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Pure braid group actions on category $\mathcal{O}$ modules

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To Corrado De Concini

Abstract: Let $\mathfrak{g}$ be a symmetrisable Kac–Moody algebra and $U_\hbar\mathfrak{g}$ its quantised enveloping algebra. Answering a question of P. Etingof, we prove that the quantum Weyl group operators of $U_\hbar\mathfrak{g}$ give rise to a canonical action of the pure braid group of $\mathfrak{g}$ on any category $\mathcal{O}$ (not necessarily integrable) $U_\hbar\mathfrak{g}$-module $\mathcal{V}$. By relying on our recent results [ATL15], we show that this action describes the monodromy of the rational Casimir connection on the $\mathfrak{g}$-module V corresponding to $\mathcal{V}$. We also extend these results to yield equivalent representations of parabolic pure braid groups on parabolic category $\mathcal{O}$ for $U_\hbar\mathfrak{g}$ and $\mathfrak{g}$.

Keywords: Quantum groups, braid groups, Casimir connection, Coxeter category, Etingof–Kazhdan quantization.

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1. Introduction

1.1.

Let $\mathfrak{g}$ be a symmetrisable Kac–Moody algebra, $U_\hbar\mathfrak{g}$ its quantized enveloping algebra and W their Weyl group. We denote by $\mathcal{O}$ the category of deformation highest weight modules of $\mathfrak{g}$, by $\mathcal{O}^{\text{int}} \subset \mathcal{O}$ the full subcategory of integrable ones, and by $\mathcal{O}_\hbar^{\text{int}} \subset \mathcal{O}_\hbar$ the corresponding categories for $U_\hbar\mathfrak{g}$. In [ATL15], we constructed an equivalence $\mathcal{O}^{\text{int}} \to \mathcal{O}_\hbar^{\text{int}}$ which intertwines the monodromy of the rational Casimir connection of $\mathfrak{g}$ and the quantum Weyl group action of the braid group $\mathcal{B}_W$ of $\mathfrak{g}$, respectively, thus extending the equivalence obtained in [TL02, TL08, TL16] when $\mathfrak{g}$ is finite-dimensional. P. Etingof asked whether this equivalence extends to suitable categories of modules which are not necessarily integrable, while remaining equivariant under the pure braid group $\mathcal{P}_W$ of $\mathfrak{g}$.

The goal of the present paper is to answer this question in the affirmative. Specifically, we prove that the quantum Weyl group action of $\mathcal{P}_W$ on category $\mathcal{O}_\hbar^{\text{int}}$ modules can be extended to all category $\mathcal{O}_\hbar$ modules. We then show that this action is equivalent to the restriction to $\mathcal{P}_W$ of the equivariant monodromy of the Casimir connection, which is defined on any category $\mathcal{O}$ module. Our results hold more generally for the category $\mathcal{O}_\infty$ of modules which are locally finite under the action of the Borel subalgebra, though for simplicity we restrict to category $\mathcal{O}$ in the Introduction.

1.2.

We turn now to a more detailed description of our results. Endow $\mathcal{O}$ with the associativity and commutativity constraints arising from the KZ equations [Dri90]. In [EK96, EK98, EK08], Etingof–Kazhdan constructed a braided tensor equivalence $\mathsf{F} : \mathcal{O} \to \mathcal{O}_\hbar$ which is Tannakian, that is endowed with a natural isomorphism α fitting in the diagram

$$\begin{tikzcd} \mathcal{O} \arrow[rr, "\mathsf{F}"] \arrow[dr, "\mathsf{f}"'] & \arrow[d, phantom, "\Leftarrow\alpha"] & \mathcal{O}_\hbar \arrow[dl, "\mathsf{f}_\hbar"] \\ & \mathrm{Vect}_\hbar & \end{tikzcd}$$

where $\mathrm{Vect}_\hbar$ is the category of topologically free modules over $\mathbb{C}[\![\hbar]\!]$, $\mathsf{f}, \mathsf{f}_\hbar$ are the forgetful functors, and f is endowed with an appropriate tensor structure. The pair (F, α) gives rise to an isomorphism $\Psi_\alpha : \mathrm{End}(\mathsf{f}_\hbar) \to \mathrm{End}(\mathsf{f})$ via the composition

$$\mathrm{End}(\mathsf{f}_\hbar) \longrightarrow \mathrm{End}(\mathsf{f}_\hbar \circ \mathsf{F}) \to \mathrm{End}(\mathsf{f})$$

where the first isomorphism is induced by F, and the second is given by $\mathrm{Ad}(\alpha)$. Note that α is only unique up to an automorphism γ of f, and that $\Psi_{\gamma\circ\alpha} = \mathrm{Ad}(\gamma) \circ \Psi_\alpha$.

1.3.

Building on our earlier work [ATL18, ATL19a, ATL19b], we constructed in [ATL15] an automorphism $\gamma \in \mathrm{Aut}(\mathsf{f})$ such that $\Psi_{\gamma\circ\alpha}$ is equivariant with respect to the action of the braid group $\mathcal{B}_W$ on integrable category $\mathcal{O}$ modules. Specifically, the Etingof–Kazhdan functor F restricts to an equivalence $\mathcal{O}^{\mathrm{int}} \to \mathcal{O}_\hbar^{\mathrm{int}}$ and therefore leads to an isomorphism $\Psi_{\alpha'}^{\mathrm{int}} : \mathrm{End}(\mathsf{f}_\hbar^{\mathrm{int}}) \to \mathrm{End}(\mathsf{f}^{\mathrm{int}})$ for any $\alpha' : \mathsf{f}_\hbar \circ \mathsf{F} \Rightarrow \mathsf{f}$. Regard the quantum Weyl group action of $\mathcal{B}_W$ on objects in $\mathcal{O}_\hbar^{\mathrm{int}}$ as a morphism $\lambda : \mathcal{B}_W \to \mathrm{End}(\mathsf{f}_\hbar^{\mathrm{int}})$, and the monodromy of the Casimir connection as a morphism $\mu : \mathcal{B}_W \to \mathrm{End}(\mathsf{f}^{\mathrm{int}})$. Then, γ may be chosen so that the following is a commutative triangle [ATL15]

$$\begin{tikzcd} & \mathcal{B}_W \arrow[dl, "\lambda"'] \arrow[dr, "\mu"] & \\ \mathrm{End}(\mathsf{f}_\hbar^{\mathrm{int}}) \arrow[rr, "\Psi_{\gamma\circ\alpha}^{\mathrm{int}}"'] & & \mathrm{End}(\mathsf{f}^{\mathrm{int}}) \end{tikzcd} \tag{1.1}$$

As a consequence, the monodromy of the Casimir connection on a module $V \in \mathcal{O}^{\mathrm{int}}$ is equivalent to the quantum Weyl group action of $\mathcal{B}_W$ on $\mathsf{F}(V)$.

1.4.

P. Etingof asked us whether such an equivalence holds for a larger class of not necessarily integrable modules, provided $\mathcal{B}_W$ is replaced by the pure braid group $\mathcal{P}_W$. The choice of the latter is suggested by the fact that $\mathcal{B}_W$ does not act on all category $\mathcal{O}$ modules for either $\mathfrak{g}$ or $U_\hbar\mathfrak{g}$, while $\mathcal{P}_W$ does on category $\mathcal{O}$ $\mathfrak{g}$-modules via the monodromy of the Casimir connection.

To the best of our knowledge, no action of $\mathcal{P}_W$ on category $\mathcal{O}_\hbar$ modules has been previously constructed. The main result of the present paper is to construct such an action, and then show the commutativity of the resulting diagram

$$\begin{tikzcd} & \mathcal{P}_W \arrow[dl, "\lambda"'] \arrow[dr, "\mu"] & \\ \mathrm{End}(\mathsf{f}_\hbar) \arrow[rr, "\Psi_{\gamma\circ\alpha}"'] & & \mathrm{End}(\mathsf{f}) \end{tikzcd} \tag{1.2}$$

1.5.

To state our results in more detail, recall first that the abelianisation $\mathcal{P}_W^{\mathrm{ab}} = \mathcal{P}_W/[\mathcal{P}_W, \mathcal{P}_W]$ of the pure braid group is isomorphic to the free abelian group with a generator p_α for each positive real root α [Tit66, Dig15]. Set $\iota = \sqrt{-1}$, and define the sign character to be the morphism

$$\varepsilon_\hbar : \mathcal{P}_W^{\mathrm{ab}} \to \mathrm{Aut}(\mathsf{f}_\hbar^{\mathrm{int}}) \qquad p_\alpha \to \exp(\pi\iota h_\alpha) \tag{1.3}$$

where $\exp(\pi\iota h_\alpha)$ acts as multiplication by $\exp(\pi\iota\nu(h_\alpha))$ on the ν-weight space of an integrable category $\mathcal{O}_\hbar$ module. The morphism $\varepsilon_\hbar$ arises as the reduction mod $\hbar$ of the quantum Weyl group action of $\mathcal{P}_W$ on category $\mathcal{O}_\hbar^{\mathrm{int}}$.

As a subgroup of $\mathcal{B}_W$, $\mathcal{P}_W$ is generated by the elements $S_{w,i}^2 = S_w S_i^2 S_w^{-1}$, where S_i is a generator of $\mathcal{B}_W$, $w \in W$ is such that $w\alpha_i$ is a positive root, and $S_w \in \mathcal{B}_W$ is the canonical lift of w [DG01]. Moreover, the quantum Weyl group action of $S_{w,i}$ on a module $\mathcal{V} \in \mathcal{O}_\hbar^{\mathrm{int}}$ is given by

$$\lambda(S_{w,i}^2) = \exp(\pi\iota h_{w\alpha_i}) q^{\mathcal{K}_{w,i}} = \varepsilon_\hbar(S_{w,i}) q^{\mathcal{K}_{w,i}} \tag{1.4}$$

where the second factor is the truncated quantum Casimir operator for the copy of $U_\hbar\mathfrak{sl}_2 \subset U_\hbar\mathfrak{g}$ corresponding to the pair (w,i) [Lus93], and $q = \exp(\hbar/2)$.

1.6.

To extend this action to an arbitrary category $\mathcal{O}_\hbar$ module, we lift the sign character $\varepsilon_\hbar$ to a morphism

$$\mathcal{P}_W^{\mathrm{ab}} \to \mathrm{Aut}(\mathsf{f}_\hbar) \qquad p_\alpha \to \exp(\pi\iota h_\alpha)$$

which we denote by the same symbol. We then prove that the quantum Casimirs $q^{\mathcal{K}_{w,i}} \in U_\hbar\mathfrak{g}$ give rise to a morphism $\mathscr{K} : \mathcal{P}_W \to (U_\hbar\mathfrak{g})^{\mathfrak{h}}$. It follows that

$$\lambda : \mathcal{P}_W \to \mathrm{Aut}(\mathsf{f}_\hbar) \qquad S_{w,i}^2 \to \exp(\pi\iota h_{w\alpha_i}) q^{\mathcal{K}_{w,i}} \tag{1.5}$$

is an extension of the quantum Weyl group action of $\mathcal{P}_W$ to all category $\mathcal{O}_\hbar$ modules.

1.7.

