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Symplectic reduction and a Darboux-Moser-Weinstein theorem for Lie algebroids

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For Victor Guillemin

Abstract: We extend the Marsden-Weinstein reduction theorem and the Darboux-Moser-Weinstein theorem to symplectic Lie algebroids. We also obtain a coisotropic embedding theorem for symplectic Lie algebroids.

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1. Introduction

In this paper we extend the Marsden-Weinstein reduction theorem and the Darboux-Moser-Weinstein theorem to the setting of log symplectic manifolds. This work is a building block in our recently established "quantization commutes with reduction" theorem for log symplectic manifolds [28].

By a log symplectic manifold we mean a real manifold equipped with a symplectic form that has first-order poles along a divisor (real hypersurface) with normal crossings. The log tangent bundle of such a manifold, i.e. the

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vector bundle whose sections are the vector fields tangent to the divisor, is an example of a Lie algebroid, and most of our arguments extend to the case of an arbitrary Lie algebroid. To underscore the utility of Lie algebroids in "desingularizing" certain Poisson structures we have chosen to formulate our results, in as far as possible, in terms of Poisson and symplectic structures on general Lie algebroids.

Poisson structures on Lie algebroids were introduced under the name of triangular Lie bialgebroids by Mackenzie and Xu [30]. Such structures include Poisson structures in the usual sense as well as triangular Lie bialgebras. Symplectic structures on Lie algebroids were introduced by Nest and Tsygan [40] in order to extend Fedosov's work on deformation quantization from symplectic manifolds to a wider class of Poisson manifolds, and by Martínez [34, 35] to develop a version of Lagrangian mechanics on Lie algebroids initiated by Weinstein [45]. Examples of symplectic Lie algebroids include symplectic manifolds, log symplectic manifolds, b^m-symplectic manifolds, complex symplectic manifolds, and constant rank Poisson structures.

Our Hamiltonian reduction theorem, Theorem 3.2.1, is a version of the Mikami-Weinstein reduction theorem [38] carried over to the context of Lie algebroids. In the special case where the target of the moment map is $\mathfrak{g}^*$, the dual of a Lie algebra $\mathfrak{g}$ equipped with the tangent Lie algebroid $T\mathfrak{g}^*$, our theorem was obtained earlier by Marrero et al. [31, Theorem 3.11]. However, Hamiltonians on log symplectic manifolds may have logarithmic poles, and one of the purposes of allowing more general momentum codomains than $\mathfrak{g}^*$ is to enable us to reduce at poles of the moment map. A novel feature of reduction "at infinity" is that it involves not only the choice of a point in the codomain, but also a choice of a subalgebra of its Lie algebroid stabilizer.

Versions of Moser's trick and the Darboux-Moser-Weinstein theorem in the context of Lie algebroids have been found by many authors, including Cavalcanti and Gualtieri [8], Cavalcanti et al. [9], Geudens and Zambon [14], Guillemin et al. [20], Kirchhof-Lukat [24], Klaasse and Lanius [25, 26, § 4.3], Miranda and Scott [39, § 2], and Smilde [42]. Our version, Theorem 4.3.3, overlaps with these results and contains some of them as special cases. It applies to situations where a Lie subalgebroid is locally a deformation retract of the ambient Lie algebroid. Our proof relies on recent work of Bischoff et al. [1] and Bursztyn et al. [6], which provides us with a method to produce Lie algebroid homotopies from so-called Euler-like sections. One corollary of our result is a normal form for transverse coisotropic submanifolds, Theorem 4.4.1, which is an ingredient in our paper [28].

We review Poisson and symplectic Lie algebroids in Section 2. The Hamiltonian reduction theorem is in Section 3. Section 4 contains a discussion of homotopies of the Lie algebroid de Rham complex, as well as the Darboux-Moser-Weinstein theorem and the coisotropic embedding theorem. We spell out some consequences for the log symplectic case in Section 5. Appendix A is a review of Lie algebroids.

1.1. Notation and terminology

See Appendix B for a notation index. By a *manifold* we mean a finite-dimensional Hausdorff second countable smooth ($\mathcal{C}^\infty$) real manifold without boundary, typically denoted by M. By a *submanifold* we mean an injectively immersed, but not necessarily embedded, submanifold. We denote the inclusion map of a submanifold N of M by i_N. Let E be a real vector bundle over M. We denote the vector bundle projection by π or π_E, the space of smooth sections over an open subset U of M by $\Gamma(U, E)$, and the space of global smooth sections by $\Gamma(E) = \Gamma(M, E)$. We denote the zero bundle over M by 0_M. By a *subbundle* we mean a (not necessarily embedded) submanifold F of E such that $N = \pi_E(F)$ is a submanifold of M and $\pi_F = \pi_E|_F \colon F \to N$ is a vector bundle. For instance, the annihilator F° of a subbundle F is a subbundle of the dual bundle E^*. By a *foliation* we mean a nonsingular (i.e. constant rank) smooth foliation. We say that the leaf space $M/\mathcal{F}$ of a foliation $\mathcal{F}$ of M *is a manifold* if it has a (necessarily unique) manifold structure that makes the quotient map $M \to M/\mathcal{F}$ a submersion. By a *Poisson Lie algebroid* we mean a Lie algebroid equipped with a Poisson structure (involutive 2-section) and by a *symplectic Lie algebroid* a Lie algebroid equipped with a nondegenerate Poisson structure.

2. Poisson Lie algebroids

In this section we review the notions of a Poisson structure and a symplectic structure on a Lie algebroid and extend some standard results of Poisson geometry to the wider context of Lie algebroids. These include symplectization and reduction theorems for presymplectic Lie algebroids, Theorems 2.4.3 and 2.5.5.

2.1. Poisson and symplectic structures on Lie algebroids

Let A be a Lie algebroid over a manifold M. We denote the projection by $\pi = \pi_A \colon A \to M$, the anchor by $\mathbf{an} = \mathbf{an}_A \colon A \to TM$, and the Lie bracket on sections by

$$[\cdot, \cdot] = [\cdot, \cdot]_A \colon \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A).$$

The Lie algebroid has a *de Rham complex* $(\Omega_A^\bullet(M), d_A)$. Its elements, which we call *Lie algebroid forms*, or *A-forms*, or just *forms*, are sections of the exterior algebra bundle $\Lambda^\bullet A^*$, and its differential d_A is defined in terms of the anchor and the Lie bracket. (See Appendix A.5 for a review.) The Lie bracket on the space of sections $\Gamma(A)$ extends to a -1-shifted graded Lie bracket $[\cdot,\cdot]_A$ on the algebra of multisections $\Gamma(\Lambda^\bullet A)$, known as the *Schouten-Nijenhuis bracket*. (See e.g. [30, § 2].) In particular, for each multisection $\sigma \in \Gamma(\Lambda^p A)$ we can form the multisection $[\sigma,\sigma] \in \Gamma(\Lambda^{2p-1}A)$. Let us call σ *involutive* if $[\sigma,\sigma]=0$.

If $A = TM$ is the ordinary tangent bundle, then a 2-section $\lambda \in \Gamma(\Lambda^2 A)$ defines a Poisson structure on M if and only if it is involutive. A result of Coste et al. [10, § III.2] says that a Poisson structure on M makes the cotangent bundle T^*M a Lie algebroid. Their result was extended by Mackenzie and Xu as follows. Item (i) of this statement is a reformulation of [30, Theorem 4.3] and item (ii) summarizes the material of [30, § 4].

2.1.1 Theorem (Mackenzie and Xu [30])**.** *Let $A \to M$ be a Lie algebroid and let $\lambda \in \Gamma(\Lambda^2 A)$ be a 2-section of A. Define the vector bundle map $\lambda^\sharp\colon A^* \to A$ by $\beta(\lambda^\sharp(\alpha)) = \lambda(\alpha,\beta)$ for α, $\beta \in \Omega_A^1(M) = \Gamma(A^*)$. Define antisymmetric bracket operations on functions and on 1-forms by*

$$\{f,g\} = \{f,g\}_\lambda = \iota_A(\lambda)(d_A f \wedge d_A g) \tag{2.1.2}$$

for f, $g \in \mathcal{C}^\infty(M)$ and

$$\{\alpha,\beta\} = \{\alpha,\beta\}_\lambda = \iota_A(\lambda^\sharp\alpha)d_A\beta - \iota_A(\lambda^\sharp\beta)d_A\alpha + d_A\iota_A(\lambda)(\alpha\wedge\beta) \tag{2.1.3}$$

for α, $\beta \in \Omega_A^1(M)$.

(i) λ is involutive if and only if $\lambda^\sharp\{\alpha,\beta\}_\lambda = [\lambda^\sharp\alpha, \lambda^\sharp\beta]_A$ for all α, $\beta \in \Omega_A^1(M)$.

(ii) Suppose λ is involutive. Then the bracket (2.1.2) is a Poisson structure on M with associated Poisson tensor $\mathbf{an}_A(\lambda) \in \Gamma(\Lambda^2 TM)$, and the bracket (2.1.3) is a Lie algebroid structure on the dual bundle A^* with anchor $\mathbf{an}_\lambda = \mathbf{an}_A \circ \lambda^\sharp$. The map $\lambda^\sharp\colon A^* \to A$ and the co-anchor $\mathbf{an}_A^*\colon T^*M \to A^*$ (the transpose of the anchor $\mathbf{an}_A$) are Lie algebroid morphisms. The differential of the Lie algebroid A^* is the operator $d_\lambda\colon \Gamma(\Lambda^\bullet A) \to \Gamma(\Lambda^{\bullet+1}A)$ given by $d_\lambda a = [\lambda, a]$.

This theorem motivates the following definition.

2.1.4 Definition. An *A-Poisson structure* on M is an involutive 2-section $\lambda \in \Gamma(\Lambda^2 A)$. A *Poisson Lie algebroid* is a pair (A,λ) consisting of a Lie algebroid A over M and an A-Poisson structure λ on M.

Thus an A-Poisson structure can be regarded as an ordinary Poisson structure on M together with a lift of the Poisson tensor to the Lie algebroid A. Unlike an ordinary Poisson structure, an A-Poisson structure is in general not determined by the Poisson bracket on functions (2.1.2) alone. Mackenzie and Xu [30] refer to the triple (A, A^*, λ) as a *triangular Lie bialgebroid*.

One use of Poisson Lie algebroids lies in the fact that sometimes an ordinary Poisson structure on M can be lifted to a "less singular" Poisson structure on a Lie algebroid A over M. For instance, it may happen that a degenerate Poisson structure lifts to a nondegenerate, i.e. symplectic, A-Poisson structure. (One such situation is described in Section 2.2.)

2.1.5 Definition. An *A-symplectic form* on M is a 2-form $\omega \in \Omega^2_A(M)$ that is d_A-closed, i.e. $d_A\omega = 0$, and non-degenerate. We write $\omega^{-1} \in \Gamma(\Lambda^2 A)$ for the 2-section and

$$\omega^\flat \colon A \xrightarrow{\cong} A^* , \qquad \omega^\sharp \colon A^* \xrightarrow{\cong} A$$

for the bundle isomorphisms determined by an A-symplectic form ω. A *symplectic Lie algebroid* is a pair (A, ω) consisting of a Lie algebroid A and an A-symplectic form ω on the base of A.

This definition follows Nest and Tsygan [40]. (The term "symplectic Lie algebroid" is used by Coste et al. [10, § III.2] to mean something different, namely the Lie algebroid of a local symplectic groupoid.) If $\omega \in \Omega^2_A(M)$ is any nondegenerate 2-form, then a calculation using Theorem 2.1.1(i) shows that $d_A\omega = 0$ if and only if $[\omega^{-1}, \omega^{-1}] = 0$. So just as in ordinary Poisson geometry an A-symplectic structure amounts to an A-Poisson structure λ such that the morphism $\lambda^\sharp \colon A^* \to A$ is invertible.

Let λ be an A-Poisson structure on M. To each function $f \in \mathcal{C}^\infty(M)$ is associated a section

$$\sigma_f = d_\lambda f = [\lambda, f] = \lambda^\sharp(d_A f) \in \Gamma(A) \tag{2.1.6}$$

called the *Hamiltonian section* of f. Conversely, if a section $\sigma \in \Gamma(A)$ is of the form $\sigma = \sigma_f$ for some function f, we say f is a *Hamiltonian* for σ. Like any section of A, the section σ_f generates a flow on the total space of A (see review in Appendix A.5), which we call the *Hamiltonian flow* of f. (In ordinary Poisson geometry, this flow is the tangent flow on $A = TM$ of what one usually calls the Hamiltonian flow of f on M.) The Poisson structure λ is invariant under the Hamiltonian flow of f, i.e.

$$\mathcal{L}_A(\sigma_f)\lambda = [\sigma_f, \lambda] = d_\lambda \lambda^\sharp(d_A f) = \lambda^\sharp(d_A^2 f) = 0,$$

where $\mathcal{L}_A$ denotes the Lie algebroid Lie derivative. The *Hamiltonian correspondence* is the map

$$\mathcal{H}\colon C^\infty(M) \longrightarrow \Gamma(A)$$

given by $\mathcal{H}(f) = \sigma_f$. The Hamiltonian correspondence is a Lie algebra homomorphism, and its kernel is the Lie ideal of A^*-invariant functions. In the symplectic case ($\lambda = \omega^{-1}$) the Poisson bracket (2.1.2) is given by $\{f, g\} = \omega(\sigma_f, \sigma_g)$.

2.2. The phase space of a Lie algebroid

The cotangent bundle ("phase space") of a manifold has a natural symplectic structure. As noted by Martínez [34] (see also de León et al. [27] and Marrero et al. [31]) this familiar fact has an analogue in the world of Lie algebroids. Let $B \to N$ be an arbitrary Lie algebroid. The projection $\pi\colon B^* \to N$ of the dual bundle B^* is a submersion, so we can form the pullback

$$A = \pi^! B = TB^* \times_{TN} B,$$

which is a Lie algebroid over B^*. (Pullbacks of Lie algebroids are reviewed in Appendix A.3.) Elements of $A = \pi^! B$ are tuples (x, p, v, b), where $x \in N$, $p \in B^*_x$, $v \in T_p B^*$, $b \in B_x$ satisfy $T_p\pi(v) = \mathbf{an}_B(b)$. We define the *Liouville form* or *canonical* 1-*form* $\alpha_{\mathrm{can}} \in \Omega^1_A(B^*)$ by

$$\alpha_{\mathrm{can}}(x, p, v, b) = p(b)$$

and the *canonical* 2-*form* $\omega_{\mathrm{can}} \in \Omega^2_A(B^*)$ by

$$\omega_{\mathrm{can}} = -d_A \alpha_{\mathrm{can}}.$$

We call the pair $(A, \omega_{\mathrm{can}})$ the *phase space* of the Lie algebroid B. The following result says that the phase space is a symplectic Lie algebroid and that the Poisson structure on B^* determined by the canonical 2-form is identical to the natural linear Poisson structure that exists on the dual of any Lie algebroid. We review the proof of item (i) because we need the details in Section 2.4. By abuse of language we say that a smooth map $f\colon P \to M$ *cleanly* (resp. *transversely*) *intersects* a Lie algebroid $C \to M$ if its tangent map $Tf\colon TP \to TM$ cleanly (resp. transversely) intersects the anchor $\mathbf{an}_C\colon C \to TM$ of A (Definition A.3.4).

2.2.1 Proposition ([31, §§ 3.2, 3.5, 7])**.** *Let $B \to N$ be a Lie algebroid, let $\pi\colon B^* \to N$ be the dual vector bundle, and let $A = \pi^! B$ be the pullback of B to B^*.*

(i) *The canonical* 2*-form* $\omega_{\mathrm{can}} \in \Omega^2_A(B^*)$ *is an* A*-symplectic form on* B^**.*
(ii) *The Poisson structure* $\mathbf{an}_A(\omega_{\mathrm{can}}^{-1}) \in \Gamma(\Lambda^2 TB^*)$ *associated with* ω_{can} *is equal to the linear Poisson structure on* B^* *determined by the Lie algebroid structure on* B*.*
(iii) *The zero section* $\zeta\colon N \to B^*$ *is transverse to* A *and we have a natural isomorphism* $B \cong \zeta^! A$*. We have* $\zeta_!^* \omega_{\mathrm{can}} = 0$*, where* $\zeta_!\colon B \cong \zeta^! A \to A$ *denotes the canonical Lie algebroid morphism* (A.3.3) *induced by* ζ*.*

Proof of (i). Let r be the rank of the vector bundle B and let $x \in N$. The fibre of the vector bundle $A = \pi^! B$ over B^* at $p \in B^*_x$ is $A_p = T_p B^* \times_{T_x N} B_x$, so the rank of A is $2r$. For a sufficiently small neighbourhood U of x we will exhibit a frame $e_1, e_2, \dots, e_r, f_1, f_2, \dots, f_r$ of A defined on $\pi^{-1}(U)$ with respect to which the matrix of the bilinear form ω_{can} is invertible. A section σ of A can be described as a pair $\sigma = (v, b)$ consisting of a vector field v on B^* and a smooth map $b\colon B^* \to B$ satisfying $T\pi \circ v = \mathbf{an}_B \circ b$. The anchor of σ is then $\mathbf{an}_A(\sigma) = v$. A section β of B gives rise to a fibrewise linear smooth function $\beta^\dagger$ on B^* defined by

$$\beta^\dagger(p) = \langle p, \beta(\pi(p)) \rangle, \tag{2.2.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between B^* and B. From (A.3.6) and (A.5.1) we obtain, for any pair of sections of A of the form $\sigma_1 = (v_1, \beta_1 \circ \pi)$, $\sigma_2 = (v_2, \beta_2 \circ \pi)$,

$$\omega_{\mathrm{can}}(\sigma_1, \sigma_2) = -v_1 \cdot \beta_2^\dagger + v_2 \cdot \beta_1^\dagger - [\beta_1, \beta_2]^\dagger. \tag{2.2.3}$$

Now choose a frame $b_1, b_2, \dots, b_r$ of the vector bundle B defined on a neighbourhood U of x. Let $b_1^*, b_2^*, \dots, b_r^*$ be the dual frame of B^*. We have a short exact sequence of vector bundles over B^*

$$\pi^* B^* \hookrightarrow TB^* \twoheadrightarrow \pi^* TN, \tag{2.2.4}$$

so each b_i^* can be thought of as a vector field on B^* tangent to the fibres of π, and the pair $f_i = (b_i^*, 0)$ defines a section of A over the open subset $\pi^{-1}(U)$ of B^*. Then

$$\omega_{\mathrm{can}}(f_i, f_j) = 0 \tag{2.2.5}$$

by (2.2.3). The frame $b_1^*, b_2^*, \dots, b_r^*$ determines a trivialization of the vector bundle B^* over U and in particular a linear connection on B^*, which gives a splitting $\theta\colon TN \to TB^*$ of the sequence (2.2.4). Each section b_i then gives rise to a vector field $v_i = \theta \circ \mathbf{an}_B(b_i)$ on B^*. For $1 \le i \le r$ the pair $e_i = (v_i, b_i \circ \pi)$ is a section of A defined over $\pi^{-1}(U)$. It follows from (2.2.3) that

$$\omega_{\mathrm{can}}(e_i, f_j) = b_j^* \cdot b_i^\dagger = \langle b_j^*, b_i \rangle = \delta_{ij}. \tag{2.2.6}$$

The vector fields v_i are horizontal and the functions $b_i^\dagger$ are covariantly constant, so

$$\omega_{\mathrm{can}}(e_i, e_j) = -v_i \cdot b_j^\dagger + v_j \cdot b_i^\dagger - [b_i, b_j]^\dagger = -[b_i, b_j]^\dagger. \tag{2.2.7}$$

We conclude that the $2r$-tuple $(e_1, e_2, \dots, e_r, f_1, f_2, \dots, f_r)$ is a frame of A and that the matrix of ω_{can} relative to this frame is $\left(\begin{smallmatrix} -C & -I_r \\ I_r & 0 \end{smallmatrix}\right)$, where C is the $r \times r$-matrix $\left([b_i, b_j]^\dagger\right)_{i,j}$. In particular ω_{can} is nondegenerate. □

2.3. Poisson morphisms and coisotropic subalgebroids

The notions of a Poisson map and a coisotropic submanifold admit straightforward extensions to the world of Poisson Lie algebroids. If $\varphi\colon A \to E$ is a Lie algebroid morphism with base map $\mathring{\varphi}\colon M \to P$, we say that multisections $u \in \Gamma(\Lambda^\bullet A)$ and $v \in \Gamma(\Lambda^\bullet E)$ are *φ-related*, notation $u \sim_\varphi v$, if $\varphi(u_x) = v_{\mathring{\varphi}(x)}$ for all $x \in M$.

2.3.1 Definition. Let $(A \to M, \lambda_A)$ and $(E \to P, \lambda_E)$ be Poisson Lie algebroids (Definition 2.1.4). A *Poisson morphism* from A to E is a Lie algebroid morphism $\varphi\colon A \to E$ such that the Poisson structures λ_A and λ_E are φ-related.

The 2-sections λ_A and λ_E are φ-related if and only if the square

$$\begin{tikzcd} A_x^* \arrow[r, "\lambda_A^\sharp"] & A_x \arrow[d, "\varphi"] \\ E_{\mathring{\varphi}(x)}^* \arrow[u, "\varphi^*"] \arrow[r, "\lambda_E^\sharp"] & E_{\mathring{\varphi}(x)} \end{tikzcd}$$

commutes for all $x \in M$, where φ^* denotes the transpose of φ.

Let V be a vector space equipped with a constant Poisson structure λ and let W be a subspace of V. We define $W^\lambda = \lambda^\sharp(W^\circ)$ of V, where $W^\circ \subseteq V^*$ denotes the annihilator of W. We say W is *coisotropic* if W^λ is contained in W. If λ is the inverse of a symplectic structure ω on V, we write $W^\lambda = W^\omega$ and call W^ω the *symplectic orthogonal* of W.

2.3.2 Definition. Let $A \to M$ be a Poisson Lie algebroid with Poisson structure $\lambda \in \Gamma(\Lambda^2 A)$. A Lie subalgebroid $B \to P$ of A is *coisotropic at* $x \in P$ if the subspace B_x of A_x is coisotropic. A submanifold N of M is *coisotropic at* $x \in N$ if the subspace $\mathbf{an}^{-1}(T_x N)$ of A_x is coisotropic. We say B, resp. N, is *coisotropic* if it is coisotropic at all $x \in P$, resp. $x \in N$. A submanifold N is *clean coisotropic* (resp. *transverse coisotropic*) if N is coisotropic and TN cleanly (resp. transversely) intersects the anchor $\mathbf{an}\colon A \to TM$.

If a submanifold N of M cleanly intersects the anchor of A, we have a well-defined pullback Lie algebroid $i_N^! A$ over N whose fibre at $x \in N$ is $\mathbf{an}_A^{-1}(T_x N)$ (see Appendix A.3), and in that case the submanifold N is coisotropic if and only if the Lie subalgebroid $i_N^! A$ is coisotropic in the sense of Definition 2.3.2.

Coisotropic subalgebroids behave in the expected way under Poisson morphisms. For submanifolds there is no distinction between being coisotropic relative to λ_A and being coisotropic relative to the underlying Poisson structure $\mathbf{an}_A(\lambda_A)$ on M.

2.3.3 Lemma. *Let $(A \to M, \lambda_A)$ and $(E \to P, \lambda_E)$ be Poisson Lie algebroids and $\varphi\colon A \to E$ a Poisson morphism.*

(i) Let $F \to Q$ be a Lie subalgebroid of E which cleanly intersects the morphism φ. Let $B = \varphi^{-1}(F)$ and $N = \mathring{\varphi}^{-1}(Q)$. The Lie subalgebroid B of A is coisotropic if and only if F is coisotropic at y for all $y \in \mathring{\varphi}(N)$.

(ii) The base map $\mathring{\varphi}\colon M \to P$ is a Poisson map in the usual sense relative to the Poisson structures $\mathbf{an}_A(\lambda_A)$ on M and $\mathbf{an}_P(\lambda_E)$ on P.

(iii) A submanifold of M is coisotropic relative to λ_A if and only if it is coisotropic in the usual sense, i.e. relative to the Poisson structure $\mathbf{an}_A(\lambda_A)$ on M.

Proof. (i) It follows from Proposition A.4.1 that B is a Lie subalgebroid of A whose base is the submanifold N of M. Now use the following straightforward linear algebra fact: if $f\colon V_1 \to V_2$ is a linear Poisson map between Poisson vector spaces (V_1, λ_1) and (V_2, λ_2), W_2 is a subspace of V_2, and $W_1 = f^{-1}(W_2)$, then

$$f(W_1^{\lambda_1}) = W_2^{\lambda_2}. \tag{2.3.4}$$

Hence W_1 is coisotropic if and only W_2 is coisotropic.

