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Spin Calogero-Moser periodic chains and two dimensional Yang-Mills theory with corners

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Dedicated to the memory of Vaughan Jones

Abstract: The Quantum Calogero-Moser spin system is a superintegrable system with the spectrum of commuting Hamiltonians that can be described entirely in terms of representation theory of the corresponding simple Lie group. Here we describe its natural generalization known as quantum Calogero-Moser spin chain or N-spin Calogero-Moser system. In the first part of this paper we show that quantum Calogero-Moser spin chain is a quantum superintegrable systems. Then we show that the Euclidean multi-time propagator for this model can be written as a partition function of a two-dimensional Yang-Mills theory on a cylinder. Then we argue that the two-dimensional Yang-Mills theory with Wilson loops with "outer ends" should be regarded as the theory on space times with non-removable corners. Partition functions of such theory satisfy non-stationary Calogero-Moser equations. In this paper the underlying Lie group G is a compact connected, simply connected simple Lie group.

Introduction

In this paper we consider the two-dimensional Yang-Mills (2D YM) theory on surfaces with open Wilson graphs. The partition function in this case is a vector in the tensor product of finite dimensional representations of a simple compact Lie group. We show that it is a solution to N-point spin Calogero-Moser (CM) evolution with initial conditions determined by the structure of the Wilson graph.

One can draw an analogy between these models and the Chern-Simons theory with Wilson lines ending on the two-dimensional boundary of a 3-manifold.

Quantum spin Calogero-Moser systems have a long history. See for example [8, 15] and references therein. In this paper we will work with so called

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N-spin version of CM systems. These models of many-particle quantum systems with internal (spin) degrees of freedom are examples of quantum superintegrable systems. For the superintegrability of quantum CM spin systems and for the notion of quantum superintegrability see [15, 19]. For the superintegrability of classical N-point spin CM chain see [18]

Two dimensional model of Yang-Mills (YM) theory is a two dimensional topological quantum field theory. A non-perturbative partition function for a disc was introduced in [11]. The 2D YM theory was studied as a TQFT in papers [23, 24] where among other things a convincing argument was given that the non-perturbative formula is a version of the Duistermaat-Heckman localization. For the latest study of perturbative aspects of the YM theory on surfaces with boundary and for more references see [12]. To be more precise, the YM theory on surfaces is not exactly topological. The partition function depends on the total area of the surface (assuming it is finite), and, possibly, on more parameters which are additive with respect to gluing, when the rank of the gauge group is greater than one [24].

The partition function and expectation values of Wilson graphs (line observables in this TQFT) can be interpreted as an analytic continuation of the semiclassical limit of quantum Chern-Simons invariants [13, 24]. If the corresponding quantum group [14] has $q = \exp(i\frac{\pi}{r})$ with $r = k + h^\vee$ where k is the level of the quantum Chern-Simons theory [22], $h^\vee$ is the dual Coxeter number, the semiclassical limit is when $k \to \infty$. The quantum Chern-Simons invariant of of a circle bundle over Σ with the Chern class m in the limit when $k \to \infty, m \to \infty$ and $a = \frac{m}{r}$ is finite becomes the partition function of the 2D YM with the area ia in the limit. When $m = 0$ this was first observed in [24]. For $a \neq 0$ see [13].

In this paper we show that the partition function of the 2D YM theory on surfaces with open Wilson graphs are solutions to non-stationary Calogero-Moser type differential equations. For example the partition function on a cylinder with Wilson lines that are parallel to the cylinder is the multi-time propagator for quantum N-spin Calogero-Moser system with Euclidean time being areas of corresponding strips of the cylinder. We also argue that such partition functions can be interpreted as a $2-1-0$ TQFT with non-removable corners (points at the boundary where Wilson lines "go out").

In the first section we define quantum N-spin Calogero-Moser model for a compact simple Lie group G. In the second section we explain the relation between quantum N-spin Calogero-Moser system and the 2D YM with open Wilson graphs.

1. Quantum N-spin Calogero-Moser system and its superintegrability

1.1. Quantum superintegrable systems

Let A_h be a family of associative algebras with trivial center. We think of each algebra A from this family as the algebra of quantum observables for a quantum system.

A commutative algebra I has rank k means that it has maximum k algebraically independent elements, i.e. k is the Krull dimension of I. We will say that an associative algebra A has rank n if n is the rank of maximal commutative subalgebra in A. Here is a definition of a *quantum superintegrable system* [15].

Definition 1. A commutative subalgebra $I \subset A$ of rank k is a quantum superintegrable system on A if the rank of the centralizer of I in A is maximal possible, i.e. is equal to n where n is the rank of A.

For a superintegrable quantum system we have embeddings of associative algebras:

$$I \subset Z(I, A) \subset A$$

where $Z(I, A)$ is the centralizer of I in A and $\mathrm{rank}(Z(I, A)) = \mathrm{rank}(A)$.

An example of a superintegrable system is when A is the algebra of differential operators in variables $x_1, \dots, x_n$ with polynomial coefficients, I is the algebra of differential operators in $x_1, \dots, x_k$ with constant coefficients. In this case $Z(I, A)$ is the algebra of differential operators in $x_1, \dots, x_n$ with polynomial coefficients in $x_{k+1}, \dots, x_n$.

1.2. Algebraic preliminaries

In this section we develop an algebraic setup for defining quantum N-spin Calogero-Moser system.

1.2.1. Let us fix some notations. Throughout this paper G is a compact, simple, simply connected Lie group. We will denote by $\mathcal{R}(G)$ the algebra of functions on G generated by matrix elements of finite dimensional representations. It is a span of matrix elements of irreducible representations.[1]

[1] There is a natural isomorphism of $G \times G$-modules

$$\mathcal{R}(G) \simeq \bigoplus_{\lambda} V_\lambda \otimes V_\lambda^*$$

The space $\mathcal{R}(G)$ is an open dense subspace in the Hilbert space $L^2(G)$ of square integrable functions with respect the Haar measure on G.

We denote by $\mathbb{D}(G)$ the algebra of differential operators on G preserving $\mathcal{R}(G)$. Let $\mathrm{Vect}(G)$ be the subalgebra in the Lie algebra of all vector fields on G which correspond to first order differential operators in $\mathbb{D}(G)$.

The Lie group G acts on itself by left and right translations: $h : g \mapsto hg$ and $h : g \mapsto gh^{-1}$. These actions give two homomorphisms of Lie algebras $\mathfrak{g} \to \mathrm{Vect}(G)$ and, as a consequence, two algebra inclusions $\widehat{\mu}_\ell, \widehat{\mu}_r : U\mathfrak{g} \to \mathbb{D}(G)$ (quantum moment maps) as right and left G-invariant differential operators, respectively.

Let $Z(\mathfrak{g})$ be the center of $U\mathfrak{g}$, and S the antipode for $U\mathfrak{g}$, i.e. the anti-involution of $U\mathfrak{g}$ that acts as $x \to -x$ on elements $\mathfrak{g} \subset U\mathfrak{g}$.

1.2.2. Consider the group G^N acting on itself by left translations:

$$(h_1, \dots, h_N) : (g_1, g_2, \dots, g_N) \mapsto (h_1 g_1, h_2 g_2, \dots, h_N g_N)$$

by right translations twisted by the cyclic permutation:

$$(h_1, \dots, h_N) : (g_1, g_2, \dots, g_N) \mapsto \left(g_1 h_2^{-1}, g_2 h_3^{-1}, \dots, g_N h_1^{-1}\right)$$

The combination of these two actions is the twisted conjugation:

$$(1) \qquad (h_1, \dots, h_N) : (g_1, g_2, \dots, g_N) \mapsto \left(h_1 g_1 h_2^{-1}, h_2 g_2 h_3^{-1}, \dots, h_N g_N h_1^{-1}\right)$$

Note that the action by twisted conjugations has a natural interpretation as the gauge action on parallel transports for a connection in a trivial principal G-bundle over a circle with with marked points. In this interpretation g_i is the holonomy along the interval connecting points i and $i+1$, and h_i is the gauge transformation at the i-th point, see Fig. 1. This is why we *call (1) the gauge action.*

The left, the right and the gauge action of G^N on itself lift to actions on $\mathbb{D}(G^N)$. Quantum moment maps for these actions give injective homomorphisms of associative algebras $\widehat{\mu}_L, \widehat{\mu}_R, \widehat{\mu}_{ad} : U\mathfrak{g}^{\otimes N} \subset \mathbb{D}(G)$.

This gives the following chain of inclusions of associative algebras:

$$\mathbb{D}(G^N) \overset{\widehat{\mu}_L \otimes \widehat{\mu}_R}{\hookleftarrow} (U\mathfrak{g})^{\otimes N} \otimes (U\mathfrak{g})^{\otimes N} \hookleftarrow (U\mathfrak{g})^{\otimes N}$$

where the sum is taken over all irreducible finite dimensional representation of G and V_λ is the irreducible representation with the highest weight λ. Note that space on the right consists of finite direct sums of unbounded length.

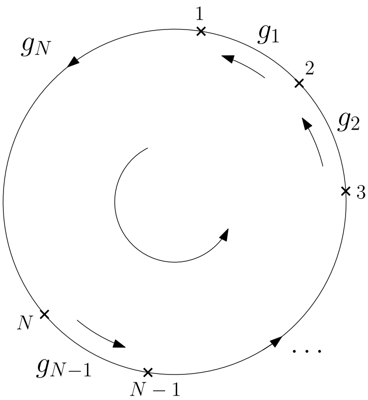


Figure 1: Holonomies along intervals and gauge transformations. Here g_i stands for the holonomy between points i and $i+1$. The gauge transformation at points $i-1$ and i acts on g_i as $g_i \mapsto h_i g_i h_{i+1}^{-1}$. Arrows show the orientation of the circle and of the disc that it bounds.

Here the right inclusion is simply mapping to the left factor in the tensor product.[2]

This maps filter through algebra homomorphisms:

$$\mathbb{D}(G^N) \overset{\widehat{\mu}_L \otimes \widehat{\mu}_R}{\hookleftarrow} (U\mathfrak{g})^{\otimes N} \otimes_{Z(\mathfrak{g})^{\otimes N}} (U\mathfrak{g})^{\otimes N} \hookleftarrow (U\mathfrak{g})^{\otimes N}$$

Here $Z(\mathfrak{g}) \subset U\mathfrak{g}$ is the center of the universal enveloping algebra and $\otimes_{Z(\mathfrak{g})^{\otimes N}}$ is the quotient of the algebraic tensor product. In this tensor product over $Z(\mathfrak{g})^{\otimes N}$ we have:

$$\begin{aligned}&(a_1 z_1 \otimes a_2 z_2 \cdots \otimes a_N z_N) \otimes_{Z(\mathfrak{g})^{\otimes N}} (b_1 \otimes b_2 \cdots \otimes b_N) = \\ &\qquad (a_1 \otimes a_2 \cdots \otimes a_N) \otimes_{Z(\mathfrak{g})^{\otimes N}} \big(S(z_1) b_1 \otimes S(z_2) b_2 \cdots \otimes S(z_N) b_N\big).\end{aligned}$$

Here $S(z)$ is the antipode of the central element z. The antipode in this case is an algebra antiautomorphism of $U(\mathfrak{g})$ which acts as $S(x) = -x$ on linear elements $x \in \mathfrak{g} \subset U(\mathfrak{g})$.

