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$RO(C_2)$-graded equivariant cohomology and classical Steenrod squares

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To Blaine on his 80^{th} birthday, with gratitude for his friendship and constant inspiration

Abstract: We investigate the *restriction to fixed-points* and *change of coefficient* functors in $RO(C_2)$-graded equivariant cohomology, with applications to the equivariant cohomology of spaces with a trivial C_2-action for $\underline{\mathbb{Z}}$ and $\underline{\mathbb{F}_2}$ coefficients. To this end, we study the nonequivariant spectra representing these theories and the corresponding functors. In particular, we show that the $RO(C_2)$-graded homology class determined by a Real submanifold Y (in the sense of Atiyah) of a Real compact manifold X encodes the total Steenrod square of the dual to Y^{C_2} in X^{C_2}.

Keywords: C_2-equivariant cohomology, Steenrod squares, real spaces.

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1. Introduction

In the foundational paper [1], M. Atiyah introduced the notions of Real spaces and *Real vector bundles* over Real spaces, which are the basic objects in the construction of Atiyah's KR-theory. In a nutshell, a *Real space in the sense of Atiyah* is a space X with an action of the cyclic group $C_2 = \{1, \varsigma\}$, and a *Real vector bundle* over X consists of a complex vector bundle $E \to X$ endowed with an antilinear involution $\varsigma\colon E \to E$ covering the action of C_2 on X.

Real algebraic geometry is a primary source of Real spaces, for a real algebraic variety X gives rise to the Real space $X(\mathbb{C})$ consisting of the set of complex points of X endowed with the analytic topology, under the complex conjugation involution $\varsigma\colon X(\mathbb{C}) \to X(\mathbb{C})$. In particular, if $Y \subset X$ is a regularly embedded real subvariety, its normal bundle $N_{Y|X}$ yields a Real vector bundle $N_{Y|X}(\mathbb{C}) \to Y(\mathbb{C})$ over $Y(\mathbb{C})$.

In analogy with algebraic geometry, we denote the fixed point set X^{C_2} of a Real space X by $X(\mathbb{R})$ and call it the set of *real points of* X. In this context, a *based Real space* is a pair (X, x_0) with $x_0 \in X(\mathbb{R})$ a real point of X and, as usual, for a Real space X we let X_+ denote the Real pair $(X \cup \{+\}, +)$.

The *equivariant Chern classes* for KR-theory take values in *ordinary $RO(C_2)$-graded equivariant cohomology*, a theory introduced in [11] for an arbitrary compact Lie group G. Given a based G-space X, the theory assigns a group $\widetilde{H}^{\alpha}_{G}(X;\mathbf{M})$ for each α in the ring of orthogonal representations of G, provided that the *coefficients system* $\mathbf{M}$ is a Mackey functor. It is common to denote the corresponding unreduced theory by $H^{\alpha}_{G}(X;\mathbf{M})$. The usual axioms of an equivariant cohomology theory are satisfied, but the suspension axiom takes the form $\widetilde{H}^{\alpha}_{G}(X;\mathbf{M}) \cong \widetilde{H}^{\alpha+V}_{G}(S^V \wedge X;\mathbf{M})$, where $S^V = V \cup \{\infty\}$ is the one-point compactification of an orthogonal representation space V.

In this paper we deal primarily with the case $G = C_2$, and write $RO(C_2) = \mathbb{Z}\cdot\mathbf{1} \oplus \mathbb{Z}\cdot\sigma$, where $\mathbf{1}$ and σ are the trivial and sign representations, respectively. Aiming at algebraic geometric applications, we use the *motivic notation* for $RO(C_2)$-graded equivariant cohomology. Namely, given $n, p \in \mathbb{Z}$ we denote

$$H^{n,p}(X;\mathbf{M}) := H^{(n-p)\cdot\mathbf{1}+p\sigma}_{C_2}(X;\mathbf{M}). \tag{1.1}$$

In this paper, we mostly consider the Mackey functors $\underline{\mathbb{Z}}$ and $\underline{\mathbb{F}_2}$, where $\underline{R}$ denotes the constant Mackey functor associated to the abelian group R. We will also restrict our attention to the subring $H^{*,*}(X;\mathbf{M})_+ \subset H^{*,*}(X;\mathbf{M})$ consisting of elements in degrees (n,p) with $n,p \geq 0$.

For the trivial group $G=\{1\}$, equivariant cohomology is just (nonequivariant) singular cohomology $H^*_{\mathrm{sing}}(X;R)$, with coefficients in R. The group inclusion $\{1\}\subset C_2$ determines a transformation of cohomology theories

$$\mathscr{R}\colon H^{n,p}(X;\underline{R}) \to H^n_{\mathrm{sing}}(X;R), \tag{1.2}$$

usually called the *restriction* or *forgetful* functor.

The cohomology of the quotient $H^*_{\mathrm{sing}}(X/C_2;R)$ is included naturally in $H^{*,\bullet}(X;\underline{R})$ as the subring of elements with bidegree $(n,0)$, $n\in\mathbb{Z}$. Under this isomorphism the restriction functor $\mathscr{R}$ is induced by the quotient map $X\to X/C_2$.

It is useful to note that the image of $\mathscr{R}$ is actually a subgroup of the invariants $H^*_{\mathrm{sing}}(X;R(p))^{C_2}$ under the simultaneous action of C_2 on X and on $R(p) := R\otimes\mathbb{Z}\cdot\sigma^p$. In other words, the image of $\mathscr{R}$ lies in the group of invariants $H^*_{\mathrm{sing}}(X;R)^+$ of singular cohomology when p is even or in the group of anti-invariants $H^*_{\mathrm{sing}}(X;R)^-$ when p is odd.

For a Real space X, the inclusion $\iota_{\mathbb{R}}\colon X(\mathbb{R})\hookrightarrow X$ along with the epimorphism $\boldsymbol{\varrho}\colon\mathbb{Z}\to\mathbb{F}_2$ induces various maps between cohomology groups that fit into the commutative cube displayed below.

$$\begin{tikzcd}
 & H^*_{\mathrm{sing}}(X;\mathbb{Z}(\bullet))^{C_2} \arrow[rr, "\iota_{\mathbb{R}}^*"] \arrow[dd, "\boldsymbol{\varrho}"'] & & H^*_{\mathrm{sing}}(X(\mathbb{R});\mathbb{Z}(\bullet))^{C_2} \arrow[dd, "\boldsymbol{\varrho}"] \\
H^{*,\bullet}(X;\underline{\mathbb{Z}})_+ \arrow[ur, "\mathscr{R}"] \arrow[rr, "\iota_{\mathbb{R}}^*"] \arrow[dd, "\boldsymbol{\varrho}"'] & & H^{*,\bullet}(X(\mathbb{R});\underline{\mathbb{Z}})_+ \arrow[ur, "\mathscr{R}"] \arrow[dd, "\boldsymbol{\varrho}"] & \\
 & H^*_{\mathrm{sing}}(X;\mathbb{F}_2)^{C_2} \arrow[rr, "\iota_{\mathbb{R}}^*"'] & & H^*_{\mathrm{sing}}(X(\mathbb{R});\mathbb{F}_2)^{C_2} \\
H^{*,\bullet}(X;\underline{\mathbb{F}_2})_+ \arrow[ur, "\mathscr{R}"'] \arrow[rr, "\iota_{\mathbb{R}}^*"'] & & H^{*,\bullet}(X(\mathbb{R});\underline{\mathbb{F}_2})_+ \arrow[ur, "\mathscr{R}"'] &
\end{tikzcd}$$

Figure 1: “The Cube”.

This paper addresses the faces of this cube, with emphasis on the information encoded in the restriction maps $\iota_{\mathbb{R}}^*\colon H^{n,p}(X;\underline{R})_+ \to H^{n,p}(X(\mathbb{R});\underline{R})_+$, with $R = \mathbb{Z}$ or $R = \mathbb{F}_2$.

A particularly pleasant example is the C_2-space BU_n under the complex conjugation action, where $\mathrm{BU}_n(\mathbb{R}) = \mathrm{BO}_n$. In this case, we explicitly describe all maps appearing in the cube.

We apply the results to study equivariant characteristic classes for Real bundles. Recall that, in the motivic notation, the *p-th equivariant Chern class* $\mathbf{c}_p(E)$ of a Real vector bundle $E \to X$ takes values in $H^{2p,p}(X;\underline{\mathbb{Z}})$, where $\mathbb{Z}$ has the trivial C_2 action. Similarly, with $\underline{\mathbb{F}_2}$ coefficients the *p-th equivariant Chern class* is denoted $\bar{\mathbf{c}}_p(E)$ and takes values in $H^{2p,p}(X;\underline{\mathbb{F}_2})$.

Even when the action of C_2 on X is trivial the equivariant Chern classes of a Real bundle $E \otimes \mathbb{C} \to X$, obtained from an ordinary vector bundle $E \to X$, offer a unifying perspective that encodes the Stiefel-Whitney classes of E, their Steenrod squares and the Pontryagin classes, all at once. To make this assertion more precise, we need to introduce some notation.

Notation 1.1 (The $RO(C_2)$-graded cohomology of a point)**.** Let $\mathbb{M}^{n,p} := H^{n,p}(*;\underline{\mathbb{Z}})$ and $\mathbb{M}_2^{n,p} := H^{n,p}(*;\underline{\mathbb{F}_2})$ denote the equivariant cohomology groups of a point with coefficients in $\underline{\mathbb{Z}}$ and $\underline{\mathbb{F}_2}$, respectively. The bigraded subring $\mathbb{M}_+ := \oplus_{n,p\geq 0}\mathbb{M}^{n,p} \subset \mathbb{M}$ can be presented as $\mathbb{M}_+ = \mathbb{Z}[a,u]$, where $\deg(a) = (1,1), \deg(u) = (0,2)$ and $2a = 0$. Here the generator a is represented by the map $a_\sigma\colon S^0 = \{S^\sigma\}^{C_2} \hookrightarrow S^\sigma$, while the generator u is determined by $\mathscr{R}(u) = 1 \in H^0(*;\mathbb{Z})$.

Similarly, $\mathbb{M}_{2+} := \oplus_{r,s\geq 0}\mathbb{M}_2^{r,s} \subset \mathbb{M}_2$ is a polynomial ring $\mathbb{M}_{2+} = \mathbb{F}_2[\rho,\tau]$, with $\deg(\rho) = (1,1)$ and $\deg(\tau) = (0,1)$, where ρ is represented by a_σ and τ is determined by $\mathscr{R}(\tau) = 1 \in H^0(*;\mathbb{F}_2)$. Furthermore, the change of coefficients homomorphism $\boldsymbol{\varrho}\colon \mathbb{M}_+ \to \mathbb{M}_{2+}$ is given by

$$\boldsymbol{\varrho}\colon a \mapsto \rho \quad \text{and} \quad \boldsymbol{\varrho}\colon u \mapsto \tau^2. \tag{1.3}$$

Now, consider a space X as a C_2-space with a trivial C_2-action, in other words, $X = X(\mathbb{R})$. It follows from the discussion in §2.2.2 that an element $\alpha \in H^{n,p}(X;\underline{\mathbb{Z}})$ can be canonically written as

$$\alpha = \sum_{s=0}^{\lfloor p/2\rfloor} \alpha_s \cdot a^{p-2s}u^s, \tag{1.4}$$

where

$$\begin{cases} \alpha_s \in H^{n-p+2s}_{\mathrm{sing}}(X;\mathbb{F}_2), & 0 \leq s < p/2 \\ \alpha_{p/2} \in H^n_{\mathrm{sing}}(X;\mathbb{Z}), & \text{if } p \text{ is even} \end{cases}$$

and a, u are the generators in the cohomology ring of a point, as in Notation 1.1. We show in Corollary 4.2 that the *n-th equivariant Chern class of* $E \otimes \mathbb{C}$ is given by

$$\mathbf{c}_n(E \otimes \mathbb{C}) = \sum_{0 \le s < n/2} \mathrm{Sq}^{2s}(w_n(E)) \cdot a^{2(n-s)} u^s \ + \ p_{n/2}(E) \cdot u^{n/2}, \tag{1.5}$$

where $\mathrm{Sq}^{2s}(w_n(E))$ is the $2s$-th Steenrod square of the n-th Stiefel-Whitney class of E and $p_{n/2}(E)$ is the $n/2$-th Pontryagin class of E when n is even and 0, otherwise.

With $\mathbb{F}_2$ coefficients the $RO(C_2)$-graded equivariant cohomology on a space with trivial action is simpler but still encodes interesting information. In [8] it is shown that $H^{*,\bullet}(X; \underline{\mathbb{F}_2}) \cong \mathbb{M}_2^{*,\bullet} \otimes_{\mathbb{F}_2} H^*_{\mathrm{sing}}(X; \mathbb{F}_2)$, where $H^*_{\mathrm{sing}}(X; \mathbb{F}_2)$ is identified with $H^{*,0}(X; \underline{\mathbb{F}_2})$, as explained above. In §2.2.3, we write an element $\beta \in H^{n,p}(X; \underline{\mathbb{F}_2})$ on a trivial Real space X canonically as

$$\beta = \sum_{k=0}^{p} \beta_k \cdot \rho^{p-k} \tau^k, \tag{1.6}$$

where $\rho^{p-k}\tau^k$ is the generator of $\mathbb{M}_2^{p-k,p}$ and $\beta_k \in H^{n-p+k}_{\mathrm{sing}}(X; \mathbb{F}_2)$. In particular, we can introduce a map $\mathrm{Sq}_{\rho,\tau} \colon H^*_{\mathrm{sing}}(X; \mathbb{F}_2) \to H^{2*,*}(X; \underline{\mathbb{F}_2})$, called the *total Steenrod square*, defined on $\beta \in H^p_{\mathrm{sing}}(X; \mathbb{F}_2)$ by

$$\mathrm{Sq}_{\rho,\tau}(\beta) = \sum_{i=0}^{p} \mathrm{Sq}^i(\beta) \cdot \rho^{p-i}\tau^i \quad \in \ H^{2p,p}(X; \underline{\mathbb{F}_2}).$$

It follows from the Cartan formulas that $\mathrm{Sq}_{\rho,\tau}$ is a ring homomorphism, and we will show that classes of the form $\mathrm{Sq}_{\rho,\tau}(\beta)$ occur often in geometric contexts as restrictions of C_2-equivariant $(2p, p)$-classes to the set of real points.

A typical example of this phenomenon is the restriction of the equivariant mod 2 Chern classes. We show in §3.2 that if $E \to X$ is a vector bundle on a trivial Real space X whose n-th Stiefel-Whitney class is ω_n, then

$$\bar{\mathbf{c}}_n(E \otimes \mathbb{C}) = \mathrm{Sq}_{\rho,\tau}(\omega_n). \tag{1.7}$$

As another example, we consider a Real submanifold $Y \subset X$ of codimension c in a Real manifold X and show in Theorem 3.19 that the restriction $(\alpha_Y)_{|X(\mathbb{R})} := \iota^*_{\mathbb{R}}(\alpha_Y)$ to the real points $X(\mathbb{R})$ of the fundamental class $\alpha_Y \in H^{2c,c}(X; \underline{\mathbb{F}_2})$ is precisely given by

$$(\alpha_Y)_{|X(\mathbb{R})} = \mathrm{Sq}_{\rho,\tau}\left(\alpha_{Y(\mathbb{R})}\right). \tag{1.8}$$

I.e., this restriction encodes all Steenrod powers of the fundamental class $\alpha_{Y(\mathbb{R})} \in H^c_{\text{sing}}(X(\mathbb{R}); \mathbb{F}_2)$.

A version of this equivariant perspective on the total Steenrod square operator appears in the literature in the context of Borel C_2-equivariant cohomology. For example, from [9] one can derive formulas that are similar to (1.5) and (1.7) when ordinary equivariant cohomology is replaced by $H^*_{\text{sing}}(- \times_{C_2} EC_2; \mathbb{F}_2)$ and $H^*_{\text{sing}}(- \times_{C_2} EC_2; \mathbb{Z})$.

Actually, one of the standard constructions of the Steenrod squares uses Borel cohomology on the *coinduced* C_2-space $\mathrm{N}^{C_2}X = X^2$ (with the transposition involution) and the restriction to its set of real points $\mathrm{N}^{C_2}X(\mathbb{R}) = X$. In Example 3.5 we approach the construction of Steenrod squares using the $RO(C_2)$-graded equivariant groups $H^{2p,p}(\mathrm{N}^{C_2}X; \underline{\mathbb{F}_2})$ instead.

In order to study the restriction maps $H^{n,p}(X; \underline{R})_+ \to H^{n,p}(X(\mathbb{R}); \underline{R})_+$, displayed in the cube and establish the properties used in applications, we provide a brief discussion of the orthogonal equivariant Eilenberg-MacLane spectra $\mathbf{H}\underline{R}$ [13] representing ordinary $RO(C_2)$-graded equivariant cohomology when R is a C_2-algebra.

Then, for each $p \geq 0$ and R-module M we consider the (non-equivariant) orthogonal spectrum

$$\mathbf{E}_M(p) := \left(\Sigma^{p\sigma - p}\mathbf{H}\underline{M}\right)^{C_2}, \tag{1.9}$$

where $\Sigma^{p\sigma-p}$ is the suspension by the virtual representation $p \cdot \sigma - p \cdot \mathbf{1}$, and $(-)^{C_2}$ is the fixed point functor; see [13, Ch. II], [17, Exs. 3.8, 3.9].

