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Global behavior at infinity of period mappings defined on algebraic surface*

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Dedicated to our long-time friend, Blaine Lawson, in appreciation of his extensive and innovative career

Abstract: The global behavior of period mappings defined on generally non-complete algebraic varieties B as well as their local behavior around points in the boundary $Z = \overline{B}\backslash B$ of smooth completions of B have been extensively investigated. In this paper we shall study the *global* behavior of period mappings in neighborhoods of the entire boundary Z when $\dim B = 2$. One method will be to decompose the dual graph of the boundary into basic building blocks of cycles and trees and analyze these separately. A main tool will be a global version of the classical nilpotent orbit theorem of Schmid.

0. Introduction

We assume given the data $(\overline{B}, Z; \Phi)$ consisting of a period mapping

$$\Phi : B \to \Gamma\backslash D \tag{0.1}$$

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*At the end of the introduction we have listed the notations used in this paper and the main assumptions that are made for the results given below.

where $B = \overline{B}\backslash Z$ is a smooth quasi-projective variety and $\overline{B}$ is a smooth projective completion of B with the boundary Z a reduced normal crossing divisor. By [1] the image $\Phi(B) := P$ is a projective algebraic variety. In this paper we are interested in the global study of Φ in neighborhoods of Z in $\overline{B}$. Typical questions we shall seek to address are

Q1. What are natural completions of the period mapping (0.1)?

By *completion* we shall mean a compact Hausdorff space $\overline{P}$ that is stratified by complex analytic varieties in which P is a dense open set. By natural we shall mean that the boundary $\overline{P}\backslash P$ is defined by Hodge-theoretic properties.

Q2. What properties do these completions have? For example are the completions defined in (0.2) below algebraic varieties? If so, what are natural line bundles over them and which are ample, semi-ample, big, nef?

Q3. What can we say about the global geometry of $\overline{B}$ along Z; e.g., properties of the normal bundle of Z?

Q4. For period mappings arising from algebraic geometry, e.g., for those mapping from the moduli space of general type surfaces, what properties may be inferred about the period mapping at infinity?

To address these questions we shall make several assumptions.

We first assume that the period mapping $(0, 1)$ is *non-degenerate*, meaning that $\dim P = \dim B$. Equivalently, the differential Φ_* is generically 1-1.

Secondly we shall restrict to the case $\dim B = 2$. One reason is that in this case the natural completions we shall define can be proved to be analytic varieties. Another is that any monodromy group is realized on a 2-dimensional smooth parameter variety. Additionally the rich structure of isolated singularities of normal algebraic surfaces will enter. Finally, the birational geometry is of course much simpler in dimension 2.

Along each irreducible component Z_i of Z the period mapping (0.1) will define a limiting mixed Hodge structure. On the Zariski open $Z_i^* := Z_i\backslash\left(\bigcup_{j\neq i} Z_i \cap Z_j\right)$ of smooth points of Z_i the associated graded of the limiting mixed Hodge structures will give an ordinary period mapping. These may be studied inductively by dimension. For this reason of particular interest is the case when these mappings are locally constant; a third assumption is that this is the case.[1] We also assume that Z is connected. Using Grauert's

[1] In general we will have $Z = Z' \cup Z''$ where along the components Z'_α of Z' the associated graded to the limiting mixed Hodge structures will vary while along the components Z''_i of Z'' it will be locally constant. With somewhat more elaborate statements the results given below when $Z' = \emptyset$ may be extended to the general case $Z' \neq \emptyset$.

theorem [9] the period mapping will contract Z to an isolated singular point on a normal analytic surface. Finally we will assume that the pair $(\overline{B}, Z)$ is minimal in the sense that any -1 curve will meet Z in at least three points.

We now establish some notations and summarize the preliminary Sections I and II. In Section I we define set-theoretic extensions Φ^T and Φ^S of the period mapping (0.1).[2] These correspond respectively to the equivalence classes of limiting mixed Hodge structures along Z and to the associated graded of these limiting mixed Hodge structures. In this sense they encode the maximal and minimal Hodge-theoretic information in the degeneration. They fit in a commutative diagram of mappings of sets

$$\begin{tikzcd} & \overline{\mathcal{P}}^T \arrow[dd] \\ \overline{B} \arrow[ur, "\Phi^T"] \arrow[dr, "\Phi^S"'] & \\ & \overline{\mathcal{P}}^S. \end{tikzcd} \tag{0.2}$$

An attempt to construct a Satake type compactification is in [12], while a toroidal type compactification is partially achieved in [16] and [13]. One point of this paper is to give a treatment of these approaches under simplifying assumptions.

The informal set-theoretic definition of $\overline{\mathcal{P}}^T$ is to take a quotient of the set of limiting mixed Hodge structures (V, Q, W, F_b) at each point $b \in \overline{B}$.[3] For $b \in B$ we have a usual polarized Hodge structure. For $b \in Z$ we have a polarizable mixed Hodge structure; for $b \in Z_i^*$ the weight filtration is $W(N_i)$ and for $b \in Z_i \cap Z_j$ it is $W(N_i + N_j)$. The quotient is with respect to

(i) identifying the fibre of the local system near to b with a fixed vector space V;
(ii) taking the equivalence class $[F_b]$ of the limit Hodge filtrations under reparametrization of the nilpotent orbit approximating Φ near $b \in Z$.

The equivalance relation (i) involves monodromy and the construction leads naturally to a fan-like object $\Sigma_\Phi \subset \mathrm{End}(V)$. The construction is basically a relative version of the one given in [16].

For the set theoretic description of $\overline{\mathcal{P}}^S$ the associated graded to a limiting mixed Hodge structure $(V, Q, W(N), F)$ is a direct sum of Hodge structures

[2]The "T" is meant to suggest toroidal and the "S" Satake-Baily-Borel.
[3]The precise definition is given at the beginning of Section I.

that depends only on the equivalence class $[F]$ and whose polarization depends on N.[4] The definition of $\overline{\mathcal{P}}^S$ is the quotient of $\overline{\mathcal{P}}^T$ induced by the map $(V,Q,W,F) \to \mathrm{Gr}^W_\bullet(V,Q,W,F)$.[5]Under a local Torelli assumption and an assumption about descent of the augmented Hodge line bundle (which is satisfied in interesting geometric cases) we shall prove that $\overline{\mathcal{P}}^S$ is a normal projective surface.[6] We may then take for $\overline{B}$ an iterated blow up of $\overline{P}^S$.

In Section II we discuss the monodromy around a single component Z_0 of Z, which meets the other components of Z in a picture given by Figure 1.

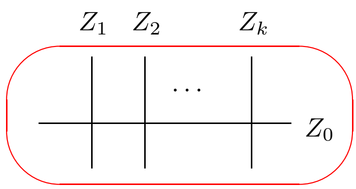


Figure 1

Referring to Theorem II.10 for the precise statement, the result is that monodromy

- preserves the weight filtration $W(N_0)$ on V;
- acts as a finite group on $\mathrm{Gr}_0^{W(N_0)}(V)$;
- acts as a discrete abelian group on the vector space $\mathrm{Gr}_{-1}^{W(N_0)}\,\mathrm{End}(V)$;
- if this action is trivial, its image in $\mathrm{Gr}_{-2}^{W(N_0)}\,\mathrm{End}(V)$ is generated by $N_1,\dots,N_k$;
- if all of the above are finite groups, then the monodromy on $\mathrm{End}(V)$ is a finite group.

[4]More precisely, along Z_i^* the associated graded is a direct sum of polarized Hodge structures. At an intersection point $Z_i \cap Z_j$ the associated graded is a direct sum of the polarized Hodge structures where the polarization depends on an element in the monodromy cone generated by N_i and N_j.

[5]In [12] it is claimed that $\overline{\mathcal{P}}^S$ has the structure of a projective algebraic variety. The argument there is incomplete in several aspects. In the case $\dim B = 2$ we will see that it follows from the Hodge index theorem, a result of Grauert that $\overline{\mathcal{P}}^S$ has the structure of a compact analytic surface, and an assumption about descending the *augmented Hodge line* bundle

$$L := \overset{p}{\otimes} \det F^p.$$

[6]It has now been proved that in any dimension this descent is always possible. The argument introduces some new Hodge theoretic techniques. The proof will appear in the updated version of [13].

The issue arises in that although the monodromy image of $\pi_1(U_i^*)$ is a finite group, we do not know that this is the case for $\pi_1(U_{ij}^*)$ where we have a picture given by Figure 2 below; U_{ij} is the red neighborhood of $Z_i \cup Z_j$ and $U_{ij}^* = U_{ij} \backslash (Z \cap U_{ij})$.

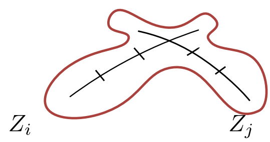


Figure 2

Thus we make the assumption

(0.3) *For any connected union Y of Z_i's whose dual graph is a tree the monodromy image of $\pi_1(U_Y^*)$ is a finite group.*

The assumption is satisfied if Z is irreducible or if all $Z_i^* = \mathbb{C}^*$, these being cases that arise in the important example of KSBA moduli of general type surfaces. With this assumption the non-finite local constituents of monodromy are representations of $\pi_1(U_C^*)$'s where C is a cycle, a cycle being a connected union of $Z_i \cong \mathbb{P}^1$'s whose dual graph is an S^1. There is then a generally infinite order monodromy element $\rho(\gamma)$ where γ is a circuit as in Figure 3. In Section V it is proved that the set $\Sigma_C \subset \operatorname{End}(V)$ of the N_i's and their monodromy conjugates lies in a 2-plane and is represented by a planar graph that as a consequence of Hodge-Riemann II has a convexity property and for $d_i = -Z_i^2$ has all $d_i \geqq 2$ and at least one $d_i \geqq 3$. In case Z consists of m cycles Σ_Z lies in a vector space of dimension $\leqq m+1$.

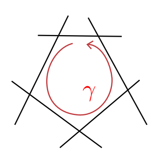


Figure 3

In summary, aspects of the familiar picture in the case of the toroidal completion of Hilbert modular surfaces [20, 21] extend to general variations of Hodge structure over a 2-dimensional parameter space.

