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To my PhD advisor Vaughan Jones

It was a memorable experience studying as a graduate student of Vaughan Jones on subfactor theory for four years at Vanderbilt University. My research taste and style has been deeply influenced by him ever since.

In the early 1980s, Jones remarkably classified the index for subfactors and soon discovered the Jones polynomial as quantum invariants of knots. It is phenomenal that the Jones index as a renormalized dimension could be non-integral and quantized below 4. Subfactors were naturally considered as quantum symmetries beyond group symmetries. Such quantum symmetries turned out to be prevalent in various areas in mathematics and physics and led to deep connections between subfactor theory and non-commutative analysis, lower dimensional topology, quantum groups, free probability, tensor categories, (algebraic, conformal, topological) quantum field theory, integrable models, topological orders, quantum information etc.

I first met Jones at a conference at Dartmouth in 2011 recommended by his former student Emily Peters. As my first impression, Jones was a high and mighty mathematician. After the conference, he brought me to a local cafe and explained the skein theory of planar algebras and how it could be a powerful method to classify and to construct subfactors. He asked me whether I would like to study with him for my Ph.D. That was a dream to me; I was highly interested in subfactor theory and planar algebras. Two months later, I solved his problem on the classification of rank-4 exchange-relation planar algebras, which I learned from that motivating discussion. Jones was very happy and invited me to present a talk at the annual meeting Subfactors in Maui 2011; this was my first academic talk. In August 2011, Jones moved to Vanderbilt University and I became his first student there. As a benefit, we had a lot time to discuss mathematics when he was not too busy at the beginning. Jones kept me polishing the methods used in the classification problem, which eventually led to the discovery of quantum Fourier analysis. Recently, my first graduate student Fan Lu and I found a systematical framework to classify exchange relation planar algebras, based on the methods of quantum Fourier analysis and newly discovered combinatoric structure behind skein theory. It is an inheritance of the spirit of Jones.

Planar Algebras I of Jones is one of my favorite papers. When I took his course on planar algebras, I was deeply impressed by his stimulating ideas and broad views which are much more extensive than the content in this paper. During my PhD, one major aim is to understand fully all the examples in this paper. In particular, my PhD thesis was classifying singly generated subfactor planar algebras with Yang-Bexter relations, generalizing the Yang-Baxter equation. It completes the first program of Jones on this topic and leads to the discovery of an unexpected one-parameter family. It exhibits the power of the skein theoretical construction of subfactors, that Jones expected.

While I was classifying annular Fuss-Catalan representations, Jones thought that the methods could be useful for the Bisch-Haagerup open problem proposed in 1994 on the classification of quotients of Fuss-Catalan planar algebras at index $3+\sqrt{5}$. Many experts had worked on this problem and believed the infinite sequence of quotients exist. I first reduced the construction of the whole sequence to solving a sequence of equations and designed an efficient algorithm to solve them. I was very confident to construct them at that time. However, some equations turned out to have no solutions. I thought I made some miscomputations. Jones had different opinions and suspected that they do not exist. I asked him why. He said that he had worked on it for twenty years, he would have constructed them if they exist. I started to use a second method to double check the computation by hand and by computer. Eventually, it was surprising that only the first four of the sequence exist and the Bisch-Haagerup problem was solved. Jones was right as always.

Recently, with several collaborators, I found an essential route to construct a 3-alterfold topological quantum field theory from a planar algebra. It is a fruitful picture language to unify various concepts and results in subfactors and related topics systematically. It provides fundamental building blocks for which many concepts can be mathematically defined in an intuitive manner and many remarkable results turned out to be apparent. It also provides powerful computational tools to prove the results in a conceptual and straightforward way.

More significantly, this essential route is independent of the dimension, so that we obtain an $n+1$-alterfold topological quantum field theory from a D^n algebra which can be fully extended down to 0 and up to infinite dimensional. It provides a mathematical framework to study higher dimensional topological quantum field theory and lattice gauge theory with quantum symmetries. It involves classical counter parts in lower dimensions and more advanced new theories in higher dimensions. It has complete skein theory, which provides efficient higher dimensional computational tools. Numerous concepts in lower dimensions blow up to more natural concepts in higher dimensions and relevant theories can be generalized, unified and proved using higher dimensional computational tools.

One month ago, I looked back at my nostalgic experience with Jones during the Tsinghua Sanya International Mathematical Forum, while preparing for the speech of its 10th anniversary. I met him in my dream and discussed with him about this new theory. He was very happy and encouraged me to develop it. We expect many areas emerging in this interdisciplinary direction in a few years in parallel to what had happened in the development of subfactors and related topics.

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