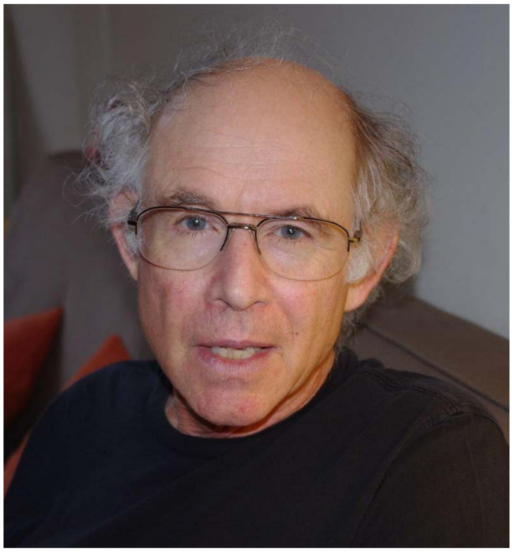
Pure and Applied Mathematics Quarterly
Volume 19, Number 4, 1679–1686, 2023

Preface



This volume is in celebration of the work and influence of Victor Guillemin.

Victor William Guillemin was born in Cambridge, Massachusetts, on October 15, 1937. Subsequently, his family moved to greater Chicago, where he attended grade school and Oak Park High School. He was an undergraduate at Harvard University, graduating with a B.A. in 1959, and completed an M.A. the next year at the University of Chicago. Returning as a graduate student to Harvard, he worked with Shlomo Sternberg, receiving his Ph.D. in 1962. Guillemin and Sternberg continued to collaborate for half a century. After two post-doctoral years at Columbia, Guillemin was appointed to a Faculty position at MIT, where he remained until retiring in 2022. While at MIT he supervised 47 Ph.D. students and a large number of postdoctoral fellows.

According to MathSciNet, Victor Guillemin has over 220 publications, including 14 books, covering an enormous breadth of mathematics. We will highlight here some of our favourite works of Victor Guillemin that we feel have had the greatest impact. Our organization of these papers and books into categories is somewhat artificial: Guillemin's work often cuts across boundaries, connecting different areas of mathematics.

Microlocal analysis and Spectral theory

Following Hörmander's use of Fourier integral operators to prove a general form of Weyl's asymptotic eigenvalue formula, the paper with Duistermaat, [2], used the global properties of these operators to give a remarkable extension of the Poisson summation formula (and Harish Chandra's trace formula) to general compact Riemannian manifolds. This refined earlier work of Chazarain and opened a path to the application of the techniques of microlocal analysis to spectral theory. For instance the work with Weinstein, [29], showed in the opposite direction how to associate a sequence of eigenvalues to a stable closed geodesic. The trace formula, and associated wave-trace spectral invariants, have been applied in numerous ways, especially to inverse spectral problems, for instance, in work with Kazhdan on the inverse problem for the Laplacian plus a potential on negatively curved manifolds discussed below. The trace formula itself was extended to manifolds with boundary in [22] with related applications to an inverse problem in [21].

In the monograph [1], Hörmander's theory of Lagrangian distributions was extended to a larger class of Hermite distributions associated to conic isotropic submanifolds of cotangent bundles. This includes the Schwartz kernels of the Szegő projectors on strictly pseudoconvex domains. An extension in a different direction, to singular Lagrangian manifolds, was made in [9]. In [14] Guillemin showed that the use of complex powers of positive operators in zeta regularization could be replaced by more general holomorphic families of complex order. Using this he computed the residue trace, which had been found earlier independently by Wodzicki, opening the way to further innovation.

Integral geometry

Guillemin has proven seminal results in integral geometry. We mention here three of them.

In his paper [4] with Kazhdan, Guillemin proved infinitesimal spectral rigidity for negatively curved compact two dimensional manifolds. One of the main ingredients of the proof is to show that the integral geometric transform consisting of integrating a symmetric two tensor over closed geodesics on the manifold is injective modulo two tensors which are in the kernel, which are identified as tensors that are the symmetric covariant derivative of a one tensor. This same method was used subsequently in several extensions including the case of Anosov manifolds. In a later article [19], Guillemin and Kazhdan proved a similar result for manifolds of higher dimension with pinched negative curvature. Croke and Sharafutdinov removed the pinched condition using similar methods as [19].

A Zoll surface is a two sphere with a metric all of whose geodesics are closed. The standard metric on the two sphere has this property, and any other metric on the two sphere is conformally equivalent to the standard metric. Hilbert asked whether there are conformal deformations of the standard metric that are Zoll. Funk showed that a necessary condition is that the conformal deformation at the initial time is an odd function. Guillemin proved in [13] that this is also sufficient. He first solved the linearized problem, which amounts to inverting (modulo odd functions) the Funk transform, which integrates a function over closed geodesics of the round metric. Guillemin then applied the Nash-Moser implicit function theorem to get the non-linear result. This is a remarkable contribution and a beautiful paper to read. The Lorentzian analog has been considered in recent work of Marquez and Neves.

