Pure and Applied Mathematics Quarterly
Volume 19, Number 6, 2841–2867, 2023

# Quantitative differentiation for $PI$ spaces and spaces with Ricci curvature bounded below

Jeff Cheeger

*Dedicated to Blaine Lawson on the occasion of his 80th birthday with friendship and great admiration*



## 1. Introduction

In this paper, we continue and sharpen the discussions of quantitive differentiation in [CKN11], and [Che12]. We begin with the most basic case, $f : [0,1] \to \mathbb{R}$, with $f$ in the Sobolev space $H^{1,2}$; see Theorem 1.3 below. While there would be no real harm in assuming that $f$ is differentiable, nothing can be assumed about the second derivative, $f''$, which might not exist. Quantitative differentiation concerns behavior on all locations and scales. For the domain, $[0,1]$, by a "location and scale" we mean a dyadic subinterval. The sum of the measures of *all* dyadic subintervals is *infinite*. Quantitative differentiation states in a precise quantitative sense, that $f$ looks as linear we like, apart from a collection of locations and scales, the sum of whose measures has a definite *finite* bound. The proof of Theorem 1.3 is given in Section 2. As explained there, it has a straightforward generalization to the

Received March 22, 2022.
**2010 Mathematics Subject Classification:** Primary 30L05; secondary 28A15, 46B85, 58C20.

case of $\mathbb{R}^m$. Sections 3, 4, deal with $PI$ spaces and riemannian manifolds with Ricci curvature bounded below.

In Section 3, we give a partial generalization of Theorem 1.3 to $PI$ spaces; see Theorem 3.8. These are metric measure spaces $(X, d, \mu)$, for which the measure $\mu$ is doubling and a Poincaré inequality holds. Here, of necessity, *linear* is weakened to *harmonic*; see Example 1.1, Theorem 3.8 and Example 3.1. Under the additional assumption that the norms on the cotangent spaces are uniformly strongly convex, there is an $\epsilon$-regularity theorem. If in addition, $(X, d)$ is a length space and $\mu$-a.e., the squared norm on cotangent spaces has uniformly bounded convexity, there is an implication for blowup limit measures $\mu_\infty$ on tangent cones.[1] By [Che99], $\mu$-a.e., every blowup limit of a Lipschitz function $f$ is a *generalized linear function*, $\ell : X_x \to \mathbb{R}$, on the corresponding tangent cone $X_x$. By definition, this means that $\ell$ is a constant multiple of a harmonic distance function. We will show that each such $\ell \not\equiv 0$ gives rise to a product decomposition, $\mu_\infty = dr \times \mathrm{Per}(X_x^a)$ of the renormalized limit measure $\mu_\infty$ on $X_x$. Here, $r$ is a harmonic distance function and Per is the perimeter measure of a fixed sublevel set $X_x^a := \{y \in X_x \,|\, r(y) \leq a\}$; compare Example 3.1. Since $|dr|^2 \equiv 1$, $\mu_\infty = dr \times \mathrm{Per}(X_x^a)$ can also be viewed as a product decomposition for the Dirichlet energy $|dr|^2\mu_\infty = \mu_\infty$ associated to the harmonic function $r$. Specializing Theorem 3.8 to the case of a distance functions leads to consideration of distance functions $r$ which can be $\epsilon$-approximated by harmonic functions $h$. The quantitative counterpart of the decomposition $dr \times \mathrm{Per}(X_x^a)$ applies to the Dirichlet energy $|dh|^2\mu$.

In Section 4, we briefly indicate how to obtain a partial generalization of the quantitative differentiation theorem for Lipschitz functions on spaces $M^m$ with Ricci curvature bounded below. The result applies to Lipschitz functions and the conclusion does involve approximation by generalized linear functions. The the key issue is that of controlling the oscillation of the gradient of the approximating harmonic function as in Theorem 3.8. The weakening involves the notion of what is considered a "good domain". Namely, rather than approximation by the generalized linear function on the whole domain, the approximation only takes place on subdomains of a definite size. Also, the notion of scale, $0 < \gamma = \gamma(m, \epsilon) < 1$, is allowed to depend on the desired closeness of the approximation. (Recall that in Sections 1–3, we can take $\gamma = 1/2$.)

In Appendix A, we briefly discuss some applications of quantitative differentiation type ideas during the past 25 years. These include a quantitative

[1] The assumptions on the norm hold if the norm on the cotangent bundle is given by an inner product.

bi-Lipschitz nonembedding theorem for the Heisenberg group into the Banach space $L_1$, quantitative structure theory for riemannian manifolds with Ricci curvature bounded below and their Gromov-Hausdorff limit spaces, and quantitative partial regularity theory for nonlinear elliptic and parabolic geometric partial differential equations.

*The basic case* For the basic case, $I = [0,1]$, we assume a definite bound on the Dirichlet energy, $\int_I |f'|^2$. We could as well assume $f$ is smooth, but with no quantitative information concerning its higher derivatives. We are interested in the behavior of $f$ when it is examined on all locations and scales, or equivalently, on all dyadic subintervals. There is an atomic measure on the collection of dyadic intervals which assigns to each such interval $I_{i_n}$, $i_n = 1, \ldots, 2^n$, a mass equal to its its *measure* $\|I_{i_n}\| = 2^{-n}$. For each $n$, we have $\sum_{i_n=1}^{2^n} \|I_{i_n}\| = 1$. Since there are countably values of $n$, the mass of this measure is *infinite*.

Let $|J|$ denote the length of the interval $J \subset I$. Although it so happens that for intervals, length equals measure, with a view towards generalizations we use distinct notations, $|J|$, $\|J\|$, as appropriate.

Let $|f|_J := \sup_{x\in J} |f(x)|$ denote the $L_\infty$ norm of $f \,|\, J$. The *secant line* $f_0$ of $f \,|\, J$, is the unique *affine linear function* which agrees with $f$ at the end points of $J$: $f_0 \,|\, \partial J = f \,|\, \partial J$.

**Definition 1.1** ($\epsilon$-linearity)**.** We say $f \,|\, J$ is *$\epsilon$-linear* if

$$\frac{|f - f_0|_J}{|J|} \leq \epsilon. \tag{1.1}$$

The condition of being $\epsilon$-linear is unchanged if for any constant $c$, we make the two rescalings, $f \to cf$, $|J| \to c|J|$. Also, since (1.1) does not involve the measure, it remains unchanged if the measure is rescaled.

**Definition 1.2** (Bad locations and scales)**.** For $\epsilon > 0$, let $\mathcal{B}_\epsilon$ denote the collection of dyadic subintervals of $I$ on which $f$ is *not* $\epsilon$-linear.

**Theorem 1.3** (Quantitative differentiation)**.** *For all $\epsilon > 0$, $f \in H^{1,2}(I)$, we have*

$$\sum_{I_{i_n}\in\mathcal{B}_\epsilon} \|I_{i_n}\| \leq 2\epsilon^{-2} \cdot \int_I |(f - f_0)'|^2. \tag{1.2}$$

*Equivalently,*

$$\sum_{I_{i_n}\in\mathcal{B}_\epsilon} \|I_{i_n}\| \leq 2\epsilon^{-2} \cdot \left(\int_I |f'|^2 - \int_I |f_0'|^2\right). \tag{1.3}$$

*Remark* 1.1 ($p > 1$). Versions of Theorem 1.3 and its generalizations hold for $H^{1,p}$, $p > 1$; see Section 3. In these cases, the sup norm must be replaced by a normalized $L^{\chi p}$ norm, for appropriate $\chi > 1$.

*Remark* 1.2 (Secants versus derivatives). Were we to assume a bound on $|f''|$, we could employ Taylor's formula with remainder to bound $|f(x) - f(x_0) - f'(x_0)(x - x_0)|/|x - x_0|$, for any $x_0 \in J$. In Theorem 1.3, we are bounding $|f - f_0|_J/|J|$ which, when $|f''|$ is large, can be much smaller than $|f(x) - f(x_0) - f'(x_0)(x - x_0)|/|x - x_0|$. This happens if $f$ has high frequency oscillations of correspondingly small amplitude e.g. the function $2^{-n} \sin 2^n x$; compare Example 2.1 below. In particular, *quantitative differentiation is not differentiation in the usual sense*. It is concerned with the accuracy of secant approximations, not derivatives.

