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A remark on calibrations and Lie groups

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Dedicated to Blaine Lawson on the occasion of his 80th birthday

Abstract: We use the notion of the principal three-dimensional subgroup of a simple Lie group to identify certain special subspaces of the Lie algebra and address the question of whether these are calibrated for invariant forms on the group.

Keywords: Calibration, Lie group, bi-invariant form, three-dimensional subgroup.

1. Introduction

The notion of a calibrated differential form φ, as introduced in [3], has become very important especially in the study of Calabi-Yau, G_2 and $Spin(7)$-manifolds, where φ is a covariant constant form. On the other hand, the manifolds which have most covariant constant forms, namely compact simple Lie groups G, have received less attention, although they are addressed in [12, 8, 9, 11].

Recall that the cohomology of a simple Lie group G of rank ℓ is an exterior algebra on ℓ generators with harmonic representatives φ_i of odd degree d_i which are covariant constant. The Cartan 3-form φ_1 is the generator of smallest degree and Tasaki [12] showed that this defines a calibration and moreover that a three-dimensional subgroup associated to the highest root is calibrated for this form and is volume-minimizing. He also showed that the Hodge dual $*\varphi_1$ calibrates the codimension 3 subspace of non-regular elements of G.

Amongst the three-dimensional subgroups there is a particularly distinguished one, the *principal* three-dimensional subgroup, and Kostant showed [6] that under the action of this group the Lie algebra decomposes $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_\ell$ into irreducible representations of $SO(3)$ whose dimensions are precisely the degrees d_i of the generators of the cohomology. The

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author conjectured in [5] that there is an exact fit here – that for each subspace V_i there exists a corresponding generator which restricts nontrivially. To the author's knowledge this has not yet been confirmed, though there is some information in [1]. In any case, if the restriction is non-zero it opens up the possibility of more complex calibrated submanifolds.

In this paper we observe first that the function defined by φ_i on the Grassmannian of oriented subspaces of $\mathfrak{g}$ of dimension d_i has a critical point on V_i. If this critical value is nonzero then any submanifold of dimension d_i tangential to a conjugate of V_i will be minimal [11]. If the non-zero value is the maximum then φ_i defines a calibration and any such submanifold is volume minimizing.

We then search for non-zero values by using the transitive action of groups on odd-dimensional spheres S^{2m+1}, and an argument initiated by X.Liu [8]. This consists of pulling back the volume form on the sphere and averaging over the group to produce an invariant form on G of degree $2m+1$. We use the well-known list of groups with transitive actions to show that in each case the pull-back of the volume form restricted to a corresponding V_i is non-negative and hence its average is non-zero, providing some evidence for the conjecture. The relevant degrees are $2n-1$ for $SO(2n)$ and $SU(n)$, $4n-1$ for $Sp(n)$, 7 for $Spin(7)$ and 15 for $Spin(9)$.

Finally we mention the entirely different context [5] in which the conjecture arose, involving the moduli space of stable bundles on a curve C.

2. Invariant forms

Let G be a compact simple Lie group. The covariant constant forms on G are the bi-invariant forms and these are defined as multilinear alternating forms α on $\mathfrak{g}$ by

$$\alpha(a_1,\ldots,a_{2m+1}) = p(a_1,[a_2,a_3],\ldots[a_{2m},a_{2m+1}])$$

where p is an adjoint-invariant polynomial of degree $m+1$. These polynomials correspond under the Chern-Weil homomorphism to characteristic classes like Chern or Pontryagin classes and we shall often label the invariant forms this way – as classes of degree $2m+2$ in the cohomology $H^*(B_G)$ of the classifying space. The Killing form is a quadratic polynomial and yields the Cartan 3-form.

The irreducible representations of the three-dimensional group $SU(2)$ are symmetric powers $\mathbf{S}^n$ of the standard complex 2-dimensional representation $\mathbf{S}$. The space $\mathbf{S}^n$ may be thought of as the action on homogeneous polynomials

$p(z_1, z_2)$ of degree n, or more conveniently the polynomial $p(z) = p(z_1/z_2, 1)$ and is therefore of dimension $n + 1$. Since $-1 \in SU(2)$ acts trivially if n is even, these are the irreducibles for $SO(3)$ and are real. When n is odd they are quaternionic representations of $SU(2)$.