The non-equivariant spectrum $\mathbf{E}_M(p)$ is defined so that the associated cohomology groups of a space X are given by

$$\begin{aligned} \mathbf{E}_M(p)^n(X) &:= [\Sigma^\infty X_+, \Sigma^n \mathbf{E}_M(p)] = \left[\Sigma^\infty X_+, \Sigma^{p\sigma+n-p}\mathbf{H}\underline{M}\right]_{C_2} \\ &= H^{n,p}(X; \underline{M}), \end{aligned} \tag{1.10}$$

where, for the second identity, X is considered a C_2-space with the trivial action. In particular, the homotopy groups of the spectra are given by

$$\pi_{-n}(\mathbf{E}_M(p)) = \mathbf{E}_M(p)^n(*) = H^{n,p}(*; \underline{M}), \quad n \in \mathbb{Z}. \tag{1.11}$$

In §2 we translate operations that occur on $\mathbf{H}\underline{R}$-modules to the non-equivariant $\mathbf{H}R^{C_2}$-modules $\mathbf{E}_M(p)$. The cases where $M = \mathbb{Z}$ or $M = \mathbb{F}_2$ are of particular interest, and from these constructions we highlight various morphisms of $\mathbf{H}\mathbb{Z}$-modules

$$\mathscr{R}_p \colon \mathbf{E}_{\mathbb{Z}}(p) \to \mathbf{H}\mathbb{Z} \qquad \text{(restriction functor)} \tag{1.12}$$

$$\boldsymbol{\varrho}\colon \mathbf{E}_{\mathbb{Z}}(p) \to \mathbf{E}_{\mathbb{F}_2}(p) \qquad \text{(change of coefficients)} \tag{1.13}$$

$$\mu_{p,q}\colon \mathbf{E}_{\mathbb{Z}}(p) \wedge_{\mathbf{H}\mathbb{Z}} \mathbf{E}_{\mathbb{Z}}(q) \to \mathbf{E}_{\mathbb{Z}}(p+q) \qquad \text{(pairings)} \tag{1.14}$$

and morphisms of $\mathbf{H}\mathbb{F}_2$-modules

$$\mathscr{R}_p\colon \mathbf{E}_{\mathbb{F}_2}(p) \to \mathbf{H}\mathbb{F}_2 \qquad \text{(restriction functor)} \tag{1.15}$$

$$\bar{\mu}_{p,q}\colon \mathbf{E}_{\mathbb{F}_2}(p) \wedge_{\mathbf{H}\mathbb{F}_2} \mathbf{E}_{\mathbb{F}_2}(q) \to \mathbf{E}_{\mathbb{F}_2}(p+q) \qquad \text{(pairings)} \tag{1.16}$$

From (1.11), we know that $\pi_{-n}\mathbf{E}_{\mathbb{Z}}(p) = \mathbb{M}^{n,p}$, and $\pi_{-n}\mathbf{E}_{\mathbb{F}_2}(p) = \mathbb{M}_2^{n,p}$. Therefore, using the equivalence between the derived category $\mathscr{D}_R$ of R-modules and the derived category of $\mathscr{D}_{\mathbf{H}R}$ of $\mathbf{H}R$-modules (see [16], [6, IV.§2] and [18, Thm. 5.1.6]), we know that $\mathbf{E}_{\mathbb{Z}}(p)$ and $\mathbf{E}_{\mathbb{F}_2}(p)$ admit decompositions as suspensions of Eilenberg-MacLane spectra

$$\mathbf{f}_p\colon \bigvee_{i\in\mathbb{Z}} \Sigma^{-i}\mathbf{H}\mathbb{M}^{i,p} = \bigvee_{0\le s\le \lfloor p/2\rfloor} \Sigma^{2s-p}\mathbf{H}\mathbb{M}^{p-2s,p} \xrightarrow{\simeq} \mathbf{E}_{\mathbb{Z}}(p) \tag{1.17}$$

and

$$\bar{\mathbf{f}}_p\colon \bigvee_{i\in\mathbb{Z}} \Sigma^{-i}\mathbf{H}\mathbb{M}_2^{i,p} = \bigvee_{0\le k\le p} \Sigma^{k-p}\mathbf{H}\mathbb{M}_2^{p-k,p} \xrightarrow{\simeq} \mathbf{E}_{\mathbb{F}_2}(p). \tag{1.18}$$

It follows that for a space X one has natural decompositions

$$\mathbf{E}^n_{\mathbb{Z}}(p)(X) \cong \bigoplus_{0\le s\le \lfloor p/2\rfloor} H^{n-p+2s}_{\text{sing}}(X, \mathbb{M}^{p-2s,p}),$$
$$\mathbf{E}^n_{\mathbb{F}_2}(p)(X) \cong \bigoplus_{0\le k\le p} H^{n-p+k}_{\text{sing}}(X; \mathbb{M}_2^{p-k,p})$$

which, together with the presentations of $\mathbb{M}^{*,*}$ and $\mathbb{M}_2^{*,*}$ in Notation 1.1, give rise to (1.4) and (1.6).

The following properties of the cohomology theories $\mathbf{E}^m_{\mathbb{Z}}(p)$ and $\mathbf{E}^m_{\mathbb{F}_2}(p)$ are shown in Corollary 2.30, Corollary 2.33 and Proposition 2.35. Here we let $\delta\colon H^*_{\text{sing}}(X;\mathbb{F}_2) \to H^{*+1}_{\text{sing}}(X;\mathbb{Z})$ denote the usual Bockstein homomorphism, and recall that $\delta(\gamma) = 0$ when γ is the reduction mod 2 of an integral class.

Properties: *Given*

$$\alpha \in \mathbf{E}^m_{\mathbb{Z}}(p)(X),\ \alpha' \in \mathbf{E}^n_{\mathbb{Z}}(q)(X),\ \beta \in \mathbf{E}^m_{\mathbb{F}_2}(p)(X) \ \text{ and } \beta' \in \mathbf{E}^n_{\mathbb{F}_2}(q)(X),$$

denote by $\alpha \star \alpha' \in \mathbf{E}_{\mathbb{Z}}^{m+n}(p+q)(X)$ *and* $\beta \star \beta' \in \mathbf{E}_{\mathbb{F}_2}^{m+n}(p+q)(X)$ *the multiplication maps induced by* $\mu_{p,q}$ *with* $\mathbb{Z}$ *and* $\mathbb{F}_2$ *coefficients, respectively. Then:*

$$\begin{aligned}
\mathscr{R}_p(\alpha) &= \mathscr{R}_p\left(\sum_{r=0}^{\lfloor p/2 \rfloor} \alpha_r \cdot a^{p-2r} u^a\right) = \begin{Bmatrix} \alpha_{p/2} & \text{, if } p \text{ is even} \\ \delta\left(\alpha_{\lfloor p/2 \rfloor}\right) & \text{, if } p \text{ is odd} \end{Bmatrix} \in H^n_{\mathrm{sing}}(X;\mathbb{Z}), \\
\mathscr{R}_p(\beta) &= \mathscr{R}_p\left(\sum_{k=0}^{p} \beta_k \cdot \rho^{p-k}\tau^k\right) = \beta_p \in H^n_{sing}(X;\mathbb{F}_2), \\
\boldsymbol{\varrho}(\alpha) &= \sum_{0 \le r < p/2} \left\{\alpha_r \cdot \rho + \mathrm{Sq}^1(\alpha_r)\cdot\tau\right\} \rho^{p-2r-1}\tau^{2r} + \bar{\alpha}_{\frac{p}{2}} \cdot \tau^p \in \mathbf{E}_{\mathbb{F}_2}(p)^n(X), \\
\alpha \star \alpha' &= \sum_{k=0}^{\frac{p+q}{2}} \left\{ \sum_{r+s=k} (\alpha_r \alpha'_s) + \sum_{r+s=k-1} \delta(\bar{\alpha}_r)\delta(\bar{\alpha}'_s)\right\} (a^{p+q-2k}u^k) + (\alpha_{\frac{p}{2}}\alpha'_{\frac{q}{2}}) \cdot u^{\frac{p+q}{2}} \\
&\quad \in \mathbf{E}_{\mathbb{Z}}(p+q)^{m+n}(X), \\
\beta \star \beta' &= \sum_{k=0}^{p+q} \sum_{r+s=k} (\beta_r\beta'_s)\cdot \rho^{p+q-k}\tau^k \in \mathbf{E}_{\mathbb{F}_2}(p+q)^{m+n}(X),
\end{aligned}$$

where, by convention, $\alpha_{p/2} = 0$ *when* p *is odd and* $\bar{\alpha}_{p/2}$ *denotes the* mod 2*-reduction of* $\alpha_{p/2}$*.*

The paper is organized as follows. In Section 2 we introduce the spectra classifying the cohomology theories $\mathbf{E}^m_{\mathbb{Z}}(p)$ and $\mathbf{E}^m_{\mathbb{F}_2}(p)$ and establish their properties. In Section 3 we apply these results to the computation of the restriction map $H^{*,*}(\mathrm{BU}_1;\underline{\mathbb{F}_2}) \to H^{*,*}(\mathrm{BO}_1;\underline{\mathbb{F}_2})$. We also show how Steenrod squares can be defined using ordinary C_2-equivariant cohomology. Finally, in Section 3.2 we compute $\iota^*_{\mathbb{R}}\colon H^{*,*}(\mathrm{BU}_n;\underline{\mathbb{Z}}) \to H^{*,*}(\mathrm{BO}_n;\underline{\mathbb{Z}})$, describe Poincaré duality for compact Real manifolds and prove the restriction formula (1.8) for Real submanifolds.

The original motivation for the work in this paper came from the questions posed in the preliminary versions of [10] and also by the results in [14] on the representation of Steenrod squares by algebro-geometric constructions.

2. The spectra $\mathbf{E}_{\mathbb{Z}}(p)$ and $\mathbf{E}_{\mathbb{F}_2}(p)$

In this section we discuss the nonequivariant cohomology theories $\mathbf{E}_{\mathbb{Z}}(p)(-)$ and $\mathbf{E}_{\mathbb{F}_2}(p)(-)$ defined in (1.10) by studying their representing spectra $\mathbf{E}_{\mathbb{Z}}(p)$ and $\mathbf{E}_{\mathbb{F}_2}(p)$. We start by recalling notation and basic definitions pertaining to C_2-spectra and reviewing the construction of $\mathbf{E}_{\mathbb{Z}}(p)$ and $\mathbf{E}_{\mathbb{F}_2}(p)$.

We work in the topological category $\mathscr{T}$ of based compactly generated weakly Hausdorff topological spaces and refer to its objects as spaces. Let $\mathscr{T}_{C_2}$ be the category whose objects are spaces equipped with a C_2-action, and morphisms are continuous maps.

Given a C_2-space X, a discrete ring R and an R-module M, let MX be the free R-module generated by X. With the appropriate topology, MX becomes a topological $R[C_2]$-module whose elements are often called 0-*cycles* on X with coefficients in R. Whenever $(X, x_o) \in \mathscr{T}_{C_2}$ is a based C_2-space, we let $M \otimes X$ be the quotient module $MX/M\{x_0\}$ and call its elements the *reduced* 0-*cycles* on X. Observe that $MX \cong M \otimes X_+$.

If M has the trivial C_2-action the fixed submodule $(M \otimes X)^{C_2}$ has two natural submodules: the 0-cycles on fixed points $M \otimes X^{C_2}$ and the submodule of *averaged cycles*:

$$(M \otimes X)^{av} := \left\{ \sum_{g \in C_2} g \cdot z \mid z \in M \otimes X \right\}. \tag{2.1}$$

Consider the countably infinite sum of the (left) regular representation $\mathcal{U}^{\text{all}} := \mathbb{R}[C_2] \oplus \mathbb{R}[C_2] \oplus \cdots$ with its usual C_2-inner product. We identify $\mathcal{U}^{\text{all}}$ with $\mathbb{C}^\infty = \mathbb{C} \oplus \mathbb{C} \oplus \cdots$ under the complex conjugation action. Using this identification let $\mathcal{U} := \mathbb{R}^\infty \subset \mathcal{U}^{\text{all}}$. In the language of [13], $\mathcal{U}^{\text{all}}$ is a complete C_2-universe and $\mathcal{U}$ is a trivial C_2-universe.

Denote by $\mathscr{I}_{C_2}$ the collection of all finite dimensional real C_2-inner product subspaces of $\mathcal{U}^{\text{all}}$. These are the objects of the topological C_2-category – also denoted $\mathscr{I}_{C_2}$ – whose C_2-space of morphisms $\mathscr{I}_{C_2}(V, W)$ are the isometric isomorphisms under the conjugation action. Similarly, the topological category obtained by considering inner product subspaces of $\mathcal{U}$ is denoted $\mathscr{I}$. An $\mathscr{I}_{C_2}$-*space* is a continuous G-functor $X : \mathscr{I}_{C_2} \to \mathscr{T}_{C_2}$. Denote the category of $\mathscr{I}_{C_2}$-spaces and natural transformations by $\mathscr{I}_{C_2}\mathscr{T}$. The corresponding nonequivariant notions of $\mathscr{I}$-functor and category $\mathscr{I}\mathscr{T}$ of $\mathscr{I}$-spaces are defined similarly.

Example 2.1. The *sphere* $\mathscr{I}$-*functor* $\mathbb{S} : \mathscr{I} \to \mathscr{T}_{C_2}$ sends V to $S^V = V \cup \{\infty\}$, with ∞ as its base-point. In the same fashion, the *sphere* $\mathscr{I}_{C_2}$-*functor* $\mathbb{S}_{C_2} : \mathscr{I}_{C_2} \to \mathscr{T}_{C_2}$ sends V to $S^V = V \cup \{\infty\}$.

Definition 2.2. *Let* X, Y *be* $\mathscr{I}_{C_2}$-*functors.*

1. *The* external smash product $X \,\overline{\wedge}\, Y := \wedge \circ (X \times Y) : \mathscr{I}_{C_2} \times \mathscr{I}_{C_2} \to \mathscr{T}_{C_2}$ *is the functor defined by* $(X \,\overline{\wedge}\, Y)(V, W) = X(V) \wedge Y(W)$.

2. *An* orthogonal C_2-spectrum *is an $\mathscr{I}_{C_2}$-space $X\colon \mathscr{I}_{C_2} \to \mathscr{T}_{C_2}$ together with a structural C_2-map $\sigma\colon X \,\overline{\wedge}\, \mathbb{S}_{C_2} \to X \circ \oplus$ satisfying unit and associativity properties. The category of orthogonal C_2-spectra is denoted by $\mathscr{I}_{C_2}\mathscr{S}$.*

Definition 2.3. *A (nonequivariant)* orthogonal spectrum *is an $\mathscr{I}$-space $X\colon \mathscr{I} \to \mathscr{T}$ together with a structural map $\sigma\colon X \,\overline{\wedge}\, \mathbb{S} \to X \circ \oplus$ satisfying unit and associativity properties. The value of X on $\mathbb{R}^n$ is usually denoted X_n instead of $X(\mathbb{R}^n)$. The category of orthogonal spectra is denoted by $\mathscr{I}\mathscr{S}$.*

Example 2.4. If N is an abelian group, the assignment $n \mapsto N \otimes S^n$ defines an orthogonal spectrum $\mathbf{H}N$ whose structural map $\mathbf{H}N \,\overline{\wedge}\, \mathbb{S} \to \mathbf{H}N \circ \oplus$ on a pair (n,m) is the map $(N \otimes S^n) \wedge S^m \longrightarrow N \otimes S^{n+m}$ induced by the smash product of spheres: $(\sum_i n_i x_i) \wedge y \mapsto \sum_i (n_i x_i \wedge y)$, for $n_i \in \mathbb{Z}$, $x_i \in S^n$ and $y \in S^m$. It is called the *Eilenberg-MacLane spectrum* associated to N.

Example 2.5. If M is a $\mathbb{Z}[C_2]$-module, the assignment $V \mapsto M \otimes S^V$ defines an orthogonal C_2-spectrum $\mathbf{H}\underline{M}$ whose structural map $\mathbf{H}\underline{M} \,\overline{\wedge}\, \mathbb{S}_{C_2} \to \mathbf{H}\underline{M} \circ \oplus$ on a pair (V,W) is the C_2-map $\left(M \otimes S^V\right) \wedge S^W \longrightarrow M \otimes S^{V\oplus W}$ coming from the smash product of spheres. It is shown in [3] that $\mathbf{H}\underline{M}$ is an *equivariant Eilenberg-MacLane* spectrum representing *ordinary $RO(C_2)$-graded equivariant cohomology* with coefficients in the Mackey functor $\underline{M}$ associated to M.

Definition 2.6. *Let $(-)^{C_2}\colon \mathscr{I}_{C_2}\mathscr{S} \to \mathscr{I}\mathscr{S}$ denote the C_2-*fixed-point functor*, from orthogonal C_2-spectra to non-equivariant spectra [13] and let $p \geq 0$. Then given a $\mathbb{Z}[C_2]$-module M, denote by $\mathbf{E}_M(p)$ the nonequivariant spectrum $\left(\varSigma^{p\cdot\sigma - p\cdot\mathbf{1}}\mathbf{H}\underline{M}\right)^{C_2}$.*

Remark 2.7. The fixed point spectrum $\mathbf{E}_M(p)$ can also be computed by first replacing the suspension $\varSigma^{p\cdot\sigma-p\cdot\mathbf{1}}\mathbf{H}\underline{M}$ by the equivalent Ω-C_2-spectrum $n \mapsto \Omega^{p\cdot\mathbf{1}}\mathbf{H}\underline{M}(n + p\cdot\sigma)$ and then taking levelwise fixed points:

$$\begin{aligned}\mathbf{E}_M(p)_n &= \left(\Omega^{p\cdot\mathbf{1}}\mathbf{H}\underline{M}(p\cdot\sigma + n\cdot\mathbf{1})\right)^{C_2} = \Omega^p\left(\mathbf{H}\underline{M}(p\cdot\sigma + n\cdot\mathbf{1})^{C_2}\right)\\ &= \Omega^p\left\{M \otimes S^{p+n,p}\right\}^{C_2}.\end{aligned}$$

Example 2.8. If $p = 0$, then $\mathbf{E}_M(0) = \mathbf{H}M^G$ is the ordinary Eilenberg-MacLane spectrum associated to the group M^G. Indeed, we have

$$\mathbf{E}_M(0)_n = (\mathbf{H}\underline{M}(n\cdot\mathbf{1}))^{C_2} = \left\{M \otimes S^{n,0}\right\}^{C_2} = M^{C_2} \otimes S^n.$$

Remark 2.9. Unravelling the definitions one sees that

$$\pi_{-n}\left(\mathbf{E}_{\mathbb{Z}}(p)\right) = \mathbb{M}^{n,p} \quad \text{and} \quad \pi_{-n}\left(\mathbf{E}_{\mathbb{F}_2}(p)\right) = \mathbb{M}_2^{n,p}.$$

2.1. First properties of $\mathbf{E}_{\mathbb{Z}}(p)$ and $\mathbf{E}_{\mathbb{F}_2}(p)$

2.1.1. Restriction functors At the level of spectra, the *forgetful* or group *restriction* functor $\mathscr{R}_p \colon \mathbf{E}_M(p) \to \mathbf{H}M$ is given by restriction to the trivial subgroup, that is, by forgetting the C_2-structure. For the spectrum $\mathbf{E}_M(p)$ this translates to the composite $\Omega^p \left\{M \otimes S^{n+p,p}\right\}^{C_2} \subset \Omega^p \left\{M \otimes S^{n+p}\right\} \cong M \otimes S^n$.

2.1.2. Pairings and mod 2 reduction Given $p, q \geq 0$ and taking $M = \mathbb{Z}$ or $M = \mathbb{F}_2$, there is a map of $(\mathscr{I}_{C_2} \times \mathscr{I}_{C_2})$-spaces

$$\mathbf{E}_M(p) \barwedge \mathbf{E}_M(q) \longrightarrow \mathbf{E}_M(p+q) \circ \oplus \tag{2.2}$$

induced by the ring product $M \times M \to M$ and by the composite

$$\begin{aligned}(S^{p\cdot\sigma} \wedge S^m) \times (S^{q\cdot\sigma} \wedge S^n) &\to S^{p\cdot\sigma} \wedge S^m \wedge S^{q\cdot\sigma} \wedge S^n \\ &\xrightarrow{\chi_{m,q\cdot\sigma}} S^{p\cdot\sigma} \wedge S^{q\cdot\sigma} \wedge S^m \wedge S^n = S^{(p+q)\cdot\sigma} \wedge S^{m+n},\end{aligned} \tag{2.3}$$

where $\chi_{m,q\cdot\sigma}$ is the transposition $S^m \wedge S^{q\cdot\sigma} \cong S^{q\cdot\sigma} \wedge S^m$. A universality argument [17, (1.6)] shows that (2.2) induces a pairing of orthogonal spectra

$$\mu_{p,q} \colon \mathbf{E}_M(p) \wedge \mathbf{E}_M(q) \to \mathbf{E}_M(p+q). \tag{2.4}$$

When $q = 0$ the pairing $\mathbf{E}_M(p) \wedge \mathbf{H}M \to \mathbf{E}_M(p)$ turns $\mathbf{E}_M(p)$ into an $\mathbf{H}M$-module and (2.4) induces $\mathbf{H}M$-module morphisms

$$\mu_{p,q} \colon \mathbf{E}_M(p) \wedge_{\mathbf{H}M} \mathbf{E}_M(q) \to \mathbf{E}_M(p+q). \tag{2.5}$$

The spectra $\mathbf{E}_{\mathbb{Z}}(p)$ and $\mathbf{E}_{\mathbb{F}_2}(p)$, for $p \geq 0$, are related by an obvious mod 2 reduction map of spectra compatible with the pairings (2.5).

