Pure and Applied Mathematics Quarterly
Volume 19, Number 5, 2525–2536, 2023

On the paving size of a subfactor*

Sorin Popa

In memory of Vaughan Jones and Mihai Pimsner

Abstract: Given an inclusion of II_1 factors $N \subset M$ with finite Jones index, $[M : N] < \infty$, we prove that for any $F \subset M$ finite and $\varepsilon > 0$, there exists a partition of 1 with $r \leq \lceil 16\varepsilon^{-2}\rceil \cdot \lceil 4[M : N]\varepsilon^{-2}\rceil$ projections $p_1, \ldots, p_r \in N$ such that $\| \sum_{i=1}^{r} p_i x p_i - E_{N' \cap M}(x)\| \leq \varepsilon \|x - E_{N' \cap M}(x)\|$, $\forall x \in F$ (where $\lceil \beta \rceil$ denotes the least integer $\geq \beta$). We consider a series of related invariants for $N \subset M$, generically called *paving size*.

Introduction

A result in [P97] shows that an inclusion of separable II_1 factors $N \subset M$ has the so-called *relative Dixmier property*, $\overline{\mathrm{co}}\{uxu^* \mid u \in \mathcal{U}(N)\} \cap N' \cap M \neq \emptyset$ (where the closure is here in operator norm), for all $x \in M$, if and only if its Jones index is finite, $[M : N] < \infty$.

Thus, if $[M : N] < \infty$ then given any $x \in M$ and any $\varepsilon > 0$, there exist unitary elements $u_1, \ldots, u_n \in \mathcal{U}(N)$ such that $\| \frac{1}{n} \sum_{i=1}^{n} u_i x u_i^* - E_{N' \cap M}(x)\| \leq \varepsilon$. Using this recursively, it follows that if $[M : N] < \infty$ then for any $F \subset M$ finite and any $\varepsilon > 0$ there exist $v_1, \ldots, v_m \in \mathcal{U}(N)$ such that $\| \frac{1}{m} \sum_{i=1}^{m} v_i x v_i^* - E_{N' \cap M}(x)\| \leq \varepsilon$, $\forall x \in F$.

We attempt to identify in this paper the optimal number n of unitaries necessary to "ε-flatten" this way an element x (more generally a finite set F), exploring its dependence on ε and on $[M : N]$. Our main result establishes an upper bound of magnitude $n \leq 64[M : N]\varepsilon^{-4}$, valid for any finite set $F \subset (M)_1$, arbitrarily large.

The corresponding n unitaries $u_1, \ldots, u_n \in N$ that we construct are in fact powers $v^k, 0 \leq k \leq n-1$, of a period n unitary element $v \in \mathcal{U}(N)$. Since an averaging by such $\{v^k\}_k$ satisfies $\frac{1}{n} \sum_{k=0}^{n-1} v^k x v^{-k} = \sum_{i=1}^{n} p_i x p_i$, where $p_i \in \mathcal{P}(N)$ is a partition of 1 with spectral projections of v, our result gives

Received October 9, 2022.
*Supported in part by NSF Grant DMS-1955812 and the Takesaki Endowed Chair at UCLA.

also an upper bound for the minimal size of a partition of 1 with projections $p_1, \dots, p_n \in N$ with the property that $\|\sum_{i=1}^{n} p_i x p_i - E_{N'\cap M}(x)\| \leq \varepsilon$, $\forall x \in F$. More precisely we get the following:

Theorem. *Let $N \subset M$ be an inclusion of II_1 factors with finite Jones index, $[M:N] < \infty$. For any $F \subset M$ finite and any $\varepsilon > 0$, there exists a partition of 1 with $r \leq \lceil 16\varepsilon^{-2}\rceil \cdot \lceil 4[M:N]\varepsilon^{-2}\rceil$ projections $e_1, \dots, e_r$ in N such that*

$$\left\|\sum_{i=1}^{r} e_i x e_i - E_{N'\cap M}(x)\right\| \leq \varepsilon \|x - E_{N'\cap M}(x)\|, \quad \forall x \in F.$$

If $x \in M$ has zero expectation onto $N' \cap M$, then an expression of the form $\sum_i p_i x p_i$, with $p_i \in \mathcal{P}(N)$ a partition of 1 with projections in N that diminishes to ε the operator norm of x is called an *ε-paving* of x over N. Taking minimal size n of partitions that can ε-pave a given $x \in M$ (or $F \subset M$), then the supremum of such n over all $x \in (M)_1$ (or over all $F \subset (M)_1$ finite), gives numerical invariants for $N \subset M$ that we generically call *paving size* of $N \subset M$. The above result gives the upper bound $64[M:N]\varepsilon^{-4}$ for all such invariants. Their exact calculation is an interesting problem. We comment on this and other related questions in Section 2 of the paper (see the definitions, remarks and Corollary 2.5 in that section). This includes a discussion of the L^2-version of paving size invariants, in Remark 2.9.

