω) be a symplectic manifold with a proper Hamiltonian action of a Lie group G and corresponding equivariant moment map $\mu : M \to \mathfrak{g}^*$. Then for any $\alpha \in \mathfrak{g}^*$ the quotient

$$M//_\alpha G := \mu^{-1}(\alpha)/G_\alpha$$

is a Hausdorff differential space with the space of smooth functions

$$C^\infty(M/_\alpha G) := C^\infty(M)^G|_{\mu^{-1}(\alpha)}.$$

Sketch of proof. Since the action of G is proper and the stabilizer G_α is closed in G, the action of G_α on M is proper. Consequently M/G_α and its closed subset $\mu^{-1}(\alpha)/G_\alpha$ are Hausdorff.

It is easy to check that the $\mathbb{R}$-algebra $C^\infty(M)^G$ of G-invariant functions is a C^∞-ring: if $a_1, \dots, a_n \in C^\infty(M)^G$ are invariant functions and $f \in C^\infty(\mathbb{R}^n)$ then the composite $f \circ (a_1, \dots, a_n)$ is also G-invariant. Since

$$\mathscr{I}_\alpha := \{f \in C^\infty(M)^G \mid f|_{\mu^{-1}(\alpha)}\} = 0$$

is an ideal in $C^\infty(M)^G$ the quotient $C^\infty(M)^G/\mathscr{I}_\alpha$ is a C^∞-ring (see Theorem 3.9). Hence

$$C^\infty(M)^G|_{\mu^{-1}(\alpha)} = C^\infty(M)^G/\mathscr{I}_\alpha$$

is a C^∞-ring.

Conditions (3.12.i) and (3.12.iii) follow from the existence of a G-invariant partition of unity on M subordinate to a G-invariant open cover of M. □

Definition 3.24. Two reduced spaces $M//_\alpha G$ and $N//_\beta H$ are isomorphic if there is an isomorphism

$$\varphi : (M//_\alpha G, C^\infty(M//_\alpha G)) \to (N//_\beta H, C^\infty(N//_\beta H))$$

of differential spaces (cf. Definition 3.14) so that

$$\varphi^* : C^\infty(N//_\beta H) \to C^\infty(M//_\alpha G)$$

is an isomorphism of Poisson algebras (see 3.0.ii).

Lemma 3.25. Suppose $\varphi : (M//_\alpha G, C^\infty(M//_\alpha G) \to (N//_\beta H, C^\infty(N//_\beta H))$ is an isomorphism of reduced spaces and $h \in C^\infty(N//_\beta H))$ a Hamiltonian. Then φ sends the integral curves of $\varphi^* h$ (meaning the integral curves of the derivation $\{\varphi^* h, \cdot\} : C^\infty(M//_\alpha G) \to C^\infty(M//_\alpha G)$) to the integral curves of h.

Proof. The proof follows easily from the definition of an isomorphism of reduced spaces and Lemma 3.22. □

Corollary 3.26. Suppose $\varphi : (M//_\alpha G, C^\infty(M//_\alpha G) \to (N//_\beta H, C^\infty(N//_\beta H))$ is an isomorphism of reduced spaces and $h \in C^\infty(N//_\beta H))$ a Hamiltonian as in Lemma 3.25 above. Then

- A point $x \in N/\!/_\beta H, C^\infty(N/\!/_\beta H)$ is a stable equilibrium of h if and only if $\varphi^{-1}(x)$ is a stable equilibrium of $\varphi^* h$.
- A point $x \in N/\!/_\beta H, C^\infty(N/\!/_\beta H)$ is an unstable equilibrium of h if and only if $\varphi^{-1}(x)$ is an unstable equilibrium of $\varphi^* h$.