The last embedding is $a \mapsto a \otimes_{Z(\mathfrak{g})^{\otimes N}} 1$. Note that we have a natural isomorphism

$$(U\mathfrak{g})^{\otimes N} \otimes_{Z(\mathfrak{g})^{\otimes N}} (U\mathfrak{g})^{\otimes N} \simeq \big((U\mathfrak{g}) \otimes_{Z(\mathfrak{g})} (U\mathfrak{g})\big)^{\otimes N}$$

[2] We could have taken the inclusion to the right factor. At the end it does not make a difference.

The action of the gauge group G^N extends naturally to its action on $\mathbb{D}(G)$. It acts by the adjoint action on each copy of $U(\mathfrak{g})^{\otimes N}$ and it acts diagonally on $(U\mathfrak{g})^{\otimes N} \otimes_{Z(\mathfrak{g})^{\otimes N}} (U\mathfrak{g})^{\otimes N}$. The G^N-invariant part of these embeddings is:

$$\mathbb{D}(G^N)^{G^N} \overset{\widehat{\mu_L}\widehat{\otimes}\widehat{\mu_R}}{\hookleftarrow} \left((U\mathfrak{g})^{\otimes N} \otimes_{Z(\mathfrak{g})^{\otimes N}} (U\mathfrak{g})^{\otimes N}\right)^{G^N} \hookleftarrow Z(\mathfrak{g})^{\otimes N} \tag{2}$$

Here $Z(\mathfrak{g})^{\otimes N}$ is the G^N-invariant part of $(U\mathfrak{g})^{\otimes N}$ with respect to the adjoint action.

1.3. Quantum N-spin Calogero-Moser system

1.3.1. The space of states Let $V_{\mu_1}, \ldots, V_{\mu_N}$ be finite dimensional irreducible representations of G with highest weights μ_i and representation maps π^{μ_i}. We will use the same notation for associated irreducible representations of $U\mathfrak{g}$, $\pi^{\mu_i} : U\mathfrak{g} \to \mathrm{End}(V_{\mu_i})$. Fix Hermitian scalar product $(\cdot,\cdot)_{\mu_i}$ on each V_{μ_i} turning it into a unitary representation. Denote $A^\dagger$ the Hermitian conjugate to $A \in \mathrm{End}(V_{\mu_i})$ relative to $(\cdot,\cdot)_{\mu_i}$. The unitarity of the representation V_{μ_i} can be written as

$$\pi_{\mu_i}(g)^\dagger = \pi_{\mu_i}(g^{-1}).$$

Now let us define the representation space for $\mathbb{D}(G^N)^{G^N}$ corresponding to representations μ_i,

Consider the following space of gauge group equivariant $V_{\mu_N} \otimes \cdots \otimes V_{\mu_1}$-valued functions of G^N:

$$\mathcal{R}_\mu = \mathcal{R}(G^N \to V_{\mu_1} \otimes \cdots \otimes V_{\mu_N}) := \left(\mathcal{R}(G^N) \otimes V_{\mu_1} \otimes \cdots \otimes V_{\mu_N}\right)^{G^N}.$$

were $\mathcal{R}(G^N)$ is the span of matrix elements of all finite dimensional representations of G^N. The G^N-equivariance condition for $f \in \mathcal{R}_\mu$ is

$$f(g^h) = \left(\pi^{\mu_1}(h_1) \otimes \cdots \otimes \pi^{\mu_N}(h_N)\right) f(g_1, \ldots, g_N) \tag{3}$$

where g^h is the gauge action of $h \in G^N$ on $g \in G^N$. Clearly this is a module over $\mathbb{D}(G^N)$ with natural action of differential operators on functions.

This space has a Hermitian scalar product

$$(f_1, f_2) = \int_{G^N} \left(f_1(g_1, \ldots, g_N), f_2(g_1, \ldots, g_N)\right)_\mu dg_1 \ldots dg_N.$$

Here $(v_1 \otimes \cdots \otimes v_N, u_1 \otimes \cdots \otimes u_N)_\mu = \prod_{i=1}^N (v_i, u_i)_{\mu_i}$ and dg is the Haar measure on G normalized such that $\int_G dg = 1$.

The completion $\mathcal{H}_\mu$ of $\mathcal{R}_\mu$ with respect to the norm $\|f\|^2 = (f, f)$ is the gauge equivariant subspace in $V_{\mu_1} \otimes \cdots \otimes V_{\mu_N}$-valued square integrable functions on G^N satisfying (3). This is the *space of states* of quantum N-spin Calogero-Moser system.

1.3.2. Quantum moment maps Let $X \in \mathfrak{g} \subset U\mathfrak{g}$, define $X_j = 1^{\otimes(j-1)} \otimes X \otimes 1^{\otimes(N-j-1)} \in (U\mathfrak{g})^{\otimes N}$. The images of X_j in $\mathbb{D}(G^N)$ with respect to the left and to the right quantum moment map act on $\mathcal{R}(G^N) \otimes V_{\mu_N} \otimes \cdots \otimes V_{\mu_1}$ as follows

$$\widehat{\mu}_R(X_j)f(g_1, \ldots, g_j, g_{j+1}, \ldots, g_N) = \frac{d}{dt} f(g_1, \ldots, g_j e^{tX}, \ldots, g_N)|_{t=0},$$
$$\widehat{\mu}_L(X_j)f(g_1, \ldots, g_j, g_{j+1}, \ldots, g_N) = \frac{d}{dt} f(g_1, \ldots, e^{-tX} g_j, \ldots, g_N)|_{t=0}.$$

where $X \mapsto e^X$ is the exponential map of G.

The image of X_j with respect to the quantum moment map for the gauge action $\widehat{\mu}(X_j) = \widehat{\mu}_L(X_j) + \widehat{\mu}_R(X_{j+1})$ acts as

$$\widehat{\mu}(X_j)f(g_1, \ldots, g_{j-1}, g_j, \ldots, g_N) = \frac{d}{dt} f(g_1, \ldots, g_{j-1}e^{-tX}, e^{tX}g_j, \ldots, g_N)|_{t=0}$$

Gauge invariance (3) implies that for each $f \in \mathcal{R}_\mu$, we have

$$\widehat{\mu}(X_j)f(g_1, \ldots, g_N) = \pi_j^{\mu_j}(X)f(g_1, \ldots, g_N) \tag{4}$$

where $\pi_j^{\mu_j} = \mathrm{id}_{V_{\mu_1}} \otimes \cdots \otimes \mathrm{id}_{V_{\mu_{j-1}}} \otimes \pi^{\mu_j} \otimes \mathrm{id}_{V_{\mu_{j+1}}} \otimes \cdots \otimes \mathrm{id}_{V_{\mu_N}}$.

1.3.3. Quantum Hamiltonians and quantum integrals Because $Z(\mathfrak{g})^{\otimes N}$, $((U\mathfrak{g} \otimes_{Z(\mathfrak{g})} \otimes U\mathfrak{g})^{\otimes N})^{G^N}$ are subalgebras of $\mathbb{D}(G^N)^{G^N}$, they act on the space $\mathcal{R}_\mu(G^G)$. Denote the images of the subalgebras $Z(\mathfrak{g})^{\otimes N}$, $((U\mathfrak{g} \otimes_{Z(\mathfrak{g})} \otimes U\mathfrak{g})^{\otimes N})^{G^N}$ and $\mathbb{D}(G^N)^{G^N}$ in $\mathrm{End}(\mathcal{R}_\mu)$ by I_μ, J_μ and A_μ respectively. They form a chain of subalgebras

$$A_\mu \hookleftarrow J_\mu \hookleftarrow I_\mu \tag{5}$$

Recall that I_μ is the subalgebra of quantum Hamiltonians and J_μ is the subalgebra of quantum integrals.

The center $Z(\mathfrak{g})$ as commutative algebra is a polynomial ring in $r = \mathrm{rank}(\mathfrak{g})$ homogeneous generators $c_{d_1}, \ldots, c_{d_r}$ of degrees $d_1 = 2 \leq d_2 \leq \cdots \leq$

d_r respectively.[3] Note that $c := c_2 \in Z(\mathfrak{g})$ is the quadratic Casimir element determined by the Killing form.

Denote by

$$H^{\mu,(j)}_{d_k} \in I_\mu, \quad 1 \le j \le N, 1 \le k \le r$$

the action of $c^{(j)}_k := 1^{\otimes(j-1)} \otimes c_k \otimes 1^{\otimes(N-j-1)} \in Z(\mathfrak{g})^{\otimes N}$ on $\mathcal{R}_\mu$. The Hamiltonians $H^{\mu,(j)}_{d_k}$ commute and generate the algebra I_μ.

We will compute the radial components of the quadratic Hamiltonians $H^{(j)} := H^{(j)}_2$ $(1 \le j \le N)$ in Subsection 1.4.

1.3.4. The spectrum of quantum Hamiltonians Fix N finite dimensional irreducible G-modules $V_{\nu_1}, \dots, V_{\nu_N}$. Denote by $\mathcal{R}^{(\nu)}(G^N)$ the subspace of $\mathcal{R}(G^N)$ spanned by matrix elements of the irreducible G^N-representation $V_{\nu_1} \otimes \cdots \otimes V_{\nu_N}$.[4] Then by Peter-Weyl theorem we have the decomposition with respect to the left and the right action of G^N:

$$\mathcal{R}(G^N) = \bigoplus_\nu \mathcal{R}^{(\nu)}(G^N) \tag{6}$$

Here ν runs over the set of N-tuples of dominant integral weights.

Subspaces $\mathcal{R}^{(\nu)}(G^N) \subset \mathcal{R}(G^N)$ are invariant with respect to the gauge action of G^N, thus we can define subspaces

$$\mathcal{R}^{(\nu)}_\mu := \left(\mathcal{R}^{(\nu)}(G^N) \otimes V_{\mu_1} \otimes \cdots \otimes V_{\mu_N}\right)^{G^N} \subset \mathcal{R}_\mu,$$

Note that this subspace is not empty only if $V_{\nu_i} \subset V_{\nu_{i-1}} \otimes V_{\mu_i}$, cyclically for $i = 1, \dots, N$.

Because $Z(\mathfrak{g})$ acts on an irreducible representation by the multiplication on the corresponding central character, the subspace $\mathcal{R}^{(\nu)}_\mu$ is an eigensubspace for I_μ with

$$H^{(j)}_{d_k} f = c_{d_k}(\nu_j) f$$

for any $f \in \mathcal{R}^{(\nu)}_\mu$. Here $c_{d_k}(\eta)$ is the value of $c_{d_k} \in Z(\mathfrak{g})$ on the irreducible finite dimensional module V_η.

The Peter-Weyl theorem for $\mathcal{R}(G^N)^{\otimes N}$ gives the decomposition of $\mathcal{R}_\mu$ in simultaneous eigensubpaces for the action of the Hamiltonians $H^{(j)}_{d_k}$:

$$\mathcal{R}_\mu = \bigoplus_\nu \mathcal{R}^{(\nu)}_\mu \tag{7}$$

[3] Numbers $d_i - 1, 1 \le i \le r$ are called the exponents of $\mathfrak{g}$.

