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On module categories related to $Sp(N-1) \subset Sl(N)$

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In memoriam of Vaughan Jones with gratitude

Abstract: Let $V = \mathbb{C}^N$ with N odd. We construct a q-deformation of $\mathrm{End}_{Sp(N-1)}(V^{\otimes n})$ which contains $\mathrm{End}_{U_q\mathfrak{sl}_N}(V^{\otimes n})$. It is a quotient of an abstract two-variable algebra which is defined by adding one more generator to the generators of the Hecke algebras H_n. These results suggest the existence of module categories of $Rep(U_q\mathfrak{sl}_N)$ which may not come from already known coideal subalgebras of $U_q\mathfrak{sl}_N$. We moreover indicate how this can be used to construct module categories of the associated fusion tensor categories as well as subfactors, along the lines of previous work for inclusions $Sp(N) \subset SL(N)$.

Keywords: Module categories.

The problem of classifying module categories of a given tensor category arises in different contexts such as conformal field theory and the study of subfac-

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tors. A lot of progress has been made for module categories of fusion categories coming from quantum groups (or Wess-Zumino-Witten models) in recent work by Edie-Michell (see [5]), building on work of Jones, Ocneanu, Gannon, Schopieray, Evans and Pugh and others. But in many cases, a detailed description of those module categories such as the fusion rules, algebras etc is still not available.

It is well-known that $Rep(H)$ is a module category of $Rep(G)$ for an inclusion of groups $H \subset G$. The basic idea here is to find a subgroup H of a Lie group G for which one can find analogs of the module $Rep(H)$ in fusion categories related to G. This was successfully carried out in [27] and [28] for inclusions $Sp(N) \subset SL(N)$, N even, and $O(N) \subset SL(N) \times \mathbb{Z}_2$ for arbitrary $N > 1$. It allowed a detailed description of these module categories in terms of the well-known combinatorics of these groups (see Section 5). The current paper roughly contains the analogous results of the paper [27] for the inclusions $Sp(N-1) \subset SL(N)$ for N odd.

Here is our approach in more detail, formulated for the group $G = SL(N)$ for simplicity. We denote the quotient tensor category of tilting modules of the quantum group $U_q\mathfrak{sl}_N$ modulo negligible modules for q a root of unity by $\overline{Rep}(U_q\mathfrak{sl}_N)$. It is often also denoted by $SU(N)_k$ for q^2 a primitive $(N+k)^{th}$ root of unity. We study the following questions: Find a subgroup $H \subset SL(N)$ for which we can find

(a) a q-deformation of $Rep(H)$ which is a module category of $Rep(U_q\mathfrak{sl}_N)$,
(b) a quotient of said q-deformation for q a root of unity which is a module category of $\overline{Rep}(U_q\mathfrak{sl}_N)$,
(c) a subfactor corresponding to the module category in (b) if it is unitarizable.

Before describing the results in this paper in more detail, we would like to make a few general remarks about this approach. It is known that Question (a) can be solved if H is the group of fixed points under a period 2 automorphism via coideal subalgebras, see work of Letzter [15] and of Noumi and Sugitani [17]. It follows from work in [16] that the examples in [28] indeed correspond to special cases of the work in [15] and [17]. However, it is also clear that not all module categories constructed via coideal algebras allow solutions of Questions (b) and (c). This can be seen e.g. for $SL(3)$, where all module categories of the corresponding fusion categories are known due to work of Gannon and Evans and Pugh, see [7].

As stated in the title, we consider the inclusion of $Sp(N-1) \subset SL(N)$ for N odd. One of the main results of this paper is the construction of a sequence of two-parameter algebras $C_n = C_n^{(N)}(q)$ which contain the Hecke algebras $H_n(q)$ of type A_{n-1} as subalgebras. For $q \neq \pm 1$ it suffices to add one more generator e, corresponding to the projection onto the trivial $Sp(N-1)$ submodule $\mathbf{1} \subset V$, where $V = \mathbb{C}^N$ is the vector representation of $SL(N)$. In order to get the correct algebra in the classical limit $q \to \pm 1$, we also need additional generators to obtain nontrivial morphisms between the two copies of the trivial representation in $V^{\otimes 2}$. We give a presentation of these algebras via generators and relations and an explicit basis. Another important result is the proof of the existence of an extension of the Markov trace for H_n to the algebras C_n which satisfies a generalized Markov condition.

We now give a more detailed description of the results of this paper which will also explain how these algebras can be used to construct module categories. We fix notations in the first section and prove a number of combinatorial and algebraic results concerning $\mathrm{End}_{Sp(N-1)}(V^{\otimes n})$. We derive relations for a q-deformation of $\mathrm{End}_{Sp(N-1)}(V^{\otimes n})$ which contains the Hecke algebra H_n as a subalgebra in the second section. These relations are essentially forced by the fusion respectively restriction rules for $Sp(N-1) \subset SL(N)$ and a generalized Markov condition (see Condition (2.9) or the discussion here for Section 4). In particular, we show that only two solutions are possible for fixed $q \neq \pm 1$ and $N > 1$ odd. Hence we have two possible choices $C_{n,\pm}$ of extensions of H_n subject to our conditions. As they are closely related, see Remark 2.9, we will often just use the notation C_n for either of these cases. We also define a version of C_n depending on two parameters q and p, which specializes to the original version for $p = q^N$. It is then shown that for $N > 2n$ the dimension of these algebras is at most $\dim \mathrm{End}_{Sp(N-1)}(V^{\otimes n})$, and explicit spanning sets are determined. We define representations of $C_n^{(N)}$ into $\mathrm{End}(V^{\otimes n})$ in the third section. It is shown that their images in the classical limit $q = 1$ concide with $\mathrm{End}_{Sp(N-1)}(V^{\otimes n})$. We conclude from this that the given spanning sets in the previous section are actually bases. Section 4 contains a proof that we can extend the Markov trace tr of the Hecke algebra H_n to a trace on C_n satisfying the generalized Markov condition $tr(cg_n) = tr(c)tr(g_n)$ for all $c \in C_n$. In the last section, we first briefly describe how this paper has been influenced by the work of Vaughan Jones, even though this will be pretty obvious to experts anyways. We then indicate how our algebras can be used to construct module categories and subfactors. To do so, we consider the quotients $\overline{H}_n$ and $\overline{C}_n$ of the algebras H_n and C_n modulo the annihilator ideals of the Markov trace tr. The objects of the category and of the module category are given by idempotents in $\overline{H}_n$ and $\overline{C}_n$ respectively, and the module action comes from the natural inclusion map $C_n \otimes H_m \to C_{n+m}$. To illustrate this, we restate results from [27] and [28] in the language of module categories which was not used there. This shows, in particular, that Problems (a)–(c) have been solved for the inclusions considered in those papers. We finally discuss how the approach in this paper can be used to give detailed descriptions for a large class of module categories.

1. Fusion rules for the embedding of $Sp(N-1) \subset SL(N)$, N odd

1.1. Fusion rules

Let $V = \mathbb{C}^N$ with N odd. We fix a symplectic bilinear form $(\ ,\ )'$ on V with 1-dimensional kernel spanned by the nonzero vector v_o and a complement V' of v_o on which the form $(\ ,\ )'$ is nondegenerate; see e.g. Lemma 3.5 for an explicit realization with $q = 1$. This defines an embedding $Sp(N-1) \subset SL(N)$ such that V decomposes into the direct sum $V' \oplus \mathbb{C}v_o$ as an $Sp(N-1)$ module. Recall that the finite-dimensional simple representations of $Sp(N-1)$ are labelled by Young diagrams with $\leq (N-1)/2$ rows, with the $(N-1)$-dimensional simple representation V' labeled by the Young diagram with one box. If V'_λ is a simple representation of $Sp(N-1)$ labeled by the Young diagram λ, we have

$$V'_\lambda \otimes V' \cong \bigoplus_{\mu \leftrightarrow \lambda} V'_\mu, \tag{1.1}$$

where μ ranges over all Young diagrams which can be obtained from λ by either adding or removing a box. One deduces from this the tensor product rule

$$V'_\lambda \otimes V \cong V'_\lambda \oplus \bigoplus_{\mu \leftrightarrow \lambda} V'_\mu, \tag{1.2}$$

with μ as in (1.1).

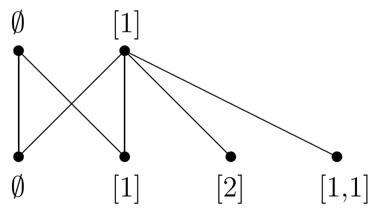
1.2. Bratteli diagrams and path bases

The inclusions of the algebras

$$\cdots \subset \mathrm{End}_{Sp(N-1)}(V^{\otimes n}) \subset \mathrm{End}_{Sp(N-1)}(V^{\otimes n+1}) \subset \cdots$$

are conveniently described by a Bratteli diagram. It follows from the tensor product rules (1.2) that its vertices at level n are labeled by Young diagrams

λ with $|\lambda| \le n$, where $|\lambda|$ denotes the number of boxes in the Young diagram. A diagram λ at level n is connected with a diagram μ at level $n+1$ if λ differs from μ by at most one box. The multiplicity of V_λ in $V^{\otimes n}$ is then given by the number of paths of length n from level 0 to level n. As we only have multiplicities 0 or 1, we obtain a basis of $\mathrm{Hom}(V'_\lambda, V^{\otimes n})$ labeled by the paths of length n which end in λ for any irreducible $Sp(N-1)$ module V_λ labeled by λ. This basis is unique up to rescaling by nonzero scalars. Below is the part of the Bratteli diagram containing level 1 and level 2.



$\mathrm{End}_{Sp(N-1)}(V)$

$\mathrm{End}_{Sp(N-1)}(V^{\otimes 2})$

Inclusion diagram

1.3. Multiplicities for large N

The tensor product rule (1.2) allows us to calculate the multiplicity $m_{n,\lambda}$ of the simple module V'_λ in $V^{\otimes n}$. If $N > 2n$, there are no restrictions on Young diagrams, and we can give closed formulas for the multiplicities $m_{n,\lambda}$. To do so, we define integers h_r inductively by $h_0 = 1$, $h_1 = 1$ and

$$h_{r+1} = h_r + r h_{r-1}. \tag{1.3}$$

We denote by d_λ the dimension of the simple S_n module labeled by the Young diagram λ, where the number of boxes $|\lambda|$ of λ is equal to n. There exists a well-known explicit formula for it in terms of the hook lengths of λ. We will need the following well-known identities

$$\sum_{\mu<\lambda} d_\mu = d_\lambda \quad \text{and} \quad \sum_{\nu>\lambda} d_\nu = (n+1) d_\lambda, \tag{1.4}$$

where $|\lambda| = n$, and μ and ν range over all Young diagrams which can be obtained by removing a box from λ (for μ) or adding a box to λ (for ν).

Proposition 1.1. (a) *The multiplicity $m_{n,\lambda}$ of the simple module V'_λ in $V^{\otimes n}$ for $N > 2n$ is equal to $h_{n-|\lambda|}\binom{n}{|\lambda|} d_\lambda$.*

(b) *We have the identity*

$$\sum_{|\lambda|=n-r} m_{n,\lambda} d_\lambda = h_r \frac{n!}{r!}$$

(c) *If* $N > 2n$, *we have* $\dim \mathrm{End}_{Sp(N-1)}(V^{\otimes n}) = \sum_{r=0}^n h_r^2 \frac{n!}{r!}\binom{n}{r}$.