The Clebsch-Gordon formula tells us how to decompose a tensor product: if $m \geq n$ then

$$\mathbf{S}^m \otimes \mathbf{S}^n = \mathbf{S}^{m+n} \oplus \mathbf{S}^{m+n-2} \oplus \cdots \oplus \mathbf{S}^{m-n}.$$

The decomposition involves contraction with the skew form on $\mathbf{S}$ and it follows then that $\mathbf{S}^n \otimes \mathbf{S}^n = \mathbf{S}^{2n} \oplus \mathbf{S}^{2n-2} \oplus \cdots$ and the skew part $\Lambda^2 \mathbf{S}^n = \mathbf{S}^{2n-2} \oplus \mathbf{S}^{2n-6} \oplus \cdots$.

The generators of the cohomology $H^*(G)$ have degrees $d_i = 2\lambda_i + 1$ where λ_i are the exponents of the Lie algebra. For completeness we list them:

A_ℓ: $1, 2, 3, \ldots, \ell$, B_ℓ: $1, 3, 5, \ldots, 2\ell - 1$, C_ℓ: $1, 3, 5, \ldots, 2\ell - 1$.

D_ℓ (ℓ odd): $1, 3, 5, \ldots, 2\ell - 3$, F_4: $1, 5, 7, 11$, G_2: $1, 5$.

E_6: $1, 4, 5, 7, 8, 11$, E_7: $1, 5, 7, 9, 11, 13, 17$, E_8: $1, 7, 11, 13, 17, 19, 23, 29$.

In this list for each group the exponents are distinct, but for D_ℓ where ℓ is even the exponent $\ell - 1$ occurs twice. In terms of $SO(4n)$ characteristic classes the two invariants can be taken to be the Euler class and a Pontryagin class of the same degree. The generators are not unique, just as we can take a basis of invariant polynomials for $SU(n)$ as $\operatorname{tr} a^k$ $(k = 2, \ldots, n)$ or the coefficients of $\det(\lambda - a)$.

Kostant's theorem [6] tells us that under the action of the principal three-dimensional subgroup, which is unique up to conjugation, $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_\ell$ where $V_i \cong \mathbf{S}^{2\lambda_i}$. Clearly $\lambda_1 = 1$ gives the Lie algebra of the subgroup.

As an example, the irreducible representation $\mathbf{S}^n$ defines a homomorphism $SU(2) \to SU(n+1)$ whose image is the principal three-dimensional subgroup and the Lie algebra $\mathfrak{su}(n+1)$ is isomorphic to the trace zero elements in $\operatorname{Hom}(\mathbf{S}^n, \mathbf{S}^n) \cong \mathbf{S}^n \otimes \mathbf{S}^n$. The Clebsch-Gordon formula gives $\mathbf{S}^2 \oplus \cdots \oplus \mathbf{S}^{2n}$ as the decomposition $V_1 \oplus V_2 \oplus \cdots \oplus V_\ell$.

3. Critical points

Given an invariant form φ_i of degree d_i we can evaluate it on an oriented d_i-dimensional subspace of $\mathfrak{g}$ to obtain a function f_i on the oriented Grassmannian $\widetilde{Gr}(d_i, \mathfrak{g})$ of such subspaces.

Theorem 3.1. *The function f_i has a critical point at $[V_i]$.*

Proof. Using the metric on the Grassmannian, the gradient of f_i at $[V_i]$ is a tangent vector which, by virtue of the adjoint invariance of φ_i, is invariant under the action of $SU(2)$ which stabilizes $[V_i]$. The tangent space of the Grassmannian at $[V_i]$ is isomorphic to $\mathrm{Hom}(V_i, \mathfrak{g}/V_i)$, but as we have seen, except for the case D_ℓ where ℓ is even, the exponents are distinct and so the irreducible V_i does not occur in the decomposition of $\mathfrak{g}/V_i$. By $SU(2)$-invariance, the homomorphism is zero and so the gradient is zero. It therefore remains to consider the case of $SO(4n)$.