(ii) This follows from the fact that the anchor map $\mathbf{an}_A$ is a Poisson morphism from (A, λ_A) to $(TM, \mathbf{an}_A(\lambda_A))$.

(iii) Let N be a submanifold of M. Then (2.3.4) yields $\mathbf{an}_A(B_x^{\lambda_A}) = (T_x N)^{\lambda_M}$ for all $x \in N$, where $B_x = \mathbf{an}_A^{-1}(T_x N)$ and $\lambda_M = \mathbf{an}_A(\lambda_A)$. Hence B_x is coisotropic if and only if $T_x N$ is coisotropic. ∎

2.4. Presymplectic Lie algebroids: symplectization

A Lie algebroid equipped with a closed 2-form of constant rank can be turned into a symplectic Lie algebroid in two “opposite” ways. In this section we explain the first method: symplectization.

2.4.1 Definition. Let $B \to N$ be a Lie algebroid. A *B-presymplectic form* on N is a 2-form $\omega_B \in \Omega^2_B(N)$ that is d_B-closed and of constant rank. A *presymplectic Lie algebroid* is a Lie algebroid equipped with a presymplectic form.

As in ordinary symplectic geometry, presymplectic Lie algebroids arise naturally from symplectic ones through coisotropic embeddings. Indeed, let $(A \to M, \omega)$ be a symplectic Lie algebroid and let B be a coisotropic Lie subalgebroid of A (Definition 2.3.2). The pullback $\omega_B = i_B^*\omega \in \Omega^2_B(N)$ is a closed 2-form. The rank of ω_B is constant equal to $\operatorname{rank}(B/B^\omega) = 2\operatorname{rank}(B) - \operatorname{rank}(A)$. Hence (B, ω_B) is a presymplectic Lie algebroid.

The symplectization theorem below asserts the converse: every presymplectic Lie algebroid $B \to N$ arises as a pullback Lie algebroid $i_N^!\mathbf{A}$ of a model symplectic Lie algebroid $\mathbf{A} \to \mathbf{M}$ via a transverse coisotropic embedding $i_N \colon N \to \mathbf{M}$.

The model $\mathbf{A}$ is constructed as follows. The input data is any presymplectic Lie algebroid $(B \to N, \omega_B)$. Let $K = \ker(\omega_B)$ be the kernel of ω_B, i.e. the bundle whose fibre at $x \in N$ is

$$K_x = \{\, b \in B_x \mid \omega_B(b, b') = 0 \text{ for all } b' \in B_x \,\}.$$

We let $\mathbf{M} = K^*$ be the dual bundle of K. The bundle projection $p \colon \mathbf{M} \to N$ is transverse to B, so we can form the pullback Lie algebroid $\mathbf{A} = p^!B$ over $\mathbf{M}$. We identify N with the zero section $\mathbf{j} \colon N \to \mathbf{M} = K^*$ and B with the Lie algebroid $\mathbf{j}^!\mathbf{A} \cong \mathbf{j}^!p^!B$. The inclusion $\mathbf{j}$ has a natural lift to a morphism $\mathbf{j}_! \colon B \to \mathbf{A}$. The definition of the symplectic structure on $\mathbf{A}$ involves the choice of a complement of the subbundle K of B, i.e. a splitting $s \colon K^* \to B^*$ of the natural surjection $B^* \to K^*$. Let $\pi \colon B^* \to N$ be the bundle projection of B^*. Then $p = \pi \circ s \colon K^* \to N$, so

$$\mathbf{A} = p^!B = (\pi \circ s)^!B = s^!\pi^!B.$$

We have canonical Lie algebroid morphisms

$$p_! \colon \mathbf{A} \longrightarrow B, \qquad \pi_! \colon \pi^!B \longrightarrow B, \qquad s_! \colon \mathbf{A} \longrightarrow \pi^!B$$

satisfying $p_! = \pi_! \circ s_!$. We define a closed $\mathbf{A}$-form of degree 2 on $\mathbf{M}$ by

$$\omega^s = p_!^* \omega_B + s_!^* \omega_{\mathrm{can}},$$

where ω_{can} is the canonical symplectic form on the phase space $\pi^! B$ (Proposition 2.2.1). We call the tuple

$$(\mathbf{A} \longrightarrow \mathbf{M}, \omega^s, \mathbf{j}_! \colon B \longrightarrow \mathbf{A}) \tag{2.4.2}$$

the *symplectization* of (B, ω_B).

2.4.3 Theorem (Symplectization)**.** *Let $(B \to N, \omega_B)$ be a presymplectic Lie algebroid. Let $(\mathbf{A}, \omega^s, \mathbf{j}_!)$ be the symplectization (2.4.2). There is an open neighbourhood $\mathbf{U}$ of N in $\mathbf{M}$ such that $\omega^s|_{\mathbf{U}}$ is symplectic. The embedding $\mathbf{j}$ is transverse coisotropic and $\mathbf{j}_!^* \omega^s = \omega_B$.*

Proof. Identify $\mathbf{M} = K^*$ with the subbundle $s(K^*)$ of B^*. Then we have $B = K \oplus L$, where $L = (K^*)^\circ$ is the annihilator of K^*. The bilinear form ω_B is nondegenerate on the subbundle L. Let $x \in N$. Choose a frame b_1, $b_2, \ldots, b_r$ of B defined in a neighbourhood U of x such that K is spanned by $b_1, b_2, \ldots, b_l$, and L is spanned by $b_{l+1}, b_{l+2}, \ldots, b_r$. Let $b_1^*, b_2^*, \ldots, b_r^*$ be the dual frame of B^*. The subbundle K^* is spanned by $b_1^*, b_2^*, \ldots, b_l^*$. The frame $b_1, b_2, \ldots, b_r$ gives rise to a frame $e_1, e_2, \ldots, e_r, f_1, f_2, \ldots, f_r$ of $\pi^! B$ over $\pi^{-1}(U)$ as in the proof of Proposition 2.2.1. The fibre $\mathbf{A}_x$ of $\mathbf{A}$ is spanned by (the values at x of) the sections $e_1, e_2, \ldots, e_l, f_1, f_2, \ldots, f_r$. The anchor of f_i is $\mathbf{an}(f_i) = b_i^*$, so the image of the anchor $\mathbf{an}_x(\mathbf{A})$ contains the span of b_1^*, $b_2^*, \ldots, b_l^*$, i.e. the fibre K_x^*. Therefore the zero section $\mathbf{j} \colon N \to \mathbf{M} = K^*$ is transverse to $\mathbf{A}$. It follows from (2.2.5)–(2.2.7) that

$$\begin{aligned}
&\omega_x^s(f_i, f_j) = 0 && \text{for } 1 \le i, j \le r,\\
&\omega_x^s(e_i, e_j) = 0 && \text{for } 1 \le i \le l,\ 1 \le j \le r,\\
&\omega_x^s(f_i, e_j) = \delta_{ij} && \text{for } 1 \le i \le l,\ 1 \le j \le r,\\
&\omega_x^s(e_i, e_j) = \omega_{B,x}(b_i, b_j) && \text{for } l+1 \le i, j \le r.
\end{aligned} \tag{2.4.4}$$

This shows that $\mathbf{j}_!^* \omega^s = \omega_B$. Also, the fibre $(\mathbf{A}_x, \omega_x^s)$ is an orthogonal direct sum of two symplectic subspaces $K_x \oplus K_x^*$ and $(L_x, \omega_{B,x})$. Hence the form ω^s is symplectic near N. The fibre B_x is spanned by $e_1, e_2, \ldots, e_r$, so its orthogonal $B_x^{\omega^s}$ is spanned by $e_1, e_2, \ldots, e_l$. This shows that the embedding $\mathbf{j}$ is transverse to $\mathbf{A}$ and coisotropic. $\square$

2.4.5 Remark. We will see in Section 4.4 that the form ω^s is independent of the splitting s up to Lie algebroid automorphisms of A fixing N. For now we note that any two splittings s_0, $s_1\colon \mathbf{M} = K^* \to B^*$ of the surjection $B^* \to K^*$ can be joined by a path $s_t = (1-t)s_0 + ts_1$ for $0 \le t \le 1$. The corresponding path of $\mathbf{A}$-symplectic forms $\omega_t = \omega^{s_t}$ satisfies $\dot{\omega}_t = -d_A\beta_t$, where $\beta_t \in \Omega^1_{\mathbf{A}}(\mathbf{M})$ is defined by

$$\beta_t = \frac{d}{dt}(s_t)_!^*\alpha_{\mathrm{can}}.$$

Since $s_t(x) = x$ for all $x \in N$ and the Liouville form $\alpha_{\mathrm{can}} \in \Omega_{\pi^! B}(N)$ vanishes along N, the form β_t vanishes along N for all t.

2.5. Presymplectic Lie algebroids: reduction

As we saw in Section 2.4, every presymplectic Lie algebroid can be symplectized, i.e. coisotropically embedded in a symplectic Lie algebroid. A second method to produce symplectic Lie algebroids out of presymplectic Lie algebroids, which works only under favourable conditions, is *symplectic reduction*, which means taking the quotient by the null foliation. This is based on the following facts.

2.5.1 Lemma. *Let $(B \to N, \omega_B)$ be a presymplectic Lie algebroid. The kernel $K = \ker(\omega_B)$ is a Lie subalgebroid of B. The form ω_B is K-basic in the sense that for all sections σ of K we have $\iota_B(\sigma)\omega_B = \mathcal{L}_B(\sigma)\omega_B = 0$.*

Proof. This follows from $d_B\omega_B = 0$ and $\mathcal{L}_B(\sigma) = [\iota_B(\sigma), d_B]$. □

We call the Lie subalgebroid K of Lemma 2.5.1 the *null Lie algebroid* of the form ω_B.

A *foliation Lie algebroid* is a Lie algebroid whose anchor is injective. A foliation Lie algebroid over a manifold P is equivalent to an involutive subbundle of TP, in other words a (nonsingular) foliation of P.

2.5.2 Lemma. *Let $(B \to N, \omega_B)$ be a presymplectic Lie algebroid. Suppose the null Lie algebroid $K = \ker(\omega_B)$ is a foliation Lie algebroid and therefore defines a foliation $\mathcal{K}$ of N. Let $i_S\colon S \to N$ be a transverse slice of $\mathcal{K}$. Then $(i_S)_!^*\omega_B$ is an $i_S^! B$-symplectic form on S.*

Proof. Since S is transverse to the leaves of $\mathcal{K}$, it is also transverse to the Lie algebroid B (Proposition A.3.5(i)), so the pullback $i_S^! B = \mathbf{an}^{-1}(TS) \subseteq B$ and the morphism $(i_S)_!\colon i_S^! B \to B$ are well-defined. Let $x \in S$. Since S is a transverse slice to the foliation, the tangent space to N is a direct sum $T_xN = \mathbf{an}(K_x) \oplus T_xS$. Hence, the anchor $\mathbf{an}\colon K_x \to T_xN$ being injective, the fibre of B is likewise a direct sum $B_x = K_x \oplus \mathbf{an}^{-1}(T_xS) = K_x \oplus (i_S^! B)_x$. It follows that ω_B restricts to a nondegenerate form on $i_S^! B$. Therefore $(i_S^! B, (i_S)_!^* \omega_B)$ is a symplectic Lie algebroid. □

We call the foliation $\mathcal{K}$ of Lemma 2.5.2 the *null foliation* of ω_B.

Let $B \to N$ and $K \to N$ be Lie algebroids over the same base and let $f \colon K \to B$ be a morphism over the identity map of N. We define a *quotient* of B by K to be a pair $(C \to Q, q)$ consisting of a Lie algebroid $C \to Q$ and a morphism $q \colon B \to C$ with $q \circ f(K) = 0$ which has the following universal property: for every Lie algebroid $A \to M$ and every morphism $g \colon B \to A$ with $g \circ f(K) = 0$ there is a unique morphism $g_C \colon C \to A$ with $g = g_C \circ q$, as in the diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "q"'] & A \\ C \arrow[ur, dashed, "g_C"'] & \end{tikzcd}$$

Clearly a quotient Lie algebroid, if it exists, is determined uniquely up to isomorphism by the Lie algebroid morphism $f \colon K \to B$. The next result, which is a special case of [22, Theorem 4.5] and which does not involve any presymplectic structures, states a sufficient condition for a quotient Lie algebroid to exist. We say that the leaf space $Q = P/\mathcal{F}$ of a foliation $\mathcal{F}$ of a manifold P *is a manifold* if Q has a manifold structure which makes the quotient map $P \to Q$ a submersion.

2.5.3 Proposition (Quotient Lie algebroids)**.** *Let $B \to N$ be a Lie algebroid and $K \to N$ a Lie subalgebroid of B. Suppose that K is a foliation Lie algebroid with associated foliation $\mathcal{K}$ and that the leaf space $Q = N/\mathcal{K}$ is a manifold. Let $\mathring{q} \colon N \to Q$ be the quotient map. Let $\bar{B}$ be the bundle B/K over N and identify sections $\bar{\tau} \in \Gamma(\bar{B}) \cong \Gamma(B)/\Gamma(K)$ with equivalence classes of sections τ of B modulo sections of K. Suppose that the flat K-connection*

$$\nabla \colon \Gamma(K) \times \Gamma(\bar{B}) \longrightarrow \Gamma(\bar{B})$$

on $\bar{B}$ defined by $\nabla_\sigma \bar{\tau} = \overline{[\sigma, \tau]}$ has trivial holonomy.

(i) The ∇-horizontal subspaces of $T\bar{B}$ define a foliation $\mathcal{L}$ of $\bar{B}$ which makes the pair $(\bar{B}, \mathcal{L})$ a foliated vector bundle over the foliated manifold $(N, \mathcal{K})$.

(ii) The leaf space $C = \bar{B}/\mathcal{L}$ is a manifold. Let $q \colon B \to C$ the quotient map and $C \to Q$ the projection induced by the bundle projection $\bar{B} \to N$. Then $(C \to Q, q)$ is a quotient Lie algebroid of B by K. The quotient morphism $q \colon B \to C$ induces an isomorphism $B \cong q^! C$.

(iii) *Let* $s\colon U \to N$ *be a section of* $\mathring{q}$ *defined over an open subset* U *of* Q*. The natural map* $q \circ s_! \colon s^! B \to B \to C$ *is a Lie algebroid isomorphism from* $s^! B$ *onto* $C|_U$*.*

(iv) *The Lie algebra of sections of* C *is isomorphic to*

$$\Gamma(C) \cong \mathfrak{n}(\Gamma(K))/\Gamma(K),$$

where $\mathfrak{n}(\Gamma(K))$ *denotes the normalizer of* $\Gamma(K)$ *in* $\Gamma(B)$*.*

Under the conditions of Proposition 2.5.3 the quotient morphism $q\colon B \to C$ induces an isomorphism of complexes

$$q^* \colon \Omega^\bullet_C(Q) \xrightarrow{\cong} \Omega^\bullet_B(N)_{K\text{-bas}}, \tag{2.5.4}$$

where the subscript "K-bas" refers to the subcomplex of Lie algebroid forms that are K-basic in the sense of Lemma 2.5.1. If $\alpha \in \Omega^\bullet_B(N)$ is K-basic, then $(q^*)^{-1}\alpha$ is determined by the following fact: for a local section $s\colon U \to N$ of q defined over an open subset U of Q we have $\big((q^*)^{-1}\alpha)\big)\big|_U = s_!^*\alpha$. Together with Lemma 2.5.2 this gives us the following criterion for when a presymplectic Lie algebroid can be reduced to a symplectic Lie algebroid.

2.5.5 Theorem (Symplectic reduction)**.** *Let* $(B \to N, \omega_B)$ *be a presymplectic Lie algebroid. Suppose that the null Lie algebroid* $K = \ker(\omega_B)$ *is a foliation Lie algebroid, that the leaf space* $N/\mathcal{K}$ *of the foliation* $\mathcal{K}$ *defined by* K *is a manifold, and that the flat* K*-connection on* B/K *has trivial holonomy. Then there is a unique form* $\omega_C \in \Omega^2_C(Q)$ *on the quotient Lie algebroid* $C \to Q$ *satisfying* $q^*\omega_C = \omega_B$*. The form* ω_C *is* C*-symplectic.*

3. Hamiltonian actions and reduction

Marsden and Weinstein [33] showed how to reduce a symplectic manifold M with respect to a moment map, i.e. a Poisson map $M \to \mathfrak{g}^*$ to the dual of a Lie algebra $\mathfrak{g}$. A version of their result for symplectic Lie algebroids was obtained by Marrero et al. [31, Theorem 3.11]. However, symplectic Lie algebroids include log symplectic manifolds, where the symplectic structure has first-order poles and Hamiltonian functions may have logarithmic poles. How do we reduce log symplectic manifolds at "infinite" values of momentum? Following Mikami and Weinstein [38], we will deal with such situations by allowing moment maps to take values in Poisson manifolds more general than $\mathfrak{g}^*$. The upshot is a Lie algebroid version of the Mikami-Weinstein reduction theorem, Theorem 3.2.1.

As noted in Section 2.1, a Poisson structure on a Lie algebroid $A \to M$ gives rise to a Poisson structure on M in the usual sense, and in that sense Theorem 3.2.1 is a special case of the Poisson reduction theorems of Marsden and Ratiu [32] and Cattaneo and Zambon [7]. But Theorem 3.2.1 offers the extra information that the quotient Poisson structure lifts to an appropriate quotient Lie algebroid over the reduced space. A Poisson structure on a Lie algebroid A can be seen as a special type of Dirac structure on the Courant algebroid $A \oplus A^*$. Presumably the Dirac reduction theorems of Bursztyn and Crainic [4, Theorem 4.11] and Bursztyn et al. [5, Proposition 3.16], which are formulated there only for standard Courant algebroids $TM \oplus T^*M$, can be extended to incorporate our setting, but we will leave that for another day.

3.1. Lie algebroid actions and Poisson morphisms

In this section we review the notion of a Lie algebroid action on a Lie algebroid and how Poisson maps give rise to such actions.

An action of a Lie algebra $\mathfrak{g}$ on a Lie algebroid $A \to M$ simply means a Lie algebra homomorphism $\mathfrak{g} \to \Gamma(A)$. Equivalently, a $\mathfrak{g}$-action on A can be defined as a $\mathfrak{g}$-action on M together with a Lie algebroid morphism $\mathfrak{g} \ltimes M \to A$ from the action Lie algebroid to A. This notion generalizes as follows.

3.1.1 Definition. Let $A \to M$ and $C \to P$ be Lie algebroids and let $\mu_0 \colon M \to P$ be a smooth map. An *action* of C on A with *anchor* μ_0 is a Lie algebra homomorphism $\rho \colon \Gamma(C) \to \Gamma(A)$ that is $\mathcal{C}^\infty$-linear with respect to μ_0 in the sense that $\rho(f\tau) = (\mu_0^* f)\rho(\tau)$ for $f \in \mathcal{C}^\infty(P)$ and $\tau \in \Gamma(C)$. The sections $\rho(\tau) \in \Gamma(A)$, where τ ranges over the space of sections of C, are called the *generating sections* of the action.

Given a C-action ρ on A there is a unique smooth vector bundle map $\bar{\rho} \colon \mu_0^* C \to A$ over the identity id_M such that the triangle

$$\begin{tikzcd} \Gamma(C) \arrow[r, "\rho"] \arrow[d, "\mu_0^*"'] & \Gamma(A) \\ \Gamma(\mu_0^* C) \arrow[ur, "\bar{\rho}_*"'] & \end{tikzcd} \tag{3.1.2}$$

commutes. The pullback bundle $\mu_0^* C$ is equipped with an anchor $\mathbf{an}_{\mu_0^* C} = \mathbf{an}_A \circ \bar{\rho} \colon \mu_0^* C \to TM$. There is a unique Lie bracket on $\Gamma(\mu_0^* C)$ that makes the maps (3.1.2) Lie algebra homomorphisms. With respect to this bracket $\mu_0^* C$ is a Lie algebroid over M, and $\bar{\rho}$ and the natural map $\mu_0^* C \to C$ are Lie algebroid morphisms. Thus an action of C on A can be alternatively defined as consisting of a smooth map $\mu_0\colon M \to P$, a Lie algebroid structure on the pullback bundle μ_0^*C such that $\mu_0^*C \to C$ is a Lie algebroid morphism, and a Lie algebroid morphism $\bar{\rho}\colon \mu_0^*C \to A$.

3.1.3 Notation and definition. To lighten the notation we will from now on denote the action map $\rho\colon \Gamma(C) \to \Gamma(A)$, the vector bundle map $\bar{\rho}\colon \mu_0^*C \to A$, and the pushforward map $\bar{\rho}_*\colon \Gamma(\mu_0^*C) \to \Gamma(A)$ all by the same letter, ρ. We will also denote by $\rho_x\colon C_{\mu_0(x)} \to A_x$ the restriction of $\bar{\rho}$ to a point $x \in M$. We say that the action ρ is *locally free* at x if ρ_x is injective and *transitive* at x if ρ_x is surjective.

The fact that $\mu_0^*C \to C$ is a Lie algebroid morphism can be viewed as an equivariance property of the anchor μ_0; it implies that the square

$$\begin{tikzcd} \mu_0^*C \arrow[r] \arrow[d, "\mathbf{an}_{\mu_0^*C}"'] & C \arrow[d, "\mathbf{an}_C"] \\ TM \arrow[r, "T\mu_0"] & TP \end{tikzcd}$$

commutes. In particular we have inclusions

$$\operatorname{stab}(\mu_0^*C, x) \subseteq \operatorname{stab}(C, \mu_0(x)) \tag{3.1.4}$$

for all $x \in M$, where we identify the fibre of μ_0^*C at x with the fibre of C at $\mu_0(x)$.

If $B \to N$ is a Lie subalgebroid of A, then the $\mathcal{C}^\infty(M)$-module of relative sections

$$\Gamma(A; B) = \{\, \tau \in \Gamma(A) \mid \tau|_N \in \Gamma(B) \,\}$$

is a Lie subalgebra of $\Gamma(A)$ and therefore the $\mathcal{C}^\infty(P)$-module $\rho^{-1}(\Gamma(A;B))$ is a Lie subalgebra of $\Gamma(C)$. It follows from the Leibniz rule that the $\mathcal{C}^\infty(P)$-module

$$\Gamma\big(C; 0_{\mu_0(N)}\big) = \{\, \sigma \in \Gamma(C) \mid \sigma_p = 0 \text{ for all } p \in \mu_0(N) \,\}$$

is a Lie ideal of $\rho^{-1}(\Gamma(A;B))$. This leads to the following definition.

3.1.5 Definition. Given an action $\rho\colon \Gamma(C) \to \Gamma(A)$ of C on A with anchor $\mu_0\colon M \to P$ and a Lie subalgebroid $B \to N$ of A, the *stabilizer* of B under the action is the Lie algebra

$$\operatorname{stab}(\rho, B) = \operatorname{stab}(C, B) = \rho^{-1}\big(\Gamma(A;B)\big)/\Gamma\big(C; 0_{\mu_0(N)}\big).$$

Let D be a Lie subalgebroid of C with base manifold $Q \subseteq P$. We say B is *stable under* D or D*-stable* if $\mu_0(N) \subseteq Q$ and $\Gamma(C;D) \subseteq \rho^{-1}(\Gamma(A;B))$.

3.1.6 Remarks. (i) The stabilizer $\mathrm{stab}(C,B)$ is a Lie-Rinehart algebra in the sense of [23], but as the next remark shows it is usually not the space of sections of a Lie subalgebroid of C.
(ii) The Lie algebroid $A \to M$ acts on the tangent bundle TM via the anchor $A \to TM$. The stabilizer of a point $x \in M$ (viewed as the zero bundle $0_x \subseteq TM$) in the sense of Definition 3.1.5 is $\mathrm{stab}(A,x) = \ker(\mathbf{an}_{A,x})$, which agrees with the usual definition (Remark A.3.8(ii)). The stabilizer of the zero subalgebroid 0_M is $\mathrm{stab}(A,0_M) = \ker(\mathbf{an}_A) \subseteq \Gamma(A)$, which is not the space of sections of a subbundle of A unless the anchor has constant rank.
(iii) Definition 3.1.5 is correct only if the submanifold N is closed and embedded, which will always be the case in the situations that concern us. (If N is not closed or embedded the definition must be modified as follows. Call a pair of open subsets $U \subseteq M$ and $V \subseteq N$ *adapted* to N if V is a closed embedded submanifold of U. There is a unique sheaf of $\mathcal{C}^\infty_N$-modules $\mathcal{S}$ such that $\mathcal{S}(V) = \mathrm{stab}(C|_U, B|_V)$ for every adapted pair (U,V). Define $\mathrm{stab}(C,B) = \mathcal{S}(N)$ to be the space of global sections of $\mathcal{S}$.)