The fact that $\mathscr{K}$ is a morphism would follow at once if $\mathrm{End}(\mathsf{f}_\hbar)$ acted faithfully on $\mathsf{f}_\hbar^{\mathrm{int}}$. This, however, is clearly false: if φ is any function on $\mathfrak{h}^*$ which vanishes on integral weights, then $\varphi \in \mathrm{End}(\mathsf{f}_\hbar)$, but φ maps to zero in $\mathrm{End}(\mathsf{f}_\hbar^{\mathrm{int}})$. To remedy this, we rely on the fact that $U_\hbar\mathfrak{g}$ acts faithfully on $\mathsf{f}_\hbar^{\mathrm{int}}$, whose proof is due to Etingof. This implies that any $\lambda(p) \in \mathrm{End}(\mathsf{f}_\hbar^{\mathrm{int}})$, $p \in \mathcal{P}_W$, arises from the action of a unique element of $U_\hbar\mathfrak{g}$, thereby yielding the required action of $\mathcal{P}_W$ on $\mathrm{End}(\mathsf{f}_\hbar)$.[1]

A similar argument works for the quantum group $U_q\mathfrak{g}$, where q is either an indeterminate, or not a root of unity. In that case, the quantum Casimirs $q^{\mathcal{K}_{w,i}}$ do not lie in $U_q\mathfrak{g}$, but in a variant $\mathcal{D}_q$ of an algebra originally introduced by Drinfeld [Dri92, Sect. 8], which consists of formal, infinite series of the form $\sum c_X X$, where X runs over a weight basis of $U_q\mathfrak{n}^+$ and $c_X \in U_q\mathfrak{b}^-$. Etingof's faithfulness result also applies to $\mathcal{D}_q$, and yields an action of $\mathcal{P}_W$ on any category $\mathcal{O}$ module for $U_q\mathfrak{g}$.

1.8.

Let now Y be the complexification of the Tits cone of $\mathfrak{g}$, $\mathsf{X} \subset \mathsf{Y}$ its set of regular points, and $x_0 \in \mathsf{X}$ a basepoint. By a theorem of van der Lek [vdL83], which generalises Brieskorn's [Bri71], the pure and full braid groups may be realised as

$$\mathcal{P}_W \cong \mathbf{\Pi}_1(\mathsf{X}; x_0) \qquad \text{and} \qquad \mathcal{B}_W \cong \mathbf{\Pi}_1(\mathsf{X}/W; [x_0])$$

The Casimir connection is the $U\mathfrak{g}$-valued formal meromorphic connection on X with logarithmic singularities on the root hyperplanes given by

$$\nabla_{\mathcal{K}} = d - \mathsf{h} \sum_{\alpha \succ 0} \frac{d\alpha}{\alpha} \cdot \mathcal{K}_\alpha^+ \tag{1.6}$$

where $\mathcal{K}_\alpha^+ = \sum_{i=1}^{\mathrm{m}_\alpha} e_{-\alpha}^{(i)} e_\alpha^{(i)}$ is the normally ordered truncated Casimir operator corresponding to the positive root α, and $\mathsf{h} = \hbar/2\pi\iota$ [MTL05, TL02,

[1] Note that this bypasses having to explicitly check that the quantum Casimirs satisfy the relations of the generators $S_{w,i}$ given in [DG01, Cor. 6].

Pro96, FMTV00]. The sum (1.6) over α is locally finite on any (not necessarily integrable) category $\mathcal{O}$ module V, and gives rise to a well-defined flat connection on the holomorphically trivial vector bundle $\mathbb{V}$ on Y with fibre V. Its monodromy therefore gives rise to a morphism

$$\mathscr{P} : \mathbf{\Pi}_1(\mathsf{X}; x_0) \to \mathrm{End}(\mathsf{f}) \tag{1.7}$$

1.9.

The normal ordering in (1.6) breaks the equivariance of $\nabla_{\mathcal{K}}$ with respect to the action of W on X and the subalgebra of $\mathfrak{h}$-invariants $U\mathfrak{g}^{\mathfrak{h}} \subset U\mathfrak{g}$, which contains the Casimirs $\mathcal{K}_\alpha^+$.

Nevertheless, it is possible to modify the monodromy of $\nabla_{\mathcal{K}}$ so that it gives rise to a representation of the braid group $\mathcal{B}_W$ on integrable category $\mathcal{O}$ modules [ATL15, Sect. 4] (see also Section 5). This relies on the equivalence of groupoids

$$\mathcal{E}_{x_0} : \mathbf{\Pi}_1(\mathsf{X}/W; [x_0]) \to W \ltimes \mathbf{\Pi}_1(\mathsf{X}; Wx_0) \tag{1.8}$$

where the right-hand side is the semi-direct product of W with the fundamental groupoid of X based at the orbit Wx_0, and $\mathcal{E}_{x_0}$ is given by the unique lifting of loops through x_0, and proceeds as follows.

- Extend the monodromy of $\nabla_{\mathcal{K}}$ to a morphism
$$\mathscr{P} : \mathbf{\Pi}_1(\mathsf{X}; Wx_0) \to \mathrm{End}(\mathsf{f}) \tag{1.9}$$
- Replace the target of $\mathscr{P}$ by a subalgebra $\mathcal{T}_{\mathfrak{g}} \subset \mathrm{End}(\mathsf{f})$ which, unlike $\mathrm{End}(\mathsf{f})$, is acted upon by W. $\mathcal{T}_{\mathfrak{g}}$ is the image of the holonomy algebra of the root arrangement of $\mathfrak{g}$, and is a completion of the subalgebra of $U\mathfrak{g}^{\mathfrak{h}}[\![\hbar]\!]$ generated by the Casimirs $\hbar\mathcal{K}_\alpha^+$ and the Cartan subalgebra $\hbar\mathfrak{h}$.
- The lack of equivariance of $\nabla_{\mathcal{K}}$ can then be measured by a 1-cocycle
$$\mathscr{A} : W \to \mathrm{Hom}(\mathbf{\Pi}_1(\mathsf{X}; Wx_0), \mathcal{T}_{\mathfrak{g}})$$
defined by $\mathscr{A}_w(\gamma) = \mathscr{P}(\gamma)^{-1} \cdot w^{-1}\mathscr{P}(w\gamma)$.
- We prove that $\mathscr{A}$ is abelian *i.e.*, takes values in
$$\mathsf{M} = \mathrm{Hom}(\mathbf{\Pi}_1(\mathsf{X}; Wx_0), \exp(\hbar\mathfrak{h}))$$
and that it is the coboundary of an essentially unique cochain $\mathscr{B} \in \mathsf{M}$ *i.e.*, that $\mathscr{A}_w = \mathscr{B} \cdot (w^{-1}\mathscr{B})^{-1}$ for any $w \in W$.

- As a consequence, $\mathscr{P}$ can be modified to a W-equivariant morphism

$$\mathscr{P}_{\mathscr{B}} : \mathbf{\Pi}_1(\mathsf{X}; Wx_0) \to \mathcal{T}_{\mathfrak{g}} \qquad\qquad \mathscr{P}_{\mathscr{B}}(\gamma) = \mathscr{P}(\gamma) \cdot \mathscr{B}(\gamma)$$

- Composing $\mathscr{P}_{\mathscr{B}}$ with the morphism $\mathcal{E}_{x_0}$ (1.8) then yields an action of $\mathcal{B}_W$ on any $W \ltimes \mathcal{T}_{\mathfrak{g}}$-module.
- It is well-known that W does not act on an integrable module V, but that the triple exponentials

$$\tau_i = \exp(e_i) \cdot \exp(-f_i) \cdot \exp(e_i) \tag{1.10}$$

  are well-defined on V, permute its weight spaces according to the W-action, and give rise to a morphism $\tau : \mathcal{B}_W \to \mathrm{Aut}(\mathsf{f}^{\mathrm{int}})$.
- Finally, lifting $\mathcal{E}_{x_0}$ to $\widetilde{\mathcal{E}}_{x_0} : \mathbf{\Pi}_1(\mathsf{X}/W; [x_0]) \to \mathcal{B}_W \ltimes \mathbf{\Pi}_1(\mathsf{X}; Wx_0)$, and composing with $\tau \otimes \mathscr{P}_{\mathscr{B}}$ yields a morphism

$$\mathscr{P}_{\tau,\mathscr{B}} : \mathcal{B}_W \to \mathrm{Aut}(\mathsf{f}^{\mathrm{int}}) \qquad\qquad \gamma \to \tau(\gamma) \cdot \mathscr{P}(\gamma) \cdot \mathscr{B}(\gamma) \tag{1.11}$$

  which we term the *equivariant monodromy* of $\nabla_{\mathcal{K}}$.

1.10.

By [ATL15], the equivariant monodromy of $\nabla_{\mathcal{K}}$ on an integrable module $V \in \mathcal{O}^{\mathrm{int}}$ is canonically equivalent to the quantum Weyl group action of $\mathcal{B}_W$ on the Etingof–Kazhdan quantisation $\mathsf{F}(V) \in \mathcal{O}_\hbar^{\mathrm{int}}$ *i.e.*, the diagram (1.1) is commutative for $\mu = \mathscr{P}_{\tau,\mathscr{B}}$. This can be used to give a monodromic description of the action λ (1.5) of $\mathcal{P}_W$ on category $\mathcal{O}_\hbar$ modules as follows.

The restriction of the triple exponential map τ (1.10) to $\mathcal{P}_W$ is the sign character

$$\varepsilon : \mathcal{P}_W^{\mathrm{ab}} \to \mathrm{Aut}(\mathsf{f}^{\mathrm{int}}) \qquad\qquad p_\alpha \to \exp(\pi\iota h_\alpha)$$

Lifting it to $\varepsilon : \mathcal{P}_W^{\mathrm{ab}} \to \mathrm{Aut}(\mathsf{f})$ as in 1.6 therefore lifts the equivariant monodromy action of $\mathcal{P}_W$ to

$$\mathscr{P}_{\varepsilon,\mathscr{B}} : \mathcal{P}_W \to \mathrm{Aut}(\mathsf{f}) \qquad\qquad \gamma \to \varepsilon(\gamma) \cdot \mathscr{P}(\gamma) \cdot \mathscr{B}(\gamma)$$

i.e., extends the restriction of $\mathscr{P}_{\tau,\mathscr{B}}$ to $\mathcal{P}_W$ to any category $\mathcal{O}$ module.

To relate $\mathscr{P}_{\varepsilon,\mathscr{B}}$ to λ, denote the restriction morphisms by

$$\mathrm{Res} : \mathrm{End}(\mathsf{f}) \to \mathrm{End}(\mathsf{f}^{\mathrm{int}}) \qquad \text{and} \qquad \mathrm{Res}_\hbar : \mathrm{End}(\mathsf{f}) \to \mathrm{End}(\mathsf{f}^{\mathrm{int}})$$

The commutativity of (1.1) implies that, for any $p \in \mathcal{P}_W$

$$\mathrm{Res} \circ \Psi_{\gamma\circ\alpha} \circ \lambda(p) = \Psi^{\mathrm{int}}_{\gamma\circ\alpha} \circ \mathrm{Res}_\hbar \circ \lambda(p) = \mathrm{Res} \circ \mathscr{P}_{\varepsilon,\mathscr{B}}(p)$$

and therefore that $\mathrm{Res} \circ \Psi_{\gamma\circ\alpha} \circ \mathscr{K}(p) = \mathrm{Res} \circ (\mathscr{P}(p)\mathscr{B}(p))$, since $\varepsilon = \Psi_{\gamma\circ\alpha}(\varepsilon_\hbar)$. In turn, this implies that $\Psi_{\gamma\circ\alpha} \circ \mathscr{K}(p) = \mathscr{P}(p)\mathscr{B}(p)$, so that $\Psi_{\gamma\circ\alpha}$ intertwines λ and $\mathscr{P}_{\varepsilon,\mathscr{B}}$, since $\Psi_{\gamma\circ\alpha}$ maps the Drinfeld algebra $\mathcal{D}_\hbar \supset U_\hbar\mathfrak{g}$ to its classical analogue $\mathcal{D}$, the latter acts faithfully on f, and the algebra $\mathcal{T}_{\mathfrak{g}} \ni \mathscr{P}(p), \mathscr{B}(p)$ is contained in $\mathcal{D}$.

1.11.

The above can also be used to give a description of the (non-equivariant) monodromy $\mathscr{P} : \mathcal{P}_W \to \mathrm{Aut}(\mathsf{f})$ of the Casimir connection $\nabla_{\mathcal{K}}$ (1.7) in terms of quantum Weyl group operators as follows.