Proof. We will prove (a) by induction on n, with the claim easily checked for $n = 1$. It follows from (1.2) that $V'_\lambda \subset V'_\mu \otimes V$ if and only if $\mu = \lambda$ or μ is obtained by removing or adding a box from/to λ. In each of theses cases V'_λ appears with multiplicity 1. It follows that

$$m_{n,\lambda} = m_{n-1,\lambda} + \sum_{\mu<\lambda} m_{n-1,\mu} + \sum_{\nu>\lambda} m_{n-1,\nu}.$$

Using the induction assumption, the identities (1.3), (1.4) and the identity $\binom{n-1}{k+1}(k+1) = \binom{n-1}{k}(n-1-k)$, we obtain

$$\begin{aligned}
m_{n,\lambda} &= h_{n-|\lambda|-1}\binom{n-1}{|\lambda|} d_\lambda + \sum_{\mu<\lambda} h_{n-|\lambda|}\binom{n-1}{|\lambda|-1} d_\mu + \sum_{\nu>\lambda} h_{n-|\lambda|-2}\binom{n-1}{|\lambda|+1} d_\nu \\
&= h_{n-|\lambda|-1}\binom{n-1}{|\lambda|} d_\lambda + h_{n-|\lambda|}\binom{n-1}{|\lambda|-1} d_\lambda + h_{n-|\lambda|-2}\binom{n-1}{|\lambda|+1} d_\lambda(|\lambda|+1) \\
&= d_\lambda[h_{n-|\lambda|-1}\binom{n-1}{|\lambda|} + h_{n-|\lambda|}\binom{n-1}{|\lambda|-1} + h_{n-|\lambda|-2}\binom{n-1}{|\lambda|}(n-1-|\lambda|) \\
&= d_\lambda[\binom{n-1}{|\lambda|}(h_{n-|\lambda|-1} + h_{n-|\lambda|-2}(n-1-|\lambda|)) + h_{n-|\lambda|}\binom{n-1}{|\lambda|-1} \\
&= d_\lambda h_{n-|\lambda|}\left(\binom{n-1}{|\lambda|} + \binom{n-1}{|\lambda|-1}\right).
\end{aligned}$$

This proves part (a). Part (b) follows from this and the identity $\sum_{|\lambda|=n-r} d_\lambda^2 = (n-r)!$. Part(c) similarly follows from this, part (a) and $\dim \mathrm{End}_{Sp(N-1)}(V^{\otimes n}) = \sum_{|\lambda|\le n} m_{n,\lambda}^2$. ☐

1.4. Description of $\mathrm{End}_{Sp(N-1)}(V^{\otimes n})$

We denote by $E \in \mathrm{End}(V)$ the projection onto the trivial representation $\mathbb{C}v_o$ of $Sp(N-1)$ with kernel V', and by U the antisymmetrization map

$$U:\ v\otimes w \mapsto v\wedge w = v\otimes w - w\otimes v, \quad v, w \in V.$$

The elements $U_i \in \mathrm{End}(V^{\otimes n})$ are defined for $1 \leq i < n$ by

$$U_i = 1 \otimes 1 \otimes \cdots \otimes U \otimes \cdots \otimes 1, \tag{1.5}$$

where U acts on the i-th and $(i+1)$-st factors in $V^{\otimes n}$. Observe that the flip $G : v \otimes w \in V^{\otimes 2} \mapsto w \otimes v$ is related to U by the simple formula $G = 1 - U$. We now extend the bilinear form $(\ , \ )'$ to a non-degenerate bilinear form $(\ , \ )$ on V by defining

$$(v_o, v_o) = 1 \quad \text{and} \quad (v_o, v') = 0 = (v', v_o) \quad \text{for all } v' \in V'. \tag{1.6}$$

If (v_i') and (w_i') are dual bases of V' with respect to $(\ , \ )'$, we obtain the canonical vector $v_o \otimes v_o + \sum_i v_i' \otimes w_i' \in V^{\otimes 2}$ for the form $(\ , \ )$. We define the element $F \in \mathrm{End}(V^{\otimes 2})$ by

$$F(v \otimes w) = (v, w)\ (v_o \otimes v_o + \sum_i v_i' \otimes w_i').$$

The elements F_i and G_i in $\mathrm{End}(V^{\otimes n})$ are defined in the same way as U_i in (1.5). It is well-known that the element F can be used to calculate the trace of an element $A \in \mathrm{End}(V)$ by

$$Tr(A)F = F(A \otimes 1)F.$$

Let us now decompose $V^{\otimes n}$ into the direct sum of three $Sp(N-1)$ submodules as follows. Write $V^{\otimes n} = \bigoplus_\lambda m_\lambda V_\lambda$, where m_λ is the multiplicity of the simple module V_λ in $V^{\otimes n}$. For given n, we call the diagram λ an *old / recent / new diagram* in $V^{\otimes n}$ if V_λ has appeared for the first time in $V^{\otimes m}$ with $m \leq n-2$ / $m = n-1$ / $m = n$. We define

$$V_{old}^{\otimes n} = \bigoplus_{\lambda \ old} m_\lambda V_\lambda',$$

and $V_{rec}^{\otimes n}$ and $V_{new}^{\otimes n}$ accordingly. Then it follows from the tensor product rule (1.2) that λ is an old / recent/ new diagram if $|\lambda| \leq n-2/|\lambda| = n-1$ / $|\lambda| = n$.

Theorem 1.2 (See e.g. [26], Proposition 4.10)**.** *Let V be a finite-dimensional self-dual G-module. Then $\mathrm{End}_G(V_{old}^{\otimes n})$ is given by a Jones basic construction for $\mathrm{End}_G(V^{\otimes n-2}) \subset \mathrm{End}_G(V^{\otimes n-1})$. Moreover, it coincides with the two-sided ideal in $\mathrm{End}_G(V^{\otimes n})$ generated by F_{n-1}, which is spanned by elements of the form $aF_{n-1}b$, with $a, b \in \mathrm{End}_G(V^{\otimes n-1} \otimes 1)$.*

Theorem 1.3. *The algebra* $\mathrm{End}_{Sp(N-1)}(V^{\otimes n})$ *is generated by the symmetric group* S_n*, acting via permutations of the factors of* $V^{\otimes n}$*, the element* $E\otimes 1_{n-1}$ *and the element* F_1*.*

Proof. The proof goes by induction on n, with $n=1$ obviously true. The claim follows for $\mathrm{End}(V^{\otimes n})_{old}$ from Theorem 1.2 and the induction assumption. If $V'_\lambda \subset V^{\otimes n}_{new}$, i.e. $|\lambda| = n$, it follows from the tensor product rules that $V'_\lambda \subset (V')^{\otimes n}$. In this case, the claim follows from Brauer's classical result. It also implies that $W_\lambda = \mathrm{Hom}(V'_\lambda, V^{\otimes n})$ is an irreducible S_n module. To deal with the remaining cases, observe that the elements G_i, $1 \le i < n$ and $T = (1-2E)\otimes 1_{n-1}$ satisfy the relations of the Weyl group $W(B_n)$ of type B_n. Hence the quotient modulo the ideal generated by F_{n-1} is also a quotient of the group algebra of $W(B_n)$. We will finish the proof in the next section after a brief review of the representation theory of $W(B_n)$. □

1.5. Weyl group of type B_n

The Weyl group $W(B_n)$ of type B_n can be defined via generators t and s_i, $1 \le i < n$ and relations such that the s_i are simple reflections of the symmetric group (e.g. we can take $s_i = (i, i+1)$) and such that t commutes with s_i for $i > 1$ and satisfies $s_1 t s_1 t = t s_1 t s_1$. It is well-known that it is isomorphic to the semidirect product $(\mathbb{Z}/2)^n \rtimes S_n$, with t corresponding to $(1, 0, \dots, 0) \in (\mathbb{Z}/2)^n$, and with S_n permuting the coordinates of elements of $(\mathbb{Z}/2)^n$. The irreducible representations of $W(B_n)$ are labeled by pairs of Young diagrams (λ, μ) with $|\lambda| + |\mu| = n$. They can be constructed as follows (see e.g. [21], Chapter 8 for details):

Let ϕ be a character of $(\mathbb{Z}/2)^n$ such that $\phi(\varepsilon_i) = -1$ for $i \le r$ and $\phi(\varepsilon_i) = 1$ for $i > r$; here ε_i is the i-th unit vector in $(\mathbb{Z}/2)^n$. The centralizer of ϕ consists of all $g \in W(B_n)$ such $\phi(gxg^{-1}) = \phi(x)$ for all $x \in (\mathbb{Z}/2)^n$. It is easy to see that for our choice of ϕ the centralizer is equal to $(\mathbb{Z}/2)^n \rtimes (S_r \times S_{n-r})$. Let W_λ and W_μ be irreducible representations of S_r and S_{n-r}. Then $W_\lambda \otimes W_\mu$ becomes an irreducible representation of $(\mathbb{Z}/2)^n \rtimes (S_r \times S_{n-r})$, where the action of $x \in (\mathbb{Z}/2)^n$ is given by the scalar $\phi(x)$. It can then be shown that inducing this representation up to $W(B_n)$ yields an irreducible representation of $W(B_n)$ of dimension $\binom{n}{r} d_\lambda d_\mu$.

Conclusion of Proof of Theorem 1.3. Let $|\lambda| = n-1$. Then $E\otimes E\otimes 1_{n-2}$, and hence also F_1 acts as 0 on $\mathrm{Hom}(V_\lambda, V^{\otimes n})$, with V_λ an irreducible $Sp(N-1)$-module. Hence we can view $\mathrm{Hom}(V_\lambda, V^{\otimes n})$ as a $W(B_n)$-module on which $E\otimes 1_{n-1}$ acts nontrivially; indeed, the module $\mathbb{C}v_o \otimes V_\lambda \cong V_\lambda$ is in the image of $E\otimes 1_{n-1}$. As $\mathbb{C}v_0 \otimes V_\lambda \subset \mathbb{C}v_o \otimes (V')^{\otimes n-1}$, it also follows that E_i acts as

0 on $\mathbb{C}v_o \otimes V_\lambda$ for $i > 1$. Hence the action of $(\mathbb{Z}/2)^n$ on $\mathbb{C}v_o \otimes V_\lambda$ is given by the functional $\phi : x \in (\mathbb{Z}/2)^n \mapsto (-1)^{x_1}$. We obtain that $\mathrm{Hom}(V_\lambda, V^{\otimes n})$ contains an irreducible $W(B_n)$-module labeled by $([1], \lambda)$. By the previous discussion it has dimension $n\, d_\lambda = w_{n,\lambda} = \dim \mathrm{Hom}(V_\lambda, V^{\otimes n})$. Hence $W(B_n)$ and therefore also C_n acts irreducibly on $\mathrm{Hom}(V_\lambda, V^{\otimes n})$. $\square$

2. A q-deformation of $\mathrm{End}_{Sp(N-1)}(V^{\otimes n})$

The goal of this paper is to study q deformations of $Rep(Sp(N-1))$ at the categorical level which are compatible with the deformation of $Rep(Sl(N))$ to $U_q\mathfrak{s}l_N$ and with the embedding $Sp(N-1) \subset Sl(N)$. As we shall see, this leads to a structure different from $Rep(U_q\mathfrak{s}p_{N-1})$.

