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On three homework problems from Vaughan Jones*

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Abstract: This paper contains my previously unpublished work on three problems proposed by Vaughan Jones.

1. Introduction

Vaughan had the amazing ability to come up with simply stated yet thought-provoking problems. Often he presented his problems to his friends in mathematics in an entertaining and casual manner, like a playful professor assigning homework problems to his students, but always in a stimulating way. Over the years I have had my share of his "homework problems". Occasionally I managed to make progress on these problems, and I still remember the excitement I had when sharing my work with him. A few months after his unexpected death, I came across my notes on some of his problems. It brought back to me many fond memories of Vaughan, but also a tremendous sense of loss when reality sunk in. For this special issue in memory of Vaughan, I have decided to include my contributions to three of Vaughan's problems. Besides completing and updating references and adding a few remarks, I have largely kept my original notes intact.

The reader may ask: "how original are the mathematical contents of these notes?" Although I am certain that my notes on Problem Two are original, I am not sure that I can say the same about my notes on Problems Three, and my main contribution to Problem One is mostly digging out known results in the literature. Nevertheless I hope that the connections with subfactors in these notes may still be interesting to some readers. One thing that I am sure is that these notes have not been published before. However, my published work in [19] was partially inspired by my notes on Problem One, and my notes on Problem Two have been circulated in a small group of people (see Appendix in [2] for a categorical account and [8] for a planar algebra construction of a related intermediate subfator). Finally, Vaughan has written a better proof based on my notes on Problem Three in his course on planar algebras (cf. [12]).

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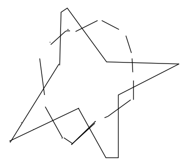

Figure 1: A configuration in meander problem: the dotted non-self intersecting curve intersects with the solid curve transversely at 8 points.

2. Problem One: A counting problem

This section is essentially a hand written letter I wrote to Vaughan on March 1, 1996. This was when people still mailed letters to each other. Vaughan's problem stated in the letter was clearly motivated by his work on Annual Algebras (cf. [10]). In that paper he found a formula for dimensions of his Annual Algebras by counting the number of non-intersecting pairings that can be drawn on an annulus. Here we refrain from discussing more background in subfactor theory and instead refer the reader to more details on his results to [10].

After receiving this letter, Vaughan invited me to give a talk at his famous Friday afternoon seminar some time in the spring of 1996, where I talked about this letter and also my related work in [19].

2.1. My letter to Vaughan on March 1, 1996

Dear Vaughan,

The purpose of this letter is to explain an answer to a question you asked me about two years ago when you were in Geneva. The question is to count all possible parings of $2n$ points on the boundary of a disk on a genus g Riemann Surface with the constraint that the curves connecting these points are drawn on the Riemann surface without intersections. I will use $I_g(n)$ to denote such number.

I came across a solution when I was reading a paper about meander problem and matrix models (in fact I had the dream of using matrix models to do some calculations in Picu's free probability theory). I made a copy of the relevant part of that paper for your convenience. You can also find it online hep-th/9506030.

As you can see from that paper, a matrix model is devised to count all possible configurations in Figure 1.

The point is that each intersection is of the form of 4-valent vertex and hence the potential is of the form $B^{(\alpha)}W^{(\beta)}B^{(\alpha)}W^{(\beta)}$ (2 blacks and 2 whites) in Figure 2.

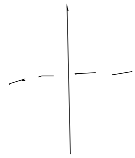

Figure 2: 4-valent vertex that is related to the potential in a matrix model.

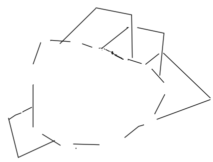

Figure 3: A specific configuration.

In your counting problem we look for configuration as in Figure 3 where one can think of black as solid line and white as dotted line.

Hence the potential is $W^{(\beta)}B^{(\alpha)}W^{(\beta)}$, i.e., one black and two whites like in Figure 4.

