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Bohr-Sommerfeld quantization of b-symplectic toric manifolds

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Dedicated to Victor Guillemin

Abstract: We introduce a Bohr-Sommerfeld quantization for b-symplectic toric manifolds and show that it coincides with the formal geometric quantization of [GMW18b]. In particular, we prove that its dimension is given by a signed count of the integral points in the moment polytope of the torus action on the manifold.

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1. Introduction

Singular symplectic manifolds appear in the investigation of geometrical and dynamical facets of non-compact manifolds as their natural compactifications. One example is the work undertaken by [KMS16] and [MO21] on the restricted three-body problem. They also appear in the realm of quantization, where new procedures are required to extend classic ideas of geometric quantization.

One approach by Guillemin, Miranda and Weitsman used the formal geometric quantization of [Wei01] and [Par09]. They proved in [GMW18b] that the formal geometric quantization of a b-symplectic manifold is a finite-dimensional vector space. This raised the natural question of whether there is a true geometric quantization of such a space. An answer was given in the affirmative in [BLS21] and [LLSS21], where virtual modules agreeing with the formal geometric quantization of [GMW18b] were constructed analytically using index theory.

The purpose of this paper is to revisit the question in the context of Bohr-Sommerfeld quantization, following [GS83] but restricting ourselves to the case of toric manifolds.

We start by revisiting a result of Guillemin and Sternberg [GS83], which identifies the Bohr-Sommerfeld leaves of a symplectic manifold endowed with a densely defined torus action with the integer points in the image of its moment map. This allows us to read the geometric quantization of a symplectic toric manifold from its Delzant polytope. Then, we prove that the same method applies to b-symplectic toric manifolds.

The main result of this paper, Theorem 5.2, states that, for any integral b-symplectic toric manifold, the Bohr-Sommerfeld quantization with sign agrees with the formal geometric quantization of [GMW18b]. For this, we need to introduce the definition of Bohr-Sommerfeld quantization with sign.

This paper is organized as follows. In Section 2 we briefly review the geometry of b-symplectic manifolds, Bohr-Sommerfeld quantization and formal geometric quantization. In Section 3, following the idea of [GS83], we prove that the Bohr-Sommerfeld leaves of an integral b-symplectic toric manifold can be obtained from the image of the moment map of the torus action. In Section 4 we introduce the Bohr-Sommerfeld quantization with sign via T-modules. In Section 5 we prove the equivalence between Bohr-Sommerfeld quantization and formal geometric quantization both for integral symplectic and integral b-symplectic toric manifolds.

2. Preliminaries. *b*-Symplectic manifolds, Bohr-Sommerfeld quantization and formal geometric quantization

In this section we review the definitions of b-symplectic manifolds, Bohr-Sommerfeld quantization and formal geometric quantization.

2.1. *b*-Symplectic manifolds

b-Symplectic geometry is a generalization of symplectic geometry to Poisson manifolds which are symplectic on the complement of a hypersurface Z. It is possible to associate a tangent and a cotangent bundle to such b-manifolds and apply classical symplectic tools. We briefly review the work of Guillemin, Miranda, Pires and Scott in [GMP11], [GMP14] and [GMPS15], we summarize the necessary definitions in b-symplectic geometry and we refer to the three articles for details.

Recall that a *b-manifold* is a pair (M, Z) where Z is a hypersurface in a manifold M and a *b-map* is a map $f : (M_1, Z_1) \longrightarrow (M_2, Z_2)$ between b-manifolds with f transverse to Z_2 and $Z_1 = f^{-1}(Z_2)$.

Definition 2.1 (b-vector field)**.** A vector field on a b-manifold (M, Z) is called a *b-vector field* if it is tangent to Z at every point $p \in Z$.

Let (M^n, Z) be a b-manifold. If x is a local defining function for Z on an open set $U \subset M$ and $(x, y_1, \ldots, y_{n-1})$ is a chart on U, then the set of b-vector fields on U is a free $C^\infty(M)$-module with basis

$$\left(x\frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right).$$

There exists a vector bundle associated to this module called the *b-tangent bundle* and denoted by ${}^bTM$. The *b-cotangent bundle* ${}^bT^*M$ of M is defined to be the vector bundle dual to ${}^bTM$.

For each $k > 0$, let ${}^b\Omega^k(M)$ denote the space of sections of the vector bundle $\Lambda^k({}^bT^*M)$, called *b-de Rham k-forms*. For any defining function f of Z, every b-de Rham k-form can be written as

$$\omega = \alpha \wedge \frac{df}{f} + \beta, \text{ with } \alpha \in \Omega^{k-1}(M) \text{ and } \beta \in \Omega^k(M). \tag{1}$$

This decomposition enables us to extend the exterior operator d to ${}^b\Omega(M)$ by setting

$$d\omega = d\alpha \wedge \frac{df}{f} + d\beta.$$

The right hand side agrees with the usual exterior operator d on $M \setminus Z$ and extends smoothly over M as a section of $\Lambda^{k+1}({}^bT^*M)$. The fact that $d^2 = 0$ allows us to define a complex of b-forms, the *b-de Rham complex*. The cohomology associated to this complex is the *b-cohomology* and it is denoted by ${}^bH^*(M)$. The elements of ${}^b\Omega^0(M)$ are also called *b-functions* and the following definition characterizes them.

Definition 2.2 (b-function)**.** The set of *b-functions* ${}^bC^\infty(M)$ consists of functions with values in $\mathbb{R} \cup \{\infty\}$ of the form

$$c \log|f| + g,$$

where $c \in \mathbb{R}$, f is a defining function for Z and g is a smooth function on M. The differential operator d is defined as:

$$d(c \log|f| + g) := \frac{c\, df}{f} + dg \in {}^b\Omega^1(M),$$

where dg and df are the standard de Rham derivatives.

A special class of closed 2-forms of the complex of b-forms is the class of *b-symplectic forms* as defined in [GMP14].

Definition 2.3 (b-symplectic manifold)**.** Let (M^{2n}, Z) be a b-manifold and $\omega \in {}^b\Omega^2(M)$ a closed b-form. We say that ω is *b-symplectic* if ω_p is of maximal rank as an element of $\Lambda^2({}^bT_p^*M)$ for all $p \in M$. The triple (M, Z, ω) is called a *b-symplectic manifold*.

The Mazzeo-Melrose Theorem describes the relationship that exists between b-cohomology and de Rham cohomology. In particular, it gives rise to a natural definition of integrality for b-forms.

Theorem 2.4 (Mazzeo-Melrose)**.** *The b-cohomology groups of (M^{2n}, Z) satisfy*

$$^bH^*(M) \cong H^*(M) \oplus H^{*-1}(Z).$$

Remark 2.5. The integrality of a b-form ω in the sense of [GMW18b] implies the integrality of the form on $M \setminus Z$. Since we will work with a line bundle on M whose Chern class is given by the projection of $[\omega]$ to $H^*(M)$, its restriction to $M \setminus Z$ has Chern class $[\omega_{M\setminus Z}]$.

2.2. Symplectic and b-symplectic toric manifolds

Delzant's Theorem gives a classification of symplectic toric manifolds. It characterizes any such manifold via its Delzant polytope, the convex polytope given by the image of the moment map for the torus action.