Definition 2.10. *The ring epimorphism $\boldsymbol{\varrho} \colon \mathbb{Z} \to \mathbb{F}_2$ yields $H\mathbb{Z}$-module maps $\mathbf{E}_{\mathbb{Z}}(p) \to \mathbf{E}_{\mathbb{F}_2}(p)$, also denoted $\boldsymbol{\varrho}$, that fit into a commutative diagram of* $\mathbf{H}\mathbb{Z}$*-modules*

$$\begin{tikzcd} \mathbf{E}_{\mathbb{Z}}(p) \wedge_{\mathbf{H}\mathbb{Z}} \mathbf{E}_{\mathbb{Z}}(q) \arrow[r, "\mu_{p,q}"] \arrow[d, "\boldsymbol{\varrho}\wedge\boldsymbol{\varrho}"'] & \mathbf{E}_{\mathbb{Z}}(p+q) \arrow[d, "\boldsymbol{\varrho}"] \\ \mathbf{E}_{\mathbb{F}_2}(p) \wedge_{\mathbf{H}\mathbb{F}_2} \mathbf{E}_{\mathbb{F}_2}(q) \arrow[r, "\bar{\mu}_{p,q}"'] & \mathbf{E}_{\mathbb{F}_2}(p+q). \end{tikzcd} \tag{2.6}$$

2.1.3. The shift maps Given $p \geq q \geq 0$ and M as above, the inclusions $S^n = \{S^{q\cdot\sigma+n}\}^{C_2} \hookrightarrow S^{q\cdot\sigma+n}$ induce a morphism

$$a_{q\sigma}\colon \mathbb{S} \to \Sigma^q \mathbf{E}_M(q) \tag{2.7}$$

which, in turn, gives an $\mathbf{H}M$-module map

$$a_{q\sigma *}\colon \Sigma^{-q}\mathbf{H}M \to \mathbf{E}_M(q). \tag{2.8}$$

Definition 2.11. *Given $p, q \geq 0$ and M as above, define*

$$\hat{a}^q\colon \Sigma^{-q}\mathbf{E}_M(p) \longrightarrow \mathbf{E}_M(p+q)$$

as the map of $\mathbf{H}M$-modules given by the composite

$$\begin{tikzcd}
\Sigma^{-q}\mathbf{E}_M(p) = \mathbf{E}_M(p) \wedge_{\mathbf{H}M} \Sigma^{-q}\mathbf{H}M \arrow[r, "1\wedge a_{q\sigma *}"] \arrow[dr, bend right, "\hat{a}^q"'] & \mathbf{E}_M(p) \wedge_{\mathbf{H}M} \mathbf{E}_M(q) \arrow[d, "\mu_{p,q}"] \\
& \mathbf{E}_M(p+q).
\end{tikzcd}$$

Remark 2.12. Note that $a_{q\sigma *}$ in (2.8) is $\hat{a}^q\colon \Sigma^{-q}\mathbf{E}(0) \to \mathbf{E}_M(q)$.

Notation 2.13. To distinguish from the case $M = \mathbb{Z}$, when $M = \mathbb{F}_2$ we will write $\hat{\rho}^q$ instead of $\hat{a}^q$.

Remark 2.14. It follows that the map $\mathbb{M}^{n-q,p} \to \mathbb{M}^{n,p+q}$ represented by $\hat{a}_{q\sigma}$ is multiplication by a^q. Similarly the map $\mathbb{M}_2^{n-q,p} \to \mathbb{M}_2^{n,p+q}$ is multiplication by ρ^q. The maps (2.7) yield generators for $\pi_{-1}(\mathbf{E}_{\mathbb{Z}}(1))$ and $\pi_{-1}(\mathbf{E}_{\mathbb{F}_2}(1))$ that will be useful later on.

Proposition 2.15. *For $p = 1$ the maps (2.7) induce a commutative diagram*

$$\begin{tikzcd}
& \mathbb{S}^{-1} \arrow[dl, "\tilde{a}"'] \arrow[dr, "\tilde{\rho}"] & \\
\mathbf{E}_{\mathbb{Z}}(1) \arrow[rr, "\boldsymbol{\varrho}"'] & & \mathbf{E}_{\mathbb{F}_2}(1),
\end{tikzcd}$$

where $\tilde{a}$ and $\tilde{\rho}$ represent, respectively, the generators

$$a \in \pi_{-1}(\mathbf{E}_{\mathbb{Z}}(1)) = \mathbb{M}^{1,1} \cong \mathbb{F}_2 \quad \textit{and} \quad \rho \in \pi_{-1}(\mathbf{E}_{\mathbb{F}_2}(1)) = \mathbb{M}_2^{1,1} \cong \mathbb{F}_2.$$

Abusing notation we will also use a and ρ to denote $\tilde{a}$ and $\tilde{\rho}$, respectively.

Remark 2.16. The proposition, in particular, expresses the fact that the *change of rings* map $\boldsymbol{\varrho}\colon \mathbf{E}_{\mathbb{Z}}(1) \to \mathbf{E}_{\mathbb{F}_2}(1)$ induces an isomorphism $\mathbb{M}^{1,1} \xrightarrow{\cong} \mathbb{M}_2^{1,1}$. As for the maps $\hat{a}^q$ and $\hat{\rho}^q$ these fit in a commutative diagram

$$\begin{tikzcd}
\Sigma^{-q}\mathbf{H}\mathbb{Z} \arrow[r, "\hat{a}^q"] \arrow[d, "\Sigma^{-q}\boldsymbol{\varrho}"'] & \mathbf{E}_{\mathbb{Z}}(q) \arrow[d, "\boldsymbol{\varrho}"] \\
\Sigma^{-q}\mathbf{H}\mathbb{F}_2 \arrow[r, "\hat{\rho}^q"'] & \mathbf{E}_{\mathbb{F}_2}(q).
\end{tikzcd}$$

Definition 2.17. *Let* $\mathrm{u}\colon \mathbb{S} \to \mathbf{E}_{\mathbb{Z}}(2)$ *be a representative for* $u \in \mathbb{M}^{0,2}$. *Set* $\mathrm{u}^{q+1} := \mu_{q,2} \circ (\mathrm{u}^q \wedge \mathrm{u})\colon \mathbb{S} = \mathbb{S} \wedge \mathbb{S} \to \mathbf{E}_{\mathbb{Z}}(q+2)$ *(where* $\mathrm{u}^1 = \mathrm{u}$*).*

Note that, for each $q \geq 0$, u^q represents $u^q \in \mathbb{M}^{0,2q}$.

Definition 2.18. *For each* $q \geq 0$*, denote by* $\mathrm{u}^q_*\colon \mathbf{H}\mathbb{Z} \longrightarrow \mathbf{E}_{\mathbb{Z}}(2q)$ *the* $\mathbf{H}\mathbb{Z}$*-module map determined by* u^q*. Also, denote by* $\hat{u}^q$ *the composite*

(2.9)

$$\begin{tikzcd}
\mathbf{E}_M(p) = \mathbf{E}_M(p) \wedge_{\mathbf{H}\mathbb{Z}} \mathbf{H}\mathbb{Z} \arrow[r, "1\wedge \mathrm{u}^q_*"] \arrow[rr, bend right, "\hat{u}^q"'] & \mathbf{E}_M(p) \wedge_{\mathbf{H}\mathbb{Z}} \mathbf{E}_{\mathbb{Z}}(2q) \arrow[r, "\mu_{p,2q}"] & \mathbf{E}_M(p+2q).
\end{tikzcd}$$

It follows that $\mathbb{M}^{n,p} \to \mathbb{M}^{n,p+2q}$ represented by $\hat{u}_q$ is multiplication by u^q.

2.2. Decomposing $\mathbf{E}_{\mathbb{Z}}(p)$ and $\mathbf{E}_{\mathbb{F}_2}(p)$

In this section we construct explicit decompositions of $\mathbf{E}_{\mathbb{Z}}(p)$ and $\mathbf{E}_{\mathbb{F}_2}(p)$ as wedges of Eilenberg-MacLane spectra, utilizing particular geometric models that are suitable for our applications and bear a direct relationship with algebraic geometric constructions.

2.2.1. Fixed and averaged cycles Here we use the model for $\mathbf{H}\underline{M}$ described in Example 2.5 to exhibit representatives for the generators of $\mathbb{M}_+$ and $\mathbb{M}_{2+}$, using the subgroup of averaged cycles of (2.1), as follows. For a finite C_2-CW-complex X and $R = \mathbb{Z}$ or $\mathbb{F}_2$, the following exact sequence is also a fibration functorial on X.

$$0 \to (R \otimes X)^{\mathrm{av}} \xrightarrow{\mathrm{av}} \{R \otimes X\}^{C_2} \xrightarrow{\chi} \mathbb{F}_2 \otimes X^{C_2} \to 0. \tag{2.10}$$

When $R = \mathbb{Z}$, it turns out that $(\mathbb{Z} \otimes X)^{\mathrm{av}} \cong \mathbb{Z} \otimes (X/C_2)$ and the map av is simply the $\mathbb{Z}$-linear extension of the inclusion $X/C_2 \to (\mathbb{Z} \otimes X)^{C_2}$ that sends $[x] \in X/C_2$ to the orbit $x + \sigma \cdot x$.

When $R = \mathbb{F}_2$, then $(\mathbb{F}_2 \otimes X)^{\mathrm{av}} \cong \mathbb{F}_2 \otimes \left(\{X/C_2\}/X^{C_2}\right)$. Indeed, the subgroup $\mathbb{F}_2 \otimes X^{C_2}$ is precisely the kernel of the $\mathbb{F}_2$-linear map $\mathbb{F}_2 \otimes (X/C_2) \to (\mathbb{F}_2 \otimes X)^{C_2}$ given by $[x] \mapsto x + \sigma \cdot x$. It follows that one obtains an inclusion $\mathbb{F}_2 \otimes \{X/C_2\}/\mathbb{F}_2 \otimes (X^{C_2}) \cong \mathbb{F}_2 \otimes (\{X/C_2\}/X^{C_2}) \to (\mathbb{F}_2 \otimes X)^{C_2}$ that identifies $\mathbb{F}_2 \otimes (\{X/C_2\}/X^{C_2})$ with $(\mathbb{F}_2 \otimes X)^{\mathrm{av}}$.

Remark 2.19. In the case $R = \mathbb{F}_2$ we denote the the map χ in (2.10) by $\bar{\chi}$ and write the topological short exact sequence of 0-cycles as

$$0 \to \mathbb{F}_2 \otimes \left(\{X/C_2\}/X^{C_2}\right) \xrightarrow{\mathrm{av}} (\mathbb{F}_2 \otimes X)^{C_2} \xrightarrow{\bar{\chi}} \mathbb{F}_2 \otimes X^{C_2} \to 0. \tag{2.11}$$

Note that this sequence is naturally split, for the inclusion $\mathbb{F}_2 \otimes X^{C_2} \hookrightarrow (\mathbb{F}_2 \otimes X)^{C_2}$ is a section of $\bar{\chi}$.

2.2.2. The decomposition of $\mathbf{E}_{\mathbb{Z}}(p)$ By Remark 2.9, when $p = 1$, we have $\mathbf{E}_{\mathbb{Z}}(1) \cong \Sigma^{-1}\mathbf{H}\mathbb{F}_2$ since the only nonzero homotopy group is $\mathbb{F}_2$. Also $\mathbf{E}_{\mathbb{F}_2}(1) \cong \Sigma^{-1}\mathbf{H}\mathbb{F}_2 \vee \mathbf{H}\mathbb{F}_2$. Using the projections χ and $\bar{\chi}$ of (2.10) and (2.11) we construct an equivalence $\mathbf{E}_{\mathbb{Z}}(1) \to \Sigma^{-1}\mathbf{H}\mathbb{F}_2$, and a right homotopy inverse to $\hat{\rho}\colon \Sigma^{-1}\mathbf{H}\mathbb{F}_2 \to \mathbf{E}_{\mathbb{F}_2}(1)$.

Corollary 2.20. *For each $p \geq 1$, there are $\mathbf{H}\mathbb{Z}$-module maps $\boldsymbol{\chi}\colon \mathbf{E}_{\mathbb{Z}}(p) \to \Sigma^{-p}\mathbf{H}\mathbb{F}_2$ and $\bar{\boldsymbol{\chi}}\colon \mathbf{E}_{\mathbb{F}_2}(p) \to \Sigma^{-p}\mathbf{H}\mathbb{F}_2$ that fit in a homotopy commutative diagram of $\mathbf{H}\mathbb{Z}$-modules:*

$$\begin{tikzcd}
\Sigma^{-p}\mathbf{H}\mathbb{Z} \arrow[rr, "\hat{a}^p"] \arrow[d, "\boldsymbol{\varrho}"'] & & \mathbf{E}_{\mathbb{Z}}(p) \arrow[d, "\boldsymbol{\varrho}"] \arrow[dll, "\boldsymbol{\chi}_p"'] \\
\Sigma^{-p}\mathbf{H}\mathbb{F}_2 \arrow[rr, "\hat{\rho}^p"] & & \mathbf{E}_{\mathbb{F}_2}(p) \arrow[ll, bend left, "\bar{\boldsymbol{\chi}}_p"]
\end{tikzcd}$$

In particular, $\boldsymbol{\chi}_1$ is an equivalence. Furthermore, $\bar{\boldsymbol{\chi}}_1$ is a right homotopy inverse to $\hat{\rho}$.

Proof. Consider the projection $\chi_{p,n}\colon \{\mathbb{Z} \otimes S^{p+n,p}\}^{C_2} \to \mathbb{F}_2 \otimes S^n$ coming from (2.11). It defines an $\mathbf{H}\mathbb{Z}$-module map $\boldsymbol{\chi}_p\colon \mathbf{E}_{\mathbb{Z}}(p) \to \Sigma^{-p}\mathbf{H}\mathbb{F}_2$ as in the diagram. Similarly, the projection $\bar{\chi}_{p,n}\colon \{\mathbb{F}_2 \otimes S^{p+n,p}\}^{C_2} \to \mathbb{F}_2 \otimes S^n$ defines $\bar{\boldsymbol{\chi}}_p\colon \mathbf{E}_{\mathbb{F}_2}(p) \to \Sigma^{-p}\mathbf{H}\mathbb{F}_2$ with the desired properties.

Unravelling definitions, it follows that both $\boldsymbol{\chi}_1$ and $\bar{\boldsymbol{\chi}}$ induce isomorphisms at the level of π_{-1} and that $\bar{\boldsymbol{\chi}} \circ \hat{\rho}$ is the identity. Hence the statements about $\boldsymbol{\chi}_1$ and $\bar{\boldsymbol{\chi}}$ follow. $\square$

Before constructing the proposed decompositions we use the maps $\hat{a}^r$ and $\hat{u}^s$ to relate the spectra $\mathbf{E}_{\mathbb{Z}}(p)$ for different values of p.

Proposition 2.21. *Fix $p \geq 0$. Then*

1. *The composition $\mathbf{H}\mathbb{Z} = \mathbf{E}_{\mathbb{Z}}(0) \xrightarrow{\hat{u}} \mathbf{E}_{\mathbb{Z}}(2) \xrightarrow{\mathscr{R}_2} \mathbf{H}\mathbb{Z}$ is an equivalence.*
2. *The following diagram commutes in the derived category of $\mathbf{H}\mathbb{Z}$-modules.*

$$\begin{tikzcd} \mathbf{E}_{\mathbb{Z}}(p) \arrow[rr, "\hat{u}"] \arrow[dr, "\mathscr{R}_p"'] & & \mathbf{E}_{\mathbb{Z}}(p+2) \arrow[dl, "\mathscr{R}_{p+2}"] \\ & \mathbf{H}\mathbb{Z}. & \end{tikzcd}$$

3. *In the level of homotopy groups, the homomorphisms induced by the maps $\hat{a}^q$ in Definition 2.11 and $\hat{u}^q$ in (2.9), for each $q > 0$, correspond to the multiplication maps by powers of the generators $a \in \mathbb{M}^{1,1}$ and $u \in \mathbb{M}^{0,2}$. As a consequence, for every p, the map*

$$\hat{a}^{p+1} \vee \hat{u} \ : \quad \Sigma^{-p-1}\mathbf{E}_{\mathbb{Z}}(1) \vee \mathbf{E}_{\mathbb{Z}}(p) \longrightarrow \mathbf{E}_{\mathbb{Z}}(p+2),$$

is an isomorphism in the homotopy category of $\mathbf{H}\mathbb{Z}$-modules.

Proof. The first two statements follow from the properties of the maps $\hat{u}^q$, the generator $u \in \mathbb{M}^{0,2}$ and the restriction functor (see Definition 2.18 and Notation 1.1).

The last assertion follows from the fact that the pairings (2.2) come from the equivariant ring spectra structure on $\mathbf{H}\underline{\mathbb{Z}}$, and the ring structure of $\mathbb{M}^{*,\bullet} = H^{*,\bullet}(*;\underline{\mathbb{Z}})$, where the multiplication maps below are isomorphisms

$$u\cdot \colon \mathbb{M}^{n,p} \to \mathbb{M}^{n,p+2},\ n,p \geq 0, \quad \text{and} \quad a\cdot \colon \mathbb{M}^{n,p} \to \mathbb{M}^{n+1,p+1},\ n \geq 1, p \geq 0.$$

$\square$

The next result gives an explicit decomposition of $\mathbf{E}_{\mathbb{Z}}(p)$ as a wedge of Eilenberg-MacLane spectra.

Corollary 2.22. *One has an equivalence of $\mathbf{H}\mathbb{Z}$-modules*

$$\mathbf{f}_p \colon \left(\bigvee_{0 \leq s < p/2} \Sigma^{2s+1-p}\mathbf{E}_{\mathbb{Z}}(1) \right) \vee \bigvee_{s=p/2} \mathbf{E}_{\mathbb{Z}}(0) \xrightarrow{\vee_s(\hat{a}^{p-1-2s}\hat{u}^s) \vee \hat{u}^{\frac{p}{2}}} \mathbf{E}_{\mathbb{Z}}(p),$$

where the summand $\bigvee_{s=p/2} \mathbf{E}_{\mathbb{Z}}(0)$ is to be understood as 0 whenever p is odd.

Remark 2.23. In the case $p = 2$ one has maps

$$\hat{u}\colon \mathbf{E}_{\mathbb{Z}}(0) \to \mathbf{E}_{\mathbb{Z}}(2) \quad \text{and} \quad \hat{a}\colon \Sigma^{-1}\mathbf{E}_{\mathbb{Z}}(1) \to \mathbf{E}_{\mathbb{Z}}(2),$$

giving $\mathbf{f}_2 = \hat{a} \vee \hat{u}$. We also have $\boldsymbol{\chi}_2\colon \mathbf{E}_{\mathbb{Z}}(2) \to \Sigma^{-2}\mathbf{H}\mathbb{F}_2 \simeq \Sigma^{-1}\mathbf{E}_{\mathbb{Z}}(1)$ and $\mathscr{R}_2\colon \mathbf{E}_{\mathbb{Z}}(2) \to \mathbf{H}\mathbb{Z} = \mathbf{E}_{\mathbb{Z}}(0)$. It follows from Corollary 2.20 and Proposition 2.21 that the map $\boldsymbol{\chi}_2 \vee \mathscr{R}_2\colon \mathbf{E}_{\mathbb{Z}}(2) \to \Sigma^{-2}\mathbf{H}\mathbb{F}_2 \vee \mathbf{E}_{\mathbb{Z}}(0)$ is the inverse of $\mathbf{f}_2$ in the stable homotopy category.