To prove the above result we first use (Theorem in [P92]) to obtain a partition of 1 with $n \leq 16\varepsilon^{-2}$ projections $f_i = (f_{i,m})_m \in N^\omega$ (where N^ω is the ultrapower of N with respect to some non-principal ultrafilter on $\mathbb{N}$) such that $\{f_i\}_i$ is free independent to the given finite set $F \subset M \ominus (N'\cap M)$. By (3.5 in [PV15]), this implies $\|\sum_{i=1}^{n} f_i x f_i\| \leq \varepsilon/2$, $\forall x \in F$, and so for any $\delta > 0$, which one can take arbitrarily small independently of any other constants involved ($\delta < \varepsilon^2/(4[M:N]|F|)^2$ will do), there is m large enough such that $\|(\sum_i f_{i,m} x f_{i,m})(1-q_x)\| \leq \varepsilon/2+\delta$, where $q_x \in \mathcal{P}(M)$ are projections of trace $\leq \delta$, $\forall x \in F$. Due to the finiteness of Jones' basic construction algebra $\langle M, e_N\rangle$ [J82], $E_N(q_x)$ have supports $s(E_N(q_x))$ of trace $\leq [M:N]\tau(q) \leq [M:N]\delta$, so they are all supported by a projection $p = \vee_{x\in F} s(E_N(q_x))$ of trace $\leq [M:N]|F|\delta$, that's still very small. This leaves room to flatten p by a partition in N with $\leq 4[M:N]\varepsilon^{-2}$ many projections, to make it $\leq \varepsilon^2/4[M:N]$ in norm. Combining the two partitions, and using a key trick from (page 147 of [P98]), relying on the [PP83]-inequality $E_N(x) \geq [M:N]^{-1}x, \forall x \in M_+$, we deduce that this final partition, which has $\leq (16\varepsilon^{-2})(4[M:N]\varepsilon^{-2})$ many projections, paves all $x \in F$ to $\varepsilon/2 + \varepsilon/2 = \varepsilon$.

1. Proof of the Theorem

For notations and terminology used hereafter we send the reader to [P13, AP17], for basics in II_1 factors to [AP17], for subfactor theory to [J82].

We first recall a Kesten-type norm estimate from [PV15]:

Lemma 1.1. *Let P be a II_1 factor, $F = F^* \subset (P)_1$ a self-adjoint set of trace 0 contractions and $n \geq 1$. Assume $v \in P$ is a unitary element with $v^n = 1$, $\tau(v^k) = 0$, $1 \leq k < n$, such that $\{v\}''$ is free independent to $F \cup F^*$, i.e., $\tau(x_0 \Pi_{i=1}^m v^{k_i} x_i) = 0$ for all $m \geq 1$, $x_1, \ldots, x_{m-1} \in F$, $x_0, x_m \in F \cup \{1\}$, and $1 \leq k_1, k_2, \ldots, k_m \leq n-1$. Then $\|\frac{1}{n}\sum_{k=1}^n v^k x v^{-k}\| \leq 2\sqrt{n-1}/n$, $\forall x \in F$. Equivalently, if $p_1, \ldots, p_n$ denote the minimal projections in $\{v\}'' \simeq L(\mathbb{Z}/n\mathbb{Z})$, then $\|\sum_{k=1}^n p_k x p_k\| \leq 2\sqrt{n-1}/n$, $\forall x \in F$.*

Proof. The freeness condition between the set F and the algebra $\{v\}''$ implies that for any $x \in F$ the set $\{v^k x v^{-k} \mid 0 \leq k \leq n-1\}$ is L-free in the sense of (Definition 3.1 in [PV15]). Thus, by (Corollary 3.5 in [PV15]), we have $\|\sum_{k=1}^n v^{k-1} x v^{-k+1}\| \leq 2\sqrt{n-1}$. The proof in [PV15] is based on (Proposition 3.4 in [PV15]), which shows that any L-free set of contractions $\{x_1, \ldots, x_n\} \subset N$ can be dilated to an L-free set of unitaries $\{U_1, \ldots, U_n\}$ in a larger II_1 factor $\tilde{N} \supset N$. Thus, one has $\|\sum_{i=1}^n x_i\| \leq \|\sum_{i=1}^n U_i\|$. But the L-free condition for a set of unitaries $\{U_1, \ldots, U_n\}$ amounts to $\{U_1^* U_i\}_{i=2}^n$ being free independent Haar unitaries, for which one has $\|1 + \sum_{i=2}^n U_1^* U_i\| = 2\sqrt{n-1}$ by Kesten's Theorem [K59]. When applied to the L-free set $x_k = v^{k-1} x v^{-k+1}$, $1 \leq k \leq n$, this entails

$$\left\|\sum_{k=1}^n v^{k-1} x v^{-k+1}\right\| \leq \left\|\sum_{i=1}^n U_i\right\| = \left\|1 + \sum_{i=2}^n U_1^* U_i\right\| = 2\sqrt{n-1}. \qquad \square$$

Lemma 1.2. *Let $N \subset M$ be an inclusion of II_1 factors with $\dim(N' \cap M) < \infty$, $F \subset (M)_1$ a finite set of elements with 0 expectation onto $N' \cap M$ and $n \geq 1$. Given any $\delta > 0$ there exists a partition of 1 with projections $p_1, \ldots, p_n \in N$ and projections $q_i \in p_i M p_i$ of trace $\tau(q_i) \leq \delta$, $1 \leq i \leq n$, such that $\|p_i x (p_i - q_i)\| \leq 2\sqrt{n-1}/n + \delta$, $\forall x \in F$, $1 \leq i \leq n$.*

Proof. Let ω be a non-principal ultrafilter on $\mathbb{N}$. By (Theorem [P92]; see also Theorem 0.1 in [P13]), there exists $v \in \mathcal{U}(N^\omega)$ such that $v^n = 1$, $\tau(v^k) = 0$, $1 \leq k < n$, and such that the algebra $\{v\}''$ is free independent to $F \cup F^*$. If $f_1, \ldots, f_n \in \mathcal{P}(N^\omega)$ are the minimal projection of $\{v\}''$, then by Lemma 1.1 we have $\|f_i x f_i\| \leq 2\sqrt{n-1}/n$, $\forall x \in F$, $1 \leq i \leq n$.