Theorem 2.6 (Delzant, [Del88])**.** *There is a bijective correspondence between the following two sets, which is given by the image of the moment map μ:*

$$\begin{array}{ccc} \{\textit{toric manifolds}\} & \longrightarrow & \{\textit{Delzant polytopes}\} \\ (M^{2n}, \omega, \mu) & \longrightarrow & \mu(M) \end{array}$$

In a symplectic toric manifold (M^{2n}, ω, μ), the singularities of the moment map μ can only be of elliptic type, in the sense of Williamson [Wil36]. In fact, the singular leaves of the toric foliation correspond to points in the boundary of Delzant polytope, as is detailed in the following observation.

Remark 2.7. Let (M^{2n}, ω, μ) be a symplectic toric manifold and Δ its Delzant polytope. For $k = 1, \dots, n$, the points in the intersection of $k \leq n$ facets of Δ correspond to the leaves of M where μ has k singular elliptic components. In particular, the vertices of Δ correspond to the fixed points of μ. On the other hand, and in the appropriate coordinates, the elliptic singular components of the moment map at any singular leaf can be written as a sum of squares [Eli90].

In [GMPS15], Guillemin, Miranda, Pires and Scott established the classification of b-symplectic toric surfaces. Their result is that any b-symplectic surface is either a b-symplectic sphere or a b-symplectic torus.

Theorem 2.8 (Guillemin-Miranda-Pires-Scott, [GMPS15])**.** *A b-symplectic surface with a toric S^1-action is equivariantly b-symplectomorphic to either (S^2, Z) or (T^2, Z), where Z is a collection of latitude circles (in the T^2 case, an even number of such circles), the action is the standard rotation, and the b-symplectic form is determined by the modular periods of the critical curves and the regularized Liouville volume.*

The theorem above comes from the following result on the classification of higher-dimensional b-symplectic toric manifolds also carried out in [GMPS15].

Proposition 2.9 (Guillemin-Miranda-Pires-Scott, Remark 38 in [GMPS15])**.** *Every b-symplectic toric manifold is either the product of a b-symplectic T^2 and a symplectic toric manifold, or it can be obtained from the product of a b-symplectic S^2 and a symplectic toric manifold by a sequence of symplectic cuts performed at the north and south "polar caps", away from the critical hypersurface Z.*

The image of the moment map of a b-symplectic toric manifold is a *b-Delzant polytope* and the classification of Proposition 2.9 is a consequence of the b-Delzant theorem (Theorem 35 of [GMPS15]). The main idea of the proof is contained in the proposition below (Proposition 18 in [GMPS15]).

Proposition 2.10. *Let (M^{2n}, Z, ω, μ) be a b-symplectic toric manifold, L a leaf of its symplectic foliation and v_Z the modular weight of Z. Pick a lattice element $X \in \mathfrak{t}$ that represents a generator of $\mathfrak{t}/\mathfrak{t}_Z$ and pairs positively with v_Z. Then, there is a neighbourhood $L \times S^1 \times (-\varepsilon, \varepsilon) \cong U \subseteq M$ of Z such that the T^n-action on $U \setminus Z$ has moment map*

$$\mu_{U\setminus Z} : L \times S^1 \times ((-\varepsilon, \varepsilon) \setminus \{0\}) \to \mathfrak{t}^* \cong \mathfrak{t}_Z^* \times \mathbb{R},\ (\ell, \rho, t) \mapsto (\mu_L(\ell), c \log |t|),$$

where c is the modular period of Z, the map $\mu_L : L \to \mathfrak{t}_Z^*$ is a moment map for the T_Z^{n-1}-action on L, and the isomorphism $\mathfrak{t}^* \cong \mathfrak{t}_Z^* \times \mathbb{R}$ is induced by the splitting $\mathfrak{t} \cong \mathfrak{t}_Z \oplus \langle X \rangle$.

Since the moment map of a group action over a b-symplectic manifold is a b-function (see Definition 2.2), it can be unbounded due to the logarithm term. Hence, in general, its image is not convex (in the sense of classical analysis, for a more sophisticated notion of convexity see [GMPS17]) and unbounded (see Figure 1). This is the main issue when one tries to obtain a finite Bohr-Sommerfeld quantization of a b-symplectic toric manifold and the reason why we introduce *Bohr-Sommerfeld quantization with sign* in Section 4.

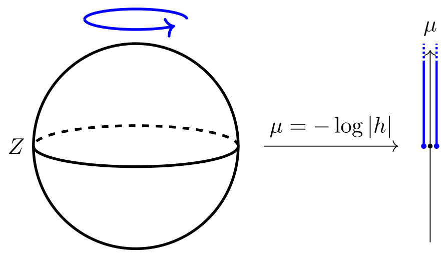

Figure 1: The moment map of the rotation action over the canonical b-symplectic sphere is unbounded in any neighbourhood of Z.

2.3. Bohr-Sommerfeld quantization

Let us recall the Bohr-Sommerfeld quantization of a compact symplectic toric manifold using the definitions of Kostant [Kos70] and Guillemin-Sternberg [GS82]. Let (M, ω) be an integral symplectic manifold and let $\mathbb{L}$ be a complex line bundle with connection ∇ whose curvature is ω. Geometric quantization is a process which associates to the quadruple $(M, \omega, \mathbb{L}, \nabla)$ a Hilbert space $Q(M)$.

Following [Kos70], we define the quantization using the additional data given by a real polarization of M, that is, a foliation of M by Lagrangian submanifolds. This foliation may be given by the fibres of a map $\pi : M \to B$ and, in this case, the quantization is given by sections $s \in \Gamma(\mathbb{L})$ satisfying

$$\nabla_X s = 0, \tag{2}$$

for any X tangent to the fibres of π. If (M, ω, μ) is a toric manifold, a Lagrangian foliation is given by the fibres of the moment map $\mu : M \to \mathfrak{t}^*$.

When M is compact, there are no smooth sections satisfying equation (2) defined globally on all M. Instead, such *leafwise constant sections* or *flat sections* are concentrated on the fibres $\pi^{-1}(b)$ such that $b \in \operatorname{Im}(\pi)$ and $\mathbb{L}|_{\pi^{-1}(b)}$ is a trivial bundle. The quantization space is defined as

$$Q(M) = \bigoplus_{b \in B_{BS}} \mathbb{C}\langle s_b \rangle, \tag{3}$$

where B_{BS} is the Bohr-Sommerfeld set, namely,

$$B_{BS} = \{b \in \operatorname{Im}(\pi) \subset B \colon \mathbb{L}|_{\pi^{-1}(b)} \text{ is trivial}\},$$

and s_b is the corresponding flat section of $\mathbb{L}|_{\pi^{-1}(b)}$.

This definition of quantization is called *Bohr-Sommerfeld quantization*. See [Ś77] or [Ham08] for an alternative definition using sheaf theory.

2.4. Formal geometric quantization

We take the basic definitions of formal geometric quantization of Hamiltonian T-spaces from [GMW18b]. See also [Wei01], [HM16] and [Par09].

Let (M, ω) be a compact symplectic manifold and let $(\mathbb{L}, \nabla)$ be a complex line bundle with connection of curvature ω. By twisting the spin-$\mathbb{C}$ Dirac operator on M by $\mathbb{L}$, we obtain an elliptic operator $\bar{\partial}_{\mathbb{L}}$. The geometric quantization $Q(M)$ of M is defined by

$$Q(M) = \operatorname{ind}(\bar{\partial}_{\mathbb{L}}),$$

and it is a virtual vector space.

If (M, ω) is a compact integral symplectic manifold, one can always find a complex line bundle $\mathbb{L}$ with connection ∇ of curvature ω and the quantization $Q(M)$ is independent of this choice.