There are line bundles naturally associated to the data $(\overline{B}, Z; \Phi)$. Assuming the local Torelli condition (II.6), which linear combinations of the line bundles $L, [Z_i], K_{\overline{B}}$ over $\overline{B}$ are ample or semi-ample? Here $L \to \overline{B}$ is the canonical extension to $\overline{B}$ of the augmented Hodge line bundle $\overset{p}{\otimes} \det F^p \to B$. Taking the case when $\dim \overline{B} = 2$ there are the following responses to this question, where for the purpose of exposition we assume that the differential Φ_* is everywhere injective on B and that the cone $\mathrm{Eff}^1(\overline{B})$ of effective algebraic 1-cycles on $\overline{B}$ is finitely generated. Although we shall not do so here the second assumption may be eliminated using Theorem 7.17 in [14].

(0.4) *There exist m_0 and $a_i > 0$ such that*

$$mL - \Sigma a_i Z_i$$

is ample for $m \geqq m_0$;[7]

(0.5) *There exists m_1 such that for $m \geqq m_1$,*

$$K_m := mL + K_{\overline{B}}$$

is ample modulo -2 curves;[8]

(0.6) $K_{\overline{B}} + Z$ *is ample modulo -2 curves in Z.*[9]

The proof will show that in general among the bundles $L, [Z_i], K_{\overline{B}}$ only these combinations have the above properties.

The remaining part of Section II pertains to the period mapping associated to a KSBA family

$$\mathfrak{X} \xrightarrow{\pi} \overline{B}$$

of general type surfaces $X_b = \pi^{-1}(b)$ ([15]). With the precise statements to be given there we assume that the singular X_b, $b \in Z$ are normal with a log-canonical singular point x_b (loc. cit.). Then there are two cases:

(i) x_b is non-Gorenstein, in which case it is a rational singularity and then the period mapping extends to give a pure Hodge structure; we shall not consider this case (cf. [11] for discussion of this situation);

[7]More specifically, in (0.4) we first may choose $a_i > 0$ such that all intersection numbers $Z_j \cdot (\sum_i a_i Z_i) < 0$. Then for such an $a := (a_1, \dots, a_k)$ there is an $m_0(a)$ such that (0.4) holds for $m \geqq m_0(a)$.

[8]This means that K_m is semi-ample and $\mathrm{Proj}(K_m)$ only contracts -2 curves in $\overline{B}$. Similarly for $K_{\overline{B}} + Z$ in (0.6).

[9]There is an extensive literature concerning the bigness and hyperbolicity of $K_{\overline{B}} + Z$ (cf. [6, 2] and the references cited there).

(ii) x_b is Gorenstein and then it is (a) a simple elliptic singularity, (b) a cusp $D = \cup D_i$ or (c) a special quotient singularity; we will consider only the cases (a) and (b).

In case (iia) we will assume that Z is irreducible and that along Z the X_b have a simple elliptic singularity where the elliptic curve C is independent of b.[10] This is equivalent to the associated graded to the limiting mixed Hodge structures being locally constant along Z, a property that is automatic in case (ii)(b). In both cases we assume a local Torelli condition. With the precise statements to be given in Section II we have

(0.7) (a) *In the simple elliptic singularity case, Z is a branched covering of C that contracts to a simple elliptic singularity with $Z^2 = C^2$;*

(b) *In the case* (iib)*, $Z = \cup Z_i$ contracts to a cusp and the Z_i^2 are determined by the D_i^2.*

Proofs are given in Section IV. They use the basic formula (0.8) below.

In Section V we discuss the basic formula together with applications. To explain it we denote by Z_0 an irreducible component of Z and by $Z_1, \dots, Z_k$ the other components of Z that meet Z_0 as in Figure 1 above. For simplicity of exposition we assume that the limiting mixed Hodge structure is constant along Z_0^*.[11] Then there is the data

(i) an abelian variety J and a morphism

$$\Phi_1 : Z_0 \to J$$

given by the level 1 extension data to the limiting mixed Hodge structures along Z_0;[12] and

(ii) a cone of ample line bundles $L_M \to J$ where $M \in \mathrm{Gr}^W_{-2} \mathrm{End}(V)^* \cong \mathrm{Gr}^W_{+2} \mathrm{End}(V)$.

The basic formula is

$$\boxed{-\Phi_1^* L_M = \sum_{i=0}^{k} \langle M, N_i \rangle \left[Z_i \right]\big|_{Z_0}.} \tag{0.8}$$

[10]This is the situation that arises for the desingularization of a KSBA family of surfaces.

[11]For the result given in Section V we need only assume that the limiting mixed Hodge structures are locally constant.

[12]Φ_1 is initially defined on Z_i^* and it extends to all of Z_i.

Both sides are elements of $\operatorname{Pic}(Z_0)$. The left-hand side gives information *along* Z_0 while the right-hand side gives information *normal* to Z_0.[13]

As a first application, if Z_0 does not meet any other components, then the assumption that Φ is non-degenerate implies that Φ_1 is non-constant along Z_0 so that

$$Z_0^2 = -\frac{\deg L_M}{\langle M, N_0\rangle} < 0.$$

Further arguments lead to the negative definiteness of the intersection matrix for a cycle C and the results that Σ_C lies in a 2-dimensional subspace of $\operatorname{End}(V)$, is a 1-dimensional convex graph, and that if Z has m cycles Σ_U lies in a vector subspace of $\operatorname{End}(V)$ having dimension $\leqq m+1$.

Notations[14]

- A *polarized Hodge structure* of weight n is (V, Q, F).
- A *mixed Hodge structure* is (V, W, F); it is understood that the mixed Hodge structures will be graded polarized (cf. [19]).
- The *associated graded* to a mixed Hodge structure is

  $$\operatorname{Gr}^W_\bullet(V) = \oplus W_k/W_{k-1} := \oplus H^k$$

  where $W_k/W_{k-1} = H^k$ is a Hodge structure of weight k.
- A mixed Hodge structure is a successive extension of pure Hodge structures; the set of mixed Hodge structures with a fixed associated graded is a complex manifold that is an iterated complex analytic fibration with successive fibres isomorphic to

  $$\bigoplus_{k\geqq 0, l\geqq 1} \operatorname{Ext}^1_{\mathrm{MHS}}(H^{k+\ell}, H^k). \tag{0.9}$$

- A nilpotent operator $N : V \to V$ defines a weight filtration $W(N)$; a *limiting mixed Hodge structure* is a mixed Hodge structure $(V, Q, W(N), F)$ with additional properties (cf. [4]).
- D denotes a *period domain*.
- A *period mapping* is given by

  $$\Phi : B \to \Gamma\backslash D. \tag{0.10}$$

- A *variation of Hodge structure* is denoted by $(\mathbb{V}, F, \nabla; B)$.

[13] A proof of this formula is given in [13]; cf. Theorem 4.3.

[14] A general reference for Hodge theory is [3].

Upon choice of a base point $b_0 \in B$, the data of a period mapping and a variation of Hodge structure are equivalent, and we shall use the two terms interchangeably.

We would like to thank the referee for an extensive constructive critique of the original version of this paper.

I. Basic definitions and localization

The objective of this section is to define, at the set-theoretic level, two natural extensions of a period mapping. We shall first do this for the local building blocks, as defined below, of the period mapping in a neighborhood $U \subset \overline{B}$ of the boundary Z of B. The construction will be done in the case of a 2-dimensional source space; it may be extended to the general case.

The basic setup is given by $(\overline{B}, Z; \Phi)$ where

- $\overline{B}$ is a projective algebraic variety;
- $Z \subset \overline{B}$ is a reduced normal crossing divisor;
- $\Phi : B \to \Gamma\backslash D$ is a variation of Hodge structure with monodromy group Γ;
- $\Phi \subset B \to P \subset \Gamma\backslash D$ is a proper holomorphic mapping with image P a quasi-projective variety and $\dim P = \dim B$;[15]
- $Z = \cup Z_i$ is connected and the logarithm N_i of the unipotent factor of the monodromy T_i around Z_i is non-zero.

One method used in this paper is to study the global variation of Hodge structure along Z by first doing this for its various pieces. Associated to $Z = \sum Z_i$ is a connected planar graph $G(Z)$.[16] A *tree* Y is a connected subgraph of $G(Z)$ with $\pi_1(Y) = \{e\}$, and a *cycle* C is a minimal subgraph with $\pi_1(C) = \mathbb{Z}$. Minimal here means that we cannot remove any subgraph and still have $\pi_1 = \mathbb{Z}$. Typical pictures are shown in Figure 4.

Notations: For the objects

$$Z_i, Y, C, Z \tag{I.1}$$

[15] Cf. [1]. We shall assume that the differential Φ_* is non-zero in B. Also what follows can be extended to the case where Φ is not locally liftable due to fixed points of Γ (cf. [11]).

[16] The usual graph notation would be $\Gamma(Z)$. However we are already using Γ for monodromy.

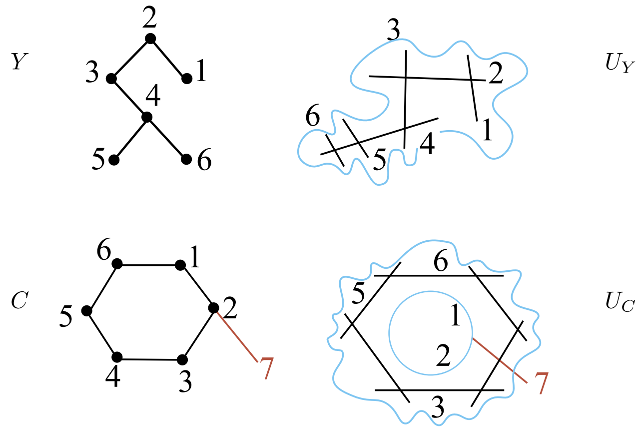


Figure 4

we denote by

$$U_i, U_Y, U_C, U \tag{I.2}$$

a neighborhood of each in $\overline{B}$, and by

$$U_i^*, U_Y^*, U_C^*, U^* \tag{I.3}$$

the open sets $U_i^* = U_i \backslash Z \cap U_i$, etc. Above we have pictured in blue a U_Y and a U_C. We will think of (I.3) as the building blocks at infinity of the global variation of Hodge structure over B. Each of the spaces in (I.1) has its own planar graph G_i, $G(Y), G(C)$ and $G(Z)$.