Let $f_i = (f_{i,m})_m$ with $f_{i,m} \in \mathcal{P}(N)$ and $\sum_i f_{i,m} = 1$, $\forall m$. Since F is finite, given any $\delta' > 0$ there exists m large enough such that the spectral projection $e_{x,i}$ of $(f_{i,m}xf_{i,m})^*(f_{i,m}xf_{i,m})$ corresponding to the interval $[4(n-1)/n^2+\delta', \infty)$ has trace satisfying $\tau(e_{x,i}) \leq \delta'$. Thus, if δ' is sufficiently small then the projection $q_i = \vee_{x\in F} e_{x,i} \in \mathcal{P}(M)$, which has trace majorized by $\sum_{x\in F} \tau(e_{x,i}) \leq \delta'|F|$, satisfies $\tau(q_i) \leq \delta$.

It follows that if we let $p_i = f_{i,m}$ and $q_i = \vee_{x\in F} e_{x,i}$, then $\tau(q_i) = \sum_{x\in F} \tau(e_{x,i}) \leq \delta$, $q_i \leq p_i$ and for each $x \in F$, $1 \leq i \leq n$ we have the norm estimate

$$\begin{aligned}\|p_i x(p_i - q_i)\| &= \left\|(p_i - q_i)x^* p_i x(p_i - q_i)\right\|^{1/2} \\ &\leq \left\|(p_i - e_{x,i})x^* p_i x(p_i - e_{x,i})\right\|^{1/2} \\ &= \left\|\big(f_{i,m}x^* f_{i,m} x f_{i,m}\big)\big(f_{i,m} - e_{x,i}\big)\right\|^{1/2} \leq 2\sqrt{n-1}/n + \delta. \quad \square\end{aligned}$$

Lemma 1.3. *Let N be a II_1 factor. For $b \geq 0$ in N, denote by $s(b)$ its support projection. Let $F \subset N$ be a finite set and $0 < \varepsilon \leq 1/2$. Assume $2(\sum_{x\in F} \tau(s(|x|))) < \varepsilon(\max\{\|x\| \mid x \in F\})^{-1}$. Let m denote the least integer greater than or equal to $\varepsilon^{-1}\max\{\|x\| \mid x \in F\}$. Then there exists a partition of 1 with m projections $q_1, \ldots, q_m \in N$ such that $\|\sum_{j=1}^m q_j x q_j\| \leq \varepsilon$.*

Proof. Let $e = \vee_{x\in F}(l(x) \vee r(x))$, where $l(x), r(x)$ denote the left and respectively right support projections of x.

The condition $2(\sum_{x\in F} \tau(s(|x|))) \leq \varepsilon(\max\{\|x\| \mid x \in F\})^{-1}$ together with the condition m satisfies, imply that there exists a partition of 1 with projections $e_1, \ldots, e_m \in N$ of trace $1/m$ such that $e \leq e_1$. Let $v \in \mathcal{U}(N)$ be a unitary element satisfying $v^m = 1$ and $v^{k-1}e_1 v^{-k+1} = e_k$, $1 \leq k \leq m$.

Let $q_1, \ldots, q_m$ denote the minimal projections of the abelian m-dimensional von Neumann algebra $\{v\}''$, with $v = \sum_{k=1}^m \alpha^{k-1} q_k$, $\alpha = \exp(2\pi i/n)$. Since all $x \in F$ are supported on e and $v^k e v^{-k}$ are mutually disjoint, it follows that $\|\frac{1}{m}\sum_k v^k x v^{-k}\| \leq \|x\|/m$, $\forall x \in F$, which by the given conditions gives $\|\frac{1}{m}\sum_k v^k x v^{-k}\| \leq \varepsilon$, $\forall x \in F$. But $\frac{1}{m}\sum_{k=0}^{m-1} v^k x v^{-k} = \sum_{k=1}^m q_k x q_k$. $\square$

Lemma 1.4. *Let $N \subset M$ be a an inclusion of II_1 factors with finite Jones index, $[M : N] < \infty$. If $q \in M$ is a projection then $\tau(s(E_N(q))) \leq \tau(q)[M : N]$.*

Proof. Let $M \subset M_1 := \langle M, e_N\rangle$ be the basic construction for $N \subset M$, with $e_N \in M_1$ denoting as usual the corresponding Jones projection. Thus, $M_1 = \mathrm{sp} M e_N M$, $[N, e_N] = 0$, $e_N x e_N = E_N(x)e_N$ and $\tau(e_N x) = \lambda\tau(x)$, $\forall x \in$, where $\lambda = [M : N]^{-1}$.