If M is equipped with a Hamiltonian action of a torus T, the action can be lifted to $\mathbb{L}$ and the almost complex structure of $\mathbb{L}$ can be chosen to be T-invariant. In this case, the quantization $Q(M)$ is a finite-dimensional virtual T-module.

For $\xi \in \mathfrak{t}^*$, denote by $M//_\xi T$ the reduced space of M at ξ. For α a weight of T and V a virtual T-module, denote by V^α the sub-module of V of weight α.

The following result states that the component of weight α of the quantization of M equals the quantization of the reduced space of M at α.

Theorem 2.11 (Quantization commutes with reduction, [Mei96])**.** *Consider a compact integral symplectic manifold (M, ω). Suppose M is equipped with a Hamiltonian action of a torus T and let α be a weight of T. Then*

$$Q(M)^\alpha = Q(M//_\alpha T). \tag{4}$$

In other words,

$$Q(M) = \bigoplus_\alpha Q(M//_\alpha T)\alpha. \tag{5}$$

Theorem 2.11 and equation (5) are valid only for regular values of the moment map of the Hamiltonian T-action. In the case where α is a singular value of the moment map, the singular quotient must be replaced by a slightly different construction using a shift of α [Mei96]. A similar caution applies in the case of Hamiltonian T-spaces which are non-compact and in the case of b-symplectic manifolds.

2.4.1. FGQ of non-compact Hamiltonian *T*-spaces In the case where M is non-compact, equation (4) may be used to *define* the quantization of such Hamiltonian T-spaces.

Definition 2.12 (Weitsman, [Wei01])**.** Let M be a Hamiltonian T-space with integral symplectic form. Suppose the moment map for the T-action is proper. Let V be an infinite-dimensional virtual T-module with finite multiplicities. We say

$$V = Q(M)$$

if for any compact Hamiltonian T-space N with integral symplectic form, we have

$$(V \otimes Q(N))^T = Q((M \times N)//_0 T). \tag{6}$$

In other words, as in (5),

$$Q(M) = \bigoplus_{\alpha} Q(M//_{\alpha}T)\alpha,$$

where the sum is taken over all weights α of T.

The fact that the moment map is proper implies that the reduced space $(M \times N)//_0 T$ is compact for any compact Hamiltonian T-space N, so that the right hand side of equation (6) is well-defined.

2.4.2. FGQ of *b*-symplectic manifolds Suppose now that (M, ω, Z) is a compact, connected, oriented, integral b-symplectic manifold. Suppose that it is equipped with a Hamiltonian action of a torus T, with nonzero modular weight [GMW18a]. Let $\mathbb{L}$ be a complex line bundle on M with connection ∇ on $\mathbb{L}|_{M\setminus Z}$ whose curvature is $\omega|_{M\setminus Z}$.

In [GMW21], the formal geometric quantization $Q(M)$ is defined as follows.

Definition 2.13. Let V be a virtual T-module with finite multiplicities. We say

$$V = Q(M)$$

if for any compact Hamiltonian T-space N with integral symplectic form, we have

$$(V \otimes Q(N))^T = \varepsilon Q((M \times N)//_0 T), \tag{7}$$

where $Q(N)$ denotes the standard geometric quantization of N, $Q((M \times N)//_0 T)$ is the geometric quantization of the compact integral symplectic manifold $(M \times N)//_0 T$, and ε is $+1$ if the symplectic orientation on the symplectic quotient $(M \times N)//_0 T$ agrees with the orientation inherited from $M \times N$ and -1 otherwise.

This means that $Q(M) = Q(M \backslash Z) = \oplus_i \varepsilon_i Q((M \backslash Z)_i)$, where the $(M \backslash Z)_i$ are the connected components of $M \setminus Z$, $Q(M \setminus Z)$ is the formal geometric quantization of the non-compact Hamiltonian T-space $M \setminus Z$, and the $\varepsilon_i \in \{\pm 1\}$ are determined by the relative orientations of the symplectic forms on the components of $M \setminus Z$ and the overall orientation of M. Alternatively,

$$Q(M) = \bigoplus_\alpha \varepsilon(\alpha) Q(M//_\alpha T)\alpha, \tag{8}$$

where $Q(M//_\alpha T)$ must be defined using the shifting trick if α is not a regular value of the moment map, and each $\varepsilon(\alpha) \in \{\pm 1\}$ is determined by the relative orientations of M and $M//_\alpha T$.

In the b-symplectic case, the condition that the modular weight is non-zero guarantees that the reduced space $(M \times N)//_0 T$ is compact and symplectic for any compact Hamiltonian T-space N, so that $Q((M \times N)//_0 T)$ is well-defined.

3. Bohr-Sommerfeld leaves via the moment map

In this section we prove that, for an integral symplectic toric manifold, the Bohr-Sommerfeld set coincides with the set of integer points in the image of the moment map of the torus action. We prove that the same result also holds for integral b-symplectic toric manifolds.

First, let us recall a result from Guillemin and Sternberg [GS83] that identifies the Bohr-Sommerfeld leaves in a symplectic manifold using the moment map of an integrable system. In particular, it proves that the count of Bohr-Sommerfeld leaves of an integral symplectic manifold equals the count of the integer points in the image of the moment map.

Theorem 3.1 (Guillemin-Sternberg, Theorem 2.4 in [GS83])**.** *Let (M, ω) be a $2n$-dimensional symplectic manifold endowed with an integrable system with* *moment map $\mu : M \to B$. Let p and q be two distinct points of B contained in an open simply connected subset B_0 of B. Then:*

- *There exists a globally defined system of action coordinates $f_1, \dots, f_n$ on B_0 such that $f_1(p) = \cdots = f_n(p) = 0$.*
- *If $p \in B$ is in the Bohr-Sommerfeld set, then $q \in B$ is in the Bohr-Sommerfeld set if and only if $f_1(q), \dots, f_n(q)$ are integers.*

In Theorem 3.1, the correspondence between the Bohr-Sommerfeld leaves and the integer points of the moment map is established after the election of a globally defined system of action coordinates. Once a Bohr-Sommerfeld leaf is identified at a point $p \in B$, the origin of all the action coordinates is set there and the other Bohr-Sommerfeld leaves correspond to the integer points in these coordinates.

As a consequence, the integer condition that Bohr-Sommerfeld leaves have to satisfy can be shifted by an additive constant as long as it is the same constant for all the leaves, since the essential implication of Theorem 3.1 is that the difference between the action variables at any two Bohr-Sommerfeld leaves is an integer. In view of this, a value of the moment map has to be fixed at some point (and leaf) of M or, equivalently, a choice of the constant in the moment map has to be made.

We will prove that this choice of a constant in the moment map is also equivalent to the choice of the connection 1-form Θ with curvature ω.

3.1. Dependence on the connection

In the following statements we show that we can always find a connection 1-form Θ with curvature ω such that the Bohr-Sommerfeld set coincides with the integer points in the image of the moment map in the appropriate coordinates.[1]

Lemma 3.2. *Let (M, ω, μ) be a toric symplectic manifold. Let $\mathbb{L}$ be a complex line bundle over M with connection ∇ whose curvature is ω. The connection 1-form Θ can always be chosen to be T-invariant.*

[1]Let $\mathbb{L} \to M$ be a complex line bundle. A choice of a Hermitian metric on $\mathbb{L}$ gives a principal S^1-bundle $P \to M$, the unit circle bundle in $\mathbb{L}$. The complex line bundle is then the bundle associated to P by the fundamental representation of S^1. A connection on this principal S^1-bundle is an element $\Theta \in \Omega^1(P)$ satisfying $i_X\Theta = 1$ and $\mathcal{L}_X\Theta = 0$, where X is the vector field on P generating the action of S^1. The form $d\Theta$ is the pullback under the bundle projection of the curvature form ω of the bundle P with connection Θ. Such a connection gives rise to a connection ∇ on $\mathbb{L}$ and this construction is functorial under bundle morphisms. For any section $s \in \Gamma(L)$, $\nabla^2 s = \omega s$. See page 20 of [Kob14] or page 486 of [Hal13] for details.