I use $\epsilon_g(n)$ to denote the number of pairings that can only be drawn on a genus g surface, i.e. the pairings that are obtained by using all g handles. Then

$$I_g(n) = \sum_{g' \leq g} \epsilon_{g'}(n).$$

Notice that $\epsilon_0(n) = \frac{(2n)!}{n!(n+1)!}$ is the Catalan number. Then use the same formalism as in the meander case we have that

$$\sum_g N^{(n+1-2g)} \epsilon_g(n) = \langle \operatorname{tr} B^{2n} \rangle \tag{1}$$

where the quantity on the right hand side of above is the expectation in Gaussian Hermitian matrices of size N. In fact, we can use (5.13) of [3] with

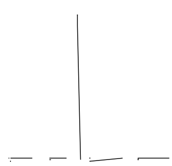

Figure 4: Trivalent vertex.

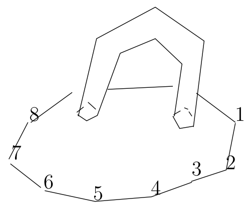

Figure 5: Counting using annulus 1.

$\det[I \otimes I - c\sum_\alpha (B^{(\alpha)})^t \otimes B^{(\alpha)}]$ replaced by $\det[I - c\sum_\alpha B^{(\alpha)}]$ and carry through the calculation of (5.14), we get

$$\sum_g N^{(n+1-2g)} \epsilon_g(n) m^{2n} = \left\langle \operatorname{tr}\left(\sum_{1\le\alpha\le m} B^{(\alpha)} \right)^{2n} \right\rangle.$$

Setting $m = 1$ we get equation (1). This is much simpler than the meander case. When I tried to calculate $\langle \operatorname{tr} B^{2n} \rangle$ directly, I realized such a calculation has already been done, notably by Hare and Zagier in [4] and Itzyson and Zuber in [5]. It is rather disappointing to see that after some reflections that $\epsilon_g(n)$ has already been explicitly determined in [4] by a recursive formula. Let me quote their formula:

$$(n+1)\epsilon_g(n) = (4n-2)\epsilon_g(n-1) + (2n-1)(n-1)(2n-3)\epsilon_{g-1}(n-2)$$

where $\epsilon_g(n) = 0$ if $2g > n$. Hence the recursion formula for $I_g(n)$ is given by

$$(n+1)I_g(n) = (4n-2)I_g(n-1) + (2n-1)(n-1)(2n-3)I_{g-1}(n-2) \tag{2}$$

with initial condition $I_g(0) = 1$ for $g \geq 0$.

It is interesting to check equation (2) in $g = 1$ case by using dimension formula $A(p, q)$ in your annular algebra (cf. [10]) paper and induction. It follows from equation (2) that

$$I_1(n) = \frac{1}{12}\frac{(2n)!}{n!(n-2)!} + \frac{(2n)!}{n!(n+1)!}, \quad n \geq 2.$$

The computation gets interesting in $I_1(4) = 84$ case $(I_1(3) = 15, I_1(2) = 3)$.

It is clear if we connect 12 or 18 in Figure 5, the best way is not through the handle, and this gives us $I_1(3) = 15$ for the number of pairings for the rest 6 points when we fix 12 or 18. When we connect 13 or 17, the number of

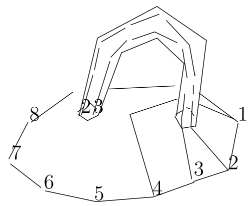

Figure 6: Counting using annulus 2.

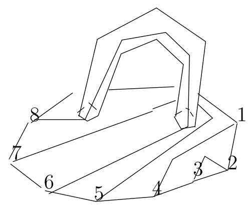

Figure 7: Counting using annulus 3.

pairings for the rest 6 points are on the annulus and given by $A(1,5) = 10$; and when we connect 15, the number of pairings for the rest 6 points are on the annulus and given by $A(3,3) = 12$. Connecting 14 (and similarly 16) case gets a little trickier. One may think that the best way is to connect 14 passing through the handle like I did the first time, this will give us the number of pairings for the rest of 6 points on the annulus and are given by $A(2,4) = 10$, by using the handle to push the points $2, 3$ to the other side as in Figure 6.

But if you count all these, you get

$$15 \times 2 + 10 \times 4 + 12 = 82.$$

In fact in connecting $1, 4$, there is another possibility: when you close points 1 and 4, 2 and 3, and let points $8, 7, 6, 5$ use the handle which gives us one more partition (86)(75) that can't be obtained otherwise as in Figure 7. This together with similar considerations for $1, 6$ gives us two more pairings, bringing the total number to 84.