[4] It does not matter which basis we use to define matrix elements, the subspace will be the same.

For intertwiners $a_i \in \mathrm{Hom}_G(V_{\nu_i}, V_{\nu_{i-1}} \otimes V_{\mu_i})$ (here $\nu_0 = \nu_N$), define the *trace function* $\Psi^{(\nu)}_{a,\mu} \in \mathcal{R}^{(\nu)}(G^N) \otimes V_{\mu_1} \otimes \cdots \otimes V_{\mu_N}$ on G^N as

$$\begin{aligned}(8)\qquad \Psi^{(\nu)}_{a,\mu}(g_1, \ldots, g_N) = \mathrm{Tr}_{V_{\nu_N}}\big((a_1\pi^{\nu_1}(g_1) \otimes \mathrm{id}_{V_{\mu_2}\otimes\cdots\otimes V_{\mu_N}})\\ \cdots (a_{N-1}\pi^{\nu_{N-1}}(g_{N-1}) \otimes \mathrm{id}_{V_{\mu_N}})a_N\pi^{\nu_N}(g_N)\big).\end{aligned}$$

It is clear that $\Psi^{(\nu)}_{a,\mu}$ is gauge invariant and that $\Psi^{(\nu)}_{a,\mu} \in \mathcal{R}^{(\nu)}_\mu$. The restriction of $\Psi^{(\nu)}_{a,\mu}$ to $1^{\times(N-1)} \times H$ gives generalized trace functions from [5, 6].

Theorem 1. *Functions $\Psi^{(\nu)}_{a,\mu}$ form a linear basis in $\mathcal{R}^{(\nu)}_\mu$ enumerated by a basis in $\otimes_{i=1}^N \mathrm{Hom}_G(V_{\nu_i}, V_{\nu_{i-1}} \otimes V_{\mu_i})$.*

Corollary 1. *We have a linear isomorphism*

$$\mathcal{R}^{(\nu)}_\mu(G^N) \simeq \otimes_{i=1}^N \mathrm{Hom}_G(V_{\nu_i}, V_{\nu_{i-1}} \otimes V_{\mu_i})$$

Let $Z(I_\mu, A_\mu)$ be the centralizer of the commutative subalgebra I_μ in A_μ. By the construction of $J_\mu \subset A_\mu$ it is a subalgebra in the centralizer of I_μ, i.e. $J_\mu \subset Z(I_\mu, A_\mu)$. Since $\mathcal{R}^{(\nu)}_\mu \subset \mathcal{R}_\mu$ is an eigensubspace of I_μ the centralizer $Z(I_\mu, A_\mu)$ acts on it. Thus, $\mathcal{R}^{(\nu)}_\mu$ is an J_μ-module.

Theorem 2. *The decomposition* (7) *is the multiplicitly free decomposition of $\mathcal{R}_\mu$ in irreducible J_μ-modules.*

The proof of this theorem and its generalization to a more general representation theoretical context will be given in [20].

Denote by $J^{(\nu)}_\mu$ and $Z(I_\mu, A_\mu)^{(\nu)}$ images of J_μ and $Z(I_\mu, A_\mu)$ in $\mathrm{End}(\mathcal{R}^{(\nu)}_\mu)$ respectively.

Corollary 2. *We have*

$$J^{(\nu)}_\mu = Z(I_\mu, A_\mu)^{(\nu)}.$$

Thus, the embeddings (5) define a simple quantum superintegrable system in the sense of [19].

Using well known identities for integrals of matrix elements of irreducible representations with respect to the Haar measure[5] it is easy to prove the following formula for the scalar product of trace functions:

$$\int_{G^N} (\Psi^{\{\mu\}}_{b,\lambda'}(g_1, \ldots, g_N), \Psi^{\{\mu\}}_{a,\lambda}(g_1, \ldots, g_N))dg_1 \ldots dg_N = \prod_{i=1}^N \delta_{\lambda_i,\lambda'_i} \prod_{i=1}^N (b_i, a_i)_{\lambda_i}$$

[5] All necessary identities are summarized in the Appendix D.

Here $(b,a)_{\lambda_i}$ is a Hermitian scalar product on $\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_i}, V_{\lambda_{i-1}} \otimes V_{\mu_i})$ defined as follows. Because all representations are equipped with Hermitian structure for each $a \in \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_i}, V_{\lambda_{i-1}} \otimes V_{\mu_i})$ we have its Hermitian conjugate $a^\dagger : V_{\lambda_{i-1}} \otimes V_{\mu_i} \to V_{\lambda_i}$. For any such a, b, by Schur's lemma the composition $ab^\dagger$ is a multiple of the identity $id_{V_{\lambda_i}}$. This defines the scalar product $(b,a)_{\lambda_i}$ as $ab^\dagger = (b,a)_{\lambda_i} id_{V_{\lambda_i}}$.

1.4. The gauge fixing

There N natural ways to identify the coset G^N/G^N for the gauge action (1) with the coset G/G with respect to the conjugation.

Indeed, fix $i \in 1, \dots, N$ and for $j \neq i+1$ set:

$$h_j = h_{i+1} g_{i+1} g_{i+2} \dots g_{j-2} g_{j-1} \tag{9}$$

Here here the product is cyclic. Denote such N-tuple $(h_1, \dots, h_N)$ by h_g.

The gauge transformation by the element h_g brings $g = (g_1, \dots, g_N)$ to $g^{h_g} = (1, \dots, h_{i+1}(g_{i+1} \dots g_N g_1 g_2 \dots g_{i-1} g_i) h_{i+1}^{-1}, \dots, 1)$. This identifies the G^N gauge orbit through $(g_1, \dots, g_N)$ with the G conjugation orbit through $g_1 \dots g_N$. Thus, we constructed the mapping

$$G^N/G^N \simeq G/G$$

which is easy to prove to be an isomorphism.

We can chose h_{i+1} in such a way that $h_{i+1}(g_{i+1} \dots g_N g_1 g_2 \dots g_{i-1} g_i) h_{i+1}^{-1} \in H$ is an element of the Cartan subgroup in G. This gives an isomorphism

$$G^N/G^N \simeq G/G \simeq H/W$$

for each $i = 1, \dots, N$.

For each $i = 1, \dots, N$, this gives an isomorphism of Hilbert spaces

$$\phi_i : \mathcal{H}_\mu \simeq \mathcal{H}_\mu^H = L_2\big(H \to (V_{\mu_1} \otimes \cdots \otimes V_{\mu_N})[0]\big)^W \tag{10}$$

Here the Weyl group W acts as $w(f)(h) = wf(w^{-1}(h))$. We took into account that the Weyl group acts naturally on the zero weight subspace of any G-module. Indeed, if $N(H)$ is the normalizer of H in G, the Cartan subgroup H is a normal subgroup in $N(H)$ and $W = N(H)/H$. The subgroup $N(H) \subset G$ acts on the zero weight subspace and because H acts trivially it acts as W.

This isomorphism acts on functions as

$$\phi_i(f)(h) = \pi_\mu(h_g) f(g) = f(1, \dots, h, \dots, 1),$$

where h_g is as above, $\pi_\mu(h)$ is as in (3) and $h=h_i(g_{i+1}\dots g_N g_1 g_2 \dots g_{i-1}g_i)h_i^{-1}$.

For the scalar product on $\mathcal{H}_\mu^H$ we have:

$$(f,g) = \int_H (f(h), g(h))|\delta(h)|^2 dh$$

Here $H \subset G$ is the Cartan subgroup and $\delta(h)$ is the denominator in the Weyl character formula.

1.4.1. Quantum Hamiltonians In the theorem below we summarized computations of the action of quantum Hamiltonians on the space $\mathcal{H}_\mu^H$ with the gauge fixing map $\phi_N : \mathcal{H}_\mu \to \mathcal{H}_\mu^H$. We show that operators $D_k = H^{(k+1)} - H^{(k)}$ are topological Knizhnik-Zamilodchikov-Bernard operators and that $H^{(N)}$ is the spin Calogero-Moser Hamiltonian.

For each positive root, let e_α, f_α be a basis in root subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ respectively. Denote h_i a basis in the Cartan subalgebra $\mathfrak{h}$ with is orthornormal with respect to the Killing form $(.,.)$ and let $\rho \in \mathfrak{h}$ be an element corresponding to half of the sum of positive roots.

Theorem 3. *Operators $H^{(N)}$ and $D_j = H^{(j+1)} - H^{(j)}$ for $j = 1, \dots, N-1$ act as*

$$H_2^{(N)} = \Delta + 2\sum_{\alpha>0} \frac{\pi^V(f_\alpha e_\alpha)}{(1-h_\alpha)(1-h_{-\alpha})} - \|\rho\|^2$$

and

$$D_j = \left(h^{(j)}, \frac{\partial}{\partial\lambda}\right) - \sum_{k=1}^{j-1} r_{kj}(\lambda) + \sum_{k=j+1}^{n} r_{jk}(\lambda)$$

where $\Delta = (\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\lambda})$ is the Laplacian on $\mathfrak{h} \subset \mathfrak{g}$ determined by the Killing form, $h_\alpha = e^{(\alpha,\lambda)}$ and $r(\lambda)$ is Felder's dynamical r-matrix [9]:

$$r(\lambda) = -\frac{1}{2}\sum_{i=1}^{r} h_i \otimes h_i - \sum_\alpha \frac{e_{-\alpha}\otimes e_\alpha}{1-h_{-\alpha}}$$

Note that we consider compact simple G which correspond to $h_\alpha = e^{iq_\alpha}$ with $q_\alpha \in \mathbb{R}$.

As always, explicit formulae for higher Hamiltonians $H_k^{(j)}$ are more complicated.

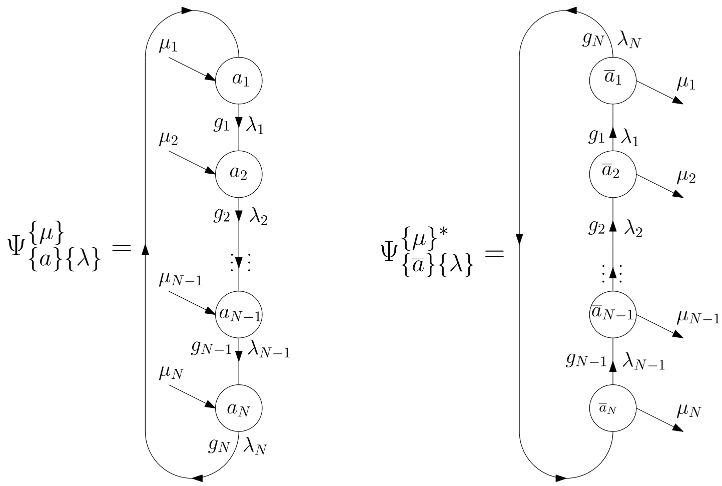


Figure 2: The N-point trace function and the dual trace function. Holonomies $g_1, \dots, g_N$ are arguments of the trace function.