The principal three-dimensional subgroup in $SO(4n)$ acts reducibly on $\mathbf{R}^{4n}$. It is the representation $1 \oplus \mathbf{S}^{4n-2}$ and so $\mathfrak{g} \cong \Lambda^2(1 \oplus \mathbf{S}^{4n-2}) = \mathbf{S}^{4n-2} \oplus \Lambda^2(\mathbf{S}^{4n-2})$. Denote by V the first subspace here. Using the Clebsch-Gordon decomposition we have $\Lambda^2(\mathbf{S}^{4n-2}) = \mathbf{S}^{8n-6} \oplus \mathbf{S}^{8n-10} \oplus \cdots \oplus \mathbf{S}^2$ which contains a copy of $\mathbf{S}^{4n-2}$ which we call V'.

If $e_0, e_1, \dots$ is an orthonormal basis of $1 \oplus \mathbf{S}^{4n-2}$ with e_0 spanning the trivial component then $(e_0, e_1, \dots) \mapsto (-e_0, e_1, \dots)$ is an orientation-reversing involution σ commuting with $SO(3)$ and acting as -1 on V and $+1$ on V'. The invariant polynomial on $\mathfrak{so}(4n)$ defined by the Pfaffian $\sqrt{\det a}$ changes sign under change of orientation so it defines an invariant form φ such that $\sigma^*\varphi = -\varphi$, hence φ evaluated on V' is zero since $\sigma = 1$ there. We therefore associate V to φ and V' to φ', defined by the Pontryagin class, and consider the corresponding functions f, f'. Pontryagin classes are of course orientation-independent. The function f' is σ-invariant and so its gradient at $[V']$ is an invariant element of $\mathrm{Hom}(V', V)$, but the action here is -1, so the gradient vanishes and this is a critical point. The case of f is similar, taking into account the fact that σ changes orientation on V. $\square$

4. Groups acting on spheres

4.1. The invariant forms

We focus now on a family of covariant constant forms which arise geometrically. If a simple group G acts transitively on an odd-dimensional sphere then we have the projection $p : G \to S^{2m+1} = G/H$ and averaging over G the pull-back $p^*\omega$ of the volume form on S^{2m+1} gives an invariant $(2m+1)$-form. Since $p^*\omega$ is H-invariant this is equivalent to averaging over the sphere as in [8]. We know in advance that this form is non-zero for, by [7] (see also [10]), the stabilizer H is not homologous to zero and so the cohomology class $[p^*\omega] \neq 0$.

The groups acting transitively on spheres are well-known, especially from their appearance as special holonomy groups. For a simple group G and an odd-dimensional sphere we have:

$$SO(2n), \quad SU(n), \quad Sp(n), \quad Spin(7) \subset SO(8), \quad Spin(9) \subset SO(16).$$

A universal multiple of the invariant form which the averaging produces can be labelled by a characteristic class which restricts to zero in the cohomology $H^*(B_H)$ of the classifying space of the stabilizer H of the action. The group H stabilizes a vector in an even-dimensional space so this is the Euler class for $SO(2n)$, the Chern class c_n for $SU(n)$, the Chern class c_{2n} for $Sp(n) \subset SU(2n)$. The last two examples in the list are stabilizers of a vector in the spin representation and expressing the Euler class for the spin representation in terms of the basic weights gives multiples of $p_1^2 - 4p_2$ for $Spin(7)$ and $p_1^4 - 8p_1^2 p_2 + 16p_2^2 - 64p_4$ in the case of $Spin(9)$ (see also [2]).

We want to prove that the invariant form is non-zero on the component $\mathbf{S}^{2m} \subset \mathfrak{g}$, the tangent space at the identity. As in [8], the translate of $p^*\omega$ from a general point g with $p(g) = v \in S^{2m+1} \subset \mathbf{R}^{2m+2}$ to the identity gives a form on the Lie algebra which, evaluated on $(a_1, \dots, a_{2m+1})$, $a_i \in \mathfrak{g}$, is $\det(v, a_1 v, a_2 v, \dots, a_{2m+1} v)$. If $(a_1, \dots, a_{2m+1})$ forms a basis for $\mathbf{S}^{2m}$ and this is nonnegative and not identically zero for all v in the sphere, then the average will be positive and the invariant form will be nonzero. We proceed to consider the different cases.