We will next construct fan-like combinatorial objects $\Sigma_i, \Sigma_Y, \Sigma_C, \Sigma_U$ in $\mathrm{End}(V)$ associated to U_i, Y, C and U. We first illustrate the construction for U_i. For this we use the exact sequence

$$0 \to \mathbb{Z} \to \pi_1(U_i^*) \to \pi_1(Z_i^*) \to 1 \tag{I.4}$$

of fundamental groups associated to the topological fibration $S^1 \to U_i^* \to Z_i^*$. For each point $b \in Z_i^*$ we choose a homotopy class of paths γ from a nearby base point $b_0 \in U_i^*$ to b and denote by $\sigma_{i,\gamma} \in \mathrm{End}(V)$ the monodromy logarithm associated to the path pictured in Figure 5. A more standard notation is $T_{i,\gamma} = \exp N_{i,\gamma}$. This generates the monodromy image of the $\mathbb{Z}$ in (I.4). Since the $\mathbb{Z}$ generates the kernel of $\pi_1(U_i^*) \to \pi_1(U_i)$, the image $N_{i,\gamma} \in \mathrm{End}(V)$ is well defined.

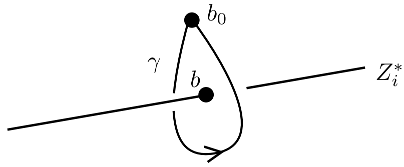


Figure 5

For a point $b \in Z_i \cap Z_j$ we denote by $\sigma_{ij,\gamma}$ the line segment generated by $\sigma_{i,\gamma}$ and $\sigma_{j,\gamma}$. We then take the union of all the $\sigma_{i,\gamma}$ and $\sigma_{ij,\gamma}$ where γ ranges over Γ_i. In this way we obtain

$$\Sigma_i \subset \mathrm{End}(V).$$

We note that Σ_i may not be a *fan*; i.e., it may have self-intersections. Sometimes it is possible to sub-divide and add finitely many vertices, but in general this will not be the case.[17]

There are corresponding constructions for U_Y, U_C and U, giving rise to Σ_Y, Σ_C, Σ_U. We denote by Σ the above construction for B itself. We note that

(I.5) $\pi_1(U_i^*), \pi_1(U_Y^*), \pi_1(U_C^*), \pi_1(U^*)$ and $\pi_1(B)$ act respectively on $\Sigma_i, \Sigma_Y, \Sigma_C, \Sigma_U$ and Σ.

For the next construction we note that for a point $b \in Z_i$ or $b \in Z_i \cap Z_j$ and a homotopy class of paths in B from a base point b_0 to b there is a well-defined identification of the fibres of the local system with the fixed vector space V. There is also a limiting mixed Hodge structure $(V, Q, W(\sigma_i), F)$ or $(V, Q, W(\sigma_{ij}), F)$ where F is well defined modulo $\exp \sigma_{i,\mathbb{C}}$ or $\exp \sigma_{ij,\mathbb{C}}$. We denote by $[F]$ the resulting equivalence class. The monodromy group Γ acts in the evident way on the set of pairs

$$(\sigma; [F]) \in \Sigma \times \left(\exp \sigma_{\mathbb{C}} \backslash \check{D}\right). \tag{I.6}$$

Definition I.7: *We denote by* $\overline{\mathcal{P}}^T$ *the quotient by* Γ *acting on* (I.6), *and define* $\Phi_T : \overline{B} \to \overline{\mathcal{P}}^T$ *to be the resulting map.*

The above construction may be localized to U and to the U_Y, U_C, U_i above, giving

$$\overline{\mathcal{P}}_U^T, \overline{\mathcal{P}}_Y^T, \overline{\mathcal{P}}_C^T, \overline{\mathcal{P}}_i^T.$$

[17] Cf. [5] for interesting results and examples concerning fans.

One may build up $\overline{\mathcal{P}}^T$ from these pieces.

Turning to the construction of $\overline{\mathcal{P}}^S$ we recall that the associated graded to an equivalence class of limiting mixed Hodge structures is a direct sum of polarizable Hodge structures. We may construct a period domain D_i from the primitive subspaces of $\mathrm{Gr}_{\bullet}^{W(N_i)}(V)$ and a period mapping

$$\Phi_i : Z_i^* \to \Gamma_i \backslash D_i \tag{I.8$_i$}$$

where $\Gamma_i \subset \mathrm{Aut}(V_i)$ is the monodromy image of $\pi_1(U_i^*)$. The period mapping (I.8)$_i$ depends on a homotopy class of mappings from b_0 to Z_i^*, and two such choices are related by an element in

$$\Gamma_i / Z(N_i) \cap \Gamma_i.$$

For points in $Z_i \cap Z_j$ by a similar construction we first have a limiting mixed Hodge structure corresponding to any N_{ij} in the closure of the interior of the monodromy cone σ_{ij}. Passing to the associated graded we obtain a point in a period domain D_{ij} defined modulo $\Gamma / Z(N_{ij}) \cap \Gamma$.[18] We write this as

$$\Phi_{ij} : Z_i \cap Z_j \to \Gamma_{ij} \backslash D_{ij}. \tag{I.8$_{ij}$}$$

We then observe that

(I.9) The fibres of the mappings (I.8)$_i$ and (I.8)$_{ij}$ are independent of the choice of the F's in their equivalence classes.

Definition I.10: *We denote by* $\overline{\mathcal{P}}^S$ *the set given by the quotient of* $\overline{\mathcal{P}}^T$ *where the fibres of the mappings* (I.8)$_i$ *are contracted to points.*

Summarizing, points of $\overline{\mathcal{P}}^T$ are (equivalence classes of) limiting mixed Hodge structures and $\overline{\mathcal{P}}^S$ is the quotient by passing to the associated graded polarizable Hodge structures. For the definition, one takes into account the action of monodromy.

As was the case above for $\overline{\mathcal{P}}^T$ the construction of $\overline{\mathcal{P}}^S$ may be localized to give $\overline{\mathcal{P}}_i^S$, $\overline{\mathcal{P}}_Y^S$, $\overline{\mathcal{P}}_C^S$ and $\overline{\mathcal{P}}_U^S$.

There are evident set-theoretic maps

$$\begin{cases} \Phi^T : \overline{B} \to \overline{\mathcal{P}}^T, \\ \Phi^S : \overline{B} \to \overline{\mathcal{P}}^S, \end{cases} \tag{I.11}$$

[18] Importantly the polarization depends on the particular N_{ij}.

that we shall refer to as the *natural extensions* of the period mapping $\Phi : B \to \Gamma\backslash D$. There is also the obvious mapping of $\overline{\mathcal{P}}^T$ to $\overline{\mathcal{P}}^S$ and a commutative diagram

$$\begin{tikzcd} & \overline{\mathcal{P}}^T \arrow[dd, "f"] \\ \overline{B} \arrow[ur, "\Phi^T"] \arrow[dr, "\Phi^S"'] & \\ & \overline{\mathcal{P}}^S . \end{tikzcd} \tag{I.12}$$

There are versions of (I.11) and (I.12) for U_i, U_Y, U_C, U.

We shall refer to $\overline{B}$ as the *global case* and to U_i, U_Y, U_C and U as the *local cases*.[19]

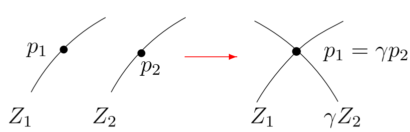
The issue of analytic and algebraic structures on $\overline{\mathcal{P}}^T$ and $\overline{\mathcal{P}}^S$ will be discussed below. We will show that under suitable assumptions, including a local Torelli one and one involving the descent of the augmented Hodge line bundle, $\overline{\mathcal{P}}^S$ is a projective algebraic variety. It seems to not yet be known if this is also the case for $\overline{\mathcal{P}}^T$ (cf. Definition II.6). What is known is that $\overline{\mathcal{P}}^T$ is an algebraic space.[20]

Finally we will comment on a particular type of discrete fibre of

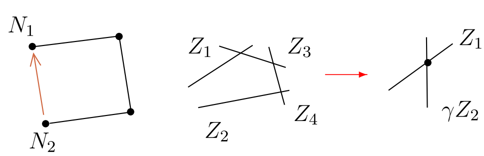
$$\Phi^T : \overline{B} \to \overline{\mathcal{P}}^T . \tag{I.13}$$

These are those that may be termed *accidental intersections*; they result from the global action of monodromy.

The first is depicted by



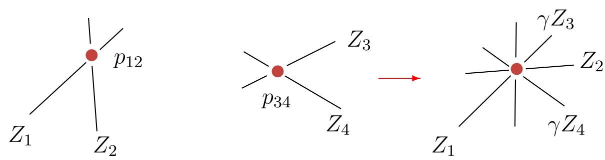
If Z is assumed to be connected, in $\Sigma \subset \mathrm{End}(V)$ we would have a picture like



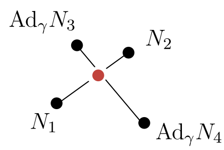
[19] A better term would be the "local around the constituents of Z" cases.

[20] This is Theorem 1.3 in arXiv:2010.06720v3; the result will not be used in this paper.

Another possibility is



The picture in Σ is



This type of accidental intersection is what prevents Σ from being a fan. If there are finitely many such, then we can blow up p_{12} and p_{34} to eliminate them. However there are higher dimensional examples where this may not be the case (cf. [5]).

II. Statement of results

Using the notations and assumptions from the introduction, in the $\dim B = 2$ case we set

$$\begin{cases} S_{ij} = Z_i \cdot Z_j \\ S = \text{matrix with entries } S_{ij}. \end{cases}$$

THEOREM II.1: *In the global case with the assumptions that the period mapping Φ is non-degenerate and that $\Phi^S(Z) =$ point,*[21] *the intersection matrix S is negative definite (written $S < 0$).*

Following arguments given in [12], a sketch of the proof of Theorem II.1 is

- the Chern form ω of the augmented Hodge bundle $L \to \overline{B}$ is positive on a Zariski open set in B;[22]

[21] This means that along each Z_i^* the Hodge structures given by associated graded to the limiting mixed Hodge structures are locally constant.

[22] More precisely, on $\overline{B}$ it is given by an integrable $(1,1)$ form where the associated current is closed and represents $c_1(L)$ in $H^2(\overline{B}, \mathbb{C})$. Moreover, as a current $\omega \geqq 0$ and $\omega > 0$ on a Zariski open set in $\overline{B}$. The form $\omega \wedge \omega$ is defined, is integrable on $\overline{B}$ and $\int_{\overline{B}} \omega \wedge \omega = c_1(L)^2[\overline{B}] > 0$. These properties and the ones in the second and third bullets are applications of [4] and are proved in [10].