The final bit of theory that we'll need to analyze the stability of tops is a theorem Montaldi [17]:

Theorem 3.27. Let (M, ω) be a symplectic manifold with a Hamiltonian action of a compact Lie group G and corresponding equivariant moment map $\mu : M \to \mathfrak{g}^*$. Let $h \in C^\infty(M)^G$ be an invariant Hamiltonian, $\alpha \in \mathfrak{g}^*$, $pr : \mu^{-1}(\alpha) \to M/\!/_\alpha G$ the quotient map and $h_\alpha \in C^\infty(M/\!/_\alpha G)$ the reduced Hamiltonian (so $pr^* h_\alpha = h|_{\mu^{-1}(\alpha)}$.) Suppose $x \in \mu^{-1}(\alpha)$ is a point so that $pr(x)$ is a local minimum or a local maximum of h_α. Then x is relative equilibrium of h which is a G-stable in M.

We end the section with a parenthetic remark on differential spaces and singular reduction. The remark will play no role in the rest of the paper.

Remark 3.28. The differential space approach to singular reduction is not a complete answer for the following annoying reason.

By a theorem of Dubuc and Kock [6] for any C^∞-ring $\mathcal{A}$ there exists a module of C^∞-Kähler differentials $\Omega^1_{\mathcal{A}}$ together with a universal derivation $d_{\mathcal{A}} : \mathcal{A} \to \Omega^1_{\mathcal{A}}$: for any $\mathcal{A}$-module $\mathcal{M}$ and any derivation $X : \mathcal{A} \to \mathcal{M}$ there exists a unique $\mathcal{A}$-module map $\varphi_X : \Omega^1_{\mathcal{A}} \to \mathcal{M}$ so that $X = \varphi_X \circ d_{\mathcal{A}}$. The universal derivation $d_{\mathcal{A}}$ is functorial in $\mathcal{A}$. One can further mimic Grothendieck's algebraic de Rham complex and produce an C^∞-algebraic de Rham complex

$$\Omega^\bullet_{\mathcal{A}} \xrightarrow{d} \Omega^{\bullet+1}_{\mathcal{A}},$$

see [11]. Moreover if $\mathcal{A} = C^\infty(M)$ for a manifold M then $\Omega^\bullet_{C^\infty(M)}$ is the usual de Rham complex (this is not obvious). Since the complex $\Omega^\bullet_{\mathcal{A}}$ is also functorial in $\mathcal{A}$, for any closed subset Z of a manifold M the the surjective restriction map

$$C^\infty(M) \to C^\infty(Z), \quad f \mapsto f|_Z$$

extends to a surjective map $\Omega(M)^\bullet \to \Omega^\bullet_{C^\infty(Z)}$ of differential graded algebras. In particular if $\mu : (M, \omega) \to \mathfrak{g}^*$ is an equivariant moment map for a Hamiltonian action of a Lie group G on a symplectic manifold then the restriction of the 2-form ω to the zero level set $\mu^{-1}(0)$ makes perfect sense: $\omega|_Z$ is a closed 2-form in the C^∞-algebraic de Rham complex of the C^∞-ring $C^\infty(\mu^{-1}(0))$. This is true regardless of whether or not $\mu^{-1}(0)$ is a manifold. Furthermore, as we have seen, the quotient $\mu^{-1}(0)/G$ is a C^∞-ring and the quotient map $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ is a smooth map of differential spaces. The trouble comes from the fact that unlike the regular case the 2-form $\omega|_{\mu^{-1}(0)}$ need not descend to a 2-form on the quotient $\mu^{-1}(0)/G$: there need not exist any 2-form $\sigma \in \Omega^2_{C^\infty(\mu^{-1}(0)/G)}$ with

