

Spectral measures for $Sp(2)$

DAVID EVANS AND MATHEW PUGH*

Dedicated to the memory of Vaughan Jones

Abstract: Spectral measures provide invariants for braided subfactors via fusion modules. In this paper we study joint spectral measures associated to the compact connected rank two Lie group $SO(5)$ and its double cover the compact connected, simply-connected rank two Lie group $Sp(2)$, including the McKay graphs for the irreducible representations of $Sp(2)$ and $SO(5)$ and their maximal tori, and fusion modules associated to the $Sp(2)$ modular invariants.

Keywords: Spectral measures, $Sp(2)$, subfactor, fusion ring, fusion module.

1. Introduction

Spectral measures associated to the compact Lie groups $SU(2)$, $SU(3)$ and G_2 , their maximal tori, nimrep graphs associated to the $SU(2)$, $SU(3)$ and G_2 modular invariants, and the McKay graphs for finite subgroups of $SU(2)$, $SU(3)$ and G_2 were studied in [1, 22, 23, 25, 26]. Spectral measures associated to other compact rank two Lie groups and their maximal tori are studied in [27].

For the $SU(2)$ and $SU(3)$ graphs, the spectral measures distill onto very special subsets of the semicircle/circle for $SU(2)$ (both are one-dimensional spaces) and discoid/torus for $SU(3)$ (which are both two-dimensional spaces), and the theory of nimreps allowed us to compute these measures precisely. Our methods gave an alternative approach to deriving the results of Banica and Bisch [1] for ADE graphs and subgroups of $SU(2)$, and explained the connection between their results for affine ADE graphs and the Kostant polynomials. In the case of G_2 , the spectral measures distill onto subsets of \mathbb{R} and

Received September 18, 2022.

2010 Mathematics Subject Classification: Primary 46L37; secondary 05C50, 43A45.

*The author was supported by the Coleg Cymraeg Cenedlaethol.

the torus \mathbb{T}^2 , which are one-dimensional and two-dimensional respectively, resulting in an infinite family of pullback measures over \mathbb{T}^2 for any spectral measure on \mathbb{R} . This ambiguity was removed by considering instead joint spectral measures for pairs of graphs corresponding to the two fundamental representations of G_2 . Such joint spectral measures, which have support in \mathbb{R}^2 , yield a unique pullback measure over \mathbb{T}^2 , and the spectral measures are obtained as pushforward measures.

In this paper we study spectral measures for the compact, connected, simply-connected rank two Lie group $Sp(2)$, the group of 4×4 unitary symplectic matrices with entries in \mathbb{C} . We also study spectral measures for the (non-simply-connected) compact rank two Lie group $SO(5)$, the group of 5×5 real orthogonal matrices, whose double cover is $Sp(2)$. In particular we determine the joint spectral measures associated to the Lie groups themselves and their maximal tori, and joint spectral measures for nimrep graphs associated to the $Sp(2)$ modular invariants.

In $C_2 = sp(2)$ (the Lie algebra of $Sp(2)$) conformal field theories, one considers the Verlinde algebra at a finite level k , which is represented by a non-degenerately braided system ${}_N\mathcal{X}_N$ of irreducible endomorphisms on a type III₁ factor N , whose fusion rules $\{N_{\lambda\nu}^\mu\}$ reproduce exactly those of the positive energy representations of the loop group of $Sp(2)$ at level k , $N_\lambda N_\mu = \sum_\nu N_{\lambda\nu}^\mu N_\nu$. The statistics generators S, T for the braided tensor category ${}_N\mathcal{X}_N$ match exactly those of the Kač-Peterson modular S, T matrices which perform the conformal character transformations (see footnote 2 in [6]). The fusion graph for these irreducible endomorphisms are truncated versions of the representation graphs of $Sp(2)$ itself (see Section 4.1). From the Verlinde formula (1) we see that this family $\{N_\lambda\}$ of commuting normal matrices can be simultaneously diagonalised:

$$(1) \quad N_\lambda = \sum_\sigma \frac{S_{\sigma,\lambda}}{S_{\sigma,0}} S_\sigma S_\sigma^*,$$

where the summation is over each $\sigma \in {}_N\mathcal{X}_N$ and 0 is the trivial representation. It is intriguing that the eigenvalues $S_{\sigma,\lambda}/S_{\sigma,0}$ and eigenvectors $S_\sigma = \{S_{\sigma,\mu}\}_\mu$ are described by the modular S matrix.

A braided subfactor is an inclusion $N \subset M$ where the dual canonical endomorphism decomposes as a finite combination of endomorphisms in ${}_N\mathcal{X}_N$, and yields a modular invariant partition function through the procedure of α -induction which allows two extensions of λ on N , depending on the use of the braiding or its opposite, to endomorphisms $\alpha_\lambda^\pm \in {}_M\mathcal{X}_M^\pm$ of M , so that the matrix $Z_{\lambda,\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle$ is a modular invariant [8, 5, 18]. The systems ${}_M\mathcal{X}_M^\pm$

are called the chiral systems, whilst the intersection ${}_M\mathcal{X}_M^0 = {}_M\mathcal{X}_M^+ \cap {}_M\mathcal{X}_M^-$ is the neutral system. Then ${}_M\mathcal{X}_M^0 \subset {}_M\mathcal{X}_M^\pm \subset {}_M\mathcal{X}_M$, where ${}_M\mathcal{X}_M \subset \text{End}(M)$ denotes a system of endomorphisms consisting of a choice of representative endomorphisms of each irreducible subsector of sectors of the form $[\iota\lambda\bar{\iota}]$, $\lambda \in {}_N\mathcal{X}_N$, where $\iota : N \hookrightarrow M$ is the inclusion map. Although ${}_N\mathcal{X}_N$ is assumed to be braided, the systems ${}_M\mathcal{X}_M^\pm$ or ${}_M\mathcal{X}_M$ are not braided in general. The action of the N - N sectors ${}_N\mathcal{X}_N$ on the M - N sectors ${}_M\mathcal{X}_N$ and produces a nimrep (non-negative integer matrix representation of the original Verlinde algebra) $G_\lambda = (\langle \xi\lambda, \xi' \rangle)_{\xi, \xi' \in {}_M\mathcal{X}_N}$, i.e. $G_\lambda G_\mu = \sum_\nu N_{\lambda\nu}^\mu G_\nu$ whose spectrum reproduces exactly the diagonal part of the modular invariant [9]. In the case of the trivial embedding of N in itself, the nimrep G is simply the trivial representation N . Since the nimreps are a family of commuting matrices, they can be simultaneously diagonalised and thus the eigenvectors ψ_σ of G_λ are the same for each $\lambda \in {}_N\mathcal{X}_N$. We have

$$(2) \quad G_\lambda = \sum_\sigma \frac{S_{\sigma,\lambda}}{S_{\sigma,0}} \psi_\sigma \psi_\sigma^*$$

where the summation is over each $\sigma \in {}_N\mathcal{X}_N$ with multiplicity given by the modular invariant, i.e. the spectrum of G_λ is given by $\{S_{\sigma,\lambda}/S_{\sigma,0}$ with multiplicity $Z_{\sigma,\sigma}\}$. We call the set $\{\mu$ with multiplicity $Z_{\mu,\mu}\}$ the set of exponents of G .

Along with the identity invariants for $Sp(2)$, there are orbifold invariants for all levels k [2]. There are three exceptional invariants due to conformal embeddings at levels 3, 7, 12 [12], $(Sp(2))_3 \subset (SO(10))_1$, $(Sp(2))_7 \subset (SO(14))_1$ and $(Sp(2))_{12} \subset (E_8)_1$. These three conformal embedding invariants correspond to *type I extensions* in [11, 5] and this list is complete by recent work of Gannon [30]. Type II extensions arise from extensions of the nets *without* locality. In general [5] for a physical modular invariant Z there are local chiral extensions $N(I) \subset M_+(I)$ and $N(I) \subset M_-(I)$ with local Q -systems naturally associated to the vacuum column $\{Z_{\lambda,0}\}$ and vacuum row $\{Z_{0,\lambda}\}$ respectively. These extensions are indeed maximal and should be regarded as the subfactor version of left and right maximal extensions of the chiral algebra. The representation theories or modular tensor categories of M_\pm are then identified. For example, the E_7 conformal net or module category is then a twist or auto-equivalence on the local D_{10} extension which form the type I parents. This reduces the analysis to understanding first local extensions and then classifying auto-equivalences to identify the two left and right local extensions. Schopieray [38], using α -induction, found bounds for levels of exceptional invariants for rank 2 Lie groups, and Gannon [30] extended this for

higher rank with improved lower bounds using Galois transformations as a further tool. Edie-Michell has undertaken extensive studies of auto-equivalences [16]. There is an exceptional Type II invariant at level 8 [42] which is a twist of the orbifold invariant at level 8. These are all the known $Sp(2)$ modular invariants.

This paper is organised as follows. In Section 2 we describe the representation theory of $Sp(2)$ and $SO(5)$, and their maximal torus \mathbb{T}^2 , and in particular focus on their fundamental representations. In Sections 2.1–2.3 we determine the (joint) spectral measures associated to the (adjacency matrices of the) McKay graphs given by the action of the irreducible characters of $Sp(2)$ on its maximal torus \mathbb{T}^2 , and the analogous results for $SO(5)$. In Section 3 we determine the (joint) spectral measures associated to the (adjacency matrices of the) McKay graphs of $Sp(2)$ and $SO(5)$ themselves. In all these cases we focus on the fundamental representations of $Sp(2)$, $SO(5)$ respectively, and determine these (joint) spectral measures over both \mathbb{T}^2 and the (joint) spectrum of these adjacency matrices. Finally in Section 4 we determine joint spectral measures over \mathbb{T}^2 for nimrep graphs arising from $Sp(2)$ braided subfactors.

2. Spectral measures for $\mathcal{W}\mathcal{A}_\infty(Sp(2))$, $\mathcal{W}\mathcal{A}_\infty(SO(5))$

The irreducible representations $\lambda_{(\mu_1, \mu_2)}$ of $Sp(2)$ are indexed by pairs $(\mu_1, \mu_2) \in \mathbb{N}^2$ such that $\mu_1 \geq \mu_2$. Let the fundamental representation $\rho_x = \lambda_{(1,0)}$ be the standard representation of $Sp(2)$, $\rho_x(Sp(2)) = Sp(2)$, the group of 4×4 unitary symplectic matrices with entries in \mathbb{C} . The maximal torus of $Sp(2)$ is $T = \text{diag}(t_1, t_2, t_1^{-1}, t_2^{-1})$, for $t_i \in \mathbb{T}$, which is isomorphic to \mathbb{T}^2 , so that the restriction of ρ_x to \mathbb{T}^2 is given by the 4×4 diagonal matrix

$$(3) \quad (\rho_x|_{\mathbb{T}^2})(\omega_1, \omega_2) = \text{diag}(\omega_1, \omega_2, \omega_1^{-1}, \omega_2^{-1}),$$

for $(\omega_1, \omega_2) \in \mathbb{T}^2$.

Let the fundamental representation $\rho_y = \lambda_{(1,1)}$ be the standard representation of $SO(5)$, $\rho_y(Sp(2)) = SO(5)$, the group of 5×5 real orthogonal matrices. The restriction of ρ_y to \mathbb{T}^2 is given by the 5×5 diagonal matrix

$$(4) \quad (\rho_y|_{\mathbb{T}^2})(\omega_1, \omega_2) = \text{diag}(\omega_1\omega_2, \omega_1^{-1}\omega_2^{-1}, \omega_1\omega_2^{-1}, \omega_1^{-1}\omega_2, 1),$$

for $(\omega_1, \omega_2) \in \mathbb{T}^2$.

The irreducible representations of $SO(5)$ are given by the representations $\lambda_{(\mu_1, \mu_2)}$ of $Sp(2)$ for which $\mu_1 + \mu_2$ is even. In order to study spectral measures

associated to $SO(5)$, we take ρ_y and a second fundamental representation $\rho_z = \lambda_{(2,0)}$ of $SO(5)$, which is the adjoint representation of $Sp(2)$ of dimension 10. The restriction of ρ_z to \mathbb{T}^2 is given by the 10×10 diagonal matrix

$$(5) \quad (\rho_z|_{\mathbb{T}^2})(\omega_1, \omega_2) = \text{diag}(\omega_1^2, \omega_2^2, \omega_1^{-2}, \omega_2^{-2}, \omega_1\omega_2, \omega_1\omega_2^{-1}, \omega_1^{-1}\omega_2, \omega_1^{-1}\omega_2^{-1}, 1, 1),$$

for $(\omega_1, \omega_2) \in \mathbb{T}^2$.

Let $\{\chi_{(\mu_1, \mu_2)}\}_{\mu_1, \mu_2 \in \mathbb{N}: \mu_1 \geq \mu_2}$, $\{\sigma_{(\mu_1, \mu_2)}\}_{\mu_1, \mu_2 \in \mathbb{Z}}$ be the irreducible characters of $Sp(2)$, \mathbb{T}^2 respectively, where $\chi_{(\mu_1, \mu_2)} := \chi_{\lambda_{(\mu_1, \mu_2)}}$. The characters $\chi_{(\mu_1, \mu_2)}$ of $Sp(2)$ are self-conjugate and thus are maps from the torus \mathbb{T}^2 to an interval $I_\mu := \chi_\mu(\mathbb{T}^2) \subset \mathbb{R}$. For $\omega_i \in \mathbb{T}$, $\mu_i \in \mathbb{Z}$, the characters of \mathbb{T}^2 are given by $\sigma_{(\mu_1, \mu_2)}(\omega_1, \omega_2) = \omega_1^{\mu_1} \omega_2^{\mu_2}$, and satisfy $\overline{\sigma_{(\mu_1, \mu_2)}} = \sigma_{(-\mu_1, -\mu_2)}$.

If σ_u is the restriction of χ_{ρ_u} to \mathbb{T}^2 , $u = x, y, z$, then from (3)–(5)

$$(6) \quad \sigma_x = \chi_{(1,0)}|_{\mathbb{T}^2} = \sigma_{(1,0)} + \sigma_{(-1,0)} + \sigma_{(0,1)} + \sigma_{(0,-1)},$$

$$(7) \quad \sigma_y = \chi_{(1,1)}|_{\mathbb{T}^2} = \sigma_{(0,0)} + \sigma_{(1,1)} + \sigma_{(-1,-1)} + \sigma_{(1,-1)} + \sigma_{(-1,1)},$$

$$(8) \quad \sigma_z = \chi_{(2,0)}|_{\mathbb{T}^2} = 2\sigma_{(0,0)} + \sigma_{(2,0)} + \sigma_{(-2,0)} + \sigma_{(0,2)} + \sigma_{(0,-2)} + \sigma_{(1,1)} \\ + \sigma_{(-1,-1)} + \sigma_{(1,-1)} + \sigma_{(-1,1)}.$$

Then

$$\sigma_x \sigma_{(\mu_1, \mu_2)} = \sigma_{(\mu_1+1, \mu_2)} + \sigma_{(\mu_1-1, \mu_2)} + \sigma_{(\mu_1, \mu_2+1)} + \sigma_{(\mu_1, \mu_2-1)},$$

for any $\mu_1, \mu_2 \in \mathbb{Z}$, where multiplication by $\sigma_x = \chi_{\rho_x}|_{\mathbb{T}^2}$ corresponds to the edges illustrated in the first diagram in Figures 1 and 2. The representation graph of \mathbb{T}^2 for the first fundamental representation ρ_x is identified with the infinite graph $\mathcal{W}\mathcal{A}_\infty^{\rho_x}(Sp(2))$, which is the first figure illustrated in Figures 3 and 4, whose vertices may be labeled by pairs $(\mu_1, \mu_2) \in \mathbb{Z}^2$ such that there is an edge from (μ_1, μ_2) to $(\mu_1 + 1, \mu_2)$, $(\mu_1 - 1, \mu_2)$, $(\mu_1, \mu_2 + 1)$ and $(\mu_1, \mu_2 - 1)$.

Similarly, the representation graph of \mathbb{T}^2 for the irreducible representation ρ_y is identified with the infinite graph $\mathcal{W}\mathcal{A}_\infty^{\rho_y}(Sp(2))$, which is the second figure illustrated in Figure 3, where multiplication by $\sigma_y = \chi_{\rho_y}|_{\mathbb{T}^2}$ corresponds to the edges illustrated in the second diagram respectively in Figure 1, and the representation graph of \mathbb{T}^2 for the irreducible representation ρ_z is identified with the infinite graph $\mathcal{W}\mathcal{A}_\infty^{\rho_z}(Sp(2))$, illustrated in Figure 4, where multiplication by $\sigma_z = \chi_{\rho_z}|_{\mathbb{T}^2}$ corresponds to the edges illustrated in Figure 2. Both these graphs are in fact a disjoint union of two infinite graphs, coloured black, grey respectively, whose vertex sets consists of all λ such that $\lambda_1 + \lambda_2$ is even, odd respectively. These graphs $\mathcal{W}\mathcal{A}_\infty^{\rho_i}(Sp(2))$ are essentially W -unfolded versions of the graphs $\mathcal{A}_\infty^{\rho_i}(Sp(2))$ (see Figures 13–16), where W denotes the Weyl group D_8 of $Sp(2)$.

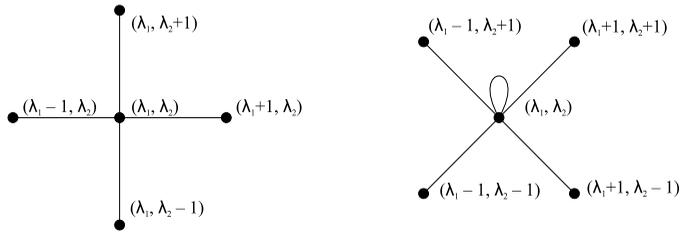


Figure 1: Multiplication by $\chi_{\rho_x}|_{\mathbb{T}^2}$ and $\chi_{\rho_y}|_{\mathbb{T}^2}$.

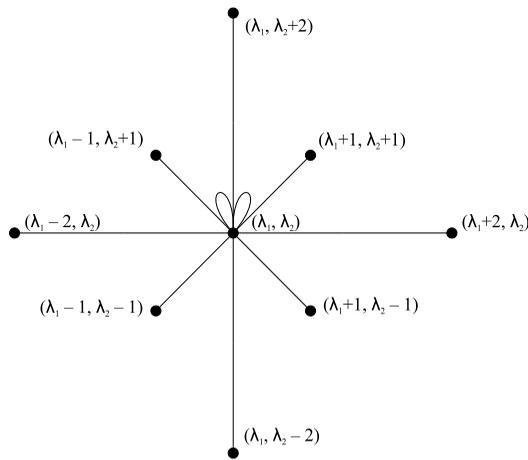


Figure 2: Multiplication by $\chi_{\rho_z}|_{\mathbb{T}^2}$.

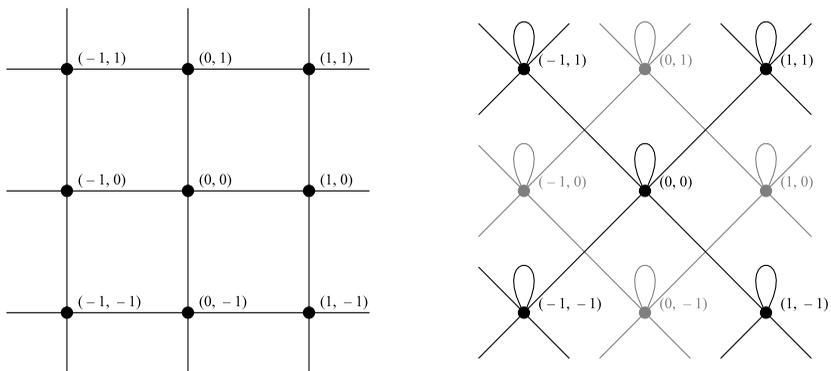


Figure 3: Infinite graphs $W\mathcal{A}_{\infty}^{\rho_x}(Sp(2))$ and $W\mathcal{A}_{\infty}^{\rho_y}(Sp(2))$.

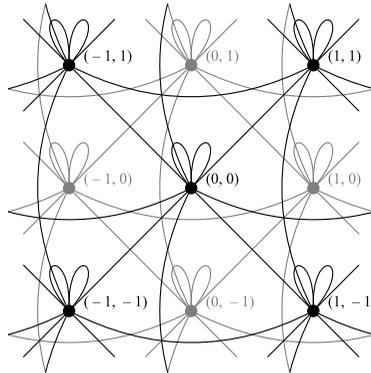


Figure 4: Infinite graph $W\mathcal{A}_\infty^{\rho_z}(Sp(2))$.

We consider first the fixed point algebra of $\otimes_{\mathbb{N}} M_4, \otimes_{\mathbb{N}} M_5, \otimes_{\mathbb{N}} M_{10}$ under the conjugate action of the torus \mathbb{T}^2 given by the restrictions of ρ_x, ρ_y, ρ_z respectively to \mathbb{T}^2 given in (3), (4), (5) respectively. Here \mathbb{T}^2 acts by conjugation on each factor in the infinite tensor product. Thus by [19, §3.5] we have $(\otimes_{\mathbb{N}} M_4)^{\mathbb{T}^2} \cong A(W\mathcal{A}_\infty^{\rho_x}(Sp(2)))$, $(\otimes_{\mathbb{N}} M_5)^{\mathbb{T}^2} \cong A(W\mathcal{A}_\infty^{\rho_y}(Sp(2)))$ and $(\otimes_{\mathbb{N}} M_{10})^{\mathbb{T}^2} \cong A(W\mathcal{A}_\infty^{\rho_z}(Sp(2)))$. Here $A(\mathcal{G}) = \bigcup_k A(\mathcal{G})_k$ is the path algebra of the graph \mathcal{G} , where $A(\mathcal{G})_k$ is the algebra generated by pairs (η_1, η_2) of paths from the distinguished vertex $*$ such that the ranges $r(\eta_1)$ and $r(\eta_2)$ are equal, and $|\eta_1| = |\eta_2| = k$, with multiplication defined by $(\eta_1, \eta_2) \cdot (\eta'_1, \eta'_2) = \delta_{\eta_2, \eta'_1}(\eta_1, \eta'_2)$.

We now define commuting self-adjoint operators which may be identified with the adjacency matrix of $W\mathcal{A}_\infty^{\rho_u}(Sp(2))$. We define operators v_Z^u in $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$, for $u = x, y, z$, by

$$\begin{aligned} v_Z^x &= s \otimes 1 + s^* \otimes 1 + 1 \otimes s + 1 \otimes s^*, \\ v_Z^y &= 1 \otimes 1 + s \otimes s + s^* \otimes s^* + s \otimes s^* + s^* \otimes s, \\ v_Z^z &= 2(1 \otimes 1) + s^2 \otimes 1 + (s^*)^2 \otimes 1 + 1 \otimes s^2 + 1 \otimes (s^*)^2 + s \otimes s + s^* \otimes s^* \\ &\quad + s \otimes s^* + s^* \otimes s, \end{aligned}$$

where s is the bilateral shift on $\ell^2(\mathbb{Z})$. Let Ω denote the vector $(\delta_{i,0})_i$. Then v_Z^u is identified with the adjacency matrix of $W\mathcal{A}_\infty^{\rho_u}(Sp(2))$, $u = x, y, z$, where we regard the vector $\Omega \otimes \Omega$ as corresponding to the vertex $(0, 0)$ of $W\mathcal{A}_\infty^{\rho_u}(Sp(2))$, and the operators of the form $s^l \otimes s^m$ which appear as terms in v_Z^u as corresponding to the edges on $W\mathcal{A}_\infty^{\rho_u}(Sp(2))$. Then $(s^{\lambda_1} \otimes s^{\lambda_2})(\Omega \otimes \Omega)$ corresponds to the vertex (λ_1, λ_2) of $W\mathcal{A}_\infty^{\rho_u}(Sp(2))$ for any $\lambda_1, \lambda_2 \in \mathbb{Z}$, and applying $(v_Z^u)^m$

to $\Omega \otimes \Omega$ gives a vector $y = (y_{(\lambda_1, \lambda_2)})$ in $\ell^2(\mathcal{WA}_\infty^{\rho_u}(Sp(2)))$, where $y_{(\lambda_1, \lambda_2)}$ gives the number of paths of length m on $\mathcal{WA}_\infty^{\rho_u}(Sp(2))$ from $(0, 0)$ to the vertex (λ_1, λ_2) .