I have checked $I_1(5) = 462$ case using similar induction and your dimension formula $A(p,q)$ for the annular diagrams with $p + q = 8$. I hope that equation (2) may be useful in generalizing your annular algebra paper to possibly higher genus case.

Best Regards, Feng

3. Problem Two: Does Haagerup subfactor come from nature?

The Haagerup subfactor and its dual are the finite depth irreducible subfactors with smallest index above 4. Its original construction as presented in [1] and later in [14] is by hand and heavily computational. Also see [9] for a different approach. The problem in the title of this section must have been on Vaughan's mind early on since the birth of Haagerup subfactor. On the other hand this may be a hard homework problem since he once remarked to me that "Maybe in fifty years subfactors like the Haagerup subfactor may be constructed naturally".

In the late 1990s I have had extensive discussions with Vaughan on this problem. Of course "nature" means different things to different people. But after having produced a large class of exotic subfactors from Conformal Field Theories (CFT) in [16], I am naturally led to search for the Haagerup subfactor or other exotic subfactors in the framework of CFT. If one interprets nature in this case as the framework of CFT, then the question in the title of this section is still open today, and there has been extensive work on this and related questions. See [7] and [20] for related work. My notes in the next section were dated November 2001, and were handed to Vaughan when I visited him in Berkeley in early 2002. I have added one formula in the proof of the only Proposition in the notes for clarity.

3.1. Haagerup subfactor on two legs

The search for Haagerup's subfactor of index $\frac{5+\sqrt{13}}{2}$ in CFT such as those in [16, 17] has proved to be disappointing. Among the negative results we'd like to point out an exotic subfactor that comes from conformal inclusions

$$(G_2)_3 \subset (E_6)_1$$

The dual principal and principal graphs of this subfactor are given by Figure 8 and Figure 9 respectively. Here all vertices are also labeled by the corresponding sectors. All vertices except distinguished ones are represented by small circles.

Note that if we remove the two small line segments from the central vertices in the graphs, we get exactly Haagerup's subfactor of index $\frac{5+\sqrt{13}}{2}$. Hence it is tempting (with apology to Uffe) to call the subfactor Haagerup subfactor on two legs.

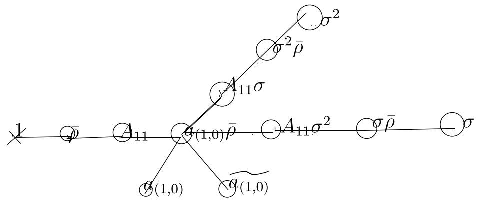

Figure 8: Principal graph of Haagerup subfactor on two legs.

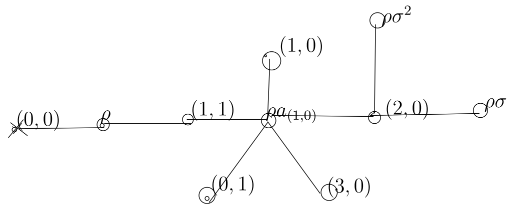

Figure 9: dual principal graph of Haagerup subfactor on two legs.

Let me say a few words on the labels of the graphs. The numbers (i,j) label the representations of G_2 at level 3, e.g., $(1,0)$ is corresponds to the 7-dimensional representation. ρ is the endomorphism corresponding to the conformal inclusion with $\bar\rho\rho = (0,0) + (1,1)$. $a_{(1,0)}$ are the braided endomorphisms introduced in [16]. To explain how we find out the above example let us remove the two legs from Figures 8 and 9 to get one of the Haagerup's subfactor's principal graphs in Figure 10, and label it as in Figure 10 (we use the same σ).

The most unusual aspect of Haagerup's subfactor is the fusion rule algebra generated by the even vertices of the above graph (cf. [1]):

$$A\sigma = \sigma^{-1}A, \quad A^2 = 1 + A + \sigma A + \sigma^2 A, \quad \sigma^3 = 1.$$

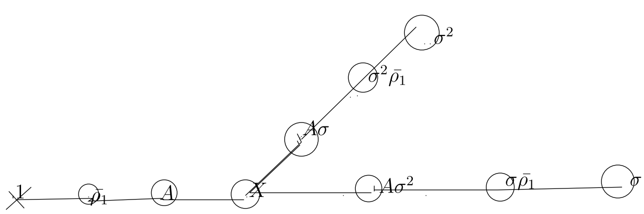

Figure 10: Principal graph of Haagerup subfactor.