$$\pi^*\sigma = \omega|_{\mu^{-1}(0)}.$$

Here is a simple example: let $(M, \omega) = (\mathbb{R}^2, dx \wedge dy)$, $G = \{\pm 1\}$ acting by $(-1) \cdot (x, y) = (-x, -y)$. Then the moment map μ is identically 0 and $\mu^{-1}(0)/G = \mathbb{R}^2/\{\pm 1\}$. The C^∞-ring of smooth functions on the quotient "is" the ring of invariant functions $C^\infty(\mathbb{R}^2)^{\{\pm 1\}}$. With this identification the pullback $\pi^* : C^\infty(\mathbb{R}^2/G) \to C^\infty(\mathbb{R}^2)$ is simply the inclusion $C^\infty(\mathbb{R}^2)^{\{\pm 1\}} \hookrightarrow C^\infty(\mathbb{R}^2)$. By a theorem of G. Schwarz the C^∞-ring of invariant functions is generated by x^2, xy and y^2. Consequently the module of 1-forms $\Omega^1_{C^\infty(\mathbb{R}^2)^{\{\pm 1\}}}$ on the quotient is generated by xdx, ydy and $xdy + ydx$. It follows that there is no 2-form $\sigma \in \Omega^2_{C^\infty(\mathbb{R}^2)^{\{\pm 1\}})}$ with $\pi^*\sigma = dx \wedge dy$.

There are several ways to fix the problem. For example on can replace the "coarse" quotient $\mathbb{R}^2/G$ by the stack quotient $[\mathbb{R}^2/G]$. Then the G-invariant 2-form $dx \wedge dy$ does descend to a closed 2-form on the stack quotient.

The example suggests to me that in order to fully understand singular symplectic quotients one would need understand Hamiltonian dynamics on stacks. For Deling-Mumford stacks over a site of manifolds this has been done. But one would need to consider Artin stacks over a site of differential spaces or maybe over a site of C^∞-affine schemes.

4. Reduction to one variable calculus

We are now back to studying the family (2.4) of $SO(2)$-symmetric Hamiltonian systems on the cotangent bundle T^*S^2 of the 2-sphere. Since we are interested in the behavior of straight up tops, we may restrict our attention to the upper hemisphere

$$S^2_+ := \{(q_1, q_2, q_3) \in \mathbb{R}^3 \mid q_3 > 0\}.$$

The projection

$$\psi : S^2_+ \to \mathbb{R}^2, \qquad \psi(q) = (q_1, q_2) \tag{4.1}$$

defines a coordinate chart on S^2. Denote the corresponding coordinates on $T^*S^2_+$ by (x, y, p_x, p_y). In these coordinates the symplectic form ω_r is given by

$$\omega_r = dx \wedge dp_x + dy \wedge dp_y + \frac{r}{\sqrt{1-x^2-y^2}} dx \wedge dy,$$

the $SO(2)$ moment map μ_r is

$$\mu_r(x, y, p_x, p_y) = xp_y - yp_x + r\sqrt{1-x^2-y^2},$$

the metric g is given by the matrix

$$g = \begin{pmatrix} 1+\frac{x^2}{z^2} & \frac{xy}{z^2} \\ \frac{xy}{z^2} & 1+\frac{y^2}{z^2} \end{pmatrix}$$

and the dual metric g^* by the inverse matrix

$$g^{-1} = z^2 \begin{pmatrix} 1+\frac{y^2}{z^2} & -\frac{xy}{z^2} \\ -\frac{xy}{z^2} & 1+\frac{x^2}{z^2} \end{pmatrix}.$$

Here and below

$$z = \sqrt{1-x^2-y^2}.$$

Consequently the Hamiltonian h of (2.4) in these coordinates is

$$h(x, y, p_x, p_y) = \frac{1}{2}\left((1-x^2)p_x^2 - 2xyp_xp_y + (1-y^2)p_y^2\right) + V(x, y, \sqrt{1-x^2-y^2}).$$

Equivalently the diffeomorphism

$$\varphi : D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\} \to S^2_+, \qquad \varphi(x, y) = (x, y, \sqrt{1-x^2-y^2})$$

induces a diffeomorphism $\Phi : T^*D^2 \to T^*S^2_+$ of the cotangent bundles,

$$\Phi^*\omega_r = dx \wedge dp_x + dy \wedge dp_y + \frac{r}{\sqrt{1-x^2-y^2}} dx \wedge dy$$

while

$$(\mu_r \circ \Phi)(x, y, p_x, p_y) = xp_y - yp_x + r\sqrt{1-x^2-y^2}$$

and

$$\begin{aligned} (4.2) \quad & h \circ \Phi\,(x, y, p_x, p_y) \\ & = \frac{1}{2}\left((1-x^2)p_x^2 - 2xyp_xp_y + (1-y^2)p_y^2\right) + V(x, y, \sqrt{1-x^2-y^2}). \end{aligned}$$