- although it is singular, the restriction of ω to Z_i may be defined and there it represents $c_1(L\big|_{Z_i})$;
- $\Phi^S(Z_i)$ = point implies that $\omega\big|_{Z_i} = 0$, and then the result follows from the Hodge index theorem (this is where the Kähler assumption is used).

We feel that the $S < 0$ result should be a local at infinity Hodge-theoretic one. To formulate this we assume given only $(U, Z; \Phi)$ where

$$\Phi : U^* \to \Gamma_U \backslash D$$

and the assumptions in the introduction and Section I are satisfied with U^* replacing B. This is the local around Z case and for it we have the

Conjecture II.2: *$S < 0$ is the local around Z case.*

We shall show that in several cases this conjecture is a consequence of the basic formula in Theorem V.8 below.

As an application of Theorem II.1 the divisor Z may be contracted to a normal singular point p on a complex analytic surface. Set theoretically this surface is $\overline{\mathcal{P}}^S$. Below we will show that under the local Torelli assumption (II.6) and descent assumption (III.10) there is a natural Hodge theoretically defined projective algebraic structure on $\overline{\mathcal{P}}^S$.

There are three possibilities for the canonically extended variations of Hodge structure along a Z_i. We shall use the notation $\Phi_k(p)$ for the map along Z to the extension data up to level k in the limiting mixed Hodge structure $\Phi^T(p)$. Thus $\Phi_0 = \Phi^S$.

If the restriction $\Phi_0\big|_{Z_i}$ is non-constant, i.e., the associated graded to the limiting mixed Hodge structure is varying along Z_i, then $\Phi^S(Z_i)$ is a curve in $\overline{\mathcal{P}}^S$. Although this case is interesting, the present paper is primarily about the situation where Φ^S has positive dimensional fibres. Part of the motivation is that, in contrast to $\overline{\mathcal{M}}_g$, moduli spaces for higher dimensional varieties are frequently singular along boundaries which may have high codimension.

(II.3) *Thus we shall assume that $\Phi_0(Z)$ is 0-dimensional.*

Then the possibilities are

$$\text{(II.4)} \qquad \begin{cases} \text{(i)} & \Phi_1\big|_{Z_i} \neq \text{ constant} \\ \text{(ii)} & \Phi_1\big|_{Z_i} = \text{ constant but } \Phi_2\big|_{Z_i} \neq \text{ constant} \\ \text{(iii)} & \Phi_1\big|_{Z_i} \text{ and } \Phi_2\big|_{Z_i} \text{ are both constant.} \end{cases}$$

In case (iii) it is proved in [13] (cf. (c) in Theorem 6.1 there) that if the extension data up to level 2 of the limiting mixed Hodge structure along Z_i is locally constant, then the entire limiting mixed Hodge structure is constant on a finite covering of Z_i.[23] Thus Z_i is a positive dimensional fibre of Φ^T.

Although we shall not do so here, in this case it may be shown that

(II.5) *Z_i is a fibre of $\overline{B} \xrightarrow{\Phi^T} \overline{\mathcal{P}}^T$ if, and only if, Z_i is not isolated and for any Z_j with $Z_j \cap Z_i \neq \emptyset$ we have $N_j = \lambda_j N_i$ for some λ_j.*

Thus the non-discrete fibres Φ^T are identified by monodromy conditions.

Amplifying (II.4) the period domain for mixed Hodge structures is a complex manifold that is an iterated analytic fibration corresponding to the filtration of a mixed Hodge structure by the level of extension data. Using the family of limiting mixed Hodge structures along Z_0^*, we have

- $\Phi_0 : Z_0^* \to \{\text{associated graded Hodge structures}\}$.

We are assuming that this map is locally constant.

- $\Phi_1 : \{\text{fibres of } \Phi_0\} \to \text{abelian variety } J$;
- $\Phi_2 : \{\text{fibres of } \Phi_1\} \to \Big\{\text{bundle whose fibres are a product of } \mathbb{C}^*\text{'s}\Big\}$.

The Φ_k for $k \geqq 3$ are determined up to "integration constants" by Φ_0, Φ_1, Φ_2 (cf. Sections 2.1, 5 and 6 in [13] for a proof).

For each step in the iterated fibration there is a corresponding exact sequence of fundamental groups; the successive maps in this are the ones in the statement of Theorem II.10. The initial map (i) is the one induced by the S^1-fibration

$$U_0^* \to Z_0^*.$$

The next map $\overline{\rho}$ is the one induced by the map Φ_0 to the associated graded polarized Hodge structures $\overset{k}{\oplus} H^k$. Under the assumption that Z is a Φ_0-fibre the next map Φ_1 maps to a translate of an abelian subvariety J of a direct sum of complex tori

$$\operatorname{Ext}^1_{\mathrm{MHS}}(H^{k+1}, H^k) = \frac{\operatorname{Hom}(H^{k+1}, H^k)}{F^0 \operatorname{Hom}(H^{k+1}, H^k) + \operatorname{Hom}(H^{k+1}_{\mathbb{Z}}, H^k_{Z})}.$$

Up to a finite group monodromy acts by a subgroup of the translations in J.

---

[23] In the classical case this is automatic; in the non-classical case it is shown in [13] that the extension data of levels $\geqq 3$ are determined up to constants of integration by the extension data of levels 1,2.

On a fibre of Φ_1, Φ_2 maps to a sub-quotient group of

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{MHS}}(H^{m+2}, H^m) &\cong \mathbb{C}^k \oplus \mathbb{C}^\ell/\mathbb{Z}^k \oplus (0)\\ &\cong (\mathbb{C}^*)^k \oplus \mathbb{C}^\ell\end{aligned}$$

The sub-quotient arises from a quotient of the first factor $\mathbb{C}^k/\mathbb{Z}^k$ where

$$\mathbb{Z}^k \cong \overset{k}{\underset{i=1}{\oplus}} \mathbb{Z} \cdot N_i \subset \mathfrak{g}_{\mathbb{C}}.$$

The details of this are given in loc. cit.[24]

We will give the following definition in general but will only use it when the condition (II.3) is satisfied.

Definition II.6: The variation of Hodge structure $(\overline{B}, Z; \Phi)$ satisfies the *local Torelli* (LT) *condition* if $Z = Z' \cup Z''$ is a disjoint union where on B the differential is injective outside a finite set of points in B, and

- $Z' = \cup Z'_\alpha$ where $\Phi_0\big|_{Z'_\alpha}$ is generically locally 1-1;
- $Z'' = \cup Z''_i$ where $\Phi_0\big|_{Z''_i} = \text{constant}$.

In particular $\omega > 0$ on Zariski open sets in B and Z'.

The obvious line bundles on $\overline{B}$ are $L = \overset{p}{\otimes} \det F^p, K_{\overline{B}}$, and the $[Z_i]$. It is of interest to know which combinations of these are ample or semi-ample.

Theorem II.7: *Assuming the LT condition* (II.6)*, there exist positive integers m_0 and a_i such that the line bundle*

$$L_{m,a} := mL - \Sigma_i a_i Z_i$$

on $\overline{B}$ is ample for $m \geqq m_0$.

The a_i may be chosen to be components of an eigenvector with maximal eigenvalue of the negative of the intersection matrix S. Only in special cases can we take the $a_i = 1$.

A refined version of this theorem is obtained by choosing any a such that $Z_i \cdot (\Sigma_j a_j Z_j) < 0$ for all i. Then we have

Theorem II.7 (bis): *For each such a there is an $m_0(a)$ such that $L_{m,a}$ is ample for $m \geqq m_0(a)$.*

Theorem II.8: *There exists m_1 such that the line bundle $K_m := mL + K_{\overline{B}}$ on $\overline{B}$ is ample modulo -2 curves for $m \geqq m_1$.*

[24] See also the discussion following (V.3) below.

Theorem II.9: *Assuming the LT condition* (II.6) $K_{\overline{B}} + Z$ *is ample modulo* -2 *curves in* Z.[25]

For the next result we use the notation $Z_0; Z_1, \dots, Z_k$ for the configuration in Figure 1 above. We choose a base point in U_0^* and set

- $\rho : \pi_1(U_0^*) \to \mathrm{Aut}(V)$ with image Γ_{U_0} the monodromy representation.

Since Γ_0 preserves the weight filtration we may define

- $\overline{\rho} : \pi_1(U_0^*) \to \bigoplus_m \mathrm{Aut}\left(\mathrm{Gr}_m^{W(N_0)}(V)\right)$ is the induced monodromy representation.

Theorem II.10:

(i) *a loop around* Z_0 *maps to* $Z(\Gamma_{U_0^*})$*;*
(ii) $\mathrm{Im}(\overline{\rho})$ *is finite;*
(iii) $\rho(\ker \overline{\rho}) \subset I + W_{-1}(N_0)\,\mathrm{End}(V)$*;*
(iv) *under our assumption that* Z_0 *is a* Φ_0*-fibre, the image of* $\rho(\ker \overline{\rho})$ *in* $\mathrm{Gr}_{-1}^{W(N_0)}\,\mathrm{End}(V)$ *is a free abelian group;*
(v) *if* Z_0 *is also a* Φ_1*-fibre, then*

$$\rho : \ker(\overline{\rho}) \to I + W_{-2}(N_0)\,\mathrm{End}(V)$$

and the map to $\mathrm{Gr}_{-2}^{W(N_0)}\,\mathrm{End}(V)$ *takes a loop around* Z_0 *to* N_0*.*

Summary: If $\Phi_0(Z_0)$ is a point, then

- the action of monodromy on $\mathrm{Gr}^{W(N_0)}(V)$ is finite;
- the monodromy action of $\ker(\overline{\rho})$ on level one extension data is explained by Φ_1;
- if $\Phi_1(Z_0)$ is a point, the monodromy action of $\ker(\overline{\rho})$ is explained by $N_1, \dots, N_k$;
- only a finite group of monodromy fixes the extension data at all levels.

Below we shall make precise the meaning of "explained."