Another striking contribution [15] has been the microlocal study of generalized Radon transforms which consists of integrating functions over more general submanifolds and measures. He showed that these transforms, under very general conditions on the submanifolds and measures, are Fourier integral operators. He then considered the microlocal analog of the double fibrations defined by Helgason and Gelfand and collaborators. Under what Guillemin called the Bolker condition, he proved that if R is the transform and R^* is a natural adjoint then the normal operator R^*R is an elliptic pseudodifferential operator, which can be inverted microlocally. Consequently, under the Bolker condition one can recover the singularities of a distribution f from the singularities of Rf. This result has had many applications, including to the study of the X-ray transform (integration along lines) with limited data. This is important for X-ray tomography, which provides the mathematical foundation of CT scans.

Geometry of the moment map

Victor Guillemin has made fundamental contributions to the field of Hamiltonian group actions. In this field, which is tightly connected with equivariant topology, geometric and topological phenomena can often be studied by combinatorial means.

Perhaps the single most influential paper in this area was [5], the first of Guillemin-Sternberg's "Convexity properties of the moment mapping" papers, where it is shown that the image of the moment map for a torus action on a compact symplectic manifold is a convex polytope.

The local building blocks for Hamiltonian group actions were spelled out in Guillemin-Sternberg's paper [26] "a normal form for the moment map". Their "Birational equivalence in the symplectic category" [8] describes the variation of the reduced spaces as we cross between components of regular reduced space, going beyond Duistermaat-Heckman's earlier paper that describes the variation within each component.

Connections with localization in equivariant cohomology were highlighted in Guillemin-Sternberg's book [30] "Supersymmetry and equivariant de Rham theory" and in "Moment maps, cobordisms, and Hamiltonian group actions" [18]. Connections with combinatorics were highlighted in Guillemin's book "Moment maps and combinatorial invariants of Hamiltonian T^n spaces" [17] for toric manifolds, and in his papers with his students Zara and Holm (e.g., [12, 3, 11]) for GKM spaces.

Guillemin-Sternberg's famous book "Symplectic techniques in physics" [27] in particular ties the geometry of the moment maps with concrete physical models.

Completely integrable systems

The Guillemin-Sternberg paper [7] introduces a classical analogue of the Gel'fand-Cetlin bases of irreducible representations of the unitary group. This results in a classical completely integrable system on every complex flag manifold. When the symplectic form is integral, the authors show that the number of Bohr-Sommerfeld joint level sets of the commuting Hamiltonians is equal to the dimension of the corresponding representation of the unitary group. This can be viewed as an instance of "independence of polarization" phenomena in geometric quantization.

Guillemin's explicit formulas [16] for the standard Kähler structure on a symplectic toric manifold, obtained from Delzant's construction, were tremendously useful in later works by many people.

In [25], Guillemin and Sternberg introduced the geometric notion of a multiplicity-free space and related it to the representation theoretic notion of a multiplicity-free representation.

Geometric quantization

In their seminal paper [6], Guillemin and Sternberg introduced the notion of "quantization commutes with reduction", and established it for Kähler quantization. This notion has become a central theme in geometric quantization, often used as testing-ground for quantization recipes.

Guillemin's paper [10] opened the door to the notion of almost-Kähler quantization, providing an actual quantization space, of the expected dimension, as a sum of lowest eigenspaces for a Laplace operator, while higher eigenspaces drift away for sufficiently high tensor powers of the prequantization line bundle.

Guillemin's book with Lerman and Sternberg [20] explains multiplicity patterns that occur in representations of compact Lie groups in terms of fibrations of the corresponding coadjoint orbits over lower dimensional coadjoint orbits. This is motivated by the correspondence between classical objects (coadjoint orbits) and corresponding quantum objects (representations). Guillemin and Sternberg's book "Geometric asymptotics" [24] continues to be a main reference for some of the central themes in geometric quantization.

Semi-classical analysis

Aspects of the quantum-classical correspondence of mechanics underlie a substantial amount of Guillemin's work. For example, much of his work on equivariant symplectic geometry can be seen as a geometric analogue of parts of the representation theory of compact Lie groups. In this light it seems natural that his attention eventually turned to semi-classical analysis, which focuses on establishing precise relationships between quantum-mechanical and classical objects as Planck's constant tends to zero. The book [28] is devoted to the theory of semi-classical pseudo-differential and Fourier integral operators on manifolds, and most of his later papers are in this area.

Pedagogy

Guillemin's lectures are famously clear and elegant, often attracting a large audience. He has also written several excellent textbooks. The most famous, Guillemin-Pollack [23], is a classic introduction to differential topology which has provided enormous inspiration to generations of young mathematicians.