Lemma 4.3. Consider the map $f : T^*(-1,1) \to D^2 \times \mathbb{R}^2 \subset \mathbb{R}^4$ given by

$$f(u, p_u) = (u, 0, p_u, \frac{r}{u}(1 - \sqrt{1 - u^2}))$$

for $u \neq 0$ and by $f(0, p_u) = (0, 0, p_u, 0)$. Then f is C^∞, $f(T^*(-1,1)) \subset \mu_r^{-1}(r)$, and f induces an isomorphism

$$f^* : C^\infty(T^*S^2_+)^{SO(2)}|_{\mu_r^{-1}(r)} \to C^\infty(T^*(-1,1))^{\mathbb{Z}/2}$$

of C^∞-rings and of Poisson algebras. Here and below $\mathbb{Z}/2 = \{\pm 1\}$ acts on $(-1,1)$ by multiplication and by the lifted action on $T^*(-1,1)$. Finally the smooth function $f : T^*(-1,1) \to \mu_r^{-1}(r)$ induces an isomorphism

$$\bar{f} : T^*(-1,1)/(\mathbb{Z}/2) \to \mu_r^{-1}(r)/SO(2) = T^*S^2_+//_r SO(2) \tag{4.4}$$

of symplectic reduced spaces. (We view $T^*(-1,1)/(\mathbb{Z}/2)$ as the reduction at zero of the cotangent bundle $(T^*(-1,1), \omega_{T^*(-1,1)})$ by the lifted action of $\mathbb{Z}/2$.)

Proof. Since $\sqrt{1+x}$ is analytic for $|x| < 1$ and since $\sqrt{1+x} = 1 + \frac{1}{2}x + h.o.t.$, the function $u \mapsto \frac{1-\sqrt{1-u^2}}{u}$ is analytic for $|u| < 1$. Hence f is C^∞.

Given $(x, y, p_x, p_y) \in T^*S^2_+$ there is $A \in SO(2)$ so that $A \cdot (x, y, p_x, p_y) = (x', 0, p'_x, p'_y)$ for some x', p'_x, p'_y with x' unique up to sign. Also if $(x, y, p_x, p_y) \in \mu_r^{-1}(r)$ and $y = 0$ then $xp_y + r\sqrt{1 - x^2} = r$ hence $p_y = r\frac{1-\sqrt{1-x^2}}{x}$. It follows that $f(T^*(-1,1)) \subset \mu_r^{-1}(r)$ and that the $SO(2)$ orbits in $\mu_r^{-1}(r) \smallsetminus \{(0,0,0,0)\}$ intersect the image of f in exactly two points. Hence the composite map $\pi \circ f : T^*(-1,1) \to \mu_r^{-1}(r)/SO(2)$ (where $\pi : \mu_r^{-1}(r) \to \mu_r^{-1}(r)/SO(2)$ is the orbit map) descends to a continuous bijection $\bar{f} : T^*(-1,1)/(\mathbb{Z}/2) \to \mu_r^{-1}(r)/SO(2)$.

Since for any $h \in C^\infty(T^*S^2_+)^{SO(2)}$ the pullback $f^*h \in C^\infty(T^*(-1,1))^{\mathbb{Z}/2}$ the map $\bar{f}$ is a map of differential spaces. Since for any $(u, p_u) \in T^*(-1,1)$

$$f((\mathbb{Z}/2) \cdot (u, p_u)) = (SO(2) \cdot f(u, p_u)) \cap \mu_r^{-1}(r)$$

the pullback map $f^* : C^\infty(T^*S^2_+)^{SO(2)}|_{\mu_r^{-1}(r)} \to C^\infty(T^*(-1,1))^{\mathbb{Z}/2}$ is injective. It remains to prove that f^* is bijective and preserves the Poisson brackets.

The preservation of brackets follows from the fact that

$$f^*\left(dx \wedge dp_x + dy \wedge dp_y + \frac{r}{\sqrt{1 - x^2 - y^2}} dx \wedge dy\right) = du \wedge dp_u$$

and (3.0.iii).