2.1. Hecke algebras

The Hecke algebra $H_n = H_n(q)$ of type A_{n-1} is defined via generators g_i, $1 \le i < n$ and relations

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad 1 \le i < n-1,$$

together with $g_i^2 = (q - q^{-1})g_i + 1$ and $g_i g_j = g_j g_i$ for $|i-j| \neq 1$. Defining

$$u_i = q1 - g_i,$$

the relations above translate to $u_i^2 = (q + q^{-1})u$, $u_i u_j = u_j u_i$ for $|i-j| \neq 1$ and

$$u_i u_{i+1} u_i - u_i = u_{i+1} u_i u_{i+1} - u_{i+1}. \tag{2.1}$$

We replace $Rep(Sl(N))$ by the representation category of the Drinfeld-Jimbo quantum group $U_q\mathfrak{s}l_N$. It was shown in [10] that the generalization of the map U of the previous subsection can then be defined with respect to a basis $\{v_i\}$ for V by $U(v_i \otimes v_i) = 0$ and by defining its restriction to the ordered basis vectors $v_i \otimes v_j$ and $v_j \otimes v_i$, $i < j$ by the matrix

$$U = \begin{bmatrix} q^{-1} & -1 \\ -1 & q \end{bmatrix}. \tag{2.2}$$

The elements $U_i \in \mathrm{End}(V^{\otimes n})$ are defined for $1 \le i < n$ as in (1.5).

2.2. Dimensions and traces

We review some basics about dimension functions for representations of quantum groups, see [22] and [12], XIV.4 for more details. Let W be an $U_q\mathfrak{sl}_N$-module with dual module W^*. Then there exist canonical morphisms

$$b_W : \mathbf{1} \to W \otimes W^* \quad \text{and} \quad d'_W : W \otimes W^* \to \mathbf{1}$$

such that for $a \in \mathrm{End}(W)$ we define

$$Tr_q(a) = d'_W(a \otimes 1)b_W. \tag{2.3}$$

We remark that while Tr_q does not satisfy the trace property $Tr_q(ab) = Tr_q(ba)$ in general, its restriction to $\mathrm{End}_{U_q\mathfrak{sl}_N}(W)$ is indeed a trace. There exists an element $q^{2\rho} \in U_q\mathfrak{sl}_N$ such that $Tr_q(a) = Tr(aq^{2\rho})$, see [20], (7.1.1). We will only need to know its action on $W = V^{\otimes n}$, where we have

$$Tr_q(a) = Tr(aD^{\otimes n}), \quad a \in \mathrm{End}(V^{\otimes n}), \quad D = diag(q^{2i-N-1}). \tag{2.4}$$

More generally, we can define a partial trace (also referred to as a contraction, or a conditional expectation) $\mathcal{E}_X : \mathrm{End}(X \otimes W) \to \mathrm{End}(X)$ by

$$\mathcal{E}_X(a) = (1_X \otimes d'_W)(a \otimes 1_{W^*})(1_X \otimes b_W), \tag{2.5}$$

which satisfies $Tr_q(a) = Tr_q(\mathcal{E}_X(a))$. If we take $X = W = V = \mathbb{C}^N$, the vector representation of $U_q\mathfrak{sl}_N$, it follows from the discussion above that $\mathcal{E}_X(U) = Tr_q(U)1_V$, as V is irreducible. One deduces from this more generally that

$$\mathcal{E}_n(U_n) = Tr(U_n)1, \tag{2.6}$$

where $\mathcal{E}_n$ is the partial trace from $\mathrm{End}(V^{\otimes n+1})$ to $\mathrm{End}(V^{\otimes n})$.

As usual, we define the q-number $[k] = (q^k - q^{-k})/(q - q^{-1})$. Then the q-dimension of the irreducible $U_q\mathfrak{sl}_N$ module V_μ with highest weight μ is given by (see e.g. [1], (3.2))

$$\dim_q V_\mu = \prod_{1\le i<j\le N} \frac{[\mu_i - \mu_j + j - i]}{[j-i]}. \tag{2.7}$$

The dimension of a simple $U_q\mathfrak{sp}_{2k}$-module V'_λ labeled by the Young diagram

λ is given by

(2.8)
$$\dim_q V'_\lambda = \prod_{i<j} \frac{[\lambda_i - \lambda_j + j - i][\lambda_i + \lambda_j + 2k + 2 - i - j]}{[j-i][2k+2-i-j]} \prod_{i=1}^{k} \frac{[2\lambda_i + 2k + 2 - 2i]}{[2k+2-2i]}.$$

Remark 2.1. Let $H_n^{(N)}$ be the image of the Hecke algebra H_n in the representation in $V^{\otimes n}$. Then we refer to the trace tr on H_n such that $tr(p_\mu) = \dim_q V_\mu/[N]^n$ (as in (2.7)) for a minimal idempotent p_μ in the direct summand of H_n labeled by μ as the Markov trace on $H_n^{(N)}$. It was shown in [24] that the quotient $\overline{H}_n$ modulo the annihilator ideal of tr is indeed isomorphic to $H_n^{(N)}$ for q not a root of unity.

2.3. Posing the question

We can now make the above mentioned problem of finding a module category $\mathcal{M}$ corresponding to a q-deformation for the embedding $Sp(N-1) \subset SL(N)$ more precise as follows: We fix N and denote by $H_n^{(N)}$ the image of the Hecke algebra H_n in the representation in $V^{\otimes n}$ for $V = \mathbb{C}^N$. Then we want to find extensions $C_n = C_n^{(N)}$ of $H_n^{(N)}$ such that

(a) $C_n \subset C_{n+1}$,
(b) $H_n^{(N)} \subset C_n \cong \mathrm{End}_{SL(N)}(V^{\otimes n}) \subset \mathrm{End}_{Sp(N-1)}(V^{\otimes n})$ at least for q not a root of unity,
(c) (Markov property) There exists a normalized trace tr on C_n extending the Markov trace on the Hecke algebra $H_n^{(N)}$ compatible with the embedding $C_n \subset C_{n+1}$ for all n such that $tr(e) = 1/[N]$ and

$$tr(cg_n) = tr(c)tr(g_n) \quad \text{for all } c \in C_n. \tag{2.9}$$

Remark 2.2. Condition (b) is a consequence of the right module action of the $\mathrm{Rep}(U_q\mathfrak{sl}_N)$ object $V^{\otimes n}$ on the $\mathcal{M}$ object $\mathbf{1}$ corresponding to the trivial represention of $\mathfrak{sp}_{N-1}$, while Condition (a) is a consequence of the right module action of the $\mathrm{Rep}(U_q\mathfrak{sl}_N)$ object V on the $\mathcal{M}$ object $\mathbf{1} \otimes V^{\otimes n}$. We require condition (c) as a necessary condition to also get a module action of the fusion tensor category, viewed as a quotient of $\mathrm{Rep}(U_q\mathfrak{sl}_N)$. Informally, a negligible $\mathrm{Rep}(U_q\mathfrak{sl}_N)$-endomorphism of $V^{\otimes n}$ should remain negligible also as an element of $\mathrm{End}_{\mathcal{M}}(\mathbf{1} \otimes V^{\otimes n})$ and of $\mathrm{End}_{\mathcal{M}}(\mathbf{1} \otimes V^{\otimes n+1})$ in the desired module category $\mathcal{M}$. This seems to be a fairly restrictive condition, which is not satisfied for arbitrary module categories of $\mathrm{Rep}(U_q\mathfrak{sl}_N)$.

The following lemma will only be needed for $n \leq 2$ in this paper (but more later).

Lemma 2.3. *If a trace tr on C_n as in (c) exists, its value on a minimal idempotent p_λ in the direct summand of C_n labeled by λ is equal to $\dim_q V'_\lambda/[N]^n$, with $\dim_q V'_\lambda$ given in (2.8).*

Proof. We view D as the representation of an element of $Sp(N-1)$ with eigenvalues $q^{\pm 2j}$, $1 \leq j \leq (N-1)/2$. These are exactly the eigenvalues of the element $q^{2\rho'}$ for ρ' half the sum of the roots of $Sp(N-1)$ in its vector representation V'. Hence the value of tr for $\tilde{p}_\lambda$ is given by the symplectic character $\chi^\lambda(q^{2\rho'})/[N]^n$, where $\tilde{p}_\lambda$ is a minimal idempotent in $\mathrm{End}_{Sp(N-1)}(V^{\otimes n})$. It is well-known that an irreducible $SL(N)$-module labeled by μ decomposes as a direct sum of $Sp(N-1)$-modules labeled by Young diagrams with fewer boxes than μ except possibly for μ itself (see [4]). By assumption $H_n \subset C_n$ has the same inclusion pattern as $\mathrm{End}_{SL(N)}(V^{\otimes n}) \subset \mathrm{End}_{Sp(N-1)}(V^{\otimes n})$. Hence $tr(p_\lambda)$ is already determined by the values $tr(p_\nu)$ for minimal idempotents $p_\nu \in C_n$ labeled by diagrams ν with $|\nu| < |\lambda|$ and by the value of tr on minimal idempotents of H_n. Hence we can show the claim by induction on $|\lambda|$. □

2.4. First structure coefficients

As a simple but important special case of the tensor product rule (1.2), we obtain that $V^{\otimes 2}$ decomposes as an $Sp(N-1)$-module into the direct sum

$$V^{\otimes 2} \cong 2 \cdot \mathbf{1} \oplus 2V' \oplus V'_{2\Lambda_1} \oplus V'_{\Lambda_2}; \tag{2.10}$$

here $\mathbf{1}$ is the trivial representation and $V'_{2\Lambda_1}$ and V'_{Λ_2} denote the nontrivial $\mathfrak{sp}_{N-1}$ representations which appear in the symmetrization respectively antisymmetrization of $(V')^{\otimes 2}$; see also the Bratteli diagram in Section 1.2. Let $V_{2\Lambda_1}$ and V_{Λ_2} denote the symmetrization and antisymmetrization of $V^{\otimes 2}$ respectively. Then one checks easily, using the dimension formulas in Section 2.2 and the tensor product formulas in Section 1.1 that

$$V_\lambda \cong V'_\lambda \oplus V' \oplus \mathbf{1}, \quad \text{for } \lambda = 2\Lambda_1, \Lambda_2,$$

as an $Sp(N-1)$ module. It follows from (2.10) that C_2 has two nonisomorphic 2-dimensional irreducible representations, and two one-dimensional representations. As the image of u_1 is equal to V_{Λ_2}, it follows from the restriction rules above that u_1 acts as a rank 1 matrix in the representations of C_2 labeled by $\mathbf{1}$, V' and V'_{Λ_2}. It follows similarly from the tensor product rules (1.1) that e

acts as a rank 1 idempotent in the two 2-dimensional representations of C_2. Hence we can assume the two 2-dimensional representations to be of the form

$$e \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.11}$$

$$u_1 \mapsto \begin{bmatrix} a & \sqrt{ab} \\ \sqrt{ab} & b \end{bmatrix} \oplus \begin{bmatrix} c & \sqrt{cd} \\ \sqrt{cd} & d \end{bmatrix}, \tag{2.12}$$

for suitable complex scalars a, b, c, d. Using (1.2) again, we obtain (for $N \geq 5$) that

$$V^{\otimes 3} \cong 4 \cdot \mathbf{1} \oplus 6V \oplus \cdots. \tag{2.13}$$

For $N = 3$, we have to make the following adjustments: In (2.10), the representation V'_{Λ_2} does not appear, and in (2.13) the representation V' only appears with multiplicity 5.

