Pure and Applied Mathematics Quarterly
Volume 19, Number 6, 2765–2785, 2023

About commuting Sasaki structures

Paul Gauduchon

En hommage à H. Blaine Lawson, Jr. pour son quatre-vingtième anniversaire, avec toute mon amitié et ma profonde considération

1. Introduction

A Kähler metric on a complex manifold is called *extremal* if its scalar curvature is a *Killing potential*, i.e. the momentum of a Hamiltonian Killing vector field. This concept was introduced in the '80s by Calabi, who also constructed extremal Kähler metrics on some compact complex manifolds admitting no Kähler metrics of constant scalar curvature, e.g. on the complex projective plane blown-up at one point. On the other hand, not all compact complex manifolds do admit extremal Kähler metrics and a number of generalisations have been proposed, among which the concept of (f, ν)-*extremal Kähler metric*, proposed by Vestislav Apostolov and David M. J. Calderbank in [3], where f is a Killing potential and ν a real number; the metric is then (f, ν)-*extremal* if the so-called (f, ν)-scalar curvature, cf. (6.25) below, is itself a Killing potential, cf. Definition 1 in [3]. As explained in [3], see also Appendix C in [1] and [2], this new concept acquires its plain significance in the Sasakian context. The present notes, largely of expository character, are mainly thought off as a self-contained introduction to this viewpoint, in particular to its treatment in [3].

Received January 28, 2022.

2. Contact structures

Let N be an oriented manifold of odd dimension $n = 2m + 1$, equipped with a *contact structure*, i.e. a totally non-integrable distribution $\mathcal{D}$ of oriented $2m$-planes, called the *contact distribution* or the *horizontal distribution*. We denote by L the quotient real line bundle $TN/\mathcal{D}$, and by p the natural projection from the tangent bundle TN to $L = TN/\mathcal{D}$. We thus get the exact sequence

$$0 \to \mathcal{D} \to TN \stackrel{p}{\to} L \to 0, \tag{2.1}$$

and the dual exact sequence

$$0 \to L^* \to T^*N \to \mathcal{D}^* \to 0, \tag{2.2}$$

where the dual line bundle L^* is viewed as a subbundle of the cotangent bundle T^*N, namely the annihilator of $\mathcal{D}$ in T^*N. The orientation of N and $\mathcal{D}$ determines an orientation on L and L^*. A *contact* 1-*form* is a positive section of L^*. Any two contact 1-forms, θ and $\tilde{\theta}$ are then related by

$$\tilde{\theta} = f^{-1}\,\theta, \tag{2.3}$$

where f is a positive function, and the contact distribution $\mathcal{D}$ is the kernel of all contact 1-forms.

A vector field Z is called a *contact vector field* if it preserves the contact distribution $\mathcal{D}$, i.e. if, for any *horizontal* vector field, i.e. any section X of $\mathcal{D}$, $\mathcal{L}_Z X = [Z, X]$ is still a horizontal vector field, where $\mathcal{L}_Z$ denotes the Lie derivative along Z. Equivalently, Z is a contact vector field if, for any contact 1-form θ, we have

$$\mathcal{L}_Z\theta = \varphi\,\theta, \tag{2.4}$$

for some function φ. This follows from the following simple computation. For any horizontal vector field X, we have: $d\theta(Z, X) = Z \cdot \theta(X) - X \cdot \theta(Z) - \theta([Z, X]) = -X \cdot \theta(Z) - \theta([Z, X])$; Z is then a contact vector field if and only if $\theta([Z, X]) = 0$ for any horizontal vector field X, hence if and only if $(\iota_Z d\theta + d(\theta(Z)))(X) = (\mathcal{L}_Z\theta)(X) = 0$, for any horizontal vector field X; this, in turn, holds if and only if $\mathcal{L}_Z\theta$ is proportional to θ.

To any contact 1-form θ is associated its *Reeb vector field*, T, determined by the following two conditions

$$\theta(T) = 1, \qquad \mathcal{L}_T\theta = \iota_T d\theta = 0. \tag{2.5}$$

In particular, T is everywhere transverse to $\mathcal{D}$ and preserves the contact distribution. If $\tilde{\theta} = f^{-1}\,\theta$ is another contact 1-form, the corresponding Reeb vector field, $\tilde{T}$, is given by

$$\tilde{T} = f\,T + \mathrm{grad}_{d\theta}\, f, \tag{2.6}$$

where $\mathrm{grad}_{d\theta}\, f$ denotes the horizontal vector field determined by

$$df(X) = -d\theta(\mathrm{grad}_{d\theta}\, f, X), \tag{2.7}$$

for any horizontal vector field X (this makes sense, as the restriction of the 2-form $d\theta$ to $\mathcal{D}$ is everywhere non-degenerate). Indeed, $\tilde{T}$ can be written $\tilde{T} = hT + Z$, for some function h, where Z is horizontal. From the condition $\tilde{\theta}(\tilde{T}) = 1$, we readily infer that $h = f$, so that $\tilde{T} = f\,T + Z$, whereas, the condition $\iota_{\tilde{T}} d\tilde{\theta} = 0$ reads: $-f^{-1}df(T)\,\theta - f^{-2}df(Z)\theta + f^{-1}df + f^{-1}\,\iota_Z d\theta = 0$. From this, we infer $df(Z) = 0$ and $df(X) + d\theta(Z, X) = 0$, for any horizontal vector field X.

The Lie algebra — for the usual bracket of vector fields — of contact vector fields on N is denoted by $\mathfrak{cont}(N, \mathcal{D})$.

Lemma 1. *The map $X \mapsto \xi_X := p(X)$ from $\mathfrak{cont}(N, \mathcal{D})$ to the space, $\Gamma(L)$, of smooth sections of L is an isomorphism.*

Proof. We first show that the map $X \mapsto \xi_X$ is injective. This amounts to showing that $\mathfrak{cont}(N, \mathcal{D})$ contains no non-zero horizontal vector field. Let Z be a horizontal vector field in $\mathfrak{cont}(N, \mathcal{D})$. For any chosen contact 1-form θ and for any horizontal vector field X, we have $d\theta(Z, X) = -\theta([Z, X]) = 0$, since $[Z, X]$ is a horizontal vector field; since the restriction of $d\theta$ to $\mathcal{D}$ is non-degenerate, il follows that $Z = 0$. We now show that any section ξ of L is the image of a (unique) contact vector field, denoted by X_ξ. For that, fix a contact 1-form θ and its Reeb vector field T; then $\xi_T := p(T)$ is a positive section of L and $\xi = \varphi\,\xi_T$, for some function φ, so that any vector field whose image by p is ξ is of the form $\varphi\,T + Z$, where Z is a horizontal vector field. Then, $\mathcal{L}_{\varphi\,T+Z}\theta = d\varphi + \iota_Z d\theta$ is proportional to θ if and only if $d\varphi(X) + d\theta(Z, X) = 0$, for any horizontal vector field, if and only if $Z = \mathrm{grad}_{d\theta}\,\varphi$. We then have:

$$X_\xi = \varphi\,T + \mathrm{grad}_{d\theta}\,\varphi. \tag{2.8}$$

□

Remark 1. If the reference contact 1-form θ is replaced by $\tilde{\theta} = f^{-1}\,\theta$, of Reeb vector field $\tilde{T} = fT + \mathrm{grad}_{d\theta}\, f$, then (2.8) is replaced by $X_\xi = \tilde{\varphi}\tilde{T} + \mathrm{grad}_{d\tilde{\theta}}\,\tilde{\varphi}$, with $\tilde{\varphi} = f^{-1}\varphi$.