We now argue that $f^* : C^\infty(T^*D^2)^{SO(2)} \to C^\infty(T^*(-1,1))^{\mathbb{Z}/2}$ is onto. Thanks to Theorem 1 of [20] we know that for a representation $G \to GL(V)$ of a compact Lie group on a finite dimensional real vector space the C^∞-ring $C^\infty(V)^G$ of invariant functions is finitely generated: there is $k > 0$ and $\sigma_1, \ldots, \sigma_k \in C^\infty(V)^G$ so that for any $a \in C^\infty(V)^G$ there is a smooth function $f \in C^\infty(\mathbb{R}^n)$ with

$$a = f_{C^\infty(V)^G}(\sigma_1, \ldots, \sigma_k) = f \circ (\sigma_1, \ldots, \sigma_k).$$

Moreover the generators may be taken to be the generators of ring of invariant polynomials $\mathbb{R}[V]^G$ on V. For the lift of the standard action of $SO(2)$ on $\mathbb{R}^2$ to $T^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ the C^∞-ring $C^\infty(T^*\mathbb{R}^2)^{SO(2)}$ is generated by four polynomials:

$$x^2 + y^2, \quad p_x^2 + p_y^2, \quad xp_x + yp_y, \quad xp_y - yp_x.$$

For the action of $\mathbb{Z}/2$ on $T^*R = \mathbb{R}^2$ the C^∞-ring $C^\infty(T^*\mathbb{R})^{\mathbb{Z}/2}$ is generated by the three polynomials:

$$u^2, \quad p_u^2, \quad up_u.$$

Observe that $u^2 = f^*(x^2 + y^2)$, $up_u = f^*(xp_x + yp_y)$ while $f^*(xp_y - yp_x) = r(1 - \sqrt{1 - u^2})$. Hence

$$p_u^2 = f^* \left((p_x^2 + p_y^2) - (xp_y - yp_x)^2(x^2 + y^2) \right).$$

It follows that

$$C^\infty(T^*\mathbb{R})^{\mathbb{Z}/2}\Big|_{T^*(-1,1)} = f^* \left(C^\infty(T^*\mathbb{R}^2)^{SO(2)}\Big|_{\mu_r^{-1}(r)} \right).$$

This doesn't quite prove surjectivity of $f^* : C^\infty(T^*D^2)^{SO(2)} \to C^\infty(T^*(-1,1))^{\mathbb{Z}/2}$ since $C^\infty(T^*(-1,1))^{\mathbb{Z}/2}$ is bigger than $C^\infty(T^*\mathbb{R})^{\mathbb{Z}/2}\Big|_{T^*(-1,1)}$. On the other hand, since $T^*(-1,1)$ is open in $T^*\mathbb{R}$ the Localization Theorem of Muñoz Díaz and Ortega [18] (see also [19, p. 28]) implies that given a function $k \in C^\infty(T^*(-1,1))^{\mathbb{Z}/2}$ there exist $g, h \in C^\infty(T^*\mathbb{R})$ so that

$$\left(h|_{T^*(-1,1)} \right) k = g|_{T^*(-1,1)}$$

and $h|_{T^*(-1,1)}$ is invertible in $C^\infty(T^*(-1,1))$. By averaging over $\mathbb{Z}/2$ if necessary we may assume that g and h are in $C^\infty(T^*\mathbb{R})^{\mathbb{Z}_2}$. This implies that there are $\tilde{g}, \tilde{h} \in C^\infty(T^*\mathbb{R}^2)^{SO(2)}|_{\mu_r^{-1}(r)}$ with

$$h|_{T^*(-1,1)} = f^*\tilde{h}, \quad g|_{T^*(-1,1)} = f^*\tilde{g}.$$

Therefore

$$k = \frac{g|_{T^*(-1,1)}}{h|_{T^*(-1,1)}} = f^*\left(\frac{\tilde{g}}{\tilde{h}}\right)$$

and we are done. □

Lemma 4.5. If $u \in (-1,1)$ is a critical point of the function

$$U_r(u) = \frac{r^2}{2}\left(\frac{1-\sqrt{1-u^2}}{u}\right)^2 + V(u, 0, \sqrt{1-u^2})$$

(where $r \in \mathbb{R}$ is a parameter) then $((u,0),(0, r\frac{1-\sqrt{1-u^2}}{u})) \in T^*S^2_+ \subseteq T^*S^2$ is a relative equilibrium of the $SO(2)$-invariant Hamiltonian system (2.4) (where we used the coordinates (4.1) on the upper hemisphere and the induced coordinates on its cotangent bundle).