When we come to trees or cycles the monodromy structure may be more complicated. To illustrate if we have a picture like Figure 2 above, then although there is a well-defined identification of $\mathrm{Gr}_\bullet^{W(N_1)}(V)$ with $\mathrm{Gr}_\bullet^{W(N_2)}(V)$ on each of which $\pi_1(U_1^*)$, $\pi_1(U_2^*)$ acts as a finite group, it does not follow that when we identify $\mathrm{Gr}_\bullet^{W(N_1)}(V)$ with $\mathrm{Gr}_\bullet^{W(N_2)}(V)$ the group they generate is finite.

[25] This is a special casee of a standard result; cf. [6] and the references cited there.

We are assuming that for any tree Y the group $\pi_1(U_Y^*)$ acts as a finite group on $\mathrm{Gr}_\bullet^W(V)$. This assumption is satisfied for a single Z_i or if each Z_i^* is isomorphic to $\mathbb{C}^*$.[26]

THEOREM II.11: *The case of a tree where the Z_i^* are $\mathbb{P}^1, \mathbb{C}$, or $\mathbb{C}^*$ and the local Torelli assumption II.6 holds cannot occur.*

THEOREM II.12: *Each of the fan-like objects $\Sigma_i, \Sigma_Y, \Sigma_C$ lies in a 2-dimensional subspace of* $\mathrm{End}(V)$ *and, after possibly finite subdivision, is a fan.*

For Σ_C what will be proved is that it has a picture of an infinite graph in an $\mathbb{R}^2$ that lies in an open radial sector and where the action of the monodromy of the circuit is translation by k steps down and to the right in the graph (see Figure 6). The graph is a straight line at the Z_i where $Z_i^2 = -2$ and is convex at the Z_i where $Z_i^2 \leqq -3$. This is a picture that is familiar from the Hilbert modular surface; the point here is that it holds in general for a cycle in the $\dim B = 2$ case.

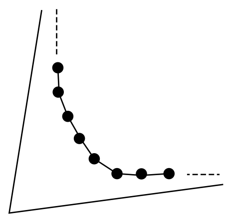


Figure 6

THEOREM II.13: *If the graph $G(Z)$ contains m cycles, then Σ lies in a vector space of dimension at most $m+1$.*

An interesting question is: Since $\overline{\mathcal{P}}^S$ is a normal surface, what is the type of its singular points? We shall give one answer to this question in the geometric case. The point is to relate the singularity type of a singular surface in a family to the singularity type of its image under the extended period mapping of the family of surfaces.

We will work in a neighborhood U of Z in $\overline{B}$. The results for KSBA degenerations $\mathcal{X} \xrightarrow{\pi} U$, as defined in [15], may be described as follows:

- The family $\mathcal{X}^* \xrightarrow{f} U^*$ is a smooth family of general type surfaces $X_u = f^{-1}(u)$.
- For $u \in Z$ the surface X_u is normal with a log canonical singular point x_u.

[26] Due to our relative minimality assumption we cannot have $Z_i^* = \mathbb{P}^1$ or $\mathbb{C}$.

- Either (a) x_u is an elliptic singularity, $(\widetilde{X}_u, C) \to (X_u, x_u)$ is the desingularization where Z is irreducible and the modulus of the elliptic curve C is constant,[27] or (b) x_u is a cusp with resolution $(\widetilde{X}_u, D_u) \to (X_u, x_u)$ where $D_u = \cup D_{i,u}$, $D_{i,u} \cong \mathbb{P}^1$ and $D^*_{iu} \cong \mathbb{P}^1 \backslash \{0, \infty\}$.
- We assume local Torelli in the form: for case (a) Φ_1 is generically locally 1-1; and in (b) the limiting mixed Hodge structure is $\{H^0, H^0(-1) \oplus H^2, H^0(-2)\}$ where $\dim H^0 = 1$, $\dim \mathrm{Hg}^1 = 1$ where Hg^1 is the Hodge part of H^2, and Φ_2 is 1-1 on each $D^*_{i,u}$.

Theorem II.14: (a) *In case* (a) *the mapping* Φ_1 *is given by a branched covering* $\Phi_1 : Z \to C$ *and*

$$Z^2 = C^2 = \text{degree of the elliptic singularity.}$$

(b) *In case* (b) Z *is a cusp and*

$$Z_i^2 = D_{i,u}^2.$$

III. Descending the Hodge line bundle

We have defined $\overline{\mathcal{P}}^S$ as a set; as a consequence of Theorem II.1 it has the structure of a normal complex surface with an isolated singular point p obtained by analytically contracting Z to a point. In the proof of Grauert's theorem the local ring $\mathcal{O}_{\overline{\mathcal{P}}^S, p}$ is defined by

$$\mathcal{O}_{\overline{\mathcal{P}}^S, p} = \lim_{U \supset Z} H^0(U, \mathcal{O}_U). \tag{III.1}$$

Here U runs through neighborhoods of Z. This result basically says that the image of $f : U \to U_0 \subset \overline{\mathcal{P}}^S$ is $\mathrm{Spec}(H^0(\mathcal{O}_U))$. A question is

> *Does the (augmented) Hodge line bundle $L \to \overline{B}$ descend to $\overline{\mathcal{P}}^S$ with the structure given by replacing $\mathcal{O}_U$ with L in* (III.1)*?*

Equivalently, for the contraction mapping $\Phi^S : U \to U_0 \subset \overline{\mathcal{P}}^S$ is there a locally free rank-1 sheaf of $\mathcal{O}_{\overline{\mathcal{P}}^S}$-modules L_0 such that $\Phi^{S*} L_0 \cong L$? A related question is

> *If the augmented Hodge line bundle descends to $L \to \overline{\mathcal{P}}^S$, is it ample?*

With the notations to explained below, we shall prove

[27] This is equivalent to Φ_0 being locally constant on Z; i.e., the associated graded to the limiting mixed Hodge structures are locally constant.

THEOREM III.2:

(i) *$L \to U$ descends to U_0 if, and only if, for some $m \geq 0$ the section $(1_Z)^m$ lifts to $H^0(U, L^m)$;*
(ii) *if $L \to \overline{B}$ descends to $L_0 \to \overline{\mathcal{P}}^B$, then it is ample.*

The proof will give the

COROLLARY III.3: *$L \to \overline{B}$ descends if $H^1(Z, \mathcal{N}^{*k}) = 0$ for $k \geqq 1$.*

For the proofs we begin with

LEMMA III.4: *For $\mathcal{N} \to Z$ the normal bundle defined as the dual of* (III.5) *below, $L \to Z$ has finite order; i.e., for some $m \in \mathbb{Z}^{>0}$*

$$L^m\big|_Z \cong \mathcal{O}_Z.$$[28]

Proof (sketch). The argument is based on two standard facts:

- the isotropy group in $G_\mathbb{Z}$ of a point in a period domain is a finite group, and
- if $\gamma \in G_\mathbb{Z}$ fixes a point F in the period domain, then γ acts with finite order on the fibre of the Hodge line bundle at F.[29] □

Denoting by

$$\mathcal{N}^* = \mathcal{I}_Z/\mathcal{I}_Z^2 \tag{III.5}$$

the co-normal bundle to $Z \subset U$ we shall use the cohomology sequences associated to the standard exact sheaf sequences

$$\begin{array}{c} 0 \longrightarrow \mathcal{I}_Z \otimes L \longrightarrow L \longrightarrow \mathcal{O}_Z \longrightarrow 0 \\ 0 \longrightarrow \mathcal{I}_Z^2 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{N}^* \longrightarrow 0 \\ 0 \longrightarrow \mathcal{I}_Z^3 \longrightarrow \mathcal{I}_Z^2 \longrightarrow \mathcal{N}^{*2} \longrightarrow 0 \\ \vdots \end{array} \tag{III.6}$$

LEMMA III.7: *There exists k_0 such that $H^1(Z, \mathcal{N}^{*k}) = 0$ for $k \geqq k_0$.*

[28]Thus $L \in \mathrm{Pic}^o(Z)$ must be a torsion point.
[29]Here the period domain corresponds to the associated graded of a limiting mixed Hodge structure.

Proof. This is a consequence of

- the cohomology sequences arising from (III.6);
- the exact sequence

$$0 \to \oplus_i N^*_{Z_i/U} \to \mathcal{N}^* \to \bigoplus_{i,j} \mathbb{C}_{Z_i \cap Z_j} \to 0;$$

  and
- $\deg N^*_{Z_i/U} > 0$. □

LEMMA III.8: *There exists a neighborhood U of Z and a k_1 such that*

$$H^1(U, \mathcal{I}^k_Z) = 0, \qquad k \geqq k_1.$$

Proof. Denoting by $(\hat{U}_Z, \hat{\mathcal{O}})$ the formal completion of $\mathcal{O}_U$ along Z,[30] by (III.6) there is a k_2 such that

$$H^1(\hat{U}_Z, \mathcal{I}^k_Z) = 0, \qquad k \geqq k_2. \tag{III.9}$$

From [9] and the fact that $S < 0$ we have

$$\dim H^1(U, \mathcal{I}_Z) < 0.$$

Using (III.9) and the arguments in loc. cit. we may infer the lemma. □

ASSUMPTION III.10: *Denoting by 1_Z a basis for $H^0(\mathcal{O}_Z) \cong \mathbb{C}$, for some $m > 0$ there exists $s \in H^0(U, L^m)$ such that in the notation of III.4*

$$s\big|_Z = (1_Z)^m.$$ [31]

We now replace L by a high power, and for simplicity of notation henceforth shall just use L. We shall also use the following consequence of Corollary III.3:

PROPOSITION III.11: *If $H^1(Z, \mathcal{N}^{*k}) = 0$ for all $k \geqq 1$, then the assumption III.10 is satisfied. This is the case if*

(i) *Z is an elliptic curve;*
(ii) *Z is a cycle of $\mathbb{P}^1$'s.*

---

[30] That means that as a set $\hat{U}_Z = Z$ and the sheaf of rings is $\hat{\mathcal{O}}_Z = \lim_{k\to\infty} \mathcal{O}_U/\mathcal{I}^k_Z$.

[31] This is what is meant by "1_Z lifts to U."

DEFINITION III.12: *With the assumption III.10, for the ring* $R = \overset{m}{\oplus} H^0(\overline{B}, L^m)$, $\overline{\mathcal{P}}^S$ *is the projective scheme* $\operatorname{Proj} R$.