4.2. The case $SO(2n)$

As noted above, the principal 3-dimensional subgroup in this case arises from a reducible representation $1 \oplus \mathbf{S}^{2n-2}$ and the subspace $V_i \subset \mathfrak{so}(2n)$ of dimension $2n-1$ is spanned by $a_i = e_0 \otimes e_i - e_i \otimes e_0$ for $1 \leq i \leq 2n-1$. Then $a_i(v) = v_i e_0 - v_0 e_i$ and, since $\|v\|^2 = 1$,

$$v \wedge a_1 v \wedge \cdots \wedge a_{2n-1} v = v_0^{2n-2} e_0 \wedge e_1 \wedge \cdots \wedge e_{2n-1}.$$

This is non-negative hence the average is non-zero.

This formula is Example 3.7 in [8], where Lemma 3.5 in that paper shows that in $\mathfrak{so}(2n)$ for general a_i

$$\det(v, a_1 v, a_2 v, \dots, a_{2n-1} v) = \|v\|^2 Q_{2n-2}(v) \tag{1}$$

where $Q_{2n-2}(v)$ is homogeneous in v of degree $2n-2$. In our situation where $a_1, \dots, a_{2n-1}$ span one of the spaces V_i, this will be an invariant of the $SU(2)$ action on $\mathbf{R}^{2n}$ and the focus of our attention in the other cases.

4.3. The case $SU(n)$

Here the principal three-dimensional subgroup is the action of $SU(2)$ in its irreducible representation $\mathbf{S}^{n-1}$, and so its image in $SU(n)$ is a copy of $SU(2)$ for n even and $SO(3)$ for n odd. The $2n-1$-dimensional subspace V_i is $\mathbf{S}^{2n-2}$ and so we have an inclusion

$$\mathbf{S}^{2n-2} \subset \mathrm{Hom}(\mathbf{S}^{n-1}, \mathbf{S}^{n-1}) \cong \mathbf{S}^{n-1} \otimes \mathbf{S}^{n-1}$$

and we can recognize this from the Clebsch-Gordon formula.

In terms of polynomials $p(z)$ it is the adjoint of the multiplication map, but a more convenient description is to identify $\mathbf{S}^m$ with $H^0(\mathrm{P}^1, \mathcal{O}(m))$, holomorphic sections of the line bundle of degree m on the projective line. Since each $\mathbf{S}^m$ has either a nondegenerate skew or symmetric form we also have an invariant identification $\mathbf{S}^m \cong H^1(\mathrm{P}^1, \mathcal{O}(-m-2))$ by Serre duality. Then we have a natural tensor product map

$$H^1(\mathrm{P}^1, \mathcal{O}(-2n)) \otimes H^0(\mathrm{P}^1, \mathcal{O}(n-1)) \to H^1(\mathrm{P}^1, \mathcal{O}(-n-1)) \cong H^0(\mathrm{P}^1, \mathcal{O}(n-1))$$

which realizes the map $\mathbf{S}^{2n-2} \otimes \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$. This is the action of $V_i \subset \mathfrak{su}(n)$ on $\mathbf{C}^n$.

Consider first the case where $n = 2m+1$ is odd, then $\mathbf{S}^{n-1} = \mathbf{S}^{2m}$ is even and has a real structure and so we can write a complex vector $v = v_1 + iv_2$ where v_1, v_2 are real. Of course $SU(n)$ does not preserve the real structure, only the three-dimensional subgroup does. Now $\mathbf{S}^{4m} \subset \mathbf{S}^{2m} \otimes \mathbf{S}^{2m}$ is symmetric and real and elements of $V_i \subset \mathfrak{su}(2m+1)$ are of the form iA for a real symmetric matrix A.