Proof. Since the group $G = T^n$ acting on M is compact, there exists a Haar measure dg such that $\int_G dg = 1$. Then, the averaging of a form Θ via $\int_G \mathcal{L}_g^* \Theta dg$ gives a G-invariant form $\tilde{\Theta}$. □

Proposition 3.3. *Let (M, ω, μ) be a symplectic toric manifold. Let $\mathbb{L}$ be a complex line bundle over M with connection ∇ whose curvature is ω. If Θ_1 and Θ_2 are two invariant connection 1-forms, the function $\langle \Theta_1 - \Theta_2, X \rangle$ is constant when X is a vector field tangent to the polarization of M by μ.*

Proof. By definition, the connection 1-forms Θ_i satisfy $\mu = \Theta_i(X)$, where μ is a moment map of the torus action and X is a vector field tangent to the polarization given by μ.

Take α such that $\pi^*\alpha = \Theta_1 - \Theta_2$, where π is the projection $\pi : \mathbb{L} \to M$. By Lemma 3.2, Θ_1 and Θ_2 can be chosen invariant so that $\mathcal{L}_X \alpha = 0$, since α is invariant under X. Then, by Cartan's magic formula, we have that $d\alpha(X) = d i_X \alpha = \mathcal{L}_X \alpha - i_X d\alpha$.

If we have two invariant connection 1-forms Θ_1 and Θ_2, we know that their difference $\Theta_1 - \Theta_2 = \pi^*\alpha$ is a constant. Then $\pi^* d\alpha = d\Theta_1 - d\Theta_2 = \omega - \omega = 0$.

Finally, if $d(\alpha(X)) = 0$, $\alpha(X)$ is a constant. □

Remark 3.4. In view of Proposition 3.3, $\alpha(X)$ is a constant for any X. On the other hand, for any Θ_1, Θ_2, we have that $\Theta_1 = \Theta_2 + \pi^*\alpha$ and $\mu_1 = \mu_2 + \pi^*\alpha(X)$. Then, fixing $\alpha(X_i)$ is equivalent to making a choice of the connection 1-form and to fixing the constant in the moment map.

In other words, the choice of a constant can be made either by selecting a specific connection 1-form or, equivalently, by setting to 0 the coordinates of the moment map at a particular Bohr-Sommerfeld leaf.

3.2. The BS set in the image of the moment map

Observe that any torus action on a symplectic manifold M of half the dimension of M defines an integrable system. Then, in a symplectic toric manifold (M, ω, μ), we can apply Theorem 3.1 to identify the Bohr-Sommerfeld leaves of M with the integer points in the image of the moment map.

We want to extend the correspondence between Bohr-Sommerfeld leaves and integer points in the image of the moment map to b-symplectic toric manifolds. To do this, we prove first Theorem 3.5, which is a particular case of Theorem 3.1 for symplectic toric manifolds. Then, we obtain Corollary 3.6, the b-symplectic toric version of Theorem 3.5.

Theorem 3.5. *Let (M, ω, μ) be an integral symplectic toric manifold. Then, the Bohr-Sommerfeld set coincides with the integer points in the image of μ.*

Proof. Suppose (M, ω, μ) is an integral symplectic toric manifold of a dimension $2n$. We will compute the Bohr-Sommerfeld leaves of M with respect to the real Lagrangian polarization given by μ and see that each one of them is mapped to an integer point of $\mu(M)$.

The Bohr-Sommerfeld leaves are the ones supporting leafwise flat sections, i.e. sections s satisfying $\nabla_X s = 0$ for any vector field X tangent to the leaf.

We use the following formula from [Kos70] (equation (4.3.2)) and also [DGMW95] (equation (1.3)):

$$X(s) = \nabla_X s + i\langle \mu, X\rangle s, \tag{9}$$

with $X \in \mathfrak{t}$.

Since M is a toric manifold, we can make use of the natural angle coordinates $\phi_1, \ldots, \phi_n$ to obtain from (9) the following equation for each $1 \leq i \leq n$:

$$\frac{\partial}{\partial \phi_i}(s) = \nabla_{\frac{\partial}{\partial \phi_i}} s + i\langle \mu_i, \phi_i\rangle s. \tag{10}$$

For leafwise flat sections, we have $\nabla_{\frac{\partial}{\partial \phi_i}} s = 0$ for each i, implying that

$$\frac{\partial}{\partial \phi_i}(s) = i\langle \mu_i, \phi_i\rangle s. \tag{11}$$

We are looking for leaves of the foliation given by μ which admit a flat section. Then, we can integrate equation (11) with respect to ϕ from 0 to 2π noting that μ_i is constant on any leaf. We obtain

$$s(\phi_i = 2\pi) = e^{i\mu_i 2\pi} \cdot s(\phi_i = 0). \tag{12}$$

In order to be a well-defined section globally (on the entire leaf), one must have $s(0) = s(2\pi)$. This condition is met if and only if $\mu_i \in \mathbb{Z}$. So the values of $\mu(M)$ at which there is a non-trivial leafwise flat section are in $\mathbb{Z}^n$. □

Corollary 3.6. *Let $({}^bM^{2n}, Z, \omega, \mu)$ be an integral b-symplectic toric manifold. Then, the Bohr-Sommerfeld set coincides with the integer points in the image of μ.*

Proof. By Proposition 2.10, in a neighbourhood $L \times S^1 \times (-\varepsilon, \varepsilon) \cong U \subseteq M$ of Z, where L is a leaf of the symplectic foliation of Z, the T^n-action on $U \setminus Z$ has moment map

$$\mu_{U\setminus Z} : L \times S^1 \times ((-\varepsilon, \varepsilon) \setminus \{0\}) \to \mathfrak{t}^* \cong \mathfrak{t}_Z^* \times \mathbb{R},\ (\ell, \rho, t) \mapsto (\mu_L(\ell), c \log |t|),$$

where c is the modular period of Z and the map $\mu_L : L \to \mathfrak{t}_Z^*$ is a moment map for the T_Z^{n-1}-action on L.

As $M \setminus Z$ is an integral symplectic toric manifold, we can apply Theorem 3.5. Then, Bohr-Sommerfeld leaves of $M \setminus Z$ correspond to the points $(\mu_L(\ell), c \log |t|)$ such that $\mu_L(\ell) \in \mathbb{Z}^{n-1}$ and $c \log |t| \in \mathbb{Z}$. $\square$

4. Bohr-Sommerfeld quantization with sign for *b*-symplectic toric manifolds

In this section we formalize the notion of counting Bohr-Sommerfeld leaves to form virtual vector spaces. We will redefine the Bohr-Sommerfeld quantization of Section 2 in order to have it defined as a sum of T-modules. We will do this first for the symplectic toric case, in which the quantization is finite. Then, using the orientation of the manifold, we will define a quantization with sign. This will allow us to construct the quantization as a direct difference of infinite-dimensional T-modules where each representation of T occurs with finite multiplicity. We will illustrate these concepts with a simple example where M is the 2-sphere.