1.5. The multi-time propagator

1.5.1. The propagator and the graphical calculus The multi-time propagator is simply an element of the semigroup generated by all commuting Hamiltonians:

$$U^{\{\mu\}}_{\{A\}} = \exp\Big(-\sum_{k,j} H_k^{(j)} A_{k,j}\Big)$$

It has Nr independent times $A_{k,j}$ and satisfies natural composition property

$$U^{\{\mu\}}_{\{A\}} U^{\{\mu\}}_{\{B\}} = U^{\{\mu\}}_{\{A+B\}}$$

where $(A+B)_{k,j} = A_{k,j} + B_{k,j}$.

The propagator is an integral operator acting on the space $\mathcal{H}_\mu$. In the usual way, from the spectral decomposition we derive its kernel:

(11)

$$U^{\{\mu\}}_{\{A\}}(\{g\}, \{g'\}) = \sum_{\{\lambda\},\{a\}} \exp\Big(-\sum_{i=1}^{N} c_2(\lambda_i) A_i\Big) \Psi^{\{\mu\}}_{\{a\},\{\lambda\}}(g_1, \dots, g_N) \Psi^{\{\mu\}}_{\{\overline{a}\},\{\lambda\}}(g'_1, \dots, g'_N)^*$$

Here $\Psi^{\{\mu\}}_{\{\overline{a}\},\{\lambda\}}(g_1,\dots,g_N)^* \in V^*_{\mu_1} \otimes \cdots \otimes V^*_{\mu_N}$ are dual trace functions defined as

$$(12) \qquad \Psi^{\{\mu\}}_{\{\overline{a}\},\{\lambda\}}(g_1,\dots,g_N)^* = Tr_{V^*_{\lambda_N}}(\pi^*_{\lambda_N}(g_N)b_1 \dots (\pi^*_{\lambda_1}(g_1) \otimes id_{V_{\mu_1}}) \\ (b_2 \otimes id_{V_{\mu_2}} \otimes id_{V_{\mu_1}}) \dots (b_{N-1} \otimes id_{V_{\mu_{N-1}}} \otimes \cdots \otimes id_{V_{\mu_1}}) \\ (\pi^*_{\lambda_1}(g_{N-1}) id_{V_{\mu_1}} \otimes \cdots \otimes id_{V_{\mu_{N-1}}}))$$

For $a_i : V_{\lambda_i} \to V_{\mu_i} \otimes V_{\lambda_{i-1}}$ the morphism $\overline{a}_i : V^*_{\lambda_i} \otimes V_{\mu_i} \to V^*_{\lambda_{i-1}}$ is a $\mathfrak{g}$ linear map obtained by partial dualizing of $a^\dagger : V_{\mu_i} \otimes V_{\lambda_{i-1}} \to V_{\lambda_i}$.

Note that this kernel pointwise is a linear operator in $End(V_{\mu_N} \otimes \cdots \otimes V_{\mu_1}) \simeq (V_{\mu_N} \otimes \cdots \otimes V_{\mu_1}) \otimes (V^*_{\mu_1} \otimes \cdots \otimes V^*_{\mu_N})$.

From now on we will focus on multi-time propagators with non-zero times for quadratic Hamiltonians only. We will also be using graphical calculus to visualize algebraic operations. It is a version of Penrose's graphical calculus. For trace functions Ψ we will write a circle marked points corresponding to vertex operators $\{a_i\}$ andwith outgoing semi-open intervals colored by representations V_{m_i}. For an edge colored by a highest weight λ and an element $g \in G$ we assign $\pi^\lambda(g)$. To a vertex colored by a G-invariant linear map $a : V_\lambda \to V_\mu \otimes V_\nu$ where λ is the color of the incoming edge and μ and ν are colors of outgoing edges, we assign the linear map a. Trace function is the contraction of these maps, which correspond to assembling (gluing) the graph on Fig. 3 from edges and vertices.

1.5.2. The integral formula

Theorem 4. *The kernel (11) can be written as the following integral*

$$(13) \qquad U^{\{\mu\}}_{\{A\}}(\{g\},\{g'\}) = \int_{G^N} Z_{A_1}({g'_1}^{-1} h_N^{-1} g_1 h_1) Z_{A_2}({g'_2}^{-1} h_1^{-1} g_2 h_2) \\ \dots Z_{A_N}({g'_{N-1}}^{-1} h_{N-1}^{-1} g_N h_N) \pi^{\mu_N}(h_N) \otimes \\ \cdots \otimes \pi^{\mu_1}(h_1) dh_1 \dots dh_N$$

Here

$$Z_A(g) = \sum_\lambda \chi_\lambda(g) \dim(\lambda) \exp(-Ac_2(\lambda))$$

is the partition function of the two-dimensional Yang-Mills theory with the boundary holonomy $g \in G$ [11, 23].

To prove this statement one should use standard integral identities from Appendix D. This integral formula also has a natural graphical representation, see Fig. 5.

$$U^{\{\mu\}}_{\{A\}}(\{g\},\{g'\}) =$$

$$= \sum_{\{a\}\{\lambda\}}$$

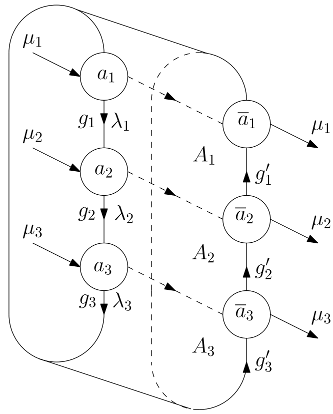
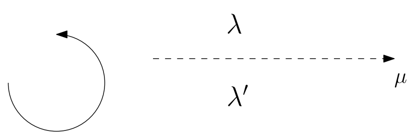


Figure 3: Here is the graphical representation of the propagator (11) for $N = 3$. We put the "color" λ_i on the i-th region. Both segments of the boundary of this region which also belong to the boundary of the cylinder inherit this color. Dashed lines are colored by representations μ_i. Colors λ_i they satisfy the Clebsch-Gordan rules at each dashed line, as it is shown below. There $V_{\lambda_i} \subset V_\mu \otimes V_{\lambda_{i+1}}$. Here we use the orientation of the cylinder to distinguish regions that are left and right to the dashed line.



$$V_\lambda \subset V_\mu \otimes V_{\lambda'}$$

Figure 4: Coloring of faces and edges are related by Clebsch-Gordan rules and uses the orientation of the surface.

The integral formula for the propagator reminds a lot the partition function of pure 2-dimensional Yang-Mills theory [11, 23]. An unusual feature of this formula are "open" Wilson lines which go to the outside of the cylinder as outer edges. We build on this analogy in the next section where will define 2-dimensional Yang-Mills theory of surfaces with corners and embedded open graphs.

$$U_{\{A\}}^{\{\mu\}}(\{g\},\{g'\}) =$$

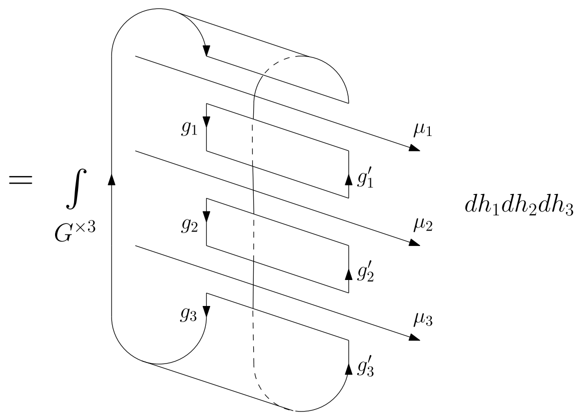


Figure 5: Here is the graphical representation of the propagator (11) in the integral form (13) for $N = 3$. Lines colored by representations μ_i should be regarded as open Wilson lines and each region which is homeomorphic to a disc contributes as the corresponding factor in (13).

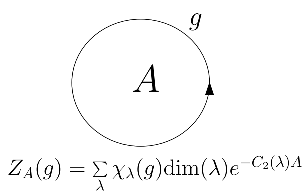


Figure 6: The partition function for a disc.

2. Quantum two dimensional Yang-Mills with corners

In this section we define two dimensional Yang-Mills theory on space time manifolds with non-removable corners. We will not use the language of 2-categories, though it is natural in such setting.

2.1. Surfaces with open graphs

Let Σ be a topological oriented compact surface, possibly with a boundary $\partial\Sigma$. Let Γ be a graph, possibly with one-valent vertices. We assume that Γ is partially embedded in Σ in the following sense: all vertices and all edges

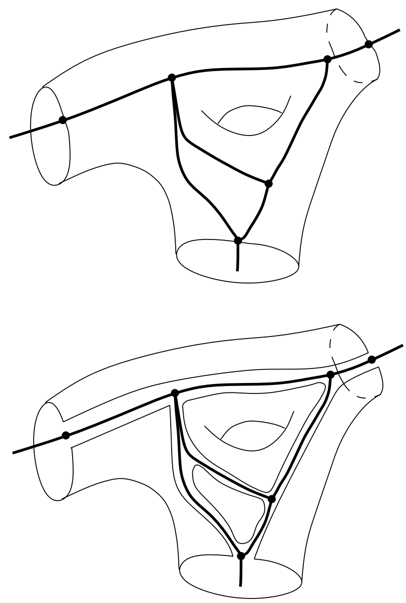


Figure 7: An example of an open graph on a surface, upper figure. The same graph but enriched is shown on a lower figure. All edges of the graph are oriented and all connected components of the boundary of the surface are oriented by the orientation of the surface ("counterclockwise").

connecting vertices of valency greater than one are embedded to Σ.[6] Some of the vertices possibly to $\partial\Gamma$, we call them boundary vertices. One-valent vertices are connected only to boundary vertices. Both, one-valent vertices and edges connecting them to boundary vertices do not belong to Σ, we will call them outer vertices and other edges.

Now let us remove outer vertices. We will be left with Γ partially embedded to Σ with outer edges being open "away from Σ". We will call such pairs (Γ, Σ) *open graphs on* Σ. An example of an open graph on a surface is shown on Fig. 7.

Note that each such graph defines a cell decomposition of Σ with cells that are not necessary contractible.

Now for each open graph Γ on Σ we will construct the *enriched graph* $\widehat{\Gamma}$ as follows:

[6] In the most general setting we can allow to have edges of Γ which also belong to the boundary of Σ.

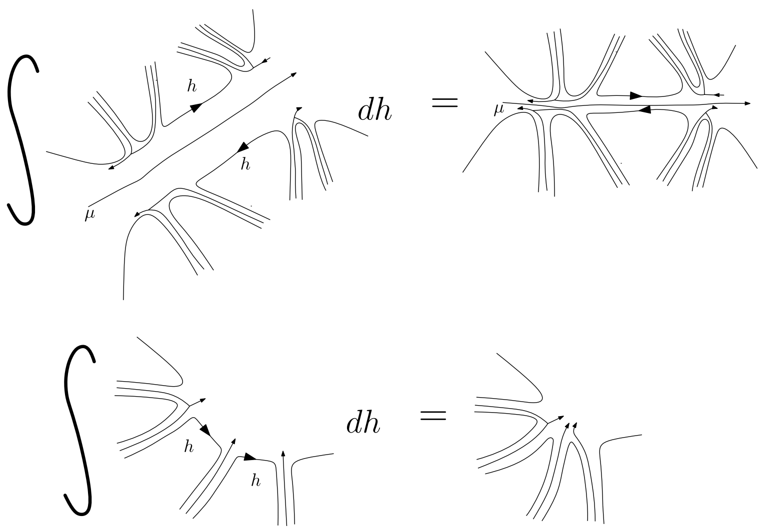


Figure 8: An illustration to how partition surfaces are glued along intervals connecting non-removable corners. This may create multiple outer edges which can still be regarded as a single outer edge colored by the tensor product of representations.

