Revisiting the Bel-Robinson tensor

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Dedicated to H. Blaine Lawson on the occasion of his 80th birthday, in recognition of all the good moments together, his friendship and the inspiration

Outline

All along history, and especially in the 19th and 20th century, Geometry and Physics have interacted in a very positive way. Of course this continues unabated in the 21st century.

This article presents such instances revolving around General Relativity leading to a specific object, the Bel-Robinson tensor. Actually, the mathematical content of this object is not yet fully elucidated. In the context of Theoretical Physics, it has reappeared a few times since its inception in 1958 in ways which were not anticipated initially. This is one reason why it may be worthwhile to try and explore its nature in more depth.

Here is an outline of the topics to be covered in the article:

– The setting of Metric Differential Geometry;
– The curvature algebra;
– The Bel-Robinson tensor;
– Some uses of the Bel-Robinson tensor;
– Concluding remarks.

1. The setting of metric differential geometry

With Carl-Friedrich Gauss’ essay “Disquisitiones circa superficies curvas” (cf. [20]) published in 1828, in which he introduces the concept of a curved space, a new world was opened to geometers. Gauss did this in his very insightful way combining an understanding of what is at stake from a geometric point of view but at the same time publishing his fundamental discovery only

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1This article develops the material of a lecture given in Beijing in 2018 at a conference celebrating Jeff Cheeger’s 75th birthday.
when he could provide a formula for the curvature that he found satisfactory, allowing him to state his Theorema egregium in an effective way.

This was followed by the $n$-dimensional generalisation of the concept by Bernhard Riemann in his famous ‘Habilitationsvorlesung’ ‘Über die Hypothesen, welche der Geometrie zu Grunde liegen’ (cf. [40]). There the key concept is that of a general scalar product $g$ on tangent vectors depending smoothly on local coordinates.

In spite of the fact that Bernhard Riemann’s essay is dealing with Geometry, the published version does not contain any figure. It actually contains only one formula, namely the one which allows to measure the deviation of a generalised geometry from Euclidean Geometry through the introduction of the Riemann curvature tensor.

In the last part of the essay entitled “Application on space”, Riemann discusses its relevance to several parts of Physics: Astronomy, the “very large”, but also the “very small”.

Discussing further in [13] the question of trying to put into correspondence two Riemannian metrics defined locally by a diffeomorphism, Elwin Bruno Christoffel develops a calculus that identifies the Riemann curvature tensor as an invariant that needs to be matched between the two metrics, hence the name Riemann-Christoffel given to it.

Gregorio Ricci-Curbastro gave a conceptual content to the objects identified by Riemann and Christoffel. He developed his ideas in several publications in the period 1888–1892. Here are the three main documents in this respect – but [35] and [37] could also be mentioned as containing relevant information:

- in “Delle derivazioni covarianti e controvarianti e del loro uso nella analisi applicati” (cf. [36]), published in Padova in 1888, he develops the key concept of a covariant derivative $D$;
- in 1892 he developed it further in an article that appeared in the Bulletin des Sciences Mathématiques (cf. [38]), where he also shows that, in presence of a Riemannian metric $g$, there is a canonical covariant derivative $D^g$ if one insists on the two conditions that it is metric and torsion free;
- later, his joint article with his student Tullio Levi-Civita “Méthodes de calcul différentiel absolu et leurs applications” published in 1900 in the Mathematische Annalen (cf. [25]) became a reference on the subject; a curvature tensor $R^D$ can be attached to any covariant derivative $D$ and the curvature tensor of $R^{D^g}$ is indeed the Riemann curvature tensor.
In the next section, we discuss at length the structure of the Riemann curvature tensor $R^g$, spontaneously a tensor of type $(3, 1)$, in particular its mathematical nature as a consequence of its symmetries.

It is only in 1904 in [39] that Ricci-Curbastro introduces the Ricci curvature $\mathcal{R}^g$, that he defines as

$$\mathcal{R}^g(X, Y) = \text{Trace}(Z \mapsto R^g_{Z,X}Y).$$

The metric trace $s^g$ of the Ricci curvature $s^g = g^{-1} \mathcal{R}^g$ is the scalar curvature, hence a function for any $n$-dimensional metric, as the Gaussian curvature is for metrics on surfaces.