We prove that the restriction to $\mathcal{P}_W$ of the cochain $\mathscr{B}$ is the map $\mathcal{P}_W^{\mathrm{ab}} \to \exp(\hbar\mathfrak{h})$ given by $\mathscr{B}(p_\alpha) = \exp(\hbar t_\alpha/2)$, where $t_\alpha \in \mathfrak{h}$ corresponds to α via the isomorphism $\mathfrak{h}^* \to \mathfrak{h}$ induced by the chosen inner product on $\mathfrak{g}$. Define the morphism

$$\lambda_{\varepsilon,\mathscr{B}} : \mathcal{P}_W \to \mathrm{Aut}(\mathsf{f}_\hbar) \qquad p \to \varepsilon_\hbar(p)^{-1} \cdot \lambda(p) \cdot \mathscr{B}(p)^{-1} = \mathscr{K}(p) \cdot \mathscr{B}(p)^{-1}$$

We refer to $\lambda_{\varepsilon,\mathscr{B}}$ as the *normally ordered* quantum Weyl group action of $\mathcal{P}_W$ on category $\mathcal{O}_\hbar$ modules. The terminology is motivated by the fact that, while $\lambda(S_i^2) = \exp(\pi\iota h_i) \cdot q^{\mathcal{K}_{\hbar,i}}$ by (1.5), $\lambda_{\varepsilon,\mathscr{B}}(S_i^2) = q^{2\mathcal{K}^+_{\hbar,i}}$, where the latter is a normally ordered version of the quantum Casimir. The commutativity of (1.1) then implies that $\lambda_{\varepsilon,\mathscr{B}}$ computes the monodromy of $\nabla_{\mathcal{K}}$, that is that $\Psi_{\gamma\circ\alpha} \circ \lambda_{\varepsilon,\mathscr{B}} = \mathscr{P}$.

1.12.

The above results can be generalised to the parabolic setting as follows. Let $\mathbf{J}$ be a subset of nodes of the Dynkin diagram of $\mathfrak{g}$, $\mathfrak{g}_{\mathbf{J}} \subseteq \mathfrak{g}$ the corresponding Lie subalgebra, $W_{\mathbf{J}} \subseteq W$ its Weyl group, and $\mathcal{PB}_{\mathbf{J}} \subseteq \mathcal{B}_W$ the parabolic pure braid group given by the preimage of $W_{\mathbf{J}}$.

We construct a quantum Weyl group action of $\mathcal{PB}_{\mathbf{J}}$ on any category $\mathcal{O}_\hbar$ module whose restriction to $U_\hbar\mathfrak{g}_{\mathbf{J}}$ is integrable. This action is such that

- its restriction to the braid group $\mathcal{B}_{W_{\mathbf{J}}}$ is the quantum Weyl group action of $\mathcal{B}_{W_{\mathbf{J}}}$ on integrable $U_\hbar\mathfrak{g}_{\mathbf{J}}$-modules

- its restriction to the pure braid group $\mathcal{P}_W$ coincides with the quantum Weyl group action (1.5) on category $\mathcal{O}_\hbar$ modules

We also define a normally ordered version of this quantum Weyl group action, in analogy with 1.11.

We then construct a monodromy action of $\mathcal{PB}_\mathbf{J}$ on any category $\mathcal{O}$ module whose restriction to $\mathfrak{g}_\mathbf{J}$ is integrable. We do so by relying on the fact that $\mathcal{PB}_\mathbf{J}$ is isomorphic to $\mathbf{\Pi}_1(\mathsf{X}/W_\mathbf{J};[x_0])$, and correcting the equivariance of the Casimir connection, as outlined in 1.9, but only with respect to $W_\mathbf{J}$. The resulting $W_\mathbf{J}$-equivariant monodromy action is such that

- its restriction to $\mathcal{B}_{W_\mathbf{J}}$ is the equivariant monodromy action of $\mathcal{B}_{W_\mathbf{J}}$ on integrable category $\mathcal{O}$ $\mathfrak{g}_\mathbf{J}$-modules
- its restriction to $\mathcal{P}_W$ coincides with the monodromy action (1.9) on category $\mathcal{O}$ modules (up to a simple correction on $\mathcal{P}_{W_\mathbf{J}}$).

Finally, we show that the above quantum Weyl group and monodromic actions of $\mathcal{PB}_\mathbf{J}$ are equivalent by relying on the fact that $\mathcal{PB}_\mathbf{J}$ is generated by $\mathcal{B}_{W_\mathbf{J}}$ and $\mathcal{P}_W$, and using the equivalences (1.1) for $\mathcal{B}_{W_\mathbf{J}}$ and (1.2) for $\mathcal{P}_W$.

1.13. Outline of the paper

In Section 2, we review the definition of quantum Weyl group operators. In Section 3, we introduce the Drinfeld algebra and prove that it acts faithfully on $\mathcal{O}_\hbar^{\text{int}}$. In Section 4, we construct the quantum Weyl group action of $\mathcal{P}_W$ on category $\mathcal{O}$. Section 5 reviews the definition of the Casimir connection, and the equivariant extension of its monodromy to a representation of the braid group $\mathcal{B}_W$. Section 6 reviews the definition of a braided Coxeter category, and Section 7 the main result of [ATL15]. In Section 8, we prove the stated equivalence. We also point out that it continues to hold if F is replaced by the Etingof–Kazhdan equivalence F^Φ corresponding to an arbitrary Lie associator Φ rather than the one arising from the KZ equations. Finally, in Section 9, we generalise these results to parabolic pure braid groups.

2. Kac–Moody algebras and quantum groups

2.1. Symmetrisable Kac–Moody algebras [Kac90]

Let $\mathbf{I}$ be a finite set and $\mathsf{A}=(a_{ij})_{i,j\in\mathbf{I}}$ a generalised Cartan matrix, *i.e.*, $a_{ii}=2$, $a_{ij}\in\mathbb{Z}_{\leqslant 0}$, $i\neq j$, and $a_{ij}=0$ implies $a_{ji}=0$. Let $(\mathfrak{h},\Pi,\Pi^\vee)$ be a realization of A, *i.e.*,

- $\mathfrak{h}$ is a finite-dimensional complex vector space[2]
- $\Pi = \{\alpha_i\}_{i\in\mathbf{I}}$ is a linearly independent subset of $\mathfrak{h}^*$
- $\Pi^\vee = \{h_i\}_{i\in\mathbf{I}}$ is a linearly independent subset of $\mathfrak{h}$
- $\alpha_i(h_j) = a_{ji}$ for any $i, j \in \mathbf{I}$

The Kac–Moody algebra corresponding to A and the realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$ is the Lie algebra $\mathfrak{g}$ generated by $\mathfrak{h}$ and elements $\{e_i, f_i\}_{i\in\mathbf{I}}$, with relations $[\mathfrak{h}, \mathfrak{h}] = 0$ and

$$[h, e_i] = \alpha_i(h)e_i \qquad [h, f_i] = -\alpha_i(h)f_i \qquad [e_i, f_j] = \delta_{ij}h_i$$

and, for any $i \neq j$,

$$\mathsf{ad}(e_i)^{1-a_{ij}}(e_j) = 0 = \mathsf{ad}(f_i)^{1-a_{ij}}(f_j)$$

Let $\mathfrak{n}^\pm \subset \mathfrak{g}$ be the Lie subalgebras generated by $\{e_i\}_{i\in\mathbf{I}}$ and $\{f_i\}_{i\in\mathbf{I}}$, respectively.

Assume that A is symmetrisable, and fix an invertible diagonal matrix $\mathsf{D} = \mathrm{diag}(d_i)_{i\in\mathbf{I}}$ with coprime entries $d_i \in \mathbb{Z}_{>0}$ such that $\mathsf{D}A$ is symmetric. Then, there is a symmetric, non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$ such that $\langle h_i, -\rangle = d_i^{-1}\alpha_i$ (see, *e.g.*, [ATL19b, Prop. 11.4]). The corresponding identification $\nu : \mathfrak{h} \to \mathfrak{h}^*$ intertwines the actions of W, satisfies $\nu(h_i) = d_i^{-1}\alpha_i$ and therefore restricts to an isomorphism $\mathfrak{h}' \xrightarrow{\sim} \mathsf{Q}\otimes_{\mathbb{Z}}\mathbb{C}$, where $\mathfrak{h}'$ is the span of $\{h_i\}_{i\in\mathbf{I}}$ and $\mathsf{Q} = \bigoplus_{i\in\mathbf{I}} \mathbb{Z}\alpha_i \subseteq \mathfrak{h}^*$ is the root lattice. Note that $\langle h_i, h_i\rangle = 2d_i^{-1}$, while the induced form on $\mathfrak{h}^*$ satisfies $\langle \alpha_i, \alpha_i\rangle = 2d_i \in 2\mathbb{Z}_{>0}$.

By [Kac90, Thm. 2.2], $\langle\cdot,\cdot\rangle$ uniquely extends to a non-degenerate, invariant symmetric bilinear form on $\mathfrak{g}$, which satisfies $\langle e_i, f_j\rangle = \delta_{ij}d_i^{-1}$ and $[x, y] = \langle x, y\rangle \cdot t_\alpha$ for any $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, where $t_\alpha = \nu^{-1}(\alpha)$.

2.2. Category $\mathcal{O}_\infty$ representations

If V is an $\mathfrak{h}$-module and $\lambda \in \mathfrak{h}^*$, we denote the corresponding weight space of V by

$$V[\lambda] = \{v \in V \,|\, h\,v = \lambda(h)v,\ h \in \mathfrak{h}\}$$

and set $P(V) = \{\lambda \in \mathfrak{h}^* |\, V[\lambda] \neq 0\}$. A $\mathfrak{g}$-module V is

(C1) a *weight module* if $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V[\lambda]$.

[2] Note that, unlike [Kac90], we do not require $\mathfrak{h}$ to have minimal dimension $2|\mathbf{I}| - \mathrm{rank}(A)$.

(C2) *integrable* if it is a weight module, and the elements $\{e_i, f_i\}_{i\in\mathbf{I}}$ act locally nilpotently. This implies that $\lambda(h_i) \in \mathbb{Z}$ for any $\lambda \in P(V)$ and $i \in \mathbf{I}$, and that V is completely reducible as a (possibly infinite) direct sum of simple finite-dimensional modules over $\mathfrak{sl}_2^{\alpha_i} = \langle e_i, h_i, f_i\rangle \subset \mathfrak{g}$.

(C3) in *category* $\mathcal{O}_{\infty,\mathfrak{g}}$ if the action of $\mathfrak{b}^+$ is locally finite, *i.e.*, any $v \in V$ is contained in a finite-dimensional $\mathfrak{b}^+$-submodule of V. This implies in particular that V is the direct sum of its generalised weight spaces and that, for any $v \in V$, $(U\mathfrak{n}^+)_\beta v = 0$ for all but finitely many $\beta \in \mathsf{Q}_+$.

(C4) in *category* $\mathcal{O}_{\mathfrak{g}}$ if it is a weight module with finite-dimensional weight spaces, such that

$$P(V) \subseteq D(\lambda_1) \cup \cdots \cup D(\lambda_m) \tag{2.1}$$

for some $\lambda_1, \ldots, \lambda_m \in \mathfrak{h}^*$, where $D(\lambda) = \{\mu \in \mathfrak{h}^* \mid \mu \leqslant \lambda\}$ and $\mu \leqslant \lambda$ iff $\lambda - \mu \in \mathsf{Q}_+ = \bigoplus_{i\in\mathbf{I}} \mathbb{N}\alpha_i$.