In the case when φ is positive in (2.8), X_ξ is the Reeb vector field of the contact 1-form $\tilde{\theta} = \varphi^{-1}\,\theta$. The set of Reeb vector fields is then an open cone in $\mathfrak{cont}(M, \mathcal{D})$.

3. Positive definite CR-structures

We now assume that the contact distribution $\mathcal{D}$ comes equipped with a complex structure, I, which is *compatible with the contact structure*, meaning that, for any horizontal vector fields X, Y, the vector field $[X, Y] - [IX, IY]$ is horizontal, or, equivalently, that $d\theta(IX, IY) = d\theta(X, Y)$. We also assume that $d\theta(X, IY)$ is a positive definite symmetric bilinear form on $\mathcal{D}$ and that I is *formally integrable*, meaning that

$$[X, Y] - [IX, IY] = -I\big([IX, Y] + [X, IY]\big), \tag{3.1}$$

for any horizontal vector fields X, Y. The resulting structure is called a *formally integrable positive definite CR-structure* or, simply in these notes, a *CR-structure*. Once given such a CR-structure, any choice of a contact 1-form θ, of Reeb vector field T, determines a (positive definite) Riemannian metric, g, an operator, J, and a 2-form ω, on N, defined by

$$g(T, T) = 1, \quad g(T, X) = 0, \quad g(X, Y) = \frac{1}{2} d\theta(X, JY), \tag{3.2}$$

$$J(T) = 0, \qquad J(X) = I(X), \tag{3.3}$$

for any horizontal vector fields X, Y, and

$$\omega = \frac{1}{2} d\theta. \tag{3.4}$$

We then have:

$$\begin{gathered} J^2 = -1 + \theta \otimes T, \\ g = \omega(\cdot, J\cdot) + \theta \otimes \theta, \\ \omega = g(J\cdot, \cdot) = -g(\cdot, J\cdot), \\ \omega(J\cdot, J\cdot) = \omega, \\ g(J\cdot, J\cdot) = g - \theta \otimes \theta. \end{gathered} \tag{3.5}$$

Notice that the Reeb vector field T is then the vector field dual to the contact 1-form θ with respect to the metric g. Also notice that T preserves θ, ω and

$\mathcal{D}$, cf. Section 2, but not g and J in general, $\mathcal{L}_T g$ and $\mathcal{L}_Y J$ being related by:

$$\mathcal{L}_T g(X,Y) = -g\big(J\mathcal{L}_T J(X), Y\big), \tag{3.6}$$

for any vector fields X, Y. In particular, $J\mathcal{L}_T J$ is symmetric with respect to g and $J\mathcal{L}_T J(T) = 0$. Denote by D^g the Levi-Civita connection of g. In general, on any Riemannian manifold of metric g, D^g is determined by the following *Koszul formula*:

$$\begin{aligned} 2g(D^g_X Y, Z) = {} & X \cdot g(Y,Z) + Y \cdot g(X,Z) - Z \cdot g(X,Y) \\ & + g\big([X,Y],Z\big) + g\big([Z,X],Y\big) + g\big(X,[Z,Y]\big), \end{aligned} \tag{3.7}$$

for any vector fields X, Y, Z. In the current setting, D^g preserves g, but not T and J; more precisely:

Lemma 2. *For a general CR-structure, $D^g T$ and $D^g J$ have the following expressions:*

$$D^g T = J - \frac{1}{2} J\mathcal{L}_T J, \tag{3.8}$$

and

$$\begin{aligned} g\big((D^g_X J)Y, Z\big) = {} & \theta(Y)\Big(g(X,Z) - \frac{1}{2} g\big((\mathcal{L}_T J)X, Z\big)\Big) \\ & - \theta(Z)\Big(g(X,Y) - \frac{1}{2} g\big((\mathcal{L}_T J)X, Y\big)\Big), \end{aligned} \tag{3.9}$$

for any vector fields X, Y, Z on N. In particular:

$$D^g_T T = 0, \qquad D^g_T J = 0. \tag{3.10}$$

Proof. (3.8) is readily derived from (3.7). In order to check (3.9), we first consider the *Schouten bracket* $[J, J] = N_J$ of J, defined by:

$$\begin{aligned} N_J(X,Y) := {} & \frac{1}{4}([JX, JY] - J[JX, Y] - J[X, JY] + J^2[X,Y]) \\ = {} & \frac{1}{4}((D^g_{JX} J)Y - J(D^g_X J)Y - (D^g_{JY} J)X + J(D_Y J)X); \end{aligned} \tag{3.11}$$

we then consider the expression:

$$d\omega(X,Y,Z) - d\omega(X, JY, JZ) \tag{3.12}$$

$$= g\big((D^g_X J)Y, Z\big) + g\big((D^g_Y J)Z, X\big) + g\big((D^g_Z J)X, Y\big) - g\big((D^g_X J)JY, JZ\big) - g\big((D^g_{JY} J)JZ, X\big) - g\big((D^g_{JZ} J)X, JY\big),$$

and the following commutation formula, derived from (3.8):

(3.13)
$$J(D^g_X J) + (D^g_X J)J = D^g_X J^2 = D^g_X \theta \otimes T + \theta \otimes D^g_X T = (JX - \frac{1}{2}((J\mathcal{L}_T J)X)^\flat \otimes T + \theta \otimes \Big(JX - \frac{1}{2}(J\mathcal{L}_T J)X\Big)$$

where $\flat$ denotes the dual 1-form with respect to g. From the above, we infer the following identity:

$$g\big((D^g_X J)Y, Z\big) = \frac{1}{2}\big(d\omega(X, Y, Z) - d\omega(X, JY, JZ)\big) + 2g\big(JX, N_J(Y, Z)\big) + \theta(Y)g(X, Z) - \theta(Z)g(X, Y), \tag{3.14}$$

for any vector fields X, Y, Z. In the current case, $d\omega = 0$ and, since I is formally integrable, N_J has the following simple expression:

$$N_J(Y, Z) = -\frac{1}{2}\omega(Y, Z)\,T - \frac{1}{4}\theta(Y)J\mathcal{L}_T J(Z) + \frac{1}{4}\theta(Z)J\mathcal{L}_T J(Y). \tag{3.15}$$

By substituting in (3.14), we eventually get (3.9). □

Lemma 3. *Let $\tilde\theta$ and θ be two contact forms, related by $\tilde\theta = f^{-1}\theta$, where f is a positive function. The corresponding Reeb vector fields, $\tilde T$ and T, are then related by*

$$\tilde T = f\,T + \frac{1}{2}J\,\mathrm{grad}_g f, \tag{3.16}$$

where $\mathrm{grad}_g f$ denotes the gradient of f with respect to the metric g. The 2-form $\tilde\omega = \frac{1}{2}d\tilde\theta$, the operator $\tilde J$ and the metric $\tilde g$ determined by $\tilde\theta$ are then related to ω, J and g by:

$$\tilde\omega = f^{-1}\Big(\omega - \frac{1}{2}f^{-1}\,df \wedge \theta\Big), \tag{3.17}$$

$$\tilde J = J + \frac{1}{2}\theta \otimes f^{-1}(\mathrm{grad}_g f - df(T)\,T), \tag{3.18}$$

and

$$\tilde{g} = f^{-1}(g - \theta \otimes \theta) + f^{-2}\,\theta \otimes \theta + \frac{1}{4} f^{-3}\big(|df|_g^2 - (df(T))^2\big)\,\theta \otimes \theta + \frac{1}{2} f^{-2}\big(\theta \otimes (df \circ J) + (df \circ J) \otimes \theta\big). \tag{3.19}$$