4.1. BS quantization via *T*-modules

We start defining a *Bohr-Sommerfeld quantization* for symplectic toric manifolds. Assume that (M, ω, μ) is a symplectic toric manifold and suppose B_{BS} is the Bohr-Sommerfeld set of the polarization given by the moment map μ.

To each $b \in B_{BS}$ regular value of μ (i.e., to each $b \in B_{BS}$ in the interior of the moment polytope $\Delta = \mu(M)$) we can associate a representation $\mathbb{C}(b)$ of T^n, where b is the weight obtained by taking the quotient with the lifted action given by μ. By the following Proposition, this representation $\mathbb{C}(b)$ is equal to the representation $\mathbb{C}\langle s_b \rangle$) that appears in equation (3).

Proposition 4.1. *For any $b = \mu(x) \in B_{BS}$ in the interior of $\Delta = \mu(M)$, $\mathbb{C}\langle s_b \rangle = \mathbb{C}(b)$.*

Proof. Suppose $b = \mu(x) \in B_{BS}$. Then, s_b is a section of the line bundle over M such that it satisfies equation (9), namely,

$$X(s_b) = \nabla_X s_b + i\langle \mu, X\rangle s_b.$$

If b is in the interior of $\Delta = \mu(M)$, it has a neighbourhood U in which all points are regular values of μ or, equivalently, in the pre-image $\mu^{-1}(U)$ the torus action has no singularities. Then, on the pre-images $x = \mu^{-1}(b)$

of these values, the line bundle $\mathbb{L}$ restricts to a line bundle with connection $\mathbb{L} \to \mu^{-1}(b)$, where $\mu^{-1}(b)$ is a torus T^n. Then, the connection on this line bundle, since $\mu(x)$ is integral, is given by the line $\mathbb{C}(\mu(x))$, where $\mathbb{C}(\mu(x))$ is the quotient $\mathbb{L}_{\mu^{-1}(\mu(x))}/T^n$ [Kos70]. $\square$

In the case of a point b in the boundary of the moment polytope Δ, the identification in Proposition 4.1 does not hold. Instead, we define the quantization for such point b using the representation $\mathbb{C}(b)$.

We have associated each $b \in B_{BS}$ a representation $\mathbb{C}(b)$, whether b is in the interior or in the boundary of Δ, and we can define the *Bohr-Sommerfeld quantization* as follows.

Definition 4.2. The Bohr-Sommerfeld quantization of a compact integral symplectic toric manifold is

$$Q(M) = \bigoplus_{b \in B_{BS}} \mathbb{C}(b). \tag{13}$$

Remark 4.3. Observe that $Q(M)$ can also be defined directly as $Q(M) = \bigoplus_{b \in \Delta \cap Z^n} \mathbb{C}(b)$. The sum $\bigoplus_{b \in \Delta \cap Z^n} \mathbb{C}(b)$ in equation (13) can be an infinite-dimensional module if M is a non-compact toric manifold (in particular, if M is a b-symplectic toric manifold) because Δ may be unbounded. Nevertheless, in all cases each weight has finite multiplicity.

By the previous remark, if we apply Definition 4.2 to a b-symplectic toric manifold we will obtain an infinite-dimensional quantization space. For this reason, we introduce the *Bohr-Sommerfeld quantization with sign* below. We again define this quantization as a T-module.

4.2. The canonical *b*-sphere

The simplest example of a b-symplectic toric manifold is the sphere with the singular hypersurface Z being a single circle at the equator and endowed with the action of rotation around its vertical axis. We are going to see that its Bohr-Sommerfeld quantization via T-modules gives an infinite-dimensional space and that, on the other hand, its Bohr-Sommerfeld quantization with sign gives a finite-dimensional space. Furthermore, in Section 5 we will prove that the latter quantization coincides with the formal geometric quantization in the general case.

Consider the b-symplectic sphere (S^2, Z, ω, μ), with the hypersurface Z on the equator and μ the moment map for the action of S^1 by rotation around the vertical axis. Away from the poles, take cylindrical polar coordinates

$\{(h,\theta)\colon -1 < h < 1, 0 \leq \theta < 2\pi\}$. Z corresponds to $\{h=0\}$, the b-symplectic form on S^2 is $\omega = \frac{1}{h} dh \wedge d\theta = d\log|h| \wedge d\theta$ and the moment map is the b-function $\mu = -\log|h|$.

Let $\mathbb{L}$ be a complex line bundle on S^2 with connection ∇ defined on $S^2 \setminus Z$ by

$$\nabla_X \sigma = X(\sigma) - \sigma i \log|h|\, d\theta(X).$$

The connection ∇ has curvature ω.

Consider the real polarization P of $S^2 \setminus Z$ given by the map $\mu = -\log|h|$. The leaves of P are the circles of the form $\{h_0\} \times S^1$, with $h_0 \in (-1,0) \cup (0,1)$, along with the two poles.

The leaf-wise flat sections, which satisfy $\nabla_X \sigma = 0$ for X tangent to the polarization P, are of the form

$$\sigma(h,\theta) = a(h) e^{i \log|h| \theta},$$

with $a(h) \in \mathbb{C}$ (see for instance [Ham10] or [MM21] for the explicit computations). The Bohr-Sommerfeld leaves on $S^2 \setminus Z$ are the leaves of the foliation by P that admit a non-trivial global leaf-wise flat section σ.

Along each leaf $\{h_0\} \times S^1$ of the foliation by P of $S^2 \setminus Z$, h is fixed at h_0. Then, a leaf is Bohr-Sommerfeld if it admits a section $\sigma(h_0,\theta)$ such that

$$\sigma(h_0,\theta) = \sigma(h_0, \theta + 2\pi).$$

Therefore, $\{h_0\} \times S^1$ is a Bohr-Sommerfeld leaf if $1 = e^{2\pi i \log|h_0|}$ or, equivalently, if $\log|h_0| \in \mathbb{Z}$. And the set of all Bohr-Sommerfeld leaves of the foliation by P of $S^2 \setminus Z$ is

$$B_{BS} = \{\{e^{-m}\} \times S^1 \subset S^2 \setminus Z\colon m \in \mathbb{N}\} \bigcup \{\{-e^{-m}\} \times S^1 \subset S^2 \setminus Z\colon m \in \mathbb{N}\}.$$

The Bohr-Sommerfeld quantization of (S^2, Z, ω, μ) is, by Definition 4.2, the following sum

$$Q(S^2) = \bigoplus_{b \in B_{BS}} \mathbb{C}(b) = \bigoplus_{b \in \mathbb{N}} \mathbb{C}(b) \oplus \mathbb{C}(b),$$

which is an infinite-dimensional space.

Observe that the quantization is infinite-dimensional because there is an infinite number of Bohr-Sommerfeld leaves arbitrarily close to Z both in the upper and the lower hemisphere. Explicitly, for any $a > 0$, there is an infinite number of values of $h \in (0,a)$ and also of $h \in (-a,0)$ satisfying the condition $\log|h| \in \mathbb{Z}$.

4.3. The b-symplectic toric sphere

In order to obtain a finite-dimensional quantization of $(S^2, Z = \{h_z\} \times S^1, \omega, \mu)$, we will define a signed sum of the quantization spaces corresponding to the Bohr-Sommerfeld leaves of S^2 that takes into account the orientation of the hemisphere in which each Bohr-Sommerfeld leaf lies. Morally, we define the quantization space by "adding" the virtual vector spaces of Bohr-Sommerfeld leaves in the northern hemisphere and "subtracting" the virtual vector spaces of Bohr-Sommerfeld leaves in the southern hemisphere. In such a way, the final sum will be a finite-dimensional virtual vector space.