The categories $\mathcal{O}_{\mathfrak{g}} \subset \mathcal{O}_{\infty,\mathfrak{g}}$ are symmetric tensor categories. Denoting by $\mathcal{O}^{\mathrm{int}}_{\mathfrak{g}} \subset \mathcal{O}_{\mathfrak{g}}$ and $\mathcal{O}^{\mathrm{int}}_{\infty,\mathfrak{g}} \subset \mathcal{O}_{\infty,\mathfrak{g}}$ the full tensor subcategories of integrable representations, we have the following inclusions

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{g}} & \subset & \mathcal{O}_{\infty,\mathfrak{g}} \\ \cup & & \cup \\ \mathcal{O}^{\mathrm{int}}_{\mathfrak{g}} & \subset & \mathcal{O}^{\mathrm{int}}_{\infty,\mathfrak{g}} \end{array}$$

2.3. Deformation category $\mathcal{O}_\infty$ representations

Similar notions can be defined for $\mathfrak{g}$-modules in the category $\mathrm{Vect}_\hbar$ of topologically free $\mathbb{C}[\![\hbar]\!]$-modules. Namely, a $\mathfrak{g}$-module $\mathcal{V} \in \mathrm{Vect}_\hbar$ is called

(D1) a *weight module* if $\mathcal{V} = \bigoplus_{\lambda\in\mathfrak{h}^*} \mathcal{V}[\lambda]$,[3] where $\bigoplus$ is the direct sum in $\mathrm{Vect}_\hbar$, *i.e.*, the completion of the algebraic direct sum in the $\hbar$-adic topology.

(D2) *integrable* if it is a weight module and, for any $i \in \mathbf{I}$ and $v \in \mathcal{V}$, $\lim_{n\to\infty} e_i^n v = 0 = \lim_{n\to\infty} f_i^n v$.

(D3) in *category* $\mathcal{O}^\hbar_{\infty,\mathfrak{g}}$ if the action of $\mathfrak{b}^+$ on $\mathcal{V}/\hbar^n\mathcal{V}$ is locally finite for any $n \geq 0$.

(D4) in *category* $\mathcal{O}^\hbar_{\mathfrak{g}}$ if it is a weight representation with finite-rank weight spaces, and such that $P(\mathcal{V})$ satisfies (2.1).

[3]Note that the eigenvalues of the action of $\mathfrak{h}$ are required to lie in $\mathfrak{h}^* \subsetneq \mathfrak{h}^*[\![\hbar]\!]$.

It is easy to see that $\mathcal{V}$ is a weight (resp. integrable) module in $\mathsf{Vect}_\hbar$ if and only if $\mathcal{V}/\hbar^n\mathcal{V}$ is a weight (resp. integrable) module in Vect for any $n \geq 0$. We denote by $\mathcal{O}^{\hbar,\mathrm{int}}_{\mathfrak{g}} \subset \mathcal{O}^{\hbar}_{\mathfrak{g}}$ and $\mathcal{O}^{\hbar,\mathrm{int}}_{\infty,\mathfrak{g}} \subset \mathcal{O}^{\hbar}_{\infty,\mathfrak{g}}$ the full tensor subcategories of integrable representations.

2.4. Braid group action

Let W be the Weyl group of $\mathfrak{g}$, and $\{s_i\}_{i\in\mathbf{I}}$ its set of simple reflections. The braid group $\mathcal{B}_W$ is the group generated by the elements $\{S_i\}_{i\in\mathbf{I}}$, with relations

$$\underbrace{S_i \cdot S_j \cdot S_i \ \cdots}_{m_{ij}} = \underbrace{S_j \cdot S_i \cdot S_j \ \cdots}_{m_{ij}} \tag{2.2}$$

for any $i \neq j$, where m_{ij} is the order of s_is_j in W. If V is an integrable $\mathfrak{g}$-module in Vect or $\mathsf{Vect}_\hbar$, the operators

$$\widetilde{s}_i = \exp(e_i)\cdot\exp(-f_i)\cdot\exp(e_i) \in GL(V) \tag{2.3}$$

are well-defined, and satisfy the braid relations (2.2) [Tit66]. The corresponding action of $\mathcal{B}_W$ on V factors through the *Tits extension* $\widetilde{W}$, an extension of W by the sign group $\mathbb{Z}_2^{\mathbf{I}}$.

2.5. The quantum group $U_\hbar\mathfrak{g}$ [Dri87, Jim85]

Let $\hbar$ be a formal variable, set $q = \exp(\hbar/2)$ and $q_i = q^{d_i}$, $i \in \mathbf{I}$. The Drinfeld–Jimbo quantum group of $\mathfrak{g}$ is the algebra $U_\hbar\mathfrak{g}$ over $\mathbb{C}[\![\hbar]\!]$ topologically generated by $\mathfrak{h}$ and the elements $\{E_i, F_i\}_{i\in\mathbf{I}}$, subject to the relations $[h,h'] = 0$,

$$[h,E_i] = \alpha_i(h)E_i \qquad [h,F_i] = -\alpha_i(h)F_i \qquad [E_i,F_j] = \delta_{ij}\frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}}$$

for any $h,h' \in \mathfrak{h}$, $i,j \in \mathbf{I}$, and the q-Serre relations

$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_i X_i^{1-a_{ij}-m} X_j X_i^m = 0$$

for $X = E, F$, $i \neq j \in \mathbf{I}$, where $[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$ and, for any $k \leqslant n$,

$$[n]_i! = [n]_i \cdot [n-1]_i \cdots [1]_i \qquad \text{and} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_i = \frac{[n]_i!}{[k]_i! \cdot [n-k]_i}$$

Define weight, integrable, category $\mathcal{O}_\infty$ and $\mathcal{O}$ modules for $U_\hbar\mathfrak{g}$ in $\mathrm{Vect}_\hbar$ analogously to Section 2.3, and denote by

$$\mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar\mathfrak{g}} \subset \mathcal{O}_{\infty,U_\hbar\mathfrak{g}} \qquad \text{and} \qquad \mathcal{O}^{\mathrm{int}}_{U_\hbar\mathfrak{g}} \subset \mathcal{O}_{U_\hbar\mathfrak{g}}$$

the subcategories of integrable modules.[4]

2.6. Quantum Weyl group operators [KR90, Lus90, Lus93, Sa94, So90]

For any $\mathcal{V} \in \mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar\mathfrak{g}}$, define the endomorphisms $\{\mathbf{S}_i\}_{i\in\mathbf{I}}$ of $\mathcal{V}$ as follows.[5] For any $v_\mu \in \mathcal{V}[\mu]$, set

$$\mathbf{S}_i\, v_\mu = \sum_{\substack{a,b,c\in\mathbb{Z}_{\geqslant 0}\\ a-b+c=-\mu(h_i)}} (-1)^b q_i^{b-ac} E_i^{(a)} F_i^{(b)} E_i^{(c)} \cdot v_\mu \tag{2.4}$$

where $X_i^{(a)} = X_i^a/[a]_i!$.

Then, $\mathbf{S}_i(\mathcal{V}[\mu]) \subseteq \mathcal{V}[s_i(\mu)]$ and the $\mathbf{S}_i$ give rise to an action of the braid group $\mathcal{B}_W$ on $\mathcal{V}$, which deforms the action by triple exponentials described in 2.4 [Lus93, Sec. 39.4].

2.7. Action of $\mathcal{B}_W$ on $U_\hbar\mathfrak{g}$ ([Lus88], [Lus93, Chaps. 37–39])

Consider the algebra automorphisms $\{\mathbf{T}_i\}_{i\in\mathbf{I}}$ of $U_\hbar\mathfrak{g}$ defined by

$$\mathbf{T}_i(h) = s_i(h) \qquad \mathbf{T}_i(E_i) = -F_i q_i^{h_i} \qquad \mathbf{T}_i(F_i) = -q_i^{-h_i} E_i$$

where $h \in \mathfrak{h}$ and, for any $i \neq j \in \mathbf{I}$,

$$\mathbf{T}_i(X_j) = \sum_{r=0}^{-a_{ij}} (-1)^r q_i^{\sigma(X)r} X_i^{-a_{ij}-r} X_j X_i^r$$

where $X = E, F$ and $\sigma(E) = -1 = -\sigma(F)$.

The automorphisms $\{\mathbf{T}_i\}_{i\in\mathbf{I}}$ define an action of the braid group $\mathcal{B}_W$ on $U_\hbar\mathfrak{g}$ which we denote by $b(X)$, $b \in \mathcal{B}_W$ and $X \in U_\hbar\mathfrak{g}$. Moreover, for any $X \in U_\hbar\mathfrak{g}$, $\mathcal{V} \in \mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar\mathfrak{g}}$, and $v \in \mathcal{V}$, one has $\mathbf{S}_i(X \cdot v) = \mathbf{T}_i(X) \cdot \mathbf{S}_i(v)$.

[4]Note in particular that a representation $\mathcal{V}$ of $U_\hbar\mathfrak{g}$ is in category $\mathcal{O}_\infty$ if the action of $U_\hbar\mathfrak{b}^+$ on $\mathcal{V}/\hbar^n\mathcal{V}$ is locally finite for any $n \geq 0$.

[5]The operator $\mathbf{S}_i$ is the operator $T''_{i,+1}$ defined in [Lus93, Sec. 5.2].

3. Faithfulness of category $\mathcal{O}$ integrable modules

Integrable $U_\hbar\mathfrak{g}$-modules are well-known to be faithful, *i.e.*, the only element of $U_\hbar\mathfrak{g}$ acting trivially on every integrable module is zero [Lus93, Prop. 3.5.4]. To the best of our knowledge, the analogous result for the more restrictive class of integrable modules in category $\mathcal{O}$ does not appear in the literature. We present here a proof due to P. Etingof, which establishes faithfulness for a larger algebra containing $U_\hbar\mathfrak{g}$.

3.1. The Drinfeld algebra $\mathcal{D}_\hbar$

For any $\beta \in \mathsf{Q}_+$, let $\mathcal{B}_\beta = \{X_{\beta,p}\}$ be a basis of $U_\hbar\mathfrak{n}^+_\beta$ and set $\mathcal{B} = \bigsqcup_{\beta\in\mathsf{Q}_+} \mathcal{B}_\beta$. Set

$$\mathcal{D}_\hbar = \left\{ \sum_{X\in\mathcal{B}} c_X X \,:\, c_X \in U_\hbar\mathfrak{b}^- \right\} = \prod_{\beta\in\mathsf{Q}_+} U_\hbar\mathfrak{b}^- \otimes U_\hbar\mathfrak{n}^+_\beta \supset U_\hbar\mathfrak{g}$$

$\mathcal{D}_\hbar$ has an algebra structure which extends that of $U_\hbar\mathfrak{g}$. Moreover, the action of $U_\hbar\mathfrak{g}$ on any module $\mathcal{V} \in \mathcal{O}_{U_\hbar\mathfrak{g}}$ extends to one of $\mathcal{D}_\hbar$ since, for any $v \in \mathcal{V}$, $U_\hbar\mathfrak{n}^+_\beta v = 0$ for all but finitely many $\beta \in \mathsf{Q}_+$.

Theorem (Etingof)**.** *Category $\mathcal{O}$ integrable $U_\hbar\mathfrak{g}$-modules are faithful for $\mathcal{D}_\hbar$.*

The proof is carried out in Sections 3.2–3.4.

Remark. A variant $\mathcal{Q}_\hbar$ of the algebra $\mathcal{D}_\hbar$ was introduced by Drinfeld in [Dri92, Sect. 8] as follows. For any $\beta \in \mathsf{Q}_+$, let $I_\beta \subset U_\hbar\mathfrak{g}$ be the left ideal generated by $U_\hbar\mathfrak{g}_{\beta'}$ for any $\beta' > \beta$, or equivalently by $\{U_\hbar\mathfrak{n}^+_{\beta'}\}_{\beta'>\beta}$, and set $\mathcal{Q}_\hbar = \lim_\beta U_\hbar\mathfrak{g}/I_\beta$. Since $U_\hbar\mathfrak{g}/I_\beta \cong \bigoplus_{\beta'\not>\beta} U_\hbar\mathfrak{b}^- \otimes U_\hbar\mathfrak{n}^+_{\beta'}$, $\mathcal{Q}_\hbar$ embeds into $\mathcal{D}_\hbar$ as the subalgebra consisting of series $\sum_{\beta\in\mathsf{Q}_+} X_\beta$, $X_\beta \in U_\hbar\mathfrak{b}^- \otimes U_\hbar\mathfrak{n}^+_\beta$, where for any $\beta \in \mathsf{Q}_+$, $X_{\beta'} = 0$ for all but finitely many $\beta' \not> \beta$. The algebra $\mathcal{Q}_\hbar$ is less natural than $\mathcal{D}_\hbar$, however. For instance, if $\emptyset \subsetneq \mathbf{J} \subsetneq \mathbf{I}$ is a proper non-empty subset, $\mathfrak{g}_\mathbf{J} \subset \mathfrak{g}$ the corresponding subalgebra, and $\mathcal{Q}_{\mathbf{J},\hbar}$ (resp. $\mathcal{D}_{\mathbf{J},\hbar}$) the analogue of $\mathcal{Q}_\hbar$ (resp. $\mathcal{D}_\hbar$) for $\mathfrak{g}_\mathbf{J}$, then $\mathcal{D}_{\mathbf{J},\hbar} \subset \mathcal{D}_\hbar$ while $\mathcal{Q}_{\mathbf{J},\hbar}$ does not map to $\mathcal{Q}_\hbar$.