We define a state φ on $C^*(v_Z^u)$ by $\varphi(\cdot) = \langle \cdot, (\Omega \otimes \Omega), \Omega \otimes \Omega \rangle$. We use the notation $(a_1, a_2, \dots, a_k)!$ to denote the multinomial coefficient

$$(a_1, a_2, \dots, a_k)! := \left(\sum_{i=1}^k a_i \right)! / \prod_{i=1}^k (a_i!).$$

Then

$$\begin{aligned} \varphi((v_Z^u)^m) &= \sum_{\substack{k_i \geq 0 \\ \sum_i k_i \leq m}} (k_1, \dots, k_{l(u)}, m - \sum_i k_i)! \varphi(s^{r_1^u} \otimes s^{r_2^u}) \\ &= \sum_{\substack{k_i \geq 0 \\ \sum_i k_i \leq m}} (k_1, \dots, k_{l(u)}, m - \sum_i k_i)! \delta_{r_1^u, 0} \delta_{r_2^u, 0}, \end{aligned}$$

where $l(u) = 3, 4, 9$ for $u = x, y, z$ respectively, and

- (9) $r_1^x = k_1 - k_2, \quad r_2^x = k_1 + k_2 + 2k_3 - m,$
- (10) $r_1^y = k_1 - k_2 + k_3 - k_4, \quad r_2^y = k_1 - k_2 - k_3 + k_4,$
- (11) $r_1^z = 2k_1 - 2k_2 + k_5 + k_6 - k_7 - k_8,$
- (12) $r_2^z = 2k_3 - 2k_4 + k_5 - k_6 + k_7 - k_8.$

When $u = x$, we get a non-zero contribution when $k_2 = k_1$ and $k_3 = -k_1 + m/2$. So we obtain

$$(13) \quad \varphi((v_Z^x)^m) = \sum_{k_1} (k_1, k_1, -k_1 + m/2, -k_1 + m/2)!$$

where the summation is over all integers $0 \leq k_1 \leq m/2$. When $u = y$, we get a non-zero contribution when $k_2 = k_1$ and $k_4 = k_3$. So we obtain

$$(14) \quad \varphi((v_Z^y)^m) = \sum_{k_1, k_3} (k_1, k_1, k_3, k_3, m - 2k_1 - 2k_3)!$$

where the summation is over all integers $k_1, k_3 \geq 0$ such that $2k_1 + 2k_3 \leq m$. When $u = z$, we get a non-zero contribution when $k_7 = k_1 - k_2 + k_3 - k_4 + k_5$ and $k_8 = k_1 - k_2 - k_3 + k_4 + k_6$. So we obtain

$$(15) \quad \varphi((v_Z^z)^m) = \sum_{k_i} (k_1, k_2, k_3, k_4, k_5, k_6, p_1, p_2, k_9, p_3)!$$

where $p_1 = k_1 - k_2 + k_3 - k_4 + k_5$, $p_2 = k_1 - k_2 - k_3 + k_4 + k_6$, $p_3 = m - 3k_1 + k_2 - k_3 - k_4 - 2k_5 - 2k_6 - k_9$, and the summation is over all integers $k_1, k_2, \dots, k_6, k_9 \geq 0$ such that $3k_1 - k_2 + k_3 + k_4 + 2k_5 + 2k_6 + k_9 \leq m$.

2.1. Joint spectral measure for $\mathcal{W}\mathcal{A}_\infty(Sp(2))$, $\mathcal{W}\mathcal{A}_\infty(SO(5))$ over \mathbb{T}^2

The ranges of the restrictions (6)–(8) of the characters χ_{ρ_u} of the irreducible representations ρ_u of $Sp(2)$ to \mathbb{T}^2 , for $u = x, y, z$, are given by

$$\begin{aligned} I_x &:= \{2\operatorname{Re}(\omega_1) + 2\operatorname{Re}(\omega_2) \mid \omega_1, \omega_2 \in \mathbb{T}\} = [-4, 4], \\ I_y &:= \{1 + 2\operatorname{Re}(\omega_1\omega_2) + 2\operatorname{Re}(\omega_1\omega_2^{-1}) \mid \omega_1, \omega_2 \in \mathbb{T}\} = [-3, 5], \\ I_z &:= \{2 + 2\operatorname{Re}(\omega_1^2) + 2\operatorname{Re}(\omega_2^2) + 2\operatorname{Re}(\omega_1\omega_2) + 2\operatorname{Re}(\omega_1\omega_2^{-1}) \mid \omega_1, \omega_2 \in \mathbb{T}\} \\ &= [-2, 10], \end{aligned}$$

where the characters χ_{ρ_u} are given by

$$\begin{aligned} \chi_{\rho_x}(\omega_1, \omega_2) &= \omega_1 + \omega_1^{-1} + \omega_2 + \omega_2^{-1} = 2\cos(2\pi\theta_1) + 2\cos(2\pi\theta_2), \\ \chi_{\rho_y}(\omega_1, \omega_2) &= 1 + \omega_1\omega_2 + \omega_1^{-1}\omega_2^{-1} + \omega_1\omega_2^{-1} + \omega_1^{-1}\omega_2 \\ &= 1 + 2\cos(2\pi(\theta_1 + \theta_2)) + 2\cos(2\pi(\theta_1 - \theta_2)), \\ \chi_{\rho_z}(\omega_1, \omega_2) &= \chi_{\rho_x}(\omega_1, \omega_2)^2 - \chi_{\rho_y}(\omega_1, \omega_2) - 1 \\ &= 2 + 2\cos(4\pi\theta_1) + 2\cos(4\pi\theta_2) + 2\cos(2\pi(\theta_1 + \theta_2)) \\ &\quad + 2\cos(2\pi(\theta_1 - \theta_2)), \end{aligned}$$

where $\omega_j = e^{2\pi i\theta_j} \in \mathbb{T}$ for $\theta_j \in [0, 1]$, $j = 1, 2$. We will write x, y, z for the elements $\chi_{\rho_x}(\omega_1, \omega_2)$, $\chi_{\rho_y}(\omega_1, \omega_2)$, $\chi_{\rho_z}(\omega_1, \omega_2)$ respectively. Since the spectrum $\sigma(s)$ of s is \mathbb{T} , the spectrum $\sigma(v_Z^u)$ of v_Z^u is I_u , $u = x, y, z$.

The Weyl group of $Sp(2)$ is the dihedral group D_8 of order 8. If we consider D_8 as the subgroup of $GL(2, \mathbb{Z})$ generated by the matrices T_2, T_4 , of orders 2, 4 respectively, given by

$$T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then the action of D_8 on \mathbb{T}^2 given by $T(\omega_1, \omega_2) = (\omega_1^{a_{11}}\omega_2^{a_{12}}, \omega_1^{a_{21}}\omega_2^{a_{22}})$, for $T = (a_{il}) \in D_8$, leaves $\chi_{\rho_u}(\omega_1, \omega_2)$ invariant, for $u = x, y, z$. Then for $u = x, y, z$, any D_8 -invariant measure ε on \mathbb{T}^2 produces a probability measure μ_u on I_u by

$$\int_{I_u} \psi(x) d\mu_u(x) = \int_{\mathbb{T}^2} \psi(\chi_{\rho_u}(\omega_1, \omega_2)) d\varepsilon(\omega_1, \omega_2),$$

for any continuous function $\psi : I_u \rightarrow \mathbb{C}$, where $d\varepsilon(\omega_1, \omega_2) = d\varepsilon(g(\omega_1, \omega_2))$ for all $g \in D_8$. There is a loss of dimension here, in the sense that the integral on the right hand side is over the two-dimensional torus \mathbb{T}^2 , whereas the spectrum of ${}^W\mathcal{A}_\infty^{\rho_u}(Sp(2))$ is real and lives on the interval I_u . We introduce an intermediate probability measure ν in Section 2.2 which lives over the joint spectrum $\mathfrak{D}_{\lambda,\mu} \subset I_\lambda \times I_\mu \subset \mathbb{R}^2$ for irreducible representations λ, μ , where there is no loss of dimension.

The spectral measure on \mathbb{T}^2 for the graph ${}^W\mathcal{A}_\infty^{\rho_u}(Sp(2))$ is easily seen to be the uniform Lebesgue measure $d\varepsilon(\omega_1, \omega_2) = d\omega_1 d\omega_2/4\pi^2$ for $u = x, y, z$, since the m^{th} moment is given by

$$\begin{aligned} & \frac{1}{4\pi^2} \int_{\mathbb{T}^2} (\chi_{\rho_u}(\omega_1, \omega_2))^m d\omega_1 d\omega_2 \\ &= \frac{1}{4\pi^2} \sum_{\substack{k_i \geq 0 \\ \sum_i k_i \leq m}} (k_1, k_2, \dots, k_{l(u)}, m - \sum_i k_i)! \int_{\mathbb{T}^2} \omega_1^{r_1^u} \omega_2^{r_2^u} d\omega_1 d\omega_2 \\ &= \sum_{\substack{k_i \geq 0 \\ \sum_i k_i \leq m}} (k_1, k_2, \dots, k_{l(u)}, m - \sum_i k_i)! \delta_{r_1^u, 0} \delta_{r_2^u, 0}, \end{aligned}$$

where r_1^u, r_2^u are as in (9)–(12) and $l(u) = 3, 4, 9$ for $u = x, y, z$ respectively, which is equal to $\varphi((v_Z^u)^m)$ given in (13)–(15).

A fundamental domain C of \mathbb{T}^2 under the action of the dihedral group D_8 is illustrated in Figure 5, where the axes are labelled by the parameters θ_1, θ_2 in $(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) \in \mathbb{T}^2$. In Figure 5, the lines $\theta_1 = 0$ and $\theta_2 = 0$ are also boundaries of copies of the fundamental domain C under the action of D_8 . The torus \mathbb{T}^2 contains 8 copies of C , so that

$$(16) \quad \int_{\mathbb{T}^2} \phi(\omega_1, \omega_2) d\varepsilon(\omega_1, \omega_2) = 8 \int_C \phi(\omega_1, \omega_2) d\varepsilon(\omega_1, \omega_2),$$

for any D_8 -invariant function $\phi : \mathbb{T}^2 \rightarrow \mathbb{C}$. The fixed points of \mathbb{T}^2 under the action of D_8 are the points $(1, 1)$ and $(-1, -1)$, which map to the points 4, -4 respectively in the interval I_x , whilst both map to the points 5, 10 in the intervals I_y, I_z respectively. The point $(-1, 1)$ (and its orbit under D_8) maps to 0, $-3, 2$ in the intervals I_x, I_y, I_z respectively.

2.2. Joint spectral measure for ${}^W\mathcal{A}_\infty(Sp(2)), {}^W\mathcal{A}_\infty(SO(5))$ on \mathbb{R}^2

Let $\Psi_{\lambda,\mu}$ be the map $(\omega_1, \omega_2) \mapsto (x_\lambda, x_\mu) = (\chi_\lambda(\omega_1, \omega_2), \chi_\mu(\omega_1, \omega_2))$. We denote by $\mathfrak{D}_{\lambda,\mu}$ the image of $\Psi_{\lambda,\mu}(C)$ ($= \Psi_{\lambda,\mu}(\mathbb{T}^2)$) in \mathbb{R}^2 . Note that we can

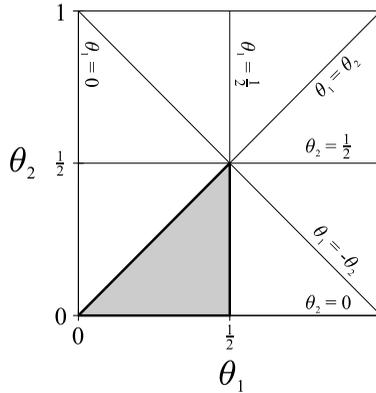


Figure 5: A fundamental domain C of \mathbb{T}^2/D_8 .

identify $\mathfrak{D}_{\lambda,\mu}$ with $\mathfrak{D}_{\mu,\lambda}$ by reflecting about the line $x_\lambda = x_\mu$. The joint spectral measure $\tilde{\nu}_{\lambda,\mu}$ is the measure on $\mathfrak{D}_{\lambda,\mu}$ uniquely determined by its cross-moments $\varsigma_{\lambda,\mu}(m, n) = \int_{\mathfrak{D}_{\lambda,\mu}} x_\lambda^m x_\mu^n d\nu_{\lambda,\mu}(x_\lambda, x_\mu)$. Then there is a unique D_8 -invariant pullback measure $\varepsilon_{\lambda,\mu}$ on \mathbb{T}^2 such that

$$\int_{\mathfrak{D}_{\lambda,\mu}} \psi(x_\lambda, x_\mu) d\tilde{\nu}_{\lambda,\mu}(x_\lambda, x_\mu) = \int_{\mathbb{T}^2} \psi(\chi_\lambda(\omega_1, \omega_2), \chi_\mu(\omega_1, \omega_2)) d\varepsilon_{\lambda,\mu}(\omega_1, \omega_2),$$

for any continuous function $\psi : \mathfrak{D}_{\lambda,\mu} \rightarrow \mathbb{C}$.

Any probability measure on $\mathfrak{D}_{\lambda,\mu}$ yields a probability measure on the interval I_λ , given by the pushforward $(p_\lambda)_*(\tilde{\nu}_{\lambda,\mu})$ of the joint spectral measure $\tilde{\nu}_{\lambda,\mu}$ under the orthogonal projection p_λ onto the spectrum $\sigma(\lambda) = I_\lambda$. In particular, when $\psi(x_\lambda, x_\mu) = \tilde{\psi}(x_\lambda)$ is only a function of one variable x_λ , then

$$\begin{aligned} \int_{\mathfrak{D}_{\lambda,\mu}} \tilde{\psi}(x_\lambda) d\tilde{\nu}_{\lambda,\mu}(x_\lambda, x_\mu) &= \int_{I_\lambda} \tilde{\psi}(x_\lambda) \int_{\mathfrak{D}_{\lambda,\mu}(x_\lambda)} d\tilde{\nu}_{\lambda,\mu}(x_\lambda, x_\mu) \\ &= \int_{I_\lambda} \tilde{\psi}(x_\lambda) d\nu_\lambda(x_\lambda), \end{aligned}$$

where the measure $d\nu_\lambda(x_\lambda) = \int_{x_\mu \in \mathfrak{D}_{\lambda,\mu}(x_\lambda)} d\tilde{\nu}_{\lambda,\mu}(x_\lambda, x_\mu)$ is given by the integral over $x_\mu \in \mathfrak{D}_{\lambda,\mu}(x_\lambda) = \{x_\mu \in I_\mu \mid (x_\lambda, x_\mu) \in \mathfrak{D}_{\lambda,\mu}\}$. Since the spectral measure ν_λ over I_λ is also uniquely determined by its (one-dimensional) moments $\tilde{\varsigma}_\lambda(m) = \int_{I_\lambda} x_\lambda^m d\nu_\lambda(x_\lambda)$ for all $m \in \mathbb{N}$, one could alternatively consider the moments $\varsigma_{\lambda,\mu}(m, 0)$ to determine the measure ν_λ over I_λ . For more detailed discussion on joint spectral measures in the context of braided subfactors associated to compact connected rank two Lie groups, see e.g. [25].

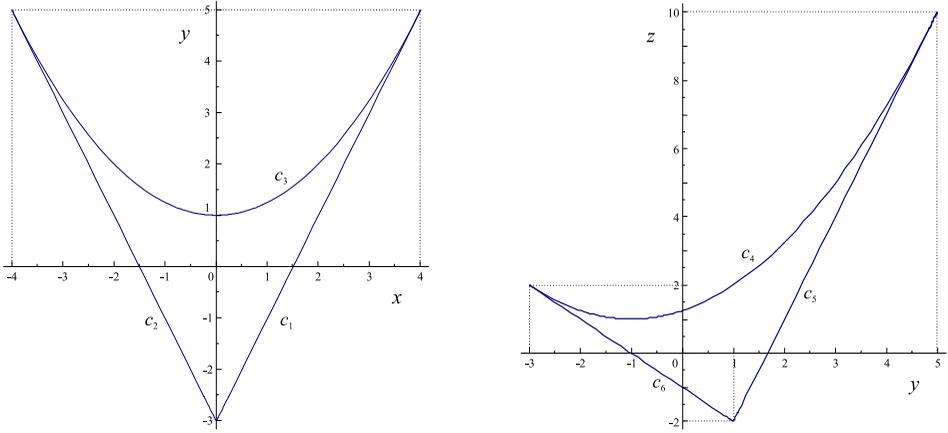


Figure 6: The domains $\mathfrak{D}_{x,y}$ and $\mathfrak{D}_{y,z}$ for $Sp(2)$.

In particular, we will consider the joint spectral measure of the fundamental representations ρ_x and ρ_y of $Sp(2)$ over $\mathfrak{D}_{x,y} := \mathfrak{D}_{\rho_x, \rho_y}$, and the joint spectral measure of the fundamental representations ρ_y and ρ_z of $SO(5)$ over $\mathfrak{D}_{y,z} := \mathfrak{D}_{\rho_y, \rho_z}$, illustrated in Figure 6.

We first describe $\mathfrak{D}_{x,y}$. The boundaries of C given by $\theta_2 = 0$, $\theta_1 = 1/2$ respectively, yield the lines c_1 , c_2 respectively, whilst the boundary $\theta_1 = \theta_2$ of C yields the curve c_3 . These curves are given by given by (cf. [41, §6.3])

$$(17) \quad c_1 : y = 2x - 3, \quad c_2 : y = -2x - 3, \quad c_3 : 4y = 4 + x^2.$$

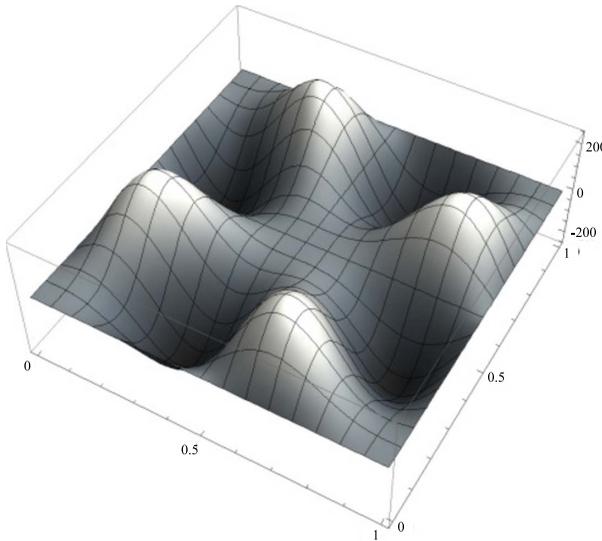
For $\mathfrak{D}_{y,z}$, the boundaries of C given by $\theta_2 = 0$ and $\theta_1 = 1/2$ both yield the curve c_4 , whilst the boundary $\theta_1 = \theta_2$ of C yields the line c_5 . Additionally, the line $\theta_2 = 1/2 - \theta_1$ which bisects C yields the third boundary of $\mathfrak{D}_{y,z}$, the line c_6 . These curves are given by

$$(18) \quad c_4 : 4z = y^2 + 2y + 5, \quad c_5 : z = 3y - 5, \quad c_6 : z = -y - 1.$$

Note that there is a two-to-one mapping from the fundamental domain C to $\mathfrak{D}_{y,z}$.

Under the change of variables $x = \chi_{\rho_x}(\omega_1, \omega_2)$, $y = \chi_{\rho_y}(\omega_1, \omega_2)$, the Jacobian $J_{x,y} = \det(\partial(x, y)/\partial(\theta_1, \theta_2))$ is given by

$$J_{x,y}(\theta_1, \theta_2) = 8\pi^2(\cos(2\pi(\theta_1 + 2\theta_2)) + \cos(2\pi(2\theta_1 - \theta_2)) - \cos(2\pi(2\theta_1 + \theta_2)) - \cos(2\pi(\theta_1 - 2\theta_2))).$$

Figure 7: The Jacobian $J_{x,y}$ over \mathbb{T}^2 .

The Jacobian $J_{x,y}$ is real and is illustrated in Figures 7, 8, where its values are plotted over the torus \mathbb{T}^2 .

With $\omega_j = e^{2\pi i \theta_j}$, $j = 1, 2$, the Jacobian is given in terms of $\omega_1, \omega_2 \in \mathbb{T}$ by

$$\begin{aligned} J_{x,y}(\omega_1, \omega_2) &= 8\pi^2 \operatorname{Re}(\omega_1 \omega_2^2 + \omega_1^2 \omega_2^{-1} - \omega_1^2 \omega_2 - \omega_1 \omega_2^{-2}) \\ &= 4\pi^2 (\omega_1 \omega_2^2 + \omega_1^{-1} \omega_2^{-2} + \omega_1^2 \omega_2^{-1} + \omega_1^{-2} \omega_2 - \omega_1^2 \omega_2 - \omega_1^{-2} \omega_2^{-1} \\ &\quad - \omega_1 \omega_2^{-2} - \omega_1^{-1} \omega_2^2). \end{aligned}$$

The Jacobian $J_{x,y}$ is invariant under $T_4^2 \in D_8$, but $T(J_{x,y}) = -J_{x,y}$ for $T = T_2, T_4$. Thus $J_{x,y}^2$ is invariant under the action of D_8 . An expression for $J_{x,y}^2$ in terms of the D_8 -invariant variables x, y may be obtained as a product of the roots appearing as the equations of the boundary of $\mathfrak{D}_{x,y}$ in (17), and is given as (see also [41])

$$(19) \quad J_{x,y}^2(x, y) = 16\pi^4 (y + 2x + 3)(y - 2x + 3)(4y - x^2 - 4),$$

for $(x, y) \in \mathfrak{D}_{x,y}$. Thus we see that the Jacobian vanishes only on the boundary of $\mathfrak{D}_{x,y}$, which is equivalent to vanishing only on the boundaries of the images of the fundamental domain in \mathbb{T}^2 under D_8 .

The factorizations of $J_{x,y}$ in (19) and the equations for the boundaries of $\mathfrak{D}_{x,y}$ given in (17) will be used in Sections 2.3, 3.2 to determine explicit

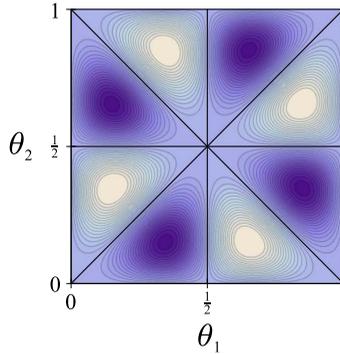


Figure 8: Contour plot of $J_{x,y}$ over \mathbb{T}^2 .

expressions for the weights which appear in the spectral measures $\mu_{v_z^u}$ over I_u in terms of elliptic integrals.