Proof. For a general contact structure, the Reeb vector field $\tilde{T}$ is related to T by (2.6); in the current CR case, we clearly have: $\mathrm{grad}_{d\theta} f = \frac{1}{2} J \,\mathrm{grad}_g f$, hence (3.16), while (3.17) readily follows from $\tilde{\omega} = \frac{1}{2} d\tilde{\theta}$. We obtain (3.18) by observing that, for any vector field X, we have: $\tilde{J}(X) = IX^{\tilde{H}} = JX^{\tilde{H}}$, where $X^{\tilde{H}} = X - \tilde{\theta}(X)\tilde{T}$ denotes the projection of X to $\mathcal{D}$ along $\tilde{T}$, which is also the orthogonal projection of X to $\mathcal{D}$ with respect to the metric $\tilde{g}$; (3.18) is then an easy consequence of (3.16) since $JT = 0$. Finally, since $\tilde{g} = \tilde{\omega}(\cdot, \tilde{J}\cdot) + \tilde{\theta} \otimes \tilde{\theta}$, (3.19) easily follows from (3.17) and (3.18). □

4. The Tanaka connection

The following statement is due to Noboru Tanaka, in [6], also cf. [7]:

Proposition 1. *For any contact 1-form θ, of Reeb vector field T, there exist an unique linear connection, ∇, on N, called the Tanaka connection attached to θ, which preserves the whole CR-structure $(g, J, \omega, \theta, T, \mathcal{D})$ and whose torsion, T^{∇}, satisfies the following two conditions:*

$$T^{\nabla}(X, Y) = 2\omega(X, Y)\,T, \qquad T^{\nabla}(T, JX) + JT^{\nabla}(T, X) = 0, \tag{4.1}$$

for any horizontal vector fields X, Y.

Proof. Since $\nabla J = 0$ and $\nabla T = 0$, from the second condition we readily infer

$$\nabla_T X = [T, X] - \frac{1}{2} J(\mathcal{L}_T J)(X), \tag{4.2}$$

for any horizontal, hence any vector field X. Taking the first condition into account, we thus get the following expression of T^{∇}:

$$T^{\nabla}(X, Y) = 2\omega(X, Y)\,T - \frac{1}{2}\theta(X)(J\mathcal{L}_T J)(Y) + \frac{1}{2}\theta(Y)(J\mathcal{L}_T J)(X), \tag{4.3}$$

for any vector fields X, Y on N. Like any connection preserving the metric g, the Tanaka connection ∇ is related to D^g by the general formula:

$$
\begin{aligned}
&g(\nabla_X Y, Z) \\
&\quad = g(D^g_X Y, Z) + \frac{1}{2}\big(g(T^\nabla(X,Y), Z) - g(T^\nabla(Y,Z), X) - g(T^\nabla(X,Z), Y)\big).
\end{aligned} \tag{4.4}
$$

In view of (4.3), we thus get:

$$
\begin{aligned}
\nabla_X Y = D^g_X Y - \theta(X) JY - \theta(Y)\left(JX - \frac{1}{2}(J\mathcal{L}_T J)(X)\right) \\
+ \left(\omega(X,Y) - \frac{1}{2} g((J\mathcal{L}_T J)X, Y)\right) T,
\end{aligned} \tag{4.5}
$$

for any vector fields X, Y on N. This proves the uniqueness of the Tanaka connection. In view of (3.8) and (3.9), it is easily checked that the linear connection ∇ on N *defined* by (4.5) actually satisfies the above conditions. This proves the existence part of the Proposition. $\square$

5. Sasaki structures

A contact 1-form θ is *of Sasakian type* or simply *Sasakian*, if its Reeb vector field T preserves the operator J, i.e. if

$$\mathcal{L}_T J = 0. \tag{5.1}$$

Equivalently, in view of (3.6), θ is of Sasakian type if T is Killing with respect to the metric g. The whole structure $(g, J, \omega, \mathcal{D})$ is then called a *Sasaki structure*. Because of (5.1), in the Sasakian case, the expression of the Tanaka connection ∇ simply becomes:

$$\nabla_X Y = D^g_X Y - \theta(X) JY - \theta(Y) JX + \omega(X,Y)\, T, \tag{5.2}$$

while (4.3) becomes:

$$\nabla_X Y - \nabla_Y X - [X,Y] = 2\,\omega(X,Y)\, T, \tag{5.3}$$

for any vector fields X, Y on N.

In particular, since $\nabla T = 0$ and $\nabla J = 0$, we have

$$D^g T = J, \tag{5.4}$$

and

$$D^g_X J = \theta \wedge X, \tag{5.5}$$

meaning that

$$(D^g_X J)(Y) = \theta(Y)X - g(Y, X)\,T, \tag{5.6}$$

for any vector fields X, Y, while

$$\nabla_T X = [T, X], \tag{5.7}$$

for any vector field X and

$$\nabla_X Y = D^g_X Y + \omega(X, Y)\,T, \tag{5.8}$$

for any *horizontal* vector fields X, Y.

Conversely, in the Sasaki situation, if the Tanaka connection ∇ is simply viewed as a connection on $\mathcal{D}$, regarded as a Hermitian complex vector bundle over N via the complex structure I and the restrictions of ω, g to $\mathcal{D}$, then ∇ may be defined as the unique linear connection on the Hermitian vector bundle $\mathcal{D}$, which preserves its Hermitian structure and satisfies the condition (5.7), for any section X of $\mathcal{D}$ as well as the condition (5.3), for any two sections X, Y, of $\mathcal{D}$.

Since T preserves the whole Sasakian structure, the quotient $M = N/T$ (locally) inherits a Kähler structure, defined as follows. Denote by π the (local) projection from N to M. Then, the pull-back ot TM coincides with the contact distribution $\mathcal{D}$, and the metric g_M, the complex structure J_M and the Kähler form ω_M on M coincide with the restriction of g, J and ω to the contact distribution $\mathcal{D}$. This makes sense, as $\mathcal{D}, g, J, \omega$ are T-invariant. Moreover, the integrability condition (3.1), of I implies the integrability of J_M, and ω_M is closed as ω is.

Remark 2. A horizontal vector field X on N descends to a vector field $\pi_*(X)$ on M, usually simply denoted by X, if and only it is T-invariant, i.e. $[T, X] = 0$ or, equivalently, $\nabla_T X = 0$. Conversely, any vector field, X, on M is the projection of a unique T-invariant horizontal vector field, still denoted by X, on N, called the *horizontal lift* of X. Beware however that, if X, Y are two vector fields on M, regarded as T-invariant horizontal vector fields on N, the bracket $[X, Y]_M$ on M, as a T-invariant horizontal vector field on N, is not $[X, Y]$ in general, but its *horizontal part* $[X, Y]^H := [X, Y] - \theta([X, Y])\,T = [X, Y] + 2\omega(X, Y)\,T$.