Lemma 2.4. *The matrix entries above satisfy $a + b = [2] = c + d$ and, if $ab \neq 0$, $(a - c)^2 = 1$. Moreover, there exists a 4-dimensional representation of the Hecke algebra H_3 with u_1 given by the matrices in (2.12) and u_2 by the same matrix blocks, with the second and third basis vectors permuted.*

Proof. The first statement follows from the fact that each of the matrix blocks in (2.12) has rank 1 and its only possible nonzero eigenvalue is [2]. We now consider the representation of the Hecke algebra H_3 on the four-dimensional space $\mathrm{Hom}(\mathbf{1}, V^{\otimes 3})$. It follows from the definition of path bases that u_1 can be represented by the 4 by 4 matrix $\rho(u_1)$ with the two diagonal blocks as in (2.12). As $\mathbb{C}v_0 \otimes V^{\otimes 2} \cong V^{\otimes 2}$ as an $Sp(N-1)$ module, we can also assume that u_2 is represented by the first matrix block in (2.12) on the span of the first and third path, while it is given by a 2 by 2 block with diagonal entries c' and d' on the remaining two paths. Checking the Hecke algebra relation (2.1) for the (1,1) entry, we deduce that $c' = c$, and hence also $d' = d$. Checking relation (2.1) for entries (1,2) and (2,1), one deduces that the off-diagonal entries in the c', d' block of u_2 are equal to $\sqrt{cd}$. This determines the matrix for u_2 as claimed. But then the (21) entry of both the left and the right hand side of relation (2.1) reads as

$$\sqrt{ab}(a^2 + bc - 1) = \sqrt{ab}(ac + cd).$$

Substituting $b = [2] - a$ and $d = [2] - c$, and dividing by $\sqrt{ab}$, we obtain $(a - c)^2 = 1$, as claimed. □

Corollary 2.5. *Let $e_{(3)}$ be the projection onto the path $0 \to 0 \to 0 \to 0$.*

(a) *If $a = c+1$, then $(u_1u_2 - u_1)e_{(3)} = (u_2u_1 - u_2)e_{(3)}$, or, equivalently, $(g_1g_2 + g_1)e_{(3)} = (g_2g_1 + g_2)e_{(3)}$,*
(b) *If $a = c+1$, then $(u_1u_2 + u_1)e_{(3)} = (u_2u_1 + u_2)e_{(3)}$, or, equivalently, $(g_1g_2 - g_1)e_{(3)} = (g_2g_1 - g_2)e_{(3)}$.*

Proof. This follows from the explicit matrices as described in the proof of Lemma 2.4. □

2.5. Relations from Markov property

Recall that the Markov property (2.9) requires $tr(cu_n) = tr(c)tr(u_n)$ for any $c \in C_n$.

Lemma 2.6. *The Markov property only holds if the matrix entries in (2.12) are as follows (where $N = 2k+1$):*

$$a = \frac{[k](q^{1/2}+q^{-1/2})}{[k+1/2]}, \qquad c = \frac{[k-1/2]}{[k+1/2]}, \qquad \text{if } a = c+1,$$
$$a = \frac{-(q^{1/2}-q^{-1/2})(q^k-q^{-k})}{q^{k+1/2}+q^{-k-1/2}}, \qquad c = \frac{q^{k-1/2}+q^{-k+1/2}}{q^{k+1/2}+q^{-k-1/2}}, \qquad \text{if } a = c-1.$$

Proof. It follows from the definitions and Lemma 2.3 that $tr(e) = 1/[N]$ and

$$tr(u) = [2]\frac{[N][N-1]}{[2]}\frac{1}{[N]^2} = \frac{[N-1]}{[N]}.$$

Using the explicit matrix representations (2.11) and (2.12) and the weights of the traces in Lemma 2.3, we obtain

$$tr(ue) = \frac{1}{[N]^2}(a + c([N]-1)).$$

It follows from the Markov property $tr(ue) = tr(u)tr(e)$, with $N = 2k+1$ that

$$a + c([2k+1]-1) = [2k].$$

By Lemma 2.4, we have $a = c \pm 1$. Let us consider the case $a = c+1$. Substituting this into the last equation we obtain

$$c = \frac{[2k]-1}{[2k+1]} = \frac{[k-1/2]}{[k+1/2]} = \frac{q^{k-1/2}-q^{-k+1/2}}{q^{k+1/2}-q^{-k-1/2}}.$$

The formula for a follows from $a = c+1$. The case $a = c-1$ goes similarly. □

2.6. Correction for $q \to 1$

We assume in this section that $a = c - 1$. It follows from Lemma 2.6 that $ab = 0$ for $q = 1$ in this case. This would make the representation in the first matrix block in (2.12) reducible. This can be avoided by introducing the elements u_{12} and u_{21} defined below; they correspond to intertwiners between the two copies of the trivial representation in $V^{\otimes 2}$. We shall show later that one can make sense of them also at $q = 1$. It will be convenient to define

$$[N]_+ = \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}} \quad \text{and} \quad [N]_- = \frac{q^{N/2} + q^{-N/2}}{q^{1/2} + q^{-1/2}}. \tag{2.14}$$

Observe that

$$\begin{aligned} b = [2] - a &= \frac{q^{k+1} + q^{-k-1}}{[N]_-}, \\ \sqrt{ab} &= \frac{-i(q^{1/2} - q^{-1/2})}{[N]_-}\sqrt{[N] - 1}, \end{aligned}$$

where we used the identity $[k](q^{k+1} + q^{-k-1}) = [N] - 1$, and where the choice of sign for $\sqrt{ab}$ will turn out to be immaterial. Let $e_{(2)}$ be the subprojection of e which is nonzero only in the first matrix block in (2.11). We define the element

$$\begin{aligned} u_{21} &= \frac{[N]_-}{-i(q^{1/2} - q^{-1/2})}(1 - e_{(2)})u_1 e_{(2)} \\ &= \frac{i[N]_-}{-q^{1/2} - q^{-1/2}} u_1 e_{(2)} + i(q^{1/2} - q^{-1/2})[k]e_{(2)}, \end{aligned} \tag{2.15}$$

where we used $e_{(2)}u_1 e_{(2)} = ae_{(2)}$. We similarly define $u_{12} = u_{21}^t$ by the same expression as in 2.15 with u_1 and $e_{(2)}$ interchanged. By construction, it follows that the elements u_{21} and u_{12} are represented by the matrices $u_{21} = \sqrt{[N] - 1}E_{21}$ and $u_{12} = \sqrt{[N] - 1}E_{12}$, where E_{ij} are matrix units in the first 2×2 matrix block in (2.12). The following results follow immediately from this.

Lemma 2.7. *We have* $u_{12}u_{21} = ([N] - 1)e_{(2)}$, $u_{21}u_{12}e_{(2)} = 0 = e_{(2)}u_{21}u_{12}$ *and the element* $P = e_{(2)} + u_{12} + u_{21} + u_{21}u_{12}$ *satisfies* $P^2 = [N]P$.

2.7. Summary of relations

The following is a preliminary definition of the algebras C_n. The precise, but less intuitive definition of the algebras C_n in various versions will be given in the following section.

Definition 2.8. Fix $N = 2k+1$. Then we define the algebra $C_{n,\pm} = C_{n,\pm}^{(N)}$ via generators e, u_i, $1 \le i < n$ with the following relations:

(a) The elements u_i satisfy the Hecke algebra relations,
(b) We have a sequence of idempotents $e_{(r)}$ defined inductively by $e_{(0)} = 1$, $e_{(1)} = e$ and

$$e_{(r+1)} = e_{(r)}u_r e_{(r)} - \frac{q^{(N-2)/2} - q^{-(N-2)/2}}{q^{N/2} - q^{-N/2}} e_{(r)}, \quad \text{for } C_{n,+},$$
$$e_{(r+1)} = \frac{q^{(N-2)/2} + q^{-(N-2)/2}}{q^{N/2} + q^{-N/2}} e_{(r)} - e_{(r)}u_r e_{(r)}, \quad \text{for } C_{n,-}.$$

(c) For $j < r$ we have $(u_{j-1}u_j - u_{j-1})e_{(r)} = (u_j u_{j-1} - u_j)e_{(r)}$ for $C_{n,+}$ and $(u_{j-1}u_j + u_{j-1})e_{(r)} = (u_j u_{j-1} + u_j)e_{(r)}$ for $C_{n,-}$.

Remark 2.9. 1. In spite of fractions in the exponents, it is not hard to check that the relations only depend on q and not on the choice of a square root $q^{1/2}$. E.g. we have

$$\frac{q^{(N-2)/2} - q^{-(N-2)/2}}{q^{N/2} - q^{-N/2}} = \frac{q^{N-1} + q^{N-2} + \cdots + q^{1-N}}{q^N + q^{N-1} + \cdots + q^{-N}}.$$

2. Let $e_\pm(q)$ and $u_{i,\pm}(q)$ be the generators of $C_{n,\pm}(q)$ for a given choice of q respectively. Then we can check that the maps

$$e_-(q) \mapsto e_+(-q), \qquad u_{i,-}(q) \mapsto -u_{i,+}(-q), \quad 1 \le i < n,$$

define an isomorphism between $C_{n,-}(q)$ and $C_{n,+}(-q)$.

2.8. Alternative definitions of the algebras C_n

We make the following adjustments for the precise definition of the algebra $C_n = C_n(p,q)$ as an algebra depending on two variables p and q. First, we substitute $p = q^N$ in the previous relations. Secondly, we introduce additional generators which will only be relevant for the important classical limit $q \to 1$ for $C_{n,-}$ (respectively for $q \to -1$ for $C_{n,+}$). It will be comparatively easy to prove for this version of the algebra that its dimension is indeed given by its upper bound, see Theorem 3.7. We also expect this fact to be true for specializations of the parameters, after possibly suitably rescaling our basis elements.

Definition 2.10 (Two variable definition)**.** We define the algebra $C_n = C_{n,\pm} = C_{n,\pm}(p,q)$ over the field $\mathbb{C}(p,q)$ of rational functions in the variables p and q as follows: We have generators u_i, $e_{(r)}$ for $1 \leq i, r < n$ with the following relations:

(a) The elements u_i satisfy the Hecke algebra relations,
(b) The elements $e_{(r)}$ are idempotents which satisfy the relations

$$e_{(r+1)} = e_{(r)} u_r e_{(r)} - \frac{p^{1/2}q^{-1} - qp^{-1/2}}{p^{1/2} - p^{-1/2}} e_{(r)}, \quad \text{for } C_{n,+},$$
$$e_{(r+1)} = \frac{p^{1/2}q^{-1} + qp^{-1/2}}{p^{1/2} + p^{-1/2}} e_{(r)} - e_{(r)} u_r e_{(r)}, \quad \text{for } C_{n,-}.$$

(c) For $j < r$ we have $(u_{j-1}u_j - u_{j-1})e_{(r)} = (u_j u_{j-1} - u_j)e_{(r)}$ for $C_{n,+}$ and $(u_{j-1}u_j + u_{j-1})e_{(r)} = (u_j u_{j-1} + u_j)e_{(r)}$ for $C_{n,-}$.

Remark 2.11*.* Similarly as e.g. for the algebras defined in [2], this definition is not convenient if we are interested in obtaining the classical limits for $q \to 1$ and $p = q^N \to 1$. This can be addressed by introducing additional generators. It will be shown in Section 3 that one can make sense of these additional elements if $q \to 1$.