Similarly, under the change of variables $y = \chi_{\rho_y}(\omega_1, \omega_2)$, $z = \chi_{\rho_z}(\omega_1, \omega_2)$, the Jacobian $J_{y,z} = \det(\partial(y, z)/\partial(\theta_1, \theta_2))$ is given by

$$J_{y,z}(\theta_1, \theta_2) = 16\pi^2(\cos(2\pi(\theta_1 - 3\theta_2)) + \cos(2\pi(3\theta_1 + \theta_2)) - \cos(2\pi(\theta_1 + 3\theta_2)) - \cos(2\pi(3\theta_1 - \theta_2))).$$

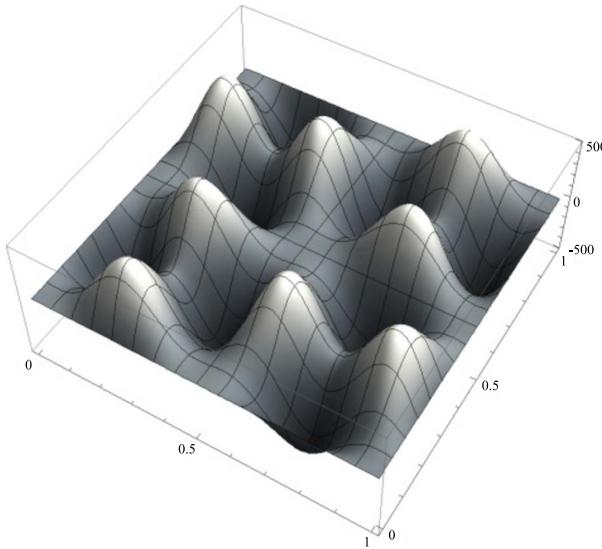
The Jacobian $J_{y,z}$ is real and is illustrated in Figures 9, 10, where its values are plotted over the torus \mathbb{T}^2 . The Jacobian is given in terms of $\omega_1, \omega_2 \in \mathbb{T}$ by

$$J_{y,z}(\omega_1, \omega_2) = 8\pi^2(\omega_1\omega_2^{-3} + \omega_1^{-1}\omega_2^3 + \omega_1^3\omega_2 + \omega_1^{-3}\omega_2^{-1} - \omega_1\omega_2^3 - \omega_1^{-1}\omega_2^{-3} - \omega_1^3\omega_2^{-1} - \omega_1^{-3}\omega_2).$$

The Jacobian $J_{y,z}$ is again invariant under $T_4^2 \in D_8$, and $T(J_{y,z}) = -J_{y,z}$ for $T = T_2, T_4$. An expression for $J_{y,z}^2$ in terms of the D_8 -invariant variables x, y may be obtained as a product of the roots appearing as the equations of the boundary of $\mathfrak{D}_{y,z}$ in (18), and is given as

$$J_{y,z}^2(y, z) = 64\pi^4(z - 3y + 5)(z + y + 1)(y^2 + 2y + 5 - 4z),$$

for $(y, z) \in \mathfrak{D}_{y,z}$. Thus we see that the Jacobian vanishes only on the boundary of $\mathfrak{D}_{y,z}$, which is equivalent to vanishing on the boundaries of the images of the fundamental domain in \mathbb{T}^2 under D_8 as well as on the lines $\theta_2 = 1/2 \pm \theta_1$. The lines $\theta_2 = 1/2 \pm \theta_1$ denote the lines of reflection of the additional symmetry

Figure 9: The Jacobian $J_{y,z}$ over \mathbb{T}^2 .

of χ_{ρ_u} , which corresponds to the fact that there is a two-to-one mapping from the fundamental domain C to $\mathfrak{D}_{y,z}$.

Note that since $z = x^2 - y - 1$, we find that

$$|J_{x,y}(y, z)| = 4\pi^2 \sqrt{(z - 3y + 5)(y^2 + 2y + 5 - 4z)},$$

and thus $J_{x,y}$ and $J_{y,z}$ are related by $J_{y,z}(y, z) = 2\sqrt{z + y + 1}J_{x,y}(y, z)$. Thus $J_{x,y}(y, z)$ is zero only on the boundaries of $\mathfrak{D}_{y,z}$ given by the curves c_4, c_5 in (18), but is not zero on the boundary given by c_6 .

Since $J_{x,y}, J_{y,z}$ are real, $J_{x,y}^2, J_{y,z}^2 \geq 0$ and we have the following expressions (cf. [22, 25] for the corresponding expressions for the Jacobian for the cases of $SU(3)$ and G_2):

$$\begin{aligned} J_{x,y}(\theta_1, \theta_2) &= 8\pi^2(\cos(2\pi(\theta_1 + 2\theta_2)) + \cos(2\pi(2\theta_1 - \theta_2)) - \cos(2\pi(2\theta_1 + \theta_2)) \\ &\quad - \cos(2\pi(\theta_1 - 2\theta_2))), \\ J_{x,y}(\omega_1, \omega_2) &= 4\pi^2(\omega_1\omega_2^2 + \omega_1^{-1}\omega_2^{-2} + \omega_1^2\omega_2^{-1} + \omega_1^{-2}\omega_2 - \omega_1^2\omega_2 - \omega_1^{-2}\omega_2^{-1} \\ &\quad - \omega_1\omega_2^{-2} - \omega_1^{-1}\omega_2^2), \\ |J_{x,y}(x, y)| &= 4\pi^2 \sqrt{(y + 2x + 3)(y - 2x + 3)(4y - x^2 - 4)}, \\ J_{y,z}(\theta_1, \theta_2) &= 16\pi^2(\cos(2\pi(\theta_1 - 3\theta_2)) + \cos(2\pi(3\theta_1 + \theta_2)) - \cos(2\pi(\theta_1 + 3\theta_2))) \end{aligned}$$

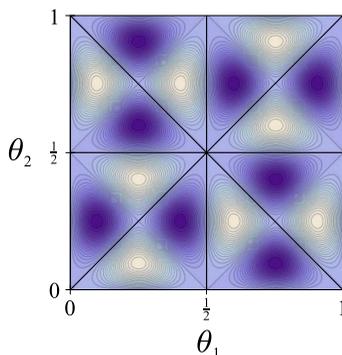


Figure 10: Contour plot of $J_{y,z}$ over \mathbb{T}^2 .

$$\begin{aligned}
 & -\cos(2\pi(3\theta_1 - \theta_2))), \\
 J_{y,z}(\omega_1, \omega_2) &= 8\pi^2(\omega_1\omega_2^{-3} + \omega_1^{-1}\omega_2^3 + \omega_1^3\omega_2 + \omega_1^{-3}\omega_2^{-1} - \omega_1\omega_2^3 - \omega_1^{-1}\omega_2^{-3} \\
 & \quad - \omega_1^3\omega_2^{-1} - \omega_1^{-3}\omega_2), \\
 |J_{y,z}(y, z)| &= 8\pi^2\sqrt{(z - 3y + 5)(z + y + 1)(y^2 + 2y + 5 - 4z)},
 \end{aligned}$$

where $0 \leq \theta_1, \theta_2 < 1$, $\omega_1, \omega_2 \in \mathbb{T}$ and $(x, y) \in \mathfrak{D}_{x,y}$, $(y, z) \in \mathfrak{D}_{y,z}$.

Then

$$(20) \quad \int_C \psi(\chi_{\rho_x}(\omega_1, \omega_2), \chi_{\rho_y}(\omega_1, \omega_2))d\omega_1 d\omega_2 = \int_{\mathfrak{D}_{x,y}} \psi(x, y)|J_{x,y}(x, y)|^{-1}dx dy,$$

$$(21) \quad \int_C \psi(\chi_{\rho_y}(\omega_1, \omega_2), \chi_{\rho_z}(\omega_1, \omega_2))d\omega_1 d\omega_2 = 2 \int_{\mathfrak{D}_{y,z}} \psi(y, z)|J_{y,z}(y, z)|^{-1}dy dz,$$

and from (16) we obtain

Theorem 1. *The joint spectral measure $\nu_{x,y}$ (over $\mathfrak{D}_{x,y}$) for $W\mathcal{A}_\infty^{\rho_x}(Sp(2))$, $W\mathcal{A}_\infty^{\rho_y}(Sp(2))$ is*

$$d\nu_{x,y}(x, y) = 8 |J_{x,y}(x, y)|^{-1}dx dy,$$

whilst the joint spectral measure $\nu_{y,z}$ (over $\mathfrak{D}_{y,z}$) for $W\mathcal{A}_\infty^{\rho_y}(Sp(2))$, $W\mathcal{A}_\infty^{\rho_z}(Sp(2))$ is

$$d\nu_{y,z}(y, z) = 16 |J_{y,z}(y, z)|^{-1}dy dz.$$

2.3. Spectral measure for $\mathcal{WA}_\infty(Sp(2))$, $\mathcal{WA}_\infty(SO(5))$ on \mathbb{R}

We now determine the spectral measure $\mu_Z^{u,G} := \mu_{v_Z^{u,G}}$ over I_u , $u = x, y, z$, where G is $Sp(2)$ or $SO(5)$, which is determined by its moments $\varphi((v_Z^{u,G})^m) = \int_{I_u} u^m d\mu_Z^{u,G}(u)$ for all $m \in \mathbb{N}$.

Thus for $\mu_Z^{x,Sp(2)}$ we set $\psi(x, y) = x^m$ in (20) and integrate with respect to y . Similarly, setting $\psi(x, y) = y^m$ in (20), the measure $\mu_Z^{y,Sp(2)}$ is obtained by integrating with respect to x . More explicitly, using the expressions for the boundaries of \mathfrak{D} given in (17), the spectral measure $\mu_Z^{x,Sp(2)}$ (over $[-4, 4]$) for the graph $\mathcal{WA}_\infty^{p_x}(Sp(2))$ is $d\mu_Z^{x,Sp(2)}(x) = J_x^{\mathbb{T}^2}(x) dx$, where $J_x^{\mathbb{T}^2}(x)$ is given by

$$J_x^{\mathbb{T}^2}(x) = \begin{cases} 8 \int_{-2x-3}^{(x^2+4)/4} |J_{x,y}(x, y)|^{-1} dy & \text{for } x \in [-4, 0], \\ 8 \int_{2x-3}^{(x^2+4)/4} |J_{x,y}(x, y)|^{-1} dy & \text{for } x \in [0, 4]. \end{cases}$$

The weight $J_x^{\mathbb{T}^2}(x)$ is the integral of the reciprocal of the square root of a cubic in y , and thus can be written in terms of the complete elliptic integral $K(m)$ of the first kind, $K(m) = \int_0^{\pi/2} (1 - m \sin^2 \theta)^{-1/2} d\theta$. Using [10, Eqn. 235.00], $J_x^{\mathbb{T}^2}(x)$ is given by

$$J_x^{\mathbb{T}^2}(x) = \frac{4}{\pi^2(4-x)} K(v(x)) = \frac{-4v(x)^{1/2}}{\pi^2(x+4)} K(v(x))$$

for $x \in [-4, 0]$, where $v(x) = (x+4)^2/(x-4)^2$, whilst for $x \in [0, 4]$, $J_x^{\mathbb{T}^2}(x)$ is given by

$$J_x^{\mathbb{T}^2}(x) = \frac{4}{\pi^2(x+4)} K(v(x)^{-1}).$$

The weight $J_x^{\mathbb{T}^2}(x)$ is illustrated in Figure 11.

The spectral measure $\mu_Z^{y,Sp(2)}$ (over $[-3, 5]$) for the graph $\mathcal{WA}_\infty^{p_y}(Sp(2))$ is $d\mu_Z^{y,Sp(2)}(y) = J_y^{\mathbb{T}^2}(y) dy$, where $J_y^{\mathbb{T}^2}(y)$ is given by

$$J_y^{\mathbb{T}^2}(y) = \begin{cases} 8 \int_{-(y+3)/2}^{(y+3)/2} |J_{x,y}(x, y)|^{-1} dx & \text{for } y \in [-3, 1], \\ 16 \int_{2\sqrt{y-1}}^{(y+3)/2} |J_{x,y}(x, y)|^{-1} dx & \text{for } y \in [1, 5], \end{cases}$$

where the value of the square root is taken to be positive. Note that the Jacobian is an even function of x . The weight $J_y^{\mathbb{T}^2}(y)$ is the integral of the reciprocal of the square root of a quadratic in x^2 and thus can also be written in terms of the complete elliptic integral of the first kind. In fact, using [10,

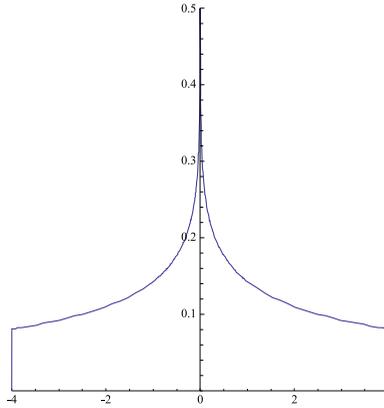


Figure 11: $J_x^{T^2}(x)$.

Eqn. 214.00] for $y \in [-3, 1]$ and [10, Eqn. 218.00] for $y \in [1, 5]$, we obtain that $J_y^{T^2}(y) = J_x^{T^2}(y - 1)$ for all $y \in [-3, 5]$. This is a surprising result, since there is no obvious symmetry between x and y in the Jacobian $J_{x,y}(x, y)$ – for one thing $J_{x,y}^2$ is a quartic in x but only a cubic in y – and yet the integral of $|J_{x,y}(x, y)|^{-1}$ over $x \in \mathbb{D}$ and over $y \in \mathbb{D}$ yields identical weights $J_x^{T^2}$ and $J_y^{T^2}$, up to a shift.

Moving to the case of $SO(5)$, the spectral measure $\mu_Z^{y,SO(5)}$ (over $[-3, 5]$) for the graph $\mathcal{W}A_\infty^{\rho y}(SO(5))$ is $d\mu_Z^{y,SO(5)}(y) = J_y^{T^2}(y) dy$, where $J_y^{T^2}(y)$ is as above, since $\mathcal{W}A_\infty^{\rho y}(SO(5))$ is simply the connected component of $(0, 0)$ in $\mathcal{W}A_\infty^{\rho y}(Sp(2))$, thus the moments $\varphi((v_Z^{y,Sp(2)})^m) = \varphi((v_Z^{y,SO(5)})^m)$. The spectral measure $\mu_Z^{z,SO(5)}$ (over $[-2, 10]$) for the graph $\mathcal{W}A_\infty^{\rho z}(SO(5))$ is $d\mu_Z^{z,SO(5)}(z) = J_z^{T^2}(z) dz$, where $J_z^{T^2}(z)$ is given by

$$J_z^{T^2}(z) = \begin{cases} 16 \int_{-z-1}^{(z+5)/3} |J_{y,z}(y, z)|^{-1} dy & \text{for } z \in [-2, 1], \\ 16Y & \text{for } y \in [1, 2], \\ 16 \int_{-1+2\sqrt{z-1}}^{(z+5)/3} |J_{y,z}(y, z)|^{-1} dy & \text{for } y \in [2, 10], \end{cases}$$

where

$$Y = \int_{-z-1}^{-1-2\sqrt{z-1}} |J_{y,z}(y, z)|^{-1} dy + \int_{-1+2\sqrt{z-1}}^{(z+5)/3} |J_{y,z}(y, z)|^{-1} dy$$

and the value of the square root is taken to be positive. A numerical plot of the weight $J_z^{T^2}(z)$ is illustrated in Figure 12.

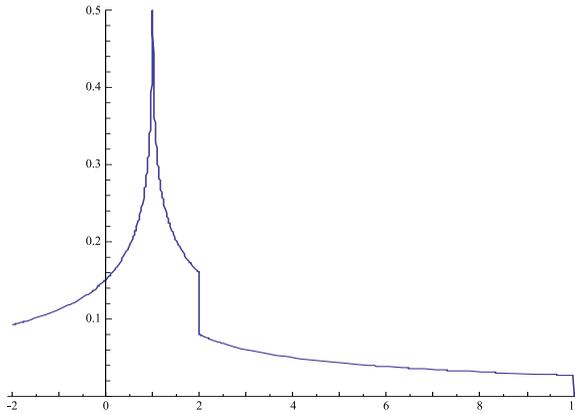


Figure 12: $J_z^{\mathbb{T}^2}(z)$.

3. Spectral measures for $\mathcal{A}_\infty(Sp(2))$, $\mathcal{A}_\infty(SO(5))$

We now consider the fixed point algebra of $\otimes_{\mathbb{N}} M_4$, $\otimes_{\mathbb{N}} M_5$ under the product action of the group $Sp(2)$ given by the fundamental representations ρ_x, ρ_y respectively, where $Sp(2)$ acts by conjugation on each factor in the infinite tensor product.

The characters $\{\chi_{(\mu_1, \mu_2)}\}_{\mu_1, \mu_2 \in \mathbb{N}; \mu_1 \geq \mu_2}$ of $Sp(2)$ satisfy the following identities, where $\chi_{(\mu_1, \mu_2)} = 0$ if $\mu_2 < 0$ or $\mu_1 < \mu_2$. For the product with $\chi_{(1,0)}$:

$$\chi_{(1,0)}\chi_{(\mu_1, \mu_2)} = \chi_{(\mu_1+1, \mu_2)} + \chi_{(\mu_1-1, \mu_2)} + \chi_{(\mu_1, \mu_2+1)} + \chi_{(\mu_1, \mu_2-1)}.$$

For the product with $\chi_{(1,1)}$:

$$\chi_{(1,1)}\chi_{(\mu_1, \mu_2)} = \chi_{(\mu_1+1, \mu_2+1)} + \chi_{(\mu_1-1, \mu_2-1)} + \chi_{(\mu_1+1, \mu_2-1)} + \chi_{(\mu_1-1, \mu_2+1)},$$

if $\mu_1 = \mu_2$, and for $\mu_1 \neq \mu_2$,

$$\begin{aligned} \chi_{(1,1)}\chi_{(\mu_1, \mu_2)} = \\ \chi_{(\mu_1, \mu_2)} + \chi_{(\mu_1+1, \mu_2+1)} + \chi_{(\mu_1-1, \mu_2-1)} + \chi_{(\mu_1+1, \mu_2-1)} + \chi_{(\mu_1-1, \mu_2+1)}. \end{aligned}$$

For the product with $\chi_{(2,0)}$:

$$\begin{aligned} \chi_{(2,0)}\chi_{(\mu_1, 0)} &= \chi_{(\mu_1, 0)} + \chi_{(\mu_1-2, 0)} + \chi_{(\mu_1+2, 0)} + \chi_{(\mu_1-1, 1)} + \chi_{(\mu_1+1, 1)}, \\ \chi_{(2,0)}\chi_{(\mu_1, \mu_2)} &= \chi_{(\mu_1, \mu_2)} + \chi_{(\mu_1+2, \mu_2)} + \chi_{(\mu_1, \mu_2-2)} + \chi_{(\mu_1+1, \mu_2-1)}, \end{aligned}$$

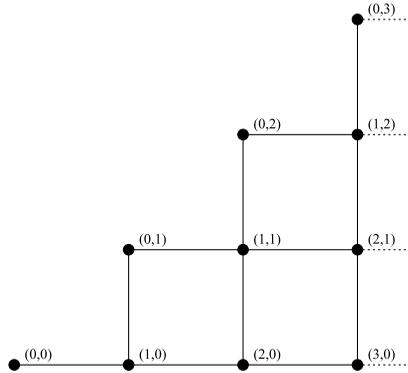


Figure 13: Infinite graph $\mathcal{A}_\infty^{\rho_x}(Sp(2))$.

if $\mu_1 = \mu_2 \neq 0$, and for all other μ_1, μ_2 ,

$$\begin{aligned} \chi_{(2,0)}\chi_{(\mu_1,\mu_2)} = & 2\chi_{(\mu_1,\mu_2)} + \chi_{(\mu_1-2,\mu_2)} + \chi_{(\mu_1+2,\mu_2)} + \chi_{(\mu_1,\mu_2-2)} + \chi_{(\mu_1,\mu_2+2)} \\ & + \chi_{(\mu_1-1,\mu_2-1)} + \chi_{(\mu_1-1,\mu_2+1)} + \chi_{(\mu_1+1,\mu_2-1)} + \chi_{(\mu_1+1,\mu_2+1)}. \end{aligned}$$

The representation graph of $Sp(2)$ for the first fundamental representation ρ_x is identified with the infinite graph $\mathcal{A}_\infty^{\rho_x}(Sp(2))$, illustrated in Figure 13, where we have made a change of labeling to the Dynkin labels $(\lambda_1, \lambda_2) = (\mu_1 - \mu_2, \mu_2)$. This labeling is more convenient in order to be able to define self-adjoint operators v_N^x, v_N^y in $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ below. The dashed lines in Figure 13 indicate edges that are removed when one restricts to the graph $\mathcal{A}_k(Sp(2))$ at finite level k , cf. Section 4.1.

Similarly, the representation graph of $Sp(2)$ for the second fundamental representation ρ_y is identified with the infinite graph $\mathcal{A}_\infty^{\rho_y}(Sp(2))$, illustrated in Figure 14, again using the Dynkin labels $(\lambda_1, \lambda_2) = (\mu_1 - \mu_2, \mu_2)$. Note that as with the infinite graph $\mathcal{W}\mathcal{A}_\infty^{\rho_y}(Sp(2))$, the graph $\mathcal{A}_\infty^{\rho_y}(Sp(2))$ is a disjoint union of two infinite graphs.

By [19, §3.5] we have $(\otimes_{\mathbb{N}} M_4)^{Sp(2)} \cong A(\mathcal{A}_\infty^{\rho_x}(Sp(2)))$ and $(\otimes_{\mathbb{N}} M_5)^{Sp(2)} \cong A(\mathcal{A}_\infty^{\rho_y}(Sp(2)))$.

We define self-adjoint operators $v_N^{x,Sp(2)}, v_N^{y,Sp(2)}$ in $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ by

$$\begin{aligned} v_N^{x,Sp(2)} &= l \otimes 1 + l^* \otimes 1 + l^* \otimes l + l \otimes l^*, \\ v_N^{y,Sp(2)} &= ll^* \otimes 1 + 1 \otimes l + 1 \otimes l^* + l^2 \otimes l^* + (l^*)^2 \otimes l, \end{aligned}$$

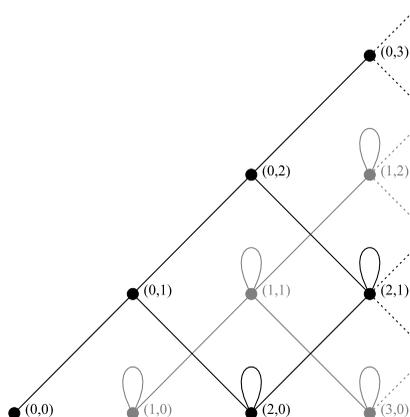


Figure 14: Infinite graph $\mathcal{A}_\infty^{\rho_y}(Sp(2))$.

identified with the adjacency matrix of $\mathcal{A}_\infty^{\rho_u}(Sp(2))$, $u = x, y$, where l is the unilateral shift to the right on $\ell^2(\mathbb{N})$.

Let Ω denote the vector $(\delta_{i,0})_i$. The vector $\Omega \otimes \Omega$ is cyclic in $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ since any vector $l^{p_1}\Omega \otimes l^{p_2}\Omega \in \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ can be written as a linear combination of elements of the form $(v_N^{x,Sp(2)})^{m_1}(v_N^{y,Sp(2)})^{m_2}(\Omega \otimes \Omega)$ so that $C^*(v_N^{x,Sp(2)}, v_N^{y,Sp(2)})(\Omega \otimes \Omega) = \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$. We define a state φ on $C^*(v_N^{x,Sp(2)}, v_N^{y,Sp(2)})$ by $\varphi(\cdot) = \langle \cdot (\Omega \otimes \Omega), \Omega \otimes \Omega \rangle$. Since $C^*(v_N^{x,Sp(2)}, v_N^{y,Sp(2)})$ is abelian and $\Omega \otimes \Omega$ is cyclic, we have that φ is a faithful state on the C^* -algebra $C^*(v_N^{x,Sp(2)}, v_N^{y,Sp(2)})$.