The reason for the assumption III.10 is that without it the sections in R may all vanish on Z. Then $\operatorname{Proj} R$ could be a blow up of the point p in $\overline{\mathcal{P}}^S$ with the scheme structure given by (III.1). By (III.11) the assumption is satisfied for a number of important geometric examples.

With this definition, $\overline{\mathcal{P}}^S$ is a normal projective scheme. The local ring at $f(Z) = p \in \overline{\mathcal{P}}^S$ reflects the type of the singularity.

PROPOSITION III.13: *Under the local Torelli assumption* (II.6) *we have a surjective morphism of projective schemes*

$$\overline{B} \to \overline{\mathcal{P}}^S.$$

Proof. By our Assumption III.10 there is a holomorphic line bundle $L_0 \to \overline{P}$ such that $\Phi_S^* L_0 \cong L$. We want to apply Nakai-Moishezon to this bundle. Denoting by ω the Chern form of $L \to \overline{B}$ then

(i) ω represents $c_1(L) = \Phi^{S*} c_1(L_0)$ in $H^2_{\mathrm{DR}}(\overline{B}, \mathbb{C})$;
(ii) $\omega \wedge \omega$ represents $c_1(L)^2$;
(iii) $\omega > 0$ on $B = \overline{B} \backslash Z$; and
(iv) for any curve $C \subset \overline{B}$ the restriction $\omega\big|_C$ is defined and $c_1(L) \cdot C = \int_C \omega$.

It follows from (i)–(ii) that $c_1(L_0)^2 > 0$. From (iii) we may infer that for any irreducible curve $C_0 \subset \overline{P}_S$ with $C = \pi^{-1}(C_0) \subset \overline{B}$,

$$\deg(L_0\big|_{C_0}) = \deg(L\big|_C) = \int_C \omega > 0. \qquad \square$$

Remark III.14: In general suppose we have

- A holomorphic mapping $f : X \to Y$ where X is a smooth algebraic surface and Y is obtained from X by contracting a normal crossing divisor $Z = \Sigma Z_i$ where the intersection matrix $Z_i \cdot Z_j$ is negative definite;
- $L \to X$ is a line bundle such that $L\big|_Z \cong \mathcal{O}_Z$ and $mL - \sum_i a_i Z_i$ is ample for some $a_i > 0$, $m > 0$.

In general L may not descend to Y. We do not know if our particular Hodge situation will imply that the descent is in fact possible.

IV. Some proofs

Proof of Theorem II.7. Recalling our notation $L_{m,a} = mL - \Sigma a_i Z_i$ and using Nakai-Moishezon we want to show the existence of m_0 and a_i such that for all $C \in \mathrm{Eff}^1(\overline{B})$

$$C \cdot L_{m,a} > 0, \quad m \geqq m_0. \tag{IV.1}$$

We first have the

Lemma IV.2: *There exists $a_i > 0$ such that for all i*

$$Z_i \cdot \left(\sum_j a_j Z_j \right) < 0.$$

Proof. This is a result about real, symmetric negative definite matrices S. Let

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$$

be an eigenvector of S with maximal eigenvalue $\mu < 0$. Then

$${}^t a S a = \mu \|a\|^2.$$

Setting $a_i^{\#} = |a_i|$ since $S_{ij} \geqq 0$ for $i \neq j$

$$\begin{aligned} {}^t a^{\#} S a^{\#} &= \sum_i a_i^2 S_{ii} + \sum_{i \neq j} |a_i|\,|a_j| S_{ij} \\ &\geqq {}^t a S a. \end{aligned}$$

Thus we may assume that all $a_i \geqq 0$. If some $a_i = 0$, then

$$\begin{aligned} \|a + \epsilon e_i\|^2 &= \|a\|^2 + \epsilon^2 \\ {}^t(a + \epsilon e_i) S (a + \epsilon e_i) &= {}^t a S a + 2\epsilon {}^t a S e_i \mod \epsilon^2 \\ &= {}^t a S a + 2\epsilon \sum_{j \neq i} a_j. \end{aligned}$$

The last term is non-negative, and if it is zero, then $a = 0$ which cannot happen because S is non-singular. Thus all $a_j > 0$, and from

$$Sa = \mu a$$

we obtain

$${}^t e_i S a = \mu a_i < 0. \qquad \square$$

Completion of the proof: If C is an irreducible curve such that $C^* := C \cap B$ is an open set in C, then

$$\omega\big|_C \geqq 0 \text{ and } \omega\big|_{C^*} > 0.$$

Thus we have

$$C \cdot L_{m,a} > 0$$

for $m \geqq m_0(C)$. By our finite generation assumption we may choose an m_0 that works for all irreducible C other than the curves Z_i.

Using the lemma, we have

$$Z_i \cdot L_{m,a} = -\sum_j a_j S_{ij} > 0$$

which gives the result. □

The proofs of Theorems II.8 and II.9 are standard arguments using the properties of L, adjunction and the assumption that the pair $(\overline{B}, Z)$ is relatively minimal.

Proof of Theorem II.14. We first treat the simple elliptic case. To begin we recall the standard method of smoothing a surface X having an elliptic singularity $x_0 \in X$ of degree d where $1 \leqq d < 9$. Let $(\widetilde{X}, C) \to (X, x_0)$ be the desingularization where $C^2 = 9 - d$. Realizing $C \subset \mathbb{P}^2$ as a cubic curve one blows up $9 - d$ points $p_i \in C$ to obtain a del Pezzo surface P containing C. Choosing the points so that the Friedman d-stability condition ([8])

$$N_{C/\widetilde{X}} \cong N^*_{C/P}$$

is satisfied the surface

$$\widetilde{X} \cup_C P$$

may to first order be smoothed.

For use below we note (loc. cit.) that associated to a first order smoothing of any normal crossing divisor we may define a limiting mixed Hodge structure that would be the usual one if a full smoothing exists. In the situation at hand the associated graded to the limiting mixed Hodge structure is $H^1, H^2, H^1(-1)$ where $H^1 = H^1(C)$.

We may assume that our KSBA family $\mathcal{X} \to U$ is a smooth fibration over $U^* = U \backslash Z$ and has canonical singularities along the elliptic singular points $x_u \in X_u$ where $u \in Z$. By standard birational geometry we may find a modification

$$\mathcal{X}' \to U$$

of the original family where for $u \in Z$

$$X'_u = \widetilde{X}_u \cup_C P_u,$$

where $(\widetilde{X}_u, C) \to (X_u, x_u)$ is a desingularization and P_u is a del Pezzo surface obtained by blowing up $C^{\#} \subset \mathbb{P}^2$ at the points $p_i(u)$. From the discussion just above and the assumption that Z is a Φ_0 fibre the elliptic curve C does not depend on the $u \in Z$.

The main observation is that the mapping Φ_1 of Z to level 1 extension data is identified with the map

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{Pic}^{9-d}(C) \\ \Cup & & \Cup \\ \{p_i(u)\} & \longrightarrow & \mathrm{AJ}_C\left(\sum_i p_i(u)\right). \end{array}$$

Thus by our Torelli along Z assumption we have that $Z \to C$ is a branched covering.

To determine Z^2 we use the basic formula (V.8) below. This gives

$$-Z^2 = \frac{\deg \Phi_1^*(L_M)}{\langle M, N\rangle}. \tag{IV.3}$$

We may take for M the $N^+ \in \mathrm{Gr}_{+2}^{W(N)} \mathrm{End}(V)$ in an sl_2 triple (cf. [4] and [13, Appendix B]). The line bundle $L_M \to \mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-1), H^1)$ is an ample line bundle of degree 1 over C. The bundle $\Phi_1^* L_M$ is then an ample line whose degree will be

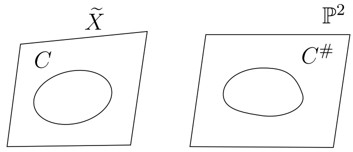
$$\deg \Phi_1^*(L_M) = d.$$

Indeed, the line bundle $N_{C/\widetilde{X}_u}$ will be

$$3\mathcal{O}_C(1) - \sum_{i=1}^{9-d} p_i(u)$$

which has degree $-d$. □

Example:[32] To construct a related example we consider



together with an isomorphism

$$f : C^{\#} \to C$$

of elliptic curves. For this we blow up the $p_i \in C^{\#}$ to obtain the del Pezzo

$$P = \mathrm{Bl}_{p_1,\dots,p_{9-d}}\mathbb{P}^2$$

where $p_1, \dots, p_{9-d} \in C^{\#}$ and

$$p_1 + \cdots + p_{9-d} \in |3\mathcal{O}_C(1) + f^* N_{C/\widetilde{X}}|.$$

Then the surface

$$\widetilde{X} \cup_f P \tag{IV.4}$$

is to first order smoothable. We can adjust f by

$$f \to t_u \cdot f$$

where t_u is translation by $u \in C$. Letting u vary over C, we obtain a non-reduced smooth surface that gives a first order smoothing of the family of surfaces (IV.4) parametrized by $u \in C$. The $p_1, \dots, p_{9-d}$ are arbitrary points of $C^{\#}$. The level 1 extension data in the limiting mixed Hodge structure is given by the extension data in $H^2(\widetilde{X}, C_u)$ where $C_u = (t_u \circ f)(C^{\#})$, which is a copy of C.

Turning to the sketch of the argument for (b) in Theorem II.14, the general issue of understanding the smoothings of a cusp singularity is more involved than the simple elliptic case. It is the subject of classical and current investigation and we refer to the introduction in [7] for an account of some history

[32] This example will be somewhat different from what was discussed above in that $\boldsymbol{\widetilde{X}, C, P}$ and the $\boldsymbol{p_i}$ will be fixed but the embedding $\boldsymbol{C \hookrightarrow P}$ will depend on $\boldsymbol{u \in C}$.

and references, and to the paper itself for recent results. From the structure of a cusp we note that standard semi-stable reduction of a family of smooth surfaces acquiring a cusp would require a normal crossing surface with triple points. So nothing so simple as glueing a del Pezzo surface P along a double curve as in the elliptic case will work. Rather for cusps the role of P is played by a smooth analytic but non-algebraic Inoue surface I. Moreover, it is not the cusp $D \subset I$ but rather the dual cusp D' where $D + D' \in |-K_I|$ that plays a central role.