- Add a circle to each connected component of $\Sigma\backslash\Gamma$ which follows the boundary of this connected component.The orientation of this circle is induced by the orientation of Σ.
- Replace intervals of $\partial\Sigma$ between boundary vertices of Γ by segments of the corresponding inserted circle.

Finally, let us define the gluing of surfaces with the enriched graphs.

A surface with an enriched graph in it can be glued to itself by identifying two intervals at the boundary as it is shown on Fig. 8. The upper figure corresponds to the case when the intervals do not shape a vertex. The lower figure corresponds to the case when the intervals share a vertex. In the first case they can belong to different connected components of the surface. Such gluing of surfaces correspond to compositions of partition functions which is also illustrated on Fig. 8 and will be discussed in the next section.

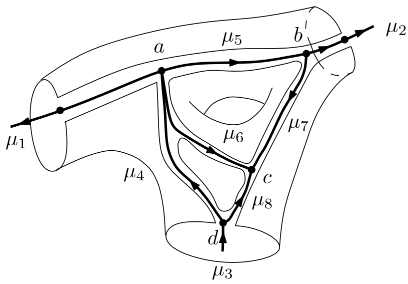


Figure 9: Enriched colored open graph on a surface. Here μ_i are highest weights of representations assigned to edges and $a, b, \ldots$ are invariant vectors assigned to vertices.

2.2. Quantum two-dimensional Yang-Mills theory on surfaces with open Wilson graphs

First fix a simple compact Lie group G and define a G-coloring of a surface with an open graph as follows:

- We assign a finite dimensional representation of G to each edge of the graph Γ.
- For each vertex of Γ fix a total ordering of adjacent edges which agrees with the cyclic ordering induced by the orientation of Σ. Enumerate adjacent edges $e_1, \ldots, e_m$ according to this total ordering. Let $V_{e_1}, \ldots, V_{e_m}$ be colors of representations assigned to adjacent edges. To the corresponding vertex we assign a G-invariant vector in the tensor product

$$a_v \in \left(V_{e_1}^{\epsilon_1} \otimes \cdots \otimes V_{e_m}^{\epsilon_m}\right)^G$$

  Here $V^+ = V$ and $V^- = V^*$ and $+$ is for edges oriented outward the vertex and $-$ is for edges oriented inward.

We will call colored graphs Wilson graphs by analogy with Wilson loops [24, 13].

Now let us assign a variable to each connected component of $\Sigma \backslash \Gamma$. We assume that this variable is additive with respect to gluing. For example if a surface would be equipped with a volume form, an area would be such additive variable. In the previous section when the surfaces are cylinders, Euclidean times evolution for spin Calogero-Moser model Hamiltonians are such variables.

Now let us describe the space of boundary states. Let $\partial_\alpha\Sigma$ be connected components of the boundary of Σ, $\partial\Sigma = \sqcup_\alpha \partial_\alpha\Sigma$. If a connected component does not have boundary vertices of Γ, it is a circle. If it does, it is a union of intervals to which boundary vertices of Γ partition this connected component. We will call these intervals and circles *parts* of $\partial\Sigma$.

To connected component $\partial_\alpha\Sigma$ we assign the following spaces of boundary states:

- If $\partial_\alpha\Sigma$ is a circle we assign to it

$$\mathcal{H}(\partial_\alpha\Sigma) = L_2(G)^G$$

  It is a Hilbert space where characters of finite dimensional irreducible representations of G form an orthonormal basis.
- If the connected component $\partial_\alpha\Sigma$ has boundary vertices of Γ on it, we assign the following space to this boundary component:

$$\mathcal{H}(\partial_\alpha\Sigma) = L_2\big(G^{N_\alpha}, \otimes_{i=1}^{N_\alpha}\big(V_{e_{1,i}}^{\epsilon_{1,i}} \otimes \cdots \otimes V_{e_{m_i,i}}^{\epsilon_{m_i,i}}\big)\big)^{G^{N_\alpha}}$$

  Here $V_{e_{1,i}}^{\epsilon_{1,i}} \otimes \cdots \otimes V_{e_{m_i,i}}^{\epsilon_{m_i,i}}$ is the tensor product of colors of outer edges adjacent to boundary vertex i, G^{N_α} acts by gauge transformations on G^{N_α}. On the tensor product it acts as

$$(g_1, \ldots, g_{N_\alpha})\big(\otimes_{i=1}^{N_\alpha}(x_{e_{1,i}} \otimes \cdots \otimes x_{e_{m_i,i}})\big) = \otimes_{i=1}^{N_\alpha}(g_i x_{e_{1,i}} \otimes \cdots \otimes g_i x_{e_{m_i,i}})$$

The partition function of the 2D Yang-Mills with for a surface with an open Wilson graphs is defined as a vector the following vector in the space of boundary states:

$$Z_{\Gamma,A}(\{g\}) = \int_{G^{E_{int}}} \prod_{D\in\Sigma\backslash\Gamma} Z_{D,A_D}(\{g,h\}_{\partial D}) W_\Gamma(\{g,h\}) \prod_{e\in E_{int}} dh_e \tag{14}$$

Here D are parts of Σ bounded by circles in the enriched open graph Γ, $hol_{\partial D}$ is the product of group elements h_e assigned to internal edges in ∂D and group elements g_e corresponding to boundary edges of ∂D in the counter clock-wise order starting from any edge.

$$Z_{D,A}(hol_{\partial D}) = \sum_\lambda \chi_\lambda(hol_{\partial D}) \dim(\lambda)^{1-2g} e^{-c_2(\lambda)A} \tag{15}$$

where g is the genus of the surface obtained from D by gluing a disc to ∂D, i.e. by "closing" the surface.

For example $hol_{\partial D} = 1$ can be interpreted as if the boundary of D does exists, i.e. shrinks to a point. Thus $Z_{D,A}(1)$ is the partition function of 2D Yang-Mills on the "closure" of D [24]. If D is a disk

$$Z_{D,A}(g) = \sum_{\lambda} \chi_\lambda(g) \dim(\lambda) e^{-c_2(\lambda)A} \tag{16}$$

is the partition function of the 2D Yang-Mills on D with boundary holonomy g. It is easy to see that for a triangulated surface D the partition function (15) can be composed from elementary partition functions (16) [23]. This is the manifestation of locality of 2D Yang-Mills theory.

The function $W_\Gamma(\{g, h\})$ is the contraction of holonomies along internal edges evaluated in the representation corresponding to this edge and invariant vectors corresponding to vertices of Γ. Such function on $G^{E_{int}}$ is know as Wilson graph evaluated on a graph connection see for example [24, 13]. Wilson loop is a particular case of such a function.

Partition functions (14) satisfy the gluing property.

Theorem 5. *Let Σ_1 and Σ_2 be two surfaces with boundary edges $e_1 \in \partial\Sigma_1$ and $e_2 \in \Sigma_2$ being identified as it is shown of Fig. 8. Let h be the holonomy along identified edges then*

$$\int_G Z(\Sigma_1; h) \otimes \pi_\mu(h) \otimes Z(\Sigma_2; h) dh = Z(\Sigma_1 \cup_{e_1 \simeq e_2} \Sigma_2)$$

where in the partition function on the right there is an extra Wilson line colored by V_μ "sandwiching" between Σ_1 and Σ_2. And a similar gluing identity we have for the partition functions in case we glue two neighboring intervals on $\partial\Sigma$.

The proof follows immediately from integral identities in the Appendix.

2.3. Point observables

Wilson graphs can be regarded as line observables in 2D Yang-Mills. They has been studied since many years ago, see for example [11, 23, 24, 3, 13].

Let $D \subset \Sigma$ be a disc. Let us assign to D the following function on G/G:

$$\mathcal{O}_A^F = \sum_{\lambda} \chi_\lambda(g) F_\lambda \dim(\lambda) e^{-Ac_2(\lambda)}$$

When $F_\lambda = 1$ for all λ this is the partition function of 2D Yang-Mills on a disc. We will call such function the partition function of a disc D with *topological* *point observables* inserted in the disc. The position of the point observable on D is not given since it is a topological observable. One should think of the function F as "sitting" at a point in the interior of D, confined to D by the Wilson loop at the boundary.

Naturally, if we glue a surface to D along a connected component $\partial_\alpha\Sigma$ of the boundary $\partial\Sigma$ of the surface Σ the resulting partition function is the partition function with the point observable F inserted in Σ.

The insertion of such observable in the partition function for a surface with an open Wilson graph Γ on it depends on into which region of $\Sigma\backslash\Gamma$ it is inserted. But it does not depend at which point of the region it is inserted.

If we put such point observable on a cylinder, the corresponding integral operators

$$U_A^F(g,h) = \sum_\lambda \chi_\lambda(g)\overline{\chi_\lambda(h)}F_\lambda \exp(-c_2(\lambda)A)$$

form commutative algebra

$$U_A^F * U_B^G = U_{A+B}^{FG}$$

where $(FG)_\lambda = F_\lambda G_\lambda$.

An example of such observable was introduced in [24] as the insertion making the partition function independent of the orientation. If $\lambda^* = -w_0(\lambda)$[7] is the highest weight of the dual representation to V_λ, such observable corresponds to $F_\lambda = \delta_{\lambda,\lambda^*}$.

3. Conclusion

Here we will list some open problems and future directions.

1. The study of large n limit in the SU_n 2D YM theory, is one of the most interesting directions for a number of well known reasons. When Σ is a sphere there is a phase transition in this limit [4] at $A=\pi^2$. Similar behavior for the cylinder and for a sphere with three holes was studied in [7]. For the relation between the 2D YM in large n limit and the string theory see [10]. It would be interesting to extend these results to the case of open Wilson graphs.
2. In the follow up paper we will describe the 2D YM theory with mixed gauge groups. Particular cases of such models are related to open spin Calogero-Moser spin chains, see [21] and for its classical counterpart [18].

[7]Here w_0 is the longest element of the Weyl group.

3. There is a natural q-deformation of the N-spin CM systems described in [2]. When q is a root of unity this system is closely related to the quantum Chern-Simons theory [22] for $U_q(\mathfrak{g})$ at roots of unity [14]. Corresponding deformations of 2D YM theory are invariants of twisted circle bundles over surfaces. Quadratic Casimirs are special in this setting because the ribbon structure in the corresponding modular tensor category is given by the exponents of quadratic Casimirs. Classical counterparts of these deformations are "relativistic" versions of spin Calogero-Moser models on moduli space of flat connections constructed in [1].
4. As it was already mentioned partition functions of 2D YM theory can be regarded as a semiclassical limit of invariants of twisted circle bundles over surfaces corresponding to the quantum Chern-Simons theory with a compact Lie group. This makes the study of a quantum 2D YM theory for a non-compact Lie group even more interesting. It may be a toy model for quantum complex Chern-Simons theory of a Chern-Simons theory for a non-compact real Lie group. Quantum N-spin systems for noncompact real forms of simple complex Lie groups were studied in [21].