The moments $\varphi((v_N^{u,Sp(2)})^m)$ count the number of closed paths of length m on the graph $\mathcal{A}_\infty^{\rho_u}(Sp(2))$ which start and end at the apex vertex $(0, 0)$.

Turning our attention to $SO(5)$, the representation graph $\mathcal{A}_\infty^{\rho_y}(SO(5))$ of $SO(5)$ for the first fundamental representation ρ_y of $SO(5)$ is identified with the connected component of the apex vertex $(0, 0)$ in the infinite graph $\mathcal{A}_\infty^{\rho_y}(Sp(2))$. The graph $\mathcal{A}_\infty^{\rho_y}(SO(5))$ is illustrated in Figure 15, where we now use the Dynkin labels for $SO(5)$, $(\lambda_1, \lambda_2) = ((\mu_1 - \mu_2)/2, \mu_2)$, where (μ_1, μ_2) label the irreducible representations of $Sp(2)$ as in Section 2. The representation graph of $SO(5)$ for the second fundamental representation ρ_z is identified with the infinite graph $\mathcal{A}_\infty^{\rho_z}(SO(5))$, illustrated in Figure 16, again using the Dynkin labels for $SO(5)$.

We define self-adjoint operators $v_N^{y,SO(5)}, v_N^{z,SO(5)}$ in $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ by

$$v_N^{y,SO(5)} = ll^* \otimes 1 + 1 \otimes l + 1 \otimes l^* + l \otimes l^* + l^* \otimes l,$$

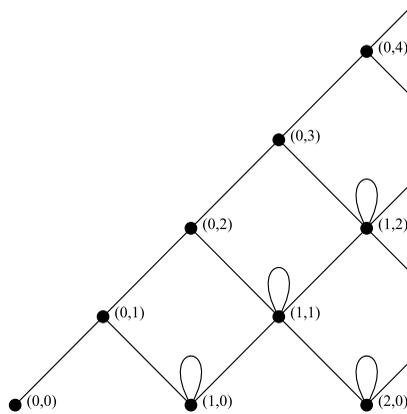


Figure 15: Infinite graph $\mathcal{A}_\infty^{\rho_y}(SO(5))$.

$$v_N^{z,SO(5)} = ll^* \otimes 1 + 1 \otimes ll^* + l \otimes 1 + l^* \otimes 1 + l \otimes l^* + l^* \otimes l + ll^* \otimes l + ll^* \otimes l^* + l^* \otimes l^2 + l \otimes (l^*)^2,$$

identified with the adjacency matrix of $\mathcal{A}_\infty^{\rho_u}(SO(5))$, $u = y, z$.

3.1. Joint spectral measure for $\mathcal{A}_\infty(Sp(2))$, $\mathcal{A}_\infty(SO(5))$ over \mathbb{T}^2

We will prove in Section 4.1 that the joint spectral measure over \mathbb{T}^2 of $v_N^{x,Sp(2)}$, $v_N^{y,Sp(2)}$ is the measure ε given by

$$d\varepsilon(\omega_1, \omega_2) = \frac{1}{128\pi^4} J_{x,y}(\omega_1, \omega_2)^2 d\omega_1 d\omega_2,$$

where $d\omega_l$ is the uniform Lebesgue measure on \mathbb{T} , $l = 1, 2$, and that the joint spectral measure over \mathbb{T}^2 of $v_N^{y,SO(5)}$, $v_N^{z,SO(5)}$ is also ε .

3.2. Spectral measure for $\mathcal{A}_\infty(Sp(2))$ on \mathbb{R}

We now determine the spectral measure $\mu_N^{u,G} := \mu_{v_N^{u,G}}$ over I_u , where $u = x, y$ for $G = Sp(2)$ and $u = y, z$ for $G = SO(5)$. We first consider the case of $Sp(2)$. From (16) and (20), with the measure given in Section 3.1, we have that

$$(22) \quad \frac{1}{128\pi^4} \int_C \psi(\chi_{\rho_u}(\omega_1, \omega_2)) J_{x,y}(\omega_1, \omega_2)^2 d\omega_1 d\omega_2 =$$

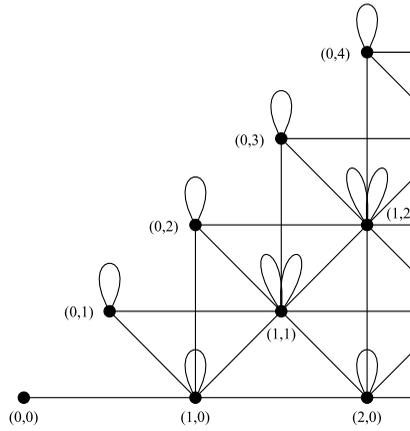


Figure 16: Infinite graph $\mathcal{A}_\infty^{\rho_z}(SO(5))$.

$$\frac{1}{16\pi^4} \int_{\mathfrak{D}_{x,y}} \psi(u) |J_{x,y}(x, y)| dx dy,$$

where C is a fundamental domain of \mathbb{T}^2/D_8 and $\mathfrak{D}_{x,y}$ is as in Section 2.3. Thus the joint spectral measure over $\mathfrak{D}_{x,y}$ is $|J_{x,y}(x, y)| dx dy / 16\pi^4$, which is the reduced Haar measure on $Sp(2)$ [41, §6.2]. The measure $\mu_N^{x,Sp(2)}$ over I_x is obtained by integrating with respect to y in (22), whilst the measure $\mu_N^{y,Sp(2)}$ over I_y is obtained by integrating with respect to x in (22). More explicitly, using the expressions for the boundaries of $\mathfrak{D}_{x,y}$ given in (17), the spectral measure $\mu_N^{x,Sp(2)}$ (over $[-4, 4]$) for the graph $\mathcal{A}_\infty^{\rho_x}(Sp(2))$ is $d\mu_N^{x,Sp(2)}(x) = J_x^{Sp(2)}(x) dx / 16\pi^4$, where $J_x^{Sp(2)}(x)$ is given by

$$\begin{aligned} & \int_{-2x-3}^{(x^2+4)/4} |J_{x,y}(x, y)| dy \quad \text{for } x \in [-4, 0], \\ & \int_{2x-3}^{(x^2+4)/4} |J_{x,y}(x, y)| dy \quad \text{for } x \in [0, 4]. \end{aligned}$$

The spectral measure $\mu_N^{y,Sp(2)}$ (over $[-3, 5]$) for $\mathcal{A}_\infty^{\rho_y}(Sp(2))$ is $d\mu_N^{y,Sp(2)}(y) = J_y^{Sp(2)}(y) dy / 16\pi^4$, where $J_y^{Sp(2)}(y)$ is given by

$$\int_{-(y+3)/2}^{(y+3)/2} |J_{x,y}(x, y)| dx,$$

for $y \in [-3, 1]$, and for $y \in [1, 5]$ by

$$\int_{-(y+3)/2}^{-2\sqrt{y-1}} |J_{x,y}(x, y)| dx + \int_{2\sqrt{y-1}}^{(y+3)/2} |J_{x,y}(x, y)| dx = 2 \int_{2\sqrt{y-1}}^{(y+3)/2} |J_{x,y}(x, y)| dx.$$

The weight $J_x^{Sp(2)}(x)$ is the integral of the square root of a cubic in y , and thus can be written in terms of the complete elliptic integrals $K(m)$, $E(m)$ of the first, second kind respectively, where $K(m) = \int_0^{\pi/2} (1 - m \sin^2 \theta)^{-1/2} d\theta$ and $E(m) = \int_0^{\pi/2} (1 - m \sin^2 \theta)^{1/2} d\theta$. Using [10, Eqn. 235.14], $J_x^{Sp(2)}(x)$ is given by

$$\frac{\pi^2}{15} (4 - x) \left[(x^4 + 224x^2 + 256) E(v(x)) + 8x(x^2 - 24x + 12) K(v(x)) \right],$$

for $x \in [-4, 0]$, where $v(x) = (x + 4)^2 / (x - 4)^2$, whilst for $x \in [0, 4]$, $J_x^{Sp(2)}(x)$ is given by

$$\frac{\pi^2}{15} (x + 4) \left[(x^4 + 224x^2 + 256) E(v(x)^{-1}) - 8x(x^2 + 24x + 12) K(v(x)^{-1}) \right],$$

The weight $J_x^{Sp(2)}(x)$ is illustrated in Figure 17.

Similarly, the weight $J_y^{Sp(2)}(y)$ is the integral of the square root of a quadratic in x^2 , and can also be written in terms of the complete elliptic integrals of the first and second kinds. Using [10, Eqn. 214.12], $J_y^{Sp(2)}(y)$ is given by

$$\frac{2\pi^2}{3} (5 - y) \left[16(1 - y) K(v(y - 1)) + (y^2 + 22y - 7) E(v(y - 1)) \right],$$

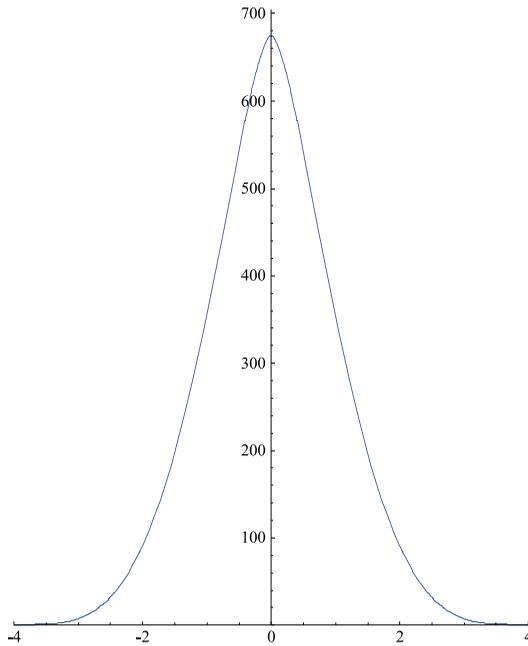
for $y \in [-3, 1]$, whilst for $y \in [1, 5]$, $J_y^{Sp(2)}(y)$ is given by

$$\frac{2\pi^2}{3} (y + 3) \left[32(1 - y) K(v(y - 1)^{-1}) + (y^2 + 22y - 7) E(v(y - 1)^{-1}) \right],$$

using [10, Eqn. 217.09]. The weight $J_y^{Sp(2)}(y)$ is illustrated in Figure 18.

We now consider the measures for $SO(5)$. From (16) and (21), with the measure given in Section 3.1, we have that

$$\frac{1}{128\pi^4} \int_C \psi(\chi_{\rho_u}(\omega_1, \omega_2)) J_{x,y}(\omega_1, \omega_2)^2 d\omega_1 d\omega_2$$

Figure 17: $J_x^{Sp(2)}(x)$.

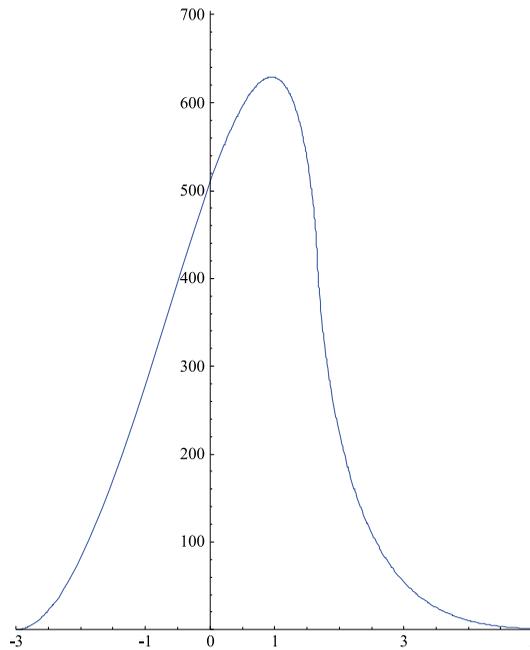
$$\begin{aligned}
 &= \frac{1}{8\pi^4} \int_{\mathfrak{D}_{y,z}} \psi(u) J_{x,y}(y, z)^2 |J_{y,z}(y, z)|^{-1} dy dz \\
 &= \frac{1}{16\pi^4} \int_{\mathfrak{D}_{y,z}} \psi(u) |J_{x,y}(y, z)| (z + y + 1)^{-1/2} dy dz.
 \end{aligned}$$

The spectral measure $\mu_N^{y,SO(5)}$ (over $[-3, 5]$) for the graph $\mathcal{A}_\infty^{\rho_y}(SO(5))$ is $d\mu_N^{y,SO(5)}(y) = J_y^{Sp(2)}(y) dy / 16\pi^4$, where $J_y^{Sp(2)}(y)$ is as above. The spectral measure $\mu_N^{z,SO(5)}$ (over $[-2, 10]$) for the graph $\mathcal{A}_\infty^{\rho_z}(SO(5))$ is $d\mu_N^{z,SO(5)}(z) = J_z^{Sp(2)}(z) dz / 8\pi^4$, where $J_z^{Sp(2)}(z)$ is given by

$$\int_{-z-1}^{(z+5)/3} |J_{x,y}(y, z)| (z + y + 1)^{-1/2} dy,$$

for $z \in [-2, 1]$, by

$$\int_{-z-1}^{-1-2\sqrt{z-1}} |J_{x,y}(y, z)| (z + y + 1)^{-1/2} dy + \int_{-1+2\sqrt{z-1}}^{(z+5)/3} |J_{x,y}(y, z)| (z + y + 1)^{-1/2} dy,$$

Figure 18: $J_y^{Sp(2)}(y)$.

for $z \in [1, 2]$, and for $z \in [2, 10]$ by

$$\int_{-1+2\sqrt{z-1}}^{(z+5)/3} |J_{x,y}(y, z)| (z + y + 1)^{-1/2} dy,$$

where the value of the square root is taken to be positive. A numerical plot of the weight $J_z^{Sp(2)}(z)$ is illustrated in Figure 19.

4. Spectral measures for nimrep graphs associated to $Sp(2)$ modular invariants

We now determine joint spectral measures for nimrep graphs associated to $Sp(2)$ modular invariants, where we will focus in particular on the nimrep graphs for the fundamental generators ρ_j , $j = 1, 2$, which have quantum dimensions $[2][6]/[3]$, $[5][6]/[2][3]$ respectively, where $[m]$ denotes the quantum integer $[m] = (q^m - q^{-m})/(q - q^{-1})$ for $q = e^{i\pi/2(k+3)}$. The nimrep graphs G_{ρ_j} were found in [14] for the conformal embeddings at levels 3, 7, 12. The realisation of modular invariants for $Sp(2)$ by braided subfactors is parallel to

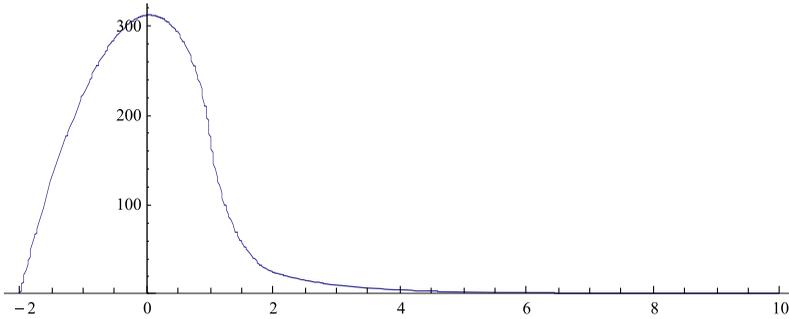


Figure 19: $J_z^{Sp(2)}(z)$.

the realisation of $SU(2)$ and $SU(3)$ modular invariants by α -induction for a suitable braided subfactors [33, 35, 44, 3, 4, 8, 9], [34, 35, 44, 3, 4, 8, 6, 7, 20, 21] respectively. The realisation of modular invariants for $SO(3)$ was done in [28], and the realisation for G_2 is also under way [25].

Let G be the nimrep associated to a braided subfactor $N \subset M$. Then the graphs G_λ , $\lambda \in {}_N\mathcal{X}_N$ are finite (undirected) graphs which share the same set of vertices ${}_N\mathcal{X}_M$. Their adjacency matrices (which we also denote by G_λ) are clearly self-adjoint. The m, n^{th} moment $\int_{\mathfrak{D}_{\lambda, \zeta}} x_\lambda^m x_\zeta^n d\mu_{\lambda, \zeta}(x_\lambda, x_\zeta)$ is given by $\langle G_\lambda^m G_\zeta^n e_1, e_1 \rangle$, where e_1 is the basis vector in $\ell^2(G_\lambda)$ ($= \ell^2(G_\zeta)$) corresponding to the distinguished vertex $*$ of G_λ with lowest Perron-Frobenius weight.

Let β_λ^ν be the eigenvalues of G_λ , indexed by $\nu \in \text{Exp}(G)$, which are ratios of the S -matrix given by $\beta_\lambda^\nu = S_{\lambda\nu}/S_{0\nu}$, with corresponding eigenvectors $(\psi_\zeta^\nu)_{\zeta \in \text{Exp}(G)}$ (note that as the nimreps are a family of commuting matrices they can be simultaneously diagonalised, and thus the eigenvectors of G_λ are the same for all λ). Then $G_\lambda^m G_\zeta^n = \mathcal{U} \Lambda_\lambda^m \Lambda_\zeta^n \mathcal{U}^*$, where Λ_ν is the diagonal matrix $\Lambda_\lambda = \text{diag}(\beta_\lambda^{\nu_1}, \beta_\lambda^{\nu_2}, \dots, \beta_\lambda^{\nu_s})$ and \mathcal{U} is the unitary matrix $\mathcal{U} = (\psi^{\nu_1}, \psi^{\nu_s}, \dots, \psi^{\nu_s})$, for $\nu_i \in \text{Exp}(G)$, so that

$$\begin{aligned}
 (23) \quad \int_{\mathbb{T}^2} (\chi_\lambda(\omega_1, \omega_2))^m (\chi_\zeta(\omega_1, \omega_2))^n d\varepsilon_{\lambda, \zeta}(\omega_1, \omega_2) &= \langle \mathcal{U} \Lambda_\lambda^m \Lambda_\zeta^n \mathcal{U}^* e_1, e_1 \rangle \\
 &= \langle \Lambda_\lambda^m \Lambda_\zeta^n \mathcal{U}^* e_1, \mathcal{U}^* e_1 \rangle \\
 &= \sum_{\nu \in \text{Exp}(G)} (\beta_\lambda^\nu)^m (\beta_\zeta^\nu)^n |\psi_\nu^*|^2.
 \end{aligned}$$

The following D_8 -invariant measure on \mathbb{T}^2 will be useful in what follows.

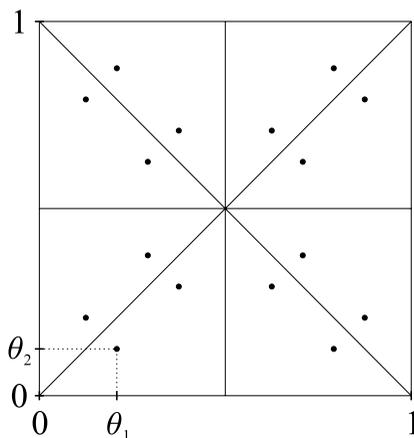


Figure 20: $\text{Supp}(d^{(\theta_1, \theta_2)})$.

Definition 4.1. We denote by $d^{(\theta_1, \theta_2)}$ the uniform Dirac measure on the D_8 -orbit of the point $(e^{2\pi i \theta_1}, e^{2\pi i \theta_2}), (e^{\pi i(2-\theta_2)}, e^{\pi i(2-\theta_1)}) \in C \subset \mathbb{T}^2$.

The set of points $(\theta_1, \theta_2) \in [0, 1]^2$ such that $(e^{2\pi i \theta_1}, e^{2\pi i \theta_2})$ is in the support of the measure $d^{(\theta_1, \theta_2)}$ is illustrated in Figure 20. For $(\theta_1, \theta_2) \notin \partial C$, $|\text{Supp}(d^{(\theta_1, \theta_2)})| = 16$, whilst $|\text{Supp}(d^{(0,0)})| = |\text{Supp}(d^{(1/2, 1/2)})| = 2$ and $|\text{Supp}(d^{(\theta_1, 1/2-\theta_1)})| = 8$. For all other $(\theta_1, \theta_2) \in \partial C$, $|\text{Supp}(d^{(\theta_1, \theta_2)})| = 4$.

4.1. Graphs $\mathcal{A}_k(Sp(2))$, $k \leq \infty$

The graphs $\mathcal{A}_k^{\rho_u}(Sp(2))$, $u = x, y$, are associated to the trivial inclusion $N \rightarrow N$, and are the trivial nimrep graphs $\mathcal{G}_\lambda = N_\lambda$, where $\lambda \in {}_N\mathcal{X}_N$ for $Sp(2)$ at level k . The graphs $\mathcal{A}_k^{\rho_u}(Sp(2))$ are illustrated in Figures 13, 14, where the set of vertices ${}_N\mathcal{X}_N = P_+^{k, Sp(2)} := \{(\lambda_1, \lambda_2) \mid \lambda_1, \lambda_2 \geq 0; \lambda_1 + \lambda_2 \leq k\}$, and the set of edges is given by the edges between these vertices.

The eigenvalues $\beta_{\rho_u}^{\nu, G}$ of $\mathcal{A}_k^{\rho_u}(G)$, where $u = x, y$ for $G = Sp(2)$ are given by the ratio $S_{\rho_u \nu} / S_{0\nu}$ with corresponding eigenvectors $\psi_\mu^\nu = S_{\nu, \mu}$ for $\mathcal{A}_k(Sp(2))$ with exponents $\text{Exp}(\mathcal{A}_k(Sp(2))) = P_+^{k, Sp(2)}$. The eigenvalues $\beta_{\rho_u}^{\lambda, G}$, $\lambda \in \text{Exp}(\mathcal{A}_k(G))$, are given by $\beta_{\rho_u}^{\lambda, G} = \chi_{\rho_u}(\omega_1, \omega_2)$, where $\omega_j = \exp^{2\pi i \theta_j}$, $j = 1, 2$ are related to $\lambda \in \text{Exp}(\mathcal{A}_k(Sp(2)))$ by

$$(24) \quad \theta_1 = \hat{\lambda}_2 / 2\kappa, \quad \theta_2 = (\hat{\lambda}_1 + \hat{\lambda}_2) / 2\kappa \Leftrightarrow \hat{\lambda}_1 = 2\kappa(\theta_2 - \theta_1), \quad \hat{\lambda}_2 = 2\kappa\theta_1.$$

The S -matrix at level k , indexed by $\lambda \in P_+^{k, Sp(2)}$, is given by [29]:

$$S_{\lambda, \mu} = \frac{1}{\kappa} \left[\begin{aligned} & \cos(\xi((\hat{\lambda}_1 + 2\hat{\lambda}_2)(\hat{\mu}_1 + 2\hat{\mu}_2) + \hat{\lambda}_1\hat{\mu}_1)) \\ & - \cos(\xi((\hat{\lambda}_1 + 2\hat{\lambda}_2)(\hat{\mu}_1 + 2\hat{\mu}_2) - \hat{\lambda}_1\hat{\mu}_1)) \\ & + \cos(\xi((\hat{\lambda}_1 + 2\hat{\lambda}_2)\hat{\mu}_1 - \hat{\lambda}_1(\hat{\mu}_1 + 2\hat{\mu}_2))) \\ & - \cos(\xi((\hat{\lambda}_1 + 2\hat{\lambda}_2)\hat{\mu}_1 + \hat{\lambda}_1(\hat{\mu}_1 + 2\hat{\mu}_2))) \end{aligned} \right],$$

where $\xi = \pi/2\kappa$, $\kappa = k + 3$, $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2)$, and $\hat{\lambda}_i = \lambda_i + 1$, $\hat{\mu}_i = \mu_i + 1$ for $i = 1, 2$. Then for μ the distinguished vertex $* = (0, 0)$, we obtain

$$(25) \quad \psi_{(0,0)}^\lambda = \frac{1}{\kappa} \left[\begin{aligned} & \cos(2\xi(2\hat{\lambda}_1 + 3\hat{\lambda}_2)) + \cos(2\xi(\hat{\lambda}_1 - \hat{\lambda}_2)) - \cos(2\xi(\hat{\lambda}_1 + 3\hat{\lambda}_2)) \\ & - \cos(2\xi(2\hat{\lambda}_1 + \hat{\lambda}_2)) \end{aligned} \right]$$

$$(26) \quad = -\frac{1}{8\kappa\pi^2} J_{x,y} \left(\hat{\lambda}_2/2\kappa, (\hat{\lambda}_1 + \hat{\lambda}_2)/2\kappa \right),$$

where in (26) we have $J_{x,y}(\theta_1, \theta_2)$ with (θ_1, θ_2) related to $\lambda \in \text{Exp}(\mathcal{A}_k(Sp(2)))$ by (24).