As in equation (1), we are concerned with the expression $v \wedge a_1 v \wedge \cdots \wedge a_{2n-1} v$ considering $\mathbf{C}^n$ as a real vector space where the a_j lie in V_i. This vanishes when some linear combination of the a_i has v as a real eigenvector. But the a_i are skew adjoint so it can only be the zero eigenvalue. Now each $a \in V_i$ is of the form iA for A real, and so $iA(v_1 + iv_2) = -Av_2 + iAv_1$ and if this vanishes then $Av_1 = 0 = Av_2$.

Represent A as an element $[\alpha]$ of $H^1(\mathrm{P}^1, \mathcal{O}(-2n))$ and v_1 as a section s of $\mathcal{O}(n-1)$ then $Av_1 = 0$ has an interpretation in algebraic geometry: consider the exact sequence of sheaves

$$0 \to \mathcal{O}(-2n) \stackrel{s}{\to} \mathcal{O}(-n-1) \to \mathcal{O}_D(-n-1) \to 0$$

where D is the divisor of zeros of s. Then the long exact cohomology sequence gives

$$0 \to H^0(D, \mathcal{O}_D(-n-1)) \stackrel{\delta}{\to} H^1(\mathrm{P}^1, \mathcal{O}(-2n)) \stackrel{s}{\to} H^1(\mathrm{P}^1, \mathcal{O}(-n-1)) \to 0$$

so that $[\alpha]s = 0$ if and only if $[\alpha] = \delta t$ for a section t of $\mathcal{O}(-n-1)$ on the zero-dimensional cycle D.

Let s_1 and s_2 be two sections representing v_1, v_2 which have a common zero x then the cycles D_1, D_2 intersect and taking t as a section of $\mathcal{O}(-n-1)$ on x defines $\delta(t) = [\alpha]$ which annihilates both s_1 and s_2. Hence $[\alpha]$ represents a linear combination of a_j such that $v \wedge a_1 v \wedge \cdots \wedge a_{2n-1} v$ vanishes when $v = v_1 + iv_2$ and v_1, v_2 are represented by s_1, s_2 which have a common zero. These are polynomials $p_1(z), p_2(z)$ of degree $n-1$ and the condition for a common zero is the vanishing of the resultant

$$R(p_1, p_2) = a_0^{n-1} b_0^{n-1} \prod_{i,j} (\lambda_i - \mu_j) = a_0^{n-1} \prod_i p_2(\lambda_i)$$

where λ_i, μ_j are the roots of $p_1(z) = a_0 z^{n-1} + \cdots + a_{n-1}$, $p_2(z) = b_0 z^{n-1} + \cdots + b_{n-1}$. This is a polynomial in $v = v_1 + iv_2$ homogeneous of degree $2n-2$. Its vanishing implies Q_{2n-2} from equation (1) vanishes, but these two invariant polynomials have the same degree and the resultant is irreducible hence they are multiples of each other.

The real structure on $\mathbf{S}^{n-1}$ is inherited from the quaternionic structure of $\mathbf{S}$ so a real polynomial of degree $2m$ satisfies $p(-1/\bar{z}) = \bar{z}^{-2m}\overline{p(z)}$ and there is a free involution $\lambda \mapsto -1/\bar{\lambda}$ on the roots of p. Let $\lambda_1, \ldots, \lambda_m, -1/\bar{\lambda}_1, \ldots, -1/\bar{\lambda}_m$ be the roots of p_1, then

$$R(p_1, p_2) = a_0^{2m} \prod_{i=1}^{m} p_2(\lambda_i) p_2(-1/\bar{\lambda}_i) = \left(a_0 \prod_{i=1}^{m} \bar{\lambda}_i^{-1} \right)^{2m} \prod_{i=1}^{m} |p_2(\lambda_i)|^2.$$

Reality implies $a_{2m} = \bar{a}_0$ so that the product of the roots is $\bar{a}_0/a_0$ and $a_0 \prod_1^m \bar{\lambda}_i^{-1}$ is real. Hence the resultant is non-negative and averaging gives a non-zero evaluation of the form.