*Remark* 1.3 (Quantitative differentiation versus telescope estimates). Telescope estimates enable one to conclude that in a quantitative sense, Sobolev functions are Lipschitz off sets of small measure (or small capacity); see [KM96]. They are derived from Neumann-Poincaré inequalities by means of maximal function estimates. They are not directly comparable to the estimates provided by quantitative differentiation. For instance, for any open set of positive measure, the corresponding set of locations and scales has infinite measure. Likewise, quantitative differentiation, has nothing directly to say about maximal functions.

*Example* 1.1. In the riemannian case, the product decomposition, $\mu = dr \times \mathrm{Per}(X^a)$, with $r$ a distance function, is equivalent to the condition that the level surfaces of $r$ are minimal. This can be seen from the formula $\Delta = \partial_r^2 + m \cdot \partial_r + \tilde{\Delta}$, where $m$ is the *mean curvature* of the level surface of $r$ and $\tilde{\Delta}$ is the Laplacian on the level surface. For instance, in $\mathbb{R}^3$, consider the doubly warped product metric

$$dr^2 + e^{2r} dx^2 + e^{-2r} dy^2. \tag{1.4}$$

The distance function $r$ is harmonic since the mean curvature of each of its level surfaces is identically zero. The gradient lines of $r$ are geodesics and we have the product decomposition $\mu = dr \times \mathrm{Per}(X^a)$ for the riemannian measure. Note, $dr$ is not parallel and there is no isometric splitting. In this case, $\mathrm{Ric}(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = -2$; for the splitting theorem in case case $\mathrm{Ric} \geq 0$, see [CG72].

*Remark* 1.4 (Some history). Versions of the fundamental phenomenon treated in Theorem 1.3 and its $n$-dimensional version, have appeared in several relatively advanced contexts; see e.g. [Dor85, Dav88, Jon90, DS93]. Theorem 1.3 is closely related to the case treated in the appendix of [Jon90]. For domains contained in $\mathbb{R}^n$, see the discussion at the end of Section 2 and [KM07, Orp21].[2] The proofs we will give are different from those in the above references. While the proof of Theorem 1.3, requires only elementary calculus and the Schwarz inequality, it is not taught in standard courses. From a pedological standpoint, this is unfortunate since it bypasses the opportunity to introduce a wide-ranging phenomenon in an elementary setting. In actuality, it seems that many experts are unaware of the statement. Perhaps as a consequence, the potential of quantitative differentiation ideas for applications in geometric analysis was slow to be fully appreciated.

## 2. The 1-dimensional case

In this section we will prove Theorem 1.3.

*Dyadic subintervals* Recall that for any interval, $J$, its *dyadic subintervals*, $J_{i_n}$, $i_n = 1, \ldots, 2^n$ are the intervals obtained by $n$ times repeated bisection of $J$. Each $J_{i_n} = J_{i_{n+1}} \cup J_{i_{n+1}+1}$ of *depth* $n$ is the union of two intervals of half the size. Here $i_{n+1} = 2i_n - 1$. The dyadic subintervals of $I := [0,1]$ are:

$$J_{i_n} = [(i_n - 1)\cdot 2^{-n}, i_n \cdot 2^{-n}], \qquad (1 \le i_n \le 2^n,\, 1 \le n < \infty).$$

Note that if $J \subset [0,1]$ and $J_{i_n}$ is a largest dyadic subinterval with $J_{i_n} \subset J$, then either $J \subset J_{i_{n-1}-1} \cup J_{i_{n-1}}$ or $J \subset J_{i_{n-1}} \cup J_{i_{n-1}+1}$. (Here, if $i_{n-1} - 1 = -1$ or $i_{n-1} = 2^{n-1}$, we define $J \subset J_{i_{n-1}-1} := \emptyset$, respectively $J_{i_{n-1}+1} := \emptyset$.) Thus, every $J$ is comparable to the union of at most two consecutive dyadic subintervals. Moreover, every dyadic interval, $J_{i_n}$ is contained in precisely $n+1$ dyadic subintervals,

$$J \supset J_{i_1} \supset J_{i_2} \supset \cdots \supset J_{i_n}. \tag{2.1}$$

*Remark* 2.1. Relation (2.1) enters crucially at a juncture in the proof where the order of summation in a double series is reversed; see (2.20).

Prior to proving Theorem 1.3, we consider a basic example first pointed out to us by Stephen Semmes.

*Example* 2.1. Set $f_k(x) = 2^{-k}\sin 2^k x$, for any $k$. Note that for any $k$, we have $|f'| \le 1$. Consider the restriction of $f_k(x) = 2^{-k}\sin 2^k x$ to some $I_{i_n}$. Note that for any $k$, we have $|f'| \le 1$.

Consider the restriction of $f_k(x) = 2^{-k}\sin 2^k x$ to some $I_{i_n}$.

[2]We thank Scott Armstrong and Guido De Philippis for pointing out these references.

If $|I_{i_n}| \gg 2^{-k}$, then a very good linear approximation to the rescaled $f_k(x) - f_k(\bar{x})$ is $\ell \equiv 0$.
If $|I_{i_n}| \sim 2^{-k}$ then the rescaled $f_k(x) - f_k(\bar{x})$ is just *not* very linear.
If $|I_{i_n}| \ll 2^{-k}$, we can take $\ell = f_k'(\bar{x})(x - \bar{x})$.

Fix $N \geq \epsilon^{-1}$ and let $k_j = 2^j$, $j = 1, \ldots N$. Consideration of the functions,

$$g_N = \sum_{j=1}^{N} \frac{1}{N} \cdot f_{k_j},$$

shows that the estimate in Theorem 1.3 is sharp. Note that since the $k_j$ grow sufficiently fast, the degree of approximability of $g_N$ by an affine linear function on a given scale, is essentially independent of its behavior on all other scales.

*Markov's inequality* To prove Theorem 1.3, it suffices to show

$$\sum_{I_{i_n} \subset I} \|I_{i_n}\| \cdot \frac{|f - f_n|^2_{I_{i_n}}}{|I_{i_n}|^2} \leq 2 \int_I |(f - f_0)'|^2. \tag{2.2}$$

To see this, note that if $I_{i_n} \in \mathcal{B}_\epsilon$, then by definition, $|f - f_n|^2_{I_{i_n}}/|I_{i_n}|^2 \geq \epsilon^2$. So restricting the sum in (2.2) to $\mathcal{B}_\epsilon$ and cross multiplying gives

$$\sum_{I_{i_n} \in \mathcal{B}_\epsilon} \|I_{i_n}\| \leq 2\epsilon^{-2} \cdot \int_I |(f - f_0)'|^2. \tag{2.3}$$

*Secant approximations*

**Definition 2.1.** The $n$-th secant approximation $f_n$ to $f$ is the function such that for all $I_{i_n}$, $f_n \,|\, I_{i_n}$ is linear and

$$f_n \,|\, \partial I_{i_n} = f \,|\, \partial I_{i_n}. \tag{2.4}$$

Note that for $m \geq n$ the ($m$-$n$)-th secant approximation to $f \,|\, I_{i_n}$ is $f_m \,|\, I_{i_n}$. If we set $f_{n+1} - f_n := \phi_n$, then

$$f - f_0 = \sum_{n=0}^{\infty} \phi_n. \tag{2.5}$$

For any $m \geq n$ and $I_{i_m} \subset I_{i_n}$, we have

$$\phi_m \,|\, \partial I_{i_m} = 0. \tag{2.6}$$

$$\phi_n \,|\, I_{i_m} \ \text{ is linear.} \tag{2.7}$$

Define $\phi_{i_m}$ by

$$\begin{aligned} \phi_{i_m} &= \phi_m \ \ (\text{on } I_{i_m}). \\ \phi_{i_m} &= 0 \quad\ (\text{on } I \setminus I_{i_m}). \end{aligned} \tag{2.8}$$

Then,

$$\begin{aligned} \phi_m \,|\, I_{i_n} &= \sum_{I_{i_m} \subset I_{i_n}} \phi_{i_m} \qquad (\text{for fixed } m \geq n). \\ f - f_n \,|\, I_{i_n} &= \sum_{m=n}^{\infty} \sum_{I_{i_m} \subset I_{i_n}} \phi_{i_m}. \end{aligned} \tag{2.9}$$

*The energy as a sum over locations and scales* It follows from (2.7) and integration by parts, that:

$$\int_I \phi'_{i_n} \cdot \phi'_{i_m} = 0 \qquad (\text{if } \phi_{i_n} \neq \phi_{i_m}). \tag{2.10}$$

Thus,

$$\int_{I_{i_n}} |df|^2 - \int_{I_{i_n}} |df_0|^2 = \int_{I_{i_n}} |(f - f_0)'|^2 = \sum_{I_{i_m} \subset I_{i_n}} \int_{I_{i_m}} |\phi'_{i_m}|^2. \tag{2.11}$$

In particular, this holds for $I_{i_n} = I$.