Denote by D^{g_M} the Levi-Civita connection of the Kähler manifold M. We then have:

Lemma 4. *The Tanaka connection ∇, viewed as a Hermitian connection on $\mathcal{D}$, coincides with the pull-back $\pi^{-1}D^{g_M}$ of D^{g_M}.*

Proof. Denote by D the pull-back $\pi^{-1}D^{g_M}$ of D^{g_M}. We then have

$$\pi_*(D_X Z) = D^{g_M}_{\pi_*(X)} Z, \tag{5.9}$$

for any vector field X on N and any vector field Z on M, regarded, in the left hand side, as a T-invariant horizontal vector field on N, cf. Remark 2 above. Then, since D^{g_M} preserves g_M, ω_M and J_M, D preserves the Hermitian structure of $\mathcal{D}$, whereas $D_T Z = 0$, since $\pi_*(T) = 0$. Since the algebraic operator $D_T - \mathcal{L}_T$ vanishes on the space of T-invariant sections of $\mathcal{D}$, it is identically equal to zero (as any element X of $\mathcal{D}$ can be locally extended to a T-invariant section of $\mathcal{D}$). It follows that D satisfies the condition (5.7). Moreover, since D^{g_M} is torsion-free, it follows from (5.9) that $D_X Y - D_Y X - [X,Y]$ is a multiple of T, for any pair of T-invariant sections X, Y of $\mathcal{D}$, hence for any sections X, Y of $\mathcal{D}$, so that $[X,Y] = \psi(X,Y)\,T$, for some 2-form ψ on $\mathcal{D}$, which is then equal to the restriction of -2ω to $\mathcal{D}$, since $\theta([X,Y]) = -d\theta(X,Y) = -2\omega(X,Y)$. It follows that D satisfies the condition (5.3) as well, hence coincides with ∇, as a connection on $\mathcal{D}$. □

Lemma 5. *Let θ be a contact 1-form of Sasakian type, T the corresponding Reeb vector field, g the induced metric on N, ∇ the corresponding Tanaka connection. Let $\tilde{T} = \varphi T + \frac{1}{2} J \operatorname{grad}_g \varphi$ be any contact vector field on N. Then, $\tilde{T}$ preserves the CR-structure $(\mathcal{D}, I)$ if and only if*

$$\nabla_{JX}(J \operatorname{grad}_g \varphi) = J\,\nabla_X(J \operatorname{grad}_g \varphi), \tag{5.10}$$

for any horizontal vector field X on N.

If, moreover, $\tilde{T}$ is T-invariant, then φ can then be locally viewed as a function defined on M and $\tilde{T}$ then preserves the CR-structure if and only if φ is a Killing potential on the Kähler manifold (M, g_M, J_M, ω_M), meaning that $J_M \operatorname{grad}_{g_M} \varphi$ is there a Hamiltonian Killing vector field.

Proof. Since $\tilde{T}$ preserves the horizontal distribution $\mathcal{D}$, it preserves the CR-structure $(\mathcal{D}, I)$ if and only if $0 = (\mathcal{L}_{\tilde{T}} I)(X) = [\tilde{T}, IX] - I[\tilde{T}, X] = [\tilde{T}, JX] - J[\tilde{T}, X] = (\mathcal{L}_{\tilde{T}} J)(X)$, for any horizontal vector field X. From (5.8), we get

$$[X,Y] = \nabla_X Y - \nabla_Y X - 2\omega(X,Y)\,T, \tag{5.11}$$

for any two horizontal vector fields X, Y. We then have

(5.12)
$$\begin{aligned}(\mathcal{L}_{\tilde{T}}J)(X) &= \Big[\varphi T + \frac{1}{2} J\, \mathrm{grad}_g\, \varphi, JX\Big] - J\Big[\varphi T + \frac{1}{2} J\, \mathrm{grad}_g\, \varphi, X\Big] \\ &= -d\varphi(JX)\, T \\ &\quad + \frac{1}{2}(\nabla_{J\, \mathrm{grad}_g \varphi} JX - \nabla_{JX}(J\, \mathrm{grad}_g\, \varphi) - 2\omega(J\, \mathrm{grad}_g\, \varphi, JX)\, T) \\ &\quad - \frac{1}{2}(J\nabla_{J\, \mathrm{grad}_g \varphi} X - J\nabla_X(J\, \mathrm{grad}_g\, \varphi)) \\ &= \frac{1}{2}(J\nabla_X(J\, \mathrm{grad}_g\, \varphi) - \nabla_{JX}(J\, \mathrm{grad}_g\, \varphi)).\end{aligned}$$

This shows the first part of Lemma 5. The contact vector field $\tilde{T}$ is T-invariant if and only if $d\varphi(T) = 0$. Indeed, $[T, \tilde{T}] = 0$ if and only if $d\varphi(T) = 0$ and $J\nabla_T\, \mathrm{grad}_g\, \varphi = 0$; now, $d\varphi(T) = 0$ if and only if $\mathrm{grad}_g\, \varphi$ is horizontal; $\nabla_T\, \mathrm{grad}_g\, \varphi$ is then horizontal as well and therefore cannot be killed by J, unless it is zero. It follows that φ can be viewed as a function defined on M, whereas the T-invariant, horizontal vector field $J\, \mathrm{grad}_g\, \varphi$ descends to the Hamiltonian vector field $J_M\, \mathrm{grad}_{g_M}\, \varphi$ on M. Then, in view of Lemma 4, by (5.12), $\tilde{T}$ preserves the CR-structure if and only if $D^{g_M}_{JX}(J_M\, \mathrm{grad}_{g_M}\, \varphi) - J_M D^{g_M}(J_M\, \mathrm{grad}_{g_M}\, \varphi = 0$, if and only if the Hamiltonian vector field $J_M\, \mathrm{grad}_{g_M}\, \varphi)$ is J_M-holomorphic, hence g_M-Killing. ☐

Denote by R^g the curvature of D^g on N, defined by

(5.13)
$$\mathrm{R}^g_{X,Y} Z = D^g_{[X,Y]} Z - D^g_X(D^g_Y Z) + D^g_Y(D^g_X Z),$$

for any vector fields X, Y, Z on N, by R^∇ the curvature of ∇, similarly defined, and by R^{g_M}, the curvature of D^{g_M} on M, defined by

(5.14)
$$\begin{aligned}\mathrm{R}^{g_M}_{X,Y} Z &= D^{g_M}_{[X,Y]_M} Z - D^{g_M}_X(D^{g_M}_Y Z) + D^{g_M}_Y(D^{g_M}_X Z) \\ &= \nabla_{[X,Y]} Z - \nabla_X(\nabla_Y Z) + \nabla_Y(\nabla_X Z) = \mathrm{R}^\nabla_{X,Y} Z,\end{aligned}$$

for any vector fields X, Y, Z on M, regarded as T-invariant horizontal vector fields in the second expression of the right hand side. Notice that $\nabla_{[X,Y]} Z = \nabla_{[X,Y]^H} Z$, since Z is T-invariant. We then have:

Lemma 6. *For any horizontal vector fields X, Y, Z, R^{g_M} is related to R^g by*

(5.15)
$$\mathrm{R}^{g_M}_{X,Y} Z = \mathrm{R}^g_{X,Y} Z + 2\,\omega(X, Y)\, JZ + \omega(X, Z)\, JY - \omega(Y, Z)\, JX.$$

Proof. Easy consequence of (5.8)–(5.7)–(5.4)–(5.5). ☐

Denote by Ric^g, resp. Ric^{g_M}, the Ricci tensor of g, resp. g_M, and by Scal^g, resp. Scal^{g_M}, the scalar curvature of g, resp. g_M. As a direct corollary of Lemma 6, we get:

Proposition 2. *For any horizontal vector fields X, Y, we have:*

$$\mathrm{Ric}^{g_M}(X,Y) = \mathrm{Ric}^g(X,Y) + 2g(X,Y), \tag{5.16}$$

and

$$\mathrm{Scal}^{g_M} = \mathrm{Scal}^g + 2m. \tag{5.17}$$

Proof. We have: $\mathrm{Ric}^{g_M}(X,Y) = \sum_{i=1}^{2m} g_M(\mathrm{R}^{g_M}_{X,e_i} Y, e_i)$, for any orthonormal horizontal frame and any horizontal vector fields X, Y. We then infer from (5.15): $\mathrm{Ric}^{g_M}(X,Y) = \mathrm{Ric}^g(X,Y) - g(\mathrm{R}^g_{X,T} Y, T) + 3g(X,Y)$. Since T is Killing with respect to g, from (5.4) and (5.5), we infer:

$$\mathrm{R}^g_{T,X} = D^g_X(D^g T) = D^g_X J = \theta \wedge X, \tag{5.18}$$

for any horizontal vector field X, where the first identity is the general *Kostant formula* for Killing vector fields. It follows that $g(\mathrm{R}^g_{X,T} Y, T) = g(X,Y)$ and we thus get (5.16). We then have: $\mathrm{Scal}^{g_M} = \sum_{i=1}^{2m} \mathrm{Ric}^{g_M}(e_i, e_i) = \sum_{i=1}^{2m} \mathrm{Ric}^g(e_i, e_i) + 4m = \mathrm{Scal}^g - \mathrm{Ric}^g(T,T) + 4m$. From (5.18), we easily infer:

$$\mathrm{Ric}^g(T) = 2m\, T, \tag{5.19}$$

where Ric^g is regarded as an endomorphism of TN. We thus get (5.17). ☐

Remark 3. The Sasaki metric g on N, determined by the Sasakian 1-form θ, is called θ-*Einstein* if its Ricci tensor Ric^g is of the form

$$\mathrm{Ric}^g = \lambda\, g + \mu\, \theta \otimes \theta, \tag{5.20}$$

for some real functions λ, μ, which are assumed to be *constant*. If $n > 3$, i.e. $m > 1$, the latter condition is automatically satisfied when (5.20) holds; this is no longer the case when $n = 3$, but then (5.20) is always satisfied. On the other hand, it follows from (5.19) that, for any $n \geq 3$, (5.20) implies that

$$\lambda + \mu = 2m. \tag{5.21}$$

It readily follows from (5.16) and (5.19) that the Kähler metric g_M is Einstein if and only if the Sasaki metric g is θ-Einstein, and we then have:

$$\lambda = \frac{\mathrm{Scal}^{g_M} - 4m}{2m}, \qquad \mu = \frac{4m(m+1) - \mathrm{Scal}^{g_M}}{2m}. \tag{5.22}$$

If g_M is Einstein, g is then Einstein if and only if $\mathrm{Scal}^{g_M} = 4m(m+1)$.

6. Commuting Sasaki structures and associated Kähler metrics

In this last section, in addition to the Sasakian contact 1-form θ as above, of Reeb vector field T, we consider another Sasakian 1-form $\tilde{\theta} = f^{-1}\theta$, of Reeb vector field $\tilde{T}$, and we assume that the two Sasakian structure *commute*, meaning that $[T, \tilde{T}] = 0$, or, equivalently, that $df(T) = 0$, i.e. that the vector field $\mathrm{grad}_g f$ is horizontal, cf. the proof of Lemma 5.

In view of Lemma 5, f is then a (positive) Killing potential with respect to the Kähler metric g_M. Also notice that f is *bi-invariant*, i.e. T- and $\tilde{T}$-invariant. As a direct consequence of Lemma 5, we have:

Proposition 3. *Let $\theta, \tilde{\theta} = f^{-1}\theta$ be two commuting contact 1-form of Sasakian type, of Reeb vector fields $T, \tilde{T} = fT + \frac{1}{2}J\,\mathrm{grad}_g f$, as above. Denote by $(M = N/T, g_M, J_M, \omega_M)$ and $(\tilde{M} = N/\tilde{T}, g_{\tilde{M}}, J_{\tilde{M}}, \omega_{\tilde{M}})$ the induced (local) Kähler structure. Denote by $C^\infty_{T,f}(N,\mathbb{R})$ the space of bi-invariant real smooth functions on N, by $\mathcal{P}_\theta$, resp. $\mathcal{P}_{\tilde{\theta}}$, the space of bi-invariant Killing potentials relative to g_M, resp, $g_{\tilde{M}}$. Then, the automorphism σ_f of $C^\infty_{T,f}(N,\mathbb{R})$ defined by*

$$\sigma_f : \varphi \mapsto f^{-1}\varphi, \tag{6.1}$$

induces an isomorphism from $\mathcal{P}_\theta$ to $\mathcal{P}_{\tilde{\theta}}$.

Proof. Direct consequence of Lemma 5 and the fact that $\varphi T + \frac{1}{2}J\,\mathrm{grad}_g\,\varphi = f^{-1}\varphi\tilde{T} + \frac{1}{2}\tilde{J}\,\mathrm{grad}_{\tilde{g}}(f^{-1}\varphi)$, cf. Remark 1. ☐

We denote by ∇ and $\tilde{\nabla}$ the Tanaka connections of θ and $\tilde{\theta}$ respectively. We then have:

Lemma 7. *Regarded as Hermitian connections on $\mathcal{D}$, $\tilde{\nabla}$ and ∇ are related by*

$$\tilde{\nabla}_X Y - \nabla_X Y = \eta_X Y, \quad \tilde{\nabla}_T Z - \nabla_T Z = \eta_T Z, \tag{6.2}$$

with:

$$\eta_X Y = \frac{1}{2} f^{-1} \big[-df(X)\, Y - df(Y)\, X + g(X,Y)\, \mathrm{grad}_g f + df(JX)\, JY + df(JY)\, JX + \omega(X,Y)\, J\, \mathrm{grad}_g f \big], \tag{6.3}$$

$$\eta_T Z = \frac{1}{2} f^{-1} \Big[f^{-1} df(JZ)\, \mathrm{grad}_g f + f^{-1} df(Z)\, J\, \mathrm{grad}_g f + \frac{1}{2} f^{-1} |df|_g^2\, JZ - J \nabla_Z\, \mathrm{grad}_g f \Big], \tag{6.4}$$

for any horizontal vector fields X, Y, Z.