Lie algebroid actions can be restricted to Lie subalgebroids in the following way.

3.1.7 Lemma. *Let $A \to M$ and $C \to P$ be Lie algebroids and let $\rho\colon \Gamma(C) \to \Gamma(A)$ be a C-action on A with anchor $\mu_0\colon M \to P$. Let $B \to N$ be a Lie subalgebroid of A. The action ρ restricts to a Lie algebra homomorphism $\mathrm{stab}(C,B) \to \Gamma(B)$. If B is stable under the action of a Lie subalgebroid D of C, we have a natural homomorphism $\Gamma(D) \to \mathrm{stab}(C,B)$ and hence a homomorphism $\rho_D\colon \Gamma(D) \to \Gamma(B)$, which is a D-action on B with anchor $\mu_0|_N$.*

Proof. If a section $\sigma \in \Gamma(C)$ vanishes on $\mu_0(N)$, then by $\mathcal{C}^\infty$-linearity $\rho(\sigma)$ vanishes on N. Therefore ρ descends to a homomorphism

$$\mathrm{stab}(C,B) = \rho^{-1}(\Gamma(A;B))/\Gamma(C;0_{\mu_0(N)}) \longrightarrow \Gamma(B) \cong \Gamma(A;B)/\Gamma(A;0_N).$$

If B is stable under $D \to Q$, then the inclusion $\Gamma(C;D) \subseteq \rho^{-1}(\Gamma(A;B))$ gives us a homomorphism

$$\Gamma(D) \cong \Gamma(C;D)/\Gamma(C;0_Q) \longrightarrow \mathrm{stab}(C,B).$$

The composition of these two homomorphisms is $\mathcal{C}^\infty$-linear over $\mu_0|_N$. □

Recall that a Poisson structure on a Lie algebroid A gives rise to a Lie algebroid structure on the dual bundle A^* (Theorem 2.1.1). The following lemma shows that a Poisson morphism gives rise to a Lie algebroid action in the same way that a moment map $M \to \mathfrak{g}^*$ gives rise to a Hamiltonian Lie algebra action $\mathfrak{g} \to \Gamma(TM)$ on a symplectic manifold.

3.1.8 Lemma. *Let $(A \to M, \lambda_A)$ and $(E \to P, \lambda_E)$ be Poisson Lie algebroids.*

(i) Let $\mu\colon A \to E$ be a Poisson morphism. The pullback map

$$\mu^*\colon \Gamma(E^*) = \Omega^1_E(P) \longrightarrow \Gamma(A^*) = \Omega^1_A(M)$$

defines an action of the Lie algebroid E^* on the Lie algebroid A^* with anchor the base map $\mathring{\mu}\colon M \to P$ of μ.

(ii) Let $\rho\colon \Omega^1_E(P) \to \Omega^1_A(M)$ be an E^*-action on A^* with anchor $\mu_0\colon M \to P$. Suppose that the conditions $\rho \circ d_E = d_A \circ \rho$ and $\rho^* \circ \lambda_A^\sharp \circ \rho = \lambda_E^\sharp$ are satisfied. Then there exists a unique Poisson morphism $\mu\colon A \to E$ with base map $\mathring{\mu} = \mu_0$ such that $\mu^* = \rho$.

Proof. (i) The pullback map μ^* is $\mathcal{C}^\infty$-linear. It is a Lie algebra homomorphism with respect to the bracket (2.1.3) because the 2-sections λ_A and λ_E are μ-related.

(ii) Let $\rho^*\colon A \to \mu_0^*E$ be the transpose of ρ and $\mu\colon A \to E$ the composition of ρ^* with the natural map $\mu_0^*E \to E$. Then μ is the unique vector bundle map with base map μ_0 such that the pullback map on sections $\mu^*\colon \Gamma(E^*) \to \Gamma(A^*)$ coincides with ρ. Since ρ commutes with the exterior derivative, by Vaintrob's theorem [43] (cf. also [37, § 12.2]) μ is a Lie algebroid morphism. The condition $\rho^* \circ \lambda_A^\sharp \circ \rho = \lambda_E^\sharp$ gives that $\mu \circ \lambda_A^\sharp \circ \mu^* = \lambda_E^\sharp$, i.e. μ is Poisson. ☐

3.1.9 Definition. Let $\mu\colon A \to E$ be a Poisson morphism of Poisson Lie algebroids $(A \to M, \lambda_A)$ and $(E \to P, \lambda_E)$ and let μ^* be the E^*-action on A^* of Lemma 3.1.8(i). The E^*-action γ on A with anchor $\mathring{\mu}$ obtained by composing the maps

$$\gamma\colon \Omega^1_E(P) \xrightarrow{\mu^*} \Omega^1_A(M) \xrightarrow{\lambda_A^\sharp} \Gamma(A)$$

is the *Hamiltonian action* of E^* on A with *moment* $\mu\colon A \to E$. A function $f \in \mathcal{C}^\infty(M)$ is *collective* for the Hamiltonian action if it is of the form $f = g \circ \mathring{\mu}$ for some $g \in \mathcal{C}^\infty(P)$.

3.1.10 Remark. Under the Hamiltonian action γ a collective function $f = g \circ \mathring{\mu}$ acts by the Hamiltonian section

$$\gamma(d_E g) = \lambda_A^\sharp \mu^* d_E g = \lambda_A^\sharp d_A \mathring{\mu}^* g = \lambda_A^\sharp d_A f = \sigma_f$$

(see (2.1.6)), which leaves λ_A invariant.

For each $x \in M$ the Hamiltonian action γ defines a linear map on the fibres $\gamma_x \colon E^*_{\mathring{\mu}(x)} \to A_x$ (notational convention 3.1.3). The following lemma describes the kernel and the image of γ_x. Recall that $W^\circ \subseteq V^*$ denotes the annihilator of a subspace W of a vector space V.

3.1.11 Lemma. *Let $(A \to M, \lambda_A)$ and $(E \to P, \lambda_E)$ be Poisson Lie algebroids. Let $\mu \colon A \to E$ be a Poisson morphism and let $\gamma \colon \Gamma(E^*) \to \Gamma(A)$ be the associated Hamiltonian action. Let $x \in M$ and $y = \mathring{\mu}(x) \in P$. Let $\gamma_x^* \colon A_x^* \to E_y$ be the transpose of $\gamma_x \colon E_y^* \to A_x$.*

(i) We have the identities

$$\gamma_x = \lambda^\sharp_{A,x} \circ \mu_x^*, \qquad \gamma_x^* = -\mu_x \circ \lambda^\sharp_{A,x}, \qquad \lambda^\sharp_{E,y} = \mu_x \circ \gamma_x = -\gamma_x^* \circ \mu_x^*.$$

(ii) For every linear subspace $\mathfrak{l}$ of E_y^* we have

$$\left(\mu_x \circ \lambda^\sharp_{A,x}\right)^{-1}(\mathfrak{l}^\circ) = (\gamma_x(\mathfrak{l}))^\circ.$$

In particular $\ker\left(\mu_x \circ \lambda^\sharp_{A,x}\right) = (\mathrm{im}(\gamma_x))^\circ$.

(iii) For every linear subspace L of A_x we have

$$\left(\mu_x \circ \lambda^\sharp_{A,x}\right)(L^\circ) = (\gamma_x^{-1}(L))^\circ.$$

In particular $\mathrm{im}\left(\mu_x \circ \lambda^\sharp_{A,x}\right) = (\ker(\gamma_x))^\circ$.

Proof. (i) This follows from the commutativity of the diagram

$$\begin{tikzcd} A_x^* \arrow[r, "\lambda^\sharp_{A,x}"] & A_x \arrow[d, "\mu_x"] \\ E_y^* \arrow[u, "\mu_x^*"] \arrow[ur, "\gamma_x"] \arrow[r, "\lambda^\sharp_{E,y}"] & E_y \end{tikzcd}$$

(Definition 2.3.1) and from the fact that $\lambda^\sharp_A$ and $\lambda^\sharp_E$ are antisymmetric.

(ii) and (iii) follow immediately from (i). □

3.1.12 Remark. The identity morphism $E \to E$ is Poisson and therefore generates a Hamiltonian action $\lambda^\sharp_E \colon \Gamma(E^*) \to \Gamma(E)$. For $E = T\mathfrak{g}^*$, the tangent bundle of the dual of a Lie algebra $\mathfrak{g}$, this is the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. The equality $\lambda^\sharp_{E,y} = \mu_x \circ \gamma_x$ of Lemma 3.1.11(i) expresses the *equivariance* of the moment μ with respect to the actions γ and $\lambda^\sharp_E$: for every 1-form

$\alpha \in \Omega^1_E(P)$ the 1-form $\gamma(\alpha) \in \Omega^1_A(M)$ and the section $\lambda^\sharp_E(\alpha) \in \Gamma(E)$ are μ-related.

3.2. Hamiltonian reduction

Let $\mu\colon A \to E$ be a Poisson morphism between two Poisson Lie algebroids $(A \to M, \lambda_A)$ and $(E \to P, \lambda_E)$ and let γ be the associated Hamiltonian action of the Lie algebroid E^* on the Lie algebroid A (Definition 3.1.9). *Reducing* the Hamiltonian action means choosing a suitable Lie subalgebroid B of A, forming the quotient of B by (a suitable restriction of) the E^*-action, and pushing the A-Poisson structure on M down to the quotient Lie algebroid B/E^*.

We will treat here the case of most interest to us, namely where the Lie algebroid A is symplectic and B is the moment fibre of a "point" of E. In the realm of Lie algebroids a "point" means a Lie algebra. Accordingly, by a "point" of E we will mean a pair $(p, \mathfrak{f})$, where p is a point of P and $\mathfrak{f}$ is a Lie subalgebra of the stabilizer $\mathrm{stab}(E, p)$. At a point p where the anchor of E is not injective, various choices for $\mathfrak{f}$ are possible, which lead to different reduced symplectic Lie algebroids.

Theorem 3.2.1 departs in two other ways from the Marsden-Weinstein theorem [33] and the Mikami-Weinstein theorem [38]. Firstly, whereas in the classical setting regularity of the moment map and freeness of the action are narrowly related, this relationship is looser in our context and these two conditions need to be imposed separately (but see Lemma 3.2.6). Secondly, the stabilizer $\mathrm{stab}(E^*, \mathfrak{f})$ of the "point" $\mathfrak{f}$ does not preserve the null foliation of the fibre, so to obtain a quotient which is symplectic we must pass to a proper subalgebra of the stabilizer.

3.2.1 Theorem (Hamiltonian reduction for symplectic Lie algebroids)**.** *Let $(A \to M, \omega)$ be a symplectic Lie algebroid and let $(E \to P, \lambda)$ be a Poisson Lie algebroid. Let $\mu\colon A \to E$ be a Poisson morphism and let $\gamma\colon \Gamma(E^*) \to \Gamma(A)$ be the associated Hamiltonian action. Let $p \in P$ and $N = \mathring{\mu}^{-1}(p)$. Suppose that p is a regular value of $\mathring{\mu}\colon M \to P$ and that the E^*-action on TM defined by $\mathbf{an}_A \circ \gamma\colon \Gamma(E^*) \to \Gamma(TM)$ is locally free at x for all $x \in N$.*

(i) Let $\mathfrak{f}$ be a Lie subalgebra of $\mathrm{stab}(E, p)$. Then μ is transverse to $\mathfrak{f}$, $B = \mu^{-1}(\mathfrak{f})$ is a Lie subalgebroid of A, whose base is the submanifold N of M, and the 2-form $\omega_B = i_B^*\omega$ on B is presymplectic. The null Lie algebroid $K = \ker(\omega_B)$ is a foliation Lie algebroid and the Lie subalgebra

$$\mathfrak{h} = (\lambda_p^\sharp)^{-1}(\mathfrak{f}) \cap \mathfrak{f}^\circ$$

of $\operatorname{stab}(E^*, \mathfrak{f})$ *acts transitively on* K.

(ii) Suppose that the leaf space $Q = N/\mathfrak{h}$ *is a manifold and that the canonical flat* K*-connection on* B/K *has trivial holonomy. Then* ω_B *descends to a* C*-symplectic structure* $\omega_C \in \Omega^2_C(Q)$ *on the quotient Lie algebroid* $C = (B/K)/\mathfrak{h}$.

Proof. (i) The stabilizer at $x \in M$ of the E^*-action on TM is

$$\ker(\mathbf{an}_A \circ \gamma_x) = \gamma_x^{-1}(\operatorname{stab}(A, x)) = \big(\mu_x(\operatorname{stab}(A, x)^\omega)\big)^\circ,$$

where we used Lemma 3.1.11(iii). Our local freeness hypothesis amounts to $\ker(\mathbf{an}_A \circ \gamma_x) = 0$, i.e.

$$\mu_x(\operatorname{stab}(A, x)^\omega) = E_p \tag{3.2.2}$$

for all $x \in N$. It now follows from our regular value hypothesis that the moment $\mu\colon A \to E$ is transverse to the Lie subalgebra $\mathfrak{f}$; see Remark A.4.2. Hence, by the regular value theorem, Proposition A.4.1, $B = \mu^{-1}(\mathfrak{f})$ is a Lie subalgebroid over $N = \mathring{\mu}^{-1}(p)$. By Definition 3.1.5 the stabilizer of B under the action γ is the Lie algebra

$$\operatorname{stab}(E^*, B) = \gamma^{-1}\big(\Gamma(A; B)\big)/\Gamma\big(E^*; 0_{\mathring{\mu}(N)}\big).$$

If N is empty, the theorem is true for trivial reasons. Suppose now that N is nonempty. Then

$$\begin{aligned}\gamma^{-1}\big(\Gamma(A; B)\big) &= \{\, \tau \in \Gamma(E^*) \mid \mu_x(\gamma_x(\tau)) \in \mathfrak{f} \text{ for all } x \in N \,\}\\ &= \{\, \tau \in \Gamma(E^*) \mid \lambda_p^\sharp(\tau) \in \mathfrak{f} \,\},\end{aligned}$$

where we used the equivariance of μ, Lemma 3.1.11(i). Also

$$\Gamma\big(E^*; 0_{\mathring{\mu}(N)}\big) = \Gamma(E^*; 0_p) = \{\, \tau \in \Gamma(E^*) \mid \tau_p = 0 \,\},$$

so

$$\operatorname{stab}(E^*, B) = (\lambda^\sharp)^{-1}(\mathfrak{f}) = \operatorname{stab}(E^*, \mathfrak{f}),$$

a Lie subalgebra of $\operatorname{stab}(E^*, p)$. By Lemma 3.1.7 this stabilizer acts on B, but it does not preserve the intersection $K = B \cap B^\omega$. The fibre at $x \in N$ of the symplectic orthogonal of B is

$$B_x^\omega = \omega_x^\sharp(B_x^\circ) = \big((\mu_x \circ \omega_x^\sharp)^{-1}(\mathfrak{f})\big)^\circ = \gamma_x(\mathfrak{f}^\circ),$$

where we used Lemma 3.1.11(i). This yields

$$K_x = B_x \cap B_x^\omega = \mu_x^{-1}(\mathfrak{f}) \cap \gamma_x(\mathfrak{f}^\circ).$$

The local freeness hypothesis implies that $\gamma_x \colon E_p^* \to A_x$ is injective for all $x \in N$. Therefore

$$\gamma_x^{-1}(K_x) = \gamma_x^{-1}(\mu_x^{-1}(\mathfrak{f}) \cap \gamma_x(\mathfrak{f}^\circ)) = \gamma_x^{-1}(\mu_x^{-1}(\mathfrak{f})) \cap \mathfrak{f}^\circ = (\lambda_p^\sharp)^{-1}(\mathfrak{f}) \cap \mathfrak{f}^\circ = \mathfrak{h},$$

where we used $\lambda_p^\sharp = \mu_x \circ \gamma_x$ (Lemma 3.1.11(i)). Thus the action of $\mathfrak{h}$ preserves the Lie algebroid K and for every $x \in N$ the map $\gamma_x \colon E_p^* \to A_x$ maps $\mathfrak{h}$ bijectively onto K_x. It follows that ω_B has constant rank and that $\mathfrak{h}$ acts locally freely and transitively on K in the sense of Definition 3.1.3. In other words, γ induces an isomorphism from the action Lie algebroid $\mathfrak{h} \ltimes N$ onto K and in particular the anchor of K is injective.

(ii) This follows from (i) and Theorem 2.5.5. ☐

3.2.3 Definition. The symplectic Lie algebroid $(C \to Q, \omega_C)$ is the *symplectic quotient* of $A \to M$ at the "point" $(p, \mathfrak{f})$.

3.2.4 Example (Reduction with respect to the identity map)**.** Let $\mu = \mathrm{id}_M$ be the identity map of M. The associated Hamiltonian action is the identity morphism $\gamma = \mathrm{id} \colon A \to A$. We can reduce M at any "point" $(x, \mathfrak{a})$, where $\mathfrak{a}$ is a Lie subalgebra of $\mathrm{stab}(A, x)$. The symplectic quotient of M at the "point" $(x, \mathfrak{a})$ is the point x equipped with the symplectic Lie algebra $\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{a}^\omega)$.

3.2.5 Remarks. (i) The hypothesis of Theorem 3.2.1(ii) is satisfied if the action of $\mathfrak{h}$ on B integrates to a proper and free action of a Lie group H with Lie algebra $\mathfrak{h}$.
(ii) See Section 5.2 for further illustrations of Theorem 3.2.1.

In the special case where the anchor of E is bijective at p, i.e. $E \cong TP$ in a neighbourhood of p, we have the following relationship between freeness of the Hamiltonian action and regularity of the moment map.

3.2.6 Lemma. *In the context of Theorem 3.2.1 suppose that the anchor of E is bijective at p. Let $x \in \mathring{\mu}^{-1}(p)$ and let L be the symplectic leaf of x with respect to the Poisson structure on M determined by the A-symplectic form ω. Then the action $\mathbf{an}_A \circ \gamma \colon \Gamma(E^*) \to \Gamma(TM)$ is locally free at x if and only if x is a regular point of the map $\mathring{\mu}|_L$.*

Proof. Since $\mathbf{an}_E$ is bijective at p and $\mathbf{an}_E \circ \mu = T\mathring{\mu} \circ \mathbf{an}_A$, condition (3.2.2) is equivalent to $T\mathring{\mu} \circ \mathbf{an}_A(\mathrm{stab}(A, x)^\omega) = T_p P$. Let λ_M be the Poisson structure determined by ω. Using (2.3.4) we see that $\mathbf{an}_A(\mathrm{stab}(A, x)^\omega) = \{0\}^{\lambda_M} = T_x L$. So the E^*-action on M is locally free at x if and only if $T_x \mathring{\mu}(T_x L) = T_p P$. ☐

4. Some normal forms for symplectic Lie algebroids

In this section we establish a Lie algebroid version of the Darboux-Moser-Weinstein theorem, Theorem 4.3.3 and of the coisotropic embedding theorem, Theorem 4.4.1. Our method is based on Lie algebroid homotopies, which were introduced (for Lie algebroid paths) by Crainic and Fernandes [11, § 1]. We review in Section 4.1 how a Lie algebroid homotopy gives rise to a cochain homotopy of the de Rham complex. Of particular interest in the context of the Moser trick are deformation retractions, which are a weak version of Lie algebroid splittings. Section 4.2 describes a technique for obtaining deformation retractions of Lie algebroids based on the notion of an Euler-like section of Bischoff et al. [1] and Bursztyn et al. [6].

4.1. Lie algebroid homotopies

Let $A \to M$ be a Lie algebroid. As in ordinary de Rham theory the operations d_A, ι_A, and $\mathcal{L}_A$ lead naturally to homotopy formulas and a Poincaré lemma for the de Rham complex of A. Let $T\mathbf{R} = \mathbf{R} \times \mathbf{R}$ be the tangent bundle of $\mathbf{R}$ and let $T\mathbf{R} \times A \to \mathbf{R} \times M$ be the direct product Lie algebroid. The Cartesian projections $\mathrm{pr}_1 \colon T\mathbf{R} \times A \to T\mathbf{R}$ and $\mathrm{pr}_2 \colon T\mathbf{R} \times A \to A$ are Lie algebroid morphisms. For each t the map

$$i_t \colon A \to T\mathbf{R} \times A \tag{4.1.1}$$

defined by $i_t(a) = (0_t, a)$ (where $a \in A$ and where 0_t denotes the origin of $T_t\mathbf{R}$) is likewise a Lie algebroid morphism. The section $\partial/\partial t$ of $T\mathbf{R}$ can be regarded as the section $(t, x) \mapsto (t, 1, 0_x)$ of $T\mathbf{R} \times A$ and as such generates a flow Υ_t. For each t the morphism

$$\Upsilon_t \colon T\mathbf{R} \times A \longrightarrow T\mathbf{R} \times A \tag{4.1.2}$$

is simply the shift by t in the $\mathbf{R}$-direction. The infinite cylinder $\mathbf{R} \times M$ contains as a submanifold the finite cylinder $[0, 1] \times M$, and the product Lie algebroid $T[0, 1] \times A$ over $[0, 1] \times M$ is a Lie subalgebroid of the product $T\mathbf{R} \times A$. The operator

$$\kappa \colon \Omega^k_{T[0,1]\times A}([0, 1] \times M) \longrightarrow \Omega^{k-1}_A(M)$$

is defined as follows: for a k-form $\alpha \in \Omega^k_{T[0,1]\times A}([0, 1] \times M)$ and sections σ_1, $\sigma_2, \ldots, \sigma_{k-1} \in \Gamma(A)$ put

$$(\kappa\alpha)(\sigma_1, \sigma_2, \ldots, \sigma_{k-1}) = \int_0^1 \alpha(\partial/\partial t, \sigma_1, \sigma_2, \ldots, \sigma_{k-1})\, dt. \tag{4.1.3}$$

It follows from (A.5.7) and (A.5.9) that

$$\begin{aligned}\kappa d_A\alpha + d_A\kappa\alpha &= \int_0^1 \iota_A(\partial/\partial t)d_A\alpha\,dt + d_A\int_0^1 \iota_A(\partial/\partial t)\alpha\,dt\\ &= \int_0^1 (\iota_A(\partial/\partial t)d_A\alpha + d_A\iota_A(\partial/\partial t)\alpha)\,dt\\ &= \int_0^1 \mathcal{L}_A(\partial/\partial t)\alpha\,dt = \int_0^1 \frac{d}{du}\Upsilon_u^*\alpha\big|_{u=0}\,dt = i_1^*\alpha - i_0^*\alpha,\end{aligned}$$

where Υ_t is as in (4.1.2). This proves

$$[\kappa, d_A] = i_1^* - i_0^*. \tag{4.1.4}$$

4.1.5 Definition. A *Lie algebroid homotopy* is a Lie algebroid morphism $\varphi\colon T[0,1]\times A \to B$, where $A \to M$ and $B \to N$ are Lie algebroids.

Let φ a Lie algebroid homotopy. The base map $\mathring{\varphi}\colon [0,1]\times M \to N$ is then a homotopy of manifolds in the usual sense. We denote the Lie algebroid morphism $i_t \circ \varphi\colon A \to B$, which is defined for $0 \le t \le 1$, by φ_t, and we say that φ is a homotopy *between* φ_0 *and* φ_1. Put

$$\kappa_\varphi = \kappa \circ \varphi^*, \tag{4.1.6}$$

where κ is the operator (4.1.3); then (4.1.4) yields the *homotopy formula*

$$[\kappa_\varphi, d_A] = \varphi_1^* - \varphi_0^*, \tag{4.1.7}$$

which tells us that $\kappa_\varphi\colon \Omega_B^\bullet(N) \to \Omega_A^{\bullet-1}(M)$ is a homotopy of complexes.

4.1.8 Definition. A Lie algebroid morphism $\varphi\colon A \to B$ is a *homotopy equivalence* if it has a *homotopy inverse*, i.e. a morphism $\psi\colon B \to A$ such that $\psi\circ\varphi$ is homotopic to id_A and $\varphi\circ\psi$ is homotopic to id_B.

The homotopy formula has the usual consequences, for instance the following lemma.