*The Schwarz inequality* Let $x_{s,t} \geq 0$, $s = 1, \ldots, 2^k$, $t = 1, \ldots, k$.

**Lemma 2.2.**

$$\sum_{s=1}^{2^k} \left( \sum_{t=1}^{k} x_{s,t} \right)^2 \leq \sum_{t=1}^{k} 2^t \cdot \left( \sum_{s=1}^{2^k} x_{s,t}^2 \right). \tag{2.12}$$

*Proof.* For fixed $s$ and $t = 1, \ldots, k$, let $v \in \mathbb{R}^k$ denote the vector whose components are $x_{s,t} 2^{t/2}$ and let $w \in \mathbb{R}^k$ denote the vector whose components are $2^{-t/2}$. Applying the Schwarz inequality, $|\langle v, w \rangle|^2 \leq |v|^2 \cdot |w|^2$, gives

$$\left( \sum_{t=1}^{k} x_{s,t} \right)^2 \leq \sum_{t=1}^{k} 2^t x_{s,t}^2. \tag{2.13}$$

Summing over $s$ and reversing the order of summation on the r.h.s. gives (2.12). $\square$

*Summation over pairs $I_{i_m} \subset I_{i_n}$* For fixed $I_{i_n}$ and $k$, set $s = i_{n+k}$, $t = m-n$, $1 \leq s \leq 2^{n+k}$, $1 \leq t \leq k$.

For $I_{i_m} \subset I_{i_n}$, set

$$x_{s,t} = |\phi_{n+t}|_{I_s} = |\phi_m|_{I_{i_{n+k}}}. \tag{2.14}$$

By using (2.9), (2.13) and letting $k \to \infty$, we get

$$|f - f_n|^2_{I_{i_n}} \leq \sum_{I_{i_m} \subset I_{i_n}} 2^{m-n} |\phi_m|^2_{I_{i_m}}. \tag{2.15}$$

Multiplying through by $\|I_{i_n}\|/|I_{i_n}|^2$and summing over $I_{i_n} \subset I$, gives

$$\sum_{I_{i_n} \subset I} \|I_{i_n}\| \cdot \frac{|f - f_n|^2_{I_{i_n}}}{|I_{i_n}|^2} \leq \sum_{I_{i_n} \subset I} \sum_{I_{i_m} \subset I_{i_n}} 2^{m-n} \cdot \frac{\|I_{i_n}\|}{|I_{i_n}|^2} \cdot |\phi_m|^2_{I_{i_m}}. \tag{2.16}$$

*Fundamental theorem of calculus* Put

$$⨍_J h = \|J\|^{-1} \cdot \int_J h.$$

If $h \,|\, \partial J = 0$, then by the *fundamental theorem of calculus*,

$$|h|_J \leq |J| \cdot ⨍_J |h'| \tag{2.17}$$

Relation (2.17), together with the Schwarz inequality, implies:

$$\frac{(|h|_J)^2}{|J|^2} \leq ⨍_J |h'|^2. \tag{2.18}$$

*Remark* 2.2. Relation (2.18) should be viewed as a *Dirichlet-Poincaré-Sobolev inequality.*

We can now prove (2.2). As noted, this will complete prove Theorem 1.3.

*Proof of Theorem 1.3.* Since $|I_{i_m}|^2/|I_{i_n}|^2 = 2^{2(n-m)}$, if we take $h = \phi_{i_m}$, then from (2.16) and (2.18) we get

$$\sum_{I_{i_n} \subset I} \|I_{i_n}\| \cdot \frac{|f - f_n|^2_{I_{i_n}}}{|I_{i_n}|^2} \leq \sum_{I_{i_n} \subset I} \sum_{I_{i_m} \subset I_{i_n}} 2^{n-m} \int_{I_{i_m}} |\phi'_{i_m}|^2. \tag{2.19}$$

Reversing the order of summation on the r.h.s. of (2.19) and using (2.1), (2.11), gives

$$\sum_{I_{i_n}\subset I} \|I_{i_n}\| \cdot \frac{|f-f_n|^2_{I_{i_n}}}{|I_{i_n}|^2} \leq \sum_{I_{i_m}\subset I} \sum_{I_{i_n}\supset I_{i_m}} 2^{n-m} \cdot \int_{I_{i_m}} |\phi'_{i_m}|^2 \tag{2.20}$$
$$= 2\int_I |(f-f_0)'|^2.$$

This proves (2.2), which suffices to complete the proof of Theorem 1.3. □

By applying (2.18) on each dyadic subinterval, $I_{i_{n+1}}$, summing over $i_{n+1}$, it follows that the secant approximations $f_n$ *converge geometrically to* $f$:

**Lemma 2.3.**

$$|f-f_n|^2_I \leq 2^{-n}\left(\int_I |f'|^2 - \int_I |f'_n|^2\right). \tag{2.21}$$

Theorem 2.4 and Corollary 2.5 below are direct consequences of Theorem 1.3 and Lemma 2.3. Their scale invariant versions can be applied on any dyadic subinterval.

**Theorem 2.4** ($\epsilon$-regularity)**.** *If for* $n \geq |\log \epsilon|$

$$\int_I |f'_n|^2 - \int_I |f'_0|^2 \leq \epsilon \cdot \left(\int_I |f'|^2 - \int_I |f'_0|^2\right), \tag{2.22}$$

*then*

$$|f-f_0|^2_I \leq \epsilon \cdot \left(\int_I |f'|^2 - \int_I |f'_0|^2\right). \tag{2.23}$$

**Corollary 2.5.** *For all* $\epsilon > 0$ *and* $0 < \eta < 1$ *there exists*

$$n \leq |\log_2 \eta| + 2\eta^{-1}\epsilon^{-2} \cdot \int_I |(f-f_0)'|^2 \tag{2.24}$$

*such that at most a fraction* $\eta$ *of the* $2^n$ *dyadic intervals,* $I_{i_n}$*, are in* $\mathcal{B}_\epsilon$*.*

*The case of* $\mathbb{R}^m$ In generalizing the above argument to the case of $\mathbb{R}^m$, one encounters the following point. Typically, there does not exist a linear function which agrees with a given (say Lipschitz) function on the boundary of a dyadic cube. One way of dealing with this is the following.

Every dyadic cube $F^m \subset \mathbb{R}^m$ can be subdivided into $2^m m!$ congruent dyadic orthoschemes. By definition, these are $m$-simplices which are in 1-1 correspondence with sequences of faces $F^0 \subset F^1 \subset \cdots \subset F^m$. The vertices of such an orthoscheme are the barycenters of the faces in the corresponding sequence. These decompositions are nested in a manner generalizing (2.1).

From the density of Lipschitz functions in $H^{1,p}$, it suffices to assume that $f$ is Lipschitz. Define $f_n$ to be the piecewise linear function whose values at the *vertices* of each such dyadic orthoscheme of depth $n$ agree with those of $f$. The proof of Theorem 3.1 below generalizes mutatis mutandis to show that $f_n \xrightarrow{H_{\chi p}} f$ geometrically. Here, $\chi > 1$ is the constant in the Poincaré-Sobolev inequality in $\mathbb{R}^n$; compare (3.4). The remainder of the argument is as in the proof of Theorem 1.3; see (3.16) in the proof of Theorem 3.8 for the relevant generalization of Lemma 2.2.

## 3. PI spaces

In this section we consider $PI$ spaces i.e. metric measure spaces, $(X, d, \mu)$, for which a doubling condition and a Poincaré inequality hold. They were introduced by Heinonen and Koskela in their fundamental work on quasiconformal and quasisymmetric maps; see [HK95]. Differentiation theory for $PI$ spaces was developed in [Che99]. The theory of Sobolev spaces was introduced via (three) different approaches in [Ha96, Che99, Sha00]. Here, we continue the quantitative discussion of Sections 15, 16 of [Che99] and of [Che12].

In [Che99], a function is said to be *generalized linear* if it is a constant multiple of a harmonic distance function. Given a Lipschitz function, $f : X \to \mathbb{R}$, it follows that for $\mu$-a.e. $x \in X$ and any tangent cone, $X_x$, any blowup limit function, $\ell : X_x \to \mathbb{R}$, of $f$ is generalized linear; see Theorem 10.2 of [Che99]. Under mild assumptions, we will show that a nonzero generalized linear function gives rise to a splitting of the measure, $dr \times \mathrm{Per}(X^a)$, where $r$ is a distance function, $X^a := \{x \,|\, \ell(x) \le a\}$ and $\mathrm{Per}(\,\cdot\,)$ is the perimeter measure. Even for riemannian manifolds, a generalized linear function does not lead to a splitting of the metric unless the Ricci curvature is nonnegative; see Example 1.1.