Proof. From (5.8), applied to $\tilde{\nabla}$, and from (3.19), we infer that, for any horizontal vector fields X, Y, Z, we have: $\tilde{g}(D^{\tilde{g}}_X Y, Z) = \tilde{g}(\tilde{\nabla}_X Y, Z) = f^{-1} g(\tilde{\nabla}_X Y, Z)$. By (3.19) and the *Koszul formula* (3.7) for $D^{\tilde{g}}$, we get:

$$\begin{aligned} 2\tilde{g}(D^{\tilde{g}}_X Y, Z) &= 2 f^{-1} g(\tilde{\nabla}_X Y, Z) \\ &= X \cdot f^{-1} g(Y,Z) + Y \cdot f^{-1} g(X,Z) - Z \cdot f^{-1} g(X,Y) \\ &\quad + \tilde{g}([X,Y],Z) + \tilde{g}([Z,X],Y) + \tilde{g}(X,[Z,Y]) \\ &= -f^{-2} df(X)\, g(Y,Z) - f^{-2} df(Y)\, g(X,Z) + f^{-2} df(Z)\, g(X,Y) \\ &\quad + f^{-1} X \cdot g(Y,Z) + f^{-1} Y \cdot g(X,Z) - f^{-1} Z \cdot g(X,Y) \\ &\quad + \tilde{g}([X,Y],Z) + \tilde{g}([Z,X],Y) + \tilde{g}(X,[Z,Y]), \end{aligned} \tag{6.5}$$

while, for any horizontal vector field X, Y, Z, it follows from (3.19) that

$$\begin{aligned} \tilde{g}([X,Y],Z) &= f^{-1} g([X,Y],Z) + \frac{1}{2} f^{-2} \theta([X,Y])\, df(JZ) \\ &= f^{-1} g([X,Y],Z) - f^{-2} \omega(X,Y)\, df(JZ). \end{aligned} \tag{6.6}$$

By using the Koszul formula again for D^g and (5.8), we eventually get:

$$\begin{aligned} & g(\tilde{\nabla}_X Y, Z) \\ &\quad = g(\nabla_X Y, Z) \\ &\qquad - \frac{1}{2} f^{-1} df(X)\, g(Y,Z) - \frac{1}{2} f^{-1} df(Y)\, g(X,Z) + \frac{1}{2} f^{-1} df(Z)\, g(X,Y) \\ &\qquad - \frac{1}{2} f^{-1} df(JZ)\, \omega(X,Y) + \frac{1}{2} f^{-1} df(JX)\, \omega(Y,Z) + \frac{1}{2} f^{-1} df(JY)\, \omega(X,Z), \end{aligned} \tag{6.7}$$

hence (6.3), since $\tilde{\nabla}_X Y$ and $\nabla_X Y$ are both horizontal. To check (6.4), we start with the identities (5.7), applied to ∇ and $\tilde{\nabla}$, and (3.16), to get

$$\eta_T Z = \frac{1}{2} f^{-1}[-2df(Z)T + [J\,\mathrm{grad}_g f, Z] - \nabla_{J\,\mathrm{grad}_g f} Z - \eta_{J\,\mathrm{grad}_g f} Z]. \tag{6.8}$$

From (5.11), we infer:

$$[J\,\mathrm{grad}_g f, Z] - \nabla_{J\,\mathrm{grad}_g f} Z - 2df(Z)\,T = -\nabla_Z J\,\mathrm{grad}_g f = -J\nabla_Z\,\mathrm{grad}_g f, \tag{6.9}$$

while, by (6.3), we get

$$\eta_{J\,\mathrm{grad}_g f} Z = -\frac{1}{2} f^{-1}[2df(JZ)\,\mathrm{grad}_g f + 2df(Z)\,J\,\mathrm{grad}_g f + |df|_g^2\,JZ]. \tag{6.10}$$

□

We denote by R^{∇} the curvature of ∇ and by $R^{\tilde{\nabla}}$ the curvature of $\tilde{\nabla}$, with, we recall, the convention $R^{\nabla}_{X,Y}Z = \nabla_{[X,Y]}Z - \nabla_X(\nabla_Y Z) + \nabla_Y(\nabla_X Z)$ and likewise for $R^{\tilde{\nabla}}_{X,Y}Z$. We then have, cf. also [5]:

Lemma 8. *For any horizontal vector fields X, Y, Z, $R^{\tilde{\nabla}}_{X,Y}Z$ and $R^{\nabla}_{X,Y}Z$ are related by:*

$$\begin{aligned} R^{\tilde{\nabla}}_{X,Y}Z &= R^{\nabla}_{X,Y}Z \\ &+ \frac{1}{4f^2}[df(Y)df(Z) + df(JY)df(JZ) + g(Y,Z)|df|^2 - 2f(\nabla_Y df)(Z)]X \\ &- \frac{1}{4f^2}[df(X)df(Z) + df(JX)df(JZ) + g(X,Z)|df|_g^2 - 2f(\nabla_X df)(Z)]Y \\ &- \frac{1}{4f^2}[df(Y)df(JZ) - df(JY)df(Z) - \omega(Y,Z)\,|df|_g^2 - 2f(\nabla_Y df)(JZ)]JX \\ &+ \frac{1}{4f^2}[df(X)df(JZ) - df(JX)df(Z) - \omega(X,Z)\,|df|_g^2 - 2f(\nabla_X df)(JZ)]JY \\ &- \frac{1}{2f^2}[df(Y)df(JX) - df(X)df(JY) + \omega(X,Y)\,|df|^2 \\ &\qquad - f(\nabla_Y df)(JX) + f(\nabla_X df)(JY)]JZ \\ &+ \frac{1}{4f^2}[df(JX)\,\omega(Y,Z) - df(JY)\,\omega(X,Z) - 2df(JZ)\,\omega(X,Y) \\ &\qquad - df(Y)\,g(X,Z) + df(X)\,g(Y,Z)]\,\mathrm{grad}_g f \end{aligned} \tag{6.11}$$

$$
\begin{aligned}
&+\frac{1}{4f^2}\big[df(X)\,\omega(Y,Z)-df(Y)\,\omega(X,Z)-2df(Z)\,\omega(X,Y)\\
&\qquad +df(JY)\,g(X,Z)-df(JX)\,g(Y,Z)\big]\,J\,\mathrm{grad}_g f\\
&-\frac{1}{2f}\big[g(Y,Z)\,\nabla_X\,\mathrm{grad}_g f-g(X,Z)\,\nabla_Y\,\mathrm{grad}_g f\big]\\
&-\frac{1}{2f}\big[\omega(Y,Z)J\nabla_X\,\mathrm{grad}_g f-\omega(X,Z)J\nabla_Y\,\mathrm{grad}_g f-2\omega(X,Y)J\nabla_Z\,\mathrm{grad}_g f\big].
\end{aligned}
$$

Proof. For any horizontal vector fields X, Y, Z, we have:

$$
\begin{aligned}
R^{\tilde{\nabla}}_{X,Y}Z &= \tilde{\nabla}_{[X,Y]}Z-\tilde{\nabla}_X(\tilde{\nabla}_Y Z)-\tilde{\nabla}_Y(\tilde{\nabla}_X Z)\\
&= R^{\nabla}_{X,Y}Z-2\omega(X,Y)\,\eta_T Z-(\nabla_X\eta)_Y Z+(\nabla_Y\eta)_X Z\\
&\quad -\eta_X(\eta_Y Z)+\eta_Y(\eta_X Z).
\end{aligned}
\tag{6.12}
$$

Then, (6.11) follows from (6.3)–(6.4) by a direct computation. □

Remark 4. The above formula holds for commuting Sasaki structures relative to the CR-structures $(\mathcal{D}, I)$, when $\mathrm{grad}_g f$ is horizontal. In the general case, the right hand-side still holds by simply replacing $\mathrm{grad}_g f$ by its horizontal part relative to T, i.e. by $-J^2\,\mathrm{grad}_g f$, and, accordingly, $|df|_g^2$ by $|df|_g^2-(df(T))^2$.