Formally, we define the *Bohr-Sommerfeld quantization with sign* of the b-symplectic toric sphere $(S^2, Z = \{h_z\} \times S^1, \omega, \mu)$, with $-1 < h_z < 1$, as the direct difference of the sum of the virtual vector spaces $\mathbb{C}(b)$ associated to the Bohr-Sommerfeld leaves in $S^2_+ = (h_z, 1) \times S^1 \subset S^2 \setminus Z$ and the sum of the virtual vector spaces $\mathbb{C}(b)$ associated to the Bohr-Sommerfeld leaves in $S^2_- = (-1, h_z) \times S^1 \subset S^2 \setminus Z$.

Definition 4.4. Let B_{BS} be the Bohr-Sommerfeld set of $(S^2, Z = \{h_z\} \times S^1, \omega, \mu)$. For each $b \in B_{BS}$, define $\epsilon(b)$ as $\epsilon(b) = +1$ if $\mu^{-1}(b)$ is a Bohr-Sommerfeld leaf in the northern hemisphere S^2_+ and $\epsilon(b) = -1$ if $\mu^{-1}(b)$ is a Bohr-Sommerfeld leaf in the southern hemisphere S^2_-. We call $\epsilon(b)$ the *sign of* b.

Definition 4.5 (BS quantization with sign of $(S^2, Z = \{h_z\} \times S^1, \omega, \mu)$)**.** Let B_{BS} be the Bohr-Sommerfeld set of $(S^2, Z = \{h_z\} \times S^1, \omega, \mu)$. The *quantization with sign of* $(S^2, Z = \{h_z\} \times S^1, \omega, \mu)$ is

$$\tilde{Q}(S^2) = \bigoplus_{b \in B_{BS}} \epsilon(b)\mathbb{C}(b).$$

Lemma 4.6. *$\tilde{Q}(S^2)$ is a finite-dimensional vector space.*

Proof. First, observe that $\bigoplus_{b \in B_{BS}} \epsilon(b)\mathbb{C}(b)$ is an infinite-dimensional module with finite multiplicities (which may be negative).

For each Bohr-Sommerfeld leaf of the form $\{h_z + \delta\} \times S^1$ in S^2_+, there is a Bohr-Sommerfeld leaf of the form $\{h_z - \delta\} \times S^1$ in S^2_- for any $\delta > 0$ small enough. Then, at any symmetric neighbourhood U of Z in $S^2 \setminus Z$, the virtual module $\bigoplus_{b \in B_{BS} \cap \mu(U)} \epsilon(b)\mathbb{C}(b)$ is exactly 0.

On the other hand, there are only finitely many Bohr-Sommerfeld leaves in $S^2 \setminus U$ and, therefore, $\bigoplus_{b \in B_{BS} \cap \mu(S^2 \setminus U)} \epsilon(b)\mathbb{C}(b)$ is finite-dimensional. Hence, $\tilde{Q}(S^2)$ is a finite-dimensional vector space. $\square$

In particular, with this definition the Bohr-Sommerfeld quantization with sign of the b-symplectic toric sphere, with singular hypersurface Z the circle at the equator, is the zero-dimensional vector space due to its symmetry (see Figure 2).

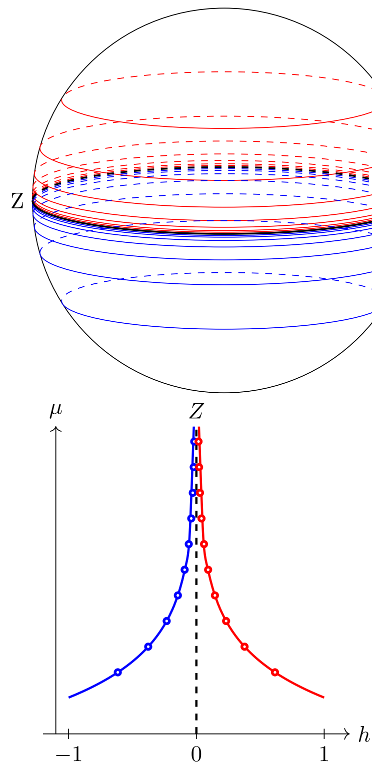

Figure 2: On the left, Bohr-Sommerfeld leaves on the northern hemisphere (in red) and the southern hemisphere (in blue) of $(S^2, Z = \{h_0 = 0\} \times S^1, \omega, \mu)$. On the right, the moment map $\mu = -\log|h|$ with dots indicating Bohr-Sommerfeld leaves.

Remark 4.7. Our quantization model resembles the computations of section 6 of [Bon22] for Bruhat-Poisson structures using symplectic groupoids. More precisely, the computations performed in [Bon22] (following [BCQM14]) for the symplectic reduction of the Lu-Weinstein groupoid integrating the standard Poisson structure on $U(2)$ yield Bohr-Sommerfeld leaves similar to those seen in our approach. We plan to address these issues elsewhere.

4.4. *b*-symplectic toric surfaces

We can naturally generalize the definition of Bohr-Sommerfeld quantization with sign of the b-symplectic toric sphere $(S^2, Z = \{h_z\} \times S^1, \omega, \mu)$ to any b-symplectic toric surface (M^2, Z, ω, μ).

By Theorem 2.8, a b-symplectic toric surface (M^2, Z, ω, μ) is equivariantly b-symplectomorphic to either (S^2, Z) or (T^2, Z), where Z is a collection of latitude circles (in T^2, an even number of them) and μ is the standard rotation.

First orient the manifold M. Since the b-symplectic form defines an orientation in each connected component of $M^2 \setminus Z$, we can associate a sign to each Bohr-Sommerfeld leaf depending on whether this orientation agrees with the orientation of M.

Definition 4.8. Let B_{BS} be the Bohr-Sommerfeld set of (M^2, Z, ω, μ). For each $b \in B_{BS}$, define $\epsilon(b)$ as $\epsilon(b) = +1$ if $\mu^{-1}(b)$ belongs to a component whose orientation agrees with that given by the b-symplectic form and as $\epsilon(b) = -1$ if $\mu^{-1}(b)$ belongs to a component whose orientation is the opposite of that given by the b-symplectic form. We call $\epsilon(b)$ the *sign of* b.

Definition 4.9 (BS quantization with sign of (M^2, Z, ω, μ))**.** Let B_{BS} be the Bohr-Sommerfeld set of (M^2, Z, ω, μ). We define the *quantization with sign of* (M^2, Z, ω, μ) as

$$\tilde{Q}(M^2) = \bigoplus_{b \in B_{BS}} \epsilon(b)\mathbb{C}(b).$$

Lemma 4.10. *$\tilde{Q}(M^2)$ is a finite-dimensional vector space.*

Proof. Take a symmetric neighbourhood $U \subset M^2 \setminus Z$ of Z; such a neighbourhood exists by Proposition 2.10. By the argument in the proof of Lemma 4.6, the contribution of leaves in $U \setminus Z$ to the sum $\bigoplus_{b \in B_{BS}} \epsilon(b)\mathbb{C}(b)$ is 0, and the contribution of leaves in $M^2 \setminus U$ is a finite-dimensional vector space. Hence, $\tilde{Q}(M^2)$ is finite-dimensional. □

Note that a b-symplectic toric surface that has an infinite number of Bohr-Sommerfeld leaves still yields a finite-dimensional quantization (see Figure 3).