Ricci-Curbastro’s motivation to do that was purely geometric. He aimed at introducing some privileged directions at every point on a manifold after one specifies a Riemannian metric. Although inspired by the relation of the Ricci curvature with the second fundamental form for hypersurfaces, this way of looking at it turned out to be rather useless... but the Ricci curvature proved itself to be a very important concept for completely other reasons connected in the first place to Theoretical Physics. Let us examine this special role of the Ricci curvature in Geometry, first in relation with the theories of Relativity.

In 1905, after a thorough reflection on what could be learned from the Michelson-Morley experiments, Albert Einstein introduces the Theory of Special Relativity. His basic idea was that the speed of light should be taken as an absolute because of the failure of the formula for the addition of velocities. For this, he elaborated on hints given by Hendrik Lorentz (cf. [27]). This led Einstein to discuss simultaneity and synchronisation of clocks on the basis of the speed of light being insurpassable for material objects. Such a change could only be achieved by challenging the absolute character of time, one of the pillars of Newton’s theory of Mechanics.

The mathematical consequences of Einstein’s new approach were soon drawn by Henri Poincaré and Hermann Minkowski. Poincaré’s approach, presented in [34], uses invariant theory to identify the geometric structure relevant in this context, namely an indefinite metric of signature $(1, 3)$. Minkowski’s approach is much more in the spirit of Euclide’s Elements: in the famous article entitled “Raum und Zeit” (cf. [28]) he states very clearly that one has to think of a 4-dimensional continuum unifying Space and Time. It is now customary to call the standard generalised metric on $\mathbb{R}^4$ with coordinates $(t, x, y, z)$ $m = -c^2 dt^2 + dx^2 + dy^2 + dz^2$ the Minkowski metric. Here $c$ refers to the speed of light. In [29] Minkowski also considers the consequences on the electromagnetic field equations.
But the story does not stop there. In 1913, EINSTEIN and Marcel GROSSMANN took a (big) step further, and introduced a Theory of General Relativity (cf. [18]) in two articles published back to back in the Zeitschrift für Mathematik und Physik: in them they replace the scalar field describing the gravitational interaction in Newton’s theory by a Lorentzian metric, a considerable conceptual jump ahead.

This new approach requires passing from Minkowski Geometry (the analog of Euclidean Geometry in the setting of Special Relativity as mentioned before) to Lorentzian Geometry, the analog of Riemannian Geometry. Note that all what was said of Metric (Riemannian) Geometry actually holds for Lorentzian metrics since the only thing that actually matters when defining the covariant derivative is actually the non-degeneracy of the metric.

Unfortunately, the 1913 article had a serious flaw in the statement of the field equations. In 1915 consistent field equations were found as the result of a joint effort by EINSTEIN (cf. [17]) and David HILBERT (cf. [21]).

What it is now customary to call the Einstein equations are as follows:

\[ r^g - \frac{1}{2} s^g g = T, \]

where \( T \) is the stress-energy tensor representing the action of all physical fields besides gravitation (in the vacuum of course \( T \equiv 0 \)). They express the Euler-Lagrange equations of the functional \( g \mapsto \int_{\text{space-time}} s^g v_g \), where \( v_g \) denotes the volume element determined by \( g \).

On several occasions EINSTEIN compared the two sides of the equation, calling the left hand side made of “imperishable marble” and the right hand side of “inferior wood”. By this, he meant in particular that the physical side appeared a bit \textit{ad hoc}, reflecting actually the physical content of the Lagrangian leading to the equations, when the internal logic of the mathematical side was much more involved and connected in depth to the geometry.

It is quite remarkable that the strong interaction of geometric ideas with Theoretical Physics of which General Relativity is an excellent example did not stop there. Indeed, in 1918, Hermann WEYL made a first attempt (cf. [44]) to unify further Physics in proposing that a scalar factor in front of the metric could allow a unification of Gravitation with Electromagnetism.