Since the S -matrix is unitary, the eigenvector ψ^* defined by (25) has norm 1. Recall that the Perron-Frobenius eigenvector for $\mathcal{A}_k(Sp(2))$ can also be written in the Kac-Weyl factorized form [14]:

$$\phi_\lambda^{(0,0)} = \frac{\sin(\hat{\lambda}_1\xi) \sin(2\hat{\lambda}_2\xi) \sin((\hat{\lambda}_1 + 2\hat{\lambda}_2)\xi) \sin((2\hat{\lambda}_1 + 2\hat{\lambda}_2)\xi)}{\sin(\xi) \sin(2\xi) \sin(3\xi) \sin(4\xi)}.$$

Now $\phi_*^* = 1$ whilst $\psi_*^* = 16 \sin(\xi) \sin(2\xi) \sin(3\xi) \sin(4\xi)/\kappa$, and thus we have $\kappa\psi_*^* = 16 \sin(\xi) \sin(2\xi) \sin(3\xi) \sin(4\xi)\phi^*$. Then from (26) we have

$$\begin{aligned} J_{x,y}(\theta_1, \theta_2) &= -8\kappa\pi^2 \psi_{*}^{\left(\hat{\lambda}_2/2\kappa, (\hat{\lambda}_1 + \hat{\lambda}_2)/2\kappa\right)} \\ &= -128\kappa\pi^2 \sin(\xi) \sin(2\xi) \sin(3\xi) \sin(4\xi) \phi_{\left(\hat{\lambda}_2/2\kappa, (\hat{\lambda}_1 + \hat{\lambda}_2)/2\kappa\right)}^* \\ &= 128\kappa\pi^2 \sin(2\pi\theta_1) \sin(2\pi\theta_2) \sin(\pi(\theta_1 + \theta_2)) \sin(\pi(\theta_1 - \theta_2)), \end{aligned}$$

so that the Jacobian $J_{x,y}(\theta_1, \theta_2)$ can also be written as a product of sine functions. A similar argument show that

$$J_{y,z}(\theta_1, \theta_2) = -4\kappa\pi^2 \psi_{*}^{\left(\hat{\lambda}_1 + 2\hat{\lambda}_2/4\kappa, -\hat{\lambda}_1/4\kappa\right)}$$

$$\begin{aligned} &= -64\kappa\pi^2 \sin(\xi) \sin(2\xi) \sin(3\xi) \sin(4\xi) \phi_{(\hat{\lambda}_1+2\hat{\lambda}_2)/4\kappa, -\hat{\lambda}_1/4\kappa}^* \\ &= 64\kappa\pi^2 \sin(2\pi\theta_1) \sin(2\pi\theta_2) \sin(2\pi(\theta_1 + \theta_2)) \sin(2\pi(\theta_1 - \theta_2)), \end{aligned}$$

so that the Jacobian $J_{y,z}(\theta_1, \theta_2)$ can also be written as a product of sine functions.

We now compute the joint spectral measure for $\mathcal{A}_k^{\rho_x}(Sp(2))$, $\mathcal{A}_k^{\rho_y}(Sp(2))$. Summing over all $(\lambda_1, \lambda_2) \in \text{Exp}(\mathcal{A}_k(Sp(2)))$ corresponds to summing over all $(\theta_1, \theta_2) \in \{(\hat{\lambda}_2/2\kappa, (\hat{\lambda}_1 + \hat{\lambda}_2)/2\kappa) \mid \hat{\lambda}_1, \hat{\lambda}_2 \geq 1, \hat{\lambda}_1 + \hat{\lambda}_2 \leq \kappa - 1\}$, or equivalently, over all $(\theta_1, \theta_2) \in M_k^{Sp(2)} = \{(q_1/2\kappa, q_2/2\kappa) \mid q_1, q_2 = 0, 1, \dots, 2\kappa - 1\}$ such that

$$\begin{aligned} \theta_1 &= \hat{\lambda}_2/2\kappa \geq 1/2\kappa, \theta_1 - \theta_2 = -\hat{\lambda}_1/2\kappa \leq -1/2\kappa \\ \theta_2 &= (\hat{\lambda}_1 + \hat{\lambda}_2)/2\kappa \leq (\kappa - 1)/2\kappa = 1/2 - 1/2\kappa. \end{aligned}$$

Denote by $\mathcal{C}_k^{Sp(2)}$ the set of all $(\omega_1, \omega_2) \in \mathbb{T}^2$ such that $(\theta_1, \theta_2) \in M_k^{Sp(2)}$ satisfies these conditions. Then from (23) and (26) we obtain

$$\begin{aligned} (27) \quad &\int_{\mathbb{T}^2} (\chi_{\rho_x}(\omega_1, \omega_2))^m (\chi_{\rho_y}(\omega_1, \omega_2))^n d\varepsilon_{x,y}(\omega_1, \omega_2) \\ &= \frac{1}{64\kappa^2\pi^4} \sum_{\lambda \in \text{Exp}(\mathcal{A}_k(Sp(2)))} (\beta_{\rho_x}^{\lambda, Sp(2)})^m (\beta_{\rho_y}^{\lambda, Sp(2)})^n J_{x,y}(\theta_1, \theta_2)^2 \\ &= \frac{1}{64\kappa^2\pi^4} \sum_{(\omega_1, \omega_2) \in \mathcal{C}_k^{Sp(2)}} (\chi_{\rho_x}(\omega_1, \omega_2))^m (\chi_{\rho_y}(\omega_1, \omega_2))^n J_{x,y}(\omega_1, \omega_2)^2 \end{aligned}$$

If we let $C^{Sp(2)}$ be the limit of $\mathcal{C}_k^{Sp(2)}$ as $k \rightarrow \infty$, then $C^{Sp(2)}$ is identified with the fundamental domain C of \mathbb{T}^2 under the action of the group D_8 , illustrated in Figure 5. Since $J_{x,y} = 0$ along the boundary of C , which is mapped to the boundary of $\mathfrak{D}_{x,y}$ under the map $\Psi_{x,y} : \mathbb{T}^2 \rightarrow \mathfrak{D}_{x,y}$, we can include points on the boundary of C in the summation in (27). Since $J_{x,y}^2$ is invariant under the action of D_8 , we have

$$\begin{aligned} (28) \quad &\int_{\mathbb{T}^2} (\chi_{\rho_x}(\omega_1, \omega_2))^m (\chi_{\rho_y}(\omega_1, \omega_2))^n d\varepsilon_{x,y}(\omega_1, \omega_2) \\ &= \frac{1}{8} \frac{1}{64\kappa^2\pi^4} \sum_{(\omega_1, \omega_2) \in \mathcal{C}_k^{W, Sp(2)}} (\chi_{\rho_x}(\omega_1, \omega_2))^m (\chi_{\rho_y}(\omega_1, \omega_2))^n J_{x,y}(\omega_1, \omega_2)^2 \end{aligned}$$

where

$$\mathcal{C}_k^{W, Sp(2)} = \{(e^{2\pi i q_1/2\kappa}, e^{2\pi i q_2/2\kappa}) \in \mathbb{T}^2 \mid q_1, q_2 = 0, 1, \dots, 2\kappa - 1\},$$

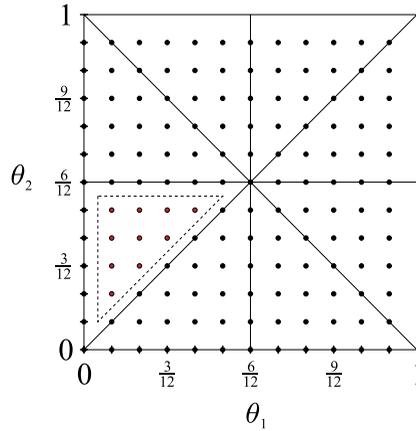


Figure 21: The points (θ_1, θ_2) such that $(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) \in \mathcal{C}_3^W$.

whose intersection with the complement of the image of the boundary of the fundamental domain C is the image of $\mathcal{C}_k^{Sp(2)}$ under the action of the Weyl group $W = D_8$. We illustrate the points (θ_1, θ_2) such that $(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) \in \mathcal{C}_3^{W, Sp(2)}$ in Figure 21. The points in the interior of the fundamental domain C , those enclosed by the dashed line, correspond to the vertices of the graph $\mathcal{A}_3(Sp(2))$.

Clearly $|\mathcal{C}_k^{W, Sp(2)}| = 4(k + 3)^2 = 4\kappa^2$. Thus from (28), we obtain (cf. [22]):

Theorem 2. *The joint spectral measure of $\mathcal{A}_k^{\rho_x}(Sp(2))$, $\mathcal{A}_k^{\rho_y}(Sp(2))$, (over \mathbb{T}^2) is given by*

$$d\varepsilon_{x,y}(\omega_1, \omega_2) = \frac{1}{128\pi^4} J_{x,y}(\omega_1, \omega_2)^2 d_{2(k+3)} \omega_1 d_{2(k+3)} \omega_2,$$

where d_m is the uniform Dirac measure over the m^{th} roots of unity.

In fact, $\varepsilon_{Sp(2)} := \varepsilon_{x,y}$ is the joint spectral measure over \mathbb{T}^2 for any $\mathcal{A}_k^\lambda(Sp(2))$, $\mathcal{A}_k^\mu(Sp(2))$.

We can now easily deduce the joint spectral measures (over \mathbb{T}^2) for the graph $\mathcal{A}_\infty(Sp(2))$ claimed in Section 3.1. Letting $k \rightarrow \infty$ in Theorem 2 we obtain:

Theorem 3. *The joint spectral measure of any pair of infinite $Sp(2)$ graphs $\mathcal{A}_\infty^\lambda(Sp(2))$, $\mathcal{A}_\infty^\mu(Sp(2))$ (over \mathbb{T}^2) are identical and are both given by*

$$d\varepsilon(\omega_1, \omega_2) = \frac{1}{128\pi^4} J_{x,y}(\omega_1, \omega_2)^2 d\omega_1 d\omega_2,$$

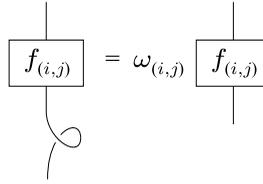


Figure 22: Statistical phase $\omega_{(i,j)}$.

where $d\omega$ is the uniform Lebesgue measure over \mathbb{T} .

4.2. Graphs $\mathcal{D}_k(Sp(2))$, $k \leq \infty$

The centre of $Sp(2)$ is \mathbb{Z}_2 . The graphs $\mathcal{D}_k^{\rho_u}(Sp(2))$, $u = x, y$, are associated to the orbifold inclusion $N \rightarrow N \rtimes_{\tau} \mathbb{Z}_2$, where $\tau = \lambda_{(0,k)}$ is a non-trivial simple current of order 2. For such an orbifold inclusion to exist, one needs an automorphism τ_0 such that $[\tau_0] = [\tau]$ and $\tau_0^2 = \text{id}$ [3, §3], which exists precisely when the statistics phase ω_{τ} of τ satisfies $\omega_{\tau}^2 = 1$ [37, Lemma 4.4]. Kuperberg’s $Sp(2)$ spider [32] involves two types of strands, $Sp(2)$ and $SO(5)$. Using this, one can construct a semisimple braided modular tensor category whose simple objects are generalised Jones-Wenzl projections $f_{(i,j)}$, $(i, j) \in P_+^{k, Sp(2)}$ (see [43] for $(SU(2))$ Jones-Wenzl projections and [39, 32, 36] for generalised $SU(3)$ Jones-Wenzl projections) and whose morphisms are intertwiners between these projections (see [40, 45, 13, 24] for a similar construction in the case of $SU(2)$ and [13, 24] for $SU(3)$). The statistics phase $\omega_{(i,j)} := \omega_{\lambda_{(i,j)}}$ is obtained by evaluating the twist applied to the generalised Jones-Wenzl projection $f_{(i,j)}$ (see Figure 22, where the single strand drawn here represents i $Sp(2)$ -strands and j $SO(5)$ -strands). Then we see that $\omega_{(0,k)} = (-1)^k$, thus the orbifold inclusion exists. Further details will be given in a future publication. See [40, Chapter XII] for a similar discussion in the case of $SO(3)$ and its double cover $SU(2)$.

Following a similar method to [9, §5.2], one finds with $[\theta] = [\lambda_{(0,0)}] \oplus [\lambda_{(0,k)}]$ that $\mathcal{D}_k^{\rho_u}(Sp(2))$, $u = x, y$, are the nimrep graphs associated to the orbifold modular invariant

$$\begin{aligned}
 Z_{\mathcal{D}_{2l}} &= \sum_{\substack{(m,n) \in P_+^{2l, Sp(2)}(0) \\ m+2n < 2l}} |\chi_{(m,n)} + \chi_{(2l-m-n, 2l-m-n)}|^2 + 2 \sum_{j=0}^l |\chi_{(2l-2j, j)}|^2, \\
 Z_{\mathcal{D}_{2l+1}} &= \sum_{(m,n) \in P_+^{2l+1, Sp(2)}(0)} |\chi_{(m,n)}|^2 + \sum_{\substack{(m,n) \in P_+^{2l+1, Sp(2)} \\ m \text{ odd}}} \chi_{(m,n)} \chi_{(m, 2l+1-m-n)}^*,
 \end{aligned}$$

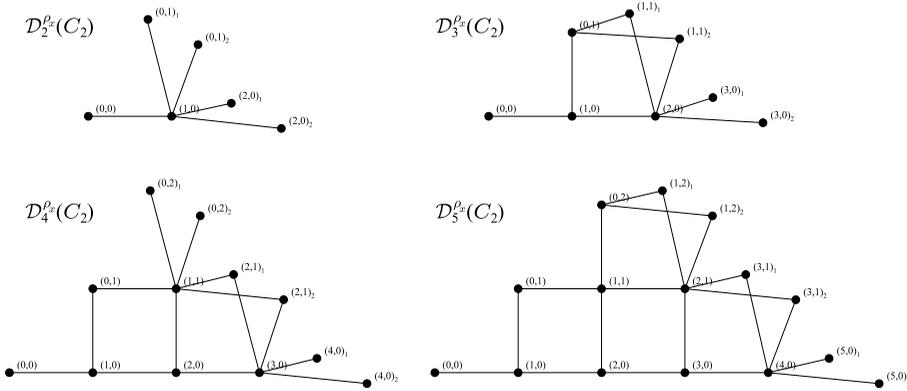


Figure 23: Orbifold graph $\mathcal{D}_k^{\rho_x}(Sp(2))$ for $k = 2, 3, 4, 5$.

where $P_+^{k,Sp(2)}(0) = \{(m, n) \in P_+^{k,Sp(2)} \mid m \text{ even}\}$. This modular invariant appeared in [2]. The graphs $\mathcal{D}_k^{\rho_u}(Sp(2))$ are \mathbb{Z}_2 -orbifolds of the graphs $\mathcal{D}_k^{\rho_u}(Sp(2))$, and are illustrated in Figures 23, 24, where we have labeled the vertices by the corresponding Dynkin labels from the $\mathcal{A}_k(Sp(2))$ graphs.

The exponents of $\mathcal{D}_k(Sp(2))$ are given by $\text{Exp}(\mathcal{D}_{2l}(Sp(2))) = \{(m, n) \in P_+^{2l,Sp(2)}(0) \mid m \neq 2l - 2n\} \cup \{\text{twice } (2l - 2j, j) \mid j = 0, 1, \dots, l\}$ for $k = 2l$ even, whilst $\text{Exp}(\mathcal{D}_{2l+1}(Sp(2))) = P_+^{2l+1,Sp(2)}(0) \cup \{(2l + 1 - 2j, j) \mid j = 0, 1, \dots, l\}$ for $k = 2l + 1$ odd. For $\lambda \in P_+^{k,Sp(2)}$ (which label the vertices of $\mathcal{A}_k(Sp(2))$) not a fixed point under the \mathbb{Z}_2 -action, i.e. $\lambda \notin \{(k - 2j, j) \mid j = 0, 1, \dots, \lfloor k/2 \rfloor\}$ where $\lfloor x \rfloor$ denotes the integer part of x , the normalized eigenvector satisfies $|\psi_*^\lambda|^2 = 2S_{*,\lambda}^2$. However for $\lambda \in \{(k - 2j, j) \mid j = 0, 1, \dots, \lfloor k/2 \rfloor\}$, $|\psi_*^{\lambda_1}| = |\psi_*^{\lambda_2}| = \sqrt{2}S_{*,\lambda}/2$, where $\lambda_j, j = 1, 2$, denote the two copies of the fixed point in the orbifold graph $\mathcal{D}_k(Sp(2))$, so that $|\psi_*^{\lambda_1}|^2 + |\psi_*^{\lambda_2}|^2 = S_{*,\lambda}^2$.

With θ_1, θ_2 as in (24), summing over all $\lambda = (\lambda_1, \lambda_2) \in P_+^{k,Sp(2)}(0)$ corresponds to summing over all $(\omega_1, \omega_2) \in C_k^{Sp(2)}$ such that $\omega_1\omega_2 = e^{2\pi i(2m+1)/2\kappa}$ for $m \in \mathbb{Z}$, where $C_k^{Sp(2)}$ is as in Section 4.1. Then from (23) and (26), with $\zeta_k = 1$ for k odd and $\zeta_k = 3/2$ for k even, we obtain

$$\int_{\mathbb{T}^2} (\chi_{\rho_x}(\omega_1, \omega_2))^m (\chi_{\rho_y}(\omega_1, \omega_2))^n d\varepsilon(\omega_1, \omega_2) = \frac{2}{64\kappa^2\pi^4} \sum_{\substack{(\omega_1, \omega_2) \in C_k^{Sp(2)}: \\ \omega_1\omega_2 = e^{2\pi i(2m+1)/2\kappa}}} (\chi_{\rho_x}(\omega_1, \omega_2))^m (\chi_{\rho_y}(\omega_1, \omega_2))^n J_{x,y}(\omega_1, \omega_2)^2$$

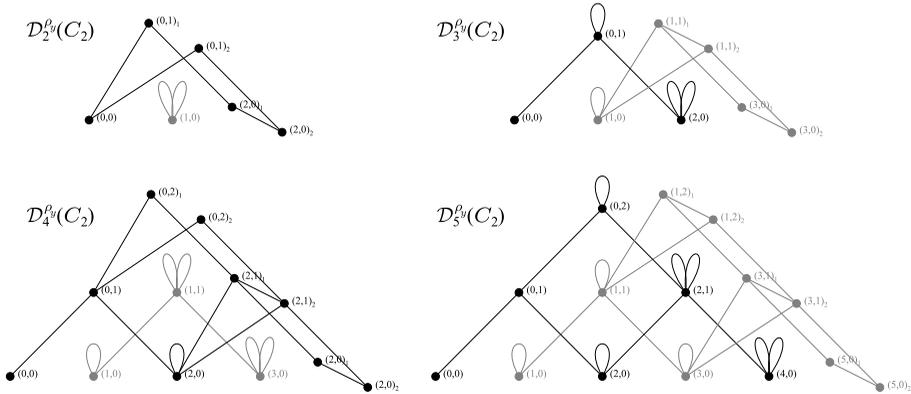


Figure 24: Orbifold graph $\mathcal{D}_k^{\rho_y}(Sp(2))$ for $k = 2, 3, 4, 5$.

$$\begin{aligned}
 & + \frac{\zeta_k}{64\kappa^2\pi^4} \sum_{\substack{(\omega_1, \omega_2) \in \mathcal{C}_k^{Sp(2)}: \\ \omega_1 = -\omega_2 \text{ or } \omega_1 = -\omega_2^{-1}}} (\chi_{\rho_x}(\omega_1, \omega_2))^m (\chi_{\rho_y}(\omega_1, \omega_2))^n J_{x,y}(\omega_1, \omega_2)^2 \\
 = & \frac{1}{8} \frac{1}{32\kappa^2\pi^4} \sum_{\substack{(\omega_1, \omega_2) \in \mathcal{C}_k^{W, Sp(2)}: \\ \omega_1 \omega_2 = e^{2\pi i(2m+1)/2\kappa}}} (\chi_{\rho_x}(\omega_1, \omega_2))^m (\chi_{\rho_y}(\omega_1, \omega_2))^n J_{x,y}(\omega_1, \omega_2)^2 \\
 & + \frac{1}{8} \frac{\zeta_k}{64\kappa^2\pi^4} \sum_{\substack{(\omega_1, \omega_2) \in \mathcal{C}_k^{W, Sp(2)}: \\ \omega_1 = -\omega_2 \text{ or } \omega_1 = -\omega_2^{-1}}} (\chi_{\rho_x}(\omega_1, \omega_2))^m (\chi_{\rho_y}(\omega_1, \omega_2))^n J_{x,y}(\omega_1, \omega_2)^2.
 \end{aligned}$$

Thus we obtain:

Theorem 4. *The joint spectral measure of $\mathcal{D}_k^{\rho_x}(Sp(2))$, $\mathcal{A}_k^{\rho_y}(Sp(2))$, (over \mathbb{T}^2) is given by*

$$\begin{aligned}
 d\varepsilon = & \frac{1}{128\pi^4} J_{x,y}(\omega_1, \omega_2)^2 (d_\kappa \times (d_{2\kappa} - d_\kappa) + (d_{2\kappa} - d_\kappa) \times d_\kappa) \\
 & + \frac{\zeta_k}{64\kappa^2\pi^4} J_{x,y}(\omega_1, \omega_2)^2 \sum_{j=1}^{k-1} d^{(j/2\kappa, (\kappa-j)/2\kappa)},
 \end{aligned}$$

where $\kappa = k + 3$, $\zeta_k = 1$ for k odd and $\zeta_k = 3/2$ for k even, $d^{(\theta_1, \theta_2)}$ is as in Definition 4.1 and d_m is the uniform Dirac measure over the m^{th} roots of unity.

Letting $k \rightarrow \infty$ we easily obtain the following corollary:

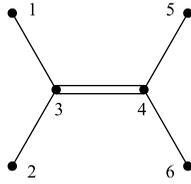


Figure 25: Exceptional graph $\mathcal{E}_3^{\rho_1}(Sp(2))$.