3.2. Verma modules

For $\lambda \in \mathfrak{h}^*$, let $M(\lambda)$ be the Verma module of highest weight λ and $v_\lambda \in M(\lambda)$ its cyclic vector. For any $\beta \in \mathsf{Q}_+$, let $M(\lambda)_\beta \subset M(\lambda)$ be the weight space of weight $\lambda - \beta$. Note that there is a natural identification $M(\lambda)_\beta \simeq (U_\hbar\mathfrak{n}^-)_\beta$. Recall that the *contragredient* Verma module $M^\vee(\lambda)$ is the pullback through the Chevalley involution of the restricted dual $M^*(\lambda) = \bigoplus_{\beta\in\mathsf{Q}_+} M(\lambda)^*_\beta$, where $M(\lambda)^*_\beta$ denotes the dual in $\mathrm{Vect}_\hbar$. The contragredient Verma module is equipped with a morphism $M(\lambda) \to M^\vee(\lambda)$, $v_\lambda \mapsto v_\lambda^*$. The Shapovalov form on $M(\lambda)$ is defined by

$$\langle\cdot,\cdot\rangle_\lambda : M(\lambda)\otimes M(\lambda) \to M(\lambda)\otimes M^\vee(\lambda) \to \mathbb{C}[\![\hbar]\!]$$

By construction, it satisfies $\langle v_\lambda, v_\lambda\rangle_\lambda = 1$, $\langle M(\lambda)_\beta, M(\lambda)_{\beta'}\rangle_\lambda = 0$ if $\beta\neq\beta'$, and $\langle xv, w\rangle_\lambda = -\langle x, \omega(x)w\rangle_\lambda$ for any $x\in\mathfrak{g}$, $v,w\in M(\lambda)$. It is well-known that $\langle\cdot,\cdot\rangle_\lambda$ is symmetric and non-degenerate only for generic $\lambda\in\mathfrak{h}^*$.

For generic $\lambda\in\mathfrak{h}^*$, let $\mathcal{B}^*_{\lambda,\beta} = \{X^*_{\beta,p}\}$ be the dual basis of $U_\hbar\mathfrak{n}^-_\beta$ with respect to the Shapovalov form. In particular, one has $\langle X^*_{\beta,i}v_\lambda, \omega(S(X_{\beta,j}))v_\lambda\rangle = \delta_{ij}$. Thus, modulo elements of weights lower than λ, $X_{\beta,j}X^*_{\beta,i}v_\lambda = \delta_{ij}v_\lambda$.

Proposition. *Verma modules are faithful for $\mathcal{D}_\hbar$.*

Proof. Assume that $x\in\mathcal{D}_\hbar$ acts trivially on every $M(\lambda)$, and write

$$x = \sum_{\mathcal{B}} x^-_{\beta,i}x^0_{\beta,i}X_{\beta,i}$$

where $x^0_{\beta,i}\in U\mathfrak{h}[\![\hbar]\!]$ and $x^-_{\beta,i}\in U_\hbar\mathfrak{n}^-$. Note that, for any $\lambda\in\mathfrak{h}^*$, the action of x on the cyclic vector $v_\lambda\in M(\lambda)$ gives

$$0 = x\cdot v_\lambda = \lambda(\varphi_0)x_0\cdot v_\lambda$$

Therefore, $x^0_0 = 0 = x^-_0$. We shall prove that, for any $X_{\beta,i}\in\mathcal{B}$, $x^0_{\beta,i} = 0 = x^-_{\beta,i}$. Proceeding by induction, we assume that $x_{\gamma,j} = 0 = x^0_{\gamma,j}$ for any $X_{\gamma,j}\in\mathcal{B}$ such that $\operatorname{ht}\gamma < n$. Fix $\beta\in\mathsf{Q}_+$ with $\operatorname{ht}\beta = n$. Then, for generic $\lambda\in\mathfrak{h}^*$, we have $X^*_{\beta,i}v_\lambda\in M(\lambda)_\beta$ and, since $X_{\beta,j}X^*_{\beta,i}v_\lambda = \delta_{ij}v_\lambda$,

$$0 = x\cdot X^*_{\beta,i}v_\lambda = \sum_j x^-_{\beta,j}x^0_{\beta,j}X_{\beta,j}X^*_{\beta,i}v_\lambda = \lambda(x^0_{\beta,i})x^-_{\beta,i}v_\lambda$$

Therefore, $x^0_{\beta,i} = 0 = x^-_{\beta,i}$. □

3.3. Regularity of the matrix coefficients on $M(\lambda)$

For any $\lambda\in\mathfrak{h}^*$, let $M^*(\lambda)$ be the (restricted) dual Verma module and $(\cdot,\cdot)_{M(\lambda)} : M(\lambda)\otimes M^*(\lambda)\to\mathbb{C}[\![\hbar]\!]$ the natural pairing.

Proposition. *For any $\lambda\in\mathfrak{h}^*$, $v\in M(\lambda)$, and $f\in M(\lambda)^*$, the matrix coefficient $(xv,f)_{M(\lambda)}$ lies in $\mathbb{C}[\lambda][\![\hbar]\!]$.*

Proof. Note that, for any $x^\pm \in U_\hbar\mathfrak{n}^\pm$, the coefficient $(x^-v, x^+f) \in \mathbb{C}[\![\hbar]\!]$ is independent of λ. We can write $x = \sum_i x_i^+ x_i^0 x_i^-$, for some $x_i^+ \in U_\hbar\mathfrak{n}^+$, $x_i^0 \in U\mathfrak{h}[\![\hbar]\!]$, and $x_i^- \in (U_\hbar\mathfrak{n}^-)_{\beta_i}$, with $\beta_i \in \mathsf{Q}_+$. Then, we have

$$(xv, f)_{M(\lambda)} = \sum_i (x_i^0 x_i^- v, S(x_i^+)f)_{M(\lambda)} = \sum_i (\lambda - \beta_i)(x_i^0)(x_i^- v, S(x_i^+)f)_{M(\lambda)}\,.$$

The result follows. ∎

3.4. Proof of Theorem 3.1

Assume that $x \in \mathcal{D}_\hbar$ acts trivially on every category $\mathcal{O}$ integrable $U_\hbar\mathfrak{g}$-module. We shall prove that x acts trivially on any Verma module, so that $x = 0$ by Proposition 3.2.

Clearly, x acts trivially on $M(\lambda)$ if and only if, for any $v \in M(\lambda)$ and $f \in M(\lambda)^*$, the matrix coefficient $(xv, f)_{M(\lambda)}$ vanishes. By Proposition 3.3, it is enough to check that this holds for λ in a Zariski open subset of $\mathfrak{h}^*$. To this end, note that, if $v \in M(\lambda)_\beta$, then $xv = x(\beta)v$, where $x(\beta) \in U\mathfrak{g}$ is the truncation of x at β. Therefore, it is possible to choose $\lambda \in \mathsf{P}_+$ large enough such that

$$(xv, f)_{M(\lambda)} = (xv, f)_{L(\lambda)} = 0$$

i.e., $(xv, f)_{M(\lambda)}$ is equal to the matrix coefficient of x on the unique irreducible quotient $L(\lambda)$ of $M(\lambda)$. By assumption on x, the latter is zero, since $L(\lambda)$ is integrable for $\lambda \in \mathsf{P}_+$. The result follows.

4. Quantum Weyl group actions of pure braid groups

4.1. Completions

Let A be an algebra, $\mathcal{C} \subset \mathrm{Rep}(A)$ a full subcategory, and $\mathrm{End}(\mathsf{f}_\mathcal{C})$ the algebra of endomorphisms of the forgetful functor $\mathsf{f}_\mathcal{C} : \mathcal{C} \to \mathrm{Vect}$. By definition, an element of $\mathrm{End}(\mathsf{f}_\mathcal{C})$ is a collection

$$\varphi = \{\varphi_V\}_{V\in\mathcal{C}} \in \prod_{V\in\mathcal{C}} \mathrm{End}(V)$$

which is natural, *i.e.*, such that $f \circ \varphi_V = \varphi_W \circ f$ for any $f : V \to W$ in $\mathcal{C}$. The action of A on any $V \in \mathcal{C}$ yields a morphism of algebras $A \to \mathrm{End}(\mathsf{f}_\mathcal{C})$, and factors through the action of $\mathrm{End}(\mathsf{f}_\mathcal{C})$ on V. We shall refer to $\mathrm{End}(\mathsf{f}_\mathcal{C})$ as the completion of A with respect to the category $\mathcal{C}$.

4.2. Braid groups and completions

The braid group actions considered in Section 2 can be concisely described in terms of completions. For instance, let $\mathrm{End}(\mathsf{f}_\hbar^{\mathrm{int}})$ be the algebra of endomorphisms of the forgetful functor $\mathsf{f}_\hbar^{\mathrm{int}} : \mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar\mathfrak{g}} \to \mathrm{Vect}_\hbar$. The quantum Weyl group operators $\mathbf{S}_i$ defined by (2.4) are elements of $\mathrm{Aut}(\mathsf{f}_\hbar^{\mathrm{int}})$, and yield a group homomorphism $\lambda : \mathcal{B}_W \to \mathrm{Aut}(\mathsf{f}_\hbar^{\mathrm{int}})$.

4.3. Sign character of the pure braid group

Let Z be the free abelian group with a generator p_α for each positive real root α, endowed with the W-action given by $w\,p_\alpha = p_{|w\alpha|}$, where $|w\alpha| = \pm w\alpha$ according to whether $w\alpha$ is positive or negative.

Let $\mathcal{P}_W \subset \mathcal{B}_W$ be the pure braid group. Its abelianisation $\mathcal{P}_W^{\mathrm{ab}} = \mathcal{P}_W/[\mathcal{P}_W, \mathcal{P}_W]$ is acted upon by $\mathcal{B}_W/\mathcal{P}_W \simeq W$. By [Tit66, Thm. 2.5] and [Dig15] the assignment $p_{\alpha_i} \to S_i^2$ uniquely extends to a W-equivariant isomorphism $Z \to \mathcal{P}_W^{\mathrm{ab}}$.

Define the sign character of $\mathcal{P}_W$ to be the morphism

$$\varepsilon_\hbar : \mathcal{P}_W^{\mathrm{ab}} \to \mathrm{Aut}(\mathsf{f}_\hbar^{\mathrm{int}}) \qquad \varepsilon_\hbar(p_\alpha) = \exp(\iota\pi h_\alpha) \tag{4.1}$$

where $\exp(\iota\pi h_\alpha)$ is the operator acting on a weight space of (integral) weight λ as multiplication by $\exp(\iota\pi\lambda(h_\alpha))$.

