Pure and Applied Mathematics Quarterly
Volume 19, Number 6, 2915–2918, 2023

Any oriented non-closed connected 4-manifold can be spread holomorphically over the complex projective plane minus a point

Dennis Sullivan

Dedicated to BLAINE LAWSON and his work on complex manifolds

Abstract: We give a 1965 era proof of the title assuming M is $\mathsf{spin^c}$. The fact that any oriented four manifold is $\mathsf{spin^c}$ is a challenging result from 1995 whose interesting argument by Teichner-Vogt is analyzed and used in the appendix to show an analogous integral lift result about the top Wu class in $\dim 4k$. This will be used in future work to study related complex structures on higher dimensional open manifolds.

Keywords: $\mathsf{spin^c}$, complex structures, 4-manifolds.

1. The 1965 proof

A lift c to integral coefficients of the second Stiefel-Whitney class can be induced by a map from the manifold M to the complex projective plane $\mathbb{CP}^2$, by cellular approximation of a map to the Eilenberg-MacLane space $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$ classifying c. This map F can miss a point because M open implies that M is homotopy equivalent to a three dimensional CW-complex.

The first three homotopy groups of the classifying space of oriented four plane bundles are respectively $0, \mathbb{Z}/2, 0$. Direct obstruction theory then shows F pulls back the tangent bundle of $\mathbb{CP}^2$ to the tangent bundle of M. Results from the 1966 thesis of Tony Phillips, published in [6] and based on earlier notes of Valentin Poénaru [8] from the University of Paris at Orsay, show that F is homotopic to a smooth map which is a local diffeomorphism.

2. The rest of the proof

The theorem of the title now follows using the result of Teichner-Vogt [9], analyzed and generalized in the appendix below, that oriented M^4 is $\mathsf{spin^c}$,

Received January 13, 2022.
2010 Mathematics Subject Classification: Primary 57R15; secondary 57D40.

namely the lift c exists. One then simply pulls the complex structure back to M by the smooth immersion homotopic to F, making it holomorphic.

Appendix A. Extending the 1995 argument of Teichner-Vogt

STEP 1. The first step in their argument is the general statement that the issue of whether a given mod 2 cohomology class lifts to integral coefficients only depends on whether or not the homomorphism of integral homology to the mod 2 periods lifts to a homomorphism corresponding to the expected integral periods. This reduction of the issue follows using the fortuitous fact that the natural map $\mathsf{Ext}(A,\mathbb{Z}) \to \mathsf{Ext}(A,\mathbb{Z}/2)$ is onto for any Abelian group A.

This fortuitous fact follows using the *First* nontrivial fact about Abelian groups that *any subgroup of a free Abelian group is free Abelian.* This gives a two-step free resolution of any Abelian group. The short resolution shows the higher Ext's are zero. This in turn means the long exact sequence of higher Ext's associated to the nontrivial extension $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ yields the fortuitous onto-ness.

STEP 2. Now, for any oriented $4k$ manifold M consider the homomorphism $H_{2k}(M,\mathbb{Z}/2) \to \mathbb{Z}/2$ given by $x \mapsto x \cdot x \mod 2$. This defines a mod 2 cohomology class ν_{2k} in degree $2k$, representing the Steenrod square Sq^{2k} in the sense that $\mathsf{Sq}^{2k}(x) = \nu_{2k} \cup x$, which we may call the top Wu class. Here one is using cohomology with compact support for x and ordinary cohomology for the Wu class.

For oriented M^4 the top Wu class is also the second Stiefel-Whitney class. (Both being equal to the class given by the inverse of Hurewicz bijection in degree two for the classifying space for oriented 4-plane bundles.) The next step in the Teichner-Vogt argument can be used to show this class lifts to an integral class.