Moreover if u is a local minimum of $U_r(u)$ then the corresponding relative equilibrium is relatively stable and if u is a local maximum of $U_r(u)$ then the corresponding relative equilibrium is unstable.

Proof. It follows from Corollary 3.26 that in order to analyze the stability of the relative equilibria of (2.4) near the straight up position it is enough to analyze relative equilibria of

$$(T^*(-1,1)/(\mathbb{Z}/2), \mathfrak{h}_r := f^*(h \circ \Phi) \in C^\infty(T^*(-1,1))^{\mathbb{Z}/2}),$$

where $h\circ\Phi \in C^\infty(T^*D^2)^{SO(2)}$ is given by (4.2). That is, it's enough to analyze the critical points of the function $\mathfrak{h}_r$ on the manifold $T^*(-1,1)$ (and remember not to double count since we want to analyze the equilibria on the quotient $T^*(-1,1)/(\mathbb{Z}/2)$). Since

$$\begin{aligned}(4.6)\quad \mathfrak{h}_r(u,p_u) &= f^*(h\circ\Phi)(u,p_u)\\ &= \frac{1}{2}\left((1-u^2)p_u^2 + r^2\left(\frac{1-\sqrt{1-u^2}}{u}\right)^2\right) + V(u,0,\sqrt{1-u^2})\\ &= \frac{1}{2}(1-u^2)p_u^2 + \frac{r^2}{2}\left(\frac{1-\sqrt{1-u^2}}{u}\right)^2 + V(u,0,\sqrt{1-u^2})\end{aligned}$$

the critical points of $\mathfrak{h}_r$ are of the form $(u, 0)$ where u is a critical point of the effective potential

$$U_r(u) = \frac{r^2}{2}\left(\frac{1-\sqrt{1-u^2}}{u}\right)^2 + V(u, 0, \sqrt{1-u^2}).$$

By Montaldi's theorem (Theorem 3.27) any extremal points of the Hamiltonian $\mathfrak{h}_r$ correspond to stable relative equilibria of the system (2.4) and any unstable equilibria of $\mathfrak{h}_r$ correspond to unstable relative equilibria of (2.2).

If u is a local minimum of U_r then $(u, 0)$ is a minimum of $\mathfrak{h}_r$, hence corresponds to a stable relative equilibrium of (2.4). On the other hand if u is a local maximum of U_r, then $(u, 0)$ is a saddle point of $\mathfrak{h}_r$ and therefore corresponds to an unstable relative equilibrium of (2.4). $\square$

Remark 4.7. Note that, as we observed in the introduction, there is a unique function $W \in C^\infty([0, 1))$ so that

$$W(u^2) = V(u, 0, \sqrt{1-u^2}).$$

Therefore the effective potential $U_r(u)$ that we need to analyze is of the form

$$U_r(u) = \frac{r^2}{2}\left(\frac{1-\sqrt{1-u^2}}{u}\right)^2 + W(u^2) \tag{4.8}$$

for some function $W \in C^\infty([0, 1))$ that depends on the top we are studying (cf. the introduction). We analyze (4.8) in the next section.