If $q \in M$ is a projection, then one has $e_N q e_N = E_N(q)e_N$. Thus, $s(e_N q e_N) = s(E_N(q))e_N$ with its trace being equal to $\lambda\tau(s(E_N(q)))$. This implies that

$$\tau(q) \geq \tau\big(s(q e_N q)\big) = \tau\big(s(e_N q e_N)\big) = \lambda\tau\big(s(E_N(q)\big),$$

and thus $\tau(s(E_N(q)) \leq \lambda^{-1}\tau(q) = [M : N]\tau(q)$. $\square$

Proof of the Theorem. Replacing F by $\{x - E_{N'\cap M}(x)/\|x - E_{N'\cap M}(x)\| \mid x \in F \setminus N' \cap M\}$, we may assume $F \subset (M \ominus N' \cap M)_1$. By Lemma 1.2, for any given integer n and any $\delta' > 0$, there exists a partition of 1 with projections $p_1, \dots, p_n$ in N of trace $1/n$ such that for each $1 \leq i \leq n$ we have a projection $q_i \in p_i M p_i$ satisfying $\tau(q_i) \leq \delta'$ and

$$\|p_i x(p_i - q_i)\| \leq \left(4(n-1)/n^2 + \delta'\right)^{1/2}, \quad \forall x \in F. \tag{1}$$

If we denote $b_{i,x} = q_i x^* p_i x q_i \in p_i M p_i$, $x \in F, 1 \leq i \leq n$, then $b_{i,x} \in (p_i M p_i)_1$ are positive elements of support $\leq q_i$. It follows that $0 \leq E_N(b_{i,x}) \leq p_i$ and by Lemma 1.4, its support has trace $\tau(s(E_N(b_{i,x}))) \leq [M : N]\tau(q_i)$.

By Lemma 1.3, given any integer $m \leq \tau(p_i)/\tau(q_i)$, there exists a partition of p_i with m projections $q_1^i, \dots, q_m^i \in \mathcal{P}(p_i N p_i)$ of trace $\tau(p_i)/m$, such that

$$\left\|\sum_{j=1}^{m} q_j^i E_N(b_{i,x}) q_j^i\right\| \leq 1/m. \tag{2}$$

Since by (Theorem 2.1 in [PP83]) we have $b \leq [M : N]E_N(b)$ for any $b \in M_+$, it follows that

$$\left\|\sum_j q_j^i b_{i,x} q_j^i\right\| \leq [M : N]\left\|\sum_{j=1}^{m} q_j^i E_N(b_{i,x}) q_j^i\right\| \leq [M : N]/m. \tag{3}$$

But since $\phi_i : p_i M p_i \to p_i M p_i$ defined by $\Phi_i(y) = \sum_j q_j^i y q_j^i$, $y \in p_i M p_i$, is unital completely positive, by Kadison's inequality we have $\phi_i(y^*)\phi_i(y) \leq \phi_i(y^* y)$, $\forall y \in p_i M p_i$. Applying this to $y = p_i x q_i$ and using (3) it follows that for each $x \in F$ and $1 \leq i \leq n$ we have

$$\begin{aligned}\left\|\sum_j q_j^i (p_i x q_i) q_j^i\right\| &\leq \left\|\sum_j q_j^i (q_i x^* p_i x q_i) q_j^i\right\|^{1/2} \\ &= \left\|\sum_j q_j^i b_{i,x} q_j^i\right\|^{1/2} \leq \left([M : N]/m\right)^{1/2}.\end{aligned} \tag{4}$$

Also, since ϕ_i are contractive, by (1) we have for each i and $x \in F$ the estimate

$$\left\|\sum_j q_j^i (p_i x (p_i - q_i)) q_j^i\right\| \leq \|p_i x (p_i - q_i)\| \leq \left(4(n-1)/n^2 + \delta'\right)^{1/2}. \tag{5}$$

This implies that the partition of 1 with $r = nm$ projections $\{e_k\}_{k=1}^r = \{q_j^i \mid 1 \le i \le n, 1 \le j \le m\}$, which refines $\{p_i\}_i$, satisfies for all $x \in F$ the inequalities

$$\begin{aligned}
(6) \qquad \Big\|\sum_k e_k x e_k\Big\| &\le \Big\|\sum_{i,j} q_j^i(p_i x(p_i - q_i))q_j^i\Big\| + \Big\|\sum_{i,j} q_j^i(p_i x q_i)q_j^i\Big\| \\
&\le \max_i \|p_i x(p_i - q_i)\| + \max_i \Big\|\sum_j q_j^i(p_i x q_i)q_j^i\Big\| \\
&\le \left(4(n-1)/n^2 + \delta'\right)^{1/2} + \left([M:N]/m\right)^{1/2}.
\end{aligned}$$

If we now take $\delta' < 4/n^2$ and the integers n, m so that $m \ge 4[M:N]\varepsilon^{-2}$, $n \ge 16\varepsilon^{-2}$, then $(4(n-1)/n^2 + \delta')^{1/2} + ([M:N]/m)^{1/2} \le \varepsilon/2 + \varepsilon/2 = \varepsilon$, ending the proof of the Theorem. □