Note that A, σ do not commute. In searching for conformal inclusions with $\mathbb{Z}_3$ symmetry and non-commutative fusion rules, we find $(G_2)_3 \subset (E_6)_1$ where $\mathbb{Z}_3$ symmetry comes from the center of E_6 which is $\mathbb{Z}_3$, and non-commutativity comes from the nontrivial multiplicities in the branching rules of $(G_2)_3 \subset (E_6)_1$. This example is also similar but simpler than the example $SU(3)_9 \subset (E_6)_1$ in [17] which is the first counter-example to the Kac-Wakimoto hypothesis. By using general results in [16] we determine the principal graphs for the subfactor that comes from conformal inclusions $(G_2)_3 \subset (E_6)_1$ as given in Figures 8 and 9. The complete fusion rules for this subfactor are given by

$$A_{11}\sigma = \sigma^{-1}A_{11}, \quad A_{11}^2 = 1 + A_{11} + \sigma A_{11} + \sigma^2 A_{11} + a_{(1,0)} + \widetilde{a_{(1,0)}},$$
$$a_{(1,0)}\widetilde{a_{(1,0)}} = A_{11} + \sigma A_{11} + \sigma^2 A_{11}, \quad a_{(1,0)}^2 = 1 + \sigma + \sigma^2 + 3a_{(1,0)},$$
$$\widetilde{a_{(1,0)}}^2 = 1 + \sigma + \sigma^2 + 3\widetilde{a_{(1,0)}}, \quad \sigma a_{(1,0)} = a_{(1,0)}, \quad \sigma\widetilde{a_{(1,0)}} = \widetilde{a_{(1,0)}}, \quad \sigma^3 = 1.$$

Note that the non-commutativity relation $A_{11}\sigma = \sigma^{-1}A_{11}$ is the same as in Haagerup's subfactor's case.

However the subfactor in Figure 8 has index $\frac{7+\sqrt{21}}{2}$ which is about 5.79 and is much bigger than $\frac{5+\sqrt{13}}{2}$. From the fusion rule for A_{11}^2 one can see that the subfactor $A_{11}(M) \subset M$ (here M is a type III factor) has an intermediate subfactor $A_{11}(M) \subset \rho_{11}(M) \subset M$ with $\rho_{11}\bar{\rho}_{11} = 1 + a_{(1,0)}$, and so the index of $\rho_{11}(M) \subset M$ is $\frac{5+\sqrt{21}}{2}$ which is about 4.79.[1] But $A(M) \subset M$ in Figure 10 has no intermediate subfactor. The similarities between Figures 8 and 10 may be misleading, but one still wonders if other techniques such as those of [11] may help one to perform "surgery" to remove the extra legs in Figure 8.

The non-commutativity of Haagerup's subfactor fusion rules also suggest to look for the system of nets of inclusions $\mathcal{A}^G \subset \mathcal{A}$ where $\mathcal{A}$ is a conformal net, where G is a non-abelian finite group acting properly on $\mathcal{A}$, and $\mathcal{A}^G$ is the fixed point subnet. One then looks for the subsectors of representations $\mathcal{A}^G$ induced to $\mathcal{A}$. This proves to be fruitless except that index computations lead us to the following:

Proposition 3.1. *Let $\mathcal{A}$ be a rational conformal net (cf. [13]). Then the set of untwisted representations of $\mathcal{A}^G$ (these are representations of $\mathcal{A}^G$ coming from the restrictions of representations of $\mathcal{A}$) is closed under fusion.*

[1]Not surprisingly, according to a referee, this so called 2221 subfactor was known to Haagerup, but constructed by different methods. See the Appendix in [2] for further discussions from the categorical point of view.

This proposition seems to be implicitly conjectured in [6] and perhaps elsewhere. Here is a proof. For unexplained notations, see [17].

Proof. We use λ, i to label irreducible representations of $\mathcal{A}^G$ and $\mathcal{A}$ on Hilbert spaces H_λ, H^i. Note that by [18], λ runs over a finite set. Assume that when restricting H^i to $\mathcal{A}^G$, H^i decomposes as $\oplus b_{i\lambda} H_\lambda$, where $b_{i\lambda}$ are non-negative integers.