Remark 2.24. Using the equivalence $\tilde{\mathbf{f}}_1\colon \Sigma^{-1}\mathbf{H}\mathbb{F}_2 \to \mathbf{E}_{\mathbb{Z}}(1)$, inverse to $\boldsymbol{\chi}_1$ in the homotopy category, we can rewrite $\mathbf{f}_p$ as

$$\begin{tikzcd}
\bigvee_{0\le s\le p/2} \mathbf{f}_{p,s}\colon \left(\bigvee_{0\le s<p/2} \Sigma^{2s-p}\mathbf{H}\mathbb{F}_2\right) \vee \bigvee_{s=p/2}\mathbf{H}\mathbb{Z} \arrow[r, "\mathbf{f}_p"] \arrow[d, equal] & \mathbf{E}_{\mathbb{Z}}(p), \\
\bigvee_{0\le s\le p/2} \Sigma^{2s-p}\mathbf{H}\mathbb{M}^{p-2s,p} \arrow[ur, "\cong"', "\mathbf{f}_p"] &
\end{tikzcd}$$

where $\mathbf{f}_{p,s} := \left(\hat{a}^{p-1-2s}\hat{u}^s\right) \circ \Sigma^{2s+1-p}(\tilde{\mathbf{f}}_1)$ for $s < p/2$, and $\mathbf{f}_{p,p/2} = \hat{u}^{p/2}$ when p is even.

2.2.3. The decomposition of $\mathbf{E}_{\mathbb{F}_2}(p)$ The decomposition of $\mathbf{E}_{\mathbb{Z}}(p)$ above is determined by the ring $\mathbb{M}_+$. The main difference in decomposing $\mathbf{E}_{\mathbb{F}_2}(p)$ lies existence of the generator $\tau \in \mathbb{M}_2^{0,1} = \pi_0\mathbf{E}_{\mathbb{F}_2}(1)$ satisfying $\tau^2 = \boldsymbol{\varrho}(u)$. Hence, to decompose $\mathbf{E}_{\mathbb{F}_2}(p)$ we first exhibit an explicit representative of τ.

Recall that $S^{1,1}/C_2 \cong S(\mathbb{P}^0) \cong [-1,1]$ and $\{S^{1,1}\}^{C_2} \equiv S^0$. Then identify $(S^{1,1}/C_2)\,/\,\{S^{1,1}\}^{C_2} \cong [-1,1]/\partial[-1,1] \cong S^1$. The "averaging" maps

$$\mathrm{av}\colon S^{1+n} \equiv \left\{\left(S^{1,1} \wedge S^{n,0}\right)/C_2\right\} / \left(S^{1,1}\right)^{C_2} \longrightarrow \left\{\mathbb{F}_2 \otimes S^{1+n,1}\right\}^{C_2}$$

induce a map of spectra $\boldsymbol{\tau}\colon \mathbb{S} \longrightarrow \mathbf{E}_{\mathbb{F}_2}(1)$ that represents the generator $\tau \in \pi_0(\mathbf{E}_{\mathbb{F}_2}(1))$. The element $\boldsymbol{\tau}^{q+1} = \bar{\mu}_{q,1} \circ (\boldsymbol{\tau}^q \wedge \boldsymbol{\tau})$ represents τ^{q+1}, for each $q \geq 0$. One gets a map of $\mathbf{H}\mathbb{F}_2$-modules

$$\tau_*^q\colon \mathbf{H}\mathbb{F}_2 \longrightarrow \mathbf{E}_{\mathbb{F}_2}(q) \tag{2.12}$$

determined by $\boldsymbol{\tau}^q$. We also denote by $\hat{\tau}^q$ the map of $\mathbf{H}\mathbb{F}_2$-modules given by the composition

$$\begin{tikzcd} \mathbf{E}_{\mathbb{F}_2}(p) = \mathbf{E}_{\mathbb{F}_2}(p) \wedge_{\mathbf{H}\mathbb{F}_2} \mathbf{H}\mathbb{F}_2 \arrow[r, "1\wedge\tau_*^q"] \arrow[rr, bend right, "\hat{\tau}^q"'] & \mathbf{E}_{\mathbb{F}_2}(p) \wedge_{\mathbf{H}\mathbb{F}_2} \mathbf{E}_{\mathbb{F}_2}(q) \arrow[r, "\bar{\mu}_{p,q}"] & \mathbf{E}_{\mathbb{F}_2}(p+q) \end{tikzcd}$$

Finally, recall from Notation 2.13 the shift maps $\hat{\rho}^p$.

Proposition 2.25. *Fix $p \geq 0$. Then,*

1. *The composition $\mathbf{H}\mathbb{F}_2 = \mathbf{E}_{\mathbb{F}_2}(0) \xrightarrow{\hat{\tau}} \mathbf{E}_{\mathbb{F}_2}(1) \xrightarrow{\bar{\mathscr{R}}_1} \mathbf{H}\mathbb{F}_2$ is an equivalence.*
2. *The following diagram commutes in $\mathscr{D}_{\mathbb{F}_2}$.*

$$\begin{tikzcd} \mathbf{E}_{\mathbb{F}_2}(p) \arrow[rr, "\hat{\tau}"] \arrow[dr, "\bar{\mathscr{R}}_p"'] & & \mathbf{E}_{\mathbb{F}_2}(p+1) \arrow[dl, "\bar{\mathscr{R}}_{p+1}"] \\ & \mathbf{H}\mathbb{F}_2. & \end{tikzcd}$$

3. *Since in the level of homotopy groups, the homomorphisms induced by $\hat{\rho}$ and $\hat{\tau}$, correspond to multiplication by the generators $\rho \in \mathbb{M}_2^{1,1}$ and $\tau \in \mathbb{M}_2^{0,1}$, the map*

$$\hat{\rho}^{p+1} \vee \hat{\tau} \ : \ \Sigma^{-p-1}\mathbf{H}\mathbb{F}_2 \vee \mathbf{E}_{\mathbb{F}_2}(p) \longrightarrow \mathbf{E}_{\mathbb{F}_2}(p+1)$$

is an equivalence of $\mathbf{H}\mathbb{F}_2$-modules.

Proof. The proof follows the arguments used to prove Proposition 2.21. ☐

As with integer coefficients, we obtain the following decomposition, which is analogous to that of Remark 2.24.

Corollary 2.26. *For each $p \geq 0$, one has a canonical equivalence of $\mathbf{H}\mathbb{F}_2$-modules*

$$\bar{\mathbf{f}}_p = \vee_{0\leq s\leq p}\, \mathbf{f}_{p,s} \colon \ \vee_{0\leq s\leq p}\, \Sigma^{s-p}\mathbf{H}\mathbb{M}_2^{p-s,p} \longrightarrow \mathbf{E}_{\mathbb{F}_2}(p), \tag{2.13}$$

with $\mathbf{f}_{p,s} = \hat{\rho}^{p-s}\hat{\tau}^s$.

2.3. The theories represented by $\mathbf{E}_{\mathbb{Z}}(p)$ and $\mathbf{E}_{\mathbb{F}_2}(p)$

In this section we study in detail the cohomology theories represented by the spectra $\mathbf{E}_{\mathbb{Z}}(p)$ and $\mathbf{E}_{\mathbb{F}_2}(p)$ and explicitly determine the *group restriction* and the *reduction of coefficients* functors.

Definition 2.27. *Given a space X and $R = \mathbb{Z}$ or $\mathbb{F}_2$, denote by*

$$\mathbf{E}_R(p)^n(X) := [X_+, \Sigma^n \mathbf{E}_R(p)] \tag{2.14}$$

the n-th cohomology group of X with values in $\mathbf{E}_R(p)$, for $n \in \mathbb{Z}$. For $\beta \in H^*_{\mathrm{sing}}(X; R)$, denote by $\bar{\beta} \in H^*_{\mathrm{sing}}(X; \mathbb{F}_2)$ the class

$$\bar{\beta} = \begin{cases} \beta, & \text{if } R = \mathbb{F}_2 \\ \boldsymbol{\varrho}(\beta), & \text{if } R = \mathbb{Z}, \end{cases}$$

where $\boldsymbol{\varrho}\colon H^*_{\mathrm{sing}}(X; \mathbb{Z}) \to H^*_{\mathrm{sing}}(X; \mathbb{F}_2)$ denotes reduction of coefficients.

1. *It follows from Remark 2.24 that one can univocally write an element*

$$\alpha \in \mathbf{E}_{\mathbb{Z}}(p)^n(X) \cong \bigoplus_{0 \le s \le \lfloor p/2 \rfloor} H^{n-p+2s}_{\mathrm{sing}}(X; \mathbb{M}^{p-2s,p})$$

in the form $\alpha = \sum_{s=0}^{\lfloor p/2 \rfloor} \alpha_s \cdot a^{p-2s} u^s$, where $a^{p-2s}u^s$ is the generator of $\mathbb{M}^{p-2s,p}$ and

$$\alpha_s \in \begin{cases} H^{n-p+2s}_{\mathrm{sing}}(X; \mathbb{F}_2), & \text{if } 0 \le s < p/2; \\ H^n_{\mathrm{sing}}(X; \mathbb{Z}), & \text{if } p \text{ is even, and } s = p/2. \end{cases}$$

2. *Similarly, using the decomposition (2.13) we can write an element*

$$\beta \in \mathbf{E}^n_{\mathbb{F}_2}(X) \cong \bigoplus_{0 \le k \le p} H^{n-p+k}_{\mathrm{sing}}(X; \mathbb{M}_2^{p-k,p})$$

in the form $\beta = \sum_{k=0}^{p} \beta_k \cdot \rho^{p-k}\tau^k$, where $\rho^{p-k}\tau^k$ is the generator of $\mathbb{M}_2^{p-k,p}$ and $\beta_k \in H^{n-p+k}_{\mathrm{sing}}(X; \mathbb{F}_2)$.

Our main interest in the spectra $\mathbf{E}_R(p)$ and associated cohomology theories $\mathbf{E}_R(p)^*$ lies in their relation to the equivariant cohomology of trivial C_2-spaces X, which by definition of these spectra becomes:

$$H^{*,\bullet}(X; \underline{\mathbb{Z}})_+ \cong \bigoplus_{p \ge 0, n \ge 0} \mathbf{E}_{\mathbb{Z}}(p)^n(X) \quad \text{and} \quad H^{*,\bullet}(X; \underline{\mathbb{F}_2})_+ \cong \bigoplus_{p \ge 0, n \ge 0} \mathbf{E}_{\mathbb{F}_2}(p)^n(X).$$

Remark 2.28. It is relevant to note that the representation of elements $\alpha \in \mathbf{E}_{\mathbb{Z}}(p)^n(X)$ given in Definition 2.27 reflects the $\mathbb{M}$-module structure of $H^{*,\bullet}(X; \underline{\mathbb{Z}})$ as follows:

$$H^{*,\bullet}(X; \underline{\mathbb{Z}})_+ \cong H^*(X; \mathbb{Z}) \otimes \mathbb{Z}[u] \ \oplus \ H^*(X; \mathbb{F}_2) \otimes a\mathbb{F}_2[a, u];$$

see Remark 2.34. Similarly, the representation of elements in $\mathbf{E}^n_{\mathbb{F}_2}(p)(X)$ reflects the $\mathbb{M}_2$-module structure of $H^{*,\bullet}(X;\underline{\mathbb{F}_2});\cong \mathbb{M}_2^{*,\bullet}\otimes_{\mathbb{F}_2} H^*_{\mathrm{sing}}(X;\mathbb{F}_2)$ proved in [8].

2.3.1. The group restriction functors Understanding the functors induced in cohomology by the restriction maps $\mathscr{R}\colon \mathbf{E}_M(p)\to \mathbf{H}M$, with $M=\mathbb{Z}$ or $\mathbb{F}_2$, amounts to determining the classes

$$\mathscr{R}_p \in [\mathbf{E}_{\mathbb{Z}}(p),\mathbf{H}\mathbb{Z}]_{\mathbf{H}\mathbb{Z}} \cong \begin{cases} \mathsf{Hom}_{\mathbb{Z}}(\mathbb{M}^{0,p},\mathbb{Z})\cong\mathbb{Z}, & \text{if } p \text{ is even}\\ \mathsf{Ext}^1_{\mathbb{Z}}(\mathbb{M}^{1,p},\mathbb{Z})\cong\mathbb{Z}/2, & \text{if } p \text{ is odd}\end{cases}, \tag{2.15}$$

and

$$\bar{\mathscr{R}}_p \in [\mathbf{E}_{\mathbb{F}_2}(p),\mathbf{H}\mathbb{F}_2]_{\mathbf{H}\mathbb{F}_2} \cong \mathsf{Hom}_{\mathbb{Z}}(\mathbb{M}_2^{0,p},\mathbb{F}_2)\cong\mathbb{Z}/2. \tag{2.16}$$

Proposition 2.29. *The maps $\mathscr{R}_p$ and $\bar{\mathscr{R}}_p$ represent the generators of the cyclic groups $[\mathbf{E}_{\mathbb{Z}}(p),\mathbf{H}\mathbb{Z}]_{\mathbf{H}\mathbb{Z}}$ and $[\mathbf{E}_{\mathbb{F}_2}(p),\mathbf{H}\mathbb{F}_2]_{\mathbf{H}\mathbb{F}_2}$, respectively, for all $p\geq 0$.*

Proof. The statement for $\bar{\mathscr{R}}_p$ follows by induction, using Proposition 2.25 and the fact that $\tau\colon \mathbb{M}_2^{j-1,p}\to\mathbb{M}_2^{j,p}$ is an isomorphism for $1\leq j\leq p$.

Similarly, it follows from Proposition 2.21 that the statement holds for $\mathscr{R}_p$ when p is even, and that the statement for p odd will follow from the case $p=1$ and induction.

To prove the statement for $p=1$, observe that $S^{1,1}\wedge S^1=S^{2,1}$ is isomorphic to the complex projective line $\mathbb{P}^1(\mathbb{C})$ as a C_2-space under complex conjugation. It is easily checked that the infinite symmetric product $SP_\infty(\mathbb{P}^1(\mathbb{C}))$ is C_2-homeomorphic to the infinite complex projective space $\mathbb{P}^\infty(\mathbb{C})$, and that the natural topological group-completion map

$$\mathbf{c}_1\colon \mathbb{P}^\infty(\mathbb{C})=SP_\infty(\mathbb{P}^1(\mathbb{C}))\hookrightarrow\mathbb{Z}\otimes\mathbb{P}^1(\mathbb{C})$$

is an equivariant homotopy equivalence. Furthermore, if $\iota_2\in H^2_{\mathrm{sing}}(\mathbb{Z}\otimes\mathbb{P}^1(\mathbb{C}),\mathbb{Z})$ is the canonical class of $\mathbb{Z}\otimes\mathbb{P}^1(\mathbb{C})$, then $\mathbf{c}_1^*(\iota_2)=c_1(\mathcal{O}(1))$ is the first Chern class of the hyperplane bundle $\mathcal{O}(1)$ over $\mathbb{P}^\infty(\mathbb{C})$. Now, consider the following commutative diagram

$$\begin{tikzcd} & \mathbb{P}^\infty(\mathbb{R}) \arrow[rr, hook, "\iota_{\mathbb{R}}"] \arrow[d, "\mathbf{c}_{1,\mathbb{R}}"', "\simeq"] & & \mathbb{P}^\infty(\mathbb{C}) \arrow[d, "\simeq"', "\mathbf{c}_1"] \\ \mathbb{F}_2\otimes S^1 & \{\mathbb{Z}\otimes\mathbb{P}^1(\mathbb{C})\}^{C_2} \arrow[l, "\simeq"', "\chi_{1,1}"] \arrow[rr, hook, "\iota_{\mathbb{R}}"] & & \mathbb{Z}\otimes\mathbb{P}^1(\mathbb{C}), \end{tikzcd} \tag{2.17}$$

where $\mathbf{c}_{1,\mathbb{R}}$ is the restriction of $\mathbf{c}_1$ to the fixed point sets and $\chi_{1,1}$ is the "fixed mod averaged" map described in the proof of Corollary 2.20. If $L \to \mathbb{P}^\infty(\mathbb{R})$ is the dual of the tautological line bundle, then

$$\begin{aligned}\mathbf{c}_{1,\mathbb{R}}^* \circ \iota_{\mathbb{R}}^*(\iota_2) = \iota_{\mathbb{R}}^* \circ \mathbf{c}_1^*(\iota_2) = \iota_{\mathbb{R}}^* c_1(\mathcal{O}(1)) \\ = c_1(\iota_{\mathbb{R}}^* \mathcal{O}(1)) = c_1(L \otimes \mathbb{C}) = \delta(w_1(L)),\end{aligned}$$

where $\delta\colon H^1_{\text{sing}}(\mathbb{P}^\infty(\mathbb{R}); \mathbb{F}_2) \to H^2_{\text{sing}}(\mathbb{P}^\infty(\mathbb{R}); \mathbb{Z})$ is the connecting homomorphism for $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{F}_2 \to 0$ and $w_1(L) \in H^1_{\text{sing}}(\mathbb{P}^1(\mathbb{R}); \mathbb{F}_2)$ is the first Stiefel-Whitney class of L. Since $\mathscr{R}_1\colon \mathbf{E}_{\mathbb{Z}}(1) \to \mathbf{H}\mathbb{Z}$ is induced by

$$\left\{\mathbb{Z} \otimes (S^{1,1} \wedge S^n)\right\}^{C_2} = \left\{\mathbb{Z} \otimes (\mathbb{P}^1(\mathbb{C}) \wedge S^{n-1})\right\}^{C_2} \hookrightarrow \mathbb{Z} \otimes (\mathbb{P}^1(\mathbb{C}) \wedge S^{n-1}),$$

one concludes that $\mathscr{R}_1$ is non-trivial and hence it represents the generator of $[\mathbf{E}_{\mathbb{Z}}(1), \mathbf{H}\mathbb{Z}]_{\mathbf{H}\mathbb{Z}} = \mathsf{Ext}^1_{\mathbb{Z}}(\mathbb{M}^{1,1}, \mathbb{Z})$. $\square$

Using the notation above, the following result expresses Proposition 2.29 in terms of cohomology functors.

Corollary 2.30. *Let X be a space and fix $p \geq 0$. Then:*

1. *The map $\mathscr{R}_p\colon \mathbf{E}_{\mathbb{Z}}(p)^n(X) \to H^n_{\text{sing}}(X; \mathbb{Z})$ is given by*

$$\mathscr{R}_p\left(\sum_{0 \leq s \leq \lfloor p/2 \rfloor} \alpha_s \cdot a^{p-2s} u^s\right) = \begin{cases} \alpha_{p/2}, & \text{if } p \text{ is even,} \\ \delta(\alpha_{\lfloor p/2 \rfloor}), & \text{if } p \text{ is odd,} \end{cases}$$

where $\delta\colon H^k_{\text{sing}}(X; \mathbb{F}_2) \to H^{k+1}_{\text{sing}}(X; \mathbb{Z})$ is the Bockstein homomorphism.