This was actually a first serious consideration of a \textit{gauge theory}, and led to further understanding of the general concept of a covariant derivative, not necessarily a metric one.

By the very way WEYL introduced the scalar field, he was in fact considering \textit{conformal classes of metrics}, i.e. metrics \( \tilde{g} = e^u g \), where \( u \) is a smooth function and \( g \) a Riemannian or Lorentzian metric, therefore giving a boost
to Conformal Geometry. This led him to identify the part of the curvature that is not affected by a conformal change of metric, now called the Weyl curvature tensor $W^g$. It is a remarkable fact that this part of the curvature tensor is precisely the part that is complementary to the part determined by the Ricci curvature, when lifted to the space of curvature tensors, as we explain in the next section.

Before moving to the Curvature Algebra, let me follow up on the role played by the Ricci curvature in Riemannian Geometry since the situation has considerably evolved since its introduction.

Here again an important further inspiration came from the Theory of General Relativity. Indeed, one possible (and successful) approach to the Einstein equations has been proposed by Richard Arnowitt, Stanley Deser and Charles Misner in [1] and is now called the ADM approach. They start from the geometric picture of space-time as foliated by a family of 3-dimensional space-like hypersurfaces parametrised by a time-like parameter, that we can call $t$. In this context, the Einstein equations take the form of an evolution equation for the induced metric $g_t$ on the hypersurfaces and their second fundamental forms. It is interesting to note that the main driver of the evolution of $g_t$ is precisely their Ricci curvature $r_{g_t}$. It is actually rather a second order effect as the Ricci curvature of the 4-dimensional space-time of course involves terms of the type $\partial^2 g_t / \partial t^2$.

This led to the idea that, in the space of Riemannian metrics $\text{Met} M$ of a manifold $M$, an object that specialists of General Relativity considered early on, sometimes calling it Superspace, it would be interesting to view the Ricci curvature as a vector field. Indeed, $r_g$ is naturally a tangent vector in $\text{Met} M$ at $g$. The question of the existence of a flow for this vector field is one of the questions I posed in [7]. In later developments, due in particular to Richard Hamilton, this was coined as the Ricci flow and became a major tool to solve very significant geometric problems (for an account, see [9]), up to the solution of the (topological) Poincaré conjecture in the hands of Grigori Perelman.

2. The curvature algebra

By its very construction, as commutator of second order covariant derivatives, the curvature tensor $R^g$ verifies the following antisymmetry property:

- $(C_0) \ R^g_{X,Y} = - R^g_{Y,X}$ for all tangent vectors $X$ and $Y$.

But $R^g$ also verifies some other properties because of the special properties of the metric covariant derivative $D^g$. 

Indeed, for all vectors $X, Y, Z$ et $U$, the curvature tensor $R^g$ of a metric $g$ also satisfies

- $(C_1)$ $g(R_{X,Y}Z,U) = -g(R_{X,Y}U,Z)$;
- $(C_2)$ $R_{X,Y}Z + R_{Y,Z}X + R_{Z,X}Y = 0$.

As $(C_1)$ suggests, it is easier to work with the $(4,0)$ version of the curvature tensor $R^g(X,Y,Z,U) = g(R^g_{X,Y}Z,U)$.

A consequence of the previous symmetry relations is the symmetry in pairs $(C_3)$ $R^g(X,Y,Z,U) = R^g(Z,U,X,Y)$.

We will be interested in 4-tensors defined on an $n$-dimensional vector space $E$ endowed with a metric $g$ satisfying identities $(C_0)$, $(C_1)$, $(C_2)$ and $(C_3)$, called algebraic curvature tensors.

They form a vector space $\mathcal{R}E$ which it is convenient, following Chern Shiing Shen, and later Katsumi Nomizu and Ravindra Kulkarni, to view as a vector subspace of $S^2(\Lambda^2 E^*)$.

Let us look at what is the dimension of $\mathcal{R}E$ in small dimensions:

- for $\dim E = 2$, $\dim \mathcal{R}E = 1$;
- for $\dim E = 3$, $\dim \mathcal{R}E = 6$;
- for $\dim E = 4$, $\dim \mathcal{R}E \leq 21$, and actually $\dim \mathcal{R}E = 20$ as explained below.