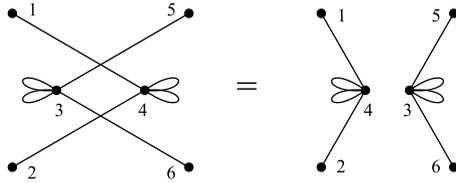


Figure 26: Exceptional graph $\mathcal{E}_3^{\rho_2}(Sp(2))$.

Corollary 1. *The joint spectral measure of $\mathcal{D}_\infty^{\rho_x}(Sp(2))$, $\mathcal{D}_\infty^{\rho_y}(Sp(2))$, (over \mathbb{T}^2) is precisely the joint spectral measure of the infinite $Sp(2)$ graphs $\mathcal{A}_\infty^{\rho_x}(Sp(2))$, $\mathcal{A}_\infty^{\rho_y}(Sp(2))$, given in Theorem 3.*

4.3. Exceptional graph $\mathcal{E}_3(Sp(2))$: $(Sp(2))_3 \rightarrow (SO(10))_1$

The graphs $\mathcal{E}_3(Sp(2))$ are associated to the conformal embedding $(Sp(2))_3 \rightarrow (SO(10))_1$ and are one of two nimreps associated to the modular invariant

$$Z_{\mathcal{E}_3} = |\chi_{(0,0)} + \chi_{(2,1)}|^2 + |\chi_{(2,0)} + \chi_{(0,3)}|^2 + 2|\chi_{(1,1)}|^2$$

which is at level 3 and has exponents

$$\text{Exp}(\mathcal{E}_3(Sp(2))) = \{(0, 0), (2, 1), (2, 0), (0, 3), \text{ and } (1, 1) \text{ twice}\}.$$

The other family $\mathcal{E}_3^M(Sp(2))$ are considered in the next section. The graphs $\mathcal{E}_3^{\rho_j}(Sp(2))$ are illustrated in Figures 25, 26. Note that $\mathcal{E}_3^{\rho_2}(Sp(2))$ has two connected components.

Following [4, §6] we can compute the principal graph and dual principal graph of the inclusion $(Sp(2))_3 \rightarrow (SO(10))_1$. The chiral induced sector bases ${}_M\mathcal{X}_M^\pm \subset \text{Sect}(M)$ and full induced sector basis ${}_M\mathcal{X}_M \subset \text{Sect}(M)$, the sector bases given by all irreducible subsectors of $[\alpha_\lambda^\pm]$ and $[\alpha_\lambda^+ \circ \alpha_{\bar{\lambda}}^-]$ respectively,

for $\lambda, \lambda' \in {}_N\mathcal{X}_N$, along with the neutral system ${}_M\mathcal{X}_M^0 = {}_M\mathcal{X}_M^+ \cap {}_M\mathcal{X}_M^-$, are given by

$$\begin{aligned} {}_M\mathcal{X}_M^\pm &= \{[\alpha_{(0,0)}], [\alpha_{(1,0)}^\pm], [\alpha_{(0,1)}^\pm], [\alpha_{(2,0)}^{(1)}], [\alpha_{(1,1)}^{(1)}], [\alpha_{(1,1)}^{(2)}]\}, \\ {}_M\mathcal{X}_M^0 &= \{[\alpha_{(0,0)}], [\alpha_{(2,0)}^{(1)}], [\alpha_{(1,1)}^{(1)}], [\alpha_{(1,1)}^{(2)}]\}, \\ {}_M\mathcal{X}_M &= \{[\alpha_{(0,0)}], [\alpha_{(1,0)}^+], [\alpha_{(1,0)}^-], [\alpha_{(0,1)}^+], [\alpha_{(0,1)}^-], [\alpha_{(2,0)}^{(1)}], [\alpha_{(1,1)}^{(1)}], [\alpha_{(1,1)}^{(2)}], [\delta_1], \\ &\quad [\delta_2], [\eta_1], [\eta_2]\}, \end{aligned}$$

where

$$\begin{aligned} [\alpha_{(2,0)}^\pm] &= [\alpha_{(0,1)}^\pm] \oplus [\alpha_{(2,0)}^{(1)}], \\ [\alpha_{(1,1)}^\pm] &= [\alpha_{(1,0)}^\pm] \oplus [\alpha_{(1,1)}^{(1)}] \oplus [\alpha_{(1,1)}^{(2)}], \\ [\alpha_{(1,0)}^+ \alpha_{(1,0)}^-] &= [\delta_1] \oplus [\delta_2], \\ [\alpha_{(1,0)}^+ \alpha_{(0,1)}^-] &= [\eta_1] \oplus [\eta_2], \end{aligned}$$

for $\alpha_{(i,j)} \equiv \alpha_{\lambda_{(i,j)}}$. The fusion graphs of $[\alpha_{(1,0)}^+]$ (solid lines) and $[\alpha_{(1,0)}^-]$ (dashed lines) are given in Figure 27, see also [14, Figure 7(a)]. The marked vertices corresponding to sectors in the neutral system ${}_M\mathcal{X}_M^0$ have been circled. These sectors obey $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules, corresponding to $SO(10)$ at level 1. Note that multiplication by $[\alpha_{(1,0)}^+]$ (or $[\alpha_{(1,0)}^-]$) does not give two copies of the nimrep graph $\mathcal{E}_3(Sp(2))$ as one might expect, but rather one copy each of $\mathcal{E}_3^{\rho_1}(Sp(2))$ and $\mathcal{E}_3^{\rho_1, M}(Sp(2))$. This is similar to the situation for the $SU(3)$ conformal embedding $SU(3)_9 \rightarrow (E_6)_1$ [17, §5.2].

Let $\iota : N \hookrightarrow M$ denote the injection map $\iota(n) = n \in M$, $n \in N$ and $\bar{\iota}$ its conjugate. The dual canonical endomorphism $\theta = \bar{\iota}\iota$ for the conformal embedding can be read from the vacuum block of the modular invariant: $[\theta] = [\lambda_{(0,0)}] \oplus [\lambda_{(2,1)}]$. By [4, Corollary 3.19] and the fact that $\langle \gamma, \gamma \rangle_M = \langle \theta, \theta \rangle_N = 2$, the canonical endomorphism $\gamma = \bar{\iota}\theta$ is given by

$$[\gamma] = [\alpha_{(0,0)}] \oplus [\delta_1].$$

Then by [4, Theorem 4.2], the principal graph of the inclusion $(Sp(2))_3 \rightarrow (SO(10))_1$ of index $3 + \sqrt{3} \approx 4.73$ is given by the connected component of $[\lambda_{(0,0)}] \in {}_N\mathcal{X}_N$ of the induction-restriction graph, and the dual principal graph is given by the connected component of $[\alpha_{(0,0)}] \in {}_M\mathcal{X}_M$ of the γ -multiplication graph. The principal graph and dual principal graph are the same, and we

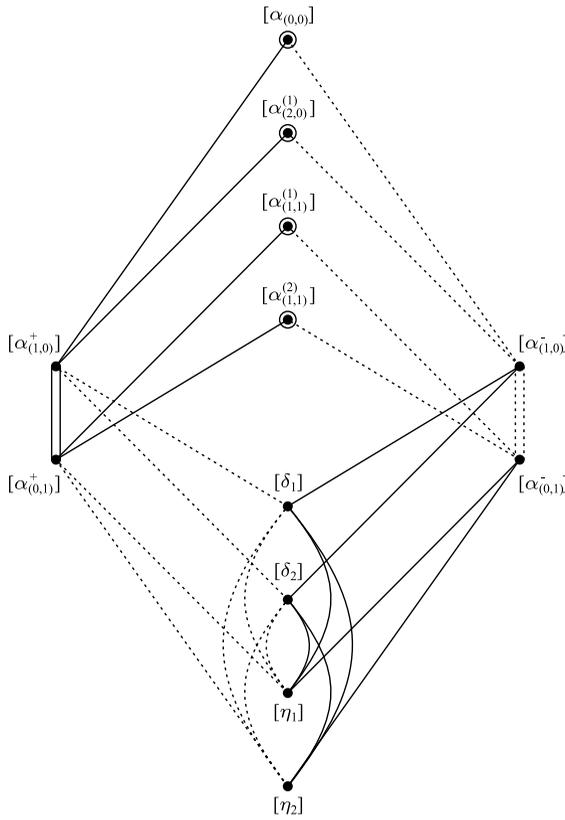


Figure 27: $\mathcal{E}_3(Sp(2))$: Multiplication by $[\alpha_{(1,0)}^+]$ (solid lines) and $[\alpha_{(1,0)}^-]$ (dashed lines).

illustrate the principal graph in Figure 28. These are the principal graphs for the 3311 Goodman-de la Harpe-Jones subfactor [31]. The principal graph in Figure 28 appears as the intertwiner for the quantum subgroup $\mathcal{E}_3(Sp(2))$ in [14, Figure 9].

One can also construct a subfactor $\alpha_{(1,0)}^\pm(M) \subset M$ with index $(1 + \sqrt{3})^2 = 2(2 + \sqrt{3}) \approx 7.46$, where M is a type III factor. Its principal graph is the nimrep graph $\mathcal{E}_3^{\rho_1}(Sp(2))$ illustrated in Figure 25. The dual principal graph is isomorphic to the principal graph as abstract graphs [44, Corollary 3.7].

We now determine the joint spectral measure of $\mathcal{E}_3^{\rho_1}(Sp(2))$, $\mathcal{E}_3^{\rho_2}(Sp(2))$. With θ_1, θ_2 as in (24) for $\lambda = (\lambda_1, \lambda_2) \in \text{Exp}(\mathcal{E}_3(Sp(2)))$, we have the following values:

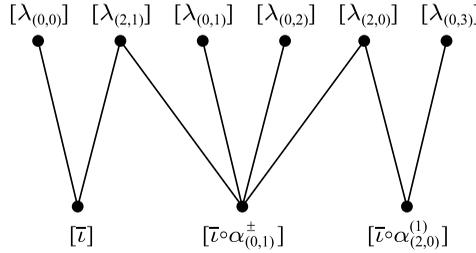


Figure 28: $\mathcal{E}_3(Sp(2))$: Principal graph of $(Sp(2))_3 \rightarrow (SO(10))_1$.

$\lambda \in \text{Exp}$	$(\theta_1, \theta_2) \in [0, 1]^2$	$ \psi_*^\lambda ^2$	$\frac{1}{8\pi^2} J(\theta_1, \theta_2) $
$(0, 0)$	$(\frac{1}{12}, \frac{2}{12})$	$\frac{3-\sqrt{3}}{24}$	$\frac{3-\sqrt{3}}{2}$
$(2, 1)$	$(\frac{2}{12}, \frac{5}{12})$	$\frac{3+\sqrt{3}}{24}$	$\frac{3+\sqrt{3}}{2}$
$(2, 0)$	$(\frac{1}{12}, \frac{4}{12})$	$\frac{3+\sqrt{3}}{24}$	$\frac{3+\sqrt{3}}{2}$
$(0, 3)$	$(\frac{4}{12}, \frac{5}{12})$	$\frac{3-\sqrt{3}}{24}$	$\frac{3-\sqrt{3}}{2}$
$(1, 1)$	$(\frac{2}{12}, \frac{4}{12})$	$\frac{1}{2}$	3

where the eigenvectors ψ^λ have been normalized so that $\|\psi^\lambda\| = 1$, and for the exponent $(1, 1)$ which has multiplicity two, the value listed in the table for $|\psi_*^{(1,1)}|^2$ is $|\psi_*^{(1,1)_1}|^2 + |\psi_*^{(1,1)_2}|^2$. Note that

$$(29) \quad |\psi_*^\lambda|^2 = \zeta_\lambda \frac{1}{12} \frac{1}{8\pi^2} |J|$$

where $\zeta_\lambda = 1$ for $\lambda \in \{(0, 0), (2, 1), (2, 0), (0, 3)\}$ and $\zeta_{(1,1)} = 2$.

The orbit of the points $(\theta_1, \theta_2) \in \{(\frac{1}{12}, \frac{2}{12}), (\frac{2}{12}, \frac{5}{12}), (\frac{1}{12}, \frac{4}{12}), (\frac{4}{12}, \frac{5}{12})\}$ under D_8 are illustrated in Figure 29, whilst the orbit of $(\frac{2}{12}, \frac{4}{12})$ is illustrated by the black points in Figure 30. The orbits of the first four points support the measure $d^{(1/12, 2/12)} + d^{(1/12, 4/12)}$, where $d^{(\theta_1, \theta_2)}$ is the discrete uniform measure given in Definition 4.1. Since the hollow points in Figure 30 lie on the boundary of the orbit of fundamental domain, $J = 0$ at these points, thus we see that the orbit of $(2/12, 4/12)$ supports the measure $|J| d_6 \times d_6$, where d_n is the uniform Dirac measure on the n^{th} roots of unity. Note that when taking the orbit under D_8 , the associated weight in (29) is now counted 8 times, thus we must divide (29) by 8. Thus the joint spectral measure for

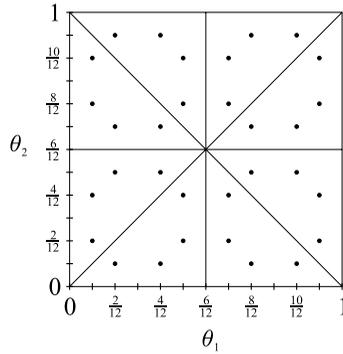


Figure 29: Orbit of $(\theta_1, \theta_2) \neq (\frac{2}{12}, \frac{4}{12})$.

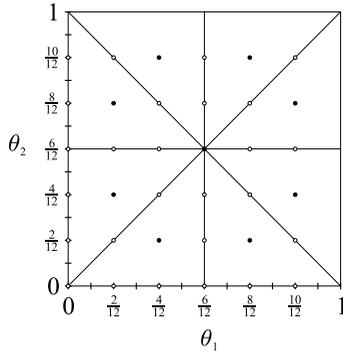


Figure 30: Orbit of $(\theta_1, \theta_2) = (\frac{2}{12}, \frac{4}{12})$.

$\mathcal{E}_3(Sp(2))$ is

$$d\varepsilon = 16 \frac{1}{8} \frac{1}{12} \frac{1}{8\pi^2} |J| \left(d^{(1/12, 2/12)} + d^{(1/12, 4/12)} \right) + 36 \frac{1}{8} \frac{2}{12} \frac{1}{8\pi^2} |J| d_6 \times d_6.$$

Then we have obtained the following result:

Theorem 5. *The joint spectral measure of $\mathcal{E}_3^{p_1}(Sp(2))$, $\mathcal{E}_3^{p_2}(Sp(2))$ (over \mathbb{T}^2) is*

$$d\varepsilon = \frac{1}{48\pi^2} |J| d^{(1/12, 2/12)} + \frac{1}{48\pi^2} |J| d^{(1/12, 4/12)} + \frac{1}{384\pi^2} |J| d_6 \times d_6,$$

where $d^{(\theta_1, \theta_2)}$ is as in Definition 4.1 and d_6 is the uniform Dirac measure on the 6th roots of unity.

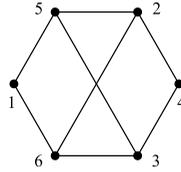


Figure 31: Graph $\mathcal{E}_3^{M, \rho_1}(Sp(2))$.

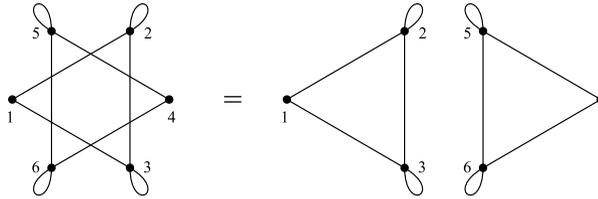


Figure 32: Graph $\mathcal{E}_3^{M, \rho_2}(Sp(2))$.

4.4. Exceptional graph $\mathcal{E}_3^M(Sp(2))$: $(Sp(2))_3 \rightarrow (SO(10))_1 \rtimes \mathbb{Z}_2$

The graphs $\mathcal{E}_3^{M, \rho_j}(Sp(2))$, illustrated in Figures 31, 32, are the nimrep graphs for the type II inclusion $(Sp(2))_3 \rightarrow (SO(10))_1 \rtimes_{\tau} \mathbb{Z}_2$ with index $2(3 + \sqrt{3}) \approx 9.46$, where $\tau = \alpha_{(2,0)}^{(1)}$ is a non-trivial simple current of order 2 in the ambichiral system ${}_M\mathcal{X}_M^0$, see Figure 27. Now $\omega_{(2,0)} = -1$ [15], thus the orbifold inclusion exists (cf. Section 4.2). Note that $[\tau^j] = [\alpha_{(1,1)}^{(j)}] \in {}_M\mathcal{X}_M^0$, $j = 1, 2$, are also simple currents of order 2 in ${}_M\mathcal{X}_M^0$, and are subsectors of $[\alpha_{(1,1)}^{\pm}]$, where $\omega_{(1,1)} = e^{7\pi i/4}$ [15]. Then $\omega_{(1,1)}^2 = e^{7\pi i/2} \neq 1$, and hence the orbifold inclusion $(Sp(2))_4 \rightarrow (SO(10))_1 \rtimes_{\tau'} \mathbb{Z}_2$ does not exist.

The principal graph for this inclusion is illustrated in Figure 33. This will be discussed in a future publication using a generalised Goodman-de la Harpe-Jones construction analogous to that for the D_{odd} and E_7 modular invariants for $SU(2)$ [9, §5.2, 5.3] and the type II inclusions for $SU(3)$ [21, §5]. It is not clear what the dual principal graph is in this case.

The associated modular invariant is again $Z_{\mathcal{E}_3}$ and the graphs are isospectral to $\mathcal{E}_3(Sp(2))$. The eigenvectors ψ^{λ} are not identical to those for $\mathcal{E}_3(Sp(2))$, however, as seen in the following table, the values of $|\psi_*^{\lambda}|^2$ are equal (up to a factor 2) to those for $\mathcal{E}_3(Sp(2))$, for $\lambda \neq (1, 1)$. With θ_1, θ_2 as in (24) for $\lambda = (\lambda_1, \lambda_2) \in \text{Exp}$, we have:

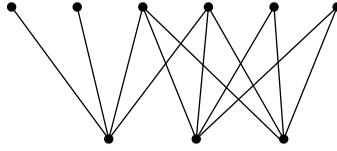


Figure 33: $\mathcal{E}_3^M(Sp(2))$: Principal graph of $(Sp(2))_3 \rightarrow (SO(10))_1 \rtimes \mathbb{Z}_2$.

$\lambda \in \text{Exp}$	$(\theta_1, \theta_2) \in [0, 1]^2$	$ \psi_*^\lambda ^2$	$\frac{1}{8\pi^2} J(\theta_1, \theta_2) $
$(0, 0)$	$(\frac{1}{12}, \frac{2}{12})$	$\frac{3-\sqrt{3}}{12}$	$\frac{3-\sqrt{3}}{2}$
$(2, 1)$	$(\frac{2}{12}, \frac{5}{12})$	$\frac{3+\sqrt{3}}{12}$	$\frac{3+\sqrt{3}}{2}$
$(2, 0)$	$(\frac{1}{12}, \frac{4}{12})$	$\frac{3+\sqrt{3}}{12}$	$\frac{3+\sqrt{3}}{2}$
$(0, 3)$	$(\frac{4}{12}, \frac{5}{12})$	$\frac{3-\sqrt{3}}{12}$	$\frac{3-\sqrt{3}}{2}$
$(1, 1)$	$(\frac{2}{12}, \frac{4}{12})$	$\frac{1}{2}$	3

where the eigenvectors ψ^λ have been normalized so that $\|\psi^\lambda\| = 1$. Then (29) becomes $|\psi_*^\lambda|^2 = \zeta_\lambda \frac{1}{6} \frac{1}{8\pi^2} |J|$, where ζ_λ is as for $\mathcal{E}_3(Sp(2))$. Thus we have the following result:

Theorem 6. *The joint spectral measure of $\mathcal{E}_3^{M,\rho_1}(Sp(2))$, $\mathcal{E}_3^{M,\rho_2}(Sp(2))$ (over \mathbb{T}^2) is*

$$d\varepsilon = \frac{1}{24\pi^2} |J| d^{(1/12, 2/12)} + \frac{1}{24\pi^2} |J| d^{(1/12, 4/12)} + \frac{1}{384\pi^2} |J| d_6 \times d_6,$$

where $d^{(\theta_1, \theta_2)}$ is as in Definition 4.1 and d_6 is the uniform Dirac measure on the 6th roots of unity.

4.5. Exceptional graph $\mathcal{E}_7(Sp(2))$: $(Sp(2))_7 \rightarrow (SO(14))_1$

The graphs $\mathcal{E}_7^{\rho_j}(G_2)$, illustrated in Figures 34, 35, are the nimrep graphs associated to the conformal embedding $(Sp(2))_7 \rightarrow (SO(14))_1$ and are one of two nimreps associated to the modular invariant

$$\begin{aligned} Z_{\mathcal{E}_7} = & |\chi_{(0,0)} + \chi_{(6,1)} + \chi_{(2,2)} + \chi_{(0,5)}|^2 + |\chi_{(6,0)} + \chi_{(0,2)} + \chi_{(2,3)} + \chi_{(0,7)}|^2 \\ & + 2|\chi_{(3,1)} + \chi_{(3,3)}|^2 \end{aligned}$$

which is at level 7 and has 12 exponents

$$\text{Exp}(\mathcal{E}_7(Sp(2))) = \{(0, 0), (6, 1), (2, 2), (0, 5), (6, 0), (0, 2), (2, 3), (0, 7)\}$$

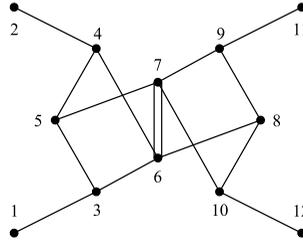


Figure 34: Graph $\mathcal{E}_7^{\rho_1}(Sp(2))$.

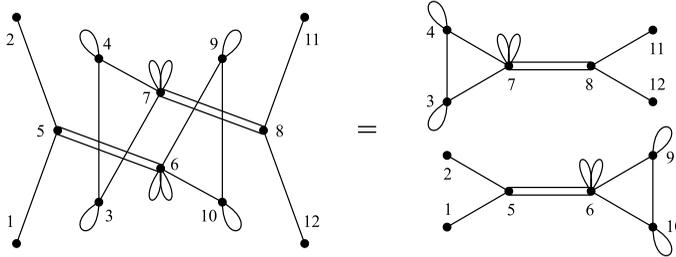


Figure 35: Graph $\mathcal{E}_7^{\rho_2}(Sp(2))$.

and twice $(3, 1), (3, 3).$

Note again that for the second fundamental representation ρ_2 , the graph (Figure 35) has two connected components.

