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Vaughan Jones – Friend, Mentor and Collaborator

My first memory of meeting Vaughan dates back to the spring of 1990 at IHES in Bures-Sur-Yvette, France. At the time, I was a graduate student of Sorin Popa. Sorin spent a sabbatical quarter at IHES, and I had the great fortune of being able to come along. It was an amazing experience, but I remember being quite intimidated by Vaughan. In hindsight, there was of course absolutely no reason to feel that way, as Vaughan has always been one of the most accessible, supportive and generous mathematicians that I have met in my entire career.

It was not until 1991 that I really got to know and work with Vaughan. After completing my Ph.D. under Sorin's supervision at UCLA in 1991, I was hired as a postdoctoral fellow at MSRI (now SLMath) and as Morrey Assistant Professor at the University of California in Berkeley. Vaughan served as my postdoctoral mentor at UC Berkeley. I ended up spending five years in Berkeley between 1991 and 1998, including two years on a Heisenberg Fellowship. I started as tenure track Assistant Professor at UC Santa Barbara in 1995 and had the opportunity to spend more time in Berkeley in fall 2000 and fall 2001 as MSRI Research Professor resp. Visiting Miller Professor. Berkeley was at that time one of the most exciting universities for mathematics in the world, and it was an absolute powerhouse in operator algebras with Arveson, Jones, Rieffel and Voiculescu on the faculty. There were many operator algebras students, four different weekly seminars in operator algebras and a functional analysis colloquium! It was amazing.

Vaughan and I would meet every day for lunch and coffee and discuss mathematics. Sometimes we sat at the Brewed Awakening for hours, scribbling ideas on napkins, and many of these napkins turned, after some hard work, into theorems and publications. I clearly remember the day when we had a capuccino at the Brewed and Vaughan asked me if I knew of an irreducible hyperfinite subfactor whose index was not an algebraic integer. Of course, I did not. I later answered Vaughan's question by explicitly constructing an appropriate infinite commuting square based on an infinite graph that is a 4-star with an A_∞ tail. I obtained in this way an irreducible, hyperfinite subfactor with Jones index 4.5. I had just learned how to do commuting square calculations from Uffe Haagerup, with whom I also collaborated a lot during that time. The work of Haagerup and Schou from the early 1990s was the key that allowed me to solve Vaughan's problem. Until a few years ago, my example was the only known example of an irreducible hyperfinite subfactor whose index was not an algebraic integer. In 2021, Hrvoje Stojanovic, one of my former Ph.D. students at Vanderbilt, constructed more examples in his dissertation and also resolved a mystery that revolved around my index 4.5 example. This is just one instance that illustrates how amazing Vaughan's instincts and inspiration for good mathematics were, and how fruitful mathematical discussions with him always were.

Sorin developed in the 1990s his theory of amenability for subfactors and proved the striking theorem that amenable subfactors of the hyperfinite II_1 factor are classified by their standard invariant. Since the hyperfinite algebras obtained from forming the union of the higher relative commutants associated to a subfactor (the so-called *core* of the subfactor) are in general not factors, the core could only be a model of the subfactor if the algebras were indeed factors. Sorin called a hyperfinite subfactor *strongly amenable* if it was amenable *and* the core algebras were factors as well. He asked if a subfactor that is amenable and irreducible is automatically strongly amenable. Vaughan suggested to Uffe and me that we should investigate certain group-type subfactors to try to answer this question. They arise via outer actions of two finite groups on the hyperfinite II_1 factor. In this set-up, one naturally obtains a subfactor by simply considering the fixed point factor under the action of one group, sitting in the crossed product factor by the action of the other group. By construction, these subactors contain an intermediate subfactor. Uffe and I established that our subfactors could be analyzed by studying the group generated by the two finite groups in the outer automorphism group of the factor on which they act, and certain double cosets associated to them. Our analysis allowed us to give a negative answer to Sorin's question. We constructed many concrete group-type subfactors, which Vaughan started calling *Bisch-Haagerup subfactors*, that were irreducible, amenable, but not strongly amenable.

In the 1990s Vaughan worked out his theory of planar algebras, an algebraic-topological way of describing the standard invariant of a subfactor. It was motivated by the (still unsolved) question what the principal graphs, or more generally the standard invariants, of so-called spin model subfactors are. (My former student Michael Montgomery made some progress on this question.) Uffe and I had discovered in our joint work that one could think of group-type subfactors as a kind of *composition of subfactors*, and we observed that there were two extreme situations – tensor product and free product. We were wondering if these two ways of composing subfactors always existed. This led to my work with Vaughan on *intermediate subfactors*.

As a postdoc at MSRI, I had worked on intermediate subfactors and proved an abstract characterization of the Jones projection onto an intermediate subfactor as a *biprojection*. A biprojection is a projection in the first higher relative commutant of a subfactor whose Fourier transform (rotation by 90 degrees in planar algebra language) is a multiple of a projection. Ocneanu had noticed that for subfactors that arise as the crossed product by a finite group, rotations are precisely the classical Fourier transform of a group and that these biprojections should be Jones projections onto intermediate subfactors. My work proves this assertion.

Since Vaughan and I wanted to figure out the basic structure of the higher relative commutants of a subfactor when an intermediate subfactor is present, it was natural to study the algebra that is generated by the Temperley-Lieb-Jones subalgebras of the higher relative commutants (i.e., the algebras generated by the Jones projections e_i in the Jones tower) and the Jones projections onto the intermediate subfactors in the tower. This is how we discovered what we called the *Fuss-Catalan algebras*, a certain 2-parameter generalization of the Temperley-Lieb-Jones algebras. We completely analyzed these algebras and found that their structure was captured by the *Fibonacci graph* (or certain subtrees of it). Our work resulted in a major paper that was published in Inventiones Math. By a result of Sorin, it followed that there are subfactors (obtained via a certain amalgamated free product construction) whose standard invariants are precisely the Fuss-Catalan algebras of Jones and myself. In particular, we obtained uncountably many infinite depth subfactors in this way.