A main result of the present section is a weakened version of Theorem 1.3 in which of necessity, $\epsilon$-linearity is replaced by *$\epsilon$-harmonicity*; see Theorem 3.8. In this connection, one issue is that of controlling the oscillation of the norm $|df|$ of the differential of $f$; compare the discussion of Section 4 where it is assumed in addition that the Ricci curvature has a definite lower bound. The issue is absent if we restrict Theorem 3.8 to the case of distance functions $\rho$. However, due to the possibility of uncontrolled oscillations of the measure $\mu$, the above restriction need not lead to an $\epsilon$-product decomposition of $\mu$ modeled on $dr \times \mathrm{Per}(X^a)$. Rather, it leads to an $\epsilon$-product decomposition of the measure $|dh|^2 \times \mu$, where $h$ is the harmonic function which $\epsilon$ approximates $\rho$. This can also be phrased in terms of the *modulus* of certain path families.

As stated above, a metric measure space, $(X, d, \mu)$ is called a $PI$ space if the measure, $\mu$, is doubling,

$$\mu(B_{2r}(p)) = \kappa \cdot \mu(B_r(p)) \qquad (\kappa = \kappa(R),\ 0 < r \leq R), \tag{3.1}$$

and a $(1,p)$-Poincaré inequality holds for some $p \geq 1$,

$$\fint_{B_r(x)} |f - f_{av}| \leq \tau r \cdot \left( \fint_{B_{2r}(x))} |df|^p \right)^{1/p}. \tag{3.2}$$

Here, $f_{av}$ denotes the average value of $f$ on $B_r(x)$ and $|df|$ is the norm of the minimal generalized upper gradient, which is known to be $\mu$-a.e. equal to the pointwise Lipschitz constant,[3]

$$|df| = \mathrm{Lip}\, f(x) = \lim_{r\to 0} \sup_{d(x,y)=r} \frac{|f(y) - f(x)|}{r}. \tag{3.3}$$

The basic differentiation theory for $PI$ spaces established in [Che99] includes the theory of Sobolev spaces $H^{1,p}$, the fact that that Lipschitz functions are generalized linear at the infinitesimal level, the existence of a measurable differentiable structure, the fact that the minimal upper gradient is equal $\mu$-a.e. to the pointwise Lipschitz constant, the theory of $p$-harmonic functions including the maximum principle and the sense in which one can solve the Dirichlet problem.

From [HK95], a $(1,p)$-Poincaré inequality implies a Poincaré-Sobolev inequality for some definite $\chi > 1$,

$$\left( \fint_{B_r(x)} |f - f_{av}|^{\chi p} \right)^{1/\chi p} \leq \tau r \cdot \left( \fint_{B_2 r(x))} |df|^p \right)^{1/p}. \tag{3.4}$$

In dimension 1, as in (2.18), we have $\chi = \infty$. More generally, for $\mathbb{R}^n$, if $p = 1$, we have $\chi = n/(n-1)$. By a standard argument, a Neuman-Poincaré-Sobolev inequality implies a Dirichlet-Poincaré-Sobolev inequality on a concentric ball of $1/3$ the radius.

[3] Part of the relevant theorem states that the limit in (3.3) exists for $\mu$-a.e. $x$.

*Locations and scales* From now on in this section we make convention that "harmonic" means *p-harmonic* for some $p$ (which our estimates depend). As in Section 2, we will be concerned with the behavior of $f$ on all locations and scales. We interpret this as follows. For some sufficiently small $\gamma$ with $\gamma \le \gamma(\kappa, \tau, p)$ and $0 < \tau\gamma < 1$, consider a sequence of maximal $\tau\gamma^n$-separated subsets $\{x_{i_n}\}$ such that $\{x_{i_n}\} \supset \{x_{i_m}\}$ if $m \ge n$. One can view the collection of locations and scales as the union over $n$ of the set of Voronoi cells, $D_{i_n}$, associated to $\{x_{i_n}\}$ for each fixed $n$.

*Remark* 3.1. For technical reasons, particularly if we do not assume that $(X, d)$ is a length space, it is better to consider the collection of balls $\{B_{\gamma^{(n-3)}}(x_{i_n})\}$ and on every scale, divide these into a definite number of families of well separated balls. Since this point plays no essential role, to simplify the exposition, we will ignore it and work with Voronoi cells (and corresponding Poincaré inequalities) rather than balls.

*Secant approximations* Let $f : B_1(x) \to \mathbb{R}$, $f \in H^{1,p}$. The solvability of the Dirichlet problem for the Voronoi cells, $D_{i_n}$, implies that as in Section 2, we can define a sequence of *piecewise p-harmonic secant approximations*, $f_n$, such that $f_n \,|\, D_{i_n}$ is $p$-harmonic and $f_n \,|\, \partial D_{i_n} = f \,|\, \partial D_{i_n}$. The corresponding sequence of energies is monotone nondecreasing since as $n$ increases, a stronger constraint on the piecewise harmonic function $f_n$ is imposed. Also, $|df_n|_{L_p} \le |df|_{L_p}$ since this holds on each $D_{i_n}$. In the following result on convergence of secant approximations, the assumption of strict convexity plays a role.

**Theorem 3.1.** *If $\mu$-a.e., the pointwise norm on cotangent spaces is strictly convex, then $f_n \xrightarrow{H^{1,p}} f$. Moreover, $f_n \xrightarrow{L_{\chi p}} f$ geometrically.*

*Proof.* As noted above, on each $D_{i_n}$, we have $|df_n|_{L_p} \le |df|_{L_p}$. Thus, for fixed $n$, applying the Poincaré-Sobolev inequality, (3.4), on each $D_{i_n}$ and summing gives:

$$\begin{aligned}
\sum_{i_n} \mu(D_{i_n}) \left( \fint_{D_{i_n}} |f - f_n|^{\chi p} \right)^{1/\chi} &\le (\tau\gamma)^{np} \cdot \sum_{i_n} \int_{D_{i_n}} |df - df_n|^p \\
&\le (\tau\gamma)^{np} \int_{B_1(x)} (|df| + |df_n|)^p \\
&\le 2^p (\tau\gamma)^{np} \int_{B_1(x)} |df|^p.
\end{aligned} \tag{3.5}$$

Since $\tau\gamma < 1$, we have $f_n \xrightarrow{L_{\chi p}} f$ and the convergence is at a geometric rate; compare (2.21).

Clearly, the sequence, $|df_n|_{L_p}$ is nondecreasing and bounded above by $|df|_{L_p}$. However, since the $H^{1,p}$ norm behaves lower semicontinuously under $L_p$ convergence, (3.5) implies that $\limsup_{n\to\infty} |df_n|_{L_p} \geq |f|_{L_p}$. Thus, $\lim_{n\to\infty} |df_n|_{L_p} = |f|_{L_p}$. By Remark 4.49 of [Che99], it now follows that $f_n \xrightarrow{H^{1,p}} f$. □

*Remark* 3.2. The *energy sequence*, $|df_n|^p_{L_p}$, can be viewed as a *monotone quantity* which, according to the Dirichlet-Poincaré-Sobolev inequality, is *coercive* in the sense that it controls the $L_{\chi p}$ norm.

*Remark* 3.3. In considerable generality one can make a bi-Lipschitz change of metric such that the new metric on the cotangent space is given by an inner product; see [Che99]. In what follows, we make a *weaker assumption*, namely, the *uniform strong convexity* of $|\cdot|^2$.