Indeed one has:

$$S^2(\Lambda^2 E^*) = \mathcal{R}E \oplus \Lambda^4 E^*,$$

which gives an easy way of computing the dimension of the space $\mathcal{R}E$ when $E$ is $n$-dimensional. It is $\frac{1}{12}n^2(n^2-1)$ since it is well known that the dimension of $\Lambda^4 E^*$ is the binomial coefficient $\binom{4}{n}$ and that of $S^2(\Lambda^2 E^*)$ is $\frac{1}{2} \frac{n^2(n-1)}{2} + 1$.

The first decomposition we will need to consider concerns the space of symmetric covariant 2-tensors where the Ricci curvature naturally lives. There, one can decompose $r^g$ into its diagonal part involving the scalar curvature $s^g$ and its trace-free part, that we denote by $z^g$; one namely has

$$r^g = \frac{1}{n} s^g g + z^g.$$

To go further and decompose the space $\mathcal{R}E$, we need to introduce the Nomizu-Kulkarni product $\otimes$ between symmetric covariant 2-tensors: namely for $h, k \in S^2 E^*$ and $X, Y, Z$ and $U \in E$, one sets:

$$(h \otimes k)(X,Y,Z,U) = h(X,U) k(Y,Z) + h(Y,Z) k(X,U)$$

$$- h(X,Z) k(Y,U) - h(Y,U) k(X,Z).$$
One verifies that $h \otimes k \in \mathcal{RE}$ for any $h, k \in S^2 E^*$. 

Then, for $R \in \mathcal{RE}$, whose trace is denoted by $r$ and iterated trace by $s$ and trace-free part by $z$, as we did earlier for the Riemann curvature tensor, the key decomposition is

$$R = \frac{1}{2n(n-1)} sg \otimes g + \frac{1}{n-2} z \otimes g + W,$$

corresponding to irreducible components of dimensions $1, \frac{1}{2} n(n+1)$ and the complement of the sum to $\frac{1}{12} n^2 (n^2 - 1)$ respectively.

For the Riemannian curvature $R^g$, one of course has $s = s^g, z = z^g, W = W^g$.

### 3. The Bel-Robinson tensor

It follows from a theorem of WEYL (cf. [45]) that the dimension of the space of quadratic forms on $\mathcal{RE}$ invariant by isometries is equal to the number of irreducible components in the decomposition of $\mathcal{RE}$ under the action of the orthogonal group. For $R \in \mathcal{RE}$, the invariant quadratic forms are $|R|^2, |r|^2$ and $s^2$.

In the late 30s, while looking for other Lagrangians for General Relativity, Cornelius LANCZOS got interested in quadratic functionals in the curvature in dimension 4 (see [24]):

$$\int_M (\alpha |R^g|^2 + \beta |r^g|^2 + \gamma (s^g)^2) \, v_g.$$

For a good choice of coefficients $\alpha, \beta$ and $\gamma$, the functional does not generate any field equation. He then concluded that this Lagrangian was not interesting.

From the point of view of a mathematician, this has a strong meaning, namely that one has put one’s hand on an invariant of $M$. Actually, this falls under the considerations that CHERN introduced when generalising the Gauß-Bonnet theorem to higher dimensions (cf. [11]). One indeed has:

$$\int_M \left( |R^g|^2 - \frac{1}{4} |r^g|^2 + (s^g)^2 \right) v_g = 8\pi^2 \chi(M),$$

where $\chi(M)$ denotes the Euler-Poincaré characteristic of the manifold $M$. 
On an oriented manifold, the volume element $v_g$ can be seen as an $n$-dimensional form. In all dimensions, contraction with the volume element and use of the duality defined by $g$ allow to define a map $\ast$ from $\Lambda^k E^*$ to $\Lambda^{n-k} E^*$ for $0 \leq k \leq n$, the so-called Hodge map.