Then by [16], $a_\lambda \succ \sum_i b_{i\lambda}\sigma_i$, so $d_\lambda \geq \sum_i b_{i\lambda} d_i$ where d_λ, d_i are statistical dimensions. Note that if $b_{1\lambda} \geq 1$, then $a_\lambda \succ b_{1\lambda}1$ and $d_\lambda \geq b_{1\lambda}$. But

$$\sum_\lambda b_{1\lambda}^2 = |G| = \sum_\lambda b_{1\lambda} d_\lambda.$$

It follows that $a_\lambda = b_{1\lambda}1$ if $b_{1\lambda} \geq 1$.

In general, let us compute

$$P_i := \sum_\lambda b_{i\lambda} d_\lambda \geq \sum_\lambda b_{i\lambda} \sum_j b_{j\lambda} d_j = \sum_{\lambda, j} b_{i\lambda} b_{j\lambda} d_j.$$

We will show that the $\geq$ above is in fact an equality.

Note by [17],

$$b_{i\lambda} = \langle \sigma_i a_{\bar\lambda} 1, 1\rangle = \sum_{\mu, j, \alpha} \frac{S_{\bar\lambda\mu}}{S_{1\mu}} \frac{S_{ij}}{S_{1j}} |\psi_1^{(j,\mu;\alpha)}|^2.$$

So

$$\sum_j b_{j\lambda} d_j = \sum_{\mu, i, \alpha, j} \frac{S_{\bar\lambda\mu}}{S_{1\mu}} \frac{S_{ij}}{S_{1i}} |\psi_1^{(i,\mu;\alpha)}|^2 \frac{S_{j1}}{S_{11}} = \sum_{\mu,\alpha} \frac{S_{\bar\lambda\mu}}{S_{1\mu}} |\psi_1^{(1,\mu;\alpha)}|^2 \frac{1}{S_{11}^2}.$$

Hence

$$\begin{aligned}\sum_{\lambda,\mu,\alpha} b_{i\lambda} \frac{S_{\bar\lambda\mu}}{S_{1\mu}} |\psi_1^{(1,\mu;\alpha)}|^2 \frac{1}{S_{11}^2} &= \sum_{\delta, k, \beta\mu, \alpha} \frac{S_{\bar\lambda\delta}}{S_{1\delta}} \frac{S_{ik}}{S_{1k}} |\psi_1^{(k,\delta;\beta)}|^2 \frac{S_{\bar\lambda\mu}}{S_{1\mu}} |\psi_1^{(1,\mu;\alpha)}|^2 \frac{1}{S_{11}^2} \\ &= \sum_{k,\beta,\mu,\alpha} \frac{1}{S_{1\mu}^2} \frac{S_{ik}}{S_{1k}} |\psi_1^{(k,\bar\mu;\beta)}|^2 |\psi_1^{(1,\mu;\alpha)}|^2 \frac{1}{S_{11}^2}.\end{aligned}$$

Note that by [17] that $|\psi_1^{(1,\mu;\alpha)}|^2 \neq 0$ implies that $b_{1\mu} > 0$, and so $a_\mu = b_{1\mu}1$, and $|\psi_1^{(k,\bar\mu;\beta)}|^2 \neq 0$ implies $k = 1$. So the above sum is equal to d_i

multiplied by a number which is independent of i. Note that when $i = 1$, $P_1 = \sum_\lambda b_{1\lambda} d_\lambda = |G|$, which is also the same as

$$\sum_{\lambda,j} b_{1\lambda} b_{j\lambda} d_j.$$

It follows that the above sum is equal to $d_i|G|$. On the other hand we have

$$\sum_\lambda b_{i\lambda} d_\lambda = \sum_{\lambda,\mu,j,\alpha} \frac{S_{\bar{\lambda}\mu}}{S_{1\mu}} \frac{S_{ij}}{S_{1j}} |\psi_1^{(j,\mu;\alpha)}|^2 \frac{S_{\lambda 1}}{S_{11}} = \sum_{1,\alpha} \frac{1}{S_{11}^2} \frac{S_{i1}}{S_{11}} |\psi_1^{(1,1;\alpha)}|^2$$

which again is d_i times a number that is independent of i. Setting $i = 1$ we see that

$$\sum_\lambda b_{i\lambda} d_\lambda = d_i|G|.$$

It follows that

$$P_i = \sum_\lambda b_{i\lambda} d_\lambda = \sum_\lambda b_{i\lambda} \sum_j b_{j\lambda} d_j.$$

Hence if $b_{i\lambda} > 0$ for some i, i.e., λ is a non-twisted representation of $\mathcal{A}^G$, then $d_\lambda = \sum_j b_{j\lambda} d_j$, and so $a_\lambda = \sum_j b_{j\lambda} \sigma_j$.