2. *The map $\bar{\mathscr{R}}_{p*}\colon \mathbf{E}_{\mathbb{F}_2}(p)^n(X) \to H^n_{\text{sing}}(X; \mathbb{F}_2)$ is given by*

$$\bar{\mathscr{R}}_p\left(\sum_{0 \leq k \leq p} \beta_k \cdot \rho^{p-k} \tau^k\right) = \beta_p.$$

Example 2.31. In the particular case of the real projective space, we obtain

$$\begin{aligned} H^{*,\bullet}(\mathbb{P}^\infty(\mathbb{R}); \underline{\mathbb{Z}})_+ &= H^*(\mathbb{P}^\infty(\mathbb{R}); \mathbb{Z}) \otimes \mathbb{Z}[u] \ \oplus \ H^*(\mathbb{P}^\infty(\mathbb{R}); \mathbb{F}_2) \otimes a\mathbb{F}_2[a, u] \\ &= \mathbb{Z}[\eta]/\langle 2\eta \rangle \otimes \mathbb{Z}[u] \ \oplus \ \mathbb{F}_2[w_1] \otimes_{\mathbb{F}_2} a\mathbb{F}_2[a, u] \\ &= \mathbb{Z}[\eta, u]/\langle 2\eta \rangle \ \oplus \ \mathbb{F}_2[w_1, a, u] \cdot a. \end{aligned}$$

Here $\eta \in H^2_{\text{sing}}(\mathbb{P}^\infty(\mathbb{R}); \mathbb{Z})$ is the generator and w_1 is the first Stiefel-Whitney class of the the tautological bundle. Note that $\eta = \delta(w_1)$, where δ is the

Bockstein homomorphism for the sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{F}_2 \to 0$. Since $\boldsymbol{\varrho} \circ \delta = \mathrm{Sq}^1$, we conclude that $\eta \cdot a = \mathrm{Sq}^1(w_1) \cdot a = w_1^2 a$, and this completely determines the bigraded ring structure of $H^{*,\bullet}(\mathbb{P}^\infty(\mathbb{R}); \underline{\mathbb{Z}})_+$.

2.3.2. The mod-2 reduction of coefficients functor We now focus on the map of $\mathbf{H}\mathbb{Z}$-modules

$$\boldsymbol{\varrho}_p \colon \mathbf{E}_{\mathbb{Z}}(p) \longrightarrow \mathbf{E}_{\mathbb{F}_2}(p), \ p \geq 1 \tag{2.18}$$

and determine the class of $\boldsymbol{\varrho}_p$ in $[\mathbf{E}_{\mathbb{Z}}(p), \mathbf{E}_{\mathbb{F}_2}(p)]_{\mathbf{H}\mathbb{Z}}$.

We have

$$[\Sigma^a \mathbf{H}M, \Sigma^b \mathbf{H}\mathbb{F}_2]_{\mathbf{H}\mathbb{Z}} = \begin{cases} \mathsf{Hom}_{\mathbb{Z}}(M, \mathbb{F}_2), & \text{if } b = a \\ \mathsf{Ext}^1_{\mathbb{Z}}(M, \mathbb{F}_2), & \text{if } b = a+1 \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} &[\mathbf{E}_{\mathbb{Z}}(p), \mathbf{E}_{\mathbb{F}_2}(p)]_{\mathbf{H}\mathbb{Z}} \\ &\quad \cong \bigoplus_{0 \leq r \leq \lfloor p/2 \rfloor} \left\{ \mathsf{Hom}_{\mathbb{Z}}(\mathbb{M}^{p-2r,p}, \mathbb{M}_2^{p-2r,p}) \ \oplus \ \mathsf{Ext}^1_{\mathbb{Z}}(\mathbb{M}^{p-2r,p}, \mathbb{M}_2^{p-1-2r,p}) \right\}. \end{aligned} \tag{2.19}$$

It is clear that all summands in the decomposition above are isomorphic to $\mathbb{Z}/2$, except when p is even and $r = p/2$, since $\mathsf{Ext}^1_{\mathbb{Z}}(\mathbb{M}^{0,p}, \mathbb{M}_2^{-1,p}) = 0$.

Proposition 2.32. *The class of $\boldsymbol{\varrho}_p$ in $[\mathbf{E}_{\mathbb{Z}}(p), \mathbf{E}_{\mathbb{F}_2}(p)]_{\mathbf{H}\mathbb{Z}}$ is precisely the element whose components in the decomposition above are all non-zero.*

Proof. First consider the case $p = 1$, where we have an isomorphism

$$\begin{aligned} [\mathbf{E}_{\mathbb{Z}}(1), \mathbf{E}_{\mathbb{F}_2}(1)]_{\mathbf{H}\mathbb{Z}} &\xrightarrow[\bar{\boldsymbol{\chi}}_{1*} \oplus \bar{\mathscr{R}}_{1*}]{\cong} [\mathbf{E}_{\mathbb{Z}}(1), \Sigma^{-1}\mathbf{H}\mathbb{F}_2]_{\mathbf{H}\mathbb{Z}} \oplus [\mathbf{E}_{\mathbb{Z}}(1), \mathbf{H}\mathbb{F}_2]_{\mathbf{H}\mathbb{Z}} \\ &\cong \mathsf{Hom}_{\mathbb{Z}}(\mathbb{M}^{1,1}, \mathbb{M}_2^{1,1}) \oplus \mathsf{Ext}^1_{\mathbb{Z}}(\mathbb{M}^{1,1}, \mathbb{M}_2^{0,1}) \end{aligned}$$

induced by the maps in the commutative diagram

$$\begin{tikzcd}
\mathbf{E}_{\mathbb{Z}}(1) \arrow[rr, "\boldsymbol{\varrho}_1"] \arrow[dr, "\simeq", "\boldsymbol{\chi}_1"'] \arrow[dd, "\mathscr{R}_1"'] & & \mathbf{E}_{\mathbb{F}_2}(1) \arrow[dl, "\bar{\boldsymbol{\chi}}_1"'] \arrow[dd, "\bar{\mathscr{R}}_1"] \\
& \Sigma^{-1}\mathbf{H}\mathbb{F}_2 \equiv \Sigma^{-1}\mathbf{H}\mathbb{M}_2{}^{1,1} & \\
\mathbf{H}\mathbb{Z} \arrow[rr, "\boldsymbol{\varrho}"'] & & \mathbf{H}\mathbb{F}_2 \equiv \mathbf{H}\mathbb{M}_2^{0,1},
\end{tikzcd}$$

since $\bar{\boldsymbol{\chi}}_1 \vee \bar{\mathscr{R}}_1$ is the homotopy inverse of $\bar{\mathbf{f}}_1$, see Corollary 2.26. Therefore,

$$(\bar{\boldsymbol{\chi}}_1, \bar{\mathscr{R}}_1) \circ \boldsymbol{\varrho}_1 = (\boldsymbol{\chi}_1, \mathscr{R}_1).$$

This concludes the proof in the case $p = 1$, since by Corollary 2.20 and Proposition 2.29 both $\boldsymbol{\chi}_1$ and $\mathscr{R}_1$ are nontrivial.

Similarly, when $p = 2$ one has isomorphisms

$$\hat{\rho}_* \colon [\Sigma^{-1}\mathbf{E}_{\mathbb{Z}}(1), \Sigma^{-1}\mathbf{E}_{\mathbb{F}_2}(1)]_{\mathbf{H}\mathbb{Z}} \xrightarrow{\cong} [\Sigma^{-1}\mathbf{E}_{\mathbb{Z}}(1), \mathbf{E}_{\mathbb{F}_2}(2)]_{\mathbf{H}\mathbb{Z}} \quad \text{and}$$
$$\hat{\tau}_*^2 \colon [\mathbf{E}_{\mathbb{Z}}(0), \mathbf{E}_{\mathbb{F}_2}(0)]_{\mathbf{H}\mathbb{Z}} \xrightarrow{\cong} [\mathbf{E}_{\mathbb{Z}}(0), \mathbf{E}_{\mathbb{F}_2}(2)]_{\mathbf{H}\mathbb{Z}}.$$

On the other hand, the commutative diagram

$$\begin{tikzcd}
\Sigma^{-1}\mathbf{E}_{\mathbb{Z}}(1) \vee \mathbf{E}_{\mathbb{Z}}(0) \arrow[r, "\mathbf{f}_1 = \hat{a} \vee \hat{u}", "\simeq"'] \arrow[d, "\Sigma^{-1}\boldsymbol{\varrho}_1 \vee \boldsymbol{\varrho}_0"'] & \mathbf{E}_{\mathbb{Z}}(2) \arrow[d, "\boldsymbol{\varrho}_2"] \\
\Sigma^{-1}\mathbf{E}_{\mathbb{F}_2}(1) \vee \mathbf{E}_{\mathbb{F}_2}(0) \arrow[r, "\hat{\rho} \vee \hat{\tau}^2"'] & \mathbf{E}_{\mathbb{F}_2}(2)
\end{tikzcd}$$

gives $\mathbf{f}_1^*(\boldsymbol{\varrho}_2) = \left(\hat{\rho}\,\Sigma^{-1}\boldsymbol{\varrho}_1, \hat{\tau}^2\boldsymbol{\varrho}_0\right)$ and the result follows from the case $p = 1$ and the fact that $\boldsymbol{\varrho}_0$ represents the generator of $[\mathbf{E}_{\mathbb{Z}}(0), \mathbf{E}_{\mathbb{F}_2}(0)]_{\mathbf{H}\mathbb{Z}} = [\mathbf{H}\mathbb{Z}, \mathbf{H}\mathbb{F}_2]_{\mathbf{H}\mathbb{Z}}$.

When $p \geq 2$ the result follows by induction along with Propositions 2.21 and 2.25, which give

$$\begin{aligned}
&[\mathbf{E}_{\mathbb{Z}}(p), \mathbf{E}_{\mathbb{F}_2}(p)]_{\mathbf{H}\mathbb{Z}} \\
&\quad \cong [\Sigma^{1-p}\mathbf{E}_{\mathbb{Z}}(1), \Sigma^{1-p}\mathbf{E}_{\mathbb{F}_2}(1)]_{\mathbf{H}\mathbb{Z}} \oplus [\mathbf{E}_{\mathbb{Z}}(p-2), \mathbf{E}_{\mathbb{F}_2}(p-2)]_{\mathbf{H}\mathbb{Z}} \\
&\quad \cong [\mathbf{E}_{\mathbb{Z}}(1), \mathbf{E}_{\mathbb{F}_2}(1)]_{\mathbf{H}\mathbb{Z}} \oplus [\mathbf{E}_{\mathbb{Z}}(p-2), \mathbf{E}_{\mathbb{F}_2}(p-2)]_{\mathbf{H}\mathbb{Z}}.
\end{aligned}$$

□

Corollary 2.33. *Given a space X, the homomorphism*

$$\boldsymbol{\varrho}\colon \mathbf{E}_{\mathbb{Z}}(p)^n(X) \to \mathbf{E}_{\mathbb{F}_2}(p)^n(X)$$

is defined on $\alpha = \sum_{s=0}^{\lfloor p/2 \rfloor} \alpha_s(a^{p-2s}u^s)$ *by*

$$\boldsymbol{\varrho}(\alpha) \;=\; \sum_{0 \le s < p/2} \left\{ \alpha_s \rho \;+\; \mathrm{Sq}^1(\alpha_s)\tau \right\} \rho^{p-2s-1}\tau^{2s} \;+\; \bar{\alpha}_{p/2} \cdot \tau^p, \tag{2.20}$$

where, by convention, $\bar{\alpha}_{p/2} = 0$ *when p is odd and otherwise* $\bar{\alpha}_{p/2}$ *denotes the* $\mathrm{mod}\,2$*-reduction of* $\alpha_{p/2}$.

2.4. On the pairings $\boldsymbol{\mu_{p,q}}$ and $\boldsymbol{\bar{\mu}_{p,q}}$

In this section we study the functors in cohomology induced by the pairings $\mu_{p,q}$ and $\bar{\mu}_{p,q}$; see Definition 2.10. This is accomplished by determining the class of $\mu_{p,q}$ in $[\mathbf{E}_{\mathbb{Z}}(p) \wedge_{\mathbf{H}\mathbb{Z}} \mathbf{E}_{\mathbb{Z}}(q); \mathbf{E}_{\mathbb{Z}}(p+q)]_{\mathbf{H}\mathbb{Z}}$ and the class of $\bar{\mu}_{p,q}$ in $[\mathbf{E}_{\mathbb{F}_2}(p) \wedge_{\mathbf{H}\mathbb{F}_2} \mathbf{E}_{\mathbb{F}_2}(q); \mathbf{E}_{\mathbb{F}_2}(p+q)]_{\mathbf{H}\mathbb{F}_2}$.

Given spaces X and Y, these pairings induce natural external products $\star$ and $\bar{\star}$ that fit into the following commutative diagram.

$$\begin{tikzcd}
\mathbf{E}_{\mathbb{Z}}(p)^m(X) \otimes \mathbf{E}_{\mathbb{Z}}(q)^n(Y) \arrow[r, "\star"] \arrow[d, "\boldsymbol{\varrho}_p \otimes \boldsymbol{\varrho}_q"'] & \mathbf{E}_{\mathbb{Z}}(p+q)^{m+n}(X \times Y) \arrow[d, "\boldsymbol{\varrho}_{p+q}"] \\
\mathbf{E}_{\mathbb{F}_2}(p)^m(X) \otimes \mathbf{E}_{\mathbb{F}_2}(q)^n(X) \arrow[r, "\bar{\star}"'] & \mathbf{E}_{\mathbb{F}_2}(p+q)^{m+n}(X \times Y)
\end{tikzcd} \tag{2.21}$$

We have seen in §2.1.2 that these pairings are compatible with change of coefficients and restriction functors. Furthermore, they are used to define the maps $\hat{u}, \hat{\tau}, \hat{a}$ and $\hat{\rho}$ that give the homotopy commutative diagrams of $\mathbf{H}\mathbb{Z}$-modules below:

$$\begin{tikzcd}
\mathbf{E}_{\mathbb{Z}}(p) \arrow[rr, "\hat{u}"] \arrow[d, "\boldsymbol{\varrho}_p"'] & & \mathbf{E}_{\mathbb{Z}}(p+2) \arrow[d, "\boldsymbol{\varrho}_{p+2}"] \\
\mathbf{E}_{\mathbb{F}_2}(p) \arrow[r, "\hat{\tau}"'] & \mathbf{E}_{\mathbb{F}_2}(p+1) \arrow[r, "\hat{\tau}"'] & \mathbf{E}_{\mathbb{F}_2}(p+2)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Sigma^{-1}\mathbf{E}_{\mathbb{Z}}(p) \arrow[r, "\hat{a}"] \arrow[d, "\Sigma^{-1}\boldsymbol{\varrho}_p"'] & \mathbf{E}_{\mathbb{Z}}(p+1) \arrow[d, "\boldsymbol{\varrho}_{p+1}"] \\
\Sigma^{-1}\mathbf{E}_{\mathbb{F}_2}(p) \arrow[r, "\hat{\rho}"'] & \mathbf{E}_{\mathbb{F}_2}(p+1).
\end{tikzcd}$$

Therefore, taking $Y = \{pt\}$ in (2.21) yields compatible pairings

$$\begin{tikzcd}
\mathbf{E}_{\mathbb{Z}}(p)^m(X) \cdot \mathbb{M}^{n,q} \arrow[r, "\star"] \arrow[d, "\varrho_p \otimes \varrho"'] & \mathbf{E}_{\mathbb{Z}}(p+q)^{m+n}(X) \arrow[d, "\varrho_{p+q}"] \\
\mathbf{E}_{\mathbb{F}_2}(p)^m(X) \cdot \mathbb{M}_2^{n,q} \arrow[r, "\bar{\star}"'] & \mathbf{E}_{\mathbb{F}_2}(p+q)^{m+n}(X)
\end{tikzcd}$$

where $\varrho \colon \mathbb{M}_+ \to \mathbb{M}_{2+}$ is the change of coefficients homomorphism (1.3).

Remark 2.34. Using Definition 2.27 one can give a simple description of the maps in the diagrams above. Namely, given $\alpha = \sum_{0 \leq k \leq \lfloor p/2 \rfloor} \alpha_k \cdot a^{p-2k} u^k \in \mathbf{E}_{\mathbb{Z}}(p)^n(X)$ then

$$\alpha \star a = \sum_{0 \leq k \leq \lfloor p/2 \rfloor} \alpha_k \cdot a^{p-2k+1} u^k \in \mathbf{E}_{\mathbb{Z}}(p+1)^{n+1}(X), \quad \text{and} \tag{2.22}$$

$$\alpha \star u = \sum_{0 \leq k \leq \lfloor p/2 \rfloor} \alpha_k \cdot a^{p-2k} u^{k+1} \in \mathbf{E}_{\mathbb{Z}}(p+2)^n(X). \tag{2.23}$$

Similarly, given $\beta = \sum_{0 \leq k \leq p} \beta_k \cdot \rho^{p-k} \tau^k \in \mathbf{E}_{\mathbb{F}_2}(p)^n(X)$ then

$$\beta \mathbin{\bar{\star}} \rho = \sum_{0 \leq k \leq p} \beta_k \cdot \rho^{p-k+1} \tau^k \in \mathbf{E}_{\mathbb{F}_2}(p+1)^{n+1}(X), \quad \text{and} \tag{2.24}$$

$$\beta \mathbin{\bar{\star}} \tau = \sum_{0 \leq k \leq p} \beta_k \cdot a^{p-k} \tau^{k+1} \in \mathbf{E}_{\mathbb{F}_2}(p+1)^n(X). \tag{2.25}$$

The general exterior product is determined in the following result.

Proposition 2.35. *Let X, Y be spaces and $p, q \geq 0$.*

1. *With $\mathbb{F}_2$-coefficients, the product in (2.21) is completely determined by the singular cohomology ring, (2.24), (2.25) and the bigraded ring $\mathbb{M}_2^{*,\bullet}$. More precisely, given $\bar{\alpha} = \sum_{r=0}^{p} \bar{\alpha}_r \cdot \rho^{p-r} \tau^r \in \mathbf{E}_{\mathbb{F}_2}(p)^m(X)$ and $\bar{\beta} = \sum_{s=0}^{q} \bar{\beta}_s \cdot \rho^{q-s} \tau^s \in \mathbf{E}_{\mathbb{F}_2}(q)^n(Y)$, then*

$$\bar{\alpha} \mathbin{\bar{\star}} \bar{\beta} \; = \sum_{k=0}^{p+q} \sum_{r+s=k} (\bar{\alpha}_r \times \bar{\beta}_s) \cdot \rho^{p+q-k} \tau^k.$$

2. *With $\mathbb{Z}$-coefficients, the product in (2.21) is completely determined by (2.22), (2.23) and the rules*

(a) If $\alpha \in H^m_{\mathrm{sing}}(X;\mathbb{Z}) = \mathbf{E}_{\mathbb{Z}}(0)^m(X)$ *and* $\beta \in H^n_{\mathrm{sing}}(Y;\mathbb{Z}) = \mathbf{E}_{\mathbb{Z}}(0)^n(Y)$ *then*

$$(\alpha \cdot 1) \star (\beta \cdot 1) = (\alpha \times \beta) \cdot 1 \ \in \mathbf{E}_{\mathbb{Z}}(0)^{m+n}(X \times Y),$$

where $\times$ *denotes the exterior product in singular cohomology.*

(b) If $\alpha \cdot a \in \mathbf{E}_{\mathbb{Z}}(1)^m(X)$ *with* $\alpha \in H^{m-1}_{\mathrm{sing}}(X;\mathbb{F}_2)$*, and* $\beta \in \mathbf{E}_{\mathbb{Z}}(0)^n(Y)$*, then*

$$(\alpha \cdot a) \star (\beta \cdot 1) = (\alpha \times \overline{\beta}) \cdot a,$$

where $\overline{\beta}$ *denotes the image of* β *under reduction of coefficients.*

(c) If $\alpha \cdot a \in \mathbf{E}_{\mathbb{Z}}(1)^m(X)$ *and* $\alpha' \cdot a \in \mathbf{E}_{\mathbb{Z}}(1)^m(Y)$ *then*

$$(\alpha \cdot a) \star (\alpha' \cdot a) = (\alpha \times \alpha') \cdot a^2 \ + \ \{\delta(\alpha) \times \delta(\alpha')\} \cdot u,$$

where δ *denotes the Bockstein map.*

Proof. The result with $\mathbb{F}_2$-coefficients is shown in [8, Lemma 3.7].