In dimension $n = 2m$, $\ast$ maps $\Lambda^m E^*$ to itself. In dimension 4 in the Riemannian setting it is an involution and $\Lambda^2 E^*$ splits as $\Lambda^2 E^* = \Lambda^+ \oplus \Lambda^-$ corresponding to the $\pm 1$-eigenvalues of $\ast$.

This introduces a further structure on $S^2(\Lambda^2 E^*)$: $g \otimes g$, being a multiple of the identity, and $W$ preserve the chirality decomposition, allowing to split $W$ as $W = W^+ \oplus W^-$, while $g \otimes z$ exchanges the factors $\Lambda^+$ and $\Lambda^-$.

On a compact orientable 4-manifold $M$ with signature $\sigma(M)$, one has:

$$\int_M (|W^g|^2 - |W^g|^2) v_g = 32\pi^2 \sigma(M).$$

In the Lorentzian setting $\ast$ is a complex structure on $\Lambda^2 E^*$.

In two Notes aux Comptes Rendus de l’Académie des Sciences de Paris in 1958 and 1959 and a short article in 1962 (cf. [2, 3] and [4]) Luis Bel introduced a quadratic expression $Q(R)$ in a curvature tensor $R$. In notes for a course (cf. [41]) Ivor Robinson introduced a similar quantity but came back to it for example in [42] published in 1997. Hence the name Bel-Robinson Tensor given to it. (Note also that it is Exercise 15.2. on page 382 of the major reference [30] for the theory of Gravitation.)

In dimension 4 a possible definition of $Q(R)$ uses $\ast$ to compose it with $R$, namely for $X, Y, Z, U \in E$ and $(e_i)$ an orthonormal basis, one can write:


The case Bel was considering was a vacuum space-time (then $R$ reduces to $W$). This is why the definition of the Bel-Robinson tensor is sometimes given only for a Weyl tensor.

An equivalent definition in dimension 4 (not involving $\ast$) is

$$Q(R)(X, Y, Z, U) = \sum_{i,j=1}^{4} (R(X, e_i, Z, e_j) R(Y, e_i, U, e_j) + R(X, e_i, U, e_j) R(Y, e_i, Z, e_j) - \frac{1}{8} |R|^2 g(X, Y) g(Z, U)).$$
This definition of the Bel-Robinson tensor works in dimension $n$.

The Bel-Robinson tensor enjoys a number of very interesting (and intriguing) properties:

- **The Bel-Robinson tensor is fully symmetric in all its arguments**, that is for $X, Y, Z, U \in E$,
  \[
  \]

- **The Bel-Robinson tensor is traceless when built on a Weyl tensor**, namely
  \[
  \sum_{i=1}^{4} Q(W)(e_i, e_i, X, Y) = 0.
  \]

- **An almost full contraction of the “square” of the Bel-Robinson tensor** (a quartic quantity in the curvature) is a multiple of the metric:
  \[
  \sum_{i,j,k=1}^{4} Q(R)(X, e_i, e_j, e_k) Q(R)(Y, e_i, e_j, e_k) = 1/4 |Q(R)|^2 g(X, Y).
  \]

In a Lorentzian setting, the Bel-Robinson tensor enjoys a positivity property, usually called the **Dominant Property**, namely, for future-oriented vectors $X, Y, Z, U$, $Q(R)(X, Y, Z, U) \geq 0$. This quantity vanishes for time-like vectors only if $R = 0$.

When defined from a Weyl curvature tensor, Joan Josep Ferrando and Juan Antonio Sáez have discussed in [19] the possibility of recovering the Weyl tensor from the Bel-Robinson tensor.

The motivation for introducing the Bel-Robinson tensor comes from the challenge of giving a local definition of the gravitational energy, something escaping since the creation of the theory. The inspiration came from electromagnetism where the energy of an electromagnetic field $\omega$ is just $|\omega|^2$ in space-time.