*Strong convexity and coercivity* If the function $f$ on the vector space $V$ is *convex*, then

$$\frac{1}{2}f(w) + \frac{1}{2}f(v) - f\left(\frac{1}{2}(v+w)\right) \geq 0.$$

Given $f$, the quantity on the left depends only on the vector space structure. Given in addition, a norm, $|\cdot|$, on $V$, the function $f$ is said to be *strongly convex* if for some $m > 0$,

$$\frac{1}{2}f(w) + \frac{1}{2}f(v) - f\left(\frac{1}{2}(v+w)\right) \geq \frac{1}{2}m \cdot |v-w|^2.$$

In the following Lemma 3.2 and in Theorem 3.5, we will assume that $\mu$-a.e. the square of the norm on the cotangent bundle, $|\cdot|^2$, is *uniformly strongly convex.* This condition can be expressed in any of the following three equivalent ways, each of which is useful. Namely, for some $m > 0$ and $\mu$-a.e. $x$, we have:

$$\frac{1}{2}|v|^2 + \frac{1}{2}|w|^2 - \left|\frac{1}{2}(v+w)\right|^2 \geq \frac{m}{2}\cdot|v-w|^2, \tag{3.6}$$

$$|w|^2 - |v|^2 \geq 2\left(\left|v + \frac{1}{2}(w-v)\right|^2 - |v|^2\right) + m|v-w|^2, \tag{3.7}$$

$$|w|^2 \geq |v|^2 + 2\left(\left|v + \frac{1}{2}(w-v)\right|^2 - |v|^2\right) + m|v-w|^2. \tag{3.8}$$

Note that if the norm comes from an inner product, then equality holds in (3.6) with $m = 1/2$.

**Lemma 3.2** (Coercivity)**.** *Assume that the norm on the cotangent bundle of $(X, d, \mu)$ is uniformly strongly convex. Let $f \in H^{1,2}$ and $f - f_0 \in H_0^{1,2}$, with $f_0$ harmonic and $f_0 \,|\, \partial\Omega = f \,|\, \partial\Omega$. Then for any $\Omega$,*

$$\int_\Omega |df|^2 - \int_\Omega |df_0|^2 \geq m \cdot \int_\Omega |df - df_0|^2. \tag{3.9}$$

*Proof.* Apply (3.7) pointwise with $v = df_0$, $w = df$. Since $2f_0 + \frac{1}{2}(f - f_0) \,|\, \partial\Omega = 2f_0 \,|\, \partial\Omega$, the global minimizing property of $2df_0$ implies (3.9). □

*Note:* The assumption of *uniform strong convexity* will be in force throughout the remainder of this section.

**Corollary 3.3.**

$$\int_\Omega |df|^{2p} - \int_\Omega |df_0|^{2p} \geq m^p \cdot \int_\Omega |df - df_0|^{2p}. \tag{3.10}$$

Let $\phi_n := f_{n+1} - f_n$. Using Vornonoi cells, $D_{i_n}$, in place of dyadic subintevals, $I_{i_n}$, define $\phi_{i_n}$ as in (2.8). Summing (3.10) over $n$ gives the following counterpart of (1.3). It states in scale invariant form that the energy difference on scale 1 bounds a definite positive multiple of the sum of the energies on all locations and scales.

**Corollary 3.4** (Sum over scales)**.**

$$\int_{D_{i_n}} |df|^{2p} - \int_{D_{i_n}} |df_0|^{2p} \geq m^p \sum_{D_{i_m} \subset D_{i_n}} \int_{D_{i_m}} |d\phi_{i_n}|^{2p}. \tag{3.11}$$

We also get the following $\epsilon$-regularity theorem.

**Theorem 3.5** ($\epsilon$-regularity)**.** *If for $n \geq |\log \epsilon|$*

$$\int_{D_{i_n}} |f_n'|^{2p} - \int_{D_{i_n}} |f_0'|^{2p} \leq \epsilon \cdot \left( \int_{D_{i_n}} |f'|^{2p} - \int_{D_{i_n}} |f_0'|^{2p} \right) \tag{3.12}$$

*then*

$$|f - f_0|^{2p}_{D_{i_n}} \leq \epsilon \cdot \left( \int_{D_{i_n}} |f'|^2 - \int_{D_{i_n}} |f_0'|^2 \right) \tag{3.13}$$

*Quantitative differentiation* For $\chi$ as in the Poincaré-Sobolev inequality (3.4), set

$$|g|_{B_r(x)} := \left( \fint_{B_r(x)} |g|^{\chi p} \right)^{1/\chi p}$$

**Definition 3.6** ($\epsilon$-harmonicity)**.** Let $f_0$ be harmonic with $f_0 \,|\, \partial B_r(x) = f\, \partial B_r(x)$. We say $f \,|\, B_r(x)$ is $\epsilon$-harmonic if

$$\frac{|f - f_0|_{B_r(x)}}{r} \leq \epsilon. \tag{3.14}$$

**Definition 3.7** (Bad locations and scales)**.** For $\epsilon > 0$, let $\mathcal{B}_\epsilon$ denote the collection of Voronoi cells on which $f$ is *not* $\epsilon$-harmonic.

**Theorem 3.8** (Quantitative differentiation)**.** *For all $\epsilon > 0$, $f \in H^{1,p}(I)$, we have*

$$\sum_{D_{i_n} \in \mathcal{B}_\epsilon} \mu(D_{i_n}) \leq \kappa^p \epsilon^{-2} \cdot \int_{B_1(x)} |d(f - f_0)|^p. \tag{3.15}$$

*Proof.* An examination of the proof of Theorem 1.3 shows the following. Suppose $\gamma$, with $0 < \gamma \leq \gamma(\kappa, p)$, is chosen sufficiently small relative to $\kappa, p$. Then (modulo a standard covering argument) the only additional required modification is the following generalization of Lemma 2.2 whose proof is completely analogous to the proof of that lemma.

Let $x_{s,t} \geq 0$, $s = 1, \dots, \kappa^k$, $t = 1, \dots, k$.

**Lemma 3.9.**

$$\sum_{s=1}^{\kappa^k} \left( \sum_{t=1}^{k} x_{s,t} \right)^2 \leq \sum_{t=1}^{k} \kappa^t \cdot \left( \sum_{s=1}^{\kappa^k} x_{s,t}^2 \right). \tag{3.16}$$

□

*Remark* 3.4 (Quantitative Hölder continuity). It is shown in [KS01] that De Giorgi's method for proving Hölder continuity in $\mathbb{R}^n$ can be adapted to the case of $p$-harmonic functions, and more generally, quasiminimizers, on PI spaces. In particular, in a quantitative sense, such functions are Hölder continuous. As a consequence, when applied to say 1-Lipschitz functions, the conclusion of Theorem 3.8 can be supplemented by adding $\epsilon$-$C^\alpha$-Hölder. (In this degree of generality, harmonic functions need not be Lipschitz unless an additional condition holds; see [KRS03].)

*Harmonic distance functions* It was observed in Section 7 of [Che99] that a harmonic distance function has a representation in terms of its boundary values; see (3.19) below. To this end, recall the definition of the upper and lower McShane extensions, $u_*, u^*$ of an $L$-Lipschitz function $u$ defined on a closed subset $E \subset X$. We define

$$u_*(x) = \max_{y_* \in E} u(y_*) - L \cdot d(x, y_*).$$
$$u^*(x) := \min_{y^* \in E} u(y^*) + L \cdot d(x, y^*). \tag{3.17}$$

Then $u_*, u^*$ are $L$-Lipschitz and

$$u_*(x) = u = u^* \qquad (\text{on } E).$$
$$u_*(x) \leq u \leq u^* \qquad (\text{on } X). \tag{3.18}$$

Thus, $u_*$ and $u^*$ are the minimal and maximal extensions of $u \,|\, E$ which are $L$-Lipschitz

For a generalized linear functions, $\ell$, and $A = \partial B_r(x)$, uniqueness for solutions of the Dirchlet problem implies that a generalized linear function has a representation in terms of its boundary values:

$$\ell_* = \ell = \ell^*. \tag{3.19}$$

*Product decompositions* $\mu = dr \times \mathrm{Per}(X^a)$ In this subsection, we consider product decompositions of $\mu$ associated to generalized linear functions, or equivalently, harmonic distance functions. We assume that $(X, d)$ is a *length space*. For brevity, we will restrict attention to $p = 2$. The general case is similar. For convenience, in the following definition we make a normalization which can always be achieved by scaling. Let $\rho : B_2(x) \to \mathbb{R}$ denote a *distance function*. Then $|d\rho| \equiv 1$. Assume further that the range of $\rho$ is $[-2, 2]$ and $\rho(x) = 0$; compare (3.19).

**Definition 3.10** (Cylinder)**.** The set $C$ is a *cylinder* if it is the union of all minimal geodesic segments, $\gamma(s) \subset B_2(x)$ of length 1 and joining $\rho^{-1}(1)$ to $\rho^{-1}(0) \cap B_{1/2}(x)$. For $0 < s \leq 1$, set $C^s := C \cap \rho^{-1}(s)$.