To prove the statement with $\mathbb{Z}$-coefficients, we use the compatibility of multiplication with the restriction and reduction of coefficients functors. Therefore, since $\mathbf{E}_{\mathbb{Z}}(0)^*$ is singular cohomology, the first assertion follows.

Now, given $\alpha \cdot a \in \mathbf{E}_{\mathbb{Z}}(1)^m(X)$, and $\beta \cdot 1 \in \mathbf{E}_{\mathbb{Z}}(0)^n(Y)$, write $(\alpha \cdot a) \star (\beta \cdot 1) = A \cdot a \in \mathbf{E}_{\mathbb{Z}}(1)^{m+n}(X \times Y)$. It follows from (2.20) that $\boldsymbol{\varrho}((\alpha \cdot a) \star (\beta \cdot 1)) = A \cdot \rho + \mathrm{Sq}^1(A) \cdot \tau$. On the other hand, with $\mathbb{F}_2$-coefficients one has

$$\begin{aligned} \boldsymbol{\varrho}\,(\alpha \cdot a \star \beta \cdot 1) &= \boldsymbol{\varrho}(\alpha \cdot a) \ \bar{\star} \ \boldsymbol{\varrho}(\beta \cdot 1) \\ &= \left\{\alpha \cdot \rho + \mathrm{Sq}^1 \alpha \cdot \tau\right\} \ \bar{\star} \ \{\bar{\beta} \cdot 1\} \\ &= (\alpha \times \bar{\beta}) \cdot \rho \ + \ (\mathrm{Sq}^1 \alpha \times \bar{\beta}) \cdot \tau. \end{aligned}$$

Hence, $A = \alpha \times \bar{\beta}$ (note that $\mathrm{Sq}^1(\bar{\beta}) = 0$) and the second statement follows.

Similarly, given $\alpha \cdot a \in \mathbf{E}_{\mathbb{Z}}(1)^m(X)$, and $\beta \cdot a \in \mathbf{E}_{\mathbb{Z}}(1)^n(Y)$, write

$$(\alpha \cdot a) \star (\beta \cdot a) = A \cdot a^2 \ + \ B \cdot u \ \in \ \mathbf{E}_{\mathbb{Z}}(2)^{m+n}(X \times Y).$$

Applying the restriction functor from §2.3.1 gives

$$B = \mathscr{R}_2\left((\alpha \cdot a) \star (\beta \cdot a)\right) = \mathscr{R}_1(\alpha \cdot a) \times \mathscr{R}_1(\beta \cdot a) = \delta(\alpha) \times \delta(\beta).$$

Now, we use reduction of coefficients and Corollary 2.33 to get

$$\boldsymbol{\varrho}\left((\alpha \cdot a) \star (\beta \cdot a)\right) = A \cdot \rho^2 + \mathrm{Sq}^1(A) \cdot \rho\tau + \bar{B} \cdot \tau^2.$$

On the other hand, with $\mathbb{F}_2$-coefficients one has

$$\begin{aligned}
\boldsymbol{\varrho}\left((\alpha \cdot a) \star (\beta \cdot a)\right) &= \boldsymbol{\varrho}(\alpha \cdot a) \,\bar{\star}\, \boldsymbol{\varrho}(\beta \cdot a) \\
&= \left\{\alpha \cdot \rho + \mathrm{Sq}^1\alpha \cdot \tau\right\} \,\bar{\star}\, \left\{\beta \cdot \rho + \mathrm{Sq}^1\beta \cdot \tau\right\} \\
&= (\alpha \times \beta) \cdot \rho^2 + (\mathrm{Sq}^1\alpha \times \beta) \cdot \rho\tau \\
&\quad + (\alpha \times \mathrm{Sq}^1\beta) \cdot \rho\tau + (\mathrm{Sq}^1\alpha \times \mathrm{Sq}^1\beta) \cdot \tau^2 \\
&= (\alpha \times \beta) \cdot \rho^2 + \mathrm{Sq}^1(\alpha \times \beta) \cdot \rho\tau + (\mathrm{Sq}^1\alpha \times \mathrm{Sq}^1\beta) \cdot \tau^2.
\end{aligned}$$

Therefore, $A = \alpha \times \beta$ (note that $\overline{\delta(\alpha) \times \delta(\beta)} = \mathrm{Sq}^1\alpha \times \mathrm{Sq}^1\beta$). $\square$

3. Applications

In this section we introduce a ring homomorphism $\mathrm{Sq}_{\rho,\tau}$ from the singular cohomology with $\mathbb{F}_2$ coefficients of a space X to its equivariant cohomology with coefficients in $\underline{\mathbb{F}_2}$, which we call the *total Steenrod square map*, for reasons that will become apparent from the definition. This homomorphism resurfaces in the study of equivariant Poincaré duality in a Real compact manifold X. In particular, it provides a necessary condition for an equivariant class in the real locus $X(\mathbb{R})$ to be dual to the class $[Y(\mathbb{R})]$ represented by the real locus $Y(\mathbb{R})$ of a Real submanifold $Y \subset X$.

3.1. Total Steenrod squares

Definition 3.1. *Let X be a Real space under the trivial action of C_2. For $\alpha \in H^k(X;\mathbb{F}_2)$ define the* total Steenrod square

$$\mathrm{Sq}_{\rho,\tau}(\alpha) = \sum_{i=0}^{k} \mathrm{Sq}^i(\alpha) \cdot \rho^{k-i}\tau^i \in H^{2k,k}(X;\underline{\mathbb{F}_2}).$$

Remark 3.2. By Cartan's formula, for $\alpha \in H^r(X;\mathbb{F}_2)$, $\beta \in H^s(X;\mathbb{F}_2)$, we have $\mathrm{Sq}_{\rho,\tau}(\alpha)\,\mathrm{Sq}_{\rho,\tau}(\beta) = \mathrm{Sq}_{\rho,\tau}(\alpha\beta)$. Therefore, the total Steenrod square gives a ring homomorphism $\mathrm{Sq}_{\rho,\tau} \;:\; H^*_{\mathrm{sing}}(X;\mathbb{F}_2) \longrightarrow H^{2*,*}(X;\underline{\mathbb{F}_2})_+$.

Example 3.3. Consider the the classifying space BU_1 with its Real space structure induced by complex conjugation. We have $\mathrm{BU}_1 = \mathbb{P}^\infty(\mathbb{C})$ and $\mathrm{BU}_1(\mathbb{R}) = \mathrm{BO}_1 = \mathbb{P}^\infty(\mathbb{R})$. The bigraded cohomology of BU_1, the classifying space for Real line bundles, is a polynomial ring in one variable just as in the nonequivarant case: $H^{*,*}(\mathrm{BU}_1;\underline{\mathbb{F}_2}) = \mathbb{M}_2^{*,*}[\bar{\mathbf{c}}_1]$, where $\bar{\mathbf{c}}_1$ is the mod 2 reduction of the Real first Chern class $\mathbf{c}_1 \in H^{2,1}(\mathrm{BU}_1;\underline{\mathbb{Z}})$; see [5].

Let $\iota_{\mathbb{R}}\colon \mathrm{BO}_1 \hookrightarrow \mathrm{BU}_1$ denote the inclusion of the real points. It follows from Proposition 2.29 that $\iota_{\mathbb{R}}^*\mathbf{c}_1 = \omega_1 \cdot a$, where $\omega_1 \in H^1(\mathrm{BO}_1;\mathbb{F}_2)$ is the generator. Now, from Corollary 2.33 we get

$$\iota_{\mathbb{R}}^*\bar{\mathbf{c}}_1 = \iota_{\mathbb{R}}^*\boldsymbol{\varrho}\mathbf{c}_1 = \boldsymbol{\varrho}\iota_{\mathbb{R}}^*\mathbf{c}_1 = \boldsymbol{\varrho}(\omega_1 \cdot a) = \omega_1\rho + \mathrm{Sq}^1\,\omega_1 \cdot \tau = \mathrm{Sq}_{\rho,\tau}(\omega_1). \tag{3.1}$$

Furthermore, it follows from Remark 3.2 that $\iota_{\mathbb{R}}^*\bar{\mathbf{c}}_1^k = \mathrm{Sq}_{\rho,\tau}(\omega_1)^k = \mathrm{Sq}_{\rho,\tau}(\omega_1^k)$. As a result, all restrictions of equivariant classes of degree $(2k,k)$ from BU_1 to BO_1 are total Steenrod squares.

Example 3.4. Consider the Real space $X = \mathbb{F}_2 \otimes S^{2,1}$. This is a $K(\mathbb{F}_2(1),2)$ space and, as such, it comes with a canonical class $\iota_{2,1} \in H^{2,1}(X;\underline{\mathbb{F}_2})$, which is classified by the identity map. It follows from Remark 2.19 that the set of real points of X can be written as the following product:

$$X(\mathbb{R}) \cong \mathbb{F}_2 \otimes S^{1,0} \times \mathbb{F}_2 \otimes S^2.$$

Denote by $\iota_1 \in H^1(\mathbb{F}_2 \otimes S^{1,0};\mathbb{F}_2)$ and $\iota_2 \in H^2(\mathbb{F}_2 \otimes S^2;\mathbb{F}_2)$ the canonical classes. We have $\iota_{\mathbb{R}}^*\iota_{2,1} = \alpha_1\rho + \alpha_2\tau$, with $\alpha_i \in H^i(X(\mathbb{R});\mathbb{F}_2)$.

From Corollary 2.30 it follows that $\alpha_2 = \mathscr{R}\iota_{\mathbb{R}}^*\iota_{2,1} = 1 \times \iota_2$, and that α_1 is classified by $\bar{\chi}_{1,1} \circ \iota_{\mathbb{R}}$. Therefore, $\alpha_1 = \iota_1 \times 1$ and one gets

$$\iota_{\mathbb{R}}^*\iota_{2,1} = (1 \times \iota_2)\rho + (\iota_1 \times 1)\tau.$$

We conclude that $\iota_{2,1}$ does not restrict to a total Steenrod square.

Example 3.5. Let X be a based space, and consider the coinduced C_2-space $\mathrm{N}^{C_2}X := X \wedge X$, with the transposition involution $\varsigma(x \wedge y) := (y \wedge x)$. Given $\alpha \in \widetilde{H}^k(X;\mathbb{F}_2)$ classified by a map $f\colon X \to \mathbb{F}_2 \otimes S^k$, consider the composition

$$X \wedge X \xrightarrow{f \wedge f} \mathbb{F}_2 \otimes S^k \wedge \mathbb{F}_2 \otimes S^k \xrightarrow{\wedge} \mathbb{F}_2 \otimes S^{2k} \xrightarrow{T_{k*}} \mathbb{F}_2 \otimes S^{2k,k},$$

where $\wedge$ denotes the additive map induced by $(x,y) \mapsto x \wedge y$ from $S^k \times S^k$ to S^{2k}, and $T_{k*}\colon \mathbb{F}_2 \otimes S^{2k} \to \mathbb{F}_2 \otimes S^{2k,k}$ is the homeomorphism induced by the linear isomorphism

$$T_k\colon \mathbb{R}^k \oplus \mathbb{R}^k \ni (v,w) \mapsto (v+w, v-w) \in \mathbb{R}^{2k,k}.$$

The result is an equivariant map $\mathcal{P}(f)\colon \mathrm{N}^{C_2}X \to \mathbb{F}_2 \otimes S^{2k,k}$ classifying a cohomology class $\mathcal{P}(\alpha) \in \widetilde{H}^{2k,k}(\mathrm{N}^{C_2}X;\underline{\mathbb{F}_2})$ that satisfies $\mathscr{R}\mathcal{P}(\alpha) = \alpha \times \alpha$.

In the universal case $X = \mathbb{F}_2 \otimes S^k$, one takes the generator $\alpha = \iota_k \in \widetilde{H}^k(\mathbb{F}_2 \otimes S^k; \mathbb{F}_2)$ and the resulting class $\mathcal{P}(\iota_k)$ is represented by the map

$$\mathbb{F}_2 \otimes S^k \wedge \mathbb{F}_2 \otimes S^k \xrightarrow{\wedge} \mathbb{F}_2 \otimes S^{2k} \xrightarrow{T_{k*}} \mathbb{F}_2 \otimes S^{2k,k}.$$

Restricting $\mathcal{P}(\alpha)$ to the fixed points $(\mathrm{N}^{C_2}X)(\mathbb{R}) = X$ and using the decomposition $\widetilde{H}^{2k,k}(X; \underline{\mathbb{F}_2}) \cong \oplus_{i=k}^{2k} \widetilde{H}^i_{\mathrm{sing}}(X; \mathbb{F}_2)$ we obtain classes in $H^{k+i}(X; \mathbb{F}_2)$ for $0 \le i \le k$ that determine the restricted class.

Denote by $\mathcal{P}_{\mathrm{bor}}(\alpha) \in \widetilde{H}^{2k,k}(\mathrm{N}^{C_2}X \wedge EC_{2+}; \underline{\mathbb{F}_2})$ the pullback of $\mathcal{P}(\alpha)$ under the projection $\pi \colon X \times EC_2 \to X$ and note that $\widetilde{H}^{*,*}(- \wedge EC_{2+}; \underline{\mathbb{F}_2})$ is $(0,1)$-periodic because $\widetilde{H}^{*,*}(S^{0,0} \wedge EC_{2+}; \underline{\mathbb{F}_2}) = \mathbb{F}_2[\rho, \tau, \tau^{-1}]$ (see [4, Prop.1.15]).

It follows that

$$\mathcal{P}_{\mathrm{bor}}(\alpha)\tau^{-k} \in \widetilde{H}^{2k,0}(\mathrm{N}^{C_2}X \wedge EC_{2+}; \underline{\mathbb{F}_2}) = \widetilde{H}^{2k}(\mathrm{N}^{C_2}X \wedge_{C_2} EC_{2+}; \mathbb{F}_2)$$

is the class $P_{C_2}(\alpha)$ discussed in [15], where it is called the *Steenrod power* of α. It is shown in [15] that the restriction of $P_{C_2}(\alpha)$ to the fixed points of $\mathrm{N}^{C_2}X \wedge_{C_2} EC_{2+}$ yields a class corresponding to total Steenrod square under the natural decomposition $\widetilde{H}^{2k}(X \wedge BC_{2+}; \mathbb{F}_2) = \oplus_{i=0}^{2k} \widetilde{H}^i(X; \mathbb{F}_2)$. Now, the pullback homomorphism

$$\pi^* \colon H^{2k,k}(\{\mathrm{N}^{C_2}X\}(\mathbb{R}); \underline{\mathbb{F}_2}) \to H^{2k,k}(\{\mathrm{N}^{C_2}X\}(\mathbb{R}) \wedge EC_{2+}; \underline{\mathbb{F}_2})$$

is determined by $\pi^* \colon \mathbb{M}_2^+ \to \widetilde{H}^{*,*}(S^{0,0} \wedge EC_{2+}; \underline{\mathbb{F}_2})$, which is the monomorphism $\rho \mapsto \rho$, $\tau \mapsto \tau$.

We conclude that under the identification of X with $\{\mathrm{N}^{C_2}X\}(\mathbb{R})$ via the diagonal map, we have

$$\iota_{\mathbb{R}}^* \mathcal{P}(\alpha) = \mathrm{Sq}_{\rho,\tau}(\alpha). \tag{3.2}$$

Example 3.6. Once again, consider the C_2-space $\mathrm{N}^{C_2}X$ for a based space X. One can produce a transformation $\mathcal{P}_{\mathbb{Z}} \colon H^k_{\mathrm{sing}}(X; \mathbb{Z}) \to H^{2k,k}(\mathrm{N}^{C_2}X; \underline{\mathbb{Z}})$ by modifying the previous example. The universal case is classified by the map

$$\mathrm{N}^{C_2}(\mathbb{Z} \otimes S^k) = \mathbb{Z} \otimes S^k \wedge \mathbb{Z} \otimes S^k \xrightarrow{\ \wedge\ } \mathbb{Z} \otimes S^{2k} \xrightarrow{\ T_{k*}\ } \mathbb{Z} \otimes S^{2k,k}.$$

Given $\alpha \in \widetilde{H}^k(X; \mathbb{Z})$ and restricting $\mathcal{P}_{\mathbb{Z}}(\alpha)$ to $X = (\mathrm{N}^{C_2}X)(\mathbb{R})$ yields classes $\mathcal{P}^i_{\mathbb{Z}}(\alpha)$, with

$$\mathcal{P}^i_{\mathbb{Z}}(\alpha) \in \begin{cases} H^{k+2i}_{\mathrm{sing}}(X; \mathbb{F}_2), & \text{if } 0 \le i < k/2; \\ H^{2k}_{\mathrm{sing}}(X; \mathbb{Z}), & \text{if } k \text{ is even, and } i = k/2. \end{cases}$$

Now apply mod 2 reduction of coefficients, as in the previous example, to obtain $\boldsymbol{\varrho}\iota_{\mathbb{R}}^{*}\mathcal{P}(\alpha) = \mathrm{Sq}_{\rho,\tau}(\boldsymbol{\varrho}\alpha) \in H^{2k,k}(X;\underline{\mathbb{F}_2})$. It follows from (2.20) and Corollary 2.30 that

$$\mathcal{P}^i_{\mathbb{Z}}(\alpha) = \begin{cases} \mathrm{Sq}^{2i}(\boldsymbol{\varrho}\alpha), & \text{if } 0 \le i < k/2; \\ \alpha^2, & \text{if } k \text{ is even, and } i = k/2. \end{cases} \tag{3.3}$$

3.2. Poincaré duality and total Steenrod squares

In this section we discuss Poincaré duality for Real manifolds. We consider a compact Real manifold X and a Real submanifold $Y \subset X$ of codimension d. In the case of $\mathbb{F}_2$ coefficients, we identify the restriction to $X(\mathbb{R})$ of the Poincaré dual $\alpha_Y \in H^{2d,d}(X;\underline{\mathbb{F}_2})$ with the total Steenrod square $\mathrm{Sq}_{\rho,\tau}\left(\alpha_{Y(\mathbb{R})}\right) \in H^{2d,d}(X(\mathbb{R});\underline{\mathbb{F}_2})$ of the dual $\alpha_{Y(\mathbb{R})}$ of $Y(\mathbb{R})$ in $X(\mathbb{R})$.