Definition 2.12 (Extended definition for $C_n(q^N, q)$)**.** We now let $p = q^N$ as before. We add to the usual generators u_i, $e_{(r)}$ for $1 \leq i, r < n$ also the elements

$$\frac{1}{q^{1/2} + q^{-1/2}} u_i e_{(r)}, \ \frac{1}{q^{1/2} + q^{-1/2}} e_{(r)} u_i, \quad 1 \leq i < r < n, \quad \text{for } C_{n,+},$$
$$\frac{1}{q^{1/2} - q^{-1/2}} u_i e_{(r)}, \ \frac{1}{q^{1/2} - q^{-1/2}} e_{(r)} u_i, \quad 1 \leq i < r < n, \quad \text{for } C_{n,-}.$$

The relations are the same as in Definition 2.8.

2.9. Basic structure results

We will prove existence of nontrivial representations of the algebras $C_{n,\pm}$ in the next section.

Proposition 2.13. (a) *The map* Φ *given by* $e \mapsto e_{(r+1)}$, $u_i \mapsto e_{(r)}u_{r+i}$ *defines a homomorphism of* C_{n-r} *onto* $e_{(r)}C_ne_{(r)}$.

(b) *The algebras* $C_{n,\pm}$ *are spanned by the* $H_n - H_n$ *bimodules* $H_ne_{(r)}H_n$, $0 \leq r \leq n$. *In particular, they are finite dimensional.*

(c) *The span* I_r *of* $\bigcup_{s\geq r} H_ne_{(s)}H_n$ *is a two-sided ideal of* C_n *for* $1 \leq s \leq n$.

Proof. The homomorphism property in (a) follows directly from the relations. For (b), we observe that the claimed spanning set contains the generators of C_n. It hence suffices to show that multiplying it by a generator from the right or left will still produce an element in the span. This is obviously true for the generators u_i. We prove the claim for multiplication by e by induction on n, with the statement obviously true for $n = 1$. Observe that the Hecke algebra H_n is spanned by elements of the form au_1b or a, with $a, b \in H_{2,n}$, where $H_{2,n}$ is the subalgebra generated by $u_2, u_3, \ldots, u_{n-1}$ and 1. We then have, using $e_{(r)} = ee_{(r)}$, and with c as in Lemma 2.6,

$$e(au_1b)e_{(r)} = a(eu_1e)be_{(r)} = c\ abe_{(r)} + ae_{(2)}be_{(r)}.$$

But now $e_{(2)}be_{(r)} \in e_{(2)}H_{2,n}e_{(2)} \subset \Phi(eH_{n-1}e)$ by (a). Hence, by induction assumption, we have

$$e_{(2)}be_{(r)} \in span \bigcup_s \Phi^{-1}(H_{n-1}e_{(s)}H_{n-1}) \subset span \bigcup_s H_{n-1}e_{(s)}H_{n-1}.$$

The proof for multiplication by e from the right goes completely analogously. This finishes the proof of statement (b). The surjectivity statement in (a) now follows from (b), as Φ maps $e_{(s)}$ to $e_{(s+1)}$. Our proof of (b) also implies statement (c). ☐

2.10. Dimension estimates

We will explicitly construct spanning sets for the algebras C_n which will later be shown to be bases. To do so, we shall use two well-known facts about Hecke algebras, here only formulated for Hecke algebras for type A (see [9] for details). Let s_i, $1 \leq i < n$ be a set of simple reflections for the symmetric group S_n (say $s_i = (i, i + 1)$). Then any element $w \in S_n$ can be written as a product of simple reflections. Any such expression for w with the minimal number of factors is called a reduced word, and the number of factors is called the length $\ell(w)$ of w. In the case of the symmetric group, the length $\ell(w)$ can also be defined as the number of pairs $i < j$ such that $w(i) > w(j)$. Replacing the elements s_i by generators g_i in such an expression defines an element $h_w \in H_n$ which does not depend on the choice of reduced expression for w.

It is easy to see that the shortest elements in the left cosets of $S_r \subset S_n$ are given by permutations w which satisfy $w(i) < w(j)$ for any $1 \leq i < j \leq r$, and that the shortest elements in the left cosets of $S_r \times S_{n-r} \subset S_n$ are given by permutations w which also satisfy the additional conditions $w(i) < w(j)$ for all $r < i < j \leq n$. It is well-known (and easy to check for these cases) that each such coset contains a unique element of lowest length.

Definition 2.14. If $w \in S_n$ and h_w the corresponding element in H_n, we define $h_w^T = h_{w^{-1}}$. If $\mathcal{S} \subset H_n$ we define $\mathcal{S}^T = \{h^T, h \in \mathcal{S}\}$. Using these conventions, we define $\mathcal{C}_{n,r} = \{h_w\}$, where w ranges over the shortest elements in the left cosets of $S_r \subset S_n$, and we define $\mathcal{D}_{n,r} = \{h_w\}$, where now w ranges over the shortest elements of the left cosets of $S_r \times S_{n-r} \subset S_n$. Finally, we define $\mathcal{B}_r$ inductively by $\mathcal{B}_0 = \emptyset$, $\mathcal{B}_1 = \{1\}$ and

$$\mathcal{B}_{r+1} = \mathcal{B}_r \cup \bigcup_{j=1}^{r} g_j g_{j+1} \cdots g_r \mathcal{B}_{r-1}.$$

We remark that $|\mathcal{C}_{n,r}| = [S_n : S_r] = n!/r!$ and that $|\mathcal{D}_{n,r}| = \binom{n}{r}$, and that $\mathcal{D}_{n,r}^T$ contains the elements h_w with w running through the shortest elements of the right cosets of $S_r \times S_{n-r} \subset S_n$.

Lemma 2.15. (a) *We have $|\mathcal{B}_r| = h_r$, with h_r as in (1.3).*
(b) *The set $\mathcal{C}_{n,r}\mathcal{B}_r e_{(r)}$ spans $H_n e_{(r)}$.*

Proof. Statement (a) follows from the definitions of h_r in (1.3) and $\mathcal{B}_r$. To establish statement (b), let us first prove it in the special case $n = r$ by induction on r. This is obviously true for $r = 1$. We now prove by downward induction from $s = r - 1$ to $s = 1$ that

$$(*) \qquad H_r g_r g_{r-1} \cdots g_s e_{(r+1)} \subset H_r e_{(r+1)} + H_r g_r e_{(r+1)}.$$

For $s = r-1$ it follows from relation (c) that $g_r g_{r-1} e_{(r+1)} = (g_{r-1}g_r - g_{r-1} + g_r)e_{(r+1)}$. This implies $H_r g_r g_{r-1} e_{(r+1)}$ is contained in the right hand side of the claim. The induction step for $s < r - 1$ is shown in the same way. As $H_{r+1} = H_r + \sum_{s=1}^{r} H_r g_r g_{r-1} \cdots g_s$, it follows that

$$(2.16) \qquad \begin{aligned} H_{r+1}e_{(r+1)} &= H_r e_{(r+1)} + H_r g_r e_{(r+1)} \\ &= H_r e_{(r+1)} + \sum_{j=1}^{r} H_{r-1} g_{r-1} g_{r-2} \cdots g_j e_{(r-1)} g_r e_{(r+1)}, \end{aligned}$$

where the summand for $j = r$ is defined to be equal to $H_{r-1}e_{(r-1)}g_r e_{(r+1)}$. The claim for $n = r$ follows from $(*)$ and the definition of $\mathcal{B}_{r+1}$. The general claim for $n > r$ follows from this and the fact that $H_n = \mathcal{C}_{n,r} H_r = \bigcup_{h_w \in \mathcal{C}_{n,r}} h_w H_r$. $\square$

Lemma 2.16. *We have the following inequalities:*

(a) $\dim H_n e_{(r)} \leq \frac{n!}{r!} h_r$,
(b) $\dim H_n e_{(r)} H_n \leq \frac{n!}{r!} \binom{n}{r} h_r^2$,
(c) *We have* $\dim C_n \leq \sum_{r=0}^n \frac{n!}{r!} \binom{n}{r} h_r^2$.

Proof. It follows from the inductive definition of $\mathcal{C}_{r,n}$ that $|\mathcal{C}_{r,n}| = n!/r!$ This and Lemma 2.15 imply claim (a). The inequality in (b) follows from (a) and the surjective map $H_n \otimes_{H_{r+1,n}} H_n \to H_n e_{(r)} H_n$. Finally, claim (c) follows from the fact that C_n is spanned by the subspaces $H_n e_{(r)} H_n$, $0 \leq r \leq n$. $\square$

2.11. An explicit spanning set

We use the notations from Definition 2.14.

Proposition 2.17. *The set* $\mathcal{B}(r) = \mathcal{C}_{r,n} \mathcal{B}_{r,n} e_{(r)} \mathcal{B}_{n,r}^T \mathcal{D}_{r,n}^T$ *spans* $H_n e_{(r)} H_n$ *for* $1 \leq r \leq n$, *and hence* $\mathcal{B} = \bigcup \mathcal{B}(r)$ *spans* C_n.

Proof. We have already proved in Lemma 2.15 that the set $\mathcal{C}_{n,r} \mathcal{B}_r e_{(r)}$ spans $H_n e_{(r)}$. One can show the same way that the set $e_{(r)} \mathcal{B}_r^T$ spans $e_{(r)} H_r$. The claim now follows from this and the fact that $\mathcal{D}_{n,r}^T$ contains the elements of minimal lengths for all right cosets of $H_r \times H_{n-r} \subset H_n$. $\square$

3. Tensor product representations

We will give explicit representations of the algebras $C_n = C_{n,+}$ in this section. In view of Remark 2.9, these representations can be easily modified to representations of the algebras $C_{n,-}$. As before, let $V = \mathbb{C}^N$ with $N = 2k+1$ and let $\{v_i\}$ denote the standard basis of $\mathbb{C}^N$. We define $v_o = \sum_{i=1}^N \alpha_i v_i$, where $\alpha_i = q^{(k+1-i)/2} / \sqrt{\sum_{i=-k}^k q^i}$. The sign of the square root is immaterial, but fixed throughout the paper. Observe that $v_o^T v_o = 1$.