**Definition 3.11** (Cross section)**.** Set $A(s) := \mathrm{Per}(C^s)(\rho^{-1}(s))$.

It follows from the coarea formula (see Proposition 4.2 of [Mir03]) that

$$\mu(C^r) = \int_0^r A(s)\, ds. \tag{3.20}$$

Assume now that $\rho$ is also *harmonic*. To indicate this special case we write $\ell$ for $\rho$, where $|d\ell|^2 \equiv 1$.

*Remark* 3.5. In this instance, the cylinder $C^r$ is a special case of what is known as a *flux tube*.

Were we in the smooth riemannian case, application of the *divergence theorem* to the vector field $\ell \cdot \nabla\ell$, would give:

$$\mu(C^r) = rA(r). \tag{3.21}$$

Note in this connection that the contribution to the r.h.s. from the "curved" part of $\partial C^r$ vanishes since the $\nabla\ell$ is tangent to this part. Thus, we are left with the contribution, $A(r)$, from $\ell^{-1}(r)$ along which the normal derivative of $\ell$ is $\equiv 1$. Differentiating (3.20), (3.21), gives $\frac{d}{dr}(r \cdot A(r)) = A(r)$. Thus,

$$\frac{d}{dr}A(r) = 0, \tag{3.22}$$

which is equivalent to the product decomposition

$$\mu = dr \times A(a) \qquad (\text{for } \mu\text{−a.e. } a). \tag{3.23}$$

A variant of the above argument can be carried out in the $PI$ case if in addition to (3.6)–(3.8), we assume that $\mu$-a.e. the squared norm on the cotangent space is $C^1$ smooth, with a definite *upper bound on the convexity*; see (3.24) below. For temporary notational convenience, set $E(v) = |v|^2$ and let $\mathbf{d}E$ denote the differential of $E$ viewed as a function on a fixed cotangent space $TX^*_x$. We assume that for some constant, $0 < M < \infty$,

$$E(w) - E(v) - \mathbf{d}E(v)(w - v) \leq M \cdot E(w - v). \tag{3.24}$$

*Remark* 3.6. Relation (3.24) holds if the norm comes from an *inner product*. In that case, we have equality for $M = 1$ and $\mathbf{d}E(v)(w - v) = 2\langle v, v - w\rangle$. In general, $dE(v)(w - v)$ is the directional derivative of the norm function $|\cdot|^2$ at the point $v$ in the direction $w - v$. Alternatively, as in the case of inner products, it is equal to $2|v|$ times the component of $(w - v)$ in the direction of $v/|v|$ with respect to the direct sum splitting of the tangent space at $v$ given by the $v$ direction and the tangent space to the level set $E^{-1}(v)$.

Under the above assumptions, for any $\Omega$ with finite perimeter, we have the following. Let $h : \Omega \to \mathbb{R}$ be harmonic and let $\phi \,|\, \partial\Omega \equiv 0$. Then for

$t > 0$, (3.24) gives

$$0 \leq \int_\Omega (E(dh \pm t d\phi) - E(dh) \leq \int_\Omega \pm t \mathbf{d}E(dh)(d\phi) + t^2 M \cdot E(d\phi). \tag{3.25}$$

Thus, $|\int_\Omega |\mathbf{d}E(dh)(d\phi)| \leq tM \int_\Omega E(d\phi)$. Since $t$ can be taken arbitrarily small, this implies

$$\int_\Omega \mathbf{d}E(dh)(d\phi) = 0. \tag{3.26}$$

Let $s(x) = d(x, \partial\Omega)$ denote distance to $\partial\Omega$. For $\delta > 0$ let $\psi \equiv 1$ on $\Omega \setminus T_\delta(\partial\Omega)$. On $T_\delta(\partial\Omega)$, put $\psi := \delta^{-1}s$. Let $\phi = \psi h$. From (3.26) we get

$$0 = \int_{\Omega \setminus T_\delta(\partial\Omega)} |dh|^2 + \int_{T_\delta(\partial\Omega)} \mathbf{d}(dh)(\psi \cdot dh + d\psi \cdot h). \tag{3.27}$$

If we use the coarea formula on $T_\delta(\partial\Omega)$, and let $\delta \to 0$, the first term on the r.h.s. goes to zero and we get:

**Proposition 3.12.**

$$\int_\Omega |dh|^2 = \int_{\partial\Omega} h \cdot \frac{dh}{ds}, \tag{3.28}$$

*where the integral on r.h.s. is with respect to the the perimeter measure* $\mathrm{Per}(\Omega)$ *on* $\partial\Omega$.

By applying (3.28) to the $h$ and to $1 - h$ we get:

**Corollary 3.13** (Equipartition for cylinders)**.** *For* $0 \leq a \leq 1$,

$$\int_{C^a} |dh|^2 = a \int_{C^1} |dh|^2. \tag{3.29}$$

Using Corollary 3.13 and arguing as in (3.20)–(3.23) gives:

$$\begin{aligned} &\int_a^r A(s) = rA(r). \\ &\frac{d}{dr} A(r) \equiv 0. \end{aligned} \tag{3.30}$$

**Theorem 3.14.** *Let* $(X, d, \mu)$ *denote a PI space which is a length space and for which* (3.6)–(3.8) *and* (3.24) *hold. Let* $C$ *denote a cylinder as in*

*Definition 3.10. Let $\ell : C \to \mathbb{R}$ denote a harmonic distance function. Then on $C$, the measure $\mu$ has a product decomposition $\mu(C^r) = r \times A(a)$, for almost all $a \in [0,1]$.*

*$\epsilon$-harmonic distance functions* In this subsection we continue the discussion of Section 15, 16, of [Che99]. Specifically, we consider *$\epsilon$-harmonic distance functions*. We show that Theorem 3.8 leads to a quantitative counterpart of the product decomposition $\mu(C^r) = r \times A(a)$ of Theorem 3.14; see Theorem 3.15. As indicated by (3.29) and Example 3.1 below, this applies to the measure $|dh|^2\mu$, though not necessarily to $\mu$ itself.

*Example* 3.1. Let $h = r + \epsilon\phi(r/\epsilon)$ where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function with $1/2 \le \phi' \le 1$, $\phi(1) = 0$. Thus, $h(1) = 1$. Then the distance function $r$ is *$\epsilon$-harmonic* if $h$ is *harmonic* with respect to the measure with density $A(r)\cdot dr$ if $h'' + (A'/A)h' = 0$. Thus, $h(s)$ determines $A(s)$ up to a constant multiple and we can choose

$$A = \frac{1}{h'} = \frac{1}{1 + \phi'(r/\epsilon)}. \tag{3.31}$$

Note that the measure with density $A^{-1}(s)\cdot ds$ is weakly close to 1 since

$$\int_0^r A^{-1}(s) = h(r) = r + \epsilon\phi(r/\epsilon). \tag{3.32}$$

On the other hand, it is easy to construct examples which show that no matter how small we take $\epsilon > 0$, the measure with density $A(s)\,ds$ need not be weakly close a measure with density $c\,ds$ for some constant $c$. For this, use the identities $(1 + x) + (1 - x) = 2$, while $1/(1 + x) + 1/(1 - x) = 2/(1 - x^2)$, for $0 < x \le 1/2$.

If for suitable $\epsilon \ll 1$, take $\phi = \frac{1}{2}\sin(r/\epsilon)$. Then $\phi'(r/\epsilon)$ is highly oscillatory. If instead we take $h = r + \epsilon^2 \sin(r/\epsilon)$ then $g$ is close 1 and the measure is close to splitting. However, the mean curvature $A'/A$ is highly oscillatory. Note that in these cases, $|dh|^2 = |h'|^2$ and

$$\int_0^1 (h')^2 A\,dr = \int_0^1 h' = 1. \tag{3.33}$$

Theorem 3.8, leads to a corresponding $\epsilon$-equidistribution result for the Dirichet energy on cylinders.

**Theorem 3.15** ($\epsilon$-equidistribution)**.** *For all $\epsilon > 0$, there exists $\delta > 0$ such that if $r : C^1 \to [0,1]$ is a distance function and $h$ is a harmonic function such that $|r - h|_{L^2 X} < \delta$, then*

$$\left|\int_{C^a} |dh|^2 - a\int_{C^1} |dh|^2\right| < \epsilon \int_{C^1} |dh|^2. \tag{3.34}$$

*Proof.* Recall the (quantitative) bound of [KS01] on the Hölder continuity of harmonic functions which was mentioned in Remark 3.4. From this and the fact that $r$ is a distance function we can assume that $|r - h| < \epsilon$ where the norm is the *uniform norm* on the cylinder $C^1$.