Definition 3.7. *Let (X,ς) be a Real space and let $E \to X$ be a vector (real or complex) bundle. A C_2-*vector bundle *structure on E is a* linear *involution $\tilde{\varsigma}\colon E \to E$ that covers ς.*

Let (X,ς) be a Real space and let $E \to X$ be a complex vector bundle of rank n. A Real vector bundle *structure on E is an* anti-linear *involution $\tilde{\varsigma}\colon E \to E$ that covers ς. We say that $(E,\tilde{\varsigma}) \to (X,\varsigma)$ is a Real vector bundle.*

Note that, in particular, a rank r *Real* vector bundle is a C_2-real vector bundle of rank $2r$.

Example 3.8. Let $\mathrm{Gr}_{r,n}$ denote the real Grassmanian scheme parametrizing r-planes in affine n-space. Then $\mathrm{Gr}_{r,n}(\mathbb{C})$ is a *Real variety* and the tautological r-plane bundle $\gamma^r \to \mathrm{Gr}_{r,n}(\mathbb{C})$ is a rank r Real vector bundle.

Definition 3.9. *A smooth manifold X is called a* Real manifold *if it has a Real space structure (X,ς) and a compatible Real vector bundle structure on the tangent bundle TX (that is, covering ς).*

Example 3.10. If X is a smooth real algebraic variety then $X(\mathbb{C})$ with the involution ς determined by complex conjugation is a Real manifold.

Example 3.11. Let X be a complex n-manifold and consider its *conjugate manifold* X^{ς}, which comes with a natural anti-holomorphic homeomorphism $\varsigma\colon X \to X^{\varsigma}$. The complex manifold $R_{\mathbb{C}/\mathbb{R}}X := X \times X^{\varsigma}$ becomes a *Real* $2n$*-manifold* under the natural anti-holomorphic involution $T_{\varsigma}\colon (x,y) \mapsto (\varsigma^{-1}y, \varsigma x)$. Furthermore, one has a diffeomorphism $\Delta_{\varsigma}\colon X \cong \{R_{\mathbb{C}/\mathbb{R}}X\}(\mathbb{R})$ given by $x \mapsto (x,\varsigma x)$. As a C_2-space $R_{\mathbb{C}/\mathbb{R}}X$ is isomorphic to $\mathrm{N}^{C_2}X_+$.

Remark 3.12. More generally, when X is an algebraic scheme over $\mathbb{C}$, one defines its *Weil restriction* $R_{\mathbb{C}/\mathbb{R}}X$ which is a real scheme satisfying $R_{\mathbb{C}/\mathbb{R}}X(\mathbb{R}) = X(\mathbb{C})$ and $R_{\mathbb{C}/\mathbb{R}}X(\mathbb{C}) = X(\mathbb{C}) \times_{\mathbb{C}} (\varsigma^* X)(\mathbb{C})$, where $\varsigma^* X = X \times_\varsigma \operatorname{Spec}(\mathbb{C})$ and ς is the automorphism of $\operatorname{Spec}(\mathbb{C})$ given by complex conjugation.

Definition 3.13 ([12]). *Let $\mathscr{E}$ be an $\mathrm{RO}(C_2)$-graded equivariant cohomology theory and let $E \to X$ be C_2-real vector bundle rank r over a Real space X. Denote by $\mathrm{Th}(E)$ the Thom space of E. An $\mathscr{E}$-orientation of E is a class $\mu \in \mathscr{E}^{r,s}(\mathrm{Th}(E))$ for some $s \geq 0$ such that, for each $K < C_2$ and, for each inclusion $i\colon C_2/K \to X$, the restriction $i^*\alpha \in \mathscr{E}^{2r,s}(\mathrm{Th}(i^*E); \underline{\mathbb{Z}})$ is a generator of the free $\mathscr{E}^{*,*}(C_2/K)$-module $\mathscr{E}^{*,*}(\mathrm{Th}(i^*E))$.*

In [4, Prop. 1.11] it is shown that Real vector bundles $E \to X$ of rank r have $H\underline{\mathbb{Z}}$-orientation classes of bidegree $(2r, r)$. Applying mod 2 reduction, it follows that Real vector bundles are $H\underline{\mathbb{F}_2}$-oriented.

Theorem 3.14. *A Real compact manifold X of dimension n satisfies equivariant Poincaré duality, which takes the following form in motivic notation:*

$$\mathrm{PD}\colon H^{r,s}(X; \underline{\mathbb{F}_2}) \to H_{2n-r,n-s}(X; \underline{\mathbb{F}_2}).$$

Furthermore, the isomorphism PD *is given by the cap product with* the fundamental equivariant homology class $[X] \in H_{2n,n}(X; \underline{\mathbb{F}_2}) \cong \mathbb{F}_2$.

Proof. Since as remarked above TX is $H\underline{\mathbb{F}_2}$-oriented with orientation in degree $(2n, n)$, the result is a direct application of [12, Prop. III.6.5]. $\square$

Remark 3.15. The restriction of a rank r Real vector bundle $E \to X$ over a Real space X to its real points $X(\mathbb{R})$ is the complexification of a rank r real vector bundle denoted E^{C_2} or $E(\mathbb{R})$.

Example 3.16. The Thom space $\mathrm{Th}(E)$ of a Real rank r vector bundle $E \to X$ is a Real space whose set of real points is the Thom space $\mathrm{Th}(E^{C_2})$ of E^{C_2}.

Proposition 3.17. *Given a Real compact submanifold $i\colon Y \hookrightarrow X$ of dimension m the cohomology class $\mathrm{PD}^{-1}(i_*[Y]) \in H^{2(n-m),n-m}(X; \underline{\mathbb{F}_2})$ is represented by the Thom class of the normal bundle of Y in X. It is called the* Poincaré dual of Y.

Proof. Let $\pi\colon N_{Y/X} \to Y$ be the normal bundle to Y in X, identified with a tubular neighborhood of Y in X. Denote by $\mathrm{Th}(N_{Y/X})$ its Thom-space, and let

$$\mu \in \widetilde{H}^{2n-2m,n-m}(\mathrm{Th}(N_{Y/X}); \underline{\mathbb{F}_2}) = H^{2n-2m,n-m}(X, X - N_{Y/X}; \underline{\mathbb{F}_2})$$

be the corresponding Thom class. Now, respectively denote by

$$[X] \in H_{2n,n}(X;\underline{\mathbb{F}_2}) \quad \text{and} \quad [Y] \in H_{2m,m}(Y;\underline{\mathbb{F}_2})$$

the fundamental classes of X and Y. We need to show that $\mu \cap [X] = i_*([Y])$.

We have $\widetilde{H}_{2n,n}(\mathrm{Th}(N_{Y/X});\underline{\mathbb{F}_2}) = H_{2n,n}(X, X - N_{Y/X};\underline{\mathbb{F}_2}) \cong \mathbb{F}_2$, and using the restriction functor $\mathscr{R}$ one sees that the image of $[X]$ in $H_{2n,n}(X, X - N_{Y/X};\underline{\mathbb{F}_2})$ is the generator. Now, taking the cap product with μ yields a map

$$\mu \cap -\colon H_{2n,n}(X, X - N_{Y/X};\underline{\mathbb{F}_2}) \to H_{2m,m}(X, X - N_{Y/X};\underline{\mathbb{F}_2})$$

that factors through $H_{2m,m}(N_{Y/X};\underline{\mathbb{F}_2}) \to H_{2m,m}(X, X - N_{Y/X};\underline{\mathbb{F}_2})$, and the composition with $\pi_*\colon H_{2m,m}(N_{Y/X};\underline{\mathbb{F}_2}) \to H_{2m,m}(Y;\underline{\mathbb{F}_2})$ gives the Thom isomorphism map. This is proved in [4, Prop.1.13] with $\underline{\mathbb{Z}}$ coefficients. The proof in the case of $\underline{\mathbb{F}_2}$ coefficients is exactly the same. □

Definition 3.18. *Let M be a compact n-dimensional manifold with connected components $M_1, \dots, M_r$. If $\theta \in H_k(M;\mathbb{F}_2)$ is given by $\theta = \sum_{i=1}^r \theta_i$ with $\theta_i \in H(M_i;\mathbb{F}_2)$, we write its* Poincaré dual *as $\mathrm{PD}^{-1}(\theta) := \sum_{i=1}^r \mathrm{PD}^{-1}(\theta_i)$, where $\mathrm{PD}^{-1}(\theta_i)$ is the Poincaré dual of θ_i in M_i.*

Theorem 3.19. *Let X be a compact Real manifold of dimension n and let $Y \hookrightarrow X$ be a Real submanifold of dimension m. Set $d = n - m$, and denote by $[Y] \in H_{2m,m}(X;\underline{\mathbb{F}_2})$ the homology class represented by Y and by $[Y(\mathbb{R})]$ the corresponding class of $Y(\mathbb{R})$ in $H_{2m}(X(\mathbb{R});\mathbb{F}_2)$. Let*

$$\alpha_Y \in H^{2d,d}(X;\underline{\mathbb{F}_2}) \quad \text{and} \quad \alpha_{Y(\mathbb{R})} \in H^d(X(\mathbb{R});\mathbb{F}_2)$$

denote the Poincaré duals of $[Y]$ and $[Y(\mathbb{R})]$, respectively. Then we have

$$\iota_{\mathbb{R}}^* \alpha_Y = \mathrm{Sq}_{\rho,\tau}\left(\alpha_{Y(\mathbb{R})}\right),$$

where $\iota_{\mathbb{R}}$ denotes the inclusion of the set of components of $X(\mathbb{R})$.

Proof. Abbreviate $H^{*,*}(-;\underline{\mathbb{F}_2})$ to $H^{*,*}(-)$. Let $\pi\colon N_{Y/X} \to Y$ denote the normal bundle to Y in X. Note that the restriction of π to the real points, $\pi(\mathbb{R})\colon N_{Y/X}(\mathbb{R}) \to Y(\mathbb{R})$ is the normal bundle to $Y(\mathbb{R})$ in $X(\mathbb{R})$.

Let $\mathrm{Th}(N_{Y/X})$ and $\mathrm{Th}(N_{Y/X}(\mathbb{R}))$ be the Thom-spaces of $N_{Y/X}$ and $N_{Y/X}(\mathbb{R})$, respectively, and let $t_{X/Y} \in H^{2d,d}(\mathrm{Th}(N_{Y/X}))$ and $t_{X/Y}(\mathbb{R}) \in H^d_{\mathrm{sing}}(\mathrm{Th}(N_{Y/X}(\mathbb{R}));\mathbb{F}_2)$ denote the corresponding Thom classes. By Proposition 3.17, the result will follow once we show that $\iota_{\mathbb{R}}^*(t_{X/Y}) = \mathrm{Sq}_{\rho,\tau}(t_{X/Y}(\mathbb{R}))$.

We have $\mathrm{Th}(N_{Y/X}) = \mathbb{P}(N_{Y/X} \oplus \mathbf{1})/\mathbb{P}(N_{Y/X})$, and the equivariant projective bundle formula yields a short exact sequence

$$0 \to \widetilde{H}^{2d,d}(\mathrm{Th}(N_{Y/X})) \to H^{2d,d}(\mathbb{P}(N_{Y/X} \oplus \mathbf{1})) \to H^{2d,d}(\mathbb{P}(N_{Y/X})) \to 0.$$

Therefore $\widetilde{H}^{2d,d}(\mathrm{Th}(N_{Y/X}))$ is identified with the kernel of the surjective map of the sequence and, if $\xi \in H^{2,1}(\mathbb{P}(N_{Y/X} \oplus \mathbf{1}))$ denotes the first Chern class of the Real line bundle $\mathcal{O}(1)$, then $t_{X/Y}$ is identified with its generator:

$$t_{X/Y} = \xi^d + \xi^{d-1}\bar{\mathbf{c}}_1(N_{Y/X}) + \cdots + \bar{\mathbf{c}}_d(N_{Y/X}).$$

Since the restriction of π to the real points $\pi(\mathbb{R})\colon N_{Y/X}(\mathbb{R}) \to Y(\mathbb{R})$ is the normal bundle to $Y(\mathbb{R})$ in $X(\mathbb{R})$, we have

$$\mathrm{Th}(N_{Y/X}(\mathbb{R})) = \mathbb{P}_{\mathbb{R}}(N_{Y/X}(\mathbb{R}) \oplus \mathbf{1}_{\mathbb{R}})/\mathbb{P}_{\mathbb{R}}(N_{Y/X}(\mathbb{R})).$$

By the usual projective bundle formula for real bundles, the Thom class $t_{X/Y}(\mathbb{R}) \in H^d_{\mathrm{sing}}(\mathrm{Th}(N_{Y/X}(\mathbb{R})); \mathbb{F}_2)$ is

$$t_{X/Y}(\mathbb{R}) = \mathsf{w}_1^d + \mathsf{w}_1^{d-1}\omega_1(N_{Y/X}(\mathbb{R})) + \cdots + \omega_d(N_{Y/X}(\mathbb{R})),$$

where $\mathsf{w}_1 \in H^1_{\mathrm{sing}}(\mathbb{P}_{\mathbb{R}}(N_{Y/X}(\mathbb{R}) \oplus \mathbf{1}_{\mathbb{R}}); \mathbb{F}_2)$ is the Stiefel-Whitney class of the real line bundle $\mathcal{O}(1)$ over $\mathbb{P}_{\mathbb{R}}(N_{Y/X}(\mathbb{R}) \oplus \mathbf{1}_{\mathbb{R}})$.

It follows from (3.1) that $\iota^*_{\mathbb{R}}\xi = \mathrm{Sq}_{\rho,\tau}(\mathsf{w}_1)$ and more generally from (4.3) below that $\iota^*_{\mathbb{R}}\bar{\mathbf{c}}_i(N_{Y/X}) = \mathrm{Sq}_{\rho,\tau}(\omega_i(N_{Y/X}(\mathbb{R})))$. Finally, since $\mathrm{Sq}_{\rho,\tau}$ is a ring homomorphism, we get

$$\begin{aligned}
\iota^*_{\mathbb{R}}(t_{X/Y}) &= \iota^*_{\mathbb{R}}\left(\xi^d + \xi^{d-1}c_1(N_{Y/X}) + \cdots + c_d(N_{Y/X})\right) \\
&= \mathrm{Sq}_{\rho,\tau}\left(\mathsf{w}_1^d + \mathsf{w}_1^{d-1}\omega_1(N_{Y/X}(\mathbb{R})) + \cdots + \omega_d(N_{Y/X}(\mathbb{R}))\right) \\
&= \mathrm{Sq}_{\rho,\tau}(t_{X/Y}(\mathbb{R})).
\end{aligned}$$ □

Example 3.20. Consider $X = \mathbb{C}/\{\mathbb{Z} + \sqrt{-1}\mathbb{Z}\}$ with the structure of Real compact manifold on dimension 1 induced by complex conjugation. It is the set of complex points of an elliptic curve defined over $\mathbb{Q}$.

As C_2-space, $X \cong S^{1,0} \times S^{1,1}$. Under this identification, we have $X(\mathbb{R}) = S^{1,0} \times \{0, \infty\}$ and the group $H^{2,1}(X; \underline{\mathbb{F}_2}) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$ is generated by $\eta := \eta_{1,0} \times \eta_{1,1}$ and $\theta := \rho\eta_{1,0} \times 1$, where

$$\eta_{1,0} \in \widetilde{H}^{1,0}(S^{1,0}; \underline{\mathbb{F}_2}) = \mathbb{F}_2, \qquad \eta_{1,1} \in \widetilde{H}^{1,1}(S^{1,1}; \underline{\mathbb{F}_2}) = \mathbb{F}_2,$$

are generators and ∞ is used as the base point in both representation spheres.

Consider the Real submanifolds of dimension zero $Y_0 := \{(0,0)\}$ and $Y_\infty := \{(0,\infty)\}$. Denote by $[Y_0]$ and $[Y_\infty]$ the corresponding generators in $H_{0,0}(X;\underline{\mathbb{F}_2}) \cong H_0(X(\mathbb{R});\mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$.

It is easy to check that $\eta_{1,0} \cap [S^{1,0}] = [\{0\}]$ and $\eta_{1,1} \cap [S^{1,1}] = [\{0\}]$, hence $\eta \cap [X] = [Y_0]$. Applying the restriction functor gives $\mathscr{R}(\theta \cap [X]) = 0$ and, therefore, $\mathrm{PD}(\theta) = [Y_0] + [Y_\infty]$. Using the notation of Theorem 3.19 gives $\alpha_{Y_0} = \eta$ and $\alpha_{Y_\infty} = \eta + \theta$. For the Poincaré duals of Y_0 and Y_∞ in $X(\mathbb{R})$, we have $\alpha_{Y_0(\mathbb{R})} = \eta_1 \times 1_{\{0\}}$ and $\alpha_{Y_\infty(\mathbb{R})} = \eta_1 \times 1_{\{\infty\}}$, where $\eta_1 = \mathscr{R}(\eta_{1,0}) \in H^1_{\mathrm{sing}}(S^{1,0};\mathbb{F}_2)$ is the generator, and $1_{\{0\}}$, $1_{\{\infty\}}$ denote the obvious idempotents in $H^0(\{0,\infty\};\mathbb{F}_2)$.

Noting that $i_0^*\eta_{1,1} = \rho$ and $i_\infty^*\eta_{1,1} = 0$, where i_0, i_∞ denote the inclusions of $\{0\}$ and $\{\infty\}$ in $S^{1,1}$, we see that restriction to the real points yields

$$\begin{aligned} \iota_{\mathbb{R}}^*\alpha_{X_0} &= \rho\eta_{1,0} \times 1_{\{0\}} = \mathrm{Sq}_{\rho,\tau}\left(\eta_1 \times 1_{\{0\}}\right), \qquad \text{and} \\ \iota_{\mathbb{R}}^*\alpha_{X_\infty} &= \rho\eta_{1,0} \times 1_{\{0\}} + \rho\eta_{1,0} \times 1_{\{0,\infty\}} = \rho\eta_{1,0} \times 1_{\{\infty\}} \\ &= \mathrm{Sq}_{\rho,\tau}\left(\eta_1 \times 1_{\{\infty\}}\right). \end{aligned}$$

4. Example: BU_n

In this section we consider the maps appearing in “the cube” (Figure 1), when $X = \mathrm{BU}_n$ under the complex conjugation action. It is well known, see [5], that the equivariant cohomology of BU_n with $\underline{\mathbb{Z}}$ or $\underline{\mathbb{F}_2}$ coefficients is free over the cohomology of a point:

$$\begin{aligned} H^{*,\bullet}(\mathrm{BU}_n;\underline{\mathbb{Z}}) &= \mathbb{M}[\mathbf{c}_1,\dots,\mathbf{c}_n] \\ H^{*,\bullet}(\mathrm{BU}_n;\underline{\mathbb{F}_2}) &= \mathbb{M}_2[\bar{\mathbf{c}}_1,\dots,\bar{\mathbf{c}}_n], \end{aligned}$$

where $\mathbf{c}_j \in H^{2j,j}(\mathrm{BU}_n;\underline{\mathbb{Z}})$, is the j-th *equivariant Chern class* of the universal quotient bundle over BU_n and $\bar{\mathbf{c}}_j \in H^{2j,j}(\mathrm{BU}_n;\underline{\mathbb{F}_2})$ is its *reduction* mod 2.