Appendix A. On quantum moment maps

A.1. Quantum moment maps for G-actions

The left and the right quantum moment maps act as:

$$\widehat{\mu}_\ell(X)f(g) = \frac{d}{dt}f\big(e^{-tX}g\big)|_{t=0}, \quad \widehat{\mu}_r(X)f(g) = \frac{d}{dt}f\big(ge^{tX}\big)|_{t=0}$$

For the left moment map acting on monomials in $U(\mathfrak{g})$ we have

$$\widehat{\mu}_\ell(x) = \widehat{\mu}_\ell(X_1)\dots\widehat{\mu}_\ell(X_k)f(g) = \frac{\partial^k}{\partial t_1\dots\partial t_k}f\big(e^{-t_kX_k}\dots e^{-t_1X_1}g\big)$$

and a similar formula for the right action.

For $X\in\mathfrak{g}\subset U(\mathfrak{g})$ the left and the right quantum moment maps are related as:

$$\widehat{\mu}_r(X)f(g) = -\widehat{\mu}_\ell\big(Ad_g(X)\big)f(g) \tag{17}$$

This relation extends to monomials $x = X_1\dots X_k$ as

$$\widehat{\mu}_r(x) = \widehat{\mu}_\ell\big(Ad_g\big(S(x)\big)$$

Indeed

$$\begin{aligned}(18)\qquad \widehat{\mu}_r(x)f(g) &= \widehat{\mu}_r(X_1)\dots\widehat{\mu}_r(X_k)f(g) = \frac{\partial^k}{\partial t_1\dots\partial t_k}f\big(ge^{t_1X_1}\dots e^{t_kX_k}\big)\\ &= (-1)^k\widehat{\mu}_\ell\big(Ad_g(X_k)\big)\dots\widehat{\mu}_\ell\big(Ad_g(X_1)\big)f(g)\\ &= \widehat{\mu}_\ell\big(Ad_g(S(x))\big)f(g)\end{aligned}$$

Lie group G acts on its Lie algebra $\mathfrak{g}$ via adjoint action $g : X \mapsto Ad_g(X)$. This action lifts to the action on $U(\mathfrak{g})$ which will also denote by Ad_g. The Lie group G also acts naturally on differential operators on G, $h : D \mapsto D^h$

$$D^h f(g) = h_L^{*\,-1}\big(Df(hg)\big)$$

Here $h_L^{*\,-1}f(g) = f(h^{-1}g)$.

Quantum moment maps are G-equivariant:

$$\widehat{\mu}_\ell^h(x) = \widehat{\mu}_\ell\big(Ad_h(x)\big)$$

Let $\{e_i\}$ be a basis in $\mathfrak{g}$ and $z = \sum_{i_1,\dots,i_k} f^{i_1,\dots,i_k}e_{i_1}\dots e_{i_k}$. It is clear that differential operator $D = \widehat{\mu}_\ell(z)$ is G-invariant iff tensor f is G-invariant, i.e. iff $z \in Z(\mathfrak{g})$.

Lemma 1. *Let $z \in Z(\mathfrak{g})$, then*

$$\widehat{\mu}_r(z) = \widehat{\mu}_\ell(S(z)$$

Proof. Indeed, we have:

$$\begin{aligned}(19)\qquad \widehat{\mu}_r f(g) &= \sum_{\{i\}} f^{\{i\}}\widehat{\mu}_r(e_{i_1})\dots\widehat{\mu}_r(e_{i_k})f(g)\\ &= \sum_{\{i\}} f^{\{i\}}\frac{\partial^k}{\partial t_1\dots\partial t_k}f\big(ge^{t_1e_{i_1}}\dots e^{t_ke_{i_k}}\big)\\ &= \sum_{\{i\}} f^{\{i\}}\widehat{\mu}_\ell(e_{i_k})\dots\widehat{\mu}_\ell(e_{i_1})f(g) = \widehat{\mu}_\ell\big(S(z)\big)f(g)\end{aligned}$$ □

A.2. Quantum moment maps for G^N actions

For the left and right action of G^N on itself:

$$\widehat{\mu}_L(X_1,\dots,X_N) = \widehat{\mu}_\ell^{(1)}(X_1) + \dots + \widehat{\mu}_\ell^{(N)}(x_N),$$

$$\widehat{\mu}_R(X_1,\dots,x_N)=\widehat{\mu}_r^{(1)}(X_1)+\cdots+\widehat{\mu}_r^{(N)}(X_N)$$

where $\widehat{\mu}_{\ell,r}^{(i)}$ are left and right moment maps for the i-th factor.

For the gauge action of G^N on G^N by gauge transformations we have:

$$\widehat{\mu}(X_1,\dots,X_N)=\widehat{\mu}_L(X_1,X_2,\dots,X_N)+\widehat{\mu}_R(X_2,X_3\dots,X_N,X_1)$$

These Lie algebra homomorphisms extend uniquely to algebra homomorphisms $\widehat{\mu},\widehat{\mu}_L,\widehat{\mu}_R:U(\mathfrak{g})^{\otimes N}\to\mathbb{D}(G^N)$

Lemma 1 implies that

$$\widehat{\mu}_R(z_1\otimes\cdots\otimes z_N)=\widehat{\mu}_L\big(S(z_1)\otimes\cdots\otimes S(z_N)\big)$$

Because of this the natural algebra homomorphism

$$\widehat{\mu}_L\otimes\widehat{\mu}_R:U(\mathfrak{g})^{\otimes N}\otimes U(\mathfrak{g})^{\otimes N}\to\mathbb{D}(G^N)$$

filters through the algebra homomorphism

$$\widehat{\mu}_L\otimes\widehat{\mu}_R:U(\mathfrak{g})^{\otimes N}\widetilde{\otimes}_{Z(\mathfrak{g})^{\otimes N}}U(\mathfrak{g})^{\otimes N}\to\mathbb{D}(G^N)$$

Here $U(\mathfrak{g})^{\otimes N}\widetilde{\otimes}_{Z(\mathfrak{g})^{\otimes N}}U(\mathfrak{g})^{\otimes N}$ is the tensor product over $Z(\mathfrak{g})^{\otimes N}$ where $(a_1z_1\otimes\cdots\otimes a_Nz_N)\otimes_{Z(\mathfrak{g})^{\otimes N}}(b_1\otimes\cdots\otimes b_N)=(a_1\otimes\cdots\otimes a_N)\otimes_{Z(\mathfrak{g})^{\otimes N}}(S(z_1)b_1\otimes\cdots\otimes S(z_N)b_N)$.

The mapping ϕ_i induces a linear isomorphism $\mathrm{End}(\mathcal{H}_\mu)\to\mathrm{End}(\mathcal{H}_\mu^H)$ for which we will use the same name ϕ_i. Let us first compute how $\widehat{\mu}_L(U(\mathfrak{g}))$ and $\widehat{\mu}_R(U(\mathfrak{g}))$ act on $\mathcal{H}_\mu^H$ and then the action of Hamiltonians $H_{d_k}^{(j)}$.

A.3. Quantum moment maps after gauge fixing

Let $X\in\mathfrak{g}\subset U(\mathfrak{g})$ and $X_j=1^{\otimes(j-1)}\otimes X\otimes 1^{\otimes(N-j)}$. The following is the result of an elementary computation:

$$\widehat{\mu_R}(X_j)f(g)=-\widehat{\mu_L}(Ad_{g_j}X_j)f(g),$$
$$\widehat{\mu_R}(X_{j+1})+\widehat{\mu_L}(X_j)=\pi_j^{\mu_j}(X)f(g)$$

for functions in $\mathcal{R}_\mu$. Applying gauge fixing at i to these identities we obtain the left action of $X^{(j)}$ on $\mathcal{H}_\mu^H$:

$$\phi_i\big(-\widehat{\mu_L}(X_{j+1})+\widehat{\mu_L}(X_j)\big)f(h)=\pi_j^{\mu_j}(X)f(h),\quad j\neq i \tag{20}$$

$$\phi_i\big(-\widehat{\mu_L}(Ad_h(X_i)) + \widehat{\mu_L}(X_{i-1})\big)f(h) = \pi_{i-1}^{\mu_{i-1}}(X)f(h), \tag{21}$$

where $f \in \mathcal{H}_\mu^H$).

Choose a basis e_α in each root subspace $\mathfrak{g}_\alpha \subset \mathfrak{g}$. From the equations above we can express the action of $e_{\alpha,j}$ in terms of $\pi_j^{\mu_j}(X)$ and $h \in H$:

$$\phi_i(\widehat{\mu_L})(e_{\alpha,j}) = \frac{1}{1-h_\alpha} S_\alpha^j,$$

where

$$S_\alpha^i = \sum_{j=1}^N \pi_j^{\mu_j}(e_\alpha) \tag{22}$$

(23)
$$S_\alpha^j = h_\alpha\big(\pi_i^{\mu_i}(e_\alpha) + \cdots + \pi_{j-1}^{\mu_{j-1}}(e_\alpha)\big) + \pi_j^{\mu_j}(e_\alpha) + \cdots + \pi_{i-1}^{\mu_{i-1}}(e_\alpha), \quad j \neq i$$

Here h_α is the coordinate function on H corresponding to the root α, $he_\alpha h^{-1} = h_\alpha e_\alpha$ and the summation is cyclic.

For the left action of the Cartan subalgebra we have

$$\phi_i\big(\widehat{\mu}_L(H_{\alpha,j}) - \widehat{\mu}_L(H_{\alpha,j+1})\big)f(g) = \pi_j(h)f(g) \tag{24}$$

Here $j+1$ is defined cyclically, i.e. $\mod N$.

After gauge fixing we have

$$\phi_i(\widehat{\mu}_L(H_{\alpha,i})f(1,\dots,h,\dots,1) = -i\frac{\partial}{\partial q_\alpha} f(1,\dots,h,\dots,1)$$

The action of $\phi_i(\widehat{\mu}_L(H_{\alpha,j})$ with $j \neq i$ can be computed from this formula and from equations (20)(21):

$$\phi_i(\widehat{\mu}_L(H_{\alpha,j}) = -\pi_i^{\mu_i}(H_\alpha) - \pi_{i+1}^{\mu_{i+1}}(H_\alpha) - \cdots - \pi_j^{\mu_j}(H_\alpha) - i\frac{\partial}{\partial q_\alpha}$$

Here the sum is cyclic.

Appendix B. Quadratic Hamiltonian for $N = 1$

For $N = 1$ the gauge invariance of functions from the space $\mathcal{R}_\mu$ is $\psi(hgh^{-1}) = \pi(h)\psi(g)$. The transformation property of functions from $\mathcal{R}_\mu$ imply:

$$\big(\widehat{\mu}_\ell(X) + \widehat{\mu}_r(X)\big)f(g) = \pi(X)f(g)$$

Combining this with (17), for functions f from $\mathcal{R}_\mu$ we have:

$$\big(\widehat{\mu}_\ell(X) - \widehat{\mu}_r\big(Ad_g(X)\big)\big)f(g) = \pi(g)f(g)$$

Now let us compute $\widehat{\mu}_\ell(Y)\widehat{\mu}_\ell(X)f(g)$. From the definition (18) we have:

$$\widehat{\mu}_\ell(Y)\big(\widehat{\mu}_\ell(X) - \widehat{\mu}_\ell\big(Ad_g(X)\big)\big)f(g) = \pi(X)\widehat{\mu}_\ell(Y)f(g)$$

similarly

$$-\widehat{\mu}_\ell\big(Ad_g(Y)\big)\big(\widehat{\mu}_\ell(X) - \widehat{\mu}_\ell\big(Ad_g(X)\big)\big)f(g) = -\pi(X)\widehat{\mu}_\ell(Ad_g(Y)f(g)$$

Combining these identities we have:

Theorem 6.