Let $h_0$ denote the harmonic function on $C^1$ such that $h_0 \,|\, C^1 = r \,|\, C^1$. By the maximum principle, it follows that $|h - h_0| < 2\epsilon$. Now, the claim follows from (3.29) of Corollary 3.13. □

*Remark* 3.7. Theorems 3.8, 3.15, are potentially of interest in connection with several previous works. Keith showed that distance functions can be used to provide an atlas for the differentiable structure; see [Kei04]. Bate showed that the existence of a Lipschitz differentiable structure on a metric measure space can be characterized in terms of the existence of sufficiently many independent *Alberti representations*; see [Bat15]. The decomposition, $\mu_\infty = dr \times \mathrm{Per}(X^a)$, is an Alberti representation of a very special type; [Alb93]. Eriksson-Bique showed that RNP-differentiability spaces are $PI$ rectifiable; see [EB19].

## 4. Ricci curvature bounded below

In the context of spaces with Ricci curvature bounded below, one can pass from Theorem 3.8, to a weakened quantitative differentiation theorem for Lipschitz functions. Here, although the approximation is by almost generalized linear functions the notion of scale $\gamma = \gamma(m, \epsilon)$ is allowed to depend on the guaranteed accuracy of the approximation. Additionally, the notion of what constitutes a good location and scale is weakened in that the generalized linear approximation is not asserted to hold on the entire domain, but only on subdomains of at least a definite size. The main issue is that of controlling the oscillation of the gradient of the approximating harmonic function; see Theorem 4.2. This is the one we will discuss.

In the context of structure theory for spaces with Ricci curvature bounded below, the quantitative discussion of harmonic functions which are close to distance functions began with [Col96a, Col96b, CC96]; see also [CG72]. A key result of [CC96] is Theorem 6.62, the quantitative splitting theorem. In present context, the pertinent hypothesis in Theorem 6.62 is the assumption that the oscillation of the norm gradient of the relevant harmonic function is sufficiently small; see (6.32) of [CC96] and compare also p. 951 of [CCM95]. This is the subject of our next result.

**Theorem 4.1.** *For all* $\epsilon > 0$ *there exists* $\gamma = \gamma(m, \epsilon)$*, with* $0 < \gamma < 1$*, such that the following holds. Let* $\mathrm{Ric}_{M^m} \geq -(m-1)$*,* $\partial B_3(x) \neq \emptyset$ *and let* $h : B_2(x) \to R$ *be harmonic with* $|h| \leq 1$*. Then for all* $n \geq 1$*,*

$$\tag{4.1} ⨍_{B_{\gamma^n}(x)} \left| |dh| - |dh|_{av} \right| < \epsilon.$$

*Proof.* By the Cheng-Yau gradient estimate, without essential loss of generality, we can assume say $|dh| \leq 1$ on $B_1(x)$. Given $\epsilon > 0$, by scaling we can reduce to the case, $\mathrm{Ric} \geq -\delta(\epsilon) > 0$ sufficiently small. To simplify the exposition we will assume $\mathrm{Ric} \geq 0$, which again entails no essential loss of generality; see (6.32) of [CC96]. Then by Bochner's formula, it follows that $|dh|^2$ is *subharmonic*. Now, (4.1) follows from Theorem 4.2 below which can be viewed as an effective version of P. Li's mean value theorem for bounded subharmonic functions. Li's theorem ([Li86]) states that that if $g : M^n \to \mathbb{R}$ is a bounded subharmonic function on a complete manifold $M^n$ with $\mathrm{Ric}_{M^n} \geq 0$, then

$$\tag{4.2} \lim_{r\to\infty} ⨍_{B_r(x)} g = \sup_{M^m} g.$$

A very short proof of (4.2) is given on p. 952 of [CCM95]. A variation on that argument gives the following effective result.

**Theorem 4.2.** *There exists* $r(m, \epsilon)$ *with* $1 \geq r(m, \epsilon) > 0$ *such that the following holds. If* $\mathrm{Ric}_{M^m} \geq -(n-1)$*, and* $k : B_1(x) \to (0, 1]$ *is subharmonic with* $\max_{\bar{B}_1(x)} k = 1$*, then there exists* $r$ *with* $1 \geq r \geq r(m, \epsilon)$ *such that*

$$\tag{4.3} ⨍_{B_r(x)} |k - k_{av}| < \epsilon.$$

*Proof.* Set $k_r = \max_{B_r(x)} k$. Since $k$ is subharmonic, by the maximum principle, $k_r$ is a monotone nondecreasing function of $r$. Let $0 < \gamma = \gamma(\epsilon, m) < 1/2$, $c(m) \geq 1$, be specified below. Put $\epsilon' = \epsilon/c(m)$. The argument on p. 952 of [CCM95] shows that $c(m)$ can be chosen such that the following holds. If for some integer $i$, we have $k_{\gamma^{2i+2}} \geq (1-\epsilon')k_{\gamma^{2i}}$, then

$$\tag{4.4} ⨍_{B_s(x)} k \geq (1 - \epsilon/2) ⨍_{B_{\gamma^{2i}}(x)} k \qquad (\text{for all } s \text{ with } \gamma^{2i+1} \leq s \leq \gamma^{2i}).$$

Let $N$ be the least integer such that $(1-\epsilon')^{2N} < \epsilon$. Then either (4.4) holds for some $i \leq N-1$, or (since $k \leq 1$) we have $k_{\gamma^{n+2}} \leq \epsilon$. In the latter case, once $\gamma$ has been specified, we are done. So assume (4.4) holds for some $i \leq N-1$. By observing $\mathrm{Vol}(B_{\gamma^{2i+1}}(x))$ from $y \in \partial B_2(x)$, it follows from relative volume comparison that there exists $c_1 = c_1(n)$ such that

$$\mathrm{Vol}(B_{\gamma^{2i+1}}(x)) \leq c_1\gamma^{-1} \cdot \mathrm{Vol}(B_{\gamma^{2i}}(x)). \tag{4.5}$$

By taking $\gamma < \epsilon/2c_1$, it follow from (4.4), (4.5) that (4.3) holds. □

This completes the proof of Theorem 4.1. □

*Remark* 4.1 (Generalizations). The above discussion extends in a straightforward way to the solutions to the solutions to elliptic equations as considered in [CC96]. A particularly significant case is the of solutions to the Poisson equation whose values on some $\partial B_r(x)$ agree with those of the distance function $r^2$.

*Remark* 4.2. It should also be possible to extend the above discussion to the synthetic Ricci context.

## Appendix A. Some recent applications

Here, we very briefly summarize some recent consequences of quantitative differentiation ideas in geometric analysis. As noted in [CKN11, Che12], there are 3 essentials for a quantitative differentiation theorem to hold.

1) A locally defined nonnegative energy $E(f)$.
2) An a priori bound on the global energy $E(f)$.
3) Coercivity: If in a quantitative sense, the energy difference $E(f_n)-E(f_0)$ is sufficiently small and $n$ is sufficiently large, then in a suitable scale invariant sense, $f$ is close to the minimizer. Hence, approximate special structure is present.

In Theorem 1.3, 3) follows from the Dirichlet-Poincaré-Sobolev inequality, (2.18), written in a form which is summable over scales.

Prior to the importation of quantitative differentiation ideas, a typical result in partial regularity theory gave a lower bound on the Hausdorff codimension of the singular set for the problem in question. (An exception was the fundamental work of Simon on rectifiability of singular sets; [Sim95].) Pioneering contributions were due to De Giorgi, Federer, Fleming, Almgren, Simons, Schoen and Uhlenbeck. These results were gotten by iterated blow up arguments, also known *as dimension reducing*; [Fed70]. In the context of partial regularity theory, quantitative differentiation can be thought of as a quantitative replacement for the iterated blow up technique.

In [CK10], by means of blow up arguments, it was proved that a 1-Lipschitz map from the 3-dimensional Heisenberg group with its Carnot-Carathéodory metric to the space $L_1$ cannot be bi-Lipschitz. Quantitative differentiation ideas were used in [CKN11] to show that such a 1-Lipschitz map must compress distances by any preassigned amount, somewhere on or above a definite scale. In this discussion, the relevant special structure turns out to be *almost monotone sets*. These are shown to be *almost half-spaces*. Both the kinematic formula of [Mon05] and the Poincaré-Sobolev inequality play a key role in establishing 1) and 2). Relation 3) is quite nontrivial.