3.1. A matrix for *e*

We define the N by N matrix $E = v_o v_o^T$. Moreover, we modify the Hecke algebra representation in (2.2) to $u \mapsto U$, where the matrix U is defined by

$$U_{|\, span\{v_i \otimes v_j, v_j \otimes v_i\}} = \begin{bmatrix} q^{-1} & 1 \\ 1 & q \end{bmatrix}, \quad i < j. \tag{3.1}$$

with the matrices U_i defined as in (1.5). We then consider the map Φ which maps the generators of C_n into $\mathrm{End}(V^{\otimes n})$ given by

$$\Phi: \quad e \mapsto E \otimes 1_{n-1}, \qquad u_i \mapsto U_i, \quad 1 \le i < n. \tag{3.2}$$

Lemma 3.1. *The map Φ is compatible with relation (b), mapping the element $e_{(n)}$ to $E^{\otimes n}$. In particular, we have*

$$E \otimes E = (E \otimes 1)U(E \otimes 1) - \frac{q^{k-1/2} - q^{-k+1/2}}{q^{k+1/2} - q^{-k-1/2}} E \otimes 1.$$

Proof. Observe that $Ev_i = \alpha_i v_o$ for all i. As the image of E is spanned by v_o, it suffices to check the claim for vectors of the form $v_o \otimes v_j$. We then calculate

$$\begin{aligned} U(v_o \otimes v_j) &= \sum_{i=1}^{j-1} \alpha_i(q^{-1} v_i \otimes v_j + v_j \otimes v_i) + \sum_{i=j+1}^{N} \alpha_i(q v_i \otimes v_j + v_j \otimes v_i), \\ (E \otimes 1)U(v_o \otimes v_j) &= \sum_{i=1}^{j-1} \alpha_i^2 q^{-1} v_o \otimes v_j + \alpha_i \alpha_j v_o \otimes v_i \\ &\quad + \sum_{i=j+1}^{N} \alpha_i^2 q v_o \otimes v_j + \alpha_i \alpha_j v_o \otimes v_i \\ &= \alpha_j \sum_{i=1}^{N} v_o \otimes \alpha_i v_i + \beta v_o \otimes e_j, \end{aligned}$$

where

$$\beta = -\alpha_j^2 + \sum_{i=1}^{j-1} \alpha_i^2 q^{-1} + \sum_{i=j+1}^{N} \alpha_i^2 q.$$

It is now straightforward to check that $\alpha_j \sum_{i=1}^{N} v_o \otimes \alpha_i v_i = (E \otimes E)(v_o \otimes v_j)$ and

$$\beta = \frac{q^{k-1} + q^{k-2} + \cdots + q^{1-k}}{\sum_{i=-k}^{k} q^i} = \frac{q^{k-1/2} - q^{-k+1/2}}{q^{k+1/2} - q^{-k-1/2}},$$

where the last equality is obtained by multiplying both numerator and denominator by $q^{1/2} - q^{-1/2}$. This proves the second statement in the claim. We can now show by induction on n that Φ can be extended to a homomorphism which maps $e_{(n)}$ to $E^{\otimes n}$. $\square$

3.2. Checking relation (c)

Let $\{v_i\}$ and v_o be as above. Then we have

Lemma 3.2.

$$(U_1U_2 - U_1)v_o^{\otimes 3} = (U_2U_1 - U_2)v_o^{\otimes 3}.$$

Proof. This is a straightforward calculation. It is manageably tedious if one checks it separately on the span of all possible permutations of $v_i \otimes v_j \otimes v_m$ for any choice of indices i, j and m. If all indices are mutually distinct, one obtains two 6×6 matrices for U_1 and U_2, both with three 2×2 blocks. Moreover, the coefficient for each of these vectors in the basis expansion of $v_o^{\otimes 3}$ is equal to $\alpha_i\alpha_j\alpha_m$. Hence it suffices to check the claim for these two 6×6 matrices, applied to the vector $(1,1,1,1,1,1)^T$, which is not very hard. The case where two indices coincide is done similarly and easier, only involving 3×3 matrices. □

Remark 3.3. If we define the algebra $\tilde{C}_3$ like the algebra C_3 without relation (c), it can be shown that $\tilde{C}_3$ modulo its radical is isomorphic to C_3.

3.3. Classical limits

It will be more convenient to consider the representations for $C_{n,-}$, i.e. we basically replace q by $-q$ in the matrix $\Phi(e)$ and in the coefficients of the vector v_o, see Remark 2.9. We are going to show that the elements $\frac{1}{q^{1/2}-q^{-1/2}}u_ie_{(r)}$ still make sense in our representation even at $q = 1$.

Lemma 3.4. (a) *The matrix coefficients of $\frac{1}{q^{1/2}-q^{-1/2}}U_iE_{(r)}$ in the tensor product representation of $C_{n,-}$ are well-defined also at $q = 1$, up to the choice of the square root $q^{1/2}$.*

(b) *The elements $E^{\otimes 2}$, $\frac{1}{q^{1/2}-q^{-1/2}}U_1E^{\otimes 2}$ and $\frac{1}{q^{1/2}-q^{-1/2}}E^{\otimes 2}U_1$ generate an algebra which is isomorphic to the 2×2 matrices if $[N] \neq 1$.*

Proof. We calculate

$$Uv_o^{\otimes 2} = \sum_{i<j} \alpha_i\alpha_j(q^{-1} - 1)v_i \otimes v_j + (q - 1)v_j \otimes v_i).$$

It follows that also the coefficients in the expression for $\frac{1}{q^{1/2}-q^{-1/2}}Uv_o^{\otimes 2}$ are in $\mathbb{Z}[q^{\pm 1/2}]$. As all columns of $E^{\otimes 2}$ are proportional to $v_o^{\otimes 2}$, claim (a) follows.

Recall the definition of u_{21} in Definition 2.15. It follows from part (a) and the equations below (2.14) that $\Phi(u_{21})$ and $\Phi(u_{12})$ are well-defined and that

$$\Phi(u_{21}u_{12}) = \frac{-[N]_-^2}{(q^{1/2}-q^{-1/2})^2}(1-E^{\otimes 2})UE^{\otimes 2}U(1-E^{\otimes 2})$$

has the nonzero eigenvalue

$$\frac{-[N]_-^2 a_-([2]-a_-)}{(q^{1/2}-q^{-1/2})^2} = [k](q^{k+1}+q^{-k-1}) = [N]-1. \qquad \square$$

3.4. An embedding of $Sp(N-1)$ into $Sl(N)$

Let A be the $N \times N$ matrix (with $N = 2k+1$ odd) defined by

$$a_{ij} = \begin{cases} (-q^{-1})^{k+1-(i+j)/2} & \text{if } i<j, \\ -(-q^{-1})^{k+1-(i+j)/2} & \text{if } i>j. \end{cases}$$

As usual, we assume fixed choices of $(-q)^{1/2}$ and of $(-q^{-1})^{1/2}$ in all these formulas such that their product is equal to -1.

Lemma 3.5. *The matrix A has rank $N-1$ in a neighborhood of $q=1$, with kernel $v_o = \sum_{j=1}^{2k+1}(-q)^{(k+1-j)/2}v_j$. Hence we obtain a symplectic form $(v,w) = v^TAw$ for $q=1$ whose restriction to any complement V' of v_o is nondegenerate.*

Proof. We check that

$$\begin{aligned}(Av_o)_i &= \sum_{j=1}^{i-1} -(-q^{-1})^{k+1-(i+j)/2}(-q)^{(k+1-j)/2} \\ &\quad + \sum_{j=i+1}^{2k+1}(-q^{-1})^{k+1-(i+j)/2}(-q)^{(k+1-j)/2} \\ &= \sum_{j=2}^{2k+1}(-q^{-1})^{(k+1-i)/2}(-1)^{k+1-j} = 0.\end{aligned}$$

To determine the rank at $q=1$, we observe that after conjugation by the diagonal matrix $D = diag((-1)^{(k+1-i)/2})$ the matrix entries become equal to $a_{ij} = (-1)^{k+1-i}$ for $i<j$ and $a_{ij} = -a_{ji}$ for $i>j$. It is now easy to see that the transformed matrix has eigenvectors $(1,0,0,\dots,\pm 1)$, $(0,1,0,\dots,\pm 1,0)$ etc. This proves the claim about the rank. $\square$

Proposition 3.6. *Fix $N = 2k+1$ and let $V = \mathbb{C}^N$. If $q = 1$, the representation of $C_{n,-}$ into* $\mathrm{End}(V^{\otimes n})$ *surjects onto* $\mathrm{End}_{Sp(N-1)}(V^{\otimes n})$*, where the embedding of $Sp(N-1) \subset Sl(N)$ is defined via the symplectic form given by the matrix A in Lemma 3.5 at $q = 1$. In particular,* $\dim_{\mathbb{C}(q)} C_{n,-}(q^N, q) \geq \dim \mathrm{End}_{Sp(N-1)}(V^{\otimes n})$.

Proof. The image of $C_{n,-}$ in $\mathrm{End}(V^{\otimes n})$ at $q = 1$ contains the usual action of the symmetric group S_n on $V^{\otimes n}$ and the projection $E \in \mathrm{End}(V)$ onto $\mathbb{C}v_o$. By Lemma 3.4, it also acts as a full 2×2 matrix algebra on $\mathrm{Hom}(\mathbf{1}, V^{\otimes 2})$. So, in particular, it also must contain the projection F. It follows from Theorem 1.3 that $C_{n,-}(q)$ maps surjectively onto $\mathrm{End}_{Sp(N-1)}(V^{\otimes n})$ for $q = 1$. This implies the estimate about the dimensions. □

3.5. A basis for $C_n(p, q)$

It will be convenient to consider the 2-variable version $C_n(p, q)$ of C_n, as defined in Definition 2.10.

Theorem 3.7. *The spanning set in Proposition 2.17 is a basis for the two-variable version $C_n(p, q)$, viewed as an algebra over the field of rational functions in p and q. In particular, we have* $\dim C_n(p, q) = \sum_{r=0}^{n} h_r^2 \frac{n!}{r!}\binom{n}{r}$.

Proof. It follows from Proposition 3.6 and Lemma 2.16(c), that $\dim C_{n,-}(q^N, q)$ is equal to $\dim \mathrm{End}_{Sp(N-1)}(V^{\otimes n})$ if $N > 2n$. Hence the spanning set $\mathcal{B}$ in Proposition 2.17 is a basis for these values, i.e. for $p = q^N$. □

If we have a linear combination $\sum_{b \in \mathcal{B}} \alpha_b b = 0$ in the $\mathbb{C}(p, q)$ algebra $C_{n,-}(p, q)$, for certain rational functions $\alpha_b \in \mathbb{C}(p, q)$, it would follow that $\alpha_b(q, q^N) = 0$ for all odd $N > 2n$ and all $b \in \mathcal{B}$. Hence $(p - q^N)|\alpha_b$ for all odd $N > 2n$. This is possible only if $\alpha_b = 0$, for all $b \in \mathcal{B}$. Hence $\mathcal{B}$ is also a basis for the algebra $C_{n,-}(p, q)$. The claim can be similarly shown for $C_{n,+}(p, q)$ using Remark 2.9.

3.6. Semisimplicity

We can now use standard techniques to also prove that the algebras C_n are semisimple for generic values. We give some details for the reader's convenience.

Theorem 3.8. *The 2 variable algebra $C_n(p, q)$ as well as the complex algebra C_n for generic values (i.e. for an open and dense subset) of the parameters are semisimple.*

Proof. Let $D(p,q) = \det(Tr(b_i b_j))$ be the discriminant for the algebra C_n; here (b_i) is the basis in Theorem 3.7 and Tr is the trace on $\mathrm{End}(C_n)$. We have seen in Proposition 3.6 and Theorem 3.7 that the image of the algebra $C_n(q^N, q)$ in the tensor product representation is semisimple and faithful in the limit $q \to 1$ for $N > 2n$. Hence the discriminant $D(q^N, q)$, and therefore also $D(p,q)$ must be a nonzero rational function. Hence C_n is semisimple as stated. □

4. Markov traces

Recall that we defined Tr_q on $\mathrm{End}(V^{\otimes n})$ in Section 2.2 by $Tr_q(a) = Tr(aD^{\otimes n})$, where $a \in \mathrm{End}(V^{\otimes n})$ and $D = diag(q^{2i-N-1})$, see (2.3) and also (2.4). We define a functional ϕ on $C_n = C_n^{(N)}$ as a normalized pull-back of Tr_q by

$$\phi(c) = \frac{1}{[N]^n} Tr_q(\Phi(c)), \quad c \in C_n. \tag{4.1}$$

Observe that $\phi(1) = 1$. It is the goal of this section to show that the functional ϕ defines a trace *tr* on C_n which satisfies the Markov condition (2.9). To do so, we first prove some basic properties of ϕ in Lemma 4.1, most of which are easy consequences of the definitions. We then prove certain algebraic identities for the ideal $H_n e_{(n)} H_n$ in Section 4.1. This allows us to prove the trace property of ϕ for that ideal in the following section. The general claim is then proved by downwards induction on r for the ideals $I_r = \oplus_{s=r}^n H_n e_{(s)} H_n$.