(25)
$$\begin{aligned}\pi(X)\pi(Y)f(g) = \big(\widehat{\mu}_\ell(Y) - \widehat{\mu}_\ell(\widetilde{Z})\big)\big(\widehat{\mu}_\ell(X) - \widehat{\mu}_\ell(Z)\big)\psi|_{Z=Ad_g(X),\widetilde{Z}=Ad_g(Y)} \\ + \widehat{\mu}_\ell\big([Y, Ad_g(X)]\big)\psi - \widehat{\mu}_\ell\big(Ad_g([Y,X])\big)f(g)\end{aligned}$$

From here, setting $X = e_\alpha, Y = e_{-\alpha}$ and specializing g to $h \in H$ we obtain:

$$\pi(e_\alpha)\pi(e_{-\alpha})f(h) = (1-h_\alpha)(1-h_{-\alpha})\widehat{\mu}_\ell(e_{-\alpha})\widehat{\mu}_\ell(e_\alpha)f(h) + (1-h_\alpha)\widehat{\mu}_\ell(H_\alpha)f(h)$$

or

$$\widehat{\mu}_\ell(e_{-\alpha})\widehat{\mu}_\ell(e_\alpha)f(h) = \frac{\pi(e_\alpha)\pi(e_{-\alpha})}{(1-h_\alpha)(1-h_{-\alpha})}f(h) + \frac{\widehat{\mu}_\ell(H_\alpha)}{(1-h_\alpha)}f(h)$$

Setting $h = e^\lambda$ we can write $\widehat{\mu}_\ell(H_\alpha) = (\alpha, \frac{\partial}{\partial\lambda})$ and for the second Casimir we obtain

$$\widehat{\mu}_\ell(c_2)f(h) = \Big(\Delta + 2\sum_{\alpha>0}\frac{\pi(e_\alpha)\pi(e_{-\alpha})}{(1-h_\alpha)(1-h_{-\alpha})} - D\Big)f(h)$$

where $D = \sum_{\alpha>0}\frac{1+h_\alpha}{1-h_\alpha}(\alpha, \frac{\partial}{\partial\lambda})$.

Appendix C. Quantum quadratic Hamiltonians for $N > 1$

Here we will compute quadratic Hamiltonians from the action of $U(\mathfrak{g})^{\otimes N}$ on trace functions by left and right quantum moment map. Since trace functions form a basis in the space $\mathcal{R}_\mu$ this is equivalent to computing quantum Hamiltonians directly from the action on functions from $\mathcal{R}_\mu$ as it was done above for $N = 1$, but it is easier.

C.1. Dynamical r-matrices

For $\lambda \in \mathfrak{h}$ define $\xi_\alpha = e^{\frac{(\lambda,\alpha)}{2}}$, $h_\alpha = \xi_\alpha^2$. The dynamical r-matrix [9] is the following $\mathfrak{g} \otimes \mathfrak{g}$-valued function on $\mathfrak{h}$

$$r(\lambda) = -\frac{1}{2}\sum_{i=1}^{r} h_i \otimes h_i - \sum_{\alpha} \frac{e_{-\alpha} \otimes e_\alpha}{1 - h_{-\alpha}}$$

or

$$r(\lambda) = -\frac{1}{2}\sum_{i=1}^{r} h_i \otimes h_i + \sum_{\alpha>0} (\frac{e_\alpha \otimes f_\alpha}{h_\alpha - 1} - \frac{h_\alpha f_\alpha \otimes e_\alpha}{h_\alpha - 1}$$

Here are some basic properties of $r(\lambda)$:

- $r(-\lambda) = r^{21}(\lambda)$,
- $r(\lambda) + r(-\lambda) = -\Omega$,
- $(e^{-\lambda})_2 r_{12}(\lambda)(e^{\lambda})_2 = -\sum_{i=1}^{r} h_i \otimes h_i - r_{21}(\lambda)$,
- $D_1 D_2 r(\lambda) D_1^{-1} D_2^{-1} = r(\lambda)$,
- The dynamical Yang-Baxter equation, see [9].

C.2. Quantum topological Knizhnik-Zamolodchikov-Bernard equation

All the computations from this sections can be done from the definition of the space $\mathcal{R}_\mu$ as in [15]. However, it is more convenient to do them using the basis of trace functions after the gauge fixing for $i = N$.

Choose a basis e_α is root subspaces of $\mathfrak{g}$ and orthonormal (with respect to the Killing form) basis in $\mathfrak{h}$. Let c_2 be the quadratic Casimir for $\mathfrak{g}$:

$$c_2 = \sum_{i=1}^{r} h_i^2 + \sum_{\alpha>0} (e_\alpha f_\alpha + f_\alpha e_\alpha)$$

where $f_\alpha = e_{-\alpha}$. If H_{α_i} is a basis in $\mathfrak{h}$ corresponding to simple roots, $[e_\alpha, f_\alpha] = H_\alpha$, the Cartan part of the Casimir element can be written as

$$\sum_{i=1}^{r} h_i^2 = \sum_{i,j=1}^{r} H_{\alpha_i} \left(B^{-1}\right)_{ij} H_{\alpha_j}$$

where B is the symmetrized Cartan matrix.

Denote by Ω the "mixed Casimir":

$$\Omega = \sum_{i=1}^{r} h_i \otimes h_i + \sum_{\alpha>0} (e_\alpha \otimes f_\alpha + f_\alpha \otimes e_\alpha)$$

If Δ is the comultiplication for $U(\mathfrak{g})$, we have

$$\Delta c_2 = c_2 \otimes 1 + 1 \otimes c_2 + 2\Omega$$

For $b \in \mathrm{Hom}_{\mathfrak{g}}(M_\nu, M_\mu \otimes V_\eta)$ where M_μ, M_ν are Verma modules and V_λ is an irreducible finite dimensional module with the highest weight η. We have

$$bc_2(\nu) = (c_2(\mu) + c_2(\eta) + 2\Omega)b$$

This identity can be written in terms of dynamical r-matrices as

$$\frac{c_2(\nu) - c_2(\mu) - c_2(\eta)}{2} b = (r_{12}(\lambda) + r_{21}(\lambda))b \tag{26}$$

Here $\lambda \in \mathfrak{h}$.

For $b_i \in \mathrm{Hom}_G(V_{\nu_i}, V_{\nu_{i-1}} \otimes V_{\mu_i})$, with $0 = N$, consider the trace function (8) evaluated at $g = (1, \dots, 1, e^\lambda)$, i.e. for the gauge fixing with $i = N$:

$$\Psi^{(\nu)}_{b,\mu}(\lambda) = Tr_a(b_1 \dots b_N (e^\lambda)_a) \tag{27}$$

Here we denote representation V_{μ_N} by index a (auxiliary space) and take trace over this space.

Denote

$$d(\lambda) = \frac{1}{2} \sum_{\alpha>0} \frac{\xi_\alpha + \xi_{-\alpha}}{\xi_\alpha - \xi_{-\alpha}} H_\alpha$$

Theorem 7. *Trace function F satisfies differential equations*

$$\left(\left(h^{(i)}, \frac{\partial}{\partial\lambda}\right) - \sum_{k=1}^{i-1} r_{ki}(\lambda) + \sum_{k=i+1}^{n} r_{ik}(\lambda) - d(\lambda)_i\right) \Psi^{(\nu)}_{b,\mu}(\lambda) \tag{28}$$
$$= \frac{c(\nu_{i+1}) - c(\nu_i)}{2} \Psi^{(\nu)}_{b,\mu}(\lambda)$$

Proof. Let us apply the identity (26) to the i-th factor in the definition of the trace function. We have:

$$\frac{c_2(\nu_i)+c_2(\mu_i)-c_2(\nu_{i+1})}{2} \Psi^{(\nu)}_{b,\mu}(\lambda) = Tr_a(b_1 \dots b_{i-1}(r_{ia}(\lambda)+r_{ai}(\lambda))b_i \dots b_N (e^\lambda)_a)$$

From the intertwining property of b_i we have:

$$b_1 \dots b_{i-1} r_{ai}(\lambda) = (r_{ai}(\lambda) + r_{1i}(\lambda) + \dots + r_{i-1,i}(\lambda) b_1 \dots b_{i-1}$$

Now let us use the cyclicity of the trace:

$$\begin{aligned}
(29) \qquad & Tr_a\big(r_{ai}(\lambda) b_1 \dots b_N (e^\lambda)_a\big) \\
& = Tr_a\big(b_1 \dots b_N (e^\lambda)_a r_{ai}(\lambda) (e^{-\lambda})_a (e^\lambda)_a\big) \\
& = Tr_a\Big(b_1 \dots b_N \Big(-\sum_k h_k^{(i)} h_k^{(a)} - r_{ia}(\lambda)\Big)(e^\lambda)_a\Big) \\
& = -\Big(h^{(i)}, \frac{\partial}{\partial \lambda}\Big) Tr_a\big(b_1 \dots b_N (e^\lambda)_a\big) - Tr_a\big(b_1 \dots b_N r_{ia}(\lambda)(e^\lambda)_a\big))
\end{aligned}$$

Here we used properties of the dynamical r-matrix listed in section C.1.

Now let us use the intertwining properties of b_j's:

$$b_{i+1} \dots b_N r_{ia}(\lambda) = (r_{i,i+1}(\lambda) + \dots + r_{iN}(\lambda) + r_{ia}(\lambda)) b_{i+1} \dots b_N$$

and

$$b_i r_{ia}(\lambda) = \big(r_{ia}(\lambda) + m(r(\lambda))_i\big) b_i$$

Here $m(a \otimes b) = ab$ for $a, b \in U(\mathfrak{g})$.

$$m(r(\lambda)) = -\frac{1}{2}\sum_i h_i^2 - \sum_\alpha \frac{e_{-\alpha} e_\alpha}{1 - h_{-\alpha}} = -\frac{1}{2} c_2 + d(\lambda)$$

where $d(\lambda)$ is as above.

Thus,

$$b_i b_{i+1} \dots b_N r_{ia}(\lambda) = \Big(r_{ia}(\lambda) - \frac{c_2(\mu_i)}{2} + d(\lambda)_i + r_{i,i+1}(\lambda) + \dots + r_{iN}(\lambda)\Big) b_i \dots b_N$$

Combining all identities we obtain (28). □

C.3. The quantum spin Calogero-Moser Hamiltonian

For $b \in \mathrm{Hom}_{\mathfrak{g}}(M_\nu, M_\nu \otimes V)$, where V is a finite dimensional representation of $\mathfrak{g}$ and M_ν is the Verma module with the highest weight ν, we define the trace function

$$\Phi_{b,V}^{(\nu)}(\lambda) = Tr_a\big(b (e^\lambda)_a\big)$$

Here the trace is takes over the space M_ν which we denote by index a (auxiliary). For our purposes it is enough to consider this trace function as a Laurent power series in e^λ.