5. Analysis of critical points of $U_r(u) = \frac{r^2}{2}(\frac{1-\sqrt{1-u^2}}{u})^2 + W(u^2)$ and a proof of Theorem 1.1

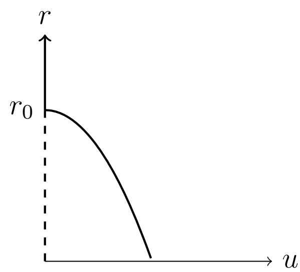
Lemma 5.1. Let $W \in C^\infty([0, 1))$ be a smooth function and $U_r \in C^\infty((-1, 1))$ be given by

$$U_r(u) = \frac{r^2}{2}\left(\frac{1-\sqrt{1-u^2}}{u}\right)^2 + W(u^2).$$

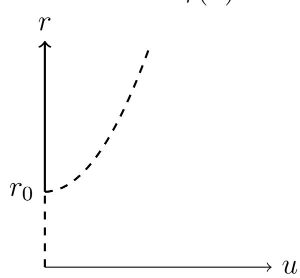
Without loss of generality assume that $0 \le r$.

5.1.i. If $W'(0) > 0$ then $u = 0$ is a local minimum of U_r for all r and there are no other critical points of U_r near $u = 0$.

5.1.ii. Suppose $W'(0)<0$ and $W''(0)>W'(0)$. Then for $r>r_0:=\sqrt{-8W'(0)}$ the point $u=0$ is a local minimum of $U_r(0)$ and there are no other critical points of $U_r(u)$ for $|u|\ll 1$. For $r<r_0$ the point $u=0$ is a local maximum of the function $U_r(u)$ and there is a continuous function $u(r)$ defined for $r<r_0$ and $|r-r_0|\ll 1$ so that $u=u(r)$ is a local minimum of $U_r(u)$:



5.1.iii. Suppose $W'(0)<0$ and $W''(0)<W'(0)$. Then for $r<r_0:=\sqrt{-8W'(0)}$ the point $u=0$ is a local maximum of $U_r(u)$. For $r>r_0$ the point $u=0$ is a local minumum and there is a continuous function $u(r)$ defined for $r>r_0$ and $|r-r_0|\ll 1$ so that $u=u(r)$ is a local maximum of $U_r(u)$:



Proof. The function

$$v(u)=\begin{cases}\frac{1-\sqrt{1-u^2}}{u} & \text{if } u\neq 0\\ 0 & \text{if } u=0\end{cases}$$

is analytic for $|u|<0$ and is invertible with the inverse $k(v)=\frac{2v}{1+v^2}$. Then

$$U_r(k(v))=\frac{r^2}{2}v^2+W(k(v)^2)=\frac{r^2}{2}v^2+W\left(\frac{4v^2}{(1+v^2)^2}\right).$$

Let

$$f(s)=W\left(\frac{4s}{(1+s)^2}\right)$$

and

$$S(r,v) := U_r(k(v)) = \frac{r^2}{2}v^2 + f(v^2).$$

Then

$$f'(0) = 4W'(0) \quad \text{and} \quad f''(0) = 16(W''(0) - W'(0)).$$

Now

$$\partial_v S = r^2 v + 2vf'(v^2) = v(r^2 + 2f'(v^2))$$

and

$$\partial_v^2 S = r^2 + 2f'(v^2) + 4v^2 f''(v^2).$$

Suppose that $f'(0) > 0$ Then $r^2 + 2f'(v^2) > 0$ for all $|v| \ll 1$ and

$$\partial_v^2 S(r,0) = r^2 + 2f'(0) > 0.$$

It follows that $v = 0$ is an isolated local minimum of $S_r(v) = S(r,v)$ for all $r > 0$. Therefore if $W'(0) > 0$ then $u = 0$ is an isolated local minimum of U_r for all r. This proves (5.1.i).

Suppose next that $f'(0) < 0$ Then for $r > r_0 = \sqrt{-2f'(0)}$ $(= \sqrt{-8W'(0)})$

$$\partial_v^2 S(r,0) = r^2 + 2f'(0) > 0$$

and $r < r_0$

$$\partial_v^2 S(r,0) < 0.$$

It follows that $u = 0$ is a local minimum of $U_r(u)$ for $r > r_0$ and a local maximum for $r < r_0$. We now break up the case $f'(0) < 0$ into two generic subcases.