2. Further remarks

Definition 2.1. If $N \subset M$ is an inclusion of II_1 factors with finite index, then for any $F \subset M$ non-empty and $\varepsilon > 0$ we denote by $\mathrm{n}(N \subset M; F, \varepsilon)$ the infimum over all n for which there exists a partition of 1 with projections $p_1, \dots, p_n \in N$ such that $\|\sum_{i=1}^n p_i x p_i - E_{N'\cap M}(x)\| \le \varepsilon \|x - E_{N'\cap M}(x)\|$, $\forall x \in F$, with the usual convention that this infimum is equal to ∞ if there exists no such finite partition. We call $\mathrm{n}(N \subset M; F, \varepsilon) \in \mathbb{N} \cup \{\infty\}$ the *ε-paving size of F in $N \subset M$.*

Definition 2.2. For each $k = 1, 2, \dots,$ we denote $\mathrm{n}_k(N \subset M; \varepsilon) = \sup\{\mathrm{n}(N \subset M; F, \varepsilon) \mid F \subset M_h, |F| \le k\}$, where $M_h = \{x \in M \mid x = x^*\}$. We also denote $\mathrm{n}_\infty(N \subset M; \varepsilon) = \sup\{\mathrm{n}(N \subset M; F, \varepsilon) \mid \emptyset \ne F \subset M \text{ finite}\}$. These numbers are obviously isomorphism invariants for $N \subset M$ and we generically refer to them as *paving size* of $N \subset M$.

Specifically, $\mathrm{n}(N \subset M; \varepsilon) = \mathrm{n}_1(N \subset M; \varepsilon)$ is called the *ε-paving size of $N \subset M$* and for each $2 \le k \le \infty$, $\mathrm{n}_k(N \subset M; \varepsilon)$ is called *(ε, k)-paving size of $N \subset M$.*

Note that these quantities are increasing in k, with $\sup_{k \ge 1} \mathrm{n}_k(N \subset M; \varepsilon) = \mathrm{n}_\infty(N \subset M; \varepsilon)$. So by the Theorem they are all bounded by an order of magnitude $64[M:N]\varepsilon^{-4}$. Also, if $N \subset P \subset M$ is an intermediate subfactor, then $\mathrm{n}_k(N \subset P; \varepsilon) \le \mathrm{n}_k(N \subset M; \varepsilon)$, $\forall 1 \le k \le \infty$.

This terminology and notations are inspired by the similar ones used for MASAs (maximal abelian *-subalgebras) in factors, $A \subset M$, in relation to the Kadison-Singer type problems (see e.g., [PV15]). Notably, the term "paving" was coined in relation with the Kadison-Singer problem and seems suitable for these quantities.

Note that if $p_1, \dots, p_n \in N$ is a partition of 1 with projections and we denote $v = \sum_{k=1}^n \alpha^{k-1} p_k$, where $\alpha = \exp(2\pi i/n)$, then for any $x \in M$ we have $\sum_{k=1}^n p_k x p_k = \frac{1}{n}\sum_{k=0}^{n-1} v^k x v^{-k}$. Thus, any "paving" of $x \in M$ with n-projections in a subfactor N of M (or in a MASA A of M) can be viewed as a "Dixmier averaging" of x by n-unitaries in N (resp. A).

Definition 2.3. In the same spirit as the pavings, for an inclusion of factors $N \subset M$, a finite set $\emptyset \neq F \subset M$ and $\varepsilon > 0$, we define the quantity $\mathrm{D}(N \subset M; F, \varepsilon)$ to be the infimum over all n for which there exist $u_1, \dots, u_n \in \mathcal{U}(N)$ such that $\|\frac{1}{n}\sum_{i=1}^n u_i x u_i^* - E_{N'\cap M}(x)\| \leq \varepsilon \|x - E_{N'\cap M}\|$, $\forall x \in F$. Then similarly to the above notations, we let $\mathrm{D}_\infty(N \subset M; \varepsilon) = \sup\{\mathrm{D}(N \subset M; F, \varepsilon) \mid \emptyset \neq F \subset M \text{ finite}\}$, $\mathrm{D}_k(N \subset M; \varepsilon) = \sup\{\mathrm{D}(N \subset M; F, \varepsilon) \mid \emptyset \neq F \subset M_h, |F| \leq k\}$, for $1 \leq k < \infty$.

We clearly have $\mathrm{D}(N \subset M; F, \varepsilon) \leq \mathrm{n}(N \subset M; F, \varepsilon)$, for any finite $F \subset M$. Also, $\mathrm{D}_k(N \subset M; \varepsilon) \leq \mathrm{n}_k(N \subset M; \varepsilon)$, for any $1 \leq k \leq \infty$. So the Theorem implies that for any subfactor of finite index $N \subset M$, these quantities are all finite, in fact bounded by the order of magnitude $64[M : N]\varepsilon^{-4}$. Like the $\mathrm{n}_*(N \subset M; \varepsilon)$-quantities, they are all isomorphism invariants for $N \subset M$. We'll still view them as *paving*-invariants for $N \subset M$, but with respect to averaging by unitaries, rather than by projections summing up to 1. Alternatively, we view them as *optimal Dixmier averaging numbers for $N \subset M$*.

In particular, for a single II_1 factor N and $1 \leq k \leq \infty$ we have $\mathrm{n}_k(N; \varepsilon) \overset{\mathsf{def}}{=} \mathrm{n}_k(N \subset N; \varepsilon) \leq 64\varepsilon^{-4}$. Consequently, $\mathrm{D}_k(N; \varepsilon) \overset{\mathsf{def}}{=} \mathrm{D}_k(N \subset N; \varepsilon) \leq 64\varepsilon^{-4}$ as well.