When $n = 2m$ is even, $\mathbf{S}^{2m-1}$ has a complex symplectic structure and a quaternionic structure: an antilinear involution J with $J^2 = -1$. Then $\mathbf{S}^{4m-2} \subset \mathbf{S}^{2m-1} \otimes \mathbf{S}^{2m-1}$ is symmetric which places it in the Lie algebra of complex symplectic transformations. But it is also real and so commutes with J. In this case if a linear combination of the a_i annihilates v it annihilates Jv so we again have a 2-dimensional kernel and the criterion is the vanishing of the resultant of two polynomials – p and its transform p^* by J where $p^*(z) = z^{2m-1}\overline{p(-1/\bar{z})}$. Then the resultant $R(p, p^*)$ is

$$(a_0 \bar{a}_{2m-1})^{2m-1} \prod_{i,j} (\lambda_i + \bar{\lambda}_j^{-1})$$

$$= (a_0\bar{a}_{2m-1})^{2m-1}\prod_i(|\lambda_i|^2+1)\prod_{i<j}|\lambda_i\bar{\lambda}_j+1|^2\Big(\prod_j\bar{\lambda}_j^{-1}\Big)^{2m-1}$$

and since $\prod_j\bar{\lambda}_j = -\bar{a}_{2m-1}/\bar{a}_0$ this expression is non-positive. Again the average is non-zero.

4.4. The case $Sp(n)$

The group $Sp(n)\subset SU(2n)$ is the subgroup which commutes with a quaternionic structure J and we have just observed that the appropriate V_i does just that, so that it lies in the Lie algebra $\mathfrak{sp}(n)$. The result follows from the previous section.

4.5. The case $Spin(7)$

Here the principal three-dimensional subgroup of $Spin(7)$ projects to the principal one in $SO(7)$. This is the irreducible representation $\mathbf{S}^6$ and from the characters we deduce that the 8-dimensional spin representation is $1\oplus\mathbf{S}^6$. This means that the subgroup fixes a spinor and so lies in the stabilizer G_2.

The Lie algebra of G_2 decomposes as $\mathbf{S}^2\oplus\mathbf{S}^{10}$ and $\mathfrak{so}(7)=\mathbf{S}^2\oplus\mathbf{S}^6\oplus\mathbf{S}^{10}$ with respect to the same 3-dimensional group. It follows that $\mathbf{S}^6$ is the orthogonal complement of $\mathfrak{g}_2$. Translated around $Spin(7)$ this is the horizontal subspace for the fibration $p: Spin(7)\to S^7$. This is a Riemannian submersion so $p^*\omega$ is always non-zero on this subspace.

4.6. The case $Spin(9)$

The defining 9-dimensional representation is here $\mathbf{S}^8$ and, from the characters again, the 16-dimensional spin representation is $\mathbf{S}^{10}\oplus\mathbf{S}^4$. In the Lie algebra $\mathfrak{so}(9)\cong\Lambda^2\mathbf{S}^8$ the 15-dimensional component is $\mathbf{S}^{14}$ and we are concerned with its action on $\mathbf{S}^{10}\oplus\mathbf{S}^4$. Since $\Lambda^2(\mathbf{S}^{10}\oplus\mathbf{S}^4)\cong\Lambda^2(\mathbf{S}^{10})\oplus(\mathbf{S}^{10}\otimes\mathbf{S}^4)\oplus\Lambda^2\mathbf{S}^4$ there are copies of $\mathbf{S}^{14}$ in the first two summands and the action is a linear combination of the two.

We consider again when a linear combination of $a_1,\dots,a_{15}\in V_i$ has a non-trivial kernel. Suppose $(p,q)\in\mathbf{S}^{10}\oplus\mathbf{S}^4$ are polynomials in the kernel of $a\in\mathbf{S}^{14}$ then we may write this as $(Ap+Bq,-B^Tp)=0$ where $a=(A,B)\in\Lambda^2(\mathbf{S}^{10})\oplus(\mathbf{S}^{10}\otimes\mathbf{S}^4)$. Now $B^T:\mathbf{S}^4\to\mathbf{S}^{10}$ is given by the map

$$H^1(\mathrm{P}^1,\mathcal{O}(-16))\otimes H^0(\mathrm{P}^1,\mathcal{O}(4))\to H^1(\mathrm{P}^1,\mathcal{O}(-12))$$

as in Section 4.3 and B by the map

$$H^1(\mathrm{P}^1, \mathcal{O}(-16)) \otimes H^0(\mathrm{P}^1, \mathcal{O}(10)) \to H^1(\mathrm{P}^1, \mathcal{O}(-6))$$

for a class $[\beta] \in H^1(\mathrm{P}^1, \mathcal{O}(-16)) \cong \mathbf{S}^{14}$. If p, q have a common zero then there exists $[\beta]$ with $Bp = 0, B^T q = 0$ represented by a class supported at a single point in P^1, the common zero. If we take this point to be $z = 0$ then $[\beta]$ can be identified with the polynomial $z^{14} \in \mathbf{S}^{14}$.