Yael Karshon
Richard Melrose
Gunther Uhlmann
Alejandro Uribe

References

[1] L. Boutet de Monvel and V. Guillemin. *The spectral theory of Toeplitz operators*, volume 99 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1981. MR0620794

[2] J. J. Duistermaat and V. W. Guillemin. The spectrum of positive elliptic operators and periodic bicharacteristics. *Invent. Math.*, **29**(1):39–79, 1975. MR0405514

[3] V. Guillemin, T. Holm, and C. Zara. A GKM description of the equivariant cohomology ring of a homogeneous space. *J. Algebraic Combin.*, **23**(1):21–41, 2006. MR2218848

[4] V. Guillemin and D. Kazhdan. Some inverse spectral results for negatively curved 2-manifolds. *Topology*, **19**(3):301–312, 1980. MR0579579

[5] V. Guillemin and S. Sternberg. Convexity properties of the moment mapping. *Invent. Math.*, **67**(3):491–513, 1982. MR0664117

[6] V. Guillemin and S. Sternberg. Geometric quantization and multiplicities of group representations. *Invent. Math.*, **67**(3):515–538, 1982. MR0664118

[7] V. Guillemin and S. Sternberg. The Gel'fand-Cetlin system and quantization of the complex flag manifolds. *J. Funct. Anal.*, **52**(1):106–128, 1983. MR0705993

[8] V. Guillemin and S. Sternberg. Birational equivalence in the symplectic category. *Invent. Math.*, **97**(3):485–522, 1989. MR1005004

[9] V. Guillemin and G. Uhlmann. Oscillatory integrals with singular symbols. *Duke Math. J.*, **48**(1):251–267, 1981. MR0610185

[10] V. Guillemin and A. Uribe. The Laplace operator on the nth tensor power of a line bundle: eigenvalues which are uniformly bounded in n. *Asymptotic Anal.*, **1**(2):105–113, 1988. MR0950009

[11] V. Guillemin and C. Zara. Equivariant de Rham theory and graphs. *Asian J. Math.*, **3**(1):49–76, 1999. MR1701922

[12] V. Guillemin and C. Zara. 1-skeleta, Betti numbers, and equivariant cohomology. *Duke Math. J.*, **107**(2):283–349, 2001. MR1823050

[13] Victor Guillemin. The Radon transform on Zoll surfaces. *Advances in Math.*, **22**(1):85–119, 1976. MR0426063

[14] Victor Guillemin. A new proof of Weyl's formula on the asymptotic distribution of eigenvalues. *Adv. in Math.*, **55**(2):131–160, 1985. MR0772612

[15] Victor Guillemin. On some results of Gel'fand in integral geometry. In *Pseudodifferential operators and applications (Notre Dame, Ind., 1984)*, volume 43 of *Proc. Sympos. Pure Math.*, pages 149–155. Amer. Math. Soc., Providence, RI, 1985. MR0812288

[16] Victor Guillemin. Kaehler structures on toric varieties. *J. Differential Geom.*, **40**(2):285–309, 1994. MR1293656

[17] Victor Guillemin. *Moment maps and combinatorial invariants of Hamiltonian T^n-spaces*, volume 122 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1994. MR1301331

[18] Victor Guillemin, Viktor Ginzburg, and Yael Karshon. *Moment maps, cobordisms, and Hamiltonian group actions*, volume 98 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002. Appendix J by Maxim Braverman. MR1929136

[19] Victor Guillemin and David Kazhdan. Some inverse spectral results for negatively curved n-manifolds. In *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, pages 153–180, 1980. MR0573432

[20] Victor Guillemin, Eugene Lerman, and Shlomo Sternberg. *Symplectic fibrations and multiplicity diagrams*. Cambridge University Press, Cambridge, 1996. MR1414677

[21] Victor Guillemin and Richard Melrose. An inverse spectral result for elliptical regions in $\mathbf{R}^2$. *Adv. in Math.*, **32**(2):128–148, 1979. MR0535619

[22] Victor Guillemin and Richard Melrose. The Poisson summation formula for manifolds with boundary. *Adv. in Math.*, **32**(3):204–232, 1979. MR0539531

[23] Victor Guillemin and Alan Pollack. *Differential topology*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974. MR0348781

[24] Victor Guillemin and Shlomo Sternberg. *Geometric asymptotics*. American Mathematical Society, Providence, R.I., 1977. MR0516965

[25] Victor Guillemin and Shlomo Sternberg. Multiplicity-free spaces. *J. Differential Geom.*, **19**(1):31–56, 1984. MR0739781

[26] Victor Guillemin and Shlomo Sternberg. A normal form for the moment map. In *Differential geometric methods in mathematical physics (Jerusalem, 1982)*, volume 6 of *Math. Phys. Stud.*, pages 161–175. Reidel, Dordrecht, 1984. MR0767835

[27] Victor Guillemin and Shlomo Sternberg. *Symplectic techniques in physics*, second edition. Cambridge University Press, Cambridge, 1990. MR1066693

[28] Victor Guillemin and Shlomo Sternberg. *Semi-classical analysis.* International Press, Boston, MA, 2013. MR3157301

[29] Victor Guillemin and Alan Weinstein. Eigenvalues associated with a closed geodesic. *Bull. Amer. Math. Soc.*, **82**(1):92–94, 1976. MR0436227

[30] Victor W. Guillemin and Shlomo Sternberg. *Supersymmetry and equivariant de Rham theory.* Mathematics Past and Present. Springer-Verlag, Berlin, 1999. With an appendix containing two reprints by Henri Cartan [MR0042426 (13,107e); MR0042427 (13,107f)].