Beginning with [CN13], beyond lower bounds on Hausdorff codimension, the importation of quantitative differentiation (and related ingredients) into nonlinear elliptic/parabolic geometric pde led to *bounds on the volumes of tubes around singular sets of a fixed positive radius*. Apart from borderline cases, these, were sharp. Crucially, there is a definite amount of regularity outside the given tube. Examples include Einstein metrics, minimal submanifolds, harmonic maps, mean curvature flow, harmonic map flow. In these instances, even after finding the relevant energy, the proof of coercivity can be highly nontrivial. Other key ingredients included the *quantitative stratification*, *quantitative cone splitting* and covering arguments; see [CN13]. Typically these results depend on the existence of some sort of elliptic or parabolic estimate in involving a coercive monotone quantity. Often (though not in the case of Lipschitz maps to $L_1$) the monotone quantity can be viewed as an energy density and the monotonicity holds pointwise.

To address the borderline cases in partial regularity theory, Naber-Valtorta and Jiang-Naber isolated the concept of a *neck region* and developed a decomposition and structure theory for such regions. Roughly speaking, these are transition regions between the regular and singular parts. The fine behavior in these borderline cases depends on the particular equation. For minimizing harmonic maps, the gradient is shown to be quantitatively in weak $L_3$; [NV17]. A well known example shows that $L_3$ is false. On the other hand, in the noncollapsed Einstein case, the full curvature tensor has a definite $L_2$ bound; [JN21]. The earlier quantitative differentiation ideas, which were not as intensively technical, gave $L_p$ for all $p < 3$ and $L_p$ for all $p < 2$ respectively. Of equal importance, the neck region theory also provides *structure theorems for singular sets* in the form of *generally applicable rectifiability theorems*. For a recent application of neck region theory to the structure of noncollapsed Gromov-Hausdorff limit spaces with Ricci curvature bounded below, see [CJN21].

## Acknowledgements

We are grateful to Scott Armstrong, Jean-Michel Bismut, Guido De Philippis and Sylvester Eriksson-Bique for a number of very helpful comments.

## References

[Alb93] Giovanni Alberti, *Rank one property for derivatives of functions with bounded variation*, Proc. Roy. Soc. Edinburgh Sect. A **123** (1993), no. 2, 239–274. MR1215412

[Bat15] David Bate, *Structure of measures in Lipschitz differentiability spaces*, J. Amer. Math. Soc. **28** (2015), no. 2, 421–482. MR3300699

[CC96] Jeff Cheeger and Tobias H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2) **144** (1996), no. 1, 189–237. MR1405949

[CCM95] J. Cheeger, T. H. Colding, and W. P. Minicozzi, II, *Linear growth harmonic functions on complete manifolds with nonnegative Ricci curvature*, Geom. Funct. Anal. **5** (1995), no. 6, 948–954. MR1361516

[CG72] Jeff Cheeger and Detlef Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geometry **6** (1971/72), 119–128. MR303460

[Che99] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. **9** (1999), no. 3, 428–517. MR1708448

[Che12] Jeff Cheeger, *Quantitative differentiation: a general formulation*, Comm. Pure Appl. Math. **65** (2012), no. 12, 1641–1670. MR2982637

[CJN21] Jeff Cheeger, Wenshuai Jiang, and Aaron Naber, *Rectifiability of singular sets of noncollapsed limit spaces with Ricci curvature bounded below*, Ann. of Math. (2) **193** (2021), no. 2, 407–538. MR4226910

[CK10] Jeff Cheeger and Bruce Kleiner, *Differentiating maps into $L^1$, and the geometry of BV functions*, Ann. of Math. (2) **171** (2010), no. 2, 1347–1385. MR2630066

[CKN11] Jeff Cheeger, Bruce Kleiner, and Assaf Naor, *Compression bounds for Lipschitz maps from the Heisenberg group to $L_1$*, Acta Math. **207** (2011), no. 2, 291–373. MR2892612

[CN13] Jeff Cheeger and Aaron Naber, *Lower bounds on Ricci curvature and quantitative behavior of singular sets*, Invent. Math. **191** (2013), no. 2, 321–339. MR3010378

[Col96a] Tobias H. Colding, *Large manifolds with positive Ricci curvature*, Invent. Math. **124** (1996), no. 1-3, 193–214. MR1369415

[Col96b] Tobias H. Colding, *Shape of manifolds with positive Ricci curvature*, Invent. Math. **124** (1996), no. 1-3, 175–191. MR1369414

[Dav88] Guy David, *Morceaux de graphes lipschitziens et intégrales singulières sur une surface*, Rev. Mat. Iberoamericana **4** (1988), no. 1, 73–114. MR1009120

[Dor85] José R. Dorronsoro, *A characterization of potential spaces*, Proc. Amer. Math. Soc. **95** (1985), no. 1, 21–31. MR796440

[DS93] Guy David and Stephen Semmes, *Analysis of and on uniformly rectifiable sets*, Mathematical Surveys and Monographs, vol. 38, American Mathematical Society, Providence, RI, 1993. MR1251061

[EB19] Sylvester Eriksson-Bique, *Characterizing spaces satisfying Poincaré inequalities and applications to differentiability*, Geom. Funct. Anal. **29** (2019), no. 1, 119–189. MR3925106

[Fed70] Herbert Federer, *The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension*, Bull. Amer. Math. Soc. **76** (1970), 767–771. MR260981

[Ha96] Piotr Hajł asz, *Sobolev spaces on an arbitrary metric space*, Potential Anal. **5** (1996), no. 4, 403–415. MR1401074

[HK95] Juha Heinonen and Pekka Koskela, *Definitions of quasiconformality*, Invent. Math. **120** (1995), no. 1, 61–79. MR1323982

[JN21] Wenshuai Jiang and Aaron Naber, *$L^2$ curvature bounds on manifolds with bounded Ricci curvature*, Ann. of Math. (2) **193** (2021), no. 1, 107–222. MR4199730

[Jon90] Peter W. Jones, *Rectifiable sets and the traveling salesman problem*, Invent. Math. **102** (1990), no. 1, 1–15. MR1069238

[Kei04] Stephen Keith, *A differentiable structure for metric measure spaces*, Adv. Math. **183** (2004), no. 2, 271–315. MR2041901

[KM96] Juha Kinnunen and Olli Martio, *The Sobolev capacity on metric spaces*, Ann. Acad. Sci. Fenn. Math. **21** (1996), no. 2, 367–382. MR1404091

[KM07] Jan Kristensen and Giuseppe Mingione, *The singular set of Lipschitzian minima of multiple integrals*, Arch. Ration. Mech. Anal. **184** (2007), no. 2, 341–369. MR2299766

[KRS03] Pekka Koskela, Kai Rajala, and Nageswari Shanmugalingam, *Lipschitz continuity of Cheeger-harmonic functions in metric measure spaces*, J. Funct. Anal. **202** (2003), no. 1, 147–173. MR1994768

[KS01] Juha Kinnunen and Nageswari Shanmugalingam, *Regularity of quasi-minimizers on metric spaces*, Manuscripta Math. **105** (2001), no. 3, 401–423. MR1856619

[Li86] Peter Li, *Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature*, Ann. of Math. (2) **124** (1986), no. 1, 1–21. MR847950

[Mir03] Michele Miranda, Jr., *Functions of bounded variation on "good" metric spaces*, J. Math. Pures Appl. (9) **82** (2003), no. 8, 975–1004. MR2005202

[Mon05] Francescopaolo Montefalcone, *Some relations among volume, intrinsic perimeter and one-dimensional restrictions of BV functions in Carnot groups*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4** (2005), no. 1, 79–128. MR2165404

[NV17] Aaron Naber and Daniele Valtorta, *Rectifiable-Reifenberg and the regularity of stationary and minimizing harmonic maps*, Ann. of Math. (2) **185** (2017), no. 1, 131–227. MR3583353

[Orp21] Tuomas Orponen, *An integralgeometric approach to Dorronsoro estimates*, Int. Math. Res. Not. IMRN (2021), no. 22, 17170–17200. MR4345824

[Sha00] Nageswari Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana **16** (2000), no. 2, 243–279. MR1809341

[Sim95] Leon Simon, *Rectifiability of the singular sets of multiplicity* 1 *minimal surfaces and energy minimizing maps*, in: Surveys in Differential Geometry, Vol. II (Cambridge, MA, 1993), Int. Press, Cambridge, MA, 1995, pp. 246–305. MR1375258

Jeff Cheeger
Silver Professor of Mathematics
New York University
USA
E-mail: cheeger@cims.nyu.edu