Consider now $A : \mathbf{S}^{10} \to \mathbf{S}^{10}$ defined by z^{14}. This consists of contracting in $\mathbf{S}^{14} \otimes \mathbf{S}^{10}$ seven pairs of terms and symmetrizing. If p vanishes at 0, contraction with z^{14} vanishes also. We deduce that the vanishing of the resultant $R(p, q)$ is a condition for the existence of $a \in V_i$ which annihilates (p, q). This is a polynomial in the coefficients of degree $4 + 10 = 14$. But $Q_{2n-2}(v) = Q_{14}(v)$ in (1) is of degree 14 and so Q_{14} is a multiple of the resultant of two real polynomials u, v of even degrees $4, 10$. As in Section 4.3, this is non-negative.

5. Conclusion

We have shown that in certain degrees and certain groups there exists an invariant form which is nonvanishing on V_i. This is true for V_1 for any G, where of course the Cartan three-form restricts non-trivially to any three-dimensional subgroup, not just the principal one. When G has rank $\ell = 2$ we have $\mathfrak{g} = V_1 \oplus V_2$, an orthogonal decomposition, and the Hodge star of the Cartan 3-form calibrates V_2 so all cases are covered. Another example is the group $SU(4)$ which acts transitively on S^7 and also on S^5 under the homomorphism $SU(4) \to SO(6)$, identifying $SU(4)$ with $Spin(6)$, so we have forms in all degrees $3, 5, 7$ in this case, but for higher rank the arguments in this article only relate to a restrictive number of forms.

6. Polyvector fields

We conclude with a brief discussion of the origin in [5] of the conjecture that for each subspace V_i there is an invariant form φ_i on $\mathfrak{g}$ which restricts non-trivially. The context is a Riemann surface C of genus $g > 1$ and the moduli space M of stable holomorphic principal G^c-bundles P on C for a complex simple Lie group G^c. The cotangent space at a point of M is isomorphic to $H^0(C, \mathrm{ad}(P) \otimes K)$ where K is the canonical bundle and evaluating an invariant polynomial p of degree k defines a holomorphic section of K^k on C. Taking the dual of $H^0(C, K^k)$ this yields a map $H^1(C, K^{1-k}) \to H^0(M, S^k T)$ which is well-known to be injective and to generate holomorphic sections of the symmetric powers S^kT of the tangent bundle which commute using the Schouten-Nijenhuis bracket [4], or equivalently define Poisson-commuting functions on the cotangent bundle T^*M.

If we now use an invariant *alternating* form φ of degree d then evaluation yields a section of K^d and dually we have a map $H^1(C, K^{1-d}) \to H^0(M, \Lambda^d T)$ into the space of polyvector fields on M and these also Schouten-commute [5]. However, whereas using the spectral curve one can see that in the symmetric case the map is injective, for the skew-symmetric case this is not apparent. Instead consider the G^c-bundle associated to a rank 2 stable bundle V by the principal homomorphism $SL(2, \mathbf{C}) \to G^c$ then we can restrict a form φ_i to the subspace $H^0(C, S^{2\lambda_i}V \otimes K) \subset H^0(C, \mathrm{ad}(P) \otimes K)$. By Riemann-Roch this has dimension $(2\lambda_i + 1)(g - 1)$ so if the conjecture held then choosing $n = 2\lambda_i + 1$ holomorphic sections s_j with $s_1 \wedge s_2 \wedge \cdots \wedge s_n$ not identically zero, we could deduce that φ_i gives a nonzero section of K^{d_i}. There may of course be simpler ways of achieving this.

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