4.1.9 Lemma. *Let $A \to M$ and $B \to N$ be Lie algebroids.*

(i) Homotopic morphisms φ_0, $\varphi_1\colon A \to B$ induce the same morphism in cohomology $\varphi_0^* = \varphi_1^*\colon H_B^\bullet(N) \to H_A^\bullet(M)$.

(ii) A homotopy equivalence $\varphi\colon A \to B$ induces an isomorphism in cohomology

$$\varphi^*\colon H_B^\bullet(N) \xrightarrow{\cong} H_A^\bullet(M).$$

4.1.10 Remark. Lemma 4.1.9 gives the following version of the Poincaré lemma: if A is homotopy equivalent to a "point", i.e. a Lie subalgebra $\mathfrak{a}$ of $\operatorname{stab}(A, x)$ for some $x \in M$, then $H^\bullet_A(M)$ is isomorphic to $H^\bullet(\mathfrak{a})$, the Lie algebra cohomology of $\mathfrak{a}$. So if $\mathfrak{a}$ is acyclic, then $H^0_A(M) = \mathbf{R}$ and $H^k_A(M) = 0$ for $k \geq 1$.

4.1.11 Definition. Let $B \to N$ be a Lie subalgebroid of A. A *weak deformation retraction of A onto B* is a homotopy $\varrho\colon T[0,1] \times A \to A$ such that $\varrho_1 = \mathrm{id}_A$, $\varrho_0(A) = B$, and $\varrho_0|_B = i_B$.

The next statement follows immediately from Lemma 4.1.9.

4.1.12 Corollary. *Let $A \to M$ be a Lie algebroid and let $B \to N$ be a Lie subalgebroid of A. Suppose there exists a weak deformation retraction $\varrho\colon T[0,1] \times A \to A$ onto B. The inclusion $i_B\colon B \to A$ induces an isomorphism $i_B^*\colon H^\bullet_A(M) \xrightarrow{\cong} H^\bullet_B(N)$ with inverse ϱ_0^*.*

4.1.13 Definition. Let $B \to N$ be a Lie subalgebroid of A. A *deformation retraction of A onto B* is a weak deformation retraction $\varrho\colon T[0,1] \times A \to A$ with the additional property that $\varrho(t, u, b) = b$ for all $(t, u) \in T[0,1] = [0,1] \times \mathbf{R}$ and all $b \in B$.

A deformation retraction ϱ satisfies $\varrho(\partial/\partial t, 0_x) = 0_x$ for all $x \in N$, where we think of $\partial/\partial t$ as the section $(t, 1)$ of $T[0,1]$ and where 0_x denotes the origin of the fibre A_x. Its base homotopy $\mathring{\varrho}\colon [0,1] \times M \to M$ restricts to the identity map $\mathring{\varrho}_t|_N = \mathrm{id}_N$ for all t, i.e. $\mathring{\varrho}$ is a deformation retraction in the usual sense. These facts have the following significance for the homotopy κ_ϱ.

4.1.14 Lemma. *Let $A \to M$ be a Lie algebroid and $B \to N$ a Lie subalgebroid. Let $\varrho\colon T[0,1] \times A \to A$ be a deformation retraction onto B. Then $(\kappa_\varrho \alpha)_x = 0$ for all A-forms $\alpha \in \Omega^k_A(M)$ and all $x \in N$. If α vanishes at a point $x_0 \in N$, then $(\mathcal{L}_A(\sigma)\kappa_\varrho\alpha)_{x_0} = 0$ for all sections σ of A.*

Proof. Let $\sigma_1, \sigma_2, \ldots, \sigma_{k-1}$ be sections of A and let f be the smooth function

$$f = \iota_A(\sigma_{k-1}) \cdots \iota_A(\sigma_2)\iota_A(\sigma_1)\kappa_\varrho\alpha$$

on M. From (4.1.3) and (4.1.6) we obtain

(4.1.15)
$$f(x) = \int_0^1 \alpha_{\mathring{\varrho}(t,x)}\big(\varrho(\partial/\partial t, 0_x), \varrho_t(\sigma_1(x)), \varrho_t(\sigma_2(x)), \ldots, \varrho_t(\sigma_{k-1}(x))\big)\, dt$$

for $x \in M$. Since ϱ is a deformation retraction, we have $\varrho(\partial/\partial t, 0_x) = 0_x$ for $x \in N$, so it follows from (4.1.15) that the function f vanishes on N. Thus $(\kappa_\varrho \alpha)_x = 0$ for all $x \in N$. If $\alpha_{x_0} = 0$ for some $x_0 \in N$, then $\alpha_{\mathring{\varrho}(t,x_0)} = \alpha_{x_0} = 0$ for all t, so we see from (4.1.15) that the function f vanishes to second order at x_0. Since $\mathcal{L}_A(\sigma)$ is a first-order differential operator, this tells us that $(\mathcal{L}_A(\sigma)f)(x_0) = 0$ for all sections σ of A. Using $[\mathcal{L}_A(\sigma), \iota_A(\sigma_i)] = \iota_A([\sigma, \sigma_i])$ we obtain $(\mathcal{L}_A(\sigma)\kappa_\varrho \alpha)_{x_0} = 0$. □

4.2. Isotopies and deformation retractions from sections

For the results of Section 4.1 to be useful we need methods for producing Lie algebroid homotopies. We are aware of two such methods: (1) the flow of a time-dependent section produces an isotopy, and (2) if the section is Euler-like near a subalgebroid in the sense of Bischoff et al. [1] and Bursztyn et al. [6], we can reparametrize the isotopy to produce a deformation retraction onto the subalgebroid. We explain both methods in turn.

4.2.1 Definition. Let $A \to M$ be a Lie algebroid. A *(global) Lie algebroid isotopy* is a Lie algebroid homotopy $\varphi\colon T[0,1] \times A \to A$ such that $\varphi_0 = \mathrm{id}_A$ and each φ_t is an isomorphism.

Just as time-dependent vector fields give rise to isotopies of manifolds, so time-dependent sections give rise to isotopies of Lie algebroids.

4.2.2 Proposition. *Let $A \to M$ be a Lie algebroid. Let σ_t be a time-dependent section of A defined for $0 \le t \le 1$, let $\mathbf{ad}_A(\sigma_t)$ be the time-dependent vector field on A defined in (A.5.3), and let Φ_t be its flow with initial condition $\Phi_0 = \mathrm{id}_A$. Suppose that Φ_t is defined globally on A for all $0 \le t \le 1$. Then $\Phi\colon [0,1] \times A \to A$ extends to a Lie algebroid isotopy $\varphi\colon T[0,1] \times A \to A$ given by*

$$\varphi(t,u,a) = u\sigma_t(\mathring{\Phi}_t(\pi_A(a))) + \Phi_t(a)$$

for $(t,u) \in T[0,1] = [0,1] \times \mathbf{R}$ and $a \in A$.

Proof. The equalities

$$\varphi_t(a) = \varphi \circ i_t(a) = \varphi(t,0,a) = \Phi_t(a),$$

where i_t is as in (4.1.1), show that $\varphi_0 = \mathrm{id}_A$, that $\varphi_t = \Phi_t$ is a Lie algebroid isomorphism for each t, and that the map φ extends Φ. It remains to show that

φ is a Lie algebroid morphism. Let $\hat{A}$ be the product Lie algebroid $T[0,1] \times A$ and define $\hat{\varphi}\colon \hat{A} \to \hat{A}$ by

$$\hat{\varphi}(t,u,a) = (t,u,\varphi(t,u,a)).$$

Then $\varphi = \mathrm{pr}_A \circ \hat{\varphi}$, where $\mathrm{pr}_A\colon \hat{A} \to A$ is the projection, so it is enough to show that $\hat{\varphi}$ is a Lie algebroid morphism. For $0 \leq s \leq 1$ define $\hat{\varphi}_s\colon \hat{A} \to \hat{A}$ by

$$\hat{\varphi}_s(t,u,a) = \Big(t,u,su\sigma_{st}(\mathring{\Phi}_{st}(\pi_A(a))) + \Phi_{st}(a)\Big).$$

Then $\hat{\varphi}_0 = \mathrm{id}_{\hat{A}}$, $\hat{\varphi}_1 = \hat{\varphi}$, and

$$\begin{aligned}\frac{d\hat{\varphi}_s}{ds}(t,u,a) = u\sigma_{st}(\mathring{\Phi}_{st}(\pi_A(a))) + stu\dot{\sigma}_{st}(\mathring{\Phi}_{st}(\pi_A(a)))+ \\ stuT\sigma_{st}\big(\mathbf{an}_A(\sigma_{st}(\pi_A(a)))\big) + t\,\mathbf{ad}_A(\sigma_{st})(\Phi_{st}(a)),\end{aligned} \tag{4.2.3}$$

where we used the differential equations

$$\frac{d\Phi_t}{dt}(a) = \mathbf{ad}_A(\sigma_t)(\Phi_t(a)) \qquad \text{and} \qquad \frac{d\mathring{\Phi}_t}{dt}(x) = \mathbf{an}_A(\sigma_t)\big(\mathring{\Phi}_t(x)\big).$$

Define an s-dependent section $\hat{\sigma}_s \in \Gamma(\hat{A})$ by $\hat{\sigma}_s(t,x) = t\sigma_{st}(x)$. A computation using (A.5.3) shows that

$$\mathbf{ad}_{\hat{A}}(\hat{\sigma}_s)(t,u,a) = t\,\mathbf{ad}_A(\sigma_{st})(a) + u\sigma_{st}(\pi_A(a)) + stu\dot{\sigma}_{st}(\pi_A(a)).$$

Comparing with (4.2.3) we see that

$$\frac{d\hat{\varphi}_s}{ds}(t,u,a) = \mathbf{ad}_{\hat{A}}(\hat{\sigma}_s)(\hat{\varphi}_s(t,u,a)).$$

Thus $\hat{\varphi}_s$ is the flow of the vector field $\mathbf{ad}_{\hat{A}}(\hat{\sigma}_s)$ on $\hat{A}$. It now follows from Lemma A.5.6(i) that $\hat{\varphi}_s$ is a Lie algebroid automorphism for each s. In particular $\hat{\varphi} = \hat{\varphi}_1$ is a Lie algebroid automorphism. □

A vector field v on a manifold M which is tangent to a submanifold N can be viewed as a smooth map of pairs $v\colon (M,N) \to (TM,TN)$. Its tangent map is then a map of pairs $Tv\colon (TM,TN) \to (TTM,TTN)$, and hence descends to a map of normal bundles

$$\mathcal{N}(v)\colon \mathcal{N}(M,N) \longrightarrow \mathcal{N}(TM,TN) \cong T\mathcal{N}(M,N),$$

i.e. a vector field on the normal bundle $\mathcal{N}(M,N)$ that is linear along the fibres. We call $\mathcal{N}(v)$ the *linear approximation* to v along N. A vector field v on M is called *Euler-like* along N if v is complete, vanishes on N, and $\mathcal{N}(v)$ is equal to the Euler vector field of the vector bundle $\mathcal{N}(M,N)$. The significance of this notion lies in the following "quantitative" version of the tubular neighbourhood theorem [6, Proposition 2.7]: for every Euler-like vector field v there exists a unique tubular neighbourhood embedding $i_v\colon \mathcal{N}(M,N) \to M$ such that $i_v^* v$ is equal to the Euler vector field of $\mathcal{N}(M,N)$. We call i_v the *tubular neighbourhood embedding associated with* v.

4.2.4 Definition. Let $A \to M$ be a Lie algebroid and $B \to N$ a Lie subalgebroid. An *Euler-like* section of A along B is a section $\varepsilon \in \Gamma(A)$ such that $\varepsilon|_N = 0$ and the vector field $\mathbf{ad}_A(\varepsilon)$ on A defined in (A.5.3) is Euler-like along the submanifold B of A.

4.2.5 Lemma. *Let $A \to M$ be a Lie algebroid, $B \to N$ a Lie subalgebroid, and $\varepsilon \in \Gamma(A)$ a section of A.*

- *(i) If the vector field $\mathbf{ad}_A(\varepsilon)$ on A is Euler-like along B, then the vector field $\mathbf{an}_A(\varepsilon)$ on M is Euler-like along N.*
- *(ii) Suppose N cleanly intersects A and let $B = i_N^! A$. If ε vanishes along N and the vector field $\mathbf{an}_A(\varepsilon)$ on M is Euler-like along N, then the section ε is Euler-like along B.*

Proof. We will use the following simple facts: (1) a linear vector field on a vector bundle $E \to M$ is complete if and only if the associated base vector field on M is complete; (2) if $F \to N$ is a second vector bundle and $\varphi\colon E \to F$ is a vector bundle map, then the Euler vector fields of E and F are φ-related; (3) if $v \in \Gamma(TE)$ and $w \in \Gamma(TF)$ are linear vector fields and $v \sim_\varphi w$, then v is uniquely determined by w if F is fibrewise injective, and w is uniquely determined by v if F is fibrewise surjective.

(i) The vector field $\mathbf{ad}_A(\varepsilon)$ is a linear lift of the vector field $\mathbf{an}_A(\varepsilon)$, so $\mathbf{an}_A(\varepsilon)$ is complete because of fact (1) above. Also $\mathbf{an}_A(\varepsilon)|_N = 0$ because $\mathbf{ad}_A(\varepsilon)|_B = 0$. From $\mathbf{ad}_A(\varepsilon) \sim_\pi \mathbf{an}_A(\varepsilon)$ we get

$$\mathcal{N}(\mathbf{ad}_A(\varepsilon)) \sim_{\mathcal{N}(\pi)} \mathcal{N}(\mathbf{an}_A(\varepsilon)).$$

Since $\mathcal{N}(\pi)\colon \mathcal{N}(A,B) \to \mathcal{N}(M,N)$ is a surjective vector bundle map and $\mathcal{N}(\mathbf{ad}_A(\varepsilon))$ is the Euler vector field of $\mathcal{N}(A,B)$, it follows from facts (2) and (3) above that $\mathcal{N}(\mathbf{an}_A(\varepsilon))$ is the Euler vector field of $\mathcal{N}(M,N)$.

(ii) We must show that the vector field $\mathbf{ad}_A(\varepsilon)$ on A is Euler-like along $B = i_N^! A = \mathbf{an}_A^{-1}(N)$. This is verified in [6, § 3.6] under the assumption that

N is transverse to A. The proof in the clean case is almost the same and goes as follows. The module of relative sections $\Gamma(A;B)$ defined in (A.1.1) is a Lie subalgebra of $\Gamma(A)$ and the submodule $\Gamma(A;0_N)$ is an ideal of $\Gamma(A;B)$. By hypothesis $\varepsilon|_N = 0$, so $[\varepsilon, v]|_N = 0$ for all $v \in \Gamma(A;B)$. It now follows from (A.5.3) that the vector field $\mathbf{ad}_A(\varepsilon)$ vanishes on B. It remains to show that its linear approximation $\mathcal{N}(\mathbf{ad}_A(\varepsilon))$ is the Euler vector field of the normal bundle $\mathcal{N}(A,B)$. Let $T\,\mathbf{an}(\varepsilon) \in \Gamma(TTM)$ be the tangent lift of $\mathbf{an}(\varepsilon) \in \Gamma(TM)$. The vector field $T\,\mathbf{an}(\varepsilon)$ is Euler-like along TN, so its linear approximation $\mathcal{N}(T\,\mathbf{an}(\varepsilon))$ is the Euler vector field of the bundle $\mathcal{N}(TM,TN)$. The vector fields $\mathbf{ad}_A(\varepsilon)$ and $T\,\mathbf{an}(\varepsilon)$ are related via the anchor: $\mathbf{ad}_A(\varepsilon) \sim_{\mathbf{an}} T\,\mathbf{an}(\varepsilon)$. Applying the normal bundle functor we obtain

$$\mathcal{N}(\mathbf{ad}_A(\varepsilon)) \sim_{\mathcal{N}(\mathbf{an})} \mathcal{N}(T\,\mathbf{an}(\varepsilon)). \tag{4.2.6}$$

The map $\mathcal{N}(\mathbf{an})$ fits into a commutative diagram of vector bundle maps

$$\begin{tikzcd}
\pi_B^*(i_N^*A/B) \arrow[r, hook] \arrow[d] & \mathcal{N}(A,B) \arrow[r, two heads] \arrow[d, "\mathcal{N}(\mathbf{an})"] & \pi_B^*\mathcal{N}(M,N) \arrow[d] \\
\pi_{TN}^*\mathcal{N}(M,N) \arrow[r, hook] & \mathcal{N}(TM,TN) \arrow[r, two heads] & \pi_{TN}^*\mathcal{N}(M,N),
\end{tikzcd}$$

the rows of which are exact. The vertical map on the right is a fibrewise isomorphism. By Lemma A.3.7(ii) the vertical map on the left is fibrewise injective because N cleanly intersects A. Hence $\mathcal{N}(\mathbf{an})$ is fibrewise injective. It now follows from (4.2.6) and from facts (2) and (3) that $\mathcal{N}(\mathbf{ad}_A(\varepsilon))$ is the Euler vector field of $\mathcal{N}(A,B)$. ☐

The utility of Euler-like sections is that their flows give rise to deformation retractions.

4.2.7 Theorem (Retraction theorem)**.** *Let $A \to M$ be a Lie algebroid and $B \to N$ a Lie subalgebroid. Suppose there exists an Euler-like section ε of A along B. Let U be the tubular neighbourhood of N associated with the Euler-like vector field $\mathbf{an}_A(\varepsilon)$. Then there exist a deformation retraction of $A|_U$ onto B. It follows that $i_B^*\colon H_A^\bullet(U) \to H_B^\bullet(N)$ is an isomorphism.*

Proof. The proof is an adaptation of [6, § 3.6]. Let Φ_t be the flow of the vector field $\mathbf{ad}_A(\varepsilon)$ and let $\varphi\colon T\mathbf{R}\times A \to A$ be the corresponding Lie algebroid isotopy as in Proposition 4.2.2. Since $\mathbf{ad}_A(\varepsilon)$ is Euler-like, near B this isotopy is exponentially expanding in the directions normal to B. We turn it into a retraction by rescaling the time variable. Put $f(s) = \log s$ for $0 < s \le 1$; then the Lie algebroid homotopy

$$\varrho\colon T(0,1]\times A \xrightarrow{Tf\times \mathrm{id}_A} T\mathbf{R}\times A \xrightarrow{\varphi} A$$

is given by the formula

$$\varrho(s,v,a)=\frac{v}{s}\varepsilon(\mathring{\Phi}_{\log s}(\pi(a)))+\Phi_{\log s}(a) \tag{4.2.8}$$

for $(s,v)\in T(0,1]=(0,1]\times\mathbf{R}$ and $a\in A$. The base homotopy $\mathring{\varrho}\colon (0,1]\times M\to M$ is given by $\mathring{\varrho}(s,x)=\mathring{\Phi}_{\log s}(x)$. By Lemma 4.2.5(i) the vector field $\mathbf{an}(\varepsilon)$ is Euler-like along N and therefore determines a tubular neighbourhood embedding $i\colon \mathcal{N}(M,N)\to U$. Let us restrict $\mathring{\varrho}$ to U and ϱ to $A|_U$. If we identify U with the normal bundle via the embedding i, the base homotopy $\mathring{\varrho}$ is given by fibrewise multiplication by s, so it extends smoothly to a homotopy $\mathring{\varrho}\colon [0,1]\times U\to U$, which retracts U onto N. Likewise, if we identify $A|_U$ with the normal bundle $\mathcal{N}(A,B)$ via the tubular neighbourhood embedding $\mathcal{N}(A,B)\to A|_U$ determined by $\mathbf{ad}_A(\varepsilon)$, the flow $\Phi_{\log s}$ is given by fibrewise multiplication by s, which retracts $A|_U$ onto B. By hypothesis ε vanishes on N, so the singularity at $s=0$ in the term $\frac{v}{s}\varepsilon\big(\mathring{\varrho}(s,\pi(a))\big)$ is removable. It follows that the homotopy (4.2.8) extends smoothly to a homotopy $T[0,1]\times A|_U\to A|_U$. We have $\varrho(1,0,a)=\Phi_0(a)=a$ for all $a\in A$, so $\varrho_1=\mathrm{id}_A$. If $b\in B$, then $\varrho(s,v,b)=b\in B$ for all $(s,v)\in T[0,1]$. Also $\varrho(0,0,a)\in B$ for all $a\in A$, so $\varrho_0(A)=B$. We have shown that ϱ is a deformation retraction of $A|_U$ onto B. The second assertion of the theorem is now an immediate consequence of Corollary 4.1.12. ☐

Given a Lie algebroid $A\to M$ and a Lie subalgebroid $B\to N$, the normal bundle $\mathcal{N}(A,B)$ has the structure of a Lie algebroid over the normal bundle $\mathcal{N}(M,N)$. The space $\mathcal{N}(A,B)$ is a double vector bundle

$$\begin{tikzcd} \mathcal{N}(A,B) \arrow[r] \arrow[d] & \mathcal{N}(M,N) \arrow[d] \\ B \arrow[r] & N \end{tikzcd}$$

and the vector bundle operations on the vertical bundle $\mathcal{N}(A,B)\to B$ are Lie algebroid morphisms of $\mathcal{N}(A,B)$; see [37, Theorem A.7].

4.2.9 Definition. The Lie algebroid $\mathcal{N}(A,B)\to\mathcal{N}(M,N)$ is the *linearization of A at B*. The Lie algebroid A is *linearizable at the Lie subalgebroid B* if there is a *linearization map* $\mathcal{N}(A,B)\to A$, i.e. a map that is simultaneously a Lie algebroid morphism and a tubular neighbourhood embedding.

Under a slightly weaker hypothesis than Theorem 4.2.7 (we don't need $\varepsilon|_N = 0$) we obtain a linearization theorem.

4.2.10 Theorem (Linearization theorem)**.** *Let $A \to M$ be a Lie algebroid and $B \to N$ a Lie subalgebroid. Suppose there exists a section ε of A such that the vector field $\mathbf{ad}_A(\varepsilon)$ on A is Euler-like along B. The tubular neighbourhood embedding $i_v \colon \mathcal{N}(A, B) \to A$ associated with $v = \mathbf{ad}_A(\varepsilon)$ on A is a linearization map, whose base map is the tubular neighbourhood embedding $\iota_{\mathring{v}} \colon \mathcal{N}(M, N) \to M$ associated with the Euler-like vector field $\mathring{v} = \mathbf{an}_A(\varepsilon)$ on M.*

Proof. Let $\mathcal{D}(M, N)$ be the deformation space of the pair (M, N). As a set this is the disjoint union

$$\mathcal{D}(M, N) = \mathcal{N}(M, N) \sqcup (\mathbf{R}^\times \times M).$$

See [1, §§ 2–3] and [37, Appendix A] for the manifold structure of $\mathcal{D}(M, N)$ and for the fact that the deformation space $\mathcal{D}(A, B)$ is a Lie algebroid over $\mathcal{D}(M, N)$. The map $\pi \colon \mathcal{D}(A, B) \to \mathbf{R}$ which sends $\mathcal{N}(A, B)$ to 0 and agrees with the projection $\mathbf{R}^\times \times M \to \mathbf{R}^\times$ on the complement of $\mathcal{N}(A, B)$ is a smooth submersion. The fibres $\pi^{-1}(t)$ are Lie algebroids isomorphic to $A \to M$ for $t \neq 0$ and to $\mathcal{N}(A, B) \to \mathcal{N}(M, N)$ for $t = 0$. In this sense $\mathcal{D}(A, B)$ is a family of Lie algebroids that deforms A to its normal bundle in B. Since the vector field $\mathbf{ad}_A(\varepsilon)$ is Euler-like, the vector field $\partial/\partial t + t^{-1}\,\mathbf{ad}(\varepsilon)$, which is defined on the complement of $\mathcal{N}(A, B)$ in $\mathcal{D}(A, B)$, extends smoothly to a complete vector field w on $\mathcal{D}(A, B)$. The vector field w is an infinitesimal Lie algebroid automorphism and is π-related to the vector field $\partial/\partial t$ on $\mathbf{R}$, so its flow at time 1 is a Lie algebroid morphism from $\pi^{-1}(0) = \mathcal{N}(A, B)$ to $\pi^{-1}(1) = A$, which agrees with the tubular neighbourhood embedding i_v. These statements are proved in [37, § A.6] when B is the pullback to a transversal N, but the same proofs work for an arbitary Lie subalgebroid B, as long as we have a section ε with $\mathbf{ad}_A(\varepsilon)$ Euler-like along B. ☐

4.2.11 Remarks. Let $A \to M$ be a Lie algebroid and $B \to N$ a Lie subalgebroid.

(i) If ε is an Euler-like section of A along N, then so is $f\varepsilon$ for any bounded smooth function f which is equal to 1 on a neighbourhood of N. Hence, by Theorem 4.2.7, there exist deformation retractions of $A|_U$ onto B for all U ranging over a basis of tubular neighbourhoods of N.