For notational simplicity, instead of using the full positive cone the equivariant cohomology of BU_n we will compute the maps of the “the cube” on subrings generated by the bases $\{\mathbf{c}_1,\dots,\mathbf{c}_n\}$ and $\{\bar{\mathbf{c}}_1,\dots,\bar{\mathbf{c}}_n\}$ over the positive cones of the coefficients:

$$H^{*,\bullet}(\mathrm{BU}_n;\underline{\mathbb{Z}})_\dagger := \mathbb{M}_+[\mathbf{c}_1,\dots,\mathbf{c}_n] = \mathbb{Z}[a,u;\mathbf{c}_1,\dots,\mathbf{c}_n] \tag{4.1}$$

$$H^{*,\bullet}(\mathrm{BU}_n;\underline{\mathbb{F}_2})_\dagger := \mathbb{M}_{2+}[\bar{\mathbf{c}}_1,\dots,\bar{\mathbf{c}}_n] = \mathbb{F}_2[\rho,\tau;\bar{\mathbf{c}}_1,\dots,\bar{\mathbf{c}}_n], \tag{4.2}$$

This avoids dealing with coefficients in the negative cones of $\mathbb{M}$ and $\mathbb{M}_2$.

On the other hand, the integral cohomology ring of BO_n can be written as a quotient $H^*_{\mathrm{sing}}(\mathrm{BO}_n;\mathbb{Z}) \cong \mathcal{R}_n/\mathcal{I}_n$ of a polynomial ring $\mathcal{R}_n[p_j, \mathbf{y}_I]$ in a certain collection of variables $\{p_j, \mathbf{y}_I \mid j = 1, \dots, n,\ I \in \mathcal{J}_n\}$; see [2, 7]. For the reader's convenience, we recall this presentation in Appendix A.

Using the notation from Theorem A.1, the various cohomology groups of BU_n and BO_n are summarized in the table below.

Table 1: Cohomology groups

	BU_n	BO_n
$H^*_{\mathrm{sing}}(-;\mathbb{Z})$	$\mathbb{Z}[c_1,\dots,c_n]$	$\mathbb{Z}[p_j,\mathbf{y}_I]/\mathcal{I}_n$
$H^{*,\bullet}(-;\underline{\mathbb{Z}})_\dagger$	$\mathbb{Z}[a,u;\mathbf{c}_1,\dots,\mathbf{c}_n]$	$\mathbb{Z}[u;p_j,\mathbf{y}_I]/\mathcal{I}_n \oplus a\mathbb{F}_2[a,u,w_1,\dots,w_n]$
$H^{*,\bullet}(-;\underline{\mathbb{F}_2})_\dagger$	$\mathbb{F}_2[\rho,\tau;\bar{\mathbf{c}}_1,\dots,\bar{\mathbf{c}}_n]$	$\mathbb{F}_2[\rho,\tau;w_1,\dots,w_n]$
$H^*_{\mathrm{sing}}(-;\mathbb{F}_2)$	$\mathbb{F}_2[\bar{c}_1,\dots,\bar{c}_n]$	$\mathbb{F}_2[w_1,\dots,w_n]$

In the next result we use the following notation:

$$w_{2I} := w_{2i_1}\cdots w_{2i_r} \in H^{2|I|}_{\mathrm{sing}}(\mathrm{BO}_n;\mathbb{F}_2), \quad \text{when} \quad I = (i_1,\dots,i_r)$$

$$\beta_{2j} := \sum_{k=0}^{j-1} \mathrm{Sq}^{2k}(w_{2j})\cdot a^{2j-2k}u^k \ + \ (-1)^j p_j \cdot u^j \in \ H^{4j,2j}(\mathrm{BO}_n;\underline{\mathbb{Z}})$$

$$\beta_{2j+1} := \sum_{k=0}^{j} \mathrm{Sq}^{2k}(w_{2j+1})\cdot a^{2j+1-2k}u^k \in H^{4j+2,2j+1}(\mathrm{BO}_n;\underline{\mathbb{Z}})$$

Proposition 4.1. *If $X = BU_n$ under complex conjugation, then $X(\mathbb{R}) = BO_n$ and all maps in Figure 1 are determined by the change of coefficient maps $\mathbb{M} \to \mathbb{M}_2$ and the diagram below, where the cohomology of BU_n is displayed on the left column and that of BO_n is on the right. (See Table 1 for notation.)*

Proof. To understand $\iota_{\mathbb{R}}^*$, consider the commutative diagram

$$\begin{tikzcd} BU_1 \times \cdots \times BU_1 \arrow[r, "\eta"] & \mathrm{BU}_n \\ BO_1 \times \cdots \times BO_1 \arrow[r, "\eta_{\mathbb{R}}"'] \arrow[u, "j_{\mathbb{R}}"] & \mathrm{BO}_n \arrow[u, "\iota_{\mathbb{R}}"'] \end{tikzcd}$$

where the horizontal maps classify n-fold direct sums of line bundles and the vertical ones denote inclusion of the real points, which classify the complex-

$$
\begin{tikzcd}
 & & \mathbf{c}_k \arrow[rrr, mapsto, dotted] & & & \beta_k & \\
 & \mathbb{Z}[a,u;\mathbf{c}_1,\dots,\mathbf{c}_n] \arrow[rrrr, "\iota_{\mathbb{R}}^*"] \arrow[ddd, "\varrho"] & & & & \frac{\mathbb{Z}[u,p_j,\mathbf{y}_I]}{\mathcal{I}_n} \oplus a\mathbb{F}_2[a,u;w_r] \arrow[ddd, "\varrho"] & \\
\mathbf{c}_k \arrow[d, mapsto, dotted] & & & \mathbf{y}_I \arrow[d, mapsto, dotted] & p_j \arrow[d, mapsto, dotted] & & w_j a \arrow[d, mapsto, dotted] \\
\bar{\mathbf{c}}_k & & & \mathrm{Sq}^1(w_{2I}) & w_{2j}^2 & & w_j\rho + \mathrm{Sq}^1(w_j)\tau \\
 & \mathbb{F}_2[\rho,\tau;\bar{\mathbf{c}}_1,\dots,\bar{\mathbf{c}}_n] \arrow[rrrr, "\iota_{\mathbb{R}}^*"'] & & & & \mathbb{F}_2[\rho,\tau;w_1,\dots,w_n] & \\
 & & \bar{\mathbf{c}}_k \arrow[rrr, mapsto, dotted] & & & \mathrm{Sq}_{\rho,\tau}(w_k) &
\end{tikzcd}
$$

Figure 2: Maps in the equivariant cohomology (front face of cube).

ification of real line bundles. Using Kunneth formula and Example 3.3 we compute the restriction map $\iota_{\mathbb{R}}^*$ with $\underline{\mathbb{F}_2}$ coefficients as follows. First observe that $j_{\mathbb{R}}^*$ is given by

$$
\begin{array}{ccc}
\mathbb{M}_2^{*,\bullet}[t_1,\dots,t_n] \cong H^{*,\bullet}(BU_1^{\times n};\underline{\mathbb{F}_2}) \supset H^{2,1}(\mathrm{BU}_1^{\times n};\underline{\mathbb{F}_2}) \ni & t_i \\
\big\downarrow j_{\mathbb{R}}^* & \big\downarrow \\
\mathbb{M}_2^{*,\bullet}[x_1,\dots,x_n] \cong H^{*,*}(BO_1^{\times n};\underline{\mathbb{F}_2}) \supset H^{2,1}(BO_1^{\times n};\underline{\mathbb{F}_2}) \ni & \mathrm{Sq}_{\rho,\tau}(x_i).
\end{array}
$$

Next, observe that (3.1) gives

$$
\begin{aligned}
\eta_{\mathbb{R}}^*\,(\iota_{\mathbb{R}}^*\,\bar{\mathbf{c}}_k) &= j_{\mathbb{R}}^*\eta^*\bar{\mathbf{c}}_k = j_{\mathbb{R}}^*\sigma_k(t_1,\dots,t_n) = \sigma_k(j_{\mathbb{R}}^*t_1,\dots,j_{\mathbb{R}}^*t_n) \\
&= \sigma_k(\mathrm{Sq}_{\rho,\tau}\,x_1,\dots,\mathrm{Sq}_{\rho,\tau}\,x_n) = \mathrm{Sq}_{\rho,\tau}\,\sigma_k(x_1,\dots,x_n) = \eta_{\mathbb{R}}^*\,\mathrm{Sq}_{\rho,\tau}\,w_k,
\end{aligned}
$$

where σ_i is the i-th elementary symmetric function. Since $\eta_{\mathbb{R}}^*\colon H^*(\mathrm{BO}_n;\mathbb{F}_2) \to H^*(BO_1^{\times n};\mathbb{F}_2)$ is injective, it follows from the decomposition (1.6) that so is the map on equivariant cohomology $\eta_{\mathbb{R}}^*\colon H^{*,\bullet}(\mathrm{BO}_n;\underline{\mathbb{F}_2}) \to H^{*,\bullet}(BO_1^{\times n};\underline{\mathbb{F}_2})$. Therefore,

$$
\iota_{\mathbb{R}}^*\bar{\mathbf{c}}_k \;=\; \mathrm{Sq}_{\rho,\tau}(w_k) = \sum_{r=0}^{k} \mathrm{Sq}^r(w_k)\cdot\rho^{k-r}\tau^r. \tag{4.3}
$$

To determine $\iota_{\mathbb{R}}^*$ with $\underline{\mathbb{Z}}$ coefficients, first observe that as a consequence of decompositions (1.4), (1.6) and the computation of ϱ in Corollary 2.33, the map

$$H^{*,\bullet}(\mathrm{BO}_n;\underline{\mathbb{Z}}) \xrightarrow{\boldsymbol{\varrho}\oplus\mathscr{R}} H^{*,\bullet}(\mathrm{BO}_n;\underline{\mathbb{F}_2}) \;\oplus\; H^*_{\mathrm{sing}}(\mathrm{BO}_n;\mathbb{Z}(\bullet))^{C_2}$$

is injective. Then, $\iota_{\mathbb{R}}^*$ is completely determined by the rest of the diagram, and a routine verification shows that $\iota_{\mathbb{R}}^*(\mathbf{c}_k) = \beta_k$. $\square$

Corollary 4.2. *Let X be a Real space and let $E \to X$ be a Real vector bundle of rank n. Denote by $\iota_{\mathbb{R}}$ the inclusion $X(\mathbb{R}) \subset X$. Then $\iota_{\mathbb{R}}^* E = E(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ where $E(\mathbb{R}) \to X(\mathbb{R})$ is a real vector bundle of rank n, and we have*

$$\iota_{\mathbb{R}}^*\mathbf{c}_n(E) = \begin{cases} \displaystyle\sum_{k=0}^{j-1} Sq^{2k}(w_{2j}(E(\mathbb{R})))a^{2j-2k}u^k + (-1)^j p_j(E(\mathbb{R}))\cdot u^j, & n = 2j \\ \displaystyle\sum_{k=0}^{j} Sq^{2k}\left(w_{2j+1}(E(\mathbb{R}))\right)\cdot a^{2j+1-2k}u^k, & n = 2j+1. \end{cases}$$

Proposition 4.1 summarizes the most relevant face of Figure 1 (the cube). For the reader's convenience, we conclude the paper with an explicit description of all maps in the cube for BU_n. The diagram below, along with Table 1, exhibits the necessary information. Note that the left column displays the cohomology of BU_n and the right column corresponds to BO_n.

The left vertical arrows have been explained in (4.1) and (4.2), and the non-equivariant $\boldsymbol{\varrho}$ is explained in Theorem A.1. The identity $\mathscr{R}\circ\boldsymbol{\varrho} = \boldsymbol{\varrho}\mathscr{R}$ on the left-hand side results from the fact that the equivariant Chern classes are sent to the non-equivariant ones under the restriction (forgetful) functor.

Recall that the reduction mod 2 of the non-equivariant Chern class c_k of the complexified universal quotient bundle over BO_n is w_k^2 and, along with the fact that $H^*_{\mathrm{sing}}(\mathrm{BO}_n;\mathbb{Z})_{\mathrm{tor}} = {}_2H^*_{\mathrm{sing}}(\mathrm{BO}_n;\mathbb{Z})$ injects into $H^*_{\mathrm{sing}}(\mathrm{BO}_n;\mathbb{F}_2)$ under $\boldsymbol{\varrho}$, one concludes that $\iota_{\mathbb{R}}^*(c_{2j}) = (-1)^j p_j$ and that $\iota_{\mathbb{R}}^*(c_{2j+1}) = p_j\mathbf{y}_{\{1/2\}} + \mathbf{y}^2_{\{j\}}$. Indeed, the latter expression is equal to $\delta\left(w_{2j}w_{2j+1}\right) = \delta\left(\mathrm{Sq}^{2j}(w_{2j+1})\right)$, see [19], for

$$\boldsymbol{\varrho}\circ\delta\left(\mathrm{Sq}^1(w_{2j}w_{2j+1})\right) = \mathrm{Sq}^1(\mathrm{Sq}^{2j}(w_{2j+1})) = \mathrm{Sq}^{2j+1}(w_{2j+1}) = w^2_{2j+1}.$$

This determines $\iota_{\mathbb{R}}^*$. The rest of the diagram is explained in Proposition 4.1.

Appendix A. On the singular cohomology of BO_n

Fix $n \in \mathbb{N}$ and denote $\mathcal{J}_n := \{\frac{1}{2}\} \cup \{1, \ldots, \frac{n}{2}\}$. Now, let $\mathcal{P}_n$ be the set of those non-empty $I \subset \mathcal{J}_n$ such that $\{1/2, n/2\} \not\subset I$, when $n > 1$, and consider the

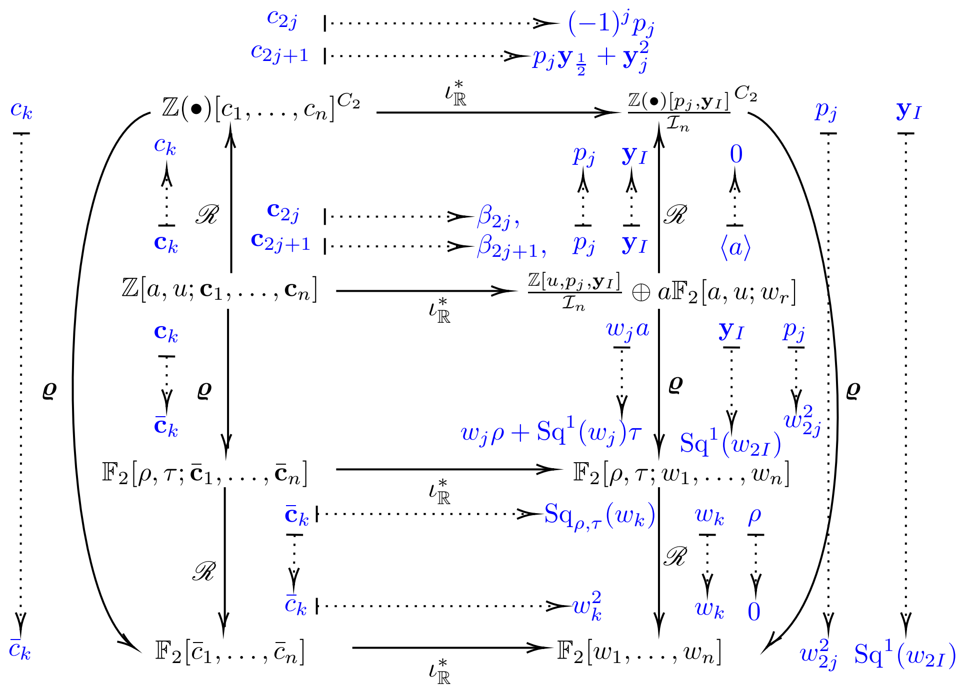

Figure 3: Cube for BU_n explained.

polynomial algebra

$$\mathcal{R}_n := \mathbb{Z}\left[p_i,\ \mathbf{y}_I \mid i \in \mathcal{J}_n,\ I \in \mathcal{P}_n\right]$$

over $\mathbb{Z}$ with generators $p_i,\ \mathbf{y}_I$, where $i \in \mathcal{J}_n$ and $I = \{i_1 < i_2 < \cdots < i_r\} \subset \mathcal{J}_n$, and having degrees $\deg(p_i) = 4i$, $\deg(\mathbf{y}_I) = 1+2|I|$, with $|I| = i_1 + \cdots + i_r$.

Define an ideal $\mathcal{I}_n \subset \mathcal{R}_n$ using the following convention. If n is even and $\{n/2, 1/2\} \subset I \cup J$ then $\mathbf{y}_{I\cup J}$ means $\mathbf{y}_{\{n/2\}}\mathbf{y}_{(I\cup J)-\{n/2,1/2\}}$. The relations generating the ideal $\mathcal{I}_n \subset \mathcal{R}_n$ are as follows.

1. $p_{\{1/2\}} = \mathbf{y}_{\{1/2\}}$
2. $2\mathbf{y}_I = 0$
3. $\mathbf{y}_I\mathbf{y}_J = \begin{cases} \mathbf{y}_{I\cup J}\mathbf{y}_{I\cap J} + \mathbf{y}_{I-J}\mathbf{y}_{J-I} \cdot \prod_{i\in I\cap J} p_i, & \text{if } I \not\subset J, I\cap J \neq \emptyset; \\ \sum_{i\in I} \mathbf{y}_{\{i\}}\mathbf{y}_{(J-I)\cup\{i\}} \cdot \prod_{j\in I-\{i\}} p_j, & \text{if } I \subset J; \\ \sum_{i\in I} \mathbf{y}_{\{i\}}\mathbf{y}_{(I\cup J)-i}, & \text{if } I\cap J = \emptyset \end{cases}$
4. $\sum_{i\in I} \mathbf{y}_{\{i\}}\mathbf{y}_{I-\{i\}} = 0;$
5. If n is even, then $\mathbf{y}_{\{1/2\}}\, p_{n/2} = \mathbf{y}_{\{n/2\}}^2$.

Theorem A.1 ([2, 7])**.** *For $n \leq \infty$, $H^*_{\text{sing}}(BO(n);\mathbb{Z}) \cong \mathcal{R}_n/\mathcal{I}_n$. Under the reduction map $\boldsymbol{\varrho}\colon H^*_{\text{sing}}(BO(n);\mathbb{Z}) \to H^*_{\text{sing}}(BO(n);\mathbb{F}_2)$ the class $\mathbf{y}_I$ maps to $\mathrm{Sq}^1(\prod_{i\in I} w_{2i})$, where w_{2i} is the $2i$-th Stiefel-Whitney class, and the class p_i maps to w_{2i}^2.*

Remark A.2. The generator $\mathbf{y}_I$ corresponds to $\delta\,(w_{2i_1}\cdots w_{2i_r})$, where $I = \{i_1 < i_2 < \cdots < i_r\}$ and δ is the Bockstein map. The last statement in the theorem follows from the identity $\rho\circ\delta = \mathrm{Sq}^1$. Note that w_1 can appear in the expression

$$w_{2I} := \prod_{i\in I} w_{2i}, \tag{A.1}$$

since $1/2 \in \mathcal{J}_n$.

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