In the coupling of gravitation with electromagnetism, the stress-energy tensor $T_{e.m.}$ is for vectors $X, Y$ and an orthonormal basis $(e_i)$ given by
\[
T_{e.m.}(X, Y) = \sum_{i,j=1}^{4} \omega(X, e_i) \omega(Y, e_i) - 1/4 |\omega|^2 g(X, Y),
\]
a definition that would make sense in any dimension.
Note that, in dimension 4, an equivalent expression of $T_{\text{e.m.}}$ can be given:

$$T_{\text{e.m.}}(X, Y) = \frac{1}{4} \sum_{i,j=1}^{4} (\omega(X, e_i) \omega(Y, e_i) + (\ast \omega)(X, e_i) (\ast \omega)(Y, e_i)).$$

Then $T_{\text{e.m.}}$ is traceless and verifies the identity:

$$\sum_{i=1}^{4} T_{\text{e.m.}}(X, e_i) T_{\text{e.m.}}(Y, e_i) = \frac{1}{4} |T_{\text{e.m.}}|^2 g(X, Y),$$

as well as a Dominant energy condition.

The inspiration for the Bel-Robinson tensor looks pretty clear.

Another feature of the Bel-Robinson tensor has to do with conservation laws. As mentioned before, Bel considered the case of a space-time satisfying $r^g = \lambda g$, for which $R$ reduces to a multiple of $g \otimes g$ and $W$.

In this case $R^g$ not only satisfies the second Bianchi identity, expressing that $R^g$ is closed as a $\Lambda^2 T^* M$-valued 2-form, but is also coclosed, making $R^g$ a harmonic $\Lambda^2 T^* M$-valued 2-form. This leads to the fact that, in this case, the Bel-Robinson tensor is also divergence-free. (In the context of harmonicity of the curvature tensor viewed as a vector-valued 2-form, one can consult [8] for an extensive use of the Curvature Algebra in the case of 4 dimensions.)

This brings another similarity of the Bel-Robinson tensor with a stress-energy tensor. Indeed, the left hand side of the Einstein equation $r^g - \frac{1}{2} s^g g$ is automatically divergence-free as the gradient of the Diff $M$-equivariant functional $\int_M s^g v_g$ on $\text{Met} M$, so that any stress-energy $T$, on the right hand side of the Einstein equations, must also be divergence free.

4. The uses of the Bel-Robinson tensor

In 1918, Theodor Kaluza sent to Einstein another approach to unify Gravitation and Electromagnetism (cf. [22]).

Starting from a Lorentzian metric $g$ on a space-time $M$ and an electromagnetic potential $\xi$ on $M$, he introduced an extended 5-dimensional Lorentzian space-time $\tilde{M} = M \times \mathbb{R}$ with metric $\tilde{g}$ obtained using the following Ansatz: if $\pi : M \times \mathbb{R} \to M$ is the natural projection, then, with $u$ the extra variable and $\phi$ a function on $M$, $\tilde{g} = \phi(\pi^* g + (du + \pi^* \xi) \otimes (du + \pi^* \xi)).$

The key formula is the one describing the 5-dimensional Ricci tensor $r^g$ in terms of the data $g$, $\xi$ and $\phi$. The striking fact is that claiming that $r^g = 0$ gives back the Maxwell equations for $\xi$ and the coupled Einstein-Maxwell equations for $g$ and $\xi$ with $T_{\text{e.m.}}$ given as before.
In the context of *Riemannian submersions*, similar formulas to compute the curvature of the natural metric on the frame bundle appear. They were developed much later by Barrett O’Neill (cf. [31]). They also involve the curvature quadratically.

In their seminal work on the global stability of Minkowski space (cf. [12]), Demetrios Christodoulou and Sergiu Klainerman use the ADM approach of looking at the Einstein equations as an evolution equation. In order to establish their results, they have to develop many estimates to control the metric.

A first strategic element they use is to think of $R$ as a harmonic vector-valued 2-form. As recalled earlier, closedness is simply the second Bianchi identity and coclosedness comes from the vacuum Einstein equations, which imply that $R$ reduces to $W$.

They use the Bel-Robinson tensor to establish some almost conservation laws for almost conformal vector fields. For this, they introduce a modified metric Lie derivative $\hat{L}$ preserving the traceless property of tensor fields.

The third idea is to use the Bel-Robinson tensor construction to develop estimates. They do it not on $Q(R)$ but on $Q(\hat{L}_X R)$ where $X$ is an approximate conformal vector field.