As already mentioned in Lemma 4, the Tanaka connection ∇, acting on $\mathcal{D}$, is the pull-back of the Levi-Civita connection D^{g_M} of the Kähler manifold $M = N/T$. For any horizontal vectors X, Y, Z, $R^{\nabla}_{X,Y}Z$ is then identified with $R^{g_M}_{X,Y}Z$, where R^{g_M} denotes the curvature of D^{g_M} and X, Y, Z are identified with $\pi_*(X), \pi_*(Y), \pi_*(Z)$. The Ricci tensor $\mathrm{Ric}^{g_M}(X,Y)$, viewed as a symmetric bilinear form on $\mathcal{D}$, and the scalar curvature Scal^{g_M} of g_M, viewed as a function on N, are then given by

$$
\mathrm{Ric}^{g_M}(X,Y)=\sum_{i=1}^{2m} g\big(R^{\nabla}_{X,e_i}Y,e_i\big),\quad \mathrm{Scal}^{g_M}=\sum_{i=1}^{2m}\mathrm{Ric}^{g_M}(e_i,e_i),
\tag{6.13}
$$

for any g-orthonormal basis $\{e_1,\dots,e_{2m}\}$ of $\mathcal{D}$. Similarly, $R^{\tilde{\nabla}}_{X,Y}Z$ is identified with $R^{g_{\tilde{M}}}_{X,Y}Z$, where $R^{g_{\tilde{M}}}$ denotes the curvature of $D^{g_{\tilde{M}}}$ and X, Y, Z are identified with $\tilde{\pi}_*(X), \tilde{\pi}_*(Y), \tilde{\pi}_*(Z)$, where $\tilde{\pi}$ denotes the natural projection from N to $\tilde{M} = N/\tilde{T}$. The Ricci tensor $\mathrm{Ric}^{g_{\tilde{M}}}(X,Y)$, viewed as a symmetric bilinear form on $\mathcal{D}$, and the scalar curvature $\mathrm{Scal}^{g_{\tilde{M}}}$ of $g_{\tilde{M}}$, viewed as a function on

N, are then given by

$$\mathrm{Ric}^{g_{\tilde M}}(X,Y) = \sum_{i=1}^{2m} \tilde g(R^{\tilde\nabla}_{X,\tilde e_i}Y, \tilde e_i) = \sum_{i=1}^{2m} g(R^{\tilde\nabla}_{X,e_i}Y, e_i), \tag{6.14}$$

and

$$\mathrm{Scal}^{g_{\tilde M}} = \sum_{i=1}^{2m} \mathrm{Ric}^{\tilde\nabla}(\tilde e_i, \tilde e_i) = f \sum_{i=1}^{2m} \mathrm{Ric}^{\tilde\nabla}(e_i, e_i), \tag{6.15}$$

where $\{\tilde{e_1} := f^{\frac12}\, e_1, \dots, \tilde{e_{2m}} := f^{\frac12}\, e_{2m}\}$ is a $\tilde g$-orthonormal basis of $\mathcal{D}$. From Lemma 8, we then infer:

(6.16)

$$\begin{aligned}\mathrm{Ric}^{g_{\tilde M}}(X,Y) = \mathrm{Ric}^{g_M}(X,Y) &\\ &- \frac{(m+2)}{2}\left(f^{-2}df(X)df(Y) + f^{-2}df(JX)df(JY)\right)\\ &- \frac{(m+2)}{2} f^{-2} g(X,Y)\, |df|^2_{g_M} + (m+2)\, f^{-1} D^{g_M}_X df(Y)\\ &- \frac{1}{2} g(X,Y)\, f^{-1}\Delta_{g_M} f,\end{aligned}$$

where $\Delta_{g_M} = -\sum_{i=1}^{2m} D^{g_M}_{e_i} df(e_i)$ denotes the Riemannian Laplacian of g_M. Observe that $\sum_{i=1}^{2m} D^{g_M}_{e_i} df(J_M e_i) = 0$, as $J_M\, \mathrm{grad}_{g_M} f$ is Killing with respect to g_M, cf. Lemma 5. We thus have:

Proposition 4. *For any CR-manifold $(N, \mathcal{D}, I)$ of dimension $n = 2m+1$, let θ and $\tilde\theta = f^{-1}\,\theta$ be two contact 1-forms of Sasakian type, of Reeb vector fields T and $\tilde T$ respectively. Denote by g, resp. $\tilde g$, the Riemannian metric induced by θ, resp. $\tilde g$, on N, by $(M = N/T, g_M, J_M, \omega_M)$, resp. $(\tilde M = N/\tilde T, g_{\tilde M}, J_{\tilde M}, \omega_{\tilde M})$ the induced (local) Kähler structures, by Scal^{g_M}, resp. $\mathrm{Scal}^{g_{\tilde M}}$, the scalar curvature of g_M, resp. $g_{\tilde M}$, viewed as functions defined on N. Assume moreover that the two Sasakian structures commute, i.e. that $[T, \tilde T] = 0$. Then, the vector field $J\,\mathrm{grad}_g f$ is a T-invariant, horizontal vector field on N, which decends to a Hamiltonian Killing vector field on M, and Scal^{g_M} and $\mathrm{Scal}^{g_{\tilde M}}$ are related by*

$$\mathrm{Scal}^{g_{\tilde M}} = f\,\mathrm{Scal}^{g_M} - 2(m+1)\,\Delta_{g_M} f - (m+1)(m+2)\, f^{-1} |df|^2_{g_M}. \tag{6.17}$$

Proof. The assertion concerning $J\,\mathrm{grad}_g f$ follows from Lemma 5, while (6.17) is a direct consequence of (6.14)–(6.15)–(6.16). ◻

Remark 5. Notice that Scal^{g_M} and $\mathrm{Scal}^{g_{\tilde{M}}}$, as functions defined on N, are both T *and* $\tilde{T}$-invariant, hence can be both regarded as functions defined on M and on $\tilde{M}$.

Denote by $\tilde{\pi}^*\rho^{g_{\tilde{M}}}$ the pull-back of the Ricci form $\rho^{g_{\tilde{M}}}$ of $\tilde{M}$ by the projection $\tilde{\pi}$ from N to $\tilde{M} = N/\tilde{T}$, and by $\pi^*\rho^{g_M}$ the pull-back of the Ricci form, ρ^{g_M}, of M, by π. From (6.16), we get:

(6.18)

$$(\tilde{\pi}^*\rho^{g_{\tilde{M}}})(X,Y) - (\pi^*\rho^{g_M})(X,Y) = \frac{(m+2)}{2}(dJd\log f)(X,Y) - \frac{1}{2}f^{-1}(\Delta_{g_M} f + (m+2)f^{-1}|df|^2_{g_M})\omega_M(X,Y),$$

for any horizontal vector fields X, Y, whereas $(\pi^*\rho^{g_M})(T,X) = 0$ and $(\tilde{\pi}^*\rho^{g_{\tilde{M}}})(\tilde{T},X) = 0$. From the latter and (3.16), we get