Lemma 4.1. *The functional ϕ has the following properties.*

(a) *The restriction of ϕ to H_n defines a trace.*
(b) $\phi(c_1 g_{n-1} c_2) = \phi(c_1 c_2)\phi(g_{n-1})$ *for all* $c \in C_{n-1}$.
(c) $\phi(e_{(r)}) = \phi(e)^r = \frac{1}{[N]^r}$.
(d) $\phi(ch) = \phi(hc)$ *for all* $h \in H_n$, $c \in C_n$.

Proof. Part (a) follows from the discussion in Section 2.2, and part (b) follows from (2.6). We also have for $n = r = 1$ and for $C_{n,+}$

$$Tr_q(E) = Tr(ED) = \sum_{i=1}^N q^{2i-N-1} q^{(N+1)/2-i}/[N]_+ = \sum_{i=1}^N q^{i-(N+1)/2}/[N]_+ = 1.$$

One deduces from this that

$$Tr_q(E^{\otimes r}) = Tr((ED)^{\otimes r}) Tr(D^{\otimes n-r}) = [N]^{n-r},$$

from which follows claim (c). As $\Phi(h)$ commutes with $D^{\otimes n}$ for all $h \in H_n$, we have

$$\begin{aligned} Tr_q(\Phi(ch)) &= Tr(\Phi(c)\Phi(h)D^{\otimes n}) = Tr_q(\Phi(c)D^{\otimes n}\Phi(h)) = Tr_q(\Phi(h)\Phi(c)D^{\otimes n}) \\ &= Tr_q(\Phi(hc)). \end{aligned}$$ □

4.1. Technical lemmas for the ideal $\boldsymbol{H_n e_{(n)} H_n}$

It follows directly from the relations that the map

$$\Theta_n : g_i \mapsto g_{n-i}, \quad 1 \leq i < n \tag{4.2}$$

induces an automorphism of the Hecke algebra H_n which will be denoted by the same letter. Also observe that if $w_o \in S_n$ is defined by $w_o(i) = n - i$, and we denote the corresponding map on $V^{\otimes n}$ given via permutation of the factors by the same letter, we have

$$w_o U_i w_o = U_{n-i}(q^{-1}), \tag{4.3}$$

where $U_i(q^{-1})$ is given by the same matrix as U_i, with every occurrence of q replaced by q^{-1}.

Lemma 4.2. *We have*

$$\begin{aligned} Tr_q(\Phi(h_1)E^{\otimes n}\Phi(h_2)) &= Tr(\Phi(h_1)E^{\otimes n}(q^{-1})\Phi(h_2)) \\ &= Tr(\Phi(\Theta_n(h_1))E^{\otimes n}\Phi(\Theta_n(h_2))). \end{aligned}$$

Proof. To avoid cumbersome notation, we denote $H_i = \Phi(h_i)$ in this proof. Then we have

$$Tr_q(H_1E^{\otimes n}H_2) = Tr(H_1(D^{1/2}ED^{1/2})^{\otimes n}H_2) = Tr(H_1E^{\otimes n}(q^{-1})H_2),$$

from which follows the first equality in the statement. Now observe that the structure coefficients in the defining relations of C_n are invariant under $q \leftrightarrow q^{-1}$. Hence we also obtain a representation of C_n via the assignment

$$e \mapsto E(q^{-1}), \qquad u_i \mapsto U_i(q^{-1}), \quad 1 \leq i < n.$$

As $E^{\otimes n}HE^{\otimes n} = \alpha E^{\otimes n}$ for some scalar α, it also follows $E^{\otimes n}(q^{-1})H(q^{-1}) \times E^{\otimes n}(q^{-1}) = \alpha E^{\otimes n}(q^{-1})$ for the same scalar. Applying this to $H = H_2H_1$, we obtain

$$Tr(H_1E^{\otimes n}H_2) = Tr(E^{\otimes n}H_2H_1E^{\otimes n}) = \alpha$$

$$= Tr(E^{\otimes n}(q^{-1})H_2H_1(q^{-1})E^{\otimes n}(q^{-1})) = Tr(\Theta_n(H_1)E^{\otimes n}(q^{-1})\Theta_n(H_2)),$$

where $\Theta_n(H_i)$ coincides with $\Phi(\Theta_n(h_i))$. This proves the second equality. $\square$

Lemma 4.3. *We have $e_{(n)}he_{(n)} = e_{(n)}\Theta_n(h)e_{(n)}$ for all $h \in H_n$.*

Proof. The claim is proved by induction on n, with $n = 1, 2$ being trivially true. For the induction step from $n-1$ to n first observe that for any $A \in \mathrm{End}(V^{\otimes n-1})$ we have

$$E^{\otimes n}(A\otimes 1)E^{\otimes n} = E^{\otimes n-1}AE^{\otimes n-1}\otimes E = E\otimes E^{\otimes n-1}AE^{\otimes n-1} = E^{\otimes n}(1\otimes A)E^{\otimes n},$$

as $E^{\otimes n-1}AE^{\otimes n-1}$ is a scalar multiple of $E^{\otimes n-1}$. We define the homomorphism $sh : H_{n-1} \to H_n$ via $sh(u_i) = u_{i+1}$. If $H \in \Phi(H_{n-1})$, it follows that $E^{\otimes n}sh(H)E^{\otimes n} = E^{\otimes n}HE^{\otimes n}$. Moreover, by induction assumption, we have

$$E^{\otimes n}(\Theta_{n-1}(H) \otimes 1)E^{\otimes n} = E^{\otimes n}(H \otimes 1)E^{\otimes n}.$$

Hence we have

$$e_{(n)}he_{(n)} = e_{(n)}sh(\Theta_{n-1}(h))e_{(n)} = e_{(n)}\Theta_n(h)e_{(n)}, \tag{4.4}$$

which proves the claim for $h \in H_{n-1}$. Let now $h = h'g_{n-1}$ with $h' \in H_{n-1}$. We first observe that for $h \in H_n$, $e_{(n-1)}he_{(n-1)}$ is a linear combination of $\alpha e_{(n-1)} + \beta e_{(n)}$. Using $e_{(n)}e_{(n-1)} = e_{(n)}$, one deduces easily that

$$e_{(n)}he_{(n-1)} = e_{(n)}he_{(n)}.$$

If $e_{(n-1)}h'e_{(n-1)} = \gamma e_{(n-1)}$, we calculate

$$e_{(n)}he_{(n)} = e_{(n)}h'e_{(n-2)}g_{n-1}e_{(n)} = \gamma e_{(n)}g_{n-1}e_{(n)} = \gamma(c'+1)e_{(n)}.$$

On the other hand,

$$e_{(n)}\Theta_n(h)e_{(n)} = e_{(n)}\Theta_n(h')g_1e_{(n)} = e_{(n)}\Theta_n(h')eg_1e_{(n)},$$

where we used that $\Theta_n(h') \in H_{2,n}$ commutes with e. But then

$$e_{(n)}\Theta_n(h)e_{(n)} = e_{(n)}\Theta_n(h')(c'+1)e_{(n)} = \gamma(c'+1)e_{(n)},$$

by (4.4). We now prove the claim for $h = h'g_{n-1}g_{n-2}\cdots g_{n-s}$ by induction on s, with the case for $s = 1$ just shown. Using the relation $(g_{n-s+1}g_{n-s} - g_{n-s+1})e_{(n)} = (g_{n-s}g_{n-s+1} - g_{n-s})e_{(n)}$, we obtain

$$e_{(n)}h'g_{n-1}g_{n-2}\cdots g_{n-s}e_{(n)}$$

$$= e_{(n)}(h'g_{n-s} - h')g_{n-1}g_{n-2}\cdots g_{n-s+1}e_{(n)} + e_{(n)}(h'g_{n-s})g_{n-1}g_{n-2}\cdots g_{n-s+2}e_{(n)}.$$

The claim now holds for each summand on the right hand side by induction assumption. After applying Θ_n to it, it can be easily shown that it is equal to $e_{(n)}\Theta(h)e_{(n)}$. □

4.2. Trace property of ϕ

We will need the following simple observation:

Remark 4.4. Let A be a semisimple algebra, and let $I \subset A$ be a two-sided ideal. If $\psi : A \to \mathbb{C}$ is a functional satisfying $\psi(cd) = \psi(dc)$ for all $c, d \in I$, then we also have $\psi(ca) = \psi(ac)$ for all $a \in A$ and $c \in I$. Indeed, we can write $a = a_I + a_J$ with $a_I \in I$ and $a_J \in J$ where J is a two-sided ideal of A such that $IJ = 0$. The claim follows from $a_J c = 0 = c a_J$.

Theorem 4.5. *The functional ϕ satisfies the Markov property $\phi(cg_{n-1}) = \phi(c)\phi(g_{n-1})$ for all $c \in C_{n-1}$ and the trace property $\phi(cd) = \phi(dc)$ for all $c, d \in C_n$. Hence there exists a trace on C_n satisfying Condition (2.9).*

Proof. The first claim follows from Lemma 4.1(b). We will prove the second claim by induction on n, which is certainly true for the abelian algebra C_1. Let $I_r = \bigoplus_{s\geq r} H_n e_{(s)} H_n$. We will prove that the restriction of ϕ to I_r satisfies the trace property by downwards induction. We define the functional $\alpha : H_n \to \mathbb{C}$ by

$$e_{(n)}he_{(n)} = \alpha(h)e_{(n)}.$$

It follows from Lemma 4.2 and Lemma 4.3 that

$$[N]^n\phi(h_1e_{(n)}h_2) = Tr(\Phi(\Theta_n(h_1))E^{\otimes n}\Phi(\Theta_n(h_2)) \\ = Tr(E^{\otimes n}\Phi(\Theta_n(h_2h_1))E^{\otimes n}) = Tr(E^{\otimes n}\Phi(h_2h_1)E^{\otimes n}) = \alpha(h_2h_1).$$

Then we calculate

$$[N]^n\phi(a_1e_{(n)}a_2b_1e_{(n)}b_2) = [N]^n\alpha(a_2b_1)\phi(a_1e_{(n)}b_2) = \alpha(a_2b_1)\alpha(b_2a_1).$$

One calculates in the same way that also $[N]^n\phi(b_1e_{(n)}b_2a_1e_{(n)}a_2) = \alpha(a_2b_1)\alpha(b_2a_1)$. This proves the claim for $r = n$. For the induction step, we first prove

$$\phi(he_{(s)}) = \phi(e_{(s)}he_{(s)}) \quad \text{for all } h \in H_n, s \geq r. \tag{4.5}$$

For $s > r$, this is clear as the restriction of ϕ to $H_n e_{(s)} H_n$ is a trace by induction assumption. For $s = r$, we will prove the claim for $h \in H_m$ by induction on $m \geq r$. If $m = r$, this follows from the fact that ϕ defines a trace on $H_r e_{(r)} H_r$ by the first part of this proof. For the induction step, it suffices to prove the claim for elements of the form $h = h_1 g_m h_2$, $h_1, h_2 \in H_m$. But by Markov property, see Lemma 4.1(b), we have

$$\phi(h_1 g_m h_2 e_{(r)}) = \phi(g_m)\phi(h_1 h_2 e_{(r)}) = \phi(g_m)\phi(e_{(r)} h_1 h_2 e_{(r)}) = \phi(e_{(r)} h_1 g_m h_2 e_{(r)}),$$

where we used the induction assumption for $h_1 h_2 \in H_m$.