4.4. Canonical lift of the sign character

Let $\mathsf{f}_\hbar : \mathcal{O}_{\infty,U_\hbar\mathfrak{g}} \to \mathrm{Vect}_\hbar$ be the forgetful functor, and consider the morphism $\mathrm{Aut}(\mathsf{f}_\hbar) \to \mathrm{Aut}(\mathsf{f}_\hbar^{\mathrm{int}})$ corresponding to the inclusion $\mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar\mathfrak{g}} \subset \mathcal{O}_{\infty,U_\hbar\mathfrak{g}}$. The sign character $\varepsilon_\hbar$ has a canonical lift

$$\mathcal{P}_W^{\mathrm{ab}} \to \mathrm{Aut}(\mathsf{f}_\hbar) \qquad p_\alpha \to \exp(\iota\pi h_\alpha)$$

which is well-defined since for any $\mathcal{V} \in \mathcal{O}_{\infty,U_\hbar\mathfrak{g}}$ and $n \geq 0$, $\mathcal{V}/\hbar^m\mathcal{V}$ is a locally finite $\mathfrak{h}$-module. We denote this lift by the same symbol.

4.5. Pure braid group action on category $\mathcal{O}_\infty$

The following is one of the main results of this paper.

Theorem. *Let $\lambda : \mathcal{B}_W \to \mathrm{End}(\mathsf{f}_\hbar^{\mathrm{int}})$ be the quantum Weyl group action of the braid group $\mathcal{B}_W$. Then, the following holds.*

(1) *For any* $p \in \mathcal{P}_W$,

$$\lambda(p) = \varepsilon_\hbar(p) \cdot \mathscr{K}(p)$$

where $\varepsilon_\hbar(p)$ *is the sign character* (4.1), *and* $\mathscr{K}(p)$ *is a unique element of* $U_\hbar\mathfrak{g}$ *which is invertible and of weight zero.*

(2) *The assignment* $p \to \mathscr{K}(p)$ *is a homomorphism* $\mathcal{P}_W \to (U_\hbar\mathfrak{g})^{\mathfrak{h}}$ *which is* $\mathcal{B}_W$*-equivariant.*

(3) *The quantum Weyl group action of the pure braid group* $\mathcal{P}_W$ *on integrable modules extends to an action*

$$\lambda : \mathcal{P}_W \to \mathrm{Aut}(\mathsf{f}_\hbar) \qquad \textit{given by} \qquad \lambda(p) = \varepsilon_\hbar(p) \cdot \mathscr{K}(p) \tag{4.2}$$

(4) *The map* λ *intertwines the inner action of* $\mathcal{P}_W$ *on* $U_\hbar\mathfrak{g}$ i.e., *for any element* $Y \in U_\hbar\mathfrak{g}$ *and* $p \in \mathcal{P}_W$

$$\lambda(p) Y \lambda(p)^{-1} = p(Y)$$

in $\mathrm{End}(\mathsf{f}_\hbar)$.

Proof. (2), (3) and (4) follow from (1).

(1) It suffices to prove the existence of $\mathscr{K}(p)$ for a set of generators of $\mathcal{P}_W$. The uniqueness of $\mathscr{K}(p)$ for any $p \in \mathcal{P}_W$ then follows from Theorem 3.1. By [DG01, Cor. 6] (see also [Dig15, Prop. 2.5]), $\mathcal{P}_W$ is generated by the elements $S_w S_i^2 S_w^{-1}$, where $i \in \mathbf{I}$, $w \in W$ is such that $w\alpha_i > 0$, and $S_w \in \mathcal{B}_W$ is the canonical lift of w.

Consider first the case $w = 1$. By [Lus93, Sec. 5.2], the square of the operator $\mathbf{S}_i$ is related to the quantum Casimir operator of $U_\hbar\mathfrak{sl}_2^{\alpha_i} = \langle E_i, F_i, h_i \rangle \subset U_\hbar\mathfrak{g}$ as follows. Let $\mathsf{f}_{\hbar,i}^{\mathrm{int}} : \mathcal{O}^{\mathrm{int}}_{\infty, U_\hbar\mathfrak{sl}_2^{\alpha_i}} \to \mathrm{Vect}_\hbar$ be the forgetful functor. An element of $\mathrm{End}(\mathsf{f}_{\hbar,i}^{\mathrm{int}})$ is determined by its action on each of the indecomposable representations $\{\mathcal{V}_r^i\}_{r \geq 0}$, where $\mathcal{V}_r^i$ is of rank $r+1$. The Casimir operator $\mathcal{C}_i$ of $U_\hbar\mathfrak{sl}_2^{\alpha_i}$ acts on $\mathcal{V}_r^i$ as multiplication by $d_i r(r+2)/2$. Set $\mathcal{K}_i = \mathcal{C}_i - d_i h_i^2/2$, so that $\mathcal{K}_i$ acts on the subspace of $\mathcal{V}_r^i$ of weight $m\alpha_i/2$ as multiplication by $d_i(r(r+2) - m^2)/2$. Then,

$$\mathbf{S}_i^2 = \exp(\iota\pi h_i) \cdot q^{\mathcal{K}_i} \tag{4.3}$$

By [Dri89, Sec. 5],

$$q^{\mathcal{C}_i} = \sum_{m \geqslant 0} F_i^m \phi_m E_i^m \tag{4.4}$$

for some explicit $\phi_m \in U\mathfrak{h}_i[\![\hbar]\!]$. It follows that $q^{\mathcal{C}_i}$ lies in $U_\hbar\mathfrak{g}$, and therefore so does $q^{\mathcal{K}_i} = q^{\mathcal{C}_i} q^{-d_i h_i^2/2}$. Thus, setting $\mathscr{K}(S_i^2) = q^{\mathcal{K}_i} \in \mathcal{D}_\hbar$, we get

$$\lambda(S_i^2) = \mathbf{S}_i^2 = \exp(\iota\pi h_i) \cdot q^{\mathcal{K}_i} = \varepsilon_\hbar(S_i^2) \cdot \mathscr{K}(S_i^2)$$

Note next that if $w \in W$ satisfies $w\alpha_i > 0$, then $\mathbf{T}_w = \mathrm{Ad}(\mathbf{S}_w)$ satisfies $\mathbf{T}_w(E_i) \in U_\hbar\mathfrak{b}^+_{w\alpha_i}$, and $\mathbf{T}_w(F_i) \in U_\hbar\mathfrak{b}^-_{-w\alpha_i}$ [Lus93, Sec. 37.1]. It follows that $q^{\mathcal{K}_{w,i}} = \mathbf{T}_w(q^{\mathcal{K}_i})$ is a weight zero element in $\mathcal{D}_\hbar$, and if we set $\mathscr{K}(S_w S_i^2 S_w^{-1}) = q^{\mathcal{K}_{w,i}}$, then

$$\lambda(S_w S_i^2 S_w^{-1}) = \mathbf{S}_w \mathbf{S}_i^2 \mathbf{S}_w^{-1} = \exp(\iota\pi h_{w,i}) \cdot q^{\mathcal{K}_{w,i}} = \varepsilon_\hbar(p) \cdot \mathscr{K}(p) \qquad \square$$

Remarks.

- The proof of Theorem 4.5 shows that the action λ on category $\mathcal{O}_\infty$ modules for $U_\hbar\mathfrak{g}$ is explicitly given on the generators of $\mathcal{P}_W$ by

$$\lambda(S_w S_i^2 S_w^{-1}) = \exp(\iota\pi h_{w,i}) \cdot q^{\mathcal{K}_{w,i}}$$

- Since $\mathscr{K}$ maps to $U_\hbar\mathfrak{g}$, it defines a (signless quantum Weyl group) action of $\mathcal{P}_W$ on *any* $U_\hbar\mathfrak{g}$-module.

4.6. The normally ordered quantum Weyl group action

We shall be interested in the following modification of the action (4.2). Let

$$\mathscr{B} : \mathcal{P}_W \to \exp(\hbar\mathfrak{h}) \subset U_\hbar\mathfrak{g} \qquad \text{be given by} \qquad \mathscr{B}(p_\alpha) = q^{t_\alpha} = \exp(\hbar t_\alpha/2)$$

(cf. Section 1.11). Define the morphism

$$\lambda_{\varepsilon,\mathscr{B}} : \mathcal{P}_W \to U_\hbar\mathfrak{g} \qquad \text{by} \qquad \lambda_{\varepsilon,\mathscr{B}}(p) = \mathscr{K}(p) \cdot \mathscr{B}(p)^{-1}$$

so that $\lambda(p) = \varepsilon_\hbar(p) \cdot \lambda_{\varepsilon,\mathscr{B}}(p) \cdot \mathscr{B}(p)$ for any $p \in \mathcal{P}_W$.

We refer to $\lambda_{\varepsilon,\mathscr{B}}$ as the *normally ordered* quantum Weyl group action of $\mathcal{P}_W$. The terminology is justified by the fact that, for any $i \in \mathbf{I}$, $\lambda_{\varepsilon,\mathscr{B}}(S_i^2)$ acts as the normally ordered quantum Casimir operator, in contrast with (4.3). Namely, one has

$$\lambda_{\varepsilon,\mathscr{B}}(S_i^2) = \mathscr{K}(S_i^2) \cdot \mathscr{B}(p_{\alpha_i})^{-1} = q^{2\mathcal{K}_i^+}$$

where $\mathcal{K}_i^+ = (\mathcal{K}_i - t_{\alpha_i})/2$. This modified action will be relevant in Theorem 8.2. Note also that for any element $Y \in U_\hbar\mathfrak{g}$ of weight $\gamma \in \mathsf{Q}$ and $p \in \mathcal{P}_W$, one has

$$\mathrm{Ad}(\lambda_{\varepsilon,\mathscr{B}}(p))(Y) = p(Y) \cdot (\varepsilon_\hbar(p), \gamma)^{-1} \cdot (\mathscr{B}(p), \gamma)^{-1}$$

in $\mathrm{End}(\mathsf{f}_\hbar)$.

4.7. Pure braid group actions for $U_q\mathfrak{g}$

Let $\mathbb{K}$ be a field of characteristic zero, $q \in \mathbb{K}^\times$ an element of infinite order, *e.g.*, $q \in \mathbb{C}^\times$ not a root of unity or $q \in \mathbb{Q}(q)$, and $U_q\mathfrak{g}$ the corresponding quantum group over $\mathbb{K}$.

The definition of (integrable) category $\mathcal{O}_\infty$ $U_q\mathfrak{g}$-modules is similar to the formal case (see *e.g.*, [Lus93, Ch. 3]). The analogues of Theorem 4.5 and Section 4.6 hold for $U_q\mathfrak{g}$ and defines actions of $\mathcal{P}_W$ on category $\mathcal{O}_\infty$ modules.

In this case, the quantum Casimirs $q^{\mathcal{K}_i}$ do not lie in $U_q\mathfrak{g}$, but in the Drinfeld algebra $\mathcal{D}_q$ of $U_q\mathfrak{g}$, and the morphism $\mathscr{K}$ takes values in $\mathcal{D}_q$. Note that the latter acts on any category $\mathcal{O}_\infty$ module $\mathcal{V}$ since, for any $v \in \mathcal{V}$, $(U_q\mathfrak{n}^+)_\beta v = 0$ for all but finitely many $\beta \in \mathsf{Q}_+$.

5. The Casimir connection

5.1. Fundamental group of root system arrangements

Let A be a symmetrisable generalised Cartan matrix, $(\mathfrak{h}_\mathbb{R}, \Pi, \Pi^\vee)$ a realisation of A over $\mathbb{R}$, and $(\mathfrak{h} = \mathbb{C} \otimes_\mathbb{R} \mathfrak{h}_\mathbb{R}, \Pi, \Pi^\vee)$ its complexification. Let $\Pi^\perp \subset \mathfrak{h}$ be the annihilator of Π, set $\mathfrak{h}^{\mathrm{e}} = \mathfrak{h}/\Pi^\perp$, and note that $\mathfrak{h}^{\mathrm{e}}$ is independent of the realisation of A. Let

$$\mathcal{C} = \{h \in \mathfrak{h}^{\mathrm{e}}_\mathbb{R} \mid \forall i \in \mathbf{I},\, \alpha_i(h) > 0\}$$

be the fundamental Weyl chamber in $\mathfrak{h}^{\mathrm{e}}_\mathbb{R}$, and $\mathsf{Y}_\mathbb{R} = \bigcup_{w \in W} w(\overline{\mathcal{C}})$ the Tits cone. $\mathsf{Y}_\mathbb{R}$ is a convex cone, and the Weyl group W acts properly discontinuously on its interior $\mathring{\mathsf{Y}}_\mathbb{R}$ and complexification $\mathsf{Y} = \mathring{\mathsf{Y}}_\mathbb{R} + \iota\mathfrak{h}^{\mathrm{e}}_\mathbb{R} \subseteq \mathfrak{h}^{\mathrm{e}}$ [Loo80, Vin71]. The regular points of this action are given by

$$\mathsf{X} = \mathsf{Y} \setminus \bigcup_{\alpha \in \Delta_+} \mathrm{Ker}(\alpha)$$

The action of W on X is proper and free, and the space X/W is a complex manifold. The following result is due to van der Lek [vdL83], and generalises Brieskorn's Theorem [Bri71] to the case of an arbitrary Weyl group.