4.5. *b*-symplectic toric manifolds

Proposition 2.9 states that any b-symplectic toric manifold (M^{2n}, Z, ω, μ) decomposes either into the product of a b-symplectic toric torus $(T^2, Z_T, \omega_T, \mu_T)$

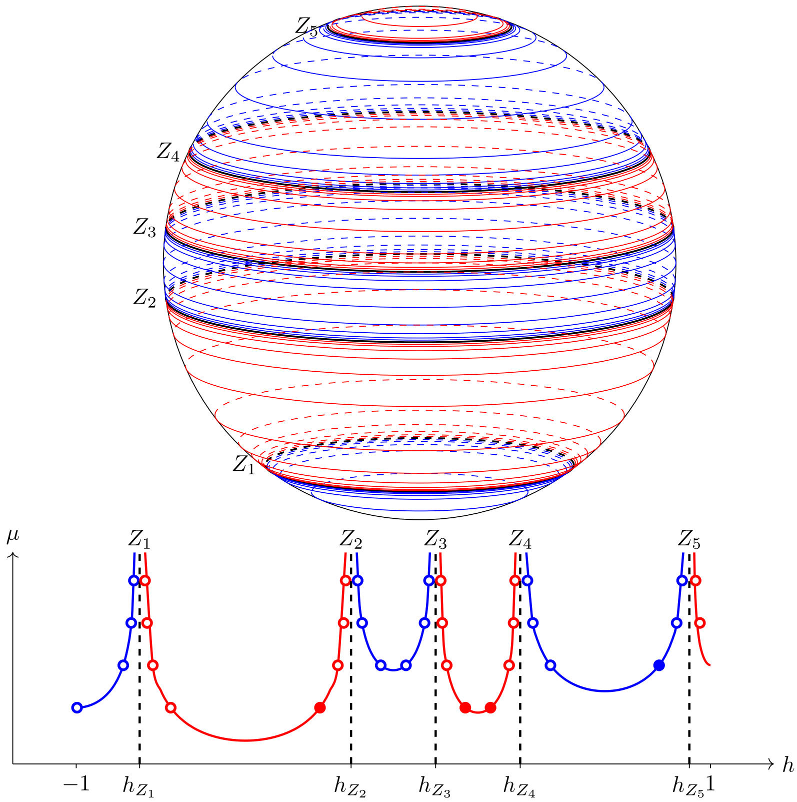

Figure 3: A b-symplectic toric sphere with Z consisting of 5 latitude circles and the image of its moment map. The blue dots denote the Bohr-Sommerfeld leaves in positively oriented components of $S^2 \setminus Z$. The red dots denote the Bohr-Sommerfeld leaves in negatively oriented components of $S^2 \setminus Z$. White-filled dots represent Bohr-Sommerfeld leaves in the neighbourhood of each Z_i whose contributions to the quantization $Q(M)$ cancel.

with a classic symplectic toric manifold, or else can be obtained from the product of a b-symplectic toric sphere $(S^2, Z_S, \omega_S, \mu_S)$ with a classic symplectic toric manifold by a sequence of symplectic cuts away from Z.

Choose an orientation of M. Then, we can define the Bohr-Sommerfeld quantization with sign for b-symplectic manifolds using the orientation given by the b-symplectic form on the connected components of M.

Definition 4.11. Let B_{BS} be the Bohr-Sommerfeld set of (M^{2n}, Z, ω, μ). For each $b \in B_{BS}$, define $\epsilon(b)$ by $\epsilon(b) = +1$ if $\pi\left(\mu^{-1}(b)\right)$ belongs to a component of $M \setminus Z$ where the orientation of M agrees with the orientation given by ω, and by $\epsilon(b) = -1$ otherwise. We call $\epsilon(b)$ the *sign of* b.

Definition 4.12 (BS quantization with sign of (M^{2n}, Z, ω, μ))**.** Let B_{BS} be the Bohr-Sommerfeld set of (M^{2n}, Z, ω, μ). The *quantization with sign of* (M^{2n}, Z, ω, μ) is

$$\tilde{Q}(M^{2n}) = \bigoplus_{b \in B_{BS}} \epsilon(b)\mathbb{C}(b).$$

Lemma 4.13. *$\tilde{Q}(M^{2n})$ is a finite-dimensional vector space.*

Proof. Take a symmetric neighbourhood $U \subset M^{2n} \setminus Z$. Such a neighbourhood always exists by Proposition 2.10. The same argument used in the proof of Lemmas 4.6 and 4.10 shows that $\tilde{Q}(M^{2n})$ is finite-dimensional. $\square$

Remark 4.14. In the definition of the Bohr-Sommerfeld quantization with sign and, in particular, in the sum $\bigoplus_{b \in B_{BS}} \epsilon(b)\mathbb{C}(b)$, we are using the fact that we have a group T acting with weights with finite multiplicity. Thus, the infinite sum $\bigoplus_{b \in B_{BS}} \epsilon(b)\mathbb{C}(b)$ is well defined.

5. The final count. Bohr-Sommerfeld quantization equals formal geometric quantization

In this section we compare Bohr-Sommerfeld quantization with formal geometric quantization. We show that both are given by the signed count of integer points in the image of the moment map. We do it first for the symplectic case (Theorem 5.1), and then the b-symplectic case (Theorem 5.2).

Theorem 5.1. *Let (M^2, ω, μ) be a symplectic toric manifold. Then, the formal geometric quantization of M coincides with the Bohr-Sommerfeld quantization.*

Proof. We compute the formal geometric quantization of a symplectic toric manifold and then we count the Bohr-Sommerfeld leaves. We see that they are the same and, in particular, they coincide with the count of the integer points in the image of the moment map of the torus action.

In view of Theorem 2.11 (Quantization commutes with reduction), the formal geometric quantization of a symplectic toric manifold (M^{2n}, ω, μ) is given by

$$Q(M) = \bigoplus_{\alpha \in \mathbb{Z}^n} Q(M//_\alpha T)\alpha. \tag{14}$$

Notice that the sum is taken over all weights α of T.

Suppose $\mu : M \to \mathfrak{t}$ is the moment map of the torus action. Then, the reduced spaces $M//_\alpha T$ are either empty if α is not in $\mu(M)$ or a point if it is. Since the quantization of a point is given by $\mathbb{C}$, we have that

$$Q(M) = \bigoplus_{\alpha \in \mathbb{Z}^n \cap \mu(M)} \mathbb{C}(\alpha). \tag{15}$$

Thus, the formal geometric quantization of M is given by as many copies of $\mathbb{C}$ as there are integer points in the image of the moment map.

On the other hand, by Theorem 3.5, the Bohr-Sommerfeld quantization is given by the count of Bohr-Sommerfeld leaves of M, which are in one to one correspondence with the integer points in the image of the moment map. $\square$

In the b-symplectic case we have

Theorem 5.2. *Let (M^{2n}, Z, ω, μ) be a b-symplectic toric manifold. Then, the formal geometric quantization of M coincides with the Bohr-Sommerfeld quantization with sign.*

Proof. Recall from [GMW18a] that for any given b-symplectic toric manifold (M, Z, ω, μ), the quantization space $Q(M)$ is defined as the vector space such that the following equality holds

$$(Q(M) \otimes Q(N))^T = \varepsilon(\alpha) Q((M \times N)//_0 T) \tag{16}$$

for any compact symplectic manifold N and any weight α of T, where T is the torus generating the action with moment map μ [GMW18b].