Dixmier's classical *averaging theorem* (see Ch. III, Sec. 5 in [D57]) amounts to $\mathrm{D}(N; \varepsilon) := \mathrm{D}_1(N; \varepsilon) < \infty$. His proof actually shows that $\mathrm{D}(N; \varepsilon) \leq \lceil \varepsilon^{-c} \rceil$, where $c = \log_{3/2} 2 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.7095 < 2$. If F is a finite set of k selfadjoint elements, then by applying consecutively Dixmier's theorem k many times, one obtains the estimate $\mathrm{D}_k(N; \varepsilon) \leq \lceil \varepsilon^{-c} \rceil^k$, which thus depends on k and gives no bound for $\mathrm{D}_\infty(N; \varepsilon)$. So Dixmier's proof gives better upper bounds for $\mathrm{D}_k(N; \varepsilon)$ if $k = 1, 2$, but a (exponentially) worse bound for $k \geq 3$, with no bound for $k = \infty$.

It would be interesting to improve the upper bound for the paving size $\mathrm{n}_k(N \subset M; \varepsilon)$, especially for $k = 1, k = \infty$, as well as for the constants $\mathrm{D}_k(N \subset M; \varepsilon)$. In particular, to determine if the order of magnitude ε^{-4} is optimal or can be lowered. Equally interesting would be to obtain some sharp lower bounds. Ideally, one would like to have exact calculation of $\mathrm{n}_*(N \subset$

$M;\varepsilon)$ or $\mathrm{D}_*(N \subset M;\varepsilon)$, for some concrete subfactors $N \subset M$ of finite index. This seems quite challenging even for $N = M$!

Another interesting problem is to determine whether these invariants only depend on the index $[M : N]$ (respectively, only on the standard invariant $\mathcal{G}_{N\subset M}$).

One can provide a (rather weak!) estimate for the lower bound of the paving size constants from the following simple observation for single II_1 factors:

Lemma 2.4. *Let N be a II_1 factor. Let $x \in N_+$ be so that $\|x\| = 1$. If $u_1, \dots, u_n \in \mathcal{U}(N)$ are so that $\|\frac{1}{n}\sum_i u_i x u_i^* - \tau(x)1\| \leq \varepsilon$, then $n \geq (\tau(x)+\varepsilon)^{-1}$. In particular, if $x = q \in \mathcal{P}(N)$ is a non-zero projection, then $\mathrm{D}(N; q, \varepsilon) \geq (\tau(q)+\varepsilon)^{-1}$.*

Proof. Since $\|\frac{1}{n}\sum_i u_i x u_i^* - \tau(x)1\| \leq \varepsilon$ and $x \geq 0$, we have $(\tau(x)+\varepsilon)1 \geq \frac{1}{n}\sum_i u_i x u_i^* \geq \frac{1}{n} u_1 x u_1^*$, so by taking norms we get $(\tau(x)+\varepsilon) \geq \|\frac{1}{n}\sum_i u_i x u_i^*\| \geq \frac{1}{n}\|x\| = \frac{1}{n}$, implying that $n \geq (\tau(x)+\varepsilon)^{-1}$. ☐

Taking $\tau(q) \to 0$ in Lemma 2.4 we get the lower bound ε^{-1} for the paving size of a single II_1 factor, and hence for any inclusion of II_1 factors. Combining with the Theorem and the above remarks, we thus get:

Corollary 2.5. *If $N \subset M$ is an inclusion of II_1 factors with finite Jones index then, with the above notations, we have for any $\varepsilon > 0$ the estimates*

$$\begin{aligned}\varepsilon^{-1} &\leq \mathrm{D}(N \subset M;\varepsilon) \leq \mathrm{n}(N \subset M;\varepsilon) \leq \mathrm{n}_\infty(N \subset M;\varepsilon)\\ &\leq \lceil 16\varepsilon^{-2}\rceil \cdot \lceil 4[M:N]\varepsilon^{-2}\rceil.\end{aligned}$$

The invariants $\mathrm{n}_*(N \subset M;\varepsilon)$, $\mathrm{D}_*(N \subset M;\varepsilon)$ can also be viewed as measuring how efficient one can "flatten" the elements in M_+ by averaging/paving with unitaries (or partitions with projections) in N. Two other quantities that measure such phenomena are the following:

Definition 2.6. Let $N \subset M$ be an inclusion of II_1 factors with finite index. Recall from (Corollary 3.1.9 in [J82]) that there exist projections $e \in M$ satisfying $E_N(e) = [M : N]^{-1}1$ and that by (Corollary 1.8 in [PP83]) any two such projections are conjugate by a unitary in N. Thus, the quantity $\mathrm{d}(N \subset M;\varepsilon) \overset{\mathsf{def}}{=} \mathrm{n}(N \subset M; e,\varepsilon)$, where e is such a "Jones projection", is well defined and it is obviously an isomorphism invariant for $N \subset M$. One clearly has $\mathrm{d}(N \subset M;\varepsilon) \leq \mathrm{n}(N \subset M;\varepsilon)$.

In a related vein, we define the invariant $\mathrm{d}_{\mathrm{ob}}(N \subset M)$ for a subfactor of finite index $N \subset M$ as the infimum of $\|\sum_j m_j^* m_j\|$ over all orthonormal basis $\{m_j\}_j$ of $N \subset M$ (as defined in Section 1 of [PP83]).