Now let λ, μ be two non-twisted irreducible representations of $\mathcal{A}^G$, and let $\delta \prec \lambda\mu$ be an irreducible representation. Then $a_\delta \prec a_\lambda a_\mu \prec \sum_i m_i \sigma_i$ for some non-negative integers m_i. It follows that $\langle a_\delta, \sigma_i \rangle \geq 1$ for some i, i.e., δ is non-twisted. $\square$

4. Problem Three: Is finite depth equivalent to rationality of generating function?

This is a problem I heard from Vaughan during a talk in 2012. Since finite depth subfactors are closely related to unitary rational CFT, if the problem has a positive answer, it seems to give a more "rational" characterization of finite depth subfactors. I was hooked immediately by this question, but was not able to make much progress until a year later. By that time, like everyone else, I had a phone which took high quality pictures, so instead of sending Vaughan a regular mail, or delivering my hand written notes to him in person, I took pictures of my hand written notes on this problem and sent it as attachment to an email. The following is essentially a copy of my handwritten notes that are appended to the email message I sent to Vaughan on May 27, 2013. The only addition is a few sentences giving a more detailed explanation of a point at the end of proof of the theorem.

Theorem 4.1. *Let $N \subset M_0$ be a subfactor with finite index and $N \subset M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots$ be its Jones tower. Then $N \subset M_0$ is of finite depth if and only if the generating function $\sum_{n\geq 0} \dim(N' \cap M_n) z^n$ is a rational function of z.*

The proof of this theorem is based on the follow two Lemmas:

Lemma 4.2. *Let $A = A^*$ be $n \times n$ Hermitian matrix, and $x, y \in \mathbb{C}$. Then $\sum_{m\geq 0} \langle x, A^m y\rangle z^m$ is a rational function of z.*

Proof. By the spectrum theorem (cf. Chapter 7 of [15]) $\langle x, A^m y\rangle = \int_{\sigma(A)} \lambda^m d\mu_{x,y}$ where $\mu_{x,y}$ is a complex measure on the spectrum of A which is $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$. Hence

$$\sum_{n\geq 0} \langle x, A^m y\rangle z^m = \sum_{1\leq i\leq n} \sum_{m\geq 0} z^m \lambda_i^m \mu_{x,y}(\lambda_i) = \sum_{1\leq i\leq n} \mu_{x,y}(\lambda_i) \frac{1}{1-\lambda_i z}. \qquad \square$$

Lemma 4.3. *Let A be a bounded self-adjoint operator on a Hilbert space H with $e \in H$ a unit vector. Denote by $\sigma(A)$ and $\rho(A)$ the spectrum and the resolvent set of A. If $\langle e, (z-A)^{-1} e\rangle$ is a rational function on $\rho(A)$, then the dimension of the subspace spanned by all vectors $A^m e, \forall m \geq 0$ is finite.*

Proof. By the spectrum theorem (cf. Chapter 7 of [15])

$$|\langle e, (z-A)^{-1} e\rangle| = \left| \int_{\sigma(A)} \frac{1}{z-\lambda} d\mu_e(\lambda) \right| \leq \int_{\sigma(A)} \frac{1}{|z-\lambda|} d\mu_e(\lambda) \leq \frac{1}{|\operatorname{Im} z|}$$