Let Δ be the Laplacian on $\mathfrak{h}$:

$$\Delta = \left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\lambda}\right)$$

where $(.,.)$ is the bilinear form on $\mathfrak{h}^*$ which is dual to the Killing from on $\mathfrak{h}$. We have:

$$\sum_i h_i^2 e^\lambda = \Delta e^\lambda$$

We also have the identity

$$c_2(\nu)\Psi^{(\nu)}_{b,V}(\lambda) = Tr_a\big(b(c_2)_a\big(e^\lambda\big)_a\big)$$

Lemma 2. *Following identities hold:*

$$Tr_{M_\nu}\big(be_\alpha e^\lambda\big) = \frac{\pi^V(e_\alpha)}{1-h_\alpha}\Psi^{(\nu)}_{b,V}(\lambda) \tag{30}$$

$$Tr_{M_\nu}\big(bf_\alpha e^\lambda\big) = \frac{\pi^V(f_\alpha)}{1-h_{-\alpha}}\Psi^{(\nu)}_{b,V}(\lambda) \tag{31}$$

Indeed, the first identity follows from the intertwining property of b and from the cyclicity of the trace:

$$Tr\big(be_\alpha e^\lambda\big) = \pi^V(e_\alpha)\Psi^{(\nu)}_{b,V}(\lambda) + Tr\big(be^\lambda e_\alpha\big) = \pi^V(e_\alpha)\Psi^{(\nu)}_{b,V}(\lambda) + h_\alpha Tr\big(be_\alpha e^\lambda\big)$$

The proof of (31) is similar.

Proposition 1.

(32)
$$Tr_{M_\nu}\big(be_\alpha f_\alpha e^\lambda\big) = \frac{\pi^V(e_\alpha f_\alpha)}{(1-h_\alpha)(1-h_{-\alpha})}\Psi^{(\nu)}_{b,V}(\lambda) - \frac{h_\alpha}{1-h_\alpha}\left(\alpha, \frac{\partial}{\partial\lambda}\right)\Psi^{(\nu)}_{b,V}(\lambda)$$

(33)
$$Tr_{M_\nu}\big(bf_\alpha e_\alpha e^\lambda\big) = \frac{\pi^V(f_\alpha e_\alpha)}{(1-h_\alpha)(1-h_{-\alpha})}\Psi^{(\nu)}_{b,V}(\lambda) - \frac{1}{1-h_\alpha}\left(\alpha, \frac{\partial}{\partial\lambda}\right)\Psi^{(\nu)}_{b,V}(\lambda)$$

Proof.

(34)

$$\begin{aligned} Tr_{M_\nu}(be_\alpha f_\alpha e^\lambda) &= \pi^V(e_\alpha)Tr_{M_\nu}(bf_\alpha e^\lambda) + Tr_{M_\nu}(bf_\alpha e^\lambda e_\alpha) \\ &= \frac{\pi^V(f_\alpha)}{1-h_{-\alpha}}\Psi^{(\nu)}_{b,V}(\lambda) + h_\alpha Tr_{M_\nu}(bf_\alpha e_\alpha e^\lambda) \\ &= \frac{\pi^V(f_\alpha)}{1-h_{-\alpha}}\Psi^{(\mu)}_{b,V}(\lambda) + h_\alpha Tr_{M_\nu}(be_\alpha f_\alpha e^\lambda) - h_\alpha\left(\alpha, \frac{\partial}{\partial\lambda}\right)\Psi^{(\mu)}_{b,V}(\lambda) \end{aligned}$$

The proof of (33) is similar. □

Corollary 3.

$$Tr_{M_\nu}(b(e_\alpha f_\alpha + f_\alpha e_\alpha)e^\lambda) = \frac{2\pi^V(f_\alpha e_\alpha)}{(1-h_\alpha)(1-h_{-\alpha})}\Psi^{(\nu)}_{b,V}(\lambda) - \frac{1+h_\alpha}{1-h_\alpha}\left(\alpha, \frac{\partial}{\partial\lambda}\right)\Psi^{(\nu)}_{b,V}(\lambda)$$

Here we use that $\Psi^{(\mu)}_{b,V}(\lambda)$ is an element of the zero weight subspace in V.
Thus, we proved the following theorem.

Theorem 8.

(35)

$$\Delta\Psi^{(\nu)}_{b,V}(\lambda) + 2\sum_{\alpha>0}\frac{\pi^V(f_\alpha e_\alpha)}{(1-h_\alpha)(1-h_{-\alpha})}\Psi^{(\nu)}_{b,V}(\lambda) - \frac{1+h_\alpha}{1-h_\alpha}\left(\alpha, \frac{\partial}{\partial\lambda}\right)\Psi^{(\nu)}_{b,V} = c_2(\mu)\Psi^{(\mu)}_{b,V}$$

It is easy to see that if we replace Verma module M_ν by an irreducible representation V_ν with the highest weight μ, the theorem and its proof hold. Note that when $V = V_{\mu_1} \otimes \cdots \otimes V_{\mu_N}$ with the diagonal action of $\mathfrak{g}$, and $b = b_1 \dots b_N$, where b_i are the same intertwines as in (27), the trace function $\Phi^{(\nu)}_{b,V}$ becomes the trace function $\Psi^{(\nu)}_{a,\mu}$ (27) with $\nu = \nu_N$.

This proves that trace functions (27) satisfy (35).

C.4. Normalization

Lemma 3.

(36) $$\Delta(\delta^{-1}) = -(\rho,\rho) - D(\delta^{-1})$$

where $D = \sum_{\alpha>0}\frac{\xi_\alpha+\xi_{-\alpha}}{\xi_\alpha-\xi_{-\alpha}}(\alpha, \frac{\partial}{\partial\lambda})$ *and* δ *is the denominator in the Weyl character formula:*

$$\delta = \prod_{\alpha>0}(\xi_\alpha - \xi_{-\alpha})$$

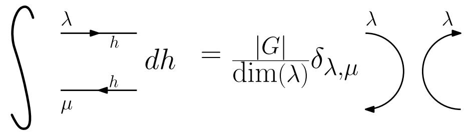


Figure 10: Graphical representation of the identity (37).

Proof. Consider the trace function for the trivial representation V with $b = id : M_\nu \to M_\nu$. In this case $\Psi^{(\nu)}_{b,V} = \frac{e^{(\nu+\rho,\lambda)}}{\delta}$ is the character of M_ν. In this case the identity (35) immediately implies (36). $\square$

Proposition 2.

$$\delta \circ (\Delta + D) \circ \delta^{-1} = \Delta - \|\rho\|^2$$
$$\delta \circ \left(h, \frac{\partial}{\partial\lambda}\right) \circ \delta^{-1} = -\sum_{\alpha>0} \frac{\xi_\alpha + \xi_{-\alpha}}{\xi_\alpha - \xi_{-\alpha}} H_\alpha$$

Define the normalized trace function as

$$F^{(\nu)}_{b,V}(\lambda) = \delta^{-1}\Psi^{(\nu)}_{b,V}$$

We proved:

Theorem 9. *Normalized trace functions satisfy differential equations*

$$H_2^{(N)} F^{(\nu)}_{b,V}(\lambda) = c_2(\nu) F^{(\nu_N)}_{b,V}(\lambda), \quad D_i F^{(\nu)}_{b,V}(\lambda) = \frac{c_2(\nu_{i+1}) - c_2(\nu_i)}{2} F^{(\nu)}_{b,V}(\lambda)$$

where $i = 1, \ldots, N$ *and*

$$H_2^{(N)} = \Delta + 2\sum_{\alpha>0} \frac{\pi^V(f_\alpha e_\alpha)}{(1-h_\alpha)(1-h_{-\alpha})} - \|\rho\|^2$$

and

$$D_i = \left(h^{(i)}, \frac{\partial}{\partial\lambda}\right) - \sum_{k=1}^{i-1} r_{ki}(\lambda) + \sum_{k=i+1}^{n} r_{ik}(\lambda)$$

Appendix D. Useful identities with the Haar measure

Choose a basis $\{e_i\}$ in the representation space V_λ and denote by $\pi^\lambda(g)^i_j$ matrix elements of linear operator $\pi_\lambda(g)$ acting on this space $\pi_\lambda(g)e_j =$

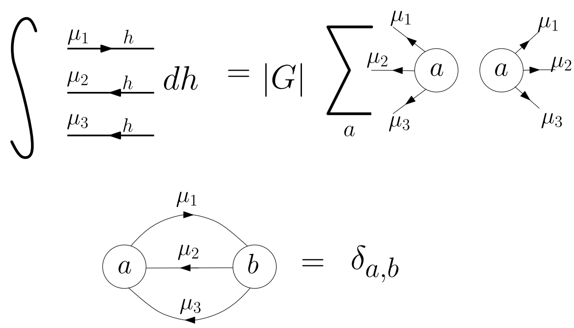


Figure 11: Graphical representation of the identity (38).

$\sum_i \pi^\lambda(g)^i_j e_i$. Let dh be the Haar measure on G, then

(37)
$$\int_{G^{\times 2}} \pi^\lambda(h^{-1})^i_j \pi^\mu(h)^k_l dh = \frac{1}{\dim(\lambda)} \delta_{\lambda,\mu} \delta^i_l \delta^k_j$$

Note that if $\{e^i\}$ is the dual basis then $\pi_\lambda(g)^*(e^i) = \sum_j \pi^\lambda(g)^i_j e^j$. In the basis free form this identity can be written as

$$\int_{G^{\times 2}} (\pi^\lambda)^*(h^{-1}) \otimes \pi^\mu(h) dh = \frac{1}{\dim(\lambda)} \delta_{\lambda,\mu} \iota \circ ev$$

where $ev : V^*_\lambda \otimes V_\lambda \to \mathbb{C}$ is the evaluation map and $\iota : \mathbb{C} \to V^*_\lambda \otimes V_\lambda$ is the injection map.

(38)
$$\int_{G^{\times 3}} \pi^{\mu_1}(h^{-1})^i_j \pi^{\mu_2}(h)^k_l \pi^{\mu_3}(h)^s_t dh = \sum_{a \in (V^*_{\mu_1} \otimes V_{\mu_2} \otimes V_{\mu_3})^G} a^{iks} \overline{a}_{jlt}$$

where $\{a\}$ is a basis in subspace of G-invariant vectors in the triple tensor product $(V^*_{\mu_1} \otimes V_{\mu_2} \otimes V_{\mu_3})^G$ and $\{\overline{a}\}$ is the dual basis in the dual vector space.

These and other similar identities can be written in a basis free form as

$$\int_{G^{\times m}} \pi^{\mu_1}(h) \otimes \pi^{\mu_2}(h) \otimes \cdots \otimes \pi^{\mu_m}(h) dh = P_0$$

where P_0 is the orthogonal projector onto the subspace $(V_{\mu_1} \otimes V_{\mu_2} \otimes \cdots \otimes V_{\mu_m})^G \subset V_{\mu_1} \otimes V_{\mu_2} \otimes \cdots \otimes V_{\mu_m}$ of G-invariant vectors.

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