(ii) Let N be an embedded submanifold that is transverse to A and let $B = i_N^! A$ be the pullback Lie algebroid. Then there automatically exists an Euler-like section of A along B, as proved in [6, Lemma 3.9], and therefore the retraction and linearization theorems apply. In fact, in this case the normal bundle is $\mathcal{N}(A, B) = \pi^!_{M,N} i^!_N A$, where $\pi_{M,N} \colon \mathcal{N}(M, N) \to N$ is the projection, so the linearization theorem amounts to

$$\pi^!_{M,N} i^!_N A \cong A|_U.$$

This is the Dufour-Fernandes-Weinstein splitting theorem. Our Theorems 4.2.7 and 4.2.10 were prompted by the treatment of the splitting theorem given by Bischoff et al. [1] and Bursztyn et al. [6, Theorem 4.1]. See also Meinrenken's notes [37, Theorem 8.13, Appendix A]. The isomorphism $H^\bullet_A(U) \cong H^\bullet_B(N)$ of Theorem 4.2.7 was obtained by Frejlich [13, Theorem B] in the transverse case. If the intersection is clean but not transverse, typically the bundle $\pi^!_{M,N} i^!_N A$ has higher rank than A, so the splitting theorem fails, but sometimes an Euler-like section still exists.
(iii) Let $A = T\mathcal{F}$ be the tangent bundle of a (regular) foliation $\mathcal{F}$ of M. Let $B = i^!_N A$ be the pullback of A to an embedded submanifold N of M that cleanly intersects A. By Proposition A.3.5 the dimension of the intersection $T_x N \cap T_x \mathcal{F}$ is independent of $x \in N$, so $\mathcal{F}$ induces a foliation $\mathcal{F}_N$ of N and B is its tangent bundle $T\mathcal{F}_N$. An Euler-like section of A along B is just an Euler-like vector field along N which is tangent to the leaves of $\mathcal{F}$. Such a vector field exists if and only if N is transverse to the leaves. In non-transverse cases (e.g. when N is a single point) Theorems 4.2.7 and 4.2.10 do not apply. But see Remark 4.2.13.

In situations where there exists no Euler-like section the following modification of the retraction and linearization theorems may be of use.

4.2.12 Corollary. *Let $\bar{A} \to M$ be a Lie algebroid. Let $A \to M$ be a Lie subalgebroid of $\bar{A}$ over the same base and let $B \to N$ be a Lie subalgebroid of A. Suppose there exists a section ε of $\bar{A}$ along B which normalizes $\Gamma(A)$. If the vector field $\mathbf{ad}_{\bar{A}}(\varepsilon)$ on $\bar{A}$ is Euler-like along B, the linearization map $\mathcal{N}(\bar{A}, B) \to \bar{A}$ given by the Euler-like vector field $\mathbf{ad}_{\bar{A}}(\varepsilon)$ on $\bar{A}$ restricts to a linearization map $\mathcal{N}(A, B) \to A$. If additionally $\varepsilon|_N = 0$, there exist a deformation retraction of $A|_U$ onto B, where U is the tubular neighbourhood of N associated with the Euler-like vector field $\mathbf{an}_{\bar{A}}(\varepsilon)$ on M, and hence $i^*_B \colon H^\bullet_A(U) \to H^\bullet_B(N)$ is an isomorphism.*

Proof. The proof of Theorem 4.2.10 tells us that the time 1 flow of the vector field $w = \partial/\partial t + t^{-1}\, \mathbf{ad}\, \bar{A}(\varepsilon)$ on the deformation space $\mathcal{D}(\bar{A}, B)$ induces a linearization $\mathcal{N}(\bar{A}, B) \to \bar{A}$. It follows from Lemma A.5.6(ii) that the flow of w preserves $\mathcal{D}(A, B)$ and therefore gives us a linearization $\mathcal{N}(A, B) \to A$.

Similarly, if $\varepsilon|_N = 0$ then Theorem 4.2.7 tells us that the flow of $\mathbf{ad}_{\bar{A}}(\varepsilon)$ induces a deformation retraction of $\bar{A}|_U$ onto B. Again by Lemma A.5.6(ii) this deformation retraction preserves A and therefore gives us a deformation retraction of $A|_U$ onto B. □

4.2.13 Remark. Let $\mathcal{F}$ be a foliation of M and let $A = T\mathcal{F}$ be the tangent bundle of $\mathcal{F}$. Let $B = i_N^! A$ be the pullback of A to an embedded submanifold N of M that cleanly intersects A. We take $\bar{A}$ to be the tangent bundle of M. A section of $\bar{A}$, i.e. vector field on M, normalizes $\Gamma(A)$ if and only if its flow maps leaves to leaves. For instance, if $N = \{x\}$ is a single point, then in a foliation chart U centred at x in which $\mathcal{F}$ is spanned by the vector fields $\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_p$ the Euler vector field $\varepsilon = \sum_i x_i \partial/\partial x_i$ normalizes $\Gamma(A)$. Hence the conclusions of Corollary 4.2.12 apply to the pair $(A, B = 0_x)$. An arbitrary clean submanifold N can be covered with foliation charts U in which $\mathcal{F}$ is spanned by the vector fields $\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_p$ and in which N is given by the equations $x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 0$. In such a chart the vector field $\varepsilon_U = \sum_{l=1}^k x_{i_l} \partial/\partial x_{i_l}$ is Euler-like along N and normalizes $\Gamma(A)$. It would be interesting to have conditions under which the ε_U can be glued to a global vector field ε that is Euler-like along N and normalizes $\Gamma(A)$.

4.3. A Darboux-Moser-Weinstein theorem for Lie algebroids

Here is a version of Moser's trick for Lie algebroids. See Definition 4.2.1 for Lie algebroid isotopies.

4.3.1 Proposition. *Let $A \to M$ be a Lie algebroid. Let $\omega_t \in \Omega_A^2(M)$ be a smooth path of A-symplectic forms defined for $0 \le t \le 1$. Let $\alpha_t \in \Omega_A^1(M)$ be a smooth path of 1-forms such that $\dot{\omega}_t = -d_A \alpha_t$.*

(i) If M is compact, there exists an isotopy φ of A such that $\varphi_t^* \omega_t = \omega_0$.

(ii) If $B \to N$ is a Lie subalgebroid of A and $(\alpha_t)_x = 0$ for all $x \in N$ and for all t, there exist a neighbourhood U of N and an isotopy φ of $A|_U$ such that $\varphi_t^* \omega_t = \omega_0$ and $\varphi_t|_B = \mathrm{id}_B$.

Proof. We omit the proof of (i), which is similar to that of (ii). The time-dependent section $\sigma_t = \omega_t^\sharp \alpha_t$ of A satisfies

$$d_A \iota_A(\sigma_t)\omega_t = d_A \alpha_t = -\dot{\omega}_t. \tag{4.3.2}$$

By hypothesis the form α_t vanishes at every point of N, and therefore so does the section σ_t. Hence the vector field $\mathbf{an}_A(\sigma_t)$ on M integrates to a flow $\mathring{\Phi}_t$ defined for $0 \le t \le 1$ on a suitable neighbourhood of N which leaves N fixed. The vector field $\mathbf{ad}_A(\sigma_t)$ on A is a linear lift of $\mathbf{an}_A(\sigma_t)$ and integrates to a flow Φ_t with initial condition $\varphi_0 = \mathrm{id}$ defined for $0 \le t \le 1$ in a neighbourhood of B. Since $\mathbf{ad}_A(\sigma_t)$ vanishes along B, the flow leaves B fixed. According to Proposition 4.2.2 the flow Φ_t determines a Lie algebroid isotopy φ. Using (A.5.8), (A.5.9), and (4.3.2) we obtain $\varphi_t^*\omega_t = \omega_0$ for $0 \le t \le 1$. $\square$

Next we give a Lie algebroid version of the Darboux-Moser-Weinstein theorem [44, § 5], which says that two symplectic forms that agree at all points of a submanifold are isomorphic in a neighbourhood of the submanifold. See Definition 4.1.13 for the notion of a Lie algebroid retraction.

4.3.3 Theorem. *Let $A \to M$ be a Lie algebroid and let $B \to N$ be a Lie subalgebroid such that $A|_U$ admits a deformation retraction onto B for some neighbourhood U of N. Let ω_0 and ω_1 be A-symplectic forms on M satisfying $\omega_{0,x} = \omega_{1,x}$ for all $x \in N$. There exist open neighbourhoods U_0 and U_1 of N in M and a Lie algebroid isomorphism $\varphi\colon A|_{U_0} \to A_{U_1}$ such that $\varphi^*\omega_1 = \omega_0$, $\mathring{\varphi}|_N = \mathrm{id}_N$, and $\varphi|_{A_x} = \mathrm{id}_{A_x}$ for all $x \in N$.*

Proof. Let $\omega_t = (1-t)\omega_0 + t\omega_1$. Choose an open neighbourhood U of N such that the form ω_t is symplectic on $A|_U$ for all t and such that $A|_U$ admits a deformation retraction $\varrho = \varrho_U\colon A|_U \to A|_U$ onto B. Put $\alpha = \kappa_\varrho(\omega_1 - \omega_0) \in \Omega^1_A(M)$, where κ_ϱ is as in (4.1.6). The homotopy formula (4.1.7) yields

$$d_A\alpha = d_A\kappa_\varrho(\omega_1 - \omega_0) = \varrho_1^*(\omega_1 - \omega_0) - \varrho_0^*(\omega_1 - \omega_0) = \omega_0 - \omega_1 = -\dot{\omega}_t.$$

By Lemma 4.1.14 the form α vanishes at every point of N, and therefore Proposition 4.3.1(ii) applies, giving us an isotopy φ_t of a possibly smaller neighbourhood U fixing B and satisfying $\varphi_t^*\omega_t = \omega_0$. Taking $\varphi = \varphi_1$ gives $\varphi^*\omega_1 = \omega_0$. Let $x \in N$. To prove that $\varphi\colon A_x \to A_x$ is the identity map we show that

$$\varphi_t(\tau(x)) = \tau(x) \tag{4.3.4}$$

for all $t \in [0,1]$ and all sections τ of A. The formula

$$\iota_A([\tau, \sigma_t]) = [\mathcal{L}_A(\tau), \iota_A(\sigma_t)]$$

yields

$$\iota_A([\tau, \sigma_t])\omega_t = \mathcal{L}_A(\tau)\iota_A(\sigma_t)\omega_t - \iota_A(\sigma_t)\mathcal{L}_A(\tau)\omega_t = \mathcal{L}_A(\tau)\alpha - \iota_A(\sigma_t)\mathcal{L}_A(\tau)\omega_t,$$

so

$$\big(\iota_A([\tau, \sigma_t])\omega_t\big)_x = (\mathcal{L}_A(\tau)\alpha)_x - (\iota_A(\sigma_t)\mathcal{L}_A(\tau)\omega_t)_x = 0,$$

where we used Lemma 4.1.14 and the fact that σ_t vanishes at x. Since ω_t is symplectic, it follows that the section $[\tau, \sigma_t]$ of A vanishes at x. In view of (A.5.4) this implies

$$\frac{d}{dt}\varphi_t(\tau(x)) = 0$$

for all t. Since $\varphi_0 = \mathrm{id}_A$, this proves (4.3.4). □

4.3.5 Remarks. (i) If A admits an Euler-like section along B, then a deformation retraction from $A|_U$ onto B exists for some neighbourhood U of N by Theorem 4.2.7 and Remark 4.2.11(i). Most importantly, this is the case if N is transverse to A and $B = i_N^! A$ (Remark 4.2.11(ii)). See Corollary 4.2.12 for another condition under which a local deformation retraction is guaranteed to exist.
(ii) A local deformation retraction is too much to ask for in many situations; all one needs to prove Theorem 4.3.3 is a suitable primitive $\alpha \in \Omega^1_A(U)$ of $\omega_1 - \omega_0$. See Scott [41, § 5], Klaasse and Lanius [25, 26, § 4.3], and Miranda and Scott [39, § 2] for instances of such situations.
(iii) The map φ of Theorem 4.3.3 is a Poisson isomorphism relative to the Poisson structures on M induced by ω_0 and ω_1.

4.4. The coisotropic embedding theorem

Theorem 4.3.3 yields Lie algebroid analogues of the familiar local normal forms of symplectic geometry. In this section we offer the Lie algebroid version of Gotay's coisotropic embedding theorem [15], which is a complement to the symplectization theorem, Theorem 2.4.3. It states that a symplectic Lie algebroid $A \to M$ is in a neighbourhood of a transverse coisotropic submanifold $i_N : N \to M$ completely determined by the pullback Lie algebroid $B = i_N^! A$ and by the B-presymplectic form on N. Our proof uses the proof of the Lie algebroid splitting theorem given by Bursztyn et al. [6]. A version of Theorem 4.4.1 for b-symplectic manifolds was established by Geudens and Zambon [14].

4.4.1 Theorem (Coisotropic embeddings)**.** *Let $(B \to N, \omega_B)$ be a presymplectic Lie algebroid. Let $(A_0 \to M_0, \omega_0)$ and $(A_1 \to M_1, \omega_1)$ be symplectic Lie algebroids and let $i_0 : N \to M_0$ and $i_1 : N \to M_1$ be transverse coisotropic embeddings such that $i_0^! A_0$ and $i_1^! A_1$ are isomorphic to B and $\omega_B = (i_0)_!^* \omega_0 = (i_1)_!^* \omega_1$. There exist open neighbourhoods U_0 of N in M_0 and U_1 of N in M_1 and an isomorphism of Lie algebroids $\varphi : A_0|_{U_0} \to A_1|_{U_1}$ such that $\varphi \circ (i_0)_! = (i_1)_!$ and $\varphi^* \omega_1 = \omega_0$.*

Proof. We will prove the theorem in two special cases, which taken together establish the general case. Let K be the Lie subalgebroid $\ker(\omega_B)$ of B. Recall the model Lie algebroid $\mathbf{A} \to \mathbf{M}$ of Section 2.4, whose base manifold $\mathbf{M}$ is the dual bundle K^* and whose total space $\mathbf{A}$ is the pullback of B to $\mathbf{M}$. We have a transverse coisotropic embedding $\mathbf{j}\colon N \to \mathbf{M}$ and a family of symplectic structures $\omega^s \in \Omega^2_{\mathbf{A}}(\mathbf{M})$ parametrized by splittings $s\colon K^* \to B^*$ of the surjective vector bundle map $B^* \to K^*$.

Case 1. Let A_0 and A_1 be two copies of the model Lie algebroid $\mathbf{A}$ and let $i_0 = i_1$ be the canonical embedding $\mathbf{j}\colon N \to \mathbf{M}$. We equip $\mathbf{A}$ with two symplectic forms $\omega_0 = \omega^{s_0}$ and $\omega_1 = \omega^{s_1}$ corresponding to two different splittings s_0, $s_1\colon K^* \to B^*$. The theorem in this case follows from Remark 2.4.5 and Proposition 4.3.1(ii).

Case 2. Let $(A \to M, \omega)$ be an arbitrary symplectic Lie algebroid equipped with a transverse coisotropic embedding $i\colon N \to M$ such that $B \cong i^! A$ and $i_!^* \omega = \omega_B$. We assert that the theorem is true for the pair

$$(A_0, \omega_0, i_0) = (\mathbf{A}, \omega^s, \mathbf{j}) \qquad \text{and} \qquad (A_1, \omega_1, i_1) = (A, \omega, i)$$

for an appropriate choice of splitting $s\colon K^* \to B^*$ depending on A. We choose s as follows. Since B is a coisotropic subbundle of the symplectic vector bundle $A|_N = i^* A$, the map

$$i^* A \xrightarrow{\omega^\sharp} (i^* A)^* \longrightarrow (B^\omega)^* = K^*$$

has kernel B, which gives a short exact sequence

$$B = i^! A \hookrightarrow i^* A \twoheadrightarrow K^* \tag{4.4.2}$$

of vector bundles on N. Now choose an isotropic subbundle L of $i^* A$ which is complementary to B. The map $i^* A \to K^*$ restricts to a vector bundle isomorphism

$$L \xrightarrow{\cong} K^*. \tag{4.4.3}$$

The subbundle $K \oplus L$ of $i^* A$ is symplectic and is isomorphic via the map (4.4.3) to $K \oplus K^*$ equipped with its standard symplectic form. We define $E = (K \oplus L)^\omega$ to be its symplectic orthogonal, so that we have an orthogonal decomposition

$$i^* A = E \oplus (K \oplus L) \cong E \oplus (K \oplus K^*). \tag{4.4.4}$$

The subbundle E of B is complementary to K and therefore defines a splitting $s\colon K^* \to B^*$ of the surjection $B^* \to K^*$. We equip $\mathbf{A}$ with the form ω^s defined by this splitting s. Next we explain how to map $\mathbf{A}$ to A. The embedding i is transverse to A, so by Lemma A.3.7(ii) the anchor induces a bundle isomorphism $i^*A/B \cong \mathcal{N}(M,N)$. Comparing with (4.4.2) we obtain a canonical identification

$$\mathbf{M} = K^* \cong \mathcal{N}(M,N) \tag{4.4.5}$$

between the model manifold and the normal bundle of N. Via this identification the model Lie algebroid $\mathbf{A}$ is just the pullback algebroid $\pi^!_{M,N}B$, where $\pi_{M,N}\colon \mathcal{N}(M,N) \to N$ is the projection. In fact, by [6, Lemma 3.8] we have a natural isomorphism $\mathbf{A} \cong \mathcal{N}(A,B)$ between $\mathbf{A}$ and the normal bundle of B in A, which is a Lie algebroid over $\mathcal{N}(M,N)$. Let $\varepsilon \in \Gamma(A)$ be a section which vanishes along N and whose normal derivative

$$\mathcal{N}(\varepsilon)\colon \mathcal{N}(M,N) \longrightarrow \mathcal{N}(i^*A, 0_M) \cong i^*A$$

is equal to the splitting of (4.4.2) given by the complement L. Such a section ε exists and is Euler-like along B; see the proof of [6, Lemma 3.9]. Let

$$\psi\colon \mathbf{A} \longrightarrow A$$

be the tubular neighbourhood embedding determined by the Euler-like vector field $\mathbf{ad}_A(\varepsilon)$ on A. The triangle

$$\begin{tikzcd} & B \arrow[dl, "\mathbf{j}_!"'] \arrow[dr, "i_!"] & \\ \mathbf{A} \arrow[rr, "\psi"] & & A \end{tikzcd} \tag{4.4.6}$$

commutes and, according to [6, Theorem 3.13, Remark 3.19, Theorem 4.1], ψ is a morphism of Lie algebroids. The base map $\mathring{\psi}\colon \mathbf{M} \to M$ is the tubular neighbourhood embedding defined by the Euler-like vector field $\mathbf{an}_A(\varepsilon)$ on M, so ψ is a Lie algebroid isomorphism $\mathbf{A} \cong A|_U$, where $U = \mathring{\psi}(\mathbf{M})$. The restriction of the bundle $\mathbf{A} \to \mathbf{M}$ to the submanifold N is $\mathbf{j}^*\mathbf{A} = B \oplus K^*$. The map

$$\psi|_N\colon \mathbf{j}^*\mathbf{A} = B \oplus K^* \longrightarrow i^*A = B \oplus L$$

is the identity on B (because of (4.4.6)) and on K^* is equal to the normal derivative $\mathcal{N}(\varepsilon)$, i.e. the inverse of the isomorphism (4.4.3). So $\psi|_N$ is identical to the symplectic bundle isomorphism (4.4.4), which shows that $\psi^*\omega_x = \omega^s_x$

for all $x \in N$. The Lie algebroid $\mathbf{A}$ deformation retracts onto B in view of Remark 4.2.11(ii). The theorem now follows from Theorem 4.3.3 applied to the Lie algebroid $\mathbf{A}$ and the symplectic forms ω^s and $\psi^*\omega$. $\square$

4.4.7 Remarks. (i) Theorems 2.4.3 and 4.4.1 together say that locally near B the symplectic Lie algebroid (A, ω) is isomorphic to its first-order approximation $(\mathbf{A}, \omega^s)$ along B.
(ii) There is no hope of obtaining a similar result for arbitrary clean coisotropic submanifolds. For instance, let $(A \to M, \omega)$ be an arbitrary symplectic Lie algebroid. Its associated *adiabatic Lie algebroid* $\tilde{A} \to \tilde{M}$ has base manifold $\tilde{M} = \mathbf{R} \times M$ and total space $\tilde{A} = \mathrm{pr}_2^* A = \mathbf{R} \times A$. The anchor $\mathbf{an}_{\tilde{A}} \colon \tilde{A} \to T\tilde{M}$ is defined by $\mathbf{an}_{\tilde{A}}(t, a) = t\,\mathbf{an}_A(a)$ for $t \in \mathbf{R}$ and $a \in A$. The Lie bracket $[\sigma, \tau]_{\tilde{A}} \in \Gamma(\tilde{A})$ for sections σ, $\tau \in \Gamma(A)$ is defined by

$$[\sigma, \tau]_{\tilde{A}}(t, x) = t[\sigma, \tau]_A(x),$$

where $(t, x) \in \tilde{M}$. This bracket extends uniquely to a bracket on $\Gamma(\tilde{A})$ satisfying the Leibniz rule and so makes $\tilde{A}$ a Lie algebroid. The form $\tilde{\omega} = \mathrm{pr}_2^* \omega \in \Omega^2_{\tilde{A}}(\tilde{M})$ is $\tilde{A}$-symplectic. Let $N = \{(0, x)\}$, where x is any point in M. Then N is an orbit of $\tilde{A}$ and hence is clean coisotropic. The induced Lie algebroid is $B = i_N^! \tilde{A} = A_x$ equipped with the zero Lie bracket and the form $\omega_B = \omega_x$. It retains no memory of the Lie bracket on A and therefore cannot determine the structure of $\tilde{A}$ in a neighbourhood of N. (Of course the adiabatic Lie algebroid is not linearizable at N. It might be possible to build a first-order model for a coisotropic Lie subalgebroid that admits an Euler-like section.)
(iii) In the Lagrangian case (i.e. $\omega_B = 0$) the model symplectic Lie algebroid $\mathbf{A}$ is the phase space $\pi^! B$ of Section 2.2. The form on $\mathbf{A}$ is the canonical symplectic form ω_{can}, which, in contrast to the general coisotropic case, is linear along the fibres and independent of any choices. Theorem 4.4.1 tells us that a neighbourhood of a transverse Lagrangian submanifold N of a symplectic Lie algebroid (A, ω) is equivalent to a neighbourhood of the zero section in $(\mathbf{A}, \omega_{\mathrm{can}})$. See Smilde [42] for further linearization theorems.

5. Log symplectic manifolds

In this section we state a symplectic reduction theorem and a local normal form theorem, Theorem 5.1.11, in the setting of log symplectic manifolds. Our results extend some of the work of Geudens and Zambon [14], Gualtieri et al. [17] and Guillemin et al. [18, § 6]. See also Braddell et al. [2, 3], and Matveeva and Miranda [36] for related recent work.

5.1. Divisors with normal crossings

The following definition is a $\mathcal{C}^\infty$ analogue of a familiar notion from algebraic geometry.

5.1.1 Definition. Let M be a manifold. A *(simple) normal crossing divisor* is a locally finite collection $\mathcal{Z}$ of hypersurfaces (connected closed embedded submanifolds of codimension 1) in M, called the *components* of $\mathcal{Z}$, which intersect transversely in the following sense: for all $x \in M$, if $Z_1, Z_2, \dots, Z_k$ is the list of all components of $\mathcal{Z}$ containing x and if f_i is a defining function for Z_i near x, then the differentials $df_1, df_2, \dots, df_k$ are independent at x. The *support* of a normal crossing divisor $\mathcal{Z}$ is the union of all its components and is denoted by $|\mathcal{Z}|$.

For the remainder of this section we will fix a manifold M with a normal crossing divisor $\mathcal{Z}$.

The collection $\mathcal{X}_{\mathcal{Z}}(M)$ of all vector fields on M that are tangent to (every component of) $\mathcal{Z}$ is a Lie subalgebra of $\mathcal{X}(M) = \Gamma(TM)$ and is a finitely generated projective $\mathcal{C}^\infty(M)$-module of rank $n = \dim(M)$. Therefore $\mathcal{X}_{\mathcal{Z}}(M)$ is the space of sections of a Lie algebroid $T_{\mathcal{Z}}M$ of rank n, which we call the *logarithmic tangent bundle* of the pair $(M, \mathcal{Z})$. If $\mathcal{Z}$ consists of a single component Z, we also speak of the *b-tangent bundle* of (M, Z) and we write $T_{\mathcal{Z}}M = T_Z M$.