Since for any orthonormal basis $\{m_j\}_j$ one has $\lambda \sum_j m_j m_j^* = 1$, where $\lambda = [M:N]^{-1}$, (cf. Proposition 1.3 in [PP83]), it follows that $1 = \tau(\lambda \sum_j m_j m_j^*) = \lambda\tau(\sum_j m_j^* m_j)$, hence $\| \sum_j m_j^* m_j\| \geq \lambda^{-1} = [M:N]$. Thus, one has $\mathrm{d}_{\mathrm{ob}}(N \subset M) \geq [M:N]$. On the other hand, one can take the orthonormal basis $\{m_j\}_j$ so that $m_1 = 1$ and so that for all but possibly one m_j to have $E_N(m_j^* m_j) = 1$, which by (Proposition 2.1 in [PP83]) implies $m_j^* m_j \leq [M:N]1$. Thus $\| \sum_j m_j^* m_j\| \leq \sum_j \|m_j^* m_j\| \leq 1 + [M:N](\lceil [M:N]\rceil - 1)$.

We have thus proved the following

Proposition 2.7. *If $N \subset M$ is an inclusion of* II_1 *factors with finite Jones index then, with the above notations, we have the estimates*

$$[M:N] \leq \mathrm{d}_{\mathrm{ob}}(N \subset M) \leq 1 + [M:N](\lceil [M:N]\rceil - 1).$$

Remark 2.8. The paving size invariants can be defined for an arbitrary inclusion of factors (not necessarily II_1), $\mathcal{N} \subset \mathcal{M}$, with exactly same formal definitions. If one has an expectation $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ with finite Pimsner-Popa index, i.e., if $\mathcal{E}(x) \geq \lambda x$, $\forall x \in \mathcal{M}_+$, for some $\lambda > 0$, and one denotes by $\mathrm{Ind}(\mathcal{E})$ the inverse λ^{-1} of the best constant λ satisfying the inequality, then the main result in [P97] shows that $\mathrm{Ind}(\mathcal{E}) < \infty$ implies $\mathrm{n}(\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}; F, \varepsilon) < \infty$, for any finite set $F \subset \mathcal{M}$. We leave it to the interested reader to adapt the proof of the Theorem in this paper, combined with the proof of the relative Dixmier property for inclusions of properly infinite factors $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ with $\mathrm{Ind}(\mathcal{E}) < \infty$ in (Section 3 of [P97]), to get estimates for $\mathrm{n}_*(\mathcal{E}; \varepsilon)$, $\mathrm{D}_*(\mathcal{E}; \varepsilon)$.

Remark 2.9. One can consider exactly the same type of definitions as we did for $\mathrm{n}_*(N \subset M; \varepsilon)$, $\mathrm{D}_*(N \subset M; \varepsilon)$, where we replace the operator norm by the Hilbert norm-$\| \ \|_2$ given by the trace. We denote these invariants of a subfactor $N \subset M$ by $\mathrm{n}_k^{(2)}(N \subset M; \varepsilon)$, $\mathrm{D}_k^{(2)}(N \subset M; \varepsilon)$, $\mathrm{d}^{(2)}(N \subset M; \varepsilon)$, respectively, and refer to them generically as L^2-*paving size* of $N \subset M$ (inspired by terminology used in Section 3 of [P13]). These invariants may be easier to calculate, but less relevant of the properties of the inclusion $N \subset M$. Recall in this respect that for any inclusion of II_1 factors $N \subset M$, the subfactor N contains a MASA $A \subset N$ such that $A' \cap M = A \vee (N' \cap M)$ (see e.g., Corollary 1.2.3 in [P16]), which by (Theorem 3.6 in [P13]) contains approximate 2-independent partitions of any size. Thus, for any $F \subset M \ominus (A \vee N' \cap M)$ finite, any $\delta > 0$ and any $n \geq 1$, one can find a partition of 1 with projections of trace $1/n$ in A, $p_1, \dots, p_n \in \mathcal{P}(A)$, such that $\| \sum_{i=1}^n p_i x p_i\|_2 \approx_\delta n^{-1/2}\|x\|_2$, $\forall x \in F$. Thus, one has the estimates $\mathrm{D}_k^{(2)}(N \subset M; \varepsilon) \leq \mathrm{n}_k^{(2)}(N \subset M; \varepsilon) \leq \mathrm{n}_\infty^{(2)}(N \subset M; \varepsilon) \leq [\varepsilon^{-2}] + 1$, for any $1 \leq k \leq \infty$, for any $N \subset M$, without even assuming $[M:N] < \infty$.