where $\mu_e(\Omega) = \langle e, P_A(\Omega) e\rangle$ is the Borel probability measure associated with the vector e. So if $\langle e, (z-A)^{-1} e\rangle$ is a rational function, and λ_i is a pole, then λ_i must be a simple pole: since when $z \in \rho(A)$ but z is close to λ_i, we have $\langle e, (z-A)^{-1} e\rangle$ grows like positive multiple of $\frac{1}{(z-\lambda)^{m_i}}$ where m_i is the order of the pole at λ_i. But $|\langle e, (z-A)^{-1} e\rangle| \leq \frac{1}{|\operatorname{Im} z|}$. It follows that $\lambda_i \in \sigma(A) \subset \mathbb{R}$ and $m_i = 1$. Therefore we must have $\langle e, (z-A)^{-1} e\rangle = \sum_{1\leq i\leq n} \frac{\alpha_i}{z-\lambda_i}$ with $\alpha_i \neq 0, \lambda_i \in \sigma(A)$. Let $d\mu$ be the point measure supported on λ_i, $1 \leq i \leq n$ with mass α_i. Then we have

$$\langle e, (z-A)^{-1} e\rangle = \int_{\sigma(A)} \frac{1}{z-\lambda} d\mu_e(\lambda) = \int \frac{1}{z-\lambda} d\mu.$$

It follows that $d\mu_e = d\mu$, and

$$\langle e, f(A) e\rangle = \int f(\lambda) d\mu$$

for all continuous functions f, so the map $f(A)e \to f(\lambda) \in L^2(\mathbb{C}, d\mu) \simeq \mathbb{C}^n$ is unitary, and the Lemma is proved. $\square$

Now we are ready to prove the theorem. Let Γ denote the principal graph, and H is the square integrable complex valued functions defined on the vertices of Γ, and e_1 the unit vector corresponding to the distinguished vertex of Γ. By abuse of notations we will also use Γ to denote the adjacency matrix of the principal graph. Then we have

$$\dim(N' \cap M_{2n}) = \langle \Gamma e_1, (\Gamma\Gamma^t)^{2n} \Gamma e_1 \rangle, \quad \dim(N' \cap M_{2n+1}) = \langle e_1, (\Gamma\Gamma^t)^{2n+2} e_1 \rangle.$$

If Γ is finite, then by Lemma 4.2 $\sum_{n\geq 0} \dim(N' \cap M_n) z^n$ is a rational function. Note that $\|\Gamma\Gamma^t\| \leq [M_0 : N] < \infty$. Suppose now that $f(z) := \sum_{n\geq 0} \dim(N' \cap M_n) z^n$ is a rational function. Then $f(z) + f(-z) = 2\sum_{n\geq 0} \dim(N' \cap M_{2n}) z^{2n}$ is also a rational, even, and so $\sum_{n\geq 0} \dim(N' \cap M_{2n}) z^n$ is a rational function. Denote by $A = (\Gamma\Gamma^t)^2$, $e = \Gamma e_1$. Then $\sum_{n\geq 0} \dim(N' \cap M_{2n}) z^n = \sum_{n\geq 0} \langle e, A^n e \rangle z^n = \langle e, (1 - zA)^{-1} e \rangle$ if $|z| < \frac{1}{\|A\|}$. Let $w = 1/z$. It follows that

$$\langle e, (1 - zA)^{-1} e \rangle = w \langle e, (w - A)^{-1} e \rangle$$

is a rational function of w for $|w| > \|A\|$. Apply Lemma 4.3 to conclude that the span of $A^m \Gamma e_1$, $\forall m \geq 0$ is finite dimensional. So there are only finitely many odd vertices, and $N \subset M_0$ is of finite depth. Here are more details on why there are only finitely many odd vertices. Note that Γ is a bipartite locally finite graph. For any odd vertex v of Γ, $\langle v, A^m \Gamma e_1 \rangle$ simply counts the number of paths of length $2m + 1$ from e_1 to v. If there are infinitely odd vertices, then one can choose infinite sequences of odd vertices v_i, $i \geq 1$ which are further away from e_1, and infinite sequence of strictly increasing integers m_i such that v_i can be reached from e_1 by a path of length $2m_i + 1$, but not by any path that has length smaller than $2m_i + 1$. It follows that $\langle v_j, A^{m_i} \Gamma e_1 \rangle = 0$ if $i < j$ and $\langle v_i, A^{m_i} \Gamma e_1 \rangle > 0$. If $\sum_{1\leq i\leq k} c_i A^{m_i} \Gamma e_1 = 0$ for some complex numbers c_i, then taking inner product with v_k we have $c_k = 0$, and taking successive inner products with $v_{k-1}, \ldots, v_1$, we conclude that $A^{m_i} \Gamma e_1$ are linearly independent, which is a contradiction.

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