Let now $a_i, b_i \in H_n$, $i = 1, 2$. Then we can write $e_{(r)} a_2 b_1 e_{(r)} = (h+k) e_{(r)} = e_{(r)}(h + k)$ with $h \in H_{r+1,n}$ and $k \in I_{r+1}$. Using $k e_{(r)} = e_{(r)} k = k$, it follows from (4.5) that

$$\phi(hk) = \phi(hk e_{(r)}) = \phi(e_{(r)} hk e_{(r)}) = \phi(e_{(r)} h e_{(r)} k), \tag{4.6}$$

where we used that ϕ is a trace on I_{r+1}. We then calculate, using (4.6)

$$\begin{aligned}
\phi(a_1 e_{(r)} a_2 b_1 e_{(r)} b_2) &= \phi(a_1 e_{(r)} (h + k) e_{(r)} b_2) \\
&= \phi(b_2 a_1 e_{(r)} (h + k) e_{(r)}) && \text{by Lemma 4.1(d)} \\
&= \phi(e_{(r)} b_2 a_1 e_{(r)} (h + k) e_{(r)}) && \text{by (4.5) and Remark 4.4} \\
&= \phi(e_{(r)} b_2 a_1 e_{(r)} a_2 b_1 e_{(r)}).
\end{aligned}$$

We similarly calculate

$$\phi(b_1 e_{(r)} b_2 a_1 e_{(r)} a_2) = \phi(e_{(r)} a_2 b_1 e_{(r)} b_2 a_1 e_{(r)}).$$

The trace property now follows from the induction assumption, using the fact that $e_{(r)} C_n e_{(r)} \cong C_{n-r}$, see Proposition 2.13. $\square$

5. Conclusions and future research

5.1. Historical context

When the author of this paper visited Columbia University as a postdoc in the first half of 1986, Vaughan suggested as a project with my host, J. Birman, that we try to find an algebraic interpretation of the Kauffman polynomial. This resulted in the definition of a new algebra (independently discovered by J. Murakami) which turned out to be a q-deformation of Brauer's centralizer algebra, see [4, 2]. This algebra was subsequently used to construct subfactors of type BCD, among other applications. A different q-deformation of Brauer's centralizer algebra was found in [27], see also [16]. It contained the Hecke algebra H_n as a subalgebra. It was shown in [28] that it could also be used to construct subfactors, as well as module categories of $Rep(U_q\mathfrak{sl}_N)$ and of the related fusion tensor categories $\overline{Rep}(U_q\mathfrak{sl}_N)$ for q a root of unity (they are often also referred to as $SU(N)_k$). This will be sketched below. In particular, we could explicitly calculate the indices and first principal graphs of these subfactors. This, in turn, also allows us to give an explicit description of the algebras corresponding to these module categories. So while the current paper is purely algebraic, it is closely related to research in which Vaughan was interested. In particular, the idea of a Markov trace which plays a crucial role in finding the relations for the algebras C_n goes back to him. It should also be noted that the first module categories for fusion categories related to $SU(2)$ were already constructed by Vaughan and his collaborators in [8].

5.2. Markov traces, module categories and subfactors

It was shown in [24] that the quotient $\overline{H}_n(q)$ of $H_n(q)$ modulo the annihilator of the Markov trace tr is semisimple for all n, with tr as in Remark 2.1. We expect the same to be true for the quotient $\overline{C}_n(q)$ modulo the annihilator ideal of its extension, which was shown to exist in Theorem 4.5. E.g. it is not hard to see that for q not a root of unity the quotient is isomorphic to the image of $C_n(q)$ in its representation in $\mathrm{End}(V^{\otimes n})$, see Section 3. Assuming this, the construction of the module category goes as follows:

It was shown in [13] (see also [23] for a variation of this construction) that $Rep(U_q\mathfrak{sl}_N)$ for q not a root of unity and $\overline{Rep}(U_q\mathfrak{sl}_N)$ for q a root of unity can be reconstructed from the quotients $\overline{H}_n(q)$ modulo the annihilator ideal of a suitable version of the Markov trace. Here the objects are given by idempotents of $\overline{H}_n(q)$. We similarly define the module category $\mathcal{M}$ whose objects are idempotents in $\overline{C}_n(q)$. If $p_M \in \overline{C}_m(q)$ and $p_H \in \overline{H}_n(q)$ are idempotents, we define the module action by

$$p_M \otimes p_H := p_M sh_m(p_H),$$

where the algebra homomorphism $sh_m : H_n(q) \to C_{n+m}(q)$ is defined by $sh_m(g_i) = g_{i+m} \in C_{n+m}(q)$. It follows from the relations that $p_M \otimes p_H$ is an idempotent in $\overline{C}_{n+m}(q)$. Finally, if the quotients $\overline{H}_n(q)$ and $\overline{C}_n(q)$ allow compatible C^* structures, we can construct subfactors $\mathcal{N} \subset \mathcal{M}$ from the inclusions $\lim_{n\to\infty} \overline{H}_n(q) \subset \overline{C}_n(q)$ following the procedure in [24], Section 1.

5.3. Results in [27] and [28]

We give an outline of the results in these papers which give a good idea of the results to be expected in the approach outlined in the previous subsection. A q-version $Br_n(q)$ of Brauer's centralizer algebra (see [4]) was defined in [27] by again adding one more generator e to the generators of the Hecke algebras $H_n(q)$. As in this paper, the relations were forced by the condition that the extension of the Markov trace on $H_n(q)$ to the algebras $Br_n(q)$ satisfy an analog of the Markov condition (2.9). Subfactors were constructed from these algebras as outlined in the previous section. Their indices and first principal graphs are given in [28] Sections 3F and 3G. Instead of copying the results there, we just state an easy consequence of these results which only appears implicitly in [28]:

Let $q = e^{\pi i/(N+k)}$. Then the category constructed from the quotients $\overline{H}_n(q)$ is equivalent to the fusion category $SU(N)_k$ (or $\overline{Rep}(U_q\mathfrak{sl}_N)$ in the notation of this paper). For simplicity, we assume a trivial twist (see [13] or [23] for details). It is well-known that the simple objects of $SU(N)_k$ are labeled by the Young diagrams λ with $\leq N-1$ rows such that $\lambda_1 \leq k$. Recall that a module category over a tensor category can be defined via an algebra object in the given tensor category (see [19]). We will reformulate the following theorem in a somewhat more conceptual way in Remark 5.2.

Theorem 5.1. *Let N be even. Then $SU(N)_k$ has an algebra object $A = Ind_{Ad}(\mathbf{1})$, where $Ind_{Ad}(\mathbf{1})$ is the direct sum of simple objects V_λ such that $N|\ |\lambda|$ and the number of boxes in each column of λ is even.*

Proof. We consider the inclusion of von Neumann factors $\mathcal{N} \subset \mathcal{M}$ constructed in [28], Theorem 3.4 for case (c) listed before that theorem. It follows from the explicit description of its principal graph in [28], Section 3G that the von Neumann algebra $\mathcal{M}$, viewed as an $\mathcal{N}-\mathcal{N}$ bimodule, decomposes into a direct sum of simple $\mathcal{N}-\mathcal{N}$ bimodules labeled by exactly the Young diagrams which appear in $Ind_{Ad}(\mathbf{1})$. As $\mathcal{M}$ has a multiplication, it follows that A in the statement is an algebra object in the category of $\mathcal{N}-\mathcal{N}$ bimodules. It is known that this category is equivalent to $Ad(SU(N)_k$, the subcategory of $SU(N)_k$ whose simple objects are labeled by Young diagrams λ such that $N|\ |\lambda|$ (see e.g. [25], Theorem 4.4). □

Remark 5.2. 1. It is well-known that the restriction to $Sp(N)$ of a simple $SU(N)$-module labeled by the Young diagram λ contains the trivial representation of $Sp(N)$ if and only if the number of boxes in each column of λ is even. Hence the algebra in Theorem 5.1 can be viewed as a natural analog in

$SU(N)_k$ of the induction of the trivial representation of $PSp(N)$ to $PSU(N)$. It would seem plausible that similar algebras exist which correspond to inducing the trivial representation of $PSp(N)$ or $Sp(N)$ to quotients of $SU(N)$ modulo a subgroup of its center. This seems to be compatible with Edie-Michell's classification results of module categories of $SU(N)_k$, see [5] and [6]. This possible generalization of our result became evident after conversations with Edie-Michell.

2. Similarly, one can construct module categories and algebras from cases (a) and (b) before [28], Theorem 3.4. As they are related to embeddings of the full orthogonal group $O(N)$, we would need as larger group the group $SU(N)\times\mathbb{Z}/2$ of unitary matrices u with $|\det(u)| = 1$. It should be possible to obtain module categories of $SU(N)_k$ from this via a $\mathbb{Z}/2$ orbifold construction. One obtains algebra objects for these cases in the same way as it was done in Theorem 5.1. Again, one would expect algebra objects and module categories corresponding to each quotient group $SU(N)/Z$, where Z is a subgroup of the center of $SU(N)$.

3. A complete realization of all module categories for all fusion tensor categories of type $SU(3)_k$ has been given in [7]. Using their results, one can show that the general approach outlined here also works in the setting of this paper for the special case $N = 3$, i.e. for the embedding of $Sp(2) \subset SL(3)$. Indeed, the explicit calculations in [7] were useful in the initial phase of finding relations for our algebras. Here the algebra object coming from the subfactor constructed there would be the direct sum of all simple objects in $SU(3)_k$ labeled by Young diagrams λ with $3|\ |\lambda|$. These subfactors seem to be closely connected to subfactors constructed by F. Xu in [30].

5.4. Classification of module categories of WZW-fusion categories

A lot of progress in classifying module categories has recently been made by Edie-Michell [5], building on the works of Ocneanu, Gannon, Schopieray, Evans and Pugh, and others. Very roughly speaking these module categories can be divided into exceptional and non-exceptional module categories. It appears that we can find realizations for all non-exceptional module categories of fusion tensor categories of type $SU(N)_k$ using the construction sketched in this paper and its generalizations in Remark 5.2 together with the orbifold construction. This was done in collaboration with Edie-Michell. It would also be interesting to find out whether non-exceptional module categories of fusion categories of other Lie types could similarly be realized via the constructions in this paper for certain subgroups in connection with orbifolds.

5.5. Co-ideal subalgebras

As mentioned in the introduction, module categories of a Drinfeld-Jimbo quantum group $U_q\mathfrak{g}$ can be defined for the sub-Lie algebra $\mathfrak{h}$ consisting of the fixed points of an order 2 Lie algebra automorphism, see [15] and [17]. It would be interesting to see whether our (proposed) module category of $U_q\mathfrak{sl}_N$ could be realized by a suitable co-ideal deformation of the universal enveloping algebras $U\mathfrak{sp}_{N-1} \subset U\mathfrak{sl}_N$.

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