(6.19)

$$(\tilde{\pi}^*\rho^{g_{\tilde{M}}})(T,X) = -\frac{1}{2}f^{-1}(\tilde{\pi}^*\rho^{g_{\tilde{M}}})(J\,\mathrm{grad}_g f, X) = \frac{1}{2}f^{-1}\,\mathrm{Ric}^{g_{\tilde{M}}}(\mathrm{grad}_g f, X),$$

for any horizontal vector field X. From (6.16), we thus infer:

$$(6.20)\quad (\tilde{\pi}^*\rho^{g_{\tilde{M}}})(T,X) = \frac{1}{2}f^{-1}\,\mathrm{Ric}^{g_M}(\mathrm{grad}_g f, X) - \frac{(m+2)}{2}f^{-3}|df|^2_g\, df(X) + \frac{(m+2)}{4}f^{-2}(d|df|^2_g)(X) - \frac{1}{4}f^{-2}\Delta_{g_M} f\, df(X),$$

for any horizontal vector field X. Now, since $J\,\mathrm{grad}_g f$ is Killing with respect to g_M, we have $\mathrm{Ric}^{g_M}(\mathrm{grad}_g f) = \frac{1}{2}\,\mathrm{grad}_g(\Delta_{g_M} f)$, whereas $f^{-2}d(|df|^2_g) - 2f^{-3}|df|^2_g\, df = d(f^{-2}|df|^2_g)$ and $f^{-1}d(\Delta_{g_M} f) - f^{-2}\Delta_{g_M} f\, df = d(f^{-1}\Delta_{g_M} f)$, so that

$$(6.21)\qquad (\tilde{\pi}^*\rho^{g_{\tilde{M}}})(T,X) = \frac{(m+2)}{4}d(f^{-2}|df|^2_g) + \frac{1}{4}d(f^{-1}\Delta_{g_M} f),$$

for any horizontal vector field X. We thus obtain:

(6.22)

$$\tilde{\pi}^*\rho^{g_{\tilde{M}}} - \pi^*\rho^{g_M} = \frac{(m+2)}{2}\,dJd\log f - \frac{1}{2}(f^{-1}\Delta_g f + (m+2)\,f^{-2}|df|^2_g)\,\omega + \frac{1}{4}\theta\wedge d(f^{-1}\Delta_g f + (m+2)\,f^{-2}|df|^2_g),$$

with the general convention that $(J\alpha)(X) = -\alpha(JX)$, for any 1-form α and any vector field X on N.

We then eventually get:

Proposition 5. *The 2-forms $\tilde{\pi}^*\rho^{g_{\tilde{M}}}$ and $\pi^*\rho^{g_M}$ are related by:*

$$\tilde{\pi}^*\rho^{g_{\tilde{M}}} - \pi^*\rho^{g_M} = d\tau_f, \tag{6.23}$$

with

$$\tau_f = \frac{(m+2)}{2} f^{-1} J df - \frac{1}{4}\left(f^{-1}\Delta_g f + (m+2)\, f^{-2}|df|_g^2\right)\theta. \tag{6.24}$$

Proof. Direct consequence of (6.22), by taking into account that $\omega = \frac{1}{2}d\theta$. □

In general, for any Kähler manifold (M, g, J, ω), any real function f on M and any real number ν, the (f,ν) *scalar curvature*, $\mathrm{Scal}^g_{f,\nu}$, is defined by

$$\mathrm{Scal}^g_{f,\nu} = f^2\,\mathrm{Scal}^g - 2(\nu - 1)\, f\Delta_g f - \nu(\nu-1)\,|df|_g^2, \tag{6.25}$$

cf. the Introduction of [3] and references therein. We hen have:

Proposition 6. *Let θ and $\tilde{\theta} = f^{-1}\theta$ be two commuting contact 1-forms of Sasakian type, as in Proposition 4. Then the $(f, m+2)$-scalar curvature $\mathrm{Scal}^{g_M}_{f,m+2}$ of g_M and the scalar curvature $\mathrm{Scal}^{g_{\tilde{M}}}$ of g_M are related by*

$$\mathrm{Scal}^{g_{\tilde{M}}} = f^{-1}\,\mathrm{Scal}^{g_M}_{f,m+2}\,. \tag{6.26}$$

In view of Lemma 5, which is essentially Lemma 2 in [3], the Definition 3 in [3] can be reformulated as follows (recall that $\mathcal{P}_\theta$, resp. $\mathcal{P}_{\tilde{\theta}}$, denotes the space of bi-invariant Killing potentials relative to the Kähler metric g_M, resp. $g_{\tilde{M}}$, cf. Proposition 3:

Definition 1. Let θ be a Sasakian contact 1-form relative to the CR-manifold $(N, \mathcal{D}, I)$, f a positive element of $\mathcal{P}_\theta$ and ν a real number. Then, the Sasaki structure $(N, \mathcal{D}, I, \theta)$ is said to be (f,ν)-*extremal* if the (f,ν)-scalar curvature $\mathrm{Scal}^{g_M}_{f,\nu}$ belongs to $\mathcal{P}_\theta$.

When $f = 1$, i.e. when $\mathrm{Scal}^{g_M}_{f,\nu} = \mathrm{Scal}^{g_M}$, we retrieve the definition of *extremal Sasaki structure* appearing in [4].

In view of Definition 1 above, Propostion 6 can then be reformulated as follows, cf. Theorem 1 in [3]:

Proposition 7. *Let θ and $\tilde{\theta} = f^{-1}\theta$ be two commuting Sasakian contact* 1-*forms relative to the CR-structure* $(\mathcal{D}, I)$. *Then, the (local) Kähler metric g_M determined by θ is $(f, m+2)$-extremal if and only the (local) Kähler metric $g_{\tilde{M}}$ determined by $\tilde{\theta}$ is extremal.*

Proof. As already observed, Scal^{g_M} and $\mathrm{Scal}^{g_{\tilde{g}}}$, as functions on N, are both bi-invariant. In view of Proposition 3, it follows from (6.26) that $\mathrm{Scal}^{g_{\tilde{M}}}$ belongs to $\mathcal{P}_{\tilde{\theta}}$ — meaning that $g_{\tilde{M}}$ is extremal — if and only if $\mathrm{Scal}^{g_M}_{f,m+2}$ belongs ro $\mathcal{P}_{\theta}$ — meaning that g_M is $(f, m+2)$-extremal. □

References

[1] V. Apostolov, D. M.J. Calderbank, P. Gauduchon, *Ambitoric geometry II: Extremal toric surfces and Einstein 4-orbifolds*, Ann. Sci. Éc. Norm. Supér. 4ème série **48** (2015), 1075–1112. MR3429476

[2] V. Apostolov, D. M.J. Calderbank, P. Gauduchon and E. Legendre, *Levi-Kähler reduction of CR structures, product of spheres, and toric geometry*, Math. Res. Lett. **27**, no. 6 (2020), 1565–1629. MR4216597

[3] V. Apostolov and D. M. J. Calderbank, *The CR geometry of weighted extremal Kähler and Sasaki metrics*, Math. Ann. **379** (2021), 1047–1088. MR4238260

[4] C. P. Boyer, K. Galicki and S. R. Simanca, *Canonical Sasakian metrics*, Comm. Math. Phys. **279** (2008), 705–733. MR2386725

[5] D. Jerison and J. M. Lee, *Extremals for the Sobolev inequality for the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc. **1** (1988), 1–13. MR0924699

[6] N. Tanaka, *A differential geometric study on strongly pseudo-convex manifolds*, Lectures in Mathematics, Kyoto University, **9**, Kinokuniya Book-Store Co, Ltd. (1975) MR0399517

[7] S. M. Webster, *Pseudohermitian structures on a real hypersurface*, J. Differential Geometry **13** (1978), 25–41. MR0520599

Paul Gauduchon
CMLS
CNRS
École polytechnique
Institut Polytechnique de Paris
91128 Palaiseau Cedex
France
E-mail: paul.gauduchon@polytechnique.edu