As in Section 4.3, we can compute the principal graph and dual principal graph of the inclusion $(Sp(2))_7 \rightarrow (SO(14))_1$. The chiral induced sector bases ${}_M\mathcal{X}_M^\pm$, the neutral system ${}_M\mathcal{X}_M^0 = {}_M\mathcal{X}_M^+ \cap {}_M\mathcal{X}_M^-$ and full induced sector basis ${}_M\mathcal{X}_M$ are given by

$$\begin{aligned}
 {}_M\mathcal{X}_M^\pm &= \{[\alpha_{(0,0)}], [\alpha_{(1,0)}^\pm], [\alpha_{(0,1)}^\pm], [\alpha_{(2,0)}^\pm], [\alpha_{(1,1)}^{(j)\pm}], [\alpha_{(0,2)}^{(1)}], [\alpha_{(3,0)}^{(1)\pm}], [\alpha_{(2,1)}^{(j)\pm}], [\alpha_{(3,1)}^{(j)}], \\
 &\quad \text{for } j = 1, 2\}, \\
 {}_M\mathcal{X}_M^0 &= \{[\alpha_{(0,0)}], [\alpha_{(0,2)}^{(1)}], [\alpha_{(3,1)}^{(1)}], [\alpha_{(3,1)}^{(2)}]\}, \\
 {}_M\mathcal{X}_M &= \{[\alpha_{(0,0)}], [\alpha_{(1,0)}^\varepsilon], [\alpha_{(0,1)}^\varepsilon], [\alpha_{(2,0)}^\varepsilon], [\alpha_{(1,1)}^{(j)\varepsilon}], [\alpha_{(0,2)}^{(1)}], [\alpha_{(3,0)}^{(1)\varepsilon}], [\alpha_{(2,1)}^{(j)\varepsilon}], [\alpha_{(3,1)}^{(j)}], \\
 &\quad [\alpha_{(1,0)}^+ \alpha_{(1,0)}^-], [\alpha_{(1,0)}^+ \alpha_{(0,1)}^-], [\alpha_{(0,1)}^+ \alpha_{(1,0)}^-], [\alpha_{(1,0)}^+ \alpha_{(2,0)}^-], [\alpha_{(2,0)}^+ \alpha_{(1,0)}^-], \\
 &\quad [\eta_j], [\zeta_j], [\psi_j], [\psi'_j], [\xi], [\xi'], [\varphi], [\gamma_j], [\gamma'_j], [\delta_j], [\lambda], [\lambda'], [\omega_j], [\phi_j], \\
 &\quad \text{for } \varepsilon = +, - \text{ and } j = 1, 2\},
 \end{aligned}$$

where $[\alpha_{(1,1)}^\pm] = [\alpha_{(1,1)}^{(1)\pm}] \oplus [\alpha_{(1,1)}^{(2)\pm}]$, $[\alpha_{(0,2)}^\pm] = [\alpha_{(2,0)}^\pm] \oplus [\alpha_{(0,2)}^{(1)}]$, $[\alpha_{(3,0)}^\pm] = [\alpha_{(1,1)}^{(1)\pm}] \oplus [\alpha_{(3,0)}^{(1)\pm}]$, $[\alpha_{(2,1)}^\pm] = [\alpha_{(0,1)}^\pm] \oplus [\alpha_{(2,0)}^\pm] \oplus [\alpha_{(2,1)}^{(1)\pm}] \oplus [\alpha_{(2,1)}^{(2)\pm}]$, $[\alpha_{(3,1)}^\pm] = [\alpha_{(1,0)}^\pm] \oplus 2[\alpha_{(1,1)}^{(1)\pm}] \oplus [\alpha_{(3,0)}^{(1)\pm}] \oplus [\alpha_{(3,1)}^{(1)}] \oplus [\alpha_{(3,1)}^{(2)}]$, $[\alpha_{(0,1)}^+ \alpha_{(0,1)}^-] = [\eta_1] \oplus [\eta_2]$, $[\alpha_{(2,0)}^+ \alpha_{(2,0)}^-] = [\zeta_1] \oplus [\zeta_2]$, $[\alpha_{(2,0)}^+ \alpha_{(0,1)}^-] = [\psi_1] \oplus [\psi_2]$, $[\alpha_{(0,1)}^+ \alpha_{(2,0)}^-] = [\psi'_1] \oplus [\psi'_2]$, $[\alpha_{(1,1)}^+ \alpha_{(1,0)}^-] = [\xi] \oplus [\varphi]$, $[\alpha_{(1,0)}^+ \alpha_{(1,1)}^-] = [\xi'] \oplus [\varphi]$, $[\alpha_{(1,1)}^+ \alpha_{(0,1)}^-] = [\alpha_{(1,0)}^+ \alpha_{(0,1)}^-] \oplus [\gamma_1] \oplus [\gamma_2]$, $[\alpha_{(0,1)}^+ \alpha_{(1,1)}^-] = [\alpha_{(0,1)}^+ \alpha_{(1,0)}^-] \oplus [\gamma'_1] \oplus [\gamma'_2]$, $[\alpha_{(1,1)}^+ \alpha_{(2,0)}^-] = [\alpha_{(1,0)}^+ \alpha_{(2,0)}^-] \oplus [\delta_1] \oplus [\delta_2]$, $[\alpha_{(3,0)}^+ \alpha_{(1,0)}^-] = [\xi] \oplus [\lambda]$, $[\alpha_{(1,0)}^+ \alpha_{(3,0)}^-] = [\xi'] \oplus [\lambda']$, $[\alpha_{(3,0)}^+ \alpha_{(0,1)}^-] = [\gamma_1] \oplus [\gamma_2] \oplus [\omega_1] \oplus [\omega_2]$ and $[\alpha_{(2,1)}^+ \alpha_{(1,0)}^-] = [\alpha_{(0,1)}^+ \alpha_{(1,0)}^-] \oplus [\alpha_{(2,0)}^+ \alpha_{(1,0)}^-] \oplus [\phi_1] \oplus [\phi_2]$.

The fusion graphs of $[\alpha_{(1,0)}^+]$ (solid lines) and $[\alpha_{(1,0)}^-]$ (dashed lines) are given in Figure 36, where we have circled the marked vertices. Here multiplication by $[\alpha_{(1,0)}^+]$ (or $[\alpha_{(1,0)}^-]$) gives two copies each of $\mathcal{E}_7(Sp(2))$ and $\mathcal{E}_7^M(Sp(2))$. The ambichiral part ${}_M\mathcal{X}_M^0$ obeys $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules, corresponding to $SO(14)$ at level 1.

We find

$$[\gamma] = [\alpha_{(0,0)}] \oplus [\alpha_{(1,0)}^+ \alpha_{(1,0)}^-] \oplus [\eta_1] \oplus [\zeta_1],$$

and the principal graph of the inclusion $(Sp(2))_7 \rightarrow (SO(14))_1$ of index $5(3 + \sqrt{5}) + \sqrt{250 + 110\sqrt{5}} \approx 48.45$ is illustrated in Figure 37, where the thick lines denote double edges and a label (a, b) of a vertex along the top row denotes the sector $[\lambda_{(a,b)}]$.

Again, we can construct a subfactor $\alpha_{(1,0)}^\pm(M) \subset M$ of index $4 + \sqrt{5} + 2\sqrt{5 + 2\sqrt{5}} \approx 12.39$, where M is a type III factor. Its principal graph is the nimrep graph $\mathcal{E}_7^{\rho^1}(Sp(2))$ illustrated in Figure 34. The dual principal graph is again isomorphic to the principal graph as abstract graphs.

We now determine the joint spectral measure of $\mathcal{E}_7^{\rho^1}(Sp(2))$, $\mathcal{E}_7^{\rho^2}(Sp(2))$. With θ_1, θ_2 as in (24) for $\lambda = (\lambda_1, \lambda_2) \in \text{Exp}(\mathcal{E}_7(Sp(2)))$, we have the following values:

$\lambda \in \text{Exp}$	$(\theta_1, \theta_2) \in [0, 1]^2$	$ \psi_*^\lambda ^2$	$\frac{1}{8\pi^2} J(\theta_1, \theta_2) $
$(0, 0), (0, 7)$	$(\frac{1}{20}, \frac{2}{20}), (\frac{8}{20}, \frac{9}{20})$	$\frac{5 - \sqrt{5} - \sqrt{10 - 2\sqrt{5}}}{80}$	$\frac{5 - \sqrt{5} - \sqrt{10 - 2\sqrt{5}}}{4}$
$(6, 1), (6, 0)$	$(\frac{2}{20}, \frac{9}{20}), (\frac{1}{20}, \frac{8}{20})$	$\frac{5 - \sqrt{5} + \sqrt{10 - 2\sqrt{5}}}{80}$	$\frac{5 - \sqrt{5} + \sqrt{10 - 2\sqrt{5}}}{4}$
$(2, 2), (2, 3)$	$(\frac{3}{20}, \frac{6}{20}), (\frac{4}{20}, \frac{7}{20})$	$\frac{5 + \sqrt{5} + \sqrt{10 + 2\sqrt{5}}}{80}$	$\frac{5 + \sqrt{5} + \sqrt{10 + 2\sqrt{5}}}{4}$
$(0, 5), (0, 2)$	$(\frac{6}{20}, \frac{7}{20}), (\frac{3}{20}, \frac{4}{20})$	$\frac{5 + \sqrt{5} - \sqrt{10 + 2\sqrt{5}}}{80}$	$\frac{5 + \sqrt{5} - \sqrt{10 + 2\sqrt{5}}}{4}$
$(3, 1), (3, 3)$	$(\frac{2}{20}, \frac{6}{20}), (\frac{4}{20}, \frac{8}{20})$	$\frac{1}{4}$	$\frac{5}{2}$

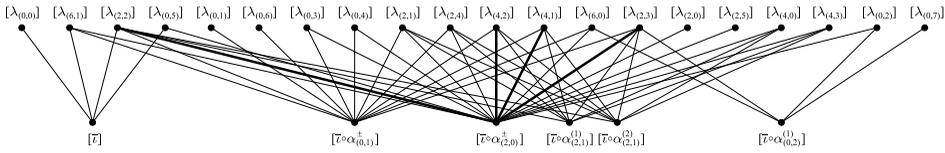


Figure 37: $\mathcal{E}_7(Sp(2))$: Principal graph of $(Sp(2))_7 \rightarrow (SO(14))_1$.

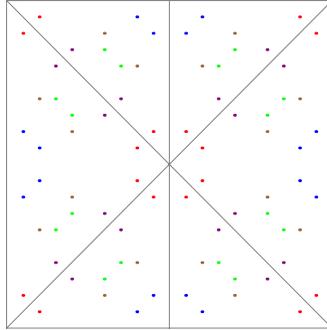


Figure 38: Orbit of (θ_1, θ_2) for $\lambda \in \text{Exp}(\mathcal{E}_7(Sp(2)))$.

is

$$d\varepsilon = 16 \frac{1}{8} \frac{1}{20} \frac{1}{8\pi^2} |J| \left(d^{(1/20,2/20)} + d^{(1/20,2/5)} + d^{(3/20,3/10)} + d^{(3/20,1/5)} + 2d^{(1/10,3/10)} \right).$$

Then we have obtained the following result:

Theorem 7. *The joint spectral measure of $\mathcal{E}_7^{\rho_1}(Sp(2))$, $\mathcal{E}_7^{\rho_2}(Sp(2))$ (over \mathbb{T}^2) is*

$$d\varepsilon = \frac{1}{80\pi^2} |J| \left(d^{(1/20,2/20)} + d^{(1/20,2/5)} + d^{(3/20,3/10)} + d^{(3/20,1/5)} + 2d^{(1/10,3/10)} \right),$$

where $d^{(\theta_1, \theta_2)}$ is as in Definition 4.1.

4.6. Exceptional graph $\mathcal{E}_7^M(Sp(2))$: $(Sp(2))_7 \rightarrow (SO(14))_1 \rtimes \mathbb{Z}_2$

The graphs $\mathcal{E}_7^{M, \rho_j}(Sp(2))$, illustrated in Figures 39 and 40 are the nimrep graphs for the type II inclusion $(Sp(2))_7 \rightarrow (SO(14))_1 \rtimes_{\tau} \mathbb{Z}_2$ with index $10(3 + \sqrt{5}) + 2\sqrt{250 + 110\sqrt{5}} \approx 96.90$, where $\tau = \alpha_{(0,2)}^{(1)}$ is a non-trivial

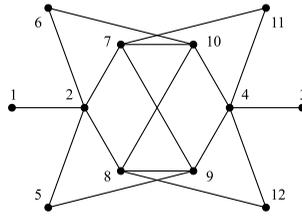


Figure 39: Graph $\mathcal{E}_7^{M, \rho_1}(Sp(2))$.

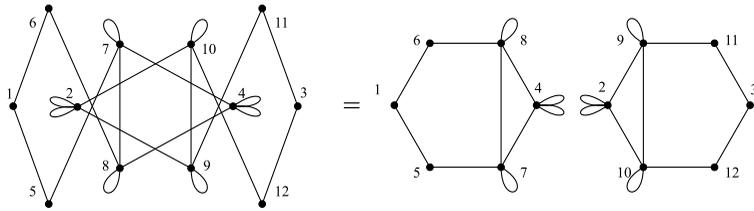


Figure 40: Graph $\mathcal{E}_7^{M, \rho_2}(Sp(2))$.

simple current of order 2 in the ambichiral system ${}_M\mathcal{X}_M^0$, see Figure 36. From Section 4.5, $[\tau]$ is a subsector of $[\alpha_{(0,2)}^\pm]$. Now $\omega_{(0,2)} = -1$ [15], which satisfies $\omega_{(0,2)}^2 = 1$, thus the orbifold inclusion exists (cf. Section 4.2). Note that $[\tau'] = [\alpha_{(3,1)}^{(j)}] \in {}_M\mathcal{X}_M^0$, $j = 1, 2$, are also non-trivial simple currents of order 2 in ${}_M\mathcal{X}_M^0$ and are subsectors of $[\alpha_{(3,1)}^\pm]$, for which $\omega_{(3,1)} = e^{7\pi i/4}$ [15]. Then $\omega_{(3,1)}^2 = e^{7\pi i/2} \neq 1$, and hence the orbifold inclusion $(Sp(2))_7 \rightarrow (SO(14))_1 \rtimes_{\tau'} \mathbb{Z}_2$ does not exist.

The principal graph for this inclusion would be the graph obtained by composing the principal graph for $(Sp(2))_7 \rightarrow (SO(14))_1$, illustrated in Figure 37 with the graph for the \mathbb{Z}_2 -action, illustrated in Figure 41. This will be discussed in a future publication using a generalised Goodman-de la Harpe-Jones construction (cf. the comments in Section 4.4). Again, it is not clear what the dual principal graph is in this case.

The associated modular invariant is again $Z_{\mathcal{E}_7}$ and the graphs are isospectral to $\mathcal{E}_7(Sp(2))$. The eigenvectors ψ^λ are not identical to those for $\mathcal{E}_3(Sp(2))$. However, as seen in the following table, the values of $|\psi_*^\lambda|^2$ are equal (up to a factor 2) to those for $\mathcal{E}_3(Sp(2))$, for $\lambda \neq (3, 1), (3, 3)$. With θ_1, θ_2 as in (24) for $\lambda = (\lambda_1, \lambda_2) \in \text{Exp}$, we have:

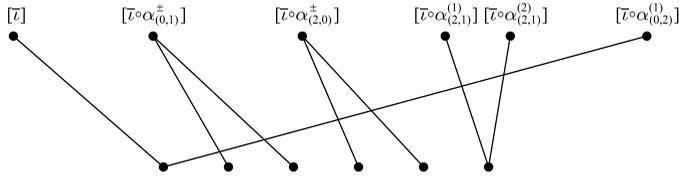


Figure 41: $\mathcal{E}_7^M(Sp(2))$: \mathbb{Z}_2 -action on $(SO(14))_1$.

$\lambda \in \text{Exp}$	$(\theta_1, \theta_2) \in [0, 1]^2$	$ \psi_\lambda ^2$	$\frac{1}{8\pi^2} J(\theta_1, \theta_2) $
$(0, 0), (0, 7)$	$(\frac{1}{20}, \frac{2}{20}), (\frac{8}{20}, \frac{9}{20})$	$\frac{5-\sqrt{5}-\sqrt{10-2\sqrt{5}}}{40}$	$\frac{5-\sqrt{5}-\sqrt{10-2\sqrt{5}}}{4}$
$(6, 1), (6, 0)$	$(\frac{2}{20}, \frac{9}{20}), (\frac{1}{20}, \frac{8}{20})$	$\frac{5-\sqrt{5}+\sqrt{10-2\sqrt{5}}}{40}$	$\frac{5-\sqrt{5}+\sqrt{10-2\sqrt{5}}}{4}$
$(2, 2), (2, 3)$	$(\frac{3}{20}, \frac{6}{20}), (\frac{4}{20}, \frac{7}{20})$	$\frac{5+\sqrt{5}+\sqrt{10+2\sqrt{5}}}{40}$	$\frac{5+\sqrt{5}+\sqrt{10+2\sqrt{5}}}{4}$
$(0, 5), (0, 2)$	$(\frac{6}{20}, \frac{7}{20}), (\frac{3}{20}, \frac{4}{20})$	$\frac{5+\sqrt{5}-\sqrt{10+2\sqrt{5}}}{40}$	$\frac{5+\sqrt{5}-\sqrt{10+2\sqrt{5}}}{4}$
$(3, 1), (3, 3)$	$(\frac{2}{20}, \frac{6}{20}), (\frac{4}{20}, \frac{8}{20})$	0	$\frac{5}{2}$

where the eigenvectors ψ^λ have been normalized so that $\|\psi^\lambda\| = 1$. Then (30) becomes $|\psi_\lambda|^2 = \zeta_\lambda \frac{1}{10} \frac{1}{8\pi^2} |J|$, where $\zeta_{(3,1)} = \zeta_{(3,3)} = 0$, and $\zeta_\lambda = 1$ for $\lambda \in \{(0, 0), (6, 1), (2, 2), (0, 5), (6, 0), (0, 2), (2, 3), (0, 7)\}$ as for $\mathcal{E}_7(Sp(2))$. Thus we have the following result:

Theorem 8. *The joint spectral measure of $\mathcal{E}_7^{M,\rho_1}(Sp(2))$, $\mathcal{E}_7^{M,\rho_2}(Sp(2))$ (over \mathbb{T}^2) is*

$$d\varepsilon = \frac{1}{80\pi^2} |J| \left(d^{(1/20, 2/20)} + d^{(1/20, 2/5)} + d^{(3/20, 3/10)} + d^{(3/20, 1/5)} \right),$$

where $d^{(\theta_1, \theta_2)}$ is as in Definition 4.1.

4.7. Exceptional graph $\mathcal{E}_8(Sp(2))$

The graphs $\mathcal{E}_8^{\rho_j}(G_2)$ are illustrated in Figures 42 and 43. To our knowledge neither graph has appeared in the literature before in the context of nimrep graphs or subfactors. The associated modular invariant is [42, (3.1)]

$$\begin{aligned} Z_{\mathcal{E}_8} = & |\chi_{(0,0)} + \chi_{(0,8)}|^2 + |\chi_{(0,2)} + \chi_{(0,6)}|^2 + |\chi_{(4,0)} + \chi_{(4,4)}|^2 + |\chi_{(4,1)} + \chi_{(4,3)}|^2 \\ & + |\chi_{(2,2)} + \chi_{(2,4)}|^2 + (\chi_{(8,0)}, \chi_{(0,1)}, \chi_{(0,7)}) + (\chi_{(6,1)}, \chi_{(0,3)}, \chi_{(0,5)}) \\ & + (\chi_{(4,2)}, \chi_{(2,1)}, \chi_{(2,5)}) + (\chi_{(2,3)}, \chi_{(6,0)}, \chi_{(6,2)}) + (\chi_{(0,4)}, \chi_{(2,0)}, \chi_{(2,6)}), \end{aligned}$$

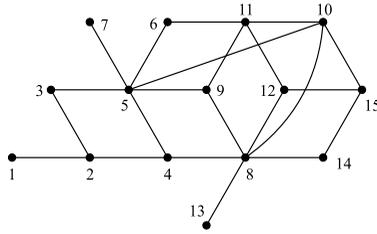


Figure 42: Graph $\mathcal{E}_8^{\rho_1}(Sp(2))$.

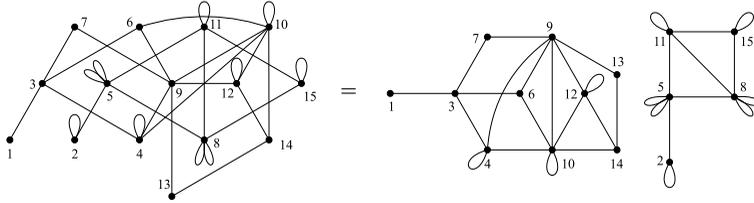


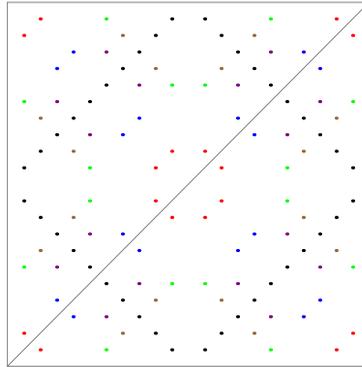
Figure 43: Graph $\mathcal{E}_8^{\rho_2}(Sp(2))$.

where $(\chi_\lambda, \chi_\mu, \chi_\nu) := |\chi_\lambda|^2 + \chi_\lambda(\chi_\mu + \chi_\nu)^* + (\chi_\mu + \chi_\nu)\chi_\lambda^*$. This modular invariant is at level 8 and has exponents

$$\text{Exp}(\mathcal{E}_8(Sp(2))) = \{(0, 0), (0, 8), (0, 2), (0, 6), (4, 0), (4, 4), (4, 1), (4, 3), (2, 2), (2, 4), (8, 0), (6, 1), (4, 2), (2, 3), (0, 4)\}.$$

Since the modular invariant associated with this family of graphs does not come from a conformal embedding, it has not yet been shown that the graphs $\mathcal{E}_{8\rho_j}(Sp(2))$ arises from a braided subfactor. This modular invariant is a twist of the $\mathcal{D}_8(Sp(2)) = \mathcal{A}_8(Sp(2))/\mathbb{Z}_2$ orbifold invariant discussed in Section 4.2, and is analogous to the E_7 modular invariant for $SU(2)$ [9, §5.3] and the Moore-Seiberg $\mathcal{E}_{MS}^{(12)}$ invariant for $SU(3)$ [21, §5.4]. The realisation of this nimrep by a braided subfactor will be discussed in a future publication, using a generalised Goodman-de la Harpe-Jones construction analogous to that for $E_7, \mathcal{E}_{MS}^{(12)}$ in [9, 21]. This construction produces $\mathcal{E}_8^{\rho_j}(G_2)$ as nimrep graphs. It is expected that $\mathcal{E}_{8\rho_j}(Sp(2))$ does indeed arise as the nimrep for a type II inclusion with index $4(\cos(\pi/11) + \cos(2\pi/11))^2 \approx 12.97$.

However, for our purposes it is sufficient to know the eigenvalues and corresponding eigenvectors for these graphs, and it is not necessary for the graph to be a nimrep graph. With θ_1, θ_2 as in (24) for $\lambda = (\lambda_1, \lambda_2) \in \text{Exp}(\mathcal{E}_8(Sp(2)))$, we have the following values:

Figure 44: Orbit of (θ_1, θ_2) for $\lambda \in \text{Exp}(\mathcal{E}_8(Sp(2)))$.

$\lambda \in \text{Exp}$	$(\theta_1, \theta_2) \in [0, 1]^2$	$ \psi_*^\lambda ^2$	$\frac{1}{64\pi^2} J_{y,z}(\theta_1, \theta_2)^2$
$(0, 0), (0, 8)$	$(\frac{1}{22}, \frac{2}{22}), (\frac{9}{22}, \frac{10}{22})$	a_5	$\frac{121}{2} a_5$
$(0, 2), (0, 6)$	$(\frac{3}{22}, \frac{4}{22}), (\frac{7}{22}, \frac{8}{22})$	a_4	$\frac{121}{2} a_4$
$(4, 0), (4, 4)$	$(\frac{1}{22}, \frac{6}{22}), (\frac{5}{22}, \frac{10}{22})$	a_3	$\frac{121}{2} a_3$
$(4, 1), (4, 3)$	$(\frac{2}{22}, \frac{7}{22}), (\frac{4}{22}, \frac{9}{22})$	a_2	$\frac{121}{2} a_2$
$(2, 2), (2, 4)$	$(\frac{3}{22}, \frac{6}{22}), (\frac{5}{22}, \frac{8}{22})$	a_1	$\frac{121}{2} a_1$
$(8, 0)$	$(\frac{1}{22}, \frac{10}{22})$	$11b_1$	0
$(6, 1)$	$(\frac{2}{22}, \frac{9}{22})$	$11b_5$	0
$(4, 2)$	$(\frac{3}{22}, \frac{8}{22})$	$11b_2$	0
$(2, 3)$	$(\frac{4}{22}, \frac{7}{22})$	$11b_4$	0
$(0, 4)$	$(\frac{5}{22}, \frac{6}{22})$	$11b_3$	0

where a_i is the i^{th} largest root of $56689952x^5 - 15460896x^4 + 1522664x^3 - 63888x^2 + 968x - 1$, b_i is the i^{th} largest root of $x^5 - 11x^4 + 44x^3 - 77x^2 + 55x - 11$, and the eigenvectors ψ^λ have been normalized so that $\|\psi^\lambda\| = 1$.