Define $\mathcal{Z}^{(k)}$ to be the collection of all points of M that are contained in exactly k distinct components of $\mathcal{Z}$. Then $\mathcal{Z}^{(k)}$ is a submanifold of codimension k of M, which we call the *codimension k stratum*, and we have inclusions

$$\overline{\mathcal{Z}^{(0)}} = M \supseteq \overline{\mathcal{Z}^{(1)}} = |\mathcal{Z}| \supseteq \overline{\mathcal{Z}^{(2)}} \supseteq \cdots .$$

The image of the anchor $\mathbf{an}\colon T_{\mathcal{Z}}M \to TM$ at $x \in \mathcal{Z}^{(k)}$ is equal to $T_x\mathcal{Z}^{(k)}$. Thus the orbits of the logarithmic tangent bundle are equal to the connected components of the strata. In particular the Lie algebroid $T_{\mathcal{Z}}M$ determines the divisor $\mathcal{Z}$.

For each $x \in M$ the tangent spaces T_xZ for $Z \in \mathcal{Z}$ define a normal crossing divisor of the tangent space T_xM. The manifold M admits an atlas consisting of *normal crossing charts*, i.e. charts $\varphi\colon U \to \mathbf{R}^n$ with the property that $\varphi(U \cap Z_i) = \varphi(U) \cap H_i$, where $Z_1, Z_2, \dots, Z_k$ are the components of $\mathcal{Z}$ intersecting U and $H_i = \{\, x \in \mathbf{R}^n \mid x_i = 0 \,\}$ is the ith coordinate hyperplane.

For each component Z of $\mathcal{Z}$ the vector bundle $T_{\mathcal{Z}}M|_Z$ has a distinguished nowhere vanishing section ξ_Z, which at generic points of Z spans the kernel of the anchor. In a neighbourhood U of a point $x \in Z$ this section is given by

$\xi_Z|_{Z\cap U} = (fv)|_{Z\cap U}$, where $f\colon U \to \mathbf{R}$ is a local defining function for Z and v is any vector field on U with $\mathcal{L}(v)(f) = 1$. In a normal crossing chart at x in which Z is given by $x_1 = 0$ we have $\xi_Z = x_1 \frac{\partial}{\partial x_1}\big|_Z$. We define

$$L_Z = \operatorname{span}(\xi_Z) \tag{5.1.2}$$

to be the trivial real line bundle on Z spanned by ξ_Z.

Under appropriate conditions the intersection of a normal crossing divisor with a submanifold N is a normal crossing divisor of N.

5.1.3 Lemma. *Let M be a manifold with normal crossing divisor $\mathcal{Z}$ and let $A = T_{\mathcal{Z}}M$ be the logarithmic tangent bundle. Let $i_N\colon N \to M$ be a connected embedded submanifold which intersects A cleanly. Let $Z_1, Z_2, \dots, Z_l$ be the list of all components Z of $\mathcal{Z}$ such that $N \subseteq Z$. Let $\mathcal{Z}_N$ be the collection consisting of all intersections $Z \cap N$ with $Z \in \mathcal{Z}\backslash\{Z_1, Z_2, \dots, Z_l\}$. Then $\mathcal{Z}_N$ is a normal crossing divisor of N. Let $L_j = L_{Z_j}\big|_N$, where L_{Z_j} is as in (5.1.2), and let V be the rank l trivial vector bundle $L_1 \oplus L_2 \oplus \cdots \oplus L_l$, considered as a Lie algebroid over N with zero anchor $V \to TN$ and trivial Lie bracket. The pullback Lie algebroid $B = i_N^! A$ is isomorphic to the direct sum $V \oplus T_{\mathcal{Z}_N}N$. If N intersects A transversely, we have $l = 0$ and $B \cong T_{\mathcal{Z}_N}N$.*

Proof. Let $x \in N$. It follows from Proposition A.3.5(i) that there is a normal crossing chart $\varphi\colon U \to \mathbf{R}^n$ at x such that $\varphi(U \cap N)$ is a coordinate subspace of $\mathbf{R}^n$. Let $Z_1, \dots, Z_l$ be those components Z of $\mathcal{Z}$ for which the image $\varphi(U \cap Z)$ is a coordinate hyperplane of $\mathbf{R}^n$ containing $\varphi(U \cap N)$. (Such Z do not exist if N intersects A transversely, so $l = 0$ in the transverse case.) Then $Z_j \cap N$ is open and closed in N for all j, so $Z_j \cap N = N$. Moreover, for all $Z \in \mathcal{Z}\backslash\{Z_1, \dots, Z_l\}$ the image $\varphi(U \cap Z)$ is either empty or a coordinate hyperplane transverse to N. Thus $Z \cap N$ is a hypersurface in N for each $Z \in \mathcal{Z}\backslash\{Z_1, Z_2, \dots, Z_l\}$. Restricting the normal crossing chart φ to N gives a normal crossing chart for the pair $(N, \mathcal{Z}_N)$. Since

$$\Gamma(i_N^* A) = \Gamma(A)/\Gamma(A; 0_N),$$

we can think of a section of $A|_N = i_N^* A$ as an equivalence class $\bar{v} = v \bmod \Gamma(A; 0_N)$ of a section $v \in \Gamma(A)$. By the definition of pullbacks we have

$$\Gamma(B) = \{\, \bar{v} \mid v \in \Gamma(A),\, v \text{ is tangent to } N \,\}.$$

Every $v \in \Gamma(A)$ which is tangent to N is tangent to $\mathcal{Z}_N$, so we have a natural C^∞-linear Lie algebra morphism $\varphi\colon \Gamma(B) \to \Gamma(T_{\mathcal{Z}_N}N)$. The kernel of this morphism is

$$\ker(\varphi) = \{\, \bar{v} \mid v \in \Gamma(A),\, v = 0 \text{ on } N \,\} = \Gamma(V).$$

The ideal $\Gamma(V)$ of $\Gamma(B)$ is central, because sections of V induce trivial flows on N. Every vector field w on N which is tangent to $\mathcal{Z}_N$ extends to a vector field $\tilde{w}$ on M which is tangent to $\mathcal{Z}$, and $\tilde{w}$ is unique modulo $\Gamma(A; 0_N)$. Thus φ has a canonical splitting $\Gamma(T_{\mathcal{Z}_N}N) \to \Gamma(B)$, which is a C^∞-linear Lie algebra morphism. We conclude that $B = V \oplus T_{\mathcal{Z}_N}N$ as Lie algebroids. □

Let P be a second manifold with normal crossing divisor $\mathcal{D}$. For a map $\varphi\colon M \to P$ to induce a Lie algebroid morphism $T_{\mathcal{Z}}M \to T_{\mathcal{D}}P$ it must map strata to strata, but that is not enough.

5.1.4 Example. Let $M = P = \mathbf{R}$ and let $\mathcal{Z} = \mathcal{D} = \{0\}$. Let $\varphi\colon M \to P$ be a smooth function with $\varphi(0) = 0$ and $\varphi(x) \neq 0$ if $x \neq 0$. Then φ maps strata to strata, but φ lifts to a Lie algebroid morphism $T_{\mathcal{Z}}M \to T_{\mathcal{D}}P$ if and only if the function $x \mapsto x\varphi'(x)$ is smoothly divisible by φ. This is the case if and only if φ is not flat at 0.

5.1.5 Definition. A *morphism* $\varphi\colon (M, \mathcal{Z}) \to (P, \mathcal{D})$ of manifolds with normal crossing divisors is a smooth map $\varphi\colon M \to P$ with the following property: every component D of $\mathcal{D}$ has a covering $\mathfrak{V}$ consisting of open subsets of P such that for each $V \in \mathfrak{V}$ either (1) $\varphi^{-1}(D \cap V) = \varphi^{-1}(V)$, or (2) there is a component Z of $\mathcal{Z}$ with $\varphi^{-1}(D \cap V) = Z \cap \varphi^{-1}(V)$. In case (2) we demand additionally that if g is a local defining function for D, then there exist an integer $\nu \geq 1$ and a local defining function f for Z such that $\varphi^*g = f^\nu$.

We call the lift $T^{\log}\varphi\colon T_{\mathcal{Z}}M \to T_{\mathcal{D}}P$ of a morphism φ guaranteed by the next lemma the *log tangent map* of φ.

5.1.6 Lemma. *Let $\varphi\colon (M, \mathcal{Z}) \to (P, \mathcal{D})$ be a morphism of manifolds with normal crossing divisors.*

(i) The map φ lifts to a unique Lie algebroid morphism $T^{\log}\varphi\colon T_{\mathcal{Z}}M \to T_{\mathcal{D}}P$ whose restriction to $M \backslash |\mathcal{Z}|$ agrees with the usual tangent map of φ. In particular, if M^0 is a connected component of M whose image $\varphi(M^0)$ is contained in a component D of $\mathcal{D}$, then $T^{\log}\varphi$ maps $T_{\mathcal{Z}}M|_{M^0}$ to the Lie subalgebroid $T_{\mathcal{D}_D}D$ of $T_{\mathcal{D}}P$, where $\mathcal{D}_D$ is the normal crossing divisor of D defined in Lemma 5.1.3.

(ii) Let Z be a component of $\mathcal{Z}$ and ξ_Z the distinguished section of $T_{\mathcal{Z}}M|_Z$. Suppose there is a component D of $\mathcal{D}$ such that $\varphi(Z) \subseteq D$ and for some local defining function g of D the function φ^*g vanishes at Z

to constant order $\nu_Z < \infty$. Then $T^{\log}\varphi(\xi_Z) = \nu_Z \eta_D$, where η_D is the distinguished section of $T_{\mathcal{D}}P|_D$. If there is no such component D, we have $T^{\log}\varphi(\xi_Z) = 0$.

Proof. (i) Let $R = C^\infty(M)$ and $S = C^\infty(P)$. We regard R as an S-module via the pullback map $\varphi^*\colon S \to R$. The usual tangent map $T\varphi\colon TM \to TP$ is the vector bundle map induced by the pushforward map

$$\varphi_*\colon \mathcal{X}(M) = \mathrm{Der}(R) \longrightarrow \Gamma(\varphi^* TP) = R \otimes_S \mathcal{X}(P) = \mathrm{Der}(S, R),$$

which maps $D \in \mathrm{Der}(R)$ to the derivation $\varphi_* D \in \mathrm{Der}(S, R)$ defined by $(\varphi_* D)(g) = D(\varphi^* g)$ for $g \in S$. We must show that φ_* lifts to an R-linear Lie algebra map $\varphi_*^{\log}$,

$$\begin{tikzcd} \mathcal{X}_{\mathcal{Z}}(M) \arrow[r, dashed, "\varphi_*^{\log}"] \arrow[d, hook, "a_M"'] & R \otimes_S \mathcal{X}_{\mathcal{D}}(P) \arrow[d, "a_P"] \\ \mathcal{X}(M) \arrow[r, "\varphi_*"] & R \otimes_S \mathcal{X}(P), \end{tikzcd}$$

where $a_M = \mathbf{an}_M$ is the anchor map for $(M, \mathcal{Z})$ and $a_P = \mathrm{id}_R \otimes \mathbf{an}_P$, with $\mathbf{an}_P$ being the anchor map for $(P, \mathcal{D})$. First suppose that M is an open neighbourhood of the origin in $\mathbf{R}^n$ with normal crossing divisor $\mathcal{Z} = \{Z_1, Z_2, \dots, Z_l\}$, where $Z_j = \{\, x \in \mathbf{R}^n \mid x_j = 0 \,\}$, and that P is an open neighbourhood of the origin in $\mathbf{R}^m$ with normal crossing divisor

$$\mathcal{D} = \{D_1, D_2, \dots, D_k\},$$

where $D_i = \{\, y \in \mathbf{R}^n \mid y_i = 0 \,\}$, and that $\varphi(0) = 0$. Let $\varphi_1, \varphi_2, \dots, \varphi_m \in R$ be the components of φ. The R-modules $\mathcal{X}(M)$ and $R \otimes_S \mathcal{X}(P)$ are free with bases

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \quad \text{resp.} \quad 1 \otimes \frac{\partial}{\partial y_1}, \dots, 1 \otimes \frac{\partial}{\partial y_m},$$

The R-modules $\mathcal{X}_{\mathcal{Z}}(M)$ and $R \otimes_S \mathcal{X}_{\mathcal{D}}(P)$ are also free with bases

$$x_1 \frac{\partial}{\partial x_1}, \dots, x_l \frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_{l+1}}, \dots, \frac{\partial}{\partial x_n},$$

resp.

$$1 \otimes y_1 \frac{\partial}{\partial y_1}, \dots, 1 \otimes y_k \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_{k+1}}, \dots, \frac{\partial}{\partial y_m}.$$

Therefore, finding the map $\varphi_*^{\log}$ amounts to finding an $m\times n$-matrix $\Psi = (\psi_{ij})$ with entries in the ring R such that $A_P\Psi = \Phi A_M$, where $\Phi = (\frac{\partial \varphi_i}{\partial x_j})$ is the matrix of φ_* and

$$\begin{aligned} A_M &= \operatorname{diag}(x_1, x_2 \dots, x_l, 1, 1, \dots, 1) \\ A_P &= \operatorname{diag}(\varphi_1, \varphi_2, \dots, \varphi_k, 1, 1, \dots, 1) \end{aligned}$$

are the matrices of a_M, resp. a_P. We obtain the following equations for the ψ_{ij}:

$$\begin{aligned} \varphi_i\psi_{ij} &= \begin{cases} x_j\partial\varphi_i/\partial x_j & \text{for } 1\le i\le k, \quad 1\le j\le l \\ \partial\varphi_i/\partial x_j & \text{for } 1\le i\le k, \quad l+1\le j\le n \end{cases} \\ \psi_{ij} &= \begin{cases} x_j\partial\varphi_i/\partial x_j & \text{for } k+1\le i\le m, \quad 1\le j\le l \\ \partial\varphi_i/\partial x_j & \text{for } k+1\le i\le m, \quad l+1\le j\le n. \end{cases} \end{aligned}$$

Clearly ψ_{ij} is uniquely determined for $k+1 \le i \le m$. Let $1 \le i \le k$. According to Definition 5.1.5 we are in either of two cases. In case 1 we have $\varphi_i = \varphi^* y_i = 0$, i.e. $\varphi(M)$ is contained in D_i. In this case we want $\varphi_*^{\log}$ to map $\mathcal{X}_{\mathcal{Z}}(M)$ to $\mathcal{X}_{\mathcal{D}_{D_i}}(D_i)$. In other words, if we take any $v \in \mathcal{X}_{\mathcal{Z}}(M)$ and express $\varphi_*^{\log}(v)$ in the basis of $\mathcal{X}_{\mathcal{D}}(P)$, we want the coefficient of $y_i\frac{\partial}{\partial y_i}$ to be equal to 0. This forces us to put $\psi_{ij} = 0$ for all j. In case 2 we have a unique $j = j_i \le l$ such that φ_i vanishes to constant finite order ν_i along Z_{j_i}. Thus we have $\varphi_i(x) = x_{j_i}^{\nu_i} f_i(x)$ for some $f_i \in R$ that vanishes nowhere on Z_{j_i}. Hence for $1 \le j \le l$

$$\psi_{ij} = \frac{x_j\partial\varphi_i/\partial x_j}{\varphi_i} = \begin{cases} \frac{x_j\partial f_i/\partial x_j}{f_i} & \text{if } j\ne j_i \\ \nu_i + \frac{x_{j_i}\partial f_i/\partial x_{j_i}}{f_i} & \text{if } j = j_i, \end{cases} \tag{5.1.7}$$

while for $l+1 \le j \le n$

$$\psi_{ij} = \frac{\partial\varphi_i/\partial x_j}{\varphi_i} = \frac{\partial f_i/\partial x_j}{f_i}.$$

We see that for $1 \le i \le m$ and $1 \le j \le n$ the function ψ_{ij} is a well-defined and uniquely determined element of R. This proves the existence of a unique R-linear lift $\varphi_*^{\log}$ locally, and hence globally by a gluing argument. The map $\varphi_*^{\log}$ is a Lie algebra homomorphism, because on the open dense set $M\backslash|\mathcal{Z}|$ it agrees with the pushforward map φ_*. It follows that the associated vector bundle map $T^{\log}\varphi$ is a Lie algebroid morphism.

(ii) Setting $x_{j_i} = 0$ in (5.1.7) yields $\psi_{ij_i} = \nu_i$ on $Z = Z_{j_i}$, i.e. $T^{\log}\varphi(\xi_Z) = \nu_i \eta_{D_i}$. □

A *log Poisson structure* on $(M, \mathcal{Z})$ is an A-Poisson structure (Definition 2.1.4) on M, where A is the log tangent bundle $T_{\mathcal{Z}}M$. A *log symplectic form* on $(M, \mathcal{Z})$ is an A-symplectic form (Definition 2.1.5) on M. If $\mathcal{Z}$ consists of a single component, we also speak of *b-Poisson* and *b-symplectic* structures.

The reduction theorem, Theorem 3.2.1, specializes to the following result for a log Poisson morphism $(M, \mathcal{Z}) \to (P, \mathcal{D})$. We can reduce M at any "point" $\mathfrak{f} \subseteq \operatorname{stab}(T_{\mathcal{D}}P, p)$, but the resulting quotient is not necessarily log symplectic unless we reduce at the trivial subalgebra $\mathfrak{f} = 0$. If we reduce at a nonzero "point", the reduced symplectic Lie algebroid will be a direct sum of a log tangent bundle and a trivial Lie algebroid.

5.1.8 Theorem (Log symplectic reduction)**.** *Let $(M, \mathcal{Z}, \omega)$ be a log symplectic manifold and let $(P, \mathcal{D}, \lambda)$ be a log Poisson manifold. Let $\mu\colon (M, \mathcal{Z}) \to (P, \mathcal{D})$ be a log Poisson morphism and let $\gamma\colon \Gamma(T^*_{\mathcal{D}}P) \to \Gamma(T_{\mathcal{Z}}M)$ be the associated Hamiltonian action. Let $p \in P$ and $N = \mu^{-1}(p)$. Suppose that p is a regular value of $\mu\colon M \to P$ and that the action γ is locally free at x for all $x \in N$.*

(i) The submanifold N of M intersects the Lie algebroid $T_{\mathcal{Z}}M$ cleanly, the 2-form $\omega_N = i_N^*\omega$ is logarithmic relative to the normal crossing divisor $\mathcal{Z}_N$ and is presymplectic. The null Lie algebroid $K = \ker(\omega_N)$ is a foliation Lie algebroid and the Lie algebra

$$\mathfrak{h} = \operatorname{stab}(T^*_{\mathcal{D}}P, p)$$

acts transitively on K.

(ii) Suppose that the leaf space $Q = N/\mathfrak{h}$ is a manifold. Then the collection $\mathcal{Z}_Q$ consisting of all quotients $Z/\mathfrak{h}$ for $Z \in \mathcal{Z}_N$ is a normal crossing divisor of Q and ω_N descends to a log symplectic structure on $(Q, \mathcal{Z}_Q)$.

Proof. Theorem 3.2.1(i), applied to the Lie algebroids $A = T_{\mathcal{Z}}M$ and $E = T_{\mathcal{D}}P$, says that $T^{\log}\mu$ is transverse to any Lie subalgebra $\mathfrak{f}$ of $\operatorname{stab}(E, p)$. Taking $\varphi = T^{\log}\mu$ and $\mathfrak{f} = \operatorname{stab}(E, p)$ in Corollary A.4.5 we find that N cleanly intersects A and that $i_N^!A = (T^{\log}\mu)^{-1}(\operatorname{stab}(E, p))$. Let $D_1, D_2, \ldots, D_k$ be the components of $\mathcal{D}$ that contain p and let η_i be the distinguished section of $T_{\mathcal{D}}P|_{D_i}$. The elements $\eta_1(p), \eta_2(p), \ldots, \eta_k(p) \in E_p$ form a basis of $\operatorname{stab}(E, p)$. Let $x \in N$ and let N_0 be the connected component of N containing x. Since μ is a morphism, there exist components $Z_1, Z_2, \ldots, Z_k$ of $\mathcal{Z}$ that contain N_0 and satisfy $\mu(Z_i) \subseteq D_i$. Since p is a regular value of μ, we have $T^{\log}\mu(\xi_i) = \eta_i$, where ξ_i is the distinguished section of $T_{\mathcal{Z}}M|_{Z_i}$. By Lemma 5.1.6, for any component Z of $\mathcal{Z}\backslash\{Z_1, Z_2, \dots, Z_k\}$ that contains x we have $T^{\log}\mu(\xi_Z) = 0$. It now follows from Lemma 5.1.3 that $i_N^! A = V \oplus T_{\mathcal{Z}_N} N$, where V is a trivial k-dimensional vector bundle equipped with a trivial Lie algebroid structure and $\mathcal{Z}_N$ is the induced divisor of N. Moreover, the preimage of the zero "point" $\mathfrak{f} = 0_p$ is the log tangent bundle of the induced normal crossing divisor $\mathcal{Z}_N$,

$$T_{\mathcal{Z}_N} N = (T^{\log}\mu)^{-1}(0_p).$$

Theorem 3.2.1(i) (with $\mathfrak{f} = 0_p$) shows that the Lie algebra $\mathfrak{h} = \mathrm{stab}(T^*_{\mathcal{D}}P, p)$ acts transitively on $K = \ker(\omega_N)$. Now suppose that the leaf space $Q = N/\mathfrak{h}$ is a manifold. To see that Z_Q is a normal crossing divisor of Q, choose an open subset U of Q such that the quotient map $q\colon N \to Q$ admits a section $s\colon U \to V = q^{-1}(U)$. Then s is transverse to the logarithmic tangent bundle $T_{\mathcal{Z}_N} N$, so by Lemma 5.1.3 s is transverse to all strata of $\mathcal{Z}_N$ and $\mathcal{Z}_Q|_U$ is a normal crossing divisor of U. Thus we have a well-defined log tangent bundle $T_{\mathcal{Z}_Q} Q$ (the holonomy condition of Theorem 3.2.1(ii) is vacuous here), which is the quotient of $T_{\mathcal{Z}_N} N$ by K, and the log presymplectic form ω_N descends to a log symplectic form on Q. ☐

5.1.9 Example. Let M be the plane $\mathbf{R}^2$, Z the y-axis, $\omega = x^{-1}\,dx \wedge dy$, and $A = T_Z M$. Let $P = \mathbf{R}$, $D = \{0\}$, $\lambda = 0$ the zero Poisson structure, and $E = T_D P$. The map $\mu(x, y) = x$ is log Poisson. We have $\mathrm{stab}(E, p) = 0$ for $p \in P\backslash\{0\}$ and $\mathrm{stab}(E, 0) = \mathbf{R}$. Let us reduce M at the "point" $\mathfrak{f} = 0_p$ for any $p \in P$. We have $N = \mu^{-1}(p) = \{p\} \times \mathbf{R}$, $(T^{\log}\mu)^{-1}(\mathfrak{f}) = TN$, $\mathfrak{h} = \mathrm{stab}(E^*, 0) = \mathbf{R}$, which acts on M as the vertical vector field $\partial/\partial y$. The reduced space $Q = N/\mathfrak{h}$ is a point equipped with a zero-dimensional Lie algebroid. However, if we reduce at $p = 0$ with respect to the "point" $\mathfrak{f} = \mathrm{stab}(E, p) = \mathbf{R}$, we have $N = \mu^{-1}(p) = Z$, $(T^{\log}\mu)^{-1}(\mathfrak{f}) = A|_Z$, and $\mathfrak{h} = 0$. The reduced space is now the divisor $Q = Z/\mathfrak{h} = Z$ equipped with the symplectic Lie algebroid $A|_Z$.

See Section 5.2 for further examples. The log symplectic version of Lemma 3.2.6 is the following.