Suppose that $f''(0) > 0$ Then $f'(t)$ is an increasing function of t for $|t| \ll 1$. Consequently $f'(t)$ is invertible on a neighborhood of 0 and $(f')^{-1}$ is also an increasing function in a small neighborhood of $f'(0) = -r_0^2/2$. It follows that

$$\text{since } -r^2/2 < -r_0^2/2 \quad \text{for } r > r_0 \geq 0, \quad (f')^{-1}(-r^2/2) < (f')^{-1}(-r_0^2/2) = 0,$$
$$\text{and for } 0 \leq r < r_0 \quad (f')^{-1}(-r^2/2) > (f')^{-1}(-r_0^2/2) = 0.$$

Hence for $r > r_0$ the equation

$$v^2 = (f')^{-1}(-r^2/2)$$

has no solutions. Consequently

$$\partial_v S(v,r) = v(r^2 + 2f'(v^2)) = 0$$

only if $v = 0$. On the other hand if $r < r_0$ then

$$v(r) = \Big((f')^{-1}(-r^2/2)\Big)^{1/2}$$

solves

$$r^2 + 2f'(v^2) = 0.$$

And then

$$(\partial_v^2 S)(r, v(r)) = r^2 + 2f'(v(r)^2) + 4v(r)^2 f''(v(r)^2) = 4v(r)^2 f''(v(r)^2).$$

Since $f''(v(r_0)^2) = f''(0) > 0$ by our assumption and since the function $v(r)$ is continuous

$$f''(v(r)^2) > 0$$

for $r < r_0$ and $r_0 - r \ll 1$. Hence $v(r)$ is a local minimum of $S_r(v) = S(r,v)$. It follows that $u(r) = \frac{2v(r)}{1+v(r)^2}$ is a local minimum of U_r if $W''(0) > W'(0)$. This proves (5.1.ii).

Suppose that $f''(0) < 0$ (which corresponds to the case of $W''(0) < W'(0)$) Then $f'(t)$ is a decreasing function for t small. Consequently f' is invertible on a neighborhood of 0 and $(f')^{-1}$ is also a decreasing function in a small neighborhood of $f'(0) = -r_0^2/2$. Then,

$$\begin{aligned} \text{since } -r^2/2 < -r_0^2/2 \quad \text{for } r > r_0 \geq 0, \quad & (f')^{-1}(-r^2/2) > (f')^{-1}(-r_0^2/2) = 0, \\ \text{and for } r < r_0 \quad & (f')^{-1}(-r^2/2) < (f')^{-1}(-r_0^2/2) = 0. \end{aligned}$$

Hence for $r < r_0$ the equation

$$v^2 = (f')^{-1}(-r^2/2)$$

has no solutions. Consequently

$$\partial_v S(v,r) = v(r^2 + 2f'(v^2)) = 0$$

only if $v = 0$. On the other hand if $r < r_0$ then

$$v(r) = \Big((f')^{-1}(-r^2/2)\Big)^{1/2}$$

solves

$$r^2 + 2f'(v^2) = 0.$$

And then

$$(\partial_v^2 S)(r, v(r)) = 4v(r)^2 f''(v(r)^2).$$

Since $f''(v(r_0)^2) = f''(0) > 0$ by our assumption,

$$f''(v(r)^2) > 0$$

for $r > r_0$ and $r - r_0 \ll 1$. Hence $v(r)$ is a local maximum of $S_r(v) = S(r, v)$. It follows that $u(r) = \frac{2v(r)}{1+v(r)^2}$ is a local maximum of U_r. This proves (5.1.iii). □

Proof of Theorem 1.1. By Lemma 4.5 local minima of the function U_r correspond to relatively stable relative equilibria of the system (1.2) and local maxima to unstable relative equilibria of (1.2). By (5.1.i) if $W'(0) > 0$ the $u = 0$ is a local minimum of U_r for all r. Hence $((u, 0), (0, 0)) \in T^*D^2 \simeq T^*S^2_+$ is a stable relative equilibrium of (1.2) for all values of r. Similarly (5.1.ii) translates into case (ii) of the theorem and (5.1.iii) translates into case (iii). □

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