The orbits under D_8 of the points $(\theta_1, \theta_2) \in \{(\frac{1}{22}, \frac{2}{22}), (\frac{9}{22}, \frac{10}{22}), (\frac{3}{22}, \frac{4}{22}), (\frac{7}{22}, \frac{8}{22}), (\frac{1}{22}, \frac{6}{22}), (\frac{5}{22}, \frac{10}{22}), (\frac{2}{22}, \frac{7}{22}), (\frac{4}{22}, \frac{9}{22}), (\frac{3}{22}, \frac{6}{22}), (\frac{5}{22}, \frac{8}{22})\}$ are shown in Figure 44. The orbits of each successive pair of points support the measures $d^{(1/20, 2/22)}, d^{(3/22, 4/22)}, d^{(1/22, 6/22)}, d^{(2/22, 7/22)}$ and $d^{(3/22, 6/22)}$ respectively. The orbits of the points $(\theta_1, \theta_2) \in \{(\frac{1}{22}, \frac{10}{22}), (\frac{2}{22}, \frac{9}{22}), (\frac{3}{22}, \frac{8}{22}), (\frac{4}{22}, \frac{7}{22}), (\frac{5}{22}, \frac{6}{22})\}$

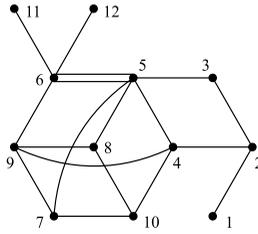


Figure 45: Graph $\mathcal{E}_{12}^{\rho_1}(Sp(2))$.

under D_8 are the black points illustrated in Figure 44. Thus we see that the joint spectral measure for $\mathcal{E}_8(Sp(2))$ is

$$\begin{aligned}
 &16 \frac{1}{8} \frac{2}{121} \frac{1}{64\pi^4} J_{y,z}^2 \left(d^{(1/22,2/22)} + d^{(3/22,4/22)} + d^{(1/22,6/22)} + d^{(3/22,6/22)} \right. \\
 &\quad \left. + 2d^{(2/22,7/22)} \right) \\
 &+ 8 \frac{1}{8} 11 \left(b_1 d^{(1/22,10/22)} + b_2 d^{(3/22,8/22)} + b_3 d^{(5/22,6/22)} + b_4 d^{(4/22,7/22)} \right. \\
 &\quad \left. + b_5 d^{(2/22,9/22)} \right).
 \end{aligned}$$

Then we have the following result:

Theorem 9. *The joint spectral measure of $\mathcal{E}_8^{\rho_1}(Sp(2))$, $\mathcal{E}_8^{\rho_2}(Sp(2))$ (over \mathbb{T}^2) is*

$$\begin{aligned}
 d\varepsilon = &\frac{1}{1936\pi^4} J_{y,z}^2 \left(d^{(1/22,2/22)} + d^{(3/22,4/22)} + d^{(1/22,6/22)} + d^{(3/22,6/22)} \right. \\
 &\quad \left. + 2d^{(2/22,7/22)} \right) \\
 &+ 11 \left(b_1 d^{(1/22,10/22)} + b_2 d^{(3/22,8/22)} + b_3 d^{(5/22,6/22)} + b_4 d^{(4/22,7/22)} \right. \\
 &\quad \left. + b_5 d^{(2/22,9/22)} \right),
 \end{aligned}$$

where b_i is the i^{th} largest root of $x^5 - 11x^4 + 44x^3 - 77x^2 + 55x - 11$, and $d^{(\theta_1, \theta_2)}$ is as in Definition 4.1.

4.8. Exceptional graph $\mathcal{E}_{12}(Sp(2))$: $(Sp(2))_{12} \rightarrow (E_8)_1$

The graphs $\mathcal{E}_{12}^{\rho_j}(Sp(2))$, illustrated in Figures 45, 46, are the nimrep graphs associated to the conformal embedding $(Sp(2))_{12} \rightarrow (E_8)_1$ and are the graphs

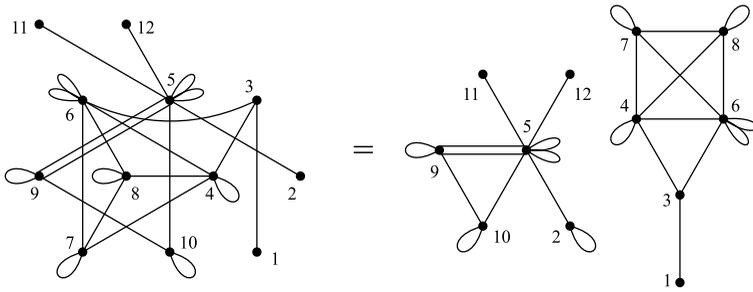


Figure 46: Graph $\mathcal{E}_{12}^{p_2}(Sp(2))$.

associated to the modular invariant

$$Z_{\mathcal{E}_{12}} = |\chi_{(0,0)} + \chi_{(6,0)} + \chi_{(8,1)} + \chi_{(2,3)} + \chi_{(8,3)} + \chi_{(6,6)} + \chi_{(2,7)} + \chi_{(0,12)} + 2\chi_{(4,4)}|^2$$

which is at level 12 and has exponents

$$\text{Exp}(\mathcal{E}_{12}(Sp(2))) = \{(0, 0), (6, 0), (8, 1), (2, 3), (8, 3), (6, 6), (2, 7), (0, 12), \\ \text{and four times } (4, 4).\}$$

The graphs $\mathcal{E}_{12}^{p_j}(Sp(2))$ are illustrated in Figures 45, 46. Note again that $\mathcal{E}_{12}^{p_2}(Sp(2))$ has two connected components.

The chiral induced sector bases ${}_M\mathcal{X}_M^\pm$ are given by

$${}_M\mathcal{X}_M^\pm = \{[\alpha_{(0,0)}], [\alpha_{(1,0)}^\pm], [\alpha_{(0,1)}^\pm], [\alpha_{(2,0)}^\pm], [\alpha_{(1,1)}^\pm], [\alpha_{(0,2)}^\pm], [\alpha_{(3,0)}^{(j)\pm}], [\alpha_{(2,1)}^{(j)\pm}], [\alpha_{(1,2)}^{(j)\pm}], \\ \text{for } j = 1, 2\},$$

where $[\alpha_{(3,0)}^\pm] = [\alpha_{(3,0)}^{(1)\pm}] \oplus [\alpha_{(3,0)}^{(2)\pm}]$, $[\alpha_{(2,1)}^\pm] = [\alpha_{(0,2)}^\pm] \oplus [\alpha_{(2,1)}^{(1)\pm}] \oplus [\alpha_{(2,1)}^{(2)\pm}]$, and $[\alpha_{(1,2)}^\pm] = [\alpha_{(1,1)}^\pm] \oplus [\alpha_{(3,0)}^{(1)\pm}] \oplus [\alpha_{(1,2)}^{(1)\pm}] \oplus [\alpha_{(1,2)}^{(2)\pm}]$.

One can in principle compute the principal graph and dual principal graph of the inclusion $(Sp(2))_{12} \rightarrow (E_8)_1$, as in Section 4.3, but we do not do that here due to their size (the principal graph for instance has 55 vertices). It is only possible to determine the edge set of the pair of vertices $[\bar{t} \circ \alpha_{(2,1)}^{(1)\pm}]$ and $[\bar{t} \circ \alpha_{(2,1)}^{(2)\pm}]$ together, but not which edges are attached to either vertex individually. However, the correct choice could be verified by the generalised Goodman-de la Harpe-Jones method referred to in Section 4.3, where the principal graph appears as the intertwining graph.

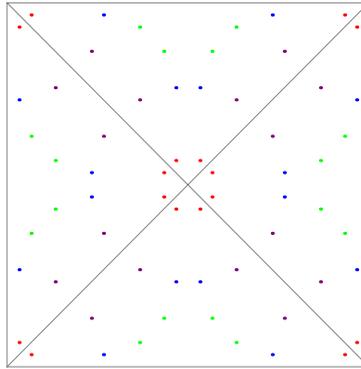


Figure 47: Orbit of (θ_1, θ_2) for $\lambda \in \text{Exp}(\mathcal{E}_{12}(Sp(2))) \setminus \{(4, 4)\}$.

The subfactor $\alpha_{(1,0)}^\pm(M) \subset M$ of index $\frac{1}{2}(10 + 3\sqrt{5} + \sqrt{75 + 30\sqrt{5}}) \approx 14.31$, where M is a type III factor, has principal graph the nimrep graph $\mathcal{E}_{12}^{\rho_1}(Sp(2))$ illustrated in Figure 45, and the dual principal graph is again isomorphic to the principal graph as abstract graphs.

We now determine the joint spectral measure of $\mathcal{E}_{12}^{\rho_1}(Sp(2))$, $\mathcal{E}_{12}^{\rho_2}(Sp(2))$. With θ_1, θ_2 as in (24) for $\lambda = (\lambda_1, \lambda_2) \in \text{Exp}(\mathcal{E}_3(Sp(2)))$, we have the following values:

$\lambda \in \text{Exp}$	$(\theta_1, \theta_2) \in [0, 1]^2$	$ \psi_*^\lambda ^2$	$\frac{1}{8\pi^2} J(\theta_1, \theta_2) $
$(0, 0), (0, 12)$	$(\frac{1}{30}, \frac{2}{30}), (\frac{13}{30}, \frac{14}{30})$	$\frac{9-\sqrt{5}-\sqrt{30+6\sqrt{5}}}{120}$	$\frac{9-\sqrt{5}-\sqrt{30+6\sqrt{5}}}{8}$
$(6, 0), (6, 6)$	$(\frac{1}{30}, \frac{8}{30}), (\frac{7}{30}, \frac{14}{30})$	$\frac{9+\sqrt{5}-\sqrt{30-6\sqrt{5}}}{120}$	$\frac{9+\sqrt{5}-\sqrt{30-6\sqrt{5}}}{8}$
$(8, 1), (8, 3)$	$(\frac{2}{30}, \frac{11}{30}), (\frac{4}{30}, \frac{13}{30})$	$\frac{9+\sqrt{5}+\sqrt{30-6\sqrt{5}}}{120}$	$\frac{9+\sqrt{5}+\sqrt{30-6\sqrt{5}}}{8}$
$(2, 3), (2, 7)$	$(\frac{4}{30}, \frac{7}{30}), (\frac{8}{30}, \frac{11}{30})$	$\frac{9-\sqrt{5}+\sqrt{30+6\sqrt{5}}}{120}$	$\frac{9-\sqrt{5}+\sqrt{30+6\sqrt{5}}}{8}$
$(4, 4)$	$(\frac{5}{30}, \frac{10}{30})$	$\frac{2}{3}$	3

where the eigenvectors ψ^λ have been normalized so that $\|\psi^\lambda\| = 1$, and for the exponent $(4, 4)$ which has multiplicity four, the value listed in the table for $|\psi_*^{(4,4)}|^2$ is $\sum_{j=1}^4 |\psi_*^{(1,1)^j}|^2$. Note that

$$(31) \quad |\psi_*^\lambda|^2 = \zeta_\lambda \frac{1}{15} \frac{1}{8\pi^2} |J|$$

where $\zeta_\lambda = 1$ for $\lambda \in \text{Exp}(\mathcal{E}_3(Sp(2))) \setminus \{(4, 4)\}$ and $\zeta_{(4,4)} = 10/3$.

The orbits under D_8 of the points $(\theta_1, \theta_2) \in \{(\frac{1}{30}, \frac{2}{30}), (\frac{13}{30}, \frac{14}{30}), (\frac{1}{30}, \frac{8}{30}), (\frac{7}{30}, \frac{14}{30}), (\frac{2}{30}, \frac{11}{30}), (\frac{4}{30}, \frac{13}{30}), (\frac{4}{30}, \frac{7}{30}), (\frac{8}{30}, \frac{11}{30})\}$ are illustrated in Figure 47. The orbits of the first four pairs of points support the measures $d^{(1/30, 2/30)}$, $d^{(1/30, 8/30)}$, $d^{(2/30, 11/30)}$ and $d^{(4/30, 7/30)}$ respectively. The orbit of the last pair has appeared before and supports the measure $|J| d_6 \times d_6$. Thus, using (31), we see that the joint spectral measure for $\mathcal{E}_{12}(Sp(2))$ is

$$d\varepsilon = 16 \frac{1}{8} \frac{1}{15} \frac{1}{8\pi^2} |J| \left(d^{(1/30, 1/15)} + d^{(1/30, 4/15)} + d^{(1/15, 11/30)} + d^{(2/15, 7/30)} \right) + 36 \frac{1}{8} \frac{10}{3} \frac{1}{15} \frac{1}{8\pi^2} |J| d_6 \times d_6.$$

Then we have obtained the following result:

Theorem 10. *The joint spectral measure of $\mathcal{E}_{12}^{\rho_1}(Sp(2))$, $\mathcal{E}_{12}^{\rho_2}(Sp(2))$ (over \mathbb{T}^2) is*

$$d\varepsilon = \frac{1}{60\pi^2} |J| \left(d^{(1/30, 1/15)} + d^{(1/30, 4/15)} + d^{(1/15, 11/30)} + d^{(2/15, 7/30)} \right) + \frac{1}{8\pi^2} |J| d_6 \times d_6,$$

where $d^{(\theta_1, \theta_2)}$ is as in Definition 4.1 and d_6 is the uniform Dirac measure on the 6th roots of unity.

References

- [1] T. BANICA AND D. BISCH, Spectral measures of small index principal graphs, *Comm. Math. Phys.* **269** (2007), 259–281. [MR2274471](#)
- [2] D. BERNARD, String characters from Kac-Moody automorphisms, *Nucl. Phys. B* **288** (1987), 628–648. [MR0892062](#)
- [3] J. BÖCKENHAUER AND D. E. EVANS, Modular invariants, graphs and α -induction for nets of subfactors. II, *Comm. Math. Phys.* **200** (1999), 57–103. [MR1671970](#)
- [4] J. BÖCKENHAUER AND D. E. EVANS, Modular invariants, graphs and α -induction for nets of subfactors. III, *Comm. Math. Phys.* **205** (1999), 183–228. [MR1706884](#)
- [5] J. BÖCKENHAUER AND D. E. EVANS, Modular invariants from subfactors: Type I coupling matrices and intermediate subfactors, *Comm. Math. Phys.* **213** (2000), 267–289. [MR1785458](#)

- [6] J. BÖCKENHAUER AND D. E. EVANS, Modular invariants and subfactors, in *Mathematical physics in mathematics and physics* (Siena, 2000), Fields Inst. Commun. **30**, 11–37, Amer. Math. Soc., Providence, RI, 2001. [MR1867545](#)
- [7] J. BÖCKENHAUER AND D. E. EVANS, Modular invariants from subfactors, in *Quantum symmetries in theoretical physics and mathematics* (Bariloche, 2000), Contemp. Math. **294**, 95–131, Amer. Math. Soc., Providence, RI, 2002. [MR1907187](#)
- [8] J. BÖCKENHAUER, D. E. EVANS AND Y. KAWAHIGASHI, On α -induction, chiral generators and modular invariants for subfactors, *Comm. Math. Phys.* **208** (1999), 429–487. [MR1729094](#)
- [9] J. BÖCKENHAUER, D. E. EVANS AND Y. KAWAHIGASHI, Chiral structure of modular invariants for subfactors, *Comm. Math. Phys.* **210** (2000), 733–784. [MR1777347](#)
- [10] P. F. BYRD AND M. D. FRIEDMAN, *Handbook of elliptic integrals for engineers and scientists*, Die Grundlehren der mathematischen Wissenschaften, **67**, Second edition, revised. Springer-Verlag, New York, 1971. [MR0277773](#)
- [11] A. CAPPELLI, C. ITZYKSON AND J.-B. ZUBER, The A - D - E classification of minimal and $A_1^{(1)}$ conformal invariant theories, *Comm. Math. Phys.* **113** (1987), 1–26. [MR0918402](#)
- [12] P. CHRISTE AND F. RAVANANI, $GN \otimes GN+L$ conformal field theories and their modular invariant partition functions, *Int. J. Mod. Phys. A* **4** (1989), 897–920. [MR1017143](#)
- [13] B. COOPER, Almost Koszul Duality and Rational Conformal Field Theory, PhD thesis, University of Bath, 2007.
- [14] R. COQUEREAUX, R. RAIS AND E. H. TAHRI, Exceptional quantum subgroups for the rank two Lie algebras B_2 and G_2 , *J. Math. Phys.* **51** (2010), 092302 (34 pages). [MR2742806](#)
- [15] R. COQUEREAUX, <http://www.cpt.univ-mrs.fr/coque/quantumfusion/FusionGraphs.html>.
- [16] C. EDIE-MICHELL, Auto-equivalences of the modular tensor categories of type A, B, C and G, *Adv. Math.* **402** (2022), 108364. [MR4401829](#)
- [17] D. E. EVANS, Fusion rules of modular invariants, *Rev. Math. Phys.* **14** (2002), 709–732. [MR1932663](#)

- [18] D. E. EVANS, Critical phenomena, modular invariants and operator algebras, in *Operator algebras and mathematical physics* (Constanța, 2001), 89–113, Theta, Bucharest, 2003. [MR2018226](#)
- [19] D. E. EVANS AND Y. KAWAHIGASHI, *Quantum symmetries on operator algebras*, Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1998. Oxford Science Publications. [MR1642584](#)
- [20] D. E. EVANS AND M. PUGH, Ocneanu Cells and Boltzmann Weights for the $SU(3)$ \mathcal{ADE} Graphs, *Münster J. Math.* **2** (2009), 95–142. [MR2545609](#)
- [21] D. E. EVANS AND M. PUGH, $SU(3)$ -Goodman-de la Harpe-Jones subfactors and the realisation of $SU(3)$ modular invariants, *Rev. Math. Phys.* **21** (2009), 877–928. [MR2553429](#)
- [22] D. E. EVANS AND M. PUGH, Spectral measures and generating series for nimrep graphs in subfactor theory, *Comm. Math. Phys.* **295** (2010), 363–413. [MR2594331](#)
- [23] D. E. EVANS AND M. PUGH, Spectral measures and generating series for nimrep graphs in subfactor theory II: $SU(3)$, *Comm. Math. Phys.* **301** (2011), 771–809. [MR2784280](#)
- [24] D. E. EVANS AND M. PUGH, The Nakayama automorphism of the almost Calabi-Yau algebras associated to $SU(3)$ modular invariants, *Comm. Math. Phys.* **312** (2012), 179–222. [MR2914061](#)
- [25] D. E. EVANS AND M. PUGH, Spectral Measures for G_2 , *Comm. Math. Phys.* **337** (2015), 1161–1197. [MR3339174](#)
- [26] D. E. EVANS AND M. PUGH, Spectral Measures for G_2 II: finite subgroups, *Rev. Math. Phys.* **32** (2020), 2050026. [MR4143016](#)
- [27] D. E. EVANS AND M. PUGH, Spectral measures associated to rank two Lie groups and finite subgroups of $GL(2, \mathbb{Z})$, *Comm. Math. Phys.* **343** (2016), 811–850. [MR3488546](#)
- [28] D. E. EVANS AND M. PUGH, Classification of module categories for $SO(3)_{2m}$, *Adv. Math.* **384** (2021), 107713. [MR4237416](#)
- [29] T. GANNON, Algorithms for affine Kac-Moody algebras. [arXiv:hep-th/0106123](#).
- [30] T. GANNON, Exotic quantum subgroups and extensions of affine Lie algebra VOAs – part I. [arXiv:2301.07287](#) [math.QA]

- [31] F. M. GOODMAN, P. DE LA HARPE AND V. F. R. JONES, *Coxeter graphs and towers of algebras*, MSRI Publications, **14**. Springer-Verlag, New York, 1989. [MR0999799](#)
- [32] G. KUPERBERG, Spiders for rank 2 Lie algebras, *Comm. Math. Phys.* **180** (1996), 109–151. [MR1403861](#)
- [33] A. OCNEANU, Paths on Coxeter diagrams: from Platonic solids and singularities to minimal models and subfactors. (Notes recorded by S. Goto), in *Lectures on operator theory*, (ed. B. V. RAJARAMA BHAT et al.), The Fields Institute Monographs, 243–323, Amer. Math. Soc., Providence, R.I., 2000.
- [34] A. OCNEANU, Higher Coxeter Systems (2000). Talk given at MSRI. <http://www.msri.org/publications/ln/msri/2000/subfactors/ocneanu>.
- [35] A. OCNEANU, The classification of subgroups of quantum $SU(N)$, in *Quantum symmetries in theoretical physics and mathematics* (Bariloche, 2000), Contemp. Math. **294**, 133–159, Amer. Math. Soc., Providence, RI, 2002. [MR1907188](#)
- [36] T. OHTSUKI AND S. YAMADA, Quantum $SU(3)$ invariant of 3-manifolds via linear skein theory, *J. Knot Theory Ramifications* **6** (1997), 373–404. [MR1457194](#)
- [37] K.-H. REHREN, Space-time fields and exchange fields, *Comm. Math. Phys.* **132** (1990), 461–483. [MR1069831](#)
- [38] A. SCHOPIERAY, Level bounds for exceptional quantum subgroups in rank two, *Int. Journal Math.* **29**, 1850034, 2018. [MR3808050](#)
- [39] L. C. SUCIU, The $SU(3)$ Wire Model, PhD thesis, The Pennsylvania State University, 1997. [MR2696263](#)
- [40] V. TURAEV, *Quantum invariants of knots and 3-manifolds*, Second revised edition, de Gruyter Studies in Mathematics, **18**. Walter de Gruyter & Co., Berlin, 2010. [MR2654259](#)
- [41] S. UHLMANN, R. MEINEL AND A. WIPF, Ward identities for invariant group integrals, *J. Phys. A* **40** (2007), 4367–4389. [MR2316711](#)
- [42] D. VERSTEGEN, New exceptional modular invariant partition functions for simple Kac-Moody algebras, *Nuclear Phys. B* **346** (1990), 349–386. [MR1080698](#)
- [43] H. WENZL, On sequences of projections, *C. R. Math. Rep. Acad. Sci. Canada* **9** (1987), 5–9. [MR0873400](#)

- [44] F. XU, New braided endomorphisms from conformal inclusions, *Comm. Math. Phys.* **192** (1998), 349–403. [MR1617550](#)
- [45] S. YAMAGAMI, A categorical and diagrammatical approach to Temperley-Lieb algebras. [arXiv:math/0405267](#) [math.QA].

David Evans
School of Mathematics
Cardiff University
Senghennydd Road
Cardiff CF24 4AG, Wales
U.K.

Mathew Pugh
School of Mathematics
Cardiff University
Senghennydd Road
Cardiff CF24 4AG, Wales
U.K.