For our purposes here we do not need to get into this interesting subject. Rather we assume that we have

$$\mathcal{X} \xrightarrow{\pi} U, \quad \pi^{-1}(u) = X_u \tag{IV.5}$$

where $\mathcal{X}^* \to U^*$ is a smooth fibration while the X_u, $u \in Z_i^*$, have a cusp x_u. Over an intersection $Z_i \cap Z_j$ we assume that the local picture is the same as for the family of Hilbert modular surfaces over the toroidal completion of the moduli space realized as the quotient of $\mathcal{H} \times \mathcal{H}$ by the Hilbert modular group (cf. [20] and [21]). The details of this are not necessary here; we only need the existence of a family (IV.5) with the structure just described.

From the structure of the limiting mixed Hodge structure for a family of smooth surfaces acquiring a cusp we see that there is no level 1 extension data. The limiting mixed Hodge structure is

$$\{H^0, H^0(-1) \oplus H^2, H^0(-2)\}.$$

Denoting by Hg^1 the subgroup of Hodge classes in $H^0(-1) \oplus H^2$, the variable part of the level 2 extension data is in a quotient of the group

$$\mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-2), \mathrm{Hg}^1).$$

Our assumptions are

(i) $\dim H^0 = 1$;
(ii) $\dim \mathrm{Hg}^1 = 1$.

Then

$$\mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-2), \mathrm{Hg}^1) \cong (\mathbb{C}^*)^2,$$

and from Section 6 in [13] there is a map

$$\left\{ \begin{matrix} \text{level 2} \\ \text{extension data} \end{matrix} \right\} \hookrightarrow \mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-2), \mathrm{Hg}^1) / \overbrace{(\mathbb{C}N_i / \mathbb{Z}N_i)}$$

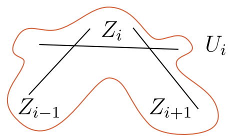
which is a $\mathbb{C}^*$. By our local Torelli assumption the resulting map

$$\Phi_2 : Z_i^* \to \mathbb{C}^* \tag{IV.6}$$

is 1-1, hence is an isomorphism. Thus $Z_i \cong \mathbb{C}^*$ and Z is a cycle a part of which we label as

$$\dots Z_{i-1}, Z_i, Z_{i+1}, \dots$$

with a picture



Since Z is complete, this chain closes up to form a cycle.

It remains to show that

$$Z_i^2 = D_i^2.$$

In fact this is fairly clear from the picture: As we travel around the cusp D in the resolved surface $\widetilde{X}$ the level 2 extension data travels 1-1 around the cusp $Z \subset U$. A formal proof may be given using the calculations as in Sections 2.3 and 2.4 in [7].

Finally we give a more detailed picture of the map (IV.6). Recalling that

$$U_i^* = U_i \backslash \{U_i \cap Z\}$$

there are generators $\gamma_{i-1}, \gamma_i, \gamma_{i+1}$ given by loops around Z_{i-1}, Z_i, Z_{i+1}, and with the generating relation

$$\gamma_{i-1}\gamma_{i+1} = \gamma_i^{d_i} \tag{IV.7}$$

where $Z_i^2 = -d_i$. Referring to Section 6 in [13] for details, the level 2 extension data is a map to

$$\frac{\mathbb{C}N_{i-1} + \mathbb{C}N_{i+1}}{\{\mathbb{Z}N_{i-1} + \mathbb{Z}N_{i+1} + R_i\}}$$

where R_i is the relation

$$N_{i-1} + N_{i+1} = d_i N_i \tag{IV.8}$$

resulting from (IV.7). This relation is a special case of the basic formula (V.8) below.

Summary The geometric point is that as we move along the locus Z where the surfaces degenerate, the extension data (level 1 in the elliptic case, level 2 in the cusp case) in the limiting mixed Hodge structures translates into geometric information normal to Z.

V. Basic formula and applications

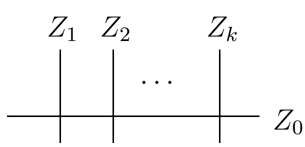
In this section we shall state and give applications of the basic formula (V.8). The classical nilpotent orbit theorem of Schmid [18] approximates a variation of Hodge structure over a punctured disc Δ^* with coordinate t and monodromy logarithm N by a nilpotent orbit

$$\exp(tN) \cdot F, \quad F \in \check{D}. \tag{V.1}$$

A scaling $t \to ct$, $c \neq 0$, replaces F in (V.1) by $\exp(cN) \cdot F$. Thus the nilpotent orbit may be thought of either as a mapping to $\exp(\mathbb{C}N)\backslash\check{D}$ or as intrinsically defined on $T_{\{0\}}\Delta$.

Globally if we have a variation of Hodge structure degenerating along a subvariety Z the basic formula (V.8) below gives a *global* interpretation of the parameter in the normal direction to Z. We will only give this in case the associated graded to the limiting mixed Hodge structures are locally constant along Z. In the general case the mapping $\Phi_1 : Z_0 \to J$ is replaced by a normal function.

Setup: We consider the case $\dim B = 2$ and a subvariety



of Z consisting of a Z_0 and the Z_i that intersect it. Denoting by U_0 a neighborhood of Z_0 in $\overline{B}$ and $U_0^* = U_0 \backslash \{\cup_i Z_i \cap U_0\}$, in U_0^* we have a local system $\mathbb{V}$ with an invariant monodromy weight filtration $W := W(N_0)$. Our Assumption III.10 is

(V.2) *The local system* $\mathrm{Gr}^W_\bullet(V)$ *is a locally constant variation of polarized Hodge structures on which* $\pi_1(U_0^*)$ *acts as a finite group* Γ_0*.*

For the purposes of exposition we shall consider the only case when Γ_0 is trivial. Then we may trivialize $\mathbb{V}$ to be a constant local system based on the vector space V.

We recall that the local system $\mathbb{V}$ is filtered by subsystems consisting of extensions of mixed Hodge structures of levels $\leqq k$. Setting $H^k = \mathrm{Gr}_k^W(V)$ the first level is

$$\bigoplus_k \mathrm{Ext}^1_{\mathrm{MHS}}(H^{k+1}, H^k). \tag{V.3}$$

Explicitly

$$\mathrm{Ext}^1_{\mathrm{MHS}}(H^{k+1}, H^k) \cong \frac{\mathrm{Hom}(H^{k+1}, H^k)}{F^0\,\mathrm{Hom}(H^{k+1}, H^k) + \mathrm{Hom}(H^{k+1}_{\mathbb{Z}}, H^k_{\mathbb{Z}})}. \tag{V.4}$$

As is well known and has been noted above this is a compact, complex torus whose complexified Lie algebra is a direct sum of Hodge structures of weight -1. Within it there is a real compact sub-torus J whose Lie algebra is the maximal subspace of type $(0,-1)\oplus(-1,0)$ that is defined over $\mathbb{Q}$. Taking the level 1 extension data along Z_0^* defines a holomorphic mapping

$$\Phi_1 : Z_0^* \to J_c$$

where $J_c = J + c$ is the translate by some c in the torus (V.4). This map extends to Z_0 and gives a diagram

$$\begin{tikzcd} Z_0 \arrow[r, "\alpha"] \arrow[dr, "\Phi_1"'] & \mathrm{Alb}(Z_0) \arrow[d] \\ & J_c. \end{tikzcd} \tag{V.5}$$

As the c will play no role we shall omit further reference to it.

The $N_0, N_1, \dots, N_m$ lie in $\mathrm{Gr}^W_{-2}\,\mathrm{End}(V)$. For each $M \in \mathrm{Gr}^W_{+2}\,\mathrm{End}(V)$ there is a line bundle $L_M \to J$. To describe its Chern class we identify the complex tangent space to J with $H_1(J,\mathbb{C})$. Using that J is a sub-torus of the complex torus (V.4) there is an inclusion

$$H_1(J,\mathbb{Z}) \hookrightarrow \mathrm{Gr}^W_{-1}\,\mathrm{End}(V_{\mathbb{Z}}). \tag{V.6}$$

A line bundle is defined, up to translation, by its Chern class. For $M \in \mathrm{Gr}^W_{+2}(V_{\mathbb{Z}})$ we define $L_M \to J$ by setting

$$\langle c_1(L_M), \eta_1 \wedge \eta_2 \rangle = \langle M, \eta_1 \wedge \eta_2 \rangle \tag{V.7}$$

where $\eta_1, \eta_2 \in H_1(J,\mathbb{Z})$ and we are using the inclusion (V.6). Then the cohomology class defined by (V.7) is rational of Hodge type (1,1), and it is positive if M is a positive linear combination of $N_0, N_1, \dots, N_m$.

Definition: For $M \in \mathrm{Gr}^W_{+2}(V_{\mathbb{Z}})$ the line bundle $\Phi_1^* L_M \to Z_0$ is defined by pulling back $L_M \to J$ in the diagram (V.5).

THEOREM V.8 (basic formula): *In* $\mathrm{Pic}(Z_0)$ *we have*

$$-\Phi_1^* L_M = \sum_{i=0}^{m} \langle M, N_i \rangle \, [Z_i]\big|_{Z_0}.$$

This formula relates the *normal* bundle of $Z_0 \subset U_0$ to the bundle L_M *along* Z_0. A proof is given in Section 5 in [13]. Here we shall only discuss some

Applications:

(V.9) *Suppose that we are in the local case and $Z_0 = Z$ is irreducible. Then the assumption (III.10) is automatically satisfied and*

$$Z^2 < 0.$$

Proof. From (V.8) we have

$$-\deg \Phi_1^* L_M = \langle M, N \rangle \, Z^2. \tag{V.10}$$

Since $N \neq 0$ we may choose $M \in \mathrm{Gr}^W_{+2} \mathrm{End}(V)$ with $\langle M, N \rangle > 0$.

Case 1: $\Phi_1\big|_Z \neq$ *constant.*
Then (V.8) gives the result.

Case 2: $\Phi_1 =$ constant.
With the local Torelli assumption (II.6) this case cannot occur. The reason is that in general if $\Phi_1\big|_Z =$ constant then $\Phi_2\big|_Z \neq$ constant. Moreover there is a non-zero and non-constant section of $L_M \to Z^*$ having non-trivial zeroes or poles at the $Z_i \cap Z$, which cannot happen if Z is irreducible. The details of this argument are given in the above reference.