By the reduction in STEP 1, we only need to study the mod 2 periods on integral classes and find the integral periods. Naïvely, one seems to be done. Because, if x is integral, then $x \cdot x$ is naturally an integer. However, this is not a linear lift, and there is a serious problem which Teichner-Vogt solve using the *Second* nontrivial fact about large abelian groups: namely, *any countable subgroup of a countable product of $\mathbb{Z}$'s is a free Abelian group.*

Consider the homomorphism

$$\mathscr{I} : H_{2k}(M,\mathbb{Z}) \longrightarrow \prod_{y \in H_{2k}(M,\mathbb{Z})} \mathbb{Z} \tag{1}$$

from $H_{2k}(M, \mathbb{Z})$ (which is countable because manifolds have a countable base) to a countable product of $\mathbb{Z}$'s with one factor for each homology class y, defined by sending each $x \in H_{2k}(M, \mathbb{Z})$ to the infinite tuple $(x \cdot y)_{y \in H_{2k}(M,\mathbb{Z})}$. This mapping $\mathscr{I}$ is linear in x. The kernel of $\mathscr{I}$ contains the kernel of the mod 2 period homomorphism for the integral cycles under consideration. Thus the lifting problem presented by the reduction of STEP 1 factors through the image of $\mathscr{I}$. By the *Second* nontrivial fact about large abelian groups we solve the problem of lifting some homomorphism of a free Abelian group to $\mathbb{Z}/2$ to a homomorphism to $\mathbb{Z}$. This is possible. QED.

Acknowledgements

These nontrivial facts about Abelian groups can be found in the treatise on abelian groups by P. Griffith [1]. I am grateful to Jiahao Hu and Claude LeBrun for discussions about the $\mathsf{spin^c}$ property of oriented 4-manifolds, and to Peter Teichner and Elmar Vogt for their wonderful argument. Also I thank Valentin Poenaru [8] and Tony Phillips [6] for their work on immersions and submersions of open manifolds and Misha Gromov for his h-principle [3] based on those ideas. Finally, André Haefliger in the early '70s [4] used the Phillips [7] and Gromov [2] independent work about maps of open manifolds transverse to a foliation to create an obstruction theory for deforming tangential structures to coordinate structures. I thank the referee for informing me that this Haefliger theory was used in the late '70s by Peter Landweber [5] to prove a general result about neighborhoods of the $(q+1)$-skeleton of an almost complex $2q$-manifold being complex that included as a special case that almost complex open four manifolds are actually complex.

N.B.: (This is a note from the editor.) In the course of handling this manuscript, the editor mediated remarkable exchanges between the author and the referee, out of which comes the following excerpt:

> "My note shows the entire story in dimension four is a one page simple proof using Phillips 65 and the Teichner-Vogt Statement discussed above. It also gives a complex structure with more geometric properties like foliations and projective coordinates. This is mathematics' usual path. Things should be simple and easy. And the history of ideas should be known to increase understanding. The original immersion idea seems powerful and simple and my goal was to provide an invitation to revisit this thought. (D. Sullivan)"

The editor would like to point out that, amidst this interaction, the author was awarded the Abel Prize 2022.

References

[1] Phillip A. Griffith. *Infinite Abelian Group Theory*. University of Chicago Press, Chicago, Ill.-London, 1970. MR0289638

[2] M. L. Gromov. Transversal mappings of foliations. *Dokl. Akad. Nauk SSSR*, **182**:255–258, 1968. MR0238352

[3] Mikhael Gromov. *Partial Differential Relations*, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1986. MR0864505

[4] A. Haefliger. Feuilletages sur les variétés ouvertes. *Topology*, **9**:183–194, 1970. MR0263104

[5] Peter S. Landweber. Complex structures on open manifolds. *Topology*, **13**:69–75, 1974. MR0339210

[6] Anthony Phillips. Submersions of open manifolds. *Topology*, **6**:171–206, 1967. MR0208611

[7] Anthony Phillips. Smooth maps transverse to a foliation. *Bull. Amer. Math. Soc.*, **76**:792–797, 1970. MR0263106

[8] V. Poenaru. *Regular Homotopy and Isotopy: I.* Harvard University, 1964.

[9] Peter Teichner and Elmar Vogt. All 4-manifolds have $\mathsf{spin}^{\mathsf{c}}$ structures. https://math.berkeley.edu/~teichner/Papers/spin.pdf.

Dennis Sullivan
Einstein Chair, Graduate Center
City University of New York
USA

Mathematics Professor
Stony Brook University
USA
E-mail: dennis@math.sunysb.edu