Alternatively, we have

$$Q(M) = \bigoplus_{\alpha \in \mathbb{Z}^n \cap \mu(M)} \varepsilon(\alpha)\mathbb{C}(\alpha), \tag{17}$$

where $\varepsilon(\alpha)$ are the signs given in equation (8)

On the other hand, by Corollary 3.6 the Bohr-Sommerfeld set of M coincides with the lattice of integer points in the image of μ. Therefore, by Definition 4.12, the Bohr-Sommerfeld quantization with sign of M is

$$\tilde{Q}(M) = \bigoplus_{b \in B_{BS}} \epsilon(b)\mathbb{C}(b) = \bigoplus_{b \in \mathbb{Z}^n \cap \mu(M)} \epsilon(b)\mathbb{C}(b). \tag{18}$$

Finally, for any point p in the Bohr-Sommerfeld set B_{BS}, the sign $\epsilon(p)$ coincides with the sign $\varepsilon(p)$ since, by definition, both of them are $+1$ if the orientation given by the symplectic form on the component of $M \setminus Z$ containing $\mu^{-1}(p)$ and the overall orientation of M agree and -1 otherwise. Hence, $Q(M) = \tilde{Q}(M)$. $\square$

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References

[BCQM14] F. Bonechi, N. Ciccoli, J. Qiu, and Tarlini M. Quantization of poisson manifolds from the integrability of the modular function. *Commun. Math. Phys.*, **331**:851–885, 2014. MR3238532

[BLS21] Maxim Braverman, Yiannis Loizides, and Yanli Song. Geometric quantization of b-symplectic manifolds. *J. Symplectic Geom.*, **19**(1):1–36, 2021. MR4325382

[Bon22] F. Bonechi. Integrability of the modular vector field and quantization, 2022.

[Del88] Thomas Delzant. Hamiltoniens périodiques et images convexes de l'application moment. *Bull. Soc. Math. France*, **116**(3):315–339, 1988. MR0984900

[DGMW95] Hans Duistermaat, Victor Guillemin, Eckhard Meinrenken, and Siye Wu. Symplectic reduction and Riemann-Roch for circle actions. *Math. Res. Lett.*, **2**(3):259–266, 1995. MR1338785

[Eli90] L. H. Eliasson. Normal forms for Hamiltonian systems with Poisson commuting integrals—elliptic case. *Comment. Math. Helv.*, **65**(1):4–35, 1990. MR1036125

[GMP11] Victor Guillemin, Eva Miranda, and Ana Rita Pires. Codimension one symplectic foliations and regular Poisson structures. *Bull. Braz. Math. Soc. (N.S.)*, **42**(4):607–623, 2011. MR2861781

[GMP14] Victor Guillemin, Eva Miranda, and Ana Rita Pires. Symplectic and Poisson geometry on b-manifolds. *Adv. Math.*, **264**:864–896, 2014. MR3250302

[GMPS15] Victor Guillemin, Eva Miranda, Ana Rita Pires, and Geoffrey Scott. Toric actions on b-symplectic manifolds. *Int. Math. Res. Not. IMRN*, pages 5818–5848, 2015. MR3384459

[GMPS17] Victor Guillemin, Eva Miranda, Ana Rita Pires, and Geoffrey Scott. Convexity for Hamiltonian torus actions on b-symplectic manifolds. *Math. Res. Lett.*, **24**(2):363–377, 2017. MR3685275

[GMW18a] Victor Guillemin, Eva Miranda, and Jonathan Weitsman. Convexity of the moment map image for torus actions on b^m-symplectic manifolds. *Philos. Trans. Roy. Soc. A*, **376**(2131), 2018. MR3868423

[GMW18b] Victor Guillemin, Eva Miranda, and Jonathan Weitsman. On geometric quantization of b-symplectic manifolds. *Adv. Math.*, **331**:941–951, 2018. MR3804693

[GMW21] Victor Guillemin, Eva Miranda, and Jonathan Weitsman. On geometric quantization of b^m-symplectic manifolds. *Math. Z.*, **298**(1-2):281–288, 2021. MR4257086

[GS82] V. Guillemin and S. Sternberg. Geometric quantization and multiplicities of group representations. *Invent. Math.*, **67**(3):515–538, 1982. MR0664118

[GS83] V. Guillemin and S. Sternberg. The Gelfand-Cetlin system and quantization of the complex flag manifolds. *J. Funct. Anal.*, **52**(1):106–128, 1983. MR0705993

[Hal13] Brian C. Hall. *Quantum Theory for Mathematicians*. Springer New York, NY, 2013. MR3112817

[Ham08] Mark D. Hamilton. The quantization of a toric manifold is given by the integer lattice points in the moment polytope. In *Toric Topology*, volume 460 of *Contemp. Math.*, pages 131–140. Amer. Math. Soc., Providence, RI, 2008. MR2428353

[Ham10] Mark D. Hamilton. Locally toric manifolds and singular Bohr-Sommerfeld leaves. *Mem. Amer. Math. Soc.*, 207(971):vi+60, 2010. MR2682223

[HM16] Peter Hochs and Varghese Mathai. Formal geometric quantisation for proper actions. *J. Homotopy Relat. Struct.*, **11**(3):409–424, 2016. MR3542091

[KMS16] Anna Kiesenhofer, Eva Miranda, and Geoffrey Scott. Action-angle variables and a KAM theorem for b-Poisson manifolds. *J. Math. Pures Appl. (9)*, **105**(1):66–85, 2016. MR3427939

[Kob14] Shoshichi Kobayashi. *Differential Geometry of Complex Vector Bundles*. Princeton University Press, Princeton, 2014. MR3643615

[Kos70] Bertram Kostant. Quantization and unitary representations. I. Prequantization. In *Lectures in modern analysis and applications, III*, pages 87–208. Lecture Notes in Math., Vol. 170. Springer-Verlag, Berlin, 1970. MR0294568

[LLSS21] Yi Lin, Yiannis Loizides, Reyer Sjamaar, and Yanli Song. Log symplectic manifolds and $[q, r] = 0$, 2021. MR4485951

[Mei96] Eckhard Meinrenken. On Riemann-Roch formulas for multiplicities. *J. Amer. Math. Soc.*, **9**(2):373–389, 1996. MR1325798

[MM21] Pau Mir and Eva Miranda. Geometric quantization via cotangent models. *Anal. Math. Phys.*, 11(3):Paper No. 118, 2021. MR4269948

[MO21] Eva Miranda and Cédric Oms. The singular Weinstein conjecture. *Adv. Math.*, 389:Paper No. 107925, 41, 2021. MR4290138

[Par09] Paul-Émile Paradan. Formal geometric quantization. *Ann. Inst. Fourier (Grenoble)*, **59**(1):199–238, 2009. MR2514864

[Ś77] Jedrzej Śniatycki. On cohomology groups appearing in geometric quantization. In *Differential Geometrical Methods in Mathematical Physics (Proc. Sympos., Univ. Bonn, Bonn, 1975)*, pages 46–66. Lecture Notes in Math., Vol. 570, 1977. MR0451304

[Wei01] Jonathan Weitsman. Non-abelian symplectic cuts and the geometric quantization of noncompact manifolds. *Lett. Math. Phys.*, **56**(1):31–40, 2001. EuroConférence Moshé Flato 2000, Part I (Dijon). MR1848164

[Wil36] John Williamson. On the algebraic problem concerning the normal forms of linear dynamical systems. *Amer. J. Math.*, **58**(1):141–163, 1936. MR1507138

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