COROLLARY V.11: *If Z is irreducible, then there is a non-constant map*

$$\Phi_1 : Z \to J \tag{V.12}$$

to an abelian variety.

Associated to (V.12) is the *Gauss map*

$$\gamma : Z \to \mathbb{P}T_{\{0\}} J. \tag{V.13}$$

If γ is constant, then Z is an elliptic curve and conversely. In this case

Z contracts to an elliptic singular point on $\overline{\mathcal{P}}^S$.

The degree of the elliptic singularity is

$$-Z^2 = \frac{c_1(\Phi_1^* L_M)}{\langle M, N \rangle}. \tag{V.14}$$

Theorem V.15: *Suppose we are in the local case and $Z_0^* = \mathbb{C}$ or $\mathbb{C}^*$. Then*

$$\Phi_1\big|_{Z_0} = \textit{constant},$$

and

$$Z_0^* = \mathbb{C}^*, \quad Z_0^2 \leqq -2.$$

Proof. Since the mapping $\Phi_1 : Z_0^* \to J$ extends to $Z_0 = \mathbb{P}^1$, we must have $\Phi_1\big|_{Z_0} =$ constant. By the local Torelli assumption

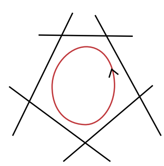
$$\Phi_2\big|_{Z_0^*} \neq \text{ constant.}$$

The non-zero sections of the $L_M \to Z_0^*$ will have zeros or poles at $0, \infty$, and by (V.8)

$$Z_0^2 = \#\{i : Z_i \cap Z_0\} \neq 0.$$

Since we are assuming that Z is minimal we must have $Z_0^2 \leqq -2$. $\square$

We now consider a cycle $C = Z_1 \cup \cdots \cup Z_m$ of $\mathbb{P}^1$'s



If C is part of the divisor at infinity of a global variation of Hodge structure, then $Z_i^2 < 0$. By the assumption of minimality, $Z_i^2 \neq -1$. Thus

$$d_i := -Z_i^2 \geqq 2. \tag{V.16}$$

By the arguments to be given below we will see that, as a consequence of the basic formula (V.8), (V.16) also holds in the local case, and moreover some

$$d_i \geqq 3.$$

Among other things this will then imply that the local intersection matrix $S_C < 0$.

We shall denote by $\gamma \in \pi_1(U_C^*)$ an element that maps to a non-trivial generator of $\pi_1(C)$. Then γ is defined up to the images of the $\pi_1(Z_i^*) \to \pi_1(U_C^*)$ given by loops around the Z_i^*. The subgroup of $\pi_1(U_C^*)$ generated by these images is a finite group[33] so that the possibly infinite part of Γ_C is generated by the circuit $\rho(\gamma)$.

In order to isolate the essential aspect of what follows we temporarily make the

Assumption V.17: *Over U_C^* the image of the $\pi_1(Z_i^*) \to \pi_1(U_C^*)$ acts trivially on the local system $\mathrm{Gr}^W_\bullet(\mathbb{V})$.*

Thus $\mathrm{Gr}^W_\bullet(V)$ and $\mathrm{Gr}^W_\bullet \mathrm{End}(V)$ are fixed vector spaces.

Case 1: *The circuit $\rho(\gamma)$ acts with finite order on $\mathrm{Gr}^W_\bullet(V)$.*

Proposition V.18: *In case 1 all $d_i = 2$ and all the N_i are equal.*

We shall give the argument in several steps, some of which will be used later. Since $\Phi_1\big|_{Z_i}$ = constant, the basic formula (V.8) is

$$\langle M, N_i\rangle [Z_{i-1}] + \langle M, N_i\rangle [Z_i] + \langle M, N_{i+1}\rangle [Z_{i+1}] = 0. \tag{V.19}$$

Taking degrees of the line bundles this gives

$$\langle M, N_i\rangle d_i = \langle M, N_{i-1}\rangle + \langle M, N_{i+1}\rangle . \tag{V.20}$$

We will see that this implies (V.16), which then shows that the results about a cycle are local ones; i.e., we need only have $(U_C, C; \Phi)$.

Set

$$R = \mathrm{span}_{\mathbb{Q}}\{N_1, \dots, N_m\}.$$

By (V.20) any three consecutive N_i are linearly dependent; thus

$$\dim R \leqq 2.$$

If $\dim R = 1$, then the N_i lie on a line through the origin in $\mathrm{Gr}^W_{-2}\mathrm{End}(V)$. Since $d_i \geqq 2$, we have on that line that

$$N_2 \geqq \frac{N_1 + N_3}{2}, N_3 \geqq \frac{N_2 + N_4}{2}, \dots, N_1 \geqq \frac{N_m + N_2}{2} \geqq \cdots .$$

This case only holds if all $d_i = 2$ and all N_i are the same. □

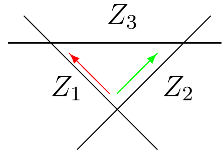
[33]This holds without making the assumption (III.10).

Case 2: $\rho(\gamma)$ *has infinite order acting on* $\mathrm{Gr}^W_{-1}|End(V)$.

This is the interesting case. We cannot assume that all the N_i are in a fixed vector space $\mathrm{Gr}^W_{-2}\,\mathrm{End}(V)$ since going around the circuit conjugates the N_i. We use the notations

$$\rho(\gamma)N_i = N_i',\ \rho(\gamma)N_i' = N_i'', \dots .$$

Taking the case $m = 3$ for illustration



the red and green arrows give different conjugates of N_3. Using the above notation the basic formula (V.8) now gives

$$\begin{aligned} N_3 - d_1N_1 + N_2 &= 0, \\ N_1 - d_2N_2 + N_3' &= 0, \\ N_2 - d_3N_3 + N_1' &= 0, \\ &\vdots \end{aligned}$$

From this we obtain

$$\begin{aligned} 1 \text{ relation on } &N_1, N_2, N_3, \\ 2 \text{ relations on } &N_1, N_2, N_3, N_3', \\ 3 \text{ relations on } &N_1, N_2, N_3, N_3', N_1', \\ &\vdots \end{aligned}$$

leading to the conclusion for

(V.21) $$R := \mathrm{span}\{N_1, \dots, N_m, N_2', \dots, N_m', N_1'', \dots\}, \quad \dim R = 2.$$

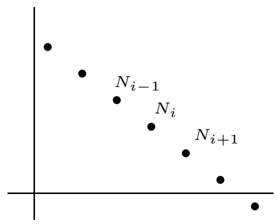
We will see now that R is not a line.

Claim: *Some* $d_i \geqq 3$*, so that the cycle is a cusp.*

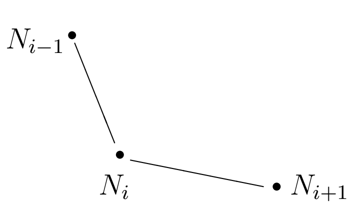
Proof. The $N_i, N_i', N_i'', \dots$ satisfy Hodge-Riemann bilinear relations of the form

(V.22) $$Q(N_i^k u, u) > 0.$$

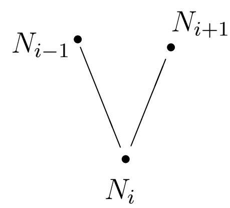
This uses that $\rho(\gamma)$ preserves the form induced by Q on $\mathrm{Gr}^W_{-2}(U)$. In $R \subset \mathrm{Gr}^W_{-2}\,\mathrm{End}(V)$ the region defined by (V.22) is a union of open sets that are radial in the sense that they contain the half line joining any point to the origin. If all $d_i = 2$, we have a picture



which cannot lie in a region defined by (V.22). Thus some $d_i \geqq 3$ and we have convexity



or even



and where the segments are of unequal length. This gives that $\dim R = 2$ and completes the proof of Theorem II.12. $\square$

Remark V.23: If we denote by S_C the part of the intersection matrix coming from the Z_i in the cycle, then

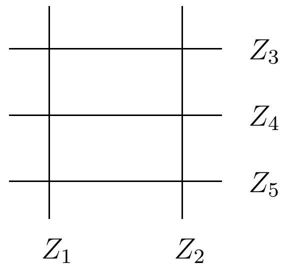
$$S_C = \begin{pmatrix} -d_1 & 1 & & \\ 1 & -d_2 & & \bigcirc \\ & & \ddots & 1 \\ \bigcirc & & 1 & -d_m \end{pmatrix}$$

where $d_i \geqq 2$ and some $d_i \geqq 3$. Any such matrix is negative definite.[34] This gives a local, Hodge-theoretic argument that the part of the intersection matrix coming from a cycle is negative definite.

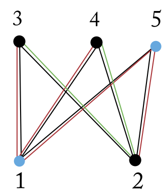
[34] In fact S_C is negative definite *only* if some $d_i \geqq 3$. In the global case when we know that $S_C < 0$ this gives an alternative proof that some $d_I \geqq 3$. It does not give that $G(C)$ is a convex curve lying in a radial sector.

Proof of Theorem II.13. We will illustrate the argument in a special case from which the general pattern will hopefully be clear. If

$Z =$



have two cycles with dual graph the black part in



The way to unambiguously assign $N_1, \ldots, N_5 \in \mathrm{Gr}^W_{-2}\, End(V)$ is to pick a spanning tree; e.g., the red part in the above. Using the basic formula (V.8) for the spanning tree we obtain a linear relation on $N_1, \ldots, N_5$; i.e.,

$N_1, \ldots, N_5$ and their conjugates span a 3-dimensional space.

Put another way we must leave out $2 =$ (# of cycles) edges (marked in green). Thus in general

$N_1, \ldots, N_m$ and their conjugates span a space of dimension $\leqq$ (# cycles) + 1. □

It remains to discuss the situation where the image of the map

$$\rho_0 : \left\{ \prod_{i=1}^{m} \pi_1(Z_i^*) \right\} \to \pi_1(U_C^*) \to \mathrm{Gr}^W_{-2}\, \mathrm{End}(V) \tag{V.24}$$

may not be trivial. In this case we may sum the basic formula (V.8) over the finitely many conjugates in the image of the map ρ_0 in (V.24) to obtain the result. Thus (V.19) becomes

$$\sum_g \langle M, \rho_0(g) N_{i-1} \rangle + \langle M, \rho_0(g) N_i \rangle + \langle M, \rho_0(g) N_{i+1} \rangle = 0.$$

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