Theorem. *The fundamental groups of X/W and X are isomorphic to $\mathcal{B}_W$ and $\mathcal{P}_W$ respectively.*

The generators $\{S_i\}_{i \in \mathbf{I}}$ of $\mathcal{B}_W$ may be described as follows. Let $p : \mathsf{X} \to \mathsf{X}/W$ be the canonical projection, fix a point $x_0 \in \mathcal{C}$ and use $[x_0] = p(x_0)$ as a base point in X/W. For any $i \in \mathbf{I}$, choose an open disk D_i in $x_0 + \mathbb{C}h_i$, centered in $x_0 - \frac{\alpha_i(x_0)}{2}h_i$, and such that $\overline{D}_i$ does not intersect any root hyperplane other than $\mathrm{Ker}(\alpha_i)$. Let $\gamma_i : [0,1] \to x_0 + \mathbb{C}h_i$ be the path from x_0 to $s_i(x_0)$ in X determined by $\gamma_i|_{[0,1/3]\cup[2/3,1]}$ is affine and lies in $x_0 + \mathbb{R}h_i \setminus D_i$, the points $\gamma_i(1/3), \gamma_i(2/3)$ are in $\partial\overline{D}_i$, and $\gamma_i|_{[1/3,2/3]}$ is a semicircular arc in $\partial\overline{D}_i$, positively oriented with respect to the natural orientation of $x_0 + \mathbb{C}h_i$. Then, $S_i = p \circ \gamma_i$.

5.2. The Casimir connection

For any positive root $\alpha \in \Delta_+$, let $\{e^{(i)}_{\pm\alpha}\}_{i=1}^{\mathrm{m}_\alpha}$ be bases of $\mathfrak{g}_{\pm\alpha}$ which are dual with respect to $\langle\cdot,\cdot\rangle$, and

$$\mathcal{K}^+_\alpha = \sum_{i=1}^{\mathrm{m}_\alpha} e^{(i)}_{-\alpha} e^{(i)}_{\alpha}$$

the corresponding truncated and normally ordered Casimir operator. Let $\mathcal{V}$ be a $\mathfrak{g}$-module in category $\mathcal{O}^\hbar_{\infty,\mathfrak{g}}$ and $\mathbb{V} = \mathsf{X} \times \mathcal{V}$ the holomorphically trivial vector bundle over X with fibre V. Finally, set $\mathsf{h} = \frac{\hbar}{2\pi\iota}$.

Definition. The Casimir connection of $\mathfrak{g}$ is the connection on $\mathbb{V}$ given by

$$\nabla_\mathcal{K} = d - \mathsf{h} \sum_{\alpha\in\Delta_+} \frac{d\alpha}{\alpha} \cdot \mathcal{K}^+_\alpha$$

Note that the sum converges in the $\hbar$-adic topology since, for any $v \in \mathcal{V}$ and $n \geq 0$, $\mathcal{K}^+_\alpha v \in \hbar^n\mathcal{V}$ for all but finitely many $\alpha \in \Delta_+$.

The Casimir connection for a semisimple Lie algebra was discovered by De Concini around '95 (unpublished, though the connection is referenced in [Pro96]) and, independently, Millson–Toledano Laredo [TL02, MTL05] and Felder–Markov–Tarasov–Varchenko [FMTV00]. In [FMTV00], the case of an arbitrary symmetrisable Kac–Moody algebra is considered.

The connection $\nabla_\mathcal{K}$ is flat (see [FMTV00] and [ATL15, Thm. 3.4]) and therefore yields a monodromy representation

$$\mathscr{P} : \mathcal{P}_W = \mathbf{\Pi}_1(\mathsf{X}; x_0) \to \mathsf{GL}(\mathcal{V})$$

Moreover, since the coefficients of $\nabla_\mathcal{K}$ have weight zero, the action of $\mathcal{P}_W$ preserves the generalised weight spaces of $\mathcal{V}$.

This is more conveniently expressed in terms of completions. Let $\mathsf{f} : \mathcal{O}^\hbar_{\infty,\mathfrak{g}} \to \mathrm{Vect}_\hbar$ be the forgetful functor. Then, the monodromy of $\nabla_\mathcal{K}$ yields an action

$$\mathscr{P} : \mathcal{P}_W = \mathbf{\Pi}_1(\mathsf{X}; x_0) \to \mathrm{Aut}(\mathsf{f})$$

5.3. The orbifold fundamental groupoid of X

Let $\mathbf{\Pi}_1(\mathsf{X}; Wx_0)$ be the fundamental groupoid of X based at the W-orbit of x_0. Then, $\mathbf{\Pi}_1(\mathsf{X}/W; [x_0])$ is equivalent to the orbifold fundamental groupoid $W \ltimes \mathbf{\Pi}_1(\mathsf{X}; Wx_0)$, which is defined as follows.

- Its set of objects is Wx_0.
- A morphism between $x, y \in Wx_0$ is a pair (w, γ), where $w \in W$ and γ is a path in X from x to $w^{-1}y$.
- The composition of $(w, \gamma) : x \to y$ and $(w', \gamma') : y \to z$ is given by

$$(w', \gamma') \circ (w, \gamma) = (w'w, w^{-1}(\gamma') \circ \gamma) : x \to z$$

The projection functor

$$P : W \ltimes \mathbf{\Pi}_1(\mathsf{X}; Wx_0) \longrightarrow \mathbf{\Pi}_1(\mathsf{X}/W; [x_0]) \tag{5.1}$$

given by $P(wx_0) = [x_0]$ and $P(w, \gamma) = [\gamma]$ is fully faithful since, for any given $x, y \in Wx_0$, a loop $[\gamma] \in \mathbf{\Pi}_1(\mathsf{X}/W; [x_0])$ lifts uniquely to a path $\gamma : x \to w^{-1}y$, for a unique $w \in W$. Any $x \in Wx_0$ therefore determines a right inverse $\mathcal{E}_x$ of P given by $\mathcal{E}_x([x_0]) = x$ and $\mathcal{E}_x([\gamma]) = (w, \gamma)$, where γ is the lift of $[\gamma]$ through x, and w is such that $\gamma(1) = w^{-1}x$.

5.4. Obstruction to W-equivariance [ATL15, Sec. 4]

Extend the monodromy of $\nabla_{\mathcal{K}}$ to $\mathbf{\Pi}_1(\mathsf{X}; Wx_0)$, and lift it to a map $\mathscr{P} : \mathbf{\Pi}_1(\mathsf{X}; Wx_0) \to \mathcal{T}_{\mathfrak{g}}$, where $\mathcal{T}_{\mathfrak{g}}$ is the holonomy algebra of the root arrangement of $\mathfrak{g}$. The lack of W-equivariance of $\nabla_{\mathcal{K}}$ can then be described by the 1-cocycle

$$\mathscr{A} : W \to \mathrm{Hom}(\mathbf{\Pi}_1(\mathsf{X}; Wx_0), \mathcal{T}_{\mathfrak{g}})$$

defined by $\mathscr{A}_w(\gamma) = \mathscr{P}(\gamma)^{-1} \cdot w^{-1}\mathscr{P}(w\gamma)$.

The following summarises the main properties of $\mathscr{A}$.

Theorem.

(1) $\mathscr{A}$ is abelian, that is takes values in $\mathsf{M} = \mathrm{Hom}(\mathbf{\Pi}_1(\mathsf{X}; Wx_0), \exp(\hbar\mathfrak{h}))$.

(2) *$\mathscr{A}$ is a coboundary, that is* $\mathscr{A}_w = d\mathscr{B}_w = \mathscr{B}\cdot(w^{-1}\mathscr{B})^{-1}$ *for some* $\mathscr{B}\in\mathsf{M}$*, and any* $w\in W$*.*
(3) *The cochain $\mathscr{B}$ can be normalised so that* $\mathscr{B}(\gamma_i) = \exp(\hbar a_i t_{\alpha_i})$ *for any given choice of* $\{a_i\}_{i\in\mathbf{I}}\subset\mathbb{C}$*, and is then unique.*

Remark. (1) follows from the fact that $w^{-1}\mathscr{P}(w\gamma)$ is the parallel transport of

$$w^*\nabla_{\mathcal{K}} = \nabla_{\mathcal{K}} - \mathsf{h}a_w \qquad \text{where} \qquad a_w = \sum_{\substack{\alpha\in\Delta_+:\\ w\alpha\in\Delta_-}} \frac{d\alpha}{\alpha}\cdot t_\alpha$$

Since $\nabla_{\mathcal{K}}$ and the $\mathfrak{h}$-valued 1-form a_w commute, $\mathscr{A}_w$ is the parallel transport of $d-\mathsf{h}a_w$, and in particular takes values in M.

5.5. Equivariant monodromy [ATL15, Sec. 4]

For any $b\in\mathcal{B}_W$, let $\tau(b)\in\mathrm{Aut}(\mathsf{f}^{\mathrm{int}})$ be its action by the triple exponentials (2.3), and $\widetilde{b}\in\mathbf{\Pi}_1(\mathsf{X};Wx_0)$ the unique lift of b through x_0. The following is a direct consequence of Theorem 5.4

Theorem. *There is a unique morphism* $\mathscr{B}:\mathbf{\Pi}_1(\mathsf{X};Wx_0)\longrightarrow\exp(\hbar\mathfrak{h})$ *such that*

(1) *The assignment*

$$\mathscr{P}_{\tau,\mathscr{B}}:\mathcal{B}_W\to\mathrm{Aut}(\mathsf{f}^{\mathrm{int}}) \qquad\qquad \mathscr{P}_{\tau,\mathscr{B}}(b) = \tau(b)\cdot\mathscr{P}(\widetilde{b})\cdot\mathscr{B}(\widetilde{b})$$

is a group homomorphism.
(2) *For any* $i\in\mathbf{I}$, $\mathscr{B}(\gamma_i) = \exp(\hbar t_{\alpha_i}/4)$.

Remarks.

- The normalisation of $\mathscr{B}(\gamma_i)$ is chosen so that, if $\mathfrak{g}=\mathfrak{sl}_2$ with simple root α_i,

$$\mathscr{P}_{\tau,\mathscr{B}}(S_i) = \widetilde{s}_i\cdot\exp(\hbar\mathcal{K}^+_{\alpha_i}/2)\cdot\exp(\hbar t_{\alpha_i}/4) = \widetilde{s}_i\cdot\exp(\hbar\mathcal{K}_{\alpha_i}/4) \tag{5.2}$$

  where $\mathcal{K}_{\alpha_i} = e_if_i + f_ie_i$ is the truncated Casimir of $\mathfrak{sl}_2$.
- We shall refer to $\mathscr{P}_{\tau,\mathscr{B}}$ as the *monodromy action* of $\mathcal{B}_W$. This is justified by the fact that, when $\mathfrak{g}$ is of finite or affine type, $\mathscr{B}$ is the monodromy of the connection $d-\mathsf{h}A$, where A is a resummation of the formal abelian 1-form

$$\widehat{A} = \frac{1}{2}\sum_{\alpha\in\Delta_+}\frac{d\alpha}{\alpha}\cdot m_\alpha t_\alpha$$

(cf. [ATL15, Prop. 4.9 and Appendix A]). Thus, in these cases, $\mathscr{P}_{\tau,\mathscr{B}}$ is the monodromy of the pushdown of the connection $\nabla_{\mathcal{K}} - \mathsf{h}A$ to the quotient X/W.