Stanley Deser is one of the theoretical physicists who has been repeatedly advocating for the use of the Bel-Robinson tensor (see e.g. [14]) up to the point of entitling one of his lectures *The Immortal Bel-Robinson Tensor* (cf. [15])!

Here is a quote from Deser taken from [15]:

“What is especially interesting is that many of the applications of the Bel-Robinson tensor have been far from the arena of $D = 4$ general relativity, for which it was created and intended, and that it has risen from its original incarnation as a would-be tensorial energy-density to its avatar as a basic member of gravitational supersymmetric multiplets and invariants; indeed it is there that it takes on precisely the stress tensor role!”

In [15] he deals with $D = 11$ Supergravity and relied on generalisations of the Bel-Robinson tensor to identify local invariants to prove the non-renormalisability of the theory.

Actually, Deser has kept advocating for the use of the Bel-Robinson tensor as a relevant tool to deal with other issues than just General Relativity (see for example his 2015 joint article [16] with Joel Franklin).

The main argument why the Bel-Robinson tensor is not a stress-energy tensor is its (physical) dimension, as it does not match what is expected from such a quantity. Bel used the expression *super-energy* for it.

One can wonder what a *super-energy* could be. In order to have a quantity with the right physical dimension, in 1997, Miguel Bonilla and José
SENOLIYA introduced in [6] a “square root” for the Bel-Robinson tensor, a traceless symmetric tensor for some special space-times and then generalised it.

This led also to several other attempts, in particular by SENOLIYA in 1999 (cf. [43]) to define also a super-energy for many fields, including the electromagnetic field $\omega$, by introducing a 4-tensor $D_\omega$ with properties similar to those of the Bel-Robinson tensor, missing the divergence-freeness... unless there is no gravitation!

Considering quantities in the curvature acting on symmetric tensors in not a new thing. In the same way as $R$ can be seen as operating on exterior 2-forms, it can be made to act on symmetric covariant 2-tensors.

A natural place where this happens is in the study of infinitesimal deformations of Einstein metrics. The linearised Ricci curvature determines a second order operator which can be compared to the operator $D^*D$, i.e. a Laplacian in Riemannian Geometry and a D’Alembertian in Lorentzian Geometry, the difference consisting in curvature terms acting on symmetric 2-tensor fields. The article Propagateurs et Commutateurs en Relativité Générale (cf. [26]) by André LICHNEROWICZ is a reference for that.

In [10] Hermann KARCHER and I discuss the action of the curvature tensor on several tensorial quantities, including symmetric 2-tensors. It should also be pointed out that, coming back to [26], Norihiro KOISO was able to obtain some rigidity results for Einstein metrics (cf. [23]).

5. Concluding remarks

The purpose of this article was to call attention to an object which appeared in the Theory of General Relativity to solve a problem, namely the definition of a local energy, which has not yet been solved in spite of repeated efforts. The fact that the Bel-Robinson tensor has undergone several rebirths calls attention. It is a fact that the nature of the Bel-Robinson tensor is not yet fully understood, and one can wonder whether it could prove useful in the context of Riemannian Geometry of fibre bundles. What could be the use of a super-energy in Riemannian Geometry?

In this article, I did not discuss the spinorial definition proposed for it by Roger PENROSE (cf. [32]) and presented on page 240 of the reference book [33] by Roger PENROSE and Wolfgang RINDLER. It was also followed by Göran BERGQVIST in [5]. This may shed another light on it and provide other clues. I leave this to another article.

Another natural question: O’NEILL formulas for the curvature of the tangent bundle viewed as a vector bundle over a Riemannian manifold produce
a quadratic expression in the curvature naturally. Can one make sense of the Bel-Robinson tensor in this way?

It is my hope that other questions will come to the minds of some of the readers.

References


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[41] Robinson Ivor, Unpublished King’s College Lectures (1958).

[42] Robinson Ivor, On the Bel-Robinson tensor, Class. Quantum Grav. 1A 14 (1997), 331–333. MR1692226


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