5.1.10 Lemma. *In the context of Theorem 5.1.8 suppose that p is contained in the complement of the divisor $\mathcal{D}$. Let $x \in \mu^{-1}(p)$ and let L be the symplectic leaf of x with respect to the Poisson structure on M determined by the log symplectic form ω. Then the Hamiltonian action γ is locally free at x if and only if x is a regular point of the map $\mu|_L$.*

The following normal form theorem, which extends a result of Guillemin and Sternberg [21, § 1], is a direct consequence of Theorems 4.4.1 and 5.1.8. It involves a log Poisson morphism $\mu\colon M \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of a Lie algebra $\mathfrak{g}$ (equipped with the empty divisor). If the infinitesimal Hamiltonian action $\gamma\colon \mathfrak{g} \to T_{\mathcal{Z}}M$ determined by μ integrates to an action of a connected Lie group G with Lie algebra $\mathfrak{g}$, we call the tuple $(M, \mathcal{Z}, \omega, \mu)$ a *log Hamiltonian G-manifold.*

5.1.11 Theorem (Normal form near zero fibre for log symplectic forms)**.** *Let $(M, \mathcal{Z}, \omega, \mu)$ be a log Hamiltonian G-manifold and let $N = \mu^{-1}(0)$ be the zero fibre of μ. Suppose that the G-action on N is proper and free. Let $Q = N/G$ be the quotient manifold, $\mathcal{Z}_Q$ its normal crossing divisor, and ω_Q its log symplectic form. Choose a connection 1-form $\theta \in \Omega^1(N, \mathfrak{g})^G$ on the principal G-bundle $q\colon N \to Q$. Let $\mathbf{M}$ be the product $N \times \mathfrak{g}^*$ with projections $\mathrm{pr}_1\colon \mathbf{M} \to N$ and $\mathrm{pr}_2\colon \mathbf{M} \to \mathfrak{g}^*$. Let $\mathcal{Z}_{\mathbf{M}}$ be the normal crossing divisor $i_N^!\mathcal{Z} \times \mathfrak{g}^*$ of $\mathbf{M}$. The closed 2-form*

$$\omega_{\mathbf{M}} = \mathrm{pr}_1^*\, q^*\omega_Q + d\langle \mathrm{pr}_2, \mathrm{pr}_1^*\,\theta\rangle \in \Omega^2(\mathbf{M}\backslash|\mathcal{Z}_{\mathbf{M}}|)$$

has logarithmic singularities at $Z_{\mathbf{M}}$ and is symplectic in a neighbourhood of $N = N \times \{0\}$ in $\mathbf{M}$. The G-action on N and the coadjoint action on $\mathfrak{g}^*$ combine to a Hamiltonian G-action $\gamma_{\mathbf{M}}$ on $\mathbf{M}$ with moment map $\mu_{\mathbf{M}} = \mathrm{pr}_2$. In an open neighbourhood of N the log Hamiltonian G-manifold (M, Z, ω, μ) is isomorphic to $(\mathbf{M}, Z_{\mathbf{M}}, \omega_{\mathbf{M}}, \mu_{\mathbf{M}})$.

5.2. Some examples of log symplectic reduction

In this section $(M, \mathcal{Z}, \omega)$ denotes a log symplectic manifold with log tangent bundle $A = T_{\mathcal{Z}}M$.

5.2.1 Example (Reduction with respect to the identity map)**.** Let $\mu = \mathrm{id}_M$ be the identity map of M. The symplectic quotient of M at a "point" $(x, \mathfrak{a})$, where $\mathfrak{a}$ is a Lie subalgebra of $\mathrm{stab}(A, x)$, is the point x equipped with the symplectic Lie algebra $\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{a}^\omega)$; see Example 3.2.4. If $x \in \mathcal{Z}^{(k)}$, then $\mathrm{stab}(A, x)$ is a k-dimensional abelian Lie algebra, spanned in a normal crossing chart by the elements $x_1\partial/\partial x_1,\ x_2\partial/\partial x_2, \ldots,\ x_k\partial/\partial x_k$. Hence $\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{a}^\omega)$ is abelian as well.

5.2.2 Example (Log cotangent reduction)**.** Let X be a manifold and $\mathcal{Z}_X$ a normal crossing divisor of X. If $\pi\colon Y \to X$ is a submersion, then $\mathcal{Z}_Y = \pi^{-1}(\mathcal{Z}_X)$ is a normal crossing divisor of Y and the log tangent bundle of $(Y, \mathcal{Z}_Y)$ is the pullback Lie algebroid $\pi^!B$, where $B = T_{\mathcal{Z}_X}X$. Taking $Y = B^*$

and π to be the bundle projection $B^* \to X$, in which case $\pi^! B$ is the phase space of the Lie algebroid B (see Section 2.2), we obtain that the log cotangent bundle $M = T^*_{\mathcal{Z}_X} X$ equipped with the divisor $\mathcal{Z} = \pi^{-1}(\mathcal{Z}_X)$ and the form $\omega_{\mathrm{can}} = -d\alpha_{\mathrm{can}}$ is log symplectic. Let G be a Lie group and $G \times X \to X$ an action that preserves $|\mathcal{Z}_X|$. The action lifts naturally to an action $G \times M \to M$, which preserves $|\mathcal{Z}|$ and leaves the Liouville form α_{can} invariant. Therefore the lifted action is log Hamiltonian with moment map $\mu\colon M \to \mathfrak{g}^*$ given by $\langle \mu, \xi \rangle = \iota_B(\xi_X)\alpha_{\mathrm{can}}$ for $\xi \in \mathfrak{g}$. If the G-action on X is proper and free, the quotient $Q = X/G$ is a manifold with normal crossing divisor $\mathcal{Z}_Q = \mathcal{Z}_X/G$, and the symplectic quotient of M at 0 is naturally isomorphic to the log cotangent bundle $T^*_{\mathcal{Z}_Q} Q$ of Q.

5.2.3 Example (Log linear Poisson structures)**.** Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra and let $\mathbf{v}\colon \mathfrak{g} \to \mathbf{R}^l$ be a surjective Lie algebra homomorphism to the abelian Lie algebra $\mathbf{R}^l$. We will regard $\mathbf{v}$ as a linearly independent l-tuple $v_1, v_2, \dots, v_l \in \mathfrak{g}^*$ of characters of $\mathfrak{g}$. Extend $v_1, v_2, \dots, v_l$ to a basis $v_1, v_2, \dots, v_n$ of $\mathfrak{g}^*$ and let $c_{ij}^k = v_k([v_i^*, v_j^*])$ be the corresponding structure constants. Then $c_{ij}^k = 0$ for $k \le l$ because v_k is a character for $k \le l$. The usual linear Poisson structure on $\mathfrak{g}^*$ is given by

$$\lambda_0 = \sum_{1 \le i < j \le n} c_{ij}(y) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j},$$

where $c_{ij}(y) = \sum_{k=l+1}^n c_{ij}^k y_k$. Substituting

$$y_k = \log|x_k| \quad \text{for } k \le l, \qquad y_k = x_k \quad \text{for } k \ge l+1 \tag{5.2.4}$$

into λ_0 yields the cubic Poisson structure

$$\lambda = \sum_{1 \le i < j \le l} x_i x_j c_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \sum_{i=1}^{l} \sum_{j=l+1}^{n} x_i c_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \sum_{l+1 \le i < j \le n} c_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \tag{5.2.5}$$

This is a priori only defined for $x_1 x_2 \cdots x_l \neq 0$, but manifestly extends to a smooth Poisson structure on $P = \mathbf{R}^n$, which is log Poisson with respect to the divisor $\mathcal{D} = \{D_1, D_2, \dots, D_l\}$, where D_k is the coordinate hyperplane $\{x_k = 0\}$. Thus $(P, \mathcal{D}, \lambda)$ is a log Poisson manifold. We will also denote this log Poisson manifold by $P(\mathfrak{g}, \mathbf{v})$. The top stratum of P, i.e. the complement of $|\mathcal{D}|$, is the disjoint union of 2^l copies of the linear Poisson space $(\mathfrak{g}^*, \lambda_0)$, so we can think of P as the result of gluing 2^l copies of $\mathfrak{g}^*$ along hyperplanes. Each of the components of the divisor $\mathcal{D}$ is a log Poisson manifold in its own right, namely $D_k = P(\mathfrak{g}_k, \mathbf{v}_k)$, where $\mathfrak{g}_k$ is the ideal $\ker(v_k)$ of $\mathfrak{g}$ and $\mathbf{v}_k$ is the $l-1$-tuple

$$v_1|_{\mathfrak{g}_k},\ v_2|_{\mathfrak{g}_k}, \dots,\ v_{k-1}|_{\mathfrak{g}_k},\ v_{k+1}|_{\mathfrak{g}_k}, \dots,\ v_l|_{\mathfrak{g}_k}$$

of characters of $\mathfrak{g}_k$. Just as the Poisson manifold $\mathfrak{g}^*$ integrates to a symplectic groupoid, so the log Poisson manifold P integrates to a log symplectic groupoid M, as follows. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ such that the characters v_k exponentiate to characters $\bar{v}_k \colon G \to \mathbf{R}$ (e.g. take G to be simply connected). We denote the tuple of characters $(\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k)$ by $\bar{\mathbf{v}}$. Making the substitution (5.2.4) into the coadjoint action of G on $\mathfrak{g}^*$ yields a smooth action of G on P that preserves the divisor $\mathcal{D}$ and the Poisson structure λ. We define

$$M = M(G, \bar{\mathbf{v}}) = G \ltimes P$$

to be the corresponding action groupoid and $\mathcal{Z}$ the divisor $G \times \mathcal{D}$ of M. Recall that the action groupoid $G \ltimes \mathfrak{g}^* = T^*G$ of the coadjoint action on $\mathfrak{g}^*$, equipped with the cotangent symplectic form $\omega_{\mathrm{can}} = -d\alpha_{\mathrm{can}}$, is a symplectic groupoid. Let $\theta \in \Omega^1(G, \mathfrak{g})$ be the left-invariant Maurer-Cartan form. We have $\alpha_{\mathrm{can}} = \sum_k y_k \theta_k$, where the θ_k are the components of θ relative to the basis v_k^*, so the Maurer-Cartan equation gives

$$\omega_{\mathrm{can}} = \sum_{k=1}^{n} (\theta_k \wedge dy_k - y_k\, d\theta_k) = \sum_{k=1}^{n} \theta_k \wedge dy_k - \frac{1}{2} \sum_{1 \le i < j \le n} c_{ij}(y) \theta_i \wedge \theta_j.$$

Substituting (5.2.4) into ω_{can} yields the form

$$\omega = \sum_{k=1}^{l} \theta_k \wedge \frac{dx_k}{x_k} + \sum_{k=l+1}^{n} \theta_k \wedge dx_k - \frac{1}{2} \sum_{1 \le i < j \le n} c_{ij}(x) \theta_i \wedge \theta_j \tag{5.2.6}$$

on M, which is log symplectic with respect to the divisor $\mathcal{Z}$. The log symplectic groupoid $M(G, \bar{\mathbf{v}})$ integrates the log Poisson manifold $P(\mathfrak{g}, \mathbf{v})$ in the sense of [10, § III]. The projection $\mu = \mathrm{pr}_2 \colon M \to P$, i.e. the source map of the groupoid, is a log Poisson map. The Hamiltonian action of $T^*_{\mathcal{D}} P = \mathfrak{g} \ltimes P$ on $T_{\mathcal{Z}} M$ with moment μ is the infinitesimal left translation action of $\mathfrak{g}$ on G. The fibre of μ at a point $p \in P$ is $\mu^{-1}(p) = G \times \{p\}$. The stabilizer $\mathfrak{h} = \mathrm{stab}(T^*_{\mathcal{D}} P, p) \subseteq \mathfrak{g}$ of $p = (p_1, p_2, \dots, p_n) \in P = \mathbf{R}^n$ is equal to the coadjoint stabilizer of the element $v(p) = \sum_{k=1}^{n} p_k v_k \in \mathfrak{g}^*$. Thus the reduced space of M with respect to μ at p is $G/\mathfrak{h}$, which is up to a covering map the coadjoint G-orbit of $v(p)$.

5.2.7 Example. The log symplectic groupoid $M(G, \bar{\mathbf{v}})$ of Example 5.2.3 can itself be obtained by a log symplectic reduction as follows. Let $\mathcal{Z}_0$ be the normal crossing divisor $q_1 q_2 \cdots q_l = 0$ of $\mathbf{R}^l$. The action of the Lie group $H = \mathbf{R}^l$ on $\mathbf{R}^l$ given by

$$h \cdot q = (e^{h_1} q_1, e^{h_2} q_2, \dots, e^{h_l} q_l)$$

preserves the divisor. By Example 5.2.2 the lift of the action to the log cotangent bundle $T^*_{\mathcal{Z}_0}\mathbf{R}^l$ is log Hamiltonian with respect to the log symplectic form $\omega_0 = \sum_{k=1}^l q_i^{-1} \, dq_i \wedge dp_i$, with moment map $\varphi_0 \colon T^*_{\mathcal{Z}_0}\mathbf{R}^l \to \mathrm{Lie}(H) = \mathbf{R}^l$ given by $\varphi_0(q, p) = p$. The action is not proper and we cannot form a good quotient. But let G be another Lie group and let $\bar{\mathbf{v}} \colon G \to \mathbf{R}^l$ be a surjective Lie group homomorphism. We regard $\bar{\mathbf{v}}$ as an l-tuple of characters $\bar{v}_k \colon G \to \mathbf{R}$; the infinitesimal characters $v_k = d_1 \bar{v}_k \colon \mathfrak{g} \to \mathbf{R}$ are then linearly independent for $1 \le k \le l$. The H-action on $T_{\mathcal{Z}_0}\mathbf{R}^l \times T^*G = T_{\mathcal{Z}_0}\mathbf{R}^l \times G \times \mathfrak{g}^*$ defined by

$$h \cdot (q, p, g, y) = \left(h \cdot q, p, g, y - \sum_{k=1}^{l} h_k v_k \right)$$

for $h \in H$, $q \in \mathbf{R}^l$, $g \in G$, and $y \in \mathfrak{g}^*$ is proper and free, because the translation action of $\mathfrak{g}^*$ on itself is proper and free. This H-action is log Hamiltonian relative to the divisor $\mathcal{Z}_0 \times T^*G$ and the log symplectic form $\omega_0 + \omega_{\mathrm{can}}$, where ω_{can} is the cotangent symplectic form on T^*G. The moment map $\varphi \colon T_{\mathcal{Z}_0}\mathbf{R}^l \times T^*G \to \mathbf{R}^l$ is given by

$$\varphi(q, p, g, y) = p + \bar{\mathbf{v}}(g),$$

the symplectic quotient at 0 is

$$(\mathbf{R}^l \times T^*G)/H = G \times (\mathbf{R}^l \times \mathfrak{g}^*)/H = M(G, \bar{\mathbf{v}}),$$

and the reduced log symplectic form agrees with (5.2.6). Similarly, the log Poisson manifold $P(\mathfrak{g}, \mathbf{v})$ is the quotient $(\mathbf{R}^l \times \mathfrak{g}^*)/H$, where we give $\mathbf{R}^l$ the zero Poisson structure and $\mathfrak{g}^*$ its linear Poisson structure λ_0. The action of $\mathfrak{g}^*$ on $\mathbf{R}^l \times \mathfrak{g}^*$ given by translation in the second factor descends to a $\mathfrak{g}^*$-action on $P(\mathfrak{g}, \mathbf{v})$, the orbits of which are the strata. The spaces $P(\mathfrak{g}, \mathbf{v})$ are a nonabelian version of the tropical welded spaces of Gualtieri et al. [17, § 3.2] and the moment codomains of Guillemin et al. [19, § 5].

Appendix A. Lie algebroids

This appendix is an exposition of some elementary properties of Lie algebroids, including fibred products, pullbacks, a regular value theorem, and differential forms. This is mostly standard material that is covered more fully in [12, 22, 29], and [37].

A.1. Lie algebroids and morphisms

A *Lie algebroid* is a real vector bundle $\pi = \pi_A : A \to M$ equipped with a Lie bracket on the space of sections, which we denote by

$$[\cdot,\cdot] = [\cdot,\cdot]_A : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A),$$

and a vector bundle map $\mathbf{an} = \mathbf{an}_A : A \to TM$ called the *anchor*. The bracket is required to satisfy the *Leibniz rule* with respect to the anchor: $[\sigma, f\tau] = f[\sigma,\tau] + (\sigma \cdot f)\tau$ for all sections σ, $\tau \in \Gamma(A)$ and functions $f \in \mathcal{C}^\infty(M)$, where $\sigma \cdot f$ denotes the derivative of f along the vector field $\mathbf{an}(\sigma)$.

For a vector bundle $E \to M$ and a (not necessarily embedded) subbundle $F \to N$ we define a *relative section* to be a section of E whose restriction to N is a section of F. We denote the $\mathcal{C}^\infty(M)$-module of relative sections by

$$\Gamma(E;F) = \{\, \sigma \in \Gamma(E) \mid \sigma|_N \in \Gamma(F)\, \}. \tag{A.1.1}$$

Let 0_N denote the zero vector bundle over N. Then $\Gamma(E;0_N)$ is the module of sections that vanish at N. We call a pair of open subsets $U \subseteq M$ and $V \subseteq N$ *adapted* to the submanifold N if V is a closed embedded submanifold of U. If the pair (U,V) is adapted, then

$$\Gamma(V,F) \cong \Gamma(E|_U; F|_V)/\Gamma(E|_U; 0_V). \tag{A.1.2}$$

A *Lie subalgebroid* of a Lie algebroid $A \to M$ is a (not necessarily embedded) subbundle $B \to N$ such that $\mathbf{an}_A(B) \subseteq TN$ and $\Gamma(A|_U; B|_V)$ is a Lie subalgebra of $\Gamma(A|_U)$ for all adapted pairs (U,V). If B is a Lie subalgebroid of A, then by the Leibniz rule $\Gamma(A|_U; 0_V)$ is an ideal of $\Gamma(A|_U; B|_V)$ for every adapted pair (U,V). Using (A.1.2) we see that

$$\Gamma(V,B) \cong \Gamma(A|_U; B|_V)/\Gamma(A|_U; 0_V)$$

is a Lie algebra. This makes B a Lie algebroid over N with anchor $\mathbf{an}_B = \mathbf{an}_A|_B$ ([37, Proposition 6.14]).

A *morphism* from A to another Lie algebroid $\pi_B\colon B \to N$ is most conveniently defined as a vector bundle morphism $\varphi\colon A \to B$ whose graph is a Lie subalgebroid of the direct product Lie algebroid $A \times B$ ([37, § 7.1]). We will denote the base map induced by a Lie algebroid morphism φ by

$$\mathring{\varphi}\colon M \longrightarrow N.$$

The anchor of A itself is a morphism of Lie algebroids $\mathbf{an}_A\colon A \to TM$ with base map $\mathring{\mathbf{an}}_A = \mathrm{id}_M$. Lie algebroids and their morphisms form a category.

A.2. Fibred products

Given Lie algebroids $A_0 \to M_0$, $A_1 \to M_1$, and $A_2 \to M_2$ and Lie algebroid morphisms $\varphi_1\colon A_1 \to A_0$ and $\varphi_2\colon A_2 \to A_0$, we form the fibred product

$$A = A_1 \times_{A_0} A_2 = \{\,(a_1, a_2) \in A_1 \times A_2 \mid \varphi_1(a_1) = \varphi_2(a_2)\,\}, \tag{A.2.1}$$

which we regard as a subspace of the direct product Lie algebroid $A_1 \times A_2$ over $M_1 \times M_2$. The image of the natural projection $A \to M_1 \times M_2$ is the fibred product $M = M_1 \times_{M_0} M_2$. Though M may not be a smooth manifold and A may not be a vector bundle, for every $x = (x_1, x_2) \in M$ the fibre of the projection $A \to M$ is a vector space, namely

$$A_x = A_{1,x_1} \times_{A_{0,x_0}} A_{2,x_2}, \tag{A.2.2}$$

where $x_0 = \mathring{\varphi}_1(x_1) = \mathring{\varphi}_2(x_2)$. The next result is taken from Meinrenken's notes [37, Proposition 7.14].

A.2.3 Proposition. *Let $A_0 \to M_0$, $A_1 \to M_1$, and $A_2 \to M_2$ be Lie algebroids and let $\varphi_1\colon A_1 \to A_0$ and $\varphi_2\colon A_2 \to A_0$ be Lie algebroid morphisms. If φ_1 and φ_2 intersect cleanly, the fibred product $A = A_1 \times_{A_0} A_2$ is a Lie subalgebroid of $A_1 \times A_2$ over the submanifold $M = M_1 \times_{M_0} M_2$ of $M_1 \times M_2$. The fibre of A at $x \in M$ is the vector space (A.2.2).*

A surprisingly recent result of Grabowski and Rotkiewicz [16] provides us with a convenient criterion for two vector bundle maps to intersect cleanly or transversely.

A.2.4 Lemma. *Let X_0, X_1, and X_2 be manifolds and let $E_0 \to X_0$, $E_1 \to X_1$, and $E_2 \to X_2$ be vector bundles. Let $\varphi_1\colon E_1 \to E_0$, resp. $\varphi_2\colon E_2 \to E_0$, be vector bundle morphisms with base maps $\mathring{\varphi}_1\colon X_1 \to X_0$, resp. $\mathring{\varphi}_2\colon X_2 \to X_0$. Form the fibred products*

$$X = X_1 \times_{X_0} X_2 = \{\,(x_1, x_2) \in X_1 \times X_2 \mid \mathring{\varphi}_1(x_1) = \mathring{\varphi}_2(x_2)\,\},$$

$$E = E_1 \times_{E_0} E_2 = \{\, (u_1, u_2) \in E_1 \times E_2 \mid \varphi_1(u_1) = \varphi_2(u_2) \,\}.$$

$$\begin{tikzcd}
 & E \arrow[rr] \arrow[dl] \arrow[dd] & & E_2 \arrow[dl, "\varphi_2"] \arrow[dd] \\
E_1 \arrow[rr, "\varphi_1"] \arrow[dd] & & E_0 \arrow[dd] & \\
 & X \arrow[rr] \arrow[dl] & & X_2 \arrow[dl, "\mathring{\varphi}_2"] \\
X_1 \arrow[rr, "\mathring{\varphi}_1"] & & X_0 &
\end{tikzcd}$$

Define $\pi_E\colon E \to X$ *by* $\pi_E(u) = (\pi_{E_1}(u_1), \pi_{E_2}(u_2))$ *for* $u = (u_1, u_2) \in E$ *and define* $E_x = \pi_E^{-1}(x)$ *for* $x \in X$.

- *(i) The morphisms* φ_1 *and* φ_2 *intersect cleanly if and only if the base maps* $\mathring{\varphi}_1$ *and* $\mathring{\varphi}_2$ *intersect cleanly and the dimension of the vector space* E_x *is independent of* $x \in X$.
- *(ii) The morphisms* φ_1 *and* φ_2 *intersect transversely if and only if* $\mathring{\varphi}_1$ *and* $\mathring{\varphi}_2$ *intersect transversely and* $E_{0,x_0} = \varphi_1(E_{1,x_1}) + \varphi_2(E_{2,x_2})$ *for all triples* $(x_0, x_1, x_2) \in X_0 \times X_1 \times X_2$ *with* $x_0 = \mathring{\varphi}_1(x_1) = \mathring{\varphi}_2(x_2)$.
- *(iii) If* φ_1 *and* φ_2 *intersect cleanly, then* X *is a submanifold of* $X_1 \times X_2$ *and* E *is a subbundle of* $E_1 \times E_2$ *with base* X *and bundle projection* π_E.