The Fuss-Catalan algebras (or *Bisch-Jones algebras* as some of my colleagues have started to call them) have played a significant role in the theory of subfactors and their planar algebras. A good number of Ph.D. theses were based on this work with Vaughan. Moreover, Fuss-Catalan algebras were shown to give rise to new solvable lattice models and solutions of the Yang-Baxter equations with spectral parameter (work of Di Francesco).

Vaughan and I observed that one could think of the Fuss-Catalan algebras as arising through a *free composition* of two Temperley-Lieb-Jones planar algebras, possibly with two different parameters. We then went on to develop a theory of *free product of planar algebras*. In another direction, we realized that our algebras are *singly generated planar algebras*, with precisely one generator in addition to Temperley-Lieb-Jones. Imposing certain skein relations, cast as a condition on the dimensions of the first and second higher relative commutants (namely, the first has dimension 3, and the second has dim ≤ 15), implied that this additional generator has to satisfy a *generalized Yang-Baxter equation*. Vaughan and I started to classify singly generated planar algebras with dimension ≤ 14, and showed that one finds only Fuss-Catalan planar algebras, a BMW planar algebra and some sporadic subfactors in small dimension. Many years later, when Vaughan had already moved to Vanderbilt, we came back to this project and completed it up to dimension 15 in joint work with Zhengwei Liu. All the BMW planar algebras now appeared as well, and we could give a complete list. Zhengwei was Vaughan's graduate student at the time and was working on various problems around subfactors and planar algebras. In particular, he tried to resolve a question of Haagerup and myself regarding the possible non-free compositions of an A_3 and an A_4 subfactor. Uffe and I had constructed a non-free composition besides the tensor product, and Izumi found another. We believed that there should be countably many non-free composition that "converge" to the free composition, but Vaughan was always skeptical that this was the case. Zhengwei completely solved the problem using planar algebra techniques. He showed that only a few non-free compositions exist. This striking result showed that Vaughan was once again right! Zhengwei went on to work on other singly generated planar algebras with generators satisfying Yang-Baxter type equations. He discovered a new 1-parameter family, which was fascinating.

I moved to Vanderbilt University in 2002 and was talked into serving as department chair in 2005. It was a very exciting time at Vanderbilt, as the university was heavily investing in mathematics and the sciences. I had conversations with Vaughan about moving to Vanderbilt in 2006, and we finally managed to attract him in 2011. His wife Wendy Jones joined Vanderbilt one year later. Vaughan's appointment was a game changer for the Vanderbilt mathematics department, and the research group in noncommutative geometry and operator algebras (NCGOA) consisting of Hughes, Jones, Kasparov, Peterson, Yu, Zheng and myself was one of the best in this area of mathematics in the world. Many Ph.D. students, visitors and postdoctoral scholars were affiliated with NCGOA, and Alain Connes spent every year several weeks at Vanderbilt as well. These were stimulating and inspiring times for operator algebras at Vanderbilt!

Over the years, Vaughan and I became good friends, and naturally we discussed mathematics frequently. It was wonderful to have him as a colleague at Vanderbilt. We ran our research seminar, the Subfactor Seminar, at Vanderbilt the same way Vaughan ran it in Berkeley in the 1990s with beer & pizza Friday night following each talk in the seminar.

Vaughan invited me to New Zealand on several occasions. My first visit was to Tolaga Bay in 1996 to attend the annual summer conference organized by what is now the New Zealand Mathematics Research Institute (NZMRI) that Vaughan co-founded. He felt strongly about bringing top level mathematics to New Zealand and helping his colleagues there to build international connections, collaborations and exchanges. It has been a highly successful endeavor. My last visit to New Zealand was in late January/early February 2020. Vaughan and I spent a week in a small beach town in the northern part of the North Island to work on problems related to commuting squares and planar algebras, and to think about quantum computing. Vaughan was very skeptical of topological quantum computing and felt that there was not enough evidence that a topological quantum computer could actually be built. I tried to convince him that quantum computers will be in our future, but I rather doubt I succeeded. During this research week in NZ, Vaughan and I drove up to Cape Reinga at New Zealand's northern tip. It is a magical place where the Tasman Sea and the Pacific Ocean meet, a place of great inspiration. After this trip to the Southern hemisphere, I met Vaughan again at MSRI in Berkeley, where I spent two weeks in early March until COVID shut down a good part of the world. This was the last time I was able to discuss mathematics (and other things) with Vaughan in person.

I now regret that I spent eleven years as department chair at Vanderbilt (2005–2016), as the job took away precious time that I would have rather liked to spend doing mathematics with Vaughan. Of course, his untimely death on September 6th, 2020, was utterly unexpected. To this day, I cannot believe he is gone. A monumental loss to mathematics and to me personally.

I think Vaughan would enjoy learning about the results his former students, colleagues and collaborators have proved in the three years since September 2020. This volume samples some of these new results. A good number of superb young mathematicians have emerged in operator algebras, and new Ph.D. students have entered the field. At Vanderbilt, two of my current Ph.D. students, Julio Cáceres and Junhwi Lim, with whom Vaughan would be pleased to work, show great promise. They have proved new cool results about graph planar algebra embeddings, quantum cellular automata using Vaughan's index for subfactors, or commuting square subfactors. Our young mathematicians are the future, and the future of the subject – Vaughan's subject – looks bright!

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