Remark 2.10. The most interesting case of inclusions of factors $\mathcal{N} \subset \mathcal{M}$ is when they are *ergodic*, i.e., $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}$. They correspond to the action $\mathcal{U}(\mathcal{N}) \curvearrowright^{\mathrm{Ad}} \mathcal{M}$ being ergodic. A strengthening of ergodicity, called *MV-ergodicity* [P19], requires that the *wo*-closure of the convex hull of $\{uxu^* \mid u \in \mathcal{U}(\mathcal{N})\}$ intersects $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}1$ (see also [P98] where this is called *weak relative Dixmier property*). Since *wo* and *so*-closures coincide on bounded convex sets, it is equivalent to $\overline{\mathrm{co}}^{so}\{uxu^* \mid u \in \mathcal{U}(\mathcal{N})\} \cap \mathbb{C}1 \neq \emptyset$. For an inclusion of II_1 factors $N \subset M$ this amounts to a von Neumann type L^2-mean value ergodicity: $\forall x \in M$, $\forall \varepsilon > 0$, $\exists u_1, \dots, u_n \in \mathcal{U}(N)$ such that $\|\frac{1}{n}\sum_{i=1}^n u_i x u_i^* - \tau(x)1\|_2 \leq \varepsilon$. Viewed from this perspective, Dixmier's averaging theorem states that for any single factor $\mathcal{N}$, the action $\mathcal{U}(\mathcal{N}) \curvearrowright^{\mathrm{Ad}} \mathcal{N}$ is L^∞-MV ergodic, while the result in (A.1 in [P96], [P97]) shows that $\mathcal{U}(N) \curvearrowright^{\mathrm{Ad}} M$ is L^∞-MV ergodic for any ergodic inclusion of II_1 factors $N \subset M$ with finite Jones index $[M : N] < \infty$ (with the converse holding true when N, M are separable, by Corollary 4.1 in [P97]). Our results in this paper can be viewed as quantitative estimates of L^∞-MV ergodicity for finite index inclusions.

Acknowledgements

Like in the proof of the relative Dixmier property for finite index inclusions in (A.1 in [P96]; the Theorem and Corollary 4.1 in [P97]; Theorem 3.1 in [P98]), an important ingredient in the proof of its quantitative version above is played by the characterization of the Jones index $[M : N]$ that Mihai Pimsner and I have discovered in our paper ([PP83], INCREST preprint 52/1983): if $N \subset M$ is an inclusion of II_1 factors then $\lambda = [M : N]^{-1}$ satisfies $E_N(x) \geq \lambda x, \forall x \in M_+$, with $\lambda = [M : N]^{-1}$ the best constant for which such inequality holds true, i.e., $[M : N] = (\sup\{c \geq 0 \mid E_N(x) \geq cx, \forall x \in M_+\})^{-1}$. We were led to this "probabilistic" characterization of $[M : N]$ while trying to elucidate some intriguing questions emanating from Vaughan Jones amazing paper *Index for subfactors* [J82], a preprint of which he sent us in the Summer of 1982. The present paper is in memory of the exciting exchanges of ideas, mathematical discussions and collaborations I had with Vaughan and with Mihai over the years. It is terribly sad to lose so dear friends. They will be greatly missed.

References

[AP17] C. Anantharaman, S. Popa: "An introduction to II_1 factors", www.math.ucla.edu/~popa/Books/IIun-v13.pdf

[D57] J. Dixmier: *Les algébres d'operateurs sur l'espace Hilbertien (Algébres de von Neumann)*, Gauthier-Villars, Paris, 1957. MR0094722

[KS59] R.V. Kadison, I.M. Singer: *Extensions of pure states*, Am. J. Math. **81** (1959), 383–400. MR0123922

[K59] H. Kesten: *Symmetric random walks on groups*, Trans. Am. Math. Soc., **92** (1959), 336–354. MR0109367

[J82] V.F.R. Jones: *Index for subfactors*, Invent. Math., **72** (1983), 1–25. MR0696688

[PP83] M. Pimsner, S. Popa: *Entropy and index for subfactors*, Ann. Sci. Éc. Norm. Supér., **19** (1986), 57–106 (INCREST preprint No. 52/1983). MR0860811

[P92] S. Popa: *Free independent sequences in type* II_1 *factors and related problems*, Astérisque, **232** (1995), 187–202. MR1372533

[P96] S. Popa: *Some properties of the symmetric enveloping algebras with applications to amenability and property T*, Doc. Math., **4** (1999), 665–744. MR1729488

[P97] S. Popa: *The relative Dixmier property for inclusions of von Neumann algebras of finite index*, Ann. Sci. Éc. Norm. Supér., **32** (1999), 743–767. MR1717575

[P98] S. Popa: *On the relative Dixmier property for inclusions of C^*-algebras*, J. Funct. Anal., **171** (2000), 139–154. MR1742862

[P13] S. Popa: *Independence properties in subalgebras of ultraproduct* II_1 *factors*, J. Funct. Anal. **266** (2014), 5818–5846. arXiv:1308.3982 [math.OA]. MR3182961

[P16] S. Popa: *Constructing MASAs with prescribed properties*, Kyoto J. Math., **59** (2019), 367–397. arXiv:1610.08945 [math.OA]. MR3960298

[P19] S. Popa: *On ergodic embeddings of factors*, Commun. Math. Phys., **384** (2021), 971–996. arXiv:1910.06923 [math.OA]. MR4259380

[PV15] S. Popa, S. Vaes: In R. Doran and E. Park (editors), *On the optimal paving over MASAs in von Neumann algebras*, Contemporary Mathematics, Volume **671**, pp. 199–208. American Mathematical Society, 2016. arXiv:1507.01072 [math.OA]. MR3546686

Sorin Popa
University of California, Los Angeles, Math. Dept.
UCLA
Los Angeles, CA 90095-1555
USA
E-mail: popa@math.ucla.edu