Proof. (i) and (iii) We view $E_1 \times E_2$ as a vector bundle over $X_1 \times X_2$ and we identify $X_1 \times X_2$ with the zero section of $E_1 \times E_2$. The intersection of E with the zero section is $E \cap (X_1 \times X_2) = X$ and E is invariant under fibrewise scalar multiplication on $E_1 \times E_2$. Suppose φ_1 and φ_2 intersect cleanly. Then E is a submanifold of $E_1 \times E_2$. In view of [16, Theorem 2.3] this implies the following two facts: (1) E cleanly intersects the zero section (so X_0 is a submanifold of $X_1 \times X_2$); and (2) E is a subbundle of $E_1 \times E_2$ over the submanifold X. It follows from fact (1) that $\mathring{\varphi}_1$ and $\mathring{\varphi}_2$ intersect cleanly. It follows from fact (2) that the fibre E_x has constant dimension for $x \in X$. To prove the converse, now suppose φ_1 and φ_2 intersect cleanly and that E_x has constant dimension for $x \in X$. Then X is a submanifold of $X_1 \times X_2$. For $i = 0$, 1, 2 let F_i be the pullback of E_i to X, and for $i = 1$, 2 let $\psi_i\colon F_i \to F_0$ be the vector bundle map induced by φ_i. Let $\psi\colon F_1 \oplus F_2 \to F_0$ be the vector bundle map defined by $\psi = (\psi_1, -\psi_2)$. Then E is the kernel of ψ, which by hypothesis has constant rank, so E is a vector bundle over X. To show that φ_1 and φ_2 intersect cleanly we must calculate the tangent space to E

at an arbitrary point $u \in E$. Let $x \in X$ be the basepoint of u; then x is a pair $x = (x_1, x_2) \in X_1 \times X_2$ with $\mathring{\varphi}_1(x_1) = \mathring{\varphi}_2(x_2)$. Likewise u is a pair $u = (u_1, u_2) \in E_1 \times E_2$ with $\varphi_1(u_1) = \varphi_2(u_2)$. Put $x_0 = \mathring{\varphi}_1(x_1) = \mathring{\varphi}_2(x_2) \in X_0$ and $u_0 = \varphi_1(u_1) = \varphi_2(u_2) \in E_0$; then x_0 is the basepoint of u_0. The tangent space T_uE fits into a short exact sequence

$$E_x \hookrightarrow T_uE \twoheadrightarrow T_xX.$$

We have similar sequences for E_0, E_1, and E_2, which combine into a commutative diagram

(A.2.5)
$$\begin{tikzcd}
E_x \arrow[r, hook] \arrow[d, hook] & T_uE \arrow[r, two heads] \arrow[d, hook] & T_xX \arrow[d, hook] \\
E_{1,x_1} \times E_{2,x_2} \arrow[r, hook] \arrow[d, "{(\varphi_1, -\varphi_2)}"'] & T_{u_1}E_1 \times T_{u_2}E_2 \arrow[r, two heads] \arrow[d, "{(T\varphi_1, -T\varphi_2)}"'] & T_{x_1}X_1 \times T_{x_2}X_2 \arrow[d, "{(T\mathring{\varphi}_1, -T\mathring{\varphi}_2)}"'] \\
E_{0,x_0} \arrow[r, hook] & T_{u_0}E_0 \arrow[r, two heads] & T_{x_0}X_0,
\end{tikzcd}$$

which has exact rows. The right column of the diagram is exact because $\mathring{\varphi}_1$ and $\mathring{\varphi}_2$ intersect cleanly. The left column is exact because $E = \ker(\psi)$. Hence the middle column is exact, which shows that $T_uE = T_{u_1}E_1 \times_{T_{u_0}E_0} T_{u_2}E_2$, proving that φ_1 and φ_2 intersect cleanly.

(ii) This follows from (i) plus the observation that the arrows $(\varphi_1, -\varphi_2)$ and $(T\mathring{\varphi}_1, -T\mathring{\varphi}_2)$ in (A.2.5) are surjective if and only if the arrow $(T\varphi_1, -T\varphi_2)$ is surjective. ☐

A.3. Pullbacks

Given a smooth map $F\colon P \to M$ to the base of the Lie algebroid A, the fibred product

(A.3.1) $$F^!A = TP \times_{TM} A = \{\, (v, a) \in TP \times A \mid TF(v) = \mathbf{an}_A(a) \,\}$$

of the Lie algebroid morphisms $TF\colon TP \to TM$ and $\mathbf{an}_A\colon A \to TM$ is called the *(Higgins-Mackenzie) pullback* of A to P. We regard $F^!A$ as a subspace of the direct sum bundle $TP \oplus F^*A$ over $P \times_M M = P$. For every $p \in P$ the fibre of the natural projection $F^!A \to P$ is the vector space

(A.3.2) $$(F^!A)_p = T_pP \times_{T_{F(p)}M} A_{F(p)}.$$

The map F lifts to a map

$$F_! \colon F^! A \to A \tag{A.3.3}$$

given by $F_!(v, a) = a$, which is linear on the fibres.

A.3.4 Definition. A smooth map $F \colon P \to M$ *cleanly intersects* the Lie algebroid $A \to M$ if the tangent map $TF \colon TP \to TM$ cleanly intersects the anchor $\mathbf{an} \colon A \to TM$. The map F *transversely intersects* A, or *is transverse to* A, if TF is transverse to $\mathbf{an}$. If P is a submanifold of M and F is the inclusion map, we say P *cleanly* (resp. *transversely*) *intersects* A if F cleanly (resp. transversely) intersects A.

The next statement follows from Proposition A.2.3 and Lemma A.2.4. The *orbits* of A are the integral manifolds of the (singular) foliation $\mathrm{im}(\mathbf{an}_A) \subseteq TM$; see e.g. [37, § 8.6].

A.3.5 Proposition. *Let $A \to M$ be a Lie algebroid and let $F \colon P \to M$ be a smooth map.*

(i) The map F intersects A cleanly if and only if the dimension of the vector space $(F^! A)_p$ given by (A.3.2) is independent of $p \in P$. The map F is transverse to A if and only if F is transverse to all orbits of A.
(ii) If F intersects A cleanly, the pullback $F^! A$ is a Lie algebroid over P, whose fibre at $p \in P$ is the vector space (A.3.2). The map $F_! \colon F^! A \to A$ given by (A.3.3) is a Lie algebroid morphism with base map equal to $\mathring{F}_! = F$.

A section of $F^! A$ is a pair (v, σ) consisting of a vector field v on P and a section of the (ordinary) pullback bundle $\sigma \in \Gamma(F^* A)$ such that $T_p F(v(p)) = \mathbf{an}_A(\sigma(p))$ for all $p \in P$. The anchor of $F^! A$ is given by $\mathbf{an}_{F^! A}(v, \sigma) = v$. Using the isomorphism

$$\Gamma(F^* A) \cong \mathcal{C}^\infty(P) \otimes_{\mathcal{C}^\infty(M)} \Gamma(A)$$

we can write any section of $F^! A$ as a pair $\left(v, \sum_i f_i \sigma_i\right)$, where $v \in \Gamma(TP)$, $f_i \in \mathcal{C}^\infty(P)$, and $\sigma_i \in \Gamma(A)$ satisfy $T_p F(v(p)) = \sum_i f_i(p)\, \mathbf{an}_A(\sigma_i(F(p)))$ for all $p \in P$. The Lie bracket of two sections (v, σ) and (w, τ) of $F^! A$ with $\sigma = \sum_i f_i \sigma_i$ and $\tau = \sum_j g_j \tau_j$ is given by

(A.3.6)
$$[(v, \sigma), (w, \tau)] = \left([v, w], \sum_{i,j} f_i g_j [\sigma_i, \tau_j] + \sum_j (v \cdot F^* g_j) \tau_j - \sum_i (w \cdot F^* f_i) \sigma_i\right).$$

For maps of constant rank we can reformulate Proposition A.3.5(i) as follows.

A.3.7 Lemma. *Let $A \to M$ be a Lie algebroid.*

(i) Let $F\colon P \to M$ be a smooth map of constant rank and let $\mathcal{N}(F) = F^*TM/\operatorname{im}(TF)$ be the normal bundle of F. The anchor $\mathbf{an}\colon A \to TM$ induces a vector bundle map $\overline{\mathbf{an}}\colon F^*A \to \mathcal{N}(F)$. The map F intersects A cleanly if and only if $\overline{\mathbf{an}}$ has constant rank. In that case we have an exact sequence of vector bundles over P,

$$\ker(TF) \hookrightarrow F^!A \longrightarrow F^*A \xrightarrow{\overline{\mathbf{an}}} \mathcal{N}(F).$$

The map F is transverse to A if and only if $\overline{\mathbf{an}}$ is surjective.

(ii) Let $i_N\colon N \to M$ be a submanifold and let $\mathcal{N}(M,N)$ be the normal bundle of N in M. The anchor $\mathbf{an}\colon A \to TM$ induces a vector bundle map $\overline{\mathbf{an}}\colon i_N^*A \to \mathcal{N}(M,N)$. The submanifold N intersects A cleanly if and only if $\overline{\mathbf{an}}$ has constant rank. In that case we have an embedding of vector bundles over N,

$$i_N^*A/i_N^!A \hookrightarrow \mathcal{N}(M,N).$$

The submanifold N is transverse to A if and only if this embedding is an isomorphism.

A.3.8 Remarks. (i) If the submanifold $i_N\colon N \to M$ is open, then N is transverse to A and $i_N^!A = i_N^*A$.
(ii) If N is an orbit of A, then N cleanly intersects A (but the intersection is not transverse unless the orbit is open) and $i_N^!A = i_N^*A$. Any submanifold of an orbit of A intersects A cleanly. In particular, if $N = \{x\}$ consists of a single point, then N cleanly intersects A (but the intersection is not transverse unless the anchor is surjective at x). So $i_N^!A = \ker(\mathbf{an}_x)$ is a Lie algebroid over x, i.e. a Lie algebra, known as the *isotropy* or *stabilizer* $\operatorname{stab}(A,x)$ of the Lie algebroid A at x.

A.4. A regular value theorem

The following special case of Proposition A.2.3 is a version of the regular value theorem for Lie algebroids.

A.4.1 Proposition. *Let $A \to M$ and $E \to P$ be Lie algebroids, let $\varphi\colon A \to E$ be a morphism, and let $F \to Q$ be a Lie subalgebroid of E. Suppose φ*

intersects F cleanly. Then $A \times_E F = \varphi^{-1}(F)$ is a Lie subalgebroid of A, whose base is the submanifold $\mathring{\varphi}^{-1}(Q)$ of M. In particular, if p is a single point of P, $\mathfrak{f}$ is a Lie subalgebra of $\operatorname{stab}(E,p)$*, and φ intersects $\mathfrak{f}$ cleanly, then $\varphi^{-1}(\mathfrak{f})$ is a Lie subalgebroid of A, whose base is the submanifold $\mathring{\varphi}^{-1}(p)$.*

A.4.2 Remark. By Lemma A.2.4(i), the map φ intersects $\mathfrak{f}$ cleanly if and only if p is a clean value of $\mathring{\varphi}$ and the subspace $\varphi_x^{-1}(\mathfrak{f})$ has constant dimension for $x \in \mathring{\varphi}^{-1}(p)$. By Lemma A.2.4(ii), φ intersects $\mathfrak{f}$ transversely if and only p is a regular value of $\mathring{\varphi}$ and $\varphi_x(A_x) + \mathfrak{f} = E_p$ for all $x \in \mathring{\varphi}^{-1}(p)$.

The next result says that "pullback commutes with fibred products".

A.4.3 Proposition. *Let $A \to M$ and $E \to P$ be Lie algebroids and $\varphi\colon A \to E$ a morphism. Let $g\colon Q \to P$ be a smooth map which intersects E cleanly (resp. transversely). Suppose the natural morphism $g_!\colon g^! E \to E$ intersects φ cleanly (resp. transversely). Then the map $\bar{g}\colon M \times_P Q \to M$ induced by the projection $M \times Q \to M$ intersects A cleanly (resp. transversely), and we have an isomorphism of Lie algebroids $\bar{g}^! A \cong A \times_E g^! E$.*

Proof. Since g intersects E cleanly, by Proposition A.3.5 the Lie algebroid $g^! E$ and the morphism $g_!$ are well-defined. Since φ intersects $g_!$ cleanly, by Lemma A.2.4(i) the base maps $\mathring{\varphi}$ and g intersect cleanly, so that the fibred product $\bar{M} = M \times_P Q$ is a manifold and the map $\bar{g}\colon \bar{M} \to M$ is smooth. Moreover, the map

$$l_{(x,q)} = (\varphi_x, -(g_!)_q)\colon A_x \times (g^! E)_q \longrightarrow E_p$$

has constant rank for all triples $(x,q,p) \in M \times Q \times P$ satisfying

$$\mathring{\varphi}(x) = g(q) = p. \tag{A.4.4}$$

The kernel of $l_{(x,q)}$ is the vector space $A_x \times_{E_p} (g^! E)_q$. By another application of Lemma A.2.4(i), to show that $\bar{g}$ intersects A cleanly it is enough to show that the map

$$\bar{l}_{(x,q)} = (\mathbf{an}_{A,x}, -T_{(x,q)}\bar{g})\colon A_x \times T_{(x,q)}\bar{M} \longrightarrow T_x M$$

has constant rank for all pairs $(x,q) \in \bar{M}$. The kernel of $\bar{l}_{(x,q)}$ is the vector space $(\bar{g}^! A)_{(x,q)}$. Using $T\bar{M} = TM \times_{TP} TQ$ we find a natural isomorphism

$$\begin{aligned}\ker(\bar{l}_{(x,q)}) = (\bar{g}^! A)_{(x,q)} &= A_x \times_{T_x M} T_{(x,q)}\bar{M} = A_x \times_{T_x M} (T_x M \times_{T_p P} T_q Q)\\ &\cong A_x \times_{T_p P} T_q Q \cong A_x \times_{E_p} (E_p \times_{T_p P} T_q Q)\end{aligned}$$

$$\cong A_x \times_{E_p} (g^! E)_q = \ker(l_{(x,q)})$$

for all triples (x, q, p) satisfying (A.4.4). So $\bar{l}$ has constant rank because l does, and we have the isomorphism $\bar{g}^! A \cong A \times_E g^! E$. This establishes the proposition in the clean case. If g intersects E transversely and $g_!$ intersects φ transversely, then by Lemma A.2.4(ii) the map

$$(\mathbf{an}_{E,p}, -T_q g) \colon E_p \times T_q Q \longrightarrow T_p P$$

and the map $l_{(x,q)}$ are surjective for all triples (x, q, p) satisfying (A.4.4). One deduces from this that $\bar{l}_{(x,q)}$ is surjective, and hence that $\bar{g}$ is transverse to A. □

Taking $F = i_Q^! E$ in Proposition A.4.1 and applying Proposition A.4.3 gives the following.

A.4.5 Corollary. *Let $A \to M$ and $E \to P$ be Lie algebroids and $\varphi \colon A \to E$ a morphism. Let Q be a submanifold of P which intersects E cleanly (resp. transversely). Suppose φ intersects the Lie subalgebroid $i_Q^! E$ of E cleanly (resp. transversely). Then $N = \mathring{\varphi}^{-1}(Q)$ is a submanifold of M which intersects the Lie algebroid A cleanly (resp. transversely), and $i_N^! A = \varphi^{-1}(i_Q^! E)$. In particular, let p be a point in P such that φ intersects the subspace* $\operatorname{stab}(E, p)$ *of E cleanly (resp. transversely). Then the submanifold $N = \mathring{\varphi}^{-1}(p)$ of M intersects A cleanly (resp. transversely), and $i_N^! A = \varphi^{-1}(\operatorname{stab}(E, p))$.*

A.5. Lie algebroid differential forms and Cartan calculus

Let $\pi \colon A \to M$ be a Lie algebroid. Let $\Lambda^\bullet A^*$ be the exterior algebra of the dual bundle A^* and let U be an open subset of M. We denote the graded vector space of sections $\Gamma(U, \Lambda^\bullet A^*)$ by $\Omega_A^\bullet(U)$ and we call elements of $\Omega_A^\bullet(U)$ *Lie algebroid differential forms*, or *A-forms*, or just *forms* on U. The *exterior derivative* of an A-form $\alpha \in \Omega_A^k(U)$ is the A-form $d_A \alpha \in \Omega_A^{k+1}(U)$ given by

$$\text{(A.5.1)} \quad d_A \alpha(\sigma_1, \sigma_2, \dots, \sigma_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \sigma_i \cdot \alpha(\sigma_1, \sigma_2, \dots, \hat{\sigma}_i, \dots, \sigma_{k+1}) + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \alpha\big([\sigma_i, \sigma_j], \sigma_1, \sigma_2, \dots, \hat{\sigma}_i, \dots, \hat{\sigma}_j, \dots, \sigma_{k+1}\big)$$

for sections $\sigma_1, \sigma_2, \dots, \sigma_{k+1} \in \Gamma(U, A)$. Here $\sigma \cdot f$ denotes the Lie derivative of a function f along the vector field $\mathbf{an}(\sigma)$. The pair $(\Omega_A^\bullet, d_A)$ is a sheaf of commutative differential graded algebras (CDGA), which we call the *de*

Rham (or *Chevalley-Eilenberg*) *complex* of A. We denote the cohomology of the complex of sections $(\Omega_A^\bullet(U), d_A)$ by $H_A^\bullet(U)$.

A Lie algebroid morphism $\varphi\colon A \to B$ induces a morphism of sheaves of CDGA

$$\Omega^\bullet(\varphi)\colon \Omega_B^\bullet \longrightarrow \varphi_*\Omega_A^\bullet.$$

On global sections this gives a pullback map $\Omega^\bullet(\varphi)\colon \Omega_B^\bullet(N) \to \Omega_A^\bullet(M)$ and an induced map in cohomology $H^\bullet(\varphi)\colon H_B^\bullet(N) \to H_A^\bullet(M)$. Usually we will denote both $\Omega^\bullet(\varphi)$ and $H^\bullet(\varphi)$ by φ^*. Taking φ to be the anchor $\mathbf{an}\colon A \to TM$ of A we get pullback maps $\mathbf{an}^*\colon \Omega^\bullet(M) \to \Omega_A^\bullet(M)$ and $\mathbf{an}^*\colon H^\bullet(M) \to H_A^\bullet(M)$, where $\Omega^\bullet(M)$ denotes the usual de Rham complex of M and $H^\bullet(M)$ its cohomology.

Let σ be a global section of A. Associated with σ are various natural objects and operations. The first is the *contraction operator* $\iota_A(\sigma)\colon \Omega_A^k \to \Omega_A^{k-1}$ defined by

$$\iota_A(\sigma)\alpha(\sigma_1, \sigma_2, \dots, \sigma_{k-1}) = \alpha(\sigma, \sigma_1, \sigma_2, \dots, \sigma_{k-1}) \tag{A.5.2}$$

for sections $\sigma_1, \sigma_2, \dots, \sigma_{k-1}$ of A. It is a derivation of degree -1 of the de Rham complex. The section σ also determines a linear vector field $\mathbf{ad}_A(\sigma)$ on the total space of A that is given by the formula

$$\mathbf{ad}_A(\sigma)(a) = T_x\upsilon(\mathbf{an}_A(\sigma)(x)) - [\sigma, \upsilon](x) \tag{A.5.3}$$

for $a \in A$ and $x = \pi_A(a)$, where $\upsilon \in \Gamma(A)$ is any section with $\upsilon(x) = a$. The vector field $\mathbf{ad}_A(\sigma)$ is characterized by the following two requirements: first, the projection $A \to M$ intertwines the flow Φ_t of $\mathbf{ad}_A(\sigma)$ with the flow $\mathring{\Phi}_t$ of the vector field $\mathbf{an}(\sigma)$ on M and second,

$$\frac{d}{dt}\Phi_t^*\tau = \Phi_t^*[\sigma, \tau] \tag{A.5.4}$$

for all sections $\tau \in \Gamma(A)$ ([37, § 7.6]). Here $\Phi_t^*\tau$ denotes the section $\Phi_t^{-1} \circ \tau \circ \mathring{\Phi}_t$ of A. We also write $[\sigma, \tau] = \mathcal{L}_A(\sigma)\tau$ and call the operator $\mathcal{L}_A(\sigma)$ the *Lie derivative*.

A time-dependent section σ_t of A (defined for t in some interval containing 0) gives rise to a time-dependent linear vector field $\mathbf{ad}_A(\sigma_t)$ and hence to a flow Φ_t on A with initial condition $\Phi_0 = \mathrm{id}_A$. The identity (A.5.4) generalizes to

$$\frac{d}{dt}\Phi_t^*\tau_t = \Phi_t^*\big([\sigma_t, \tau_t] + \dot{\tau}_t\big) \tag{A.5.5}$$

for all time-dependent sections τ_t of A.

A.5.6 Lemma. *Let σ_t be a time-dependent section of a Lie algebroid $A \to M$ and let Φ_t be the flow of the vector field $\mathbf{ad}_A(\sigma_t)$ with initial condition $\Phi_0 = \mathrm{id}_A$.*

(i) For each t the map Φ_t is an automorphism of A (where defined).
(ii) Let $B \to M$ be a Lie subalgebroid of A over the same base which is normalized by σ_t in the sense that $[\sigma_t, \tau] \in \Gamma(B)$ for all t and all $\tau \in \Gamma(B)$. Then the flow Φ_t preserves B.

Proof. (i) Reading (A.5.5) as a linear differential equation $\frac{d}{dt}\Phi_t^* = \Phi_t^* \circ \mathrm{ad}(\sigma_t)$ for the operator Φ_t^* (acting on time-independent sections τ) with initial condition $\Phi_0^* = \mathrm{id}$, we see that $\Phi_t^* = \exp(\Sigma_t)$, where $\Sigma_t = \int_0^t \mathrm{ad}(\sigma_u)\, du$. Since Σ_t is a derivation of the Lie algebra $\Gamma(A)$, Φ_t^* is an automorphism of $\Gamma(A)$, so Φ_t is an automorphism of A.

(ii) It follows from (A.5.3) that $\mathbf{ad}_A(\sigma_t)$ is tangent to B. Hence $\Phi_t(B) \subseteq B$. $\square$

Dually, for differential forms the Lie derivative $\mathcal{L}_A(\sigma)\colon \Omega_A^\bullet \to \Omega_A^\bullet$ is defined by

$$\mathcal{L}_A(\sigma)\alpha = \frac{d}{dt}\Phi_t^*\alpha\Big|_{t=0}. \tag{A.5.7}$$

It is a derivation of degree 0 of the de Rham complex. For a time-dependent section σ_t and a time-dependent A-form α_t we have

$$\frac{d}{dt}\Phi_t^*\alpha_t = \Phi_t^*(\mathcal{L}_A(\sigma_t)\alpha_t + \dot{\alpha}_t). \tag{A.5.8}$$

The exterior derivative (A.5.1), the contractions (A.5.2), and the Lie derivatives (A.5.7) obey the usual rules of the Cartan differential calculus, namely

$$\begin{aligned} [\iota_A(\sigma), \iota_A(\tau)] &= 0, & [\mathcal{L}_A(\sigma), \mathcal{L}_A(\tau)] &= \mathcal{L}_A([\sigma,\tau]),\\ [\mathcal{L}_A(\sigma), d_A] &= 0, & [\mathcal{L}_A(\sigma), \iota_A(\tau)] &= \iota_A([\sigma,\tau]),\\ [d_A, d_A] &= 0, & [\iota_A(\sigma), d_A] &= \mathcal{L}_A(\sigma) \end{aligned} \tag{A.5.9}$$

for all σ, $\tau \in \Gamma(A)$, where the square brackets denote graded commutators.

Appendix B. Notation index

! Lie algebroid pullback
$^\circ$ annihilator of subspace or subbundle

$[\cdot,\cdot]_A$ Lie algebroid bracket
$\{\cdot,\cdot\}_\lambda$ Poisson bracket

0_M zero vector bundle over M

$\mathbf{ad}_A\colon \Gamma(A)\to\Gamma(TA)$ "adjoint representation" of Lie algebroid
$\mathbf{an}_A\colon A\to TM$ anchor of Lie algebroid
$A\to M$ Lie algebroid

d_A Lie algebroid exterior derivative
d_λ Poisson differential

$F_!$ natural morphism $F^!A\to A$
$F^!A$ pullback of Lie algebroid along map

$\Gamma(E)$ global smooth sections
$\Gamma(E;F)$ relative sections of vector bundle pair
$\Gamma(U,E)$ smooth sections over U

$i_N\colon N\to M$ inclusion of immersed submanifold
ι_A Lie algebroid interior product
$i_v\colon \mathcal{N}(M,N)\to M$ tubular neighbourhood embedding associated with Euler-like vector field

$\mathcal{L}_A$ Lie algebroid Lie derivative
λ Poisson structure
$\lambda^\sharp\colon A^*\to A$ structure map of Poisson structure

$(M,\mathcal{F})$ foliated manifold
$M/\mathcal{F}$ leaf space

$\mathcal{N}(M,N)$ normal bundle of N in M
$\mathcal{N}(v)$ linearization of vector field along N

$\Omega^\bullet_A(M)$ Lie algebroid de Rham complex
ω symplectic structure
$\omega^\flat\colon A\to A^*$ structure map of ω
ω^{-1} Poisson structure inverse to ω
$\omega^\sharp\colon A^*\to A$ inverse of $\omega^\flat$

$\mathring{\varphi}$ base map of vector bundle map φ
$\pi_E\colon E\to M$ vector bundle projection

stab(A, x) Lie algebroid stabilizer of point
stab(C, B) stabilizer of B under C-action

W° annihilator of subspace or subbundle
$W^\lambda = \lambda^\sharp(W^\circ)$ Poisson "orthogonal" of subspace or subbundle
$W^\omega = \omega^\sharp(W^\circ)$ symplectic orthogonal of subspace or subbundle

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