5.6. Monodromy action of the pure braid group on category $\mathcal{O}_\infty$

Let

$$\varepsilon : \mathcal{P}_W^{\mathrm{ab}} \to \mathrm{Aut}(\mathsf{f}^{\mathrm{int}}) \qquad \varepsilon(p_\alpha) = \exp(\iota\pi h_\alpha) \tag{5.3}$$

be the sign character (cf. 4.3), $\mathsf{f} : \mathcal{O}^\hbar_{\infty,\mathfrak{g}} \to \mathrm{Vect}_\hbar$ the forgetful functor, and lift ε to a morphism $\mathcal{P}_W^{\mathrm{ab}} \to \mathrm{Aut}(\mathsf{f})$ as in 4.4.

Proposition. *The following holds.*

1. *For any $\alpha \in \Delta_+^{\mathrm{re}}$, $\tau(p_\alpha) = \varepsilon(p_\alpha)$ and $\mathscr{B}(p_\alpha) = \exp(\hbar t_\alpha/2)$.*
2. *The restriction of $\mathscr{P}_{\tau,\mathscr{B}}$ to $\mathcal{P}_W$ lifts to an action*

$$\mathscr{P}_{\varepsilon,\mathscr{B}} : \mathcal{P}_W \to \mathrm{Aut}(\mathsf{f}) \qquad \textit{given by} \qquad \mathscr{P}_{\varepsilon,\mathscr{B}}(p) = \varepsilon(p)\cdot\mathscr{P}(p)\cdot\mathscr{B}(p)$$

Proof. (1) For any $i \in \mathbf{I}$, $\tau(S_i^2) = \widetilde{s}_i^2 = \exp(\iota\pi h_i)$ so that, for any $w \in W$ such that $w\alpha_i > 0$, $\tau(S_w S_i^2 S_w^{-1}) = \exp(\iota\pi h_{w\alpha_i})$. Thus, $\tau(p) = \varepsilon(p)$ for any $p \in \mathcal{P}_W$.

For the second identity, it is enough to verify the relation on the loops $p_{w\alpha_i} = w(p_{\alpha_i}) \in \mathbf{\Pi}_1(\mathsf{X}; wx_0)$, where $p_{\alpha_i} = s_i(\gamma_i)\circ\gamma_i$, for $i \in \mathbf{I}$, and $w \in W$ is such that $w\alpha_i > 0$ (cf. Section 5.1). For $w = \mathrm{id}$, one has

$$\mathscr{B}(p_{\alpha_i}) = \mathscr{B}(s_i(\gamma_i))\mathscr{B}(\gamma_i) = s_i(\mathscr{A}_{s_i}(\gamma_i)^{-1}\mathscr{B}(\gamma_i))\mathscr{B}(\gamma_i) = s_i(\mathscr{A}_{s_i}(\gamma_i))^{-1}$$

where the second equality follows from $\mathscr{A} = d\mathscr{B}$, and the third one from $\mathscr{B}(\gamma_i) \in \exp(\mathbb{C}\hbar t_{\alpha_i})$. By Remark 5.4, $\mathscr{A}_v$ is the parallel transport of the abelian connection

$$d - \mathsf{h}\sum_{\substack{\alpha\in\Delta_+:\\ v\alpha\in\Delta_-}} \frac{d\alpha}{\alpha}\cdot t_\alpha \tag{5.4}$$

For $v = s_i$, this is $d - \mathsf{h}d\log\alpha_i \cdot t_{\alpha_i}$, so that $\mathscr{A}_{s_i}(\gamma_i) = \exp(\hbar t_{\alpha_i}/2)$.

For $w \neq \mathrm{id}$, one has

$$\mathscr{B}(w(p_{\alpha_i})) = w(\mathscr{A}_w(p_{\alpha_i})^{-1}\mathscr{B}(p_{\alpha_i})) = w(\mathscr{A}_w(p_{\alpha_i}))^{-1}\exp(\hbar t_{w\alpha_i}/2)$$

Note that $d\alpha/\alpha$ has a non-zero residue on the hyperplane $\alpha_i = 0$ only if $\alpha = \pm\alpha_i$. It follows from (5.4) for $v = w$, and $w\alpha_i \in \Delta_+$ that $\mathscr{A}_w(p_{\alpha_i}) = 1$, whence the result.

(2) follows from (1) and Theorem 5.5. □

6. Braided Coxeter categories

We review below the notion of *braided Coxeter category* introduced in [ATL19a]. Informally speaking, such an object is a collection of braided monoidal categories labelled by the subdiagrams of a given diagram $\mathbb{D}$ – in the relevant examples the Coxeter graph of $\mathfrak{g}$. These are equipped with relative fiber functors corresponding to the inclusions of subdiagrams and an additional combinatorial datum – a *maximal nested set* – which labels points at infinity in the De Concini–Procesi model of the Cartan subalgebra of $\mathfrak{g}$ [DCP95]. The functors corresponding to the inclusion $\emptyset \subset \mathbb{D}$ additionally carry distinguished automorphisms – the *local monodromies* – which give rise to an action of the generalised braid group $\mathcal{B}_W$.

For $U_\hbar\mathfrak{g}$, such a structure arises on $\mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar\mathfrak{g}}$ from the R-matrix and quantum Weyl group operators. For the category $\mathcal{O}^{\hbar,\mathrm{int}}_{\infty,\mathfrak{g}}$, it arises from the dynamical coupling of the KZ and Casimir connections of $\mathfrak{g}$ [TL16]. This is analogous to the fact that the monodromy of the KZ equations gives rise to a braided tensor category structure on $\mathcal{O}^{\hbar}_{\infty,\mathfrak{g}}$ [Dri89], and the fact that the canonical fundamental solutions of the Casimir equations constructed by Cherednik and De Concini–Procesi [Che89, DCP95] give rise to a Coxeter structure on $\mathcal{O}^{\hbar,\mathrm{int}}_{\infty,\mathfrak{g}}$ [TL08].

6.1. Nested sets [ATL15, Sec. 5]

A *diagram* is an undirected graph $\mathbb{D}$ with no multiple edges or loops. A *subdiagram* $B \subseteq \mathbb{D}$ is a full subgraph that is, a graph consisting of a (possibly empty) subset of vertices of $\mathbb{D}$, together with all edges of $\mathbb{D}$ joining any two elements of it.

Two subdiagrams $B_1, B_2 \subseteq \mathbb{D}$ are *orthogonal* if they have no vertices in common, and no two vertices $i_1 \in B_1$, $i_2 \in B_2$ are joined by an edge in $\mathbb{D}$. Two subdiagrams $B_1, B_2 \subseteq \mathbb{D}$ are *compatible* if either one contains the other or they are orthogonal.

A *nested set* on $\mathbb{D}$ is a collection H of pairwise compatible, connected subdiagrams of $\mathbb{D}$ which contains the empty subdiagram and the connected components of $\mathbb{D}$. We denote by $\mathsf{Mns}(\mathbb{D})$ the collections of maximal nested sets on $\mathbb{D}$.

More generally, if $B' \subseteq B \subseteq \mathbb{D}$ are two subdiagrams, a nested set on B *relative to* B' is a collection of pairwise compatible subdiagrams of B which contains the connected components of B and B', and in which every element is compatible with, but not properly contained in any of the connected components of B'. We denote by $\mathsf{Mns}(B, B')$ the collections of maximal nested sets on B relative to B'.

Remark. It is well-known that when $\mathbb{D}$ is a diagram of type A_{n-1}

$$\overset{1}{\bullet} \text{---} \overset{2}{\bullet} \text{---} \cdots \text{---} \overset{n-2}{\bullet} \text{---} \overset{n-1}{\bullet}$$

maximal nested sets on $\mathbb{D}$ are in bijection with complete bracketings on the non-associative monomial $x_1 x_2 \cdots x_n$. Specifically, for any $1 \leqslant i \leqslant j \leqslant n$, the connected subdiagram $[i, j] \subseteq \mathbb{D}$ corresponds to the brackets

$$x_1 \cdots (x_i \cdots x_{j+1}) \cdots x_n$$

and two subdiagrams $B_1, B_2 \subseteq \mathbb{D}$ are compatible if and only if the corresponding brackets are consistent. Similarly, maximal nested sets on $\mathbb{D}$ relative to a subdiagram $B \subset \mathbb{D}$ are in bijection with *partially complete* bracketings, *i.e.*, complete except for the monomials $(x_i \cdots x_{j+1})$, where $[i, j]$ is a connected component of B.

6.2. Braided Coxeter categories [ATL15, Sec. 9]

A *labelling* $\underline{m}$ of a diagram $\mathbb{D}$ is the assignment of an element $m_{ij} \in \{2, 3, \ldots, \infty\}$ to any pair i, j of distinct vertices of $\mathbb{D}$ such that $m_{ij} = m_{ji}$ and $m_{ij} = 2$ if i and j are orthogonal.

Let $(\mathbb{D}, \underline{m})$ be a labelled diagram. A braided Coxeter category $\mathscr{C}$ of type $(\mathbb{D}, \underline{m})$ consists of the following data

- **Diagrammatic categories.** For any subdiagram $B \subseteq \mathbb{D}$, a braided monoidal category $\mathcal{C}_B$.
- **Restriction functors.** For any pair of subdiagrams $B' \subseteq B$ and relative maximal nested set $\mathcal{F} \in \mathsf{Mns}(B, B')$, a tensor functor $F_{\mathcal{F}} : \mathcal{C}_B \to \mathcal{C}_{B'}$.[6]
- **Generalised associators.** For any pair of subdiagrams $B' \subseteq B$ and relative maximal nested sets $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$, an isomorphism of tensor functors $\Upsilon_{\mathcal{G}\mathcal{F}} : F_{\mathcal{F}} \Rightarrow F_{\mathcal{G}}$.

[6]Note that $F_{\mathcal{F}}$ is not assumed to be braided.

- **Vertical joins.** For any chain of inclusions $B'' \subseteq B' \subseteq B$, $\mathcal{F} \in \mathsf{Mns}(B, B')$, and $\mathcal{F}' \in \mathsf{Mns}(B', B'')$, an isomorphism of tensor functors $\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'} : F_{\mathcal{F}'} \circ F_{\mathcal{F}} \Rightarrow F_{\mathcal{F}' \cup \mathcal{F}}$.
- **Local monodromies.** For any vertex i of $\mathbb{D}$ with corresponding restriction functor $F_{\{i\}} : \mathcal{C}_i \to \mathcal{C}_\emptyset$, a distinguished automorphism $S_i \in \mathrm{Aut}(F_{\{i\}})$.[7]

These data are assumed to satisfy the following properties.

- **Normalisation.** If $\mathcal{F} = \{B\}$ is the unique element in $\mathsf{Mns}(B, B)$, then $F_{\mathcal{F}} = \mathrm{id}_{\mathcal{C}_B}$ with the trivial tensor structure.
- **Transitivity.** For any $B' \subseteq B$ and $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathsf{Mns}(B, B')$, $\Upsilon_{\mathcal{H}\mathcal{F}} = \Upsilon_{\mathcal{H}\mathcal{G}} \circ \Upsilon_{\mathcal{G}\mathcal{F}}$ as isomorphisms $F_{\mathcal{F}} \Rightarrow F_{\mathcal{H}}$. In particular, $\Upsilon_{\mathcal{F}\mathcal{F}} = \mathrm{id}_{F_{\mathcal{F}}}$ and $\Upsilon_{\mathcal{G}\mathcal{F}} = \Upsilon^{-1}_{\mathcal{F}\mathcal{G}}$.
- **Associativity.** For any $B''' \subseteq B'' \subseteq B' \subseteq B$, $\mathcal{F} \in \mathsf{Mns}(B, B')$, $\mathcal{F}' \in \mathsf{Mns}(B', B'')$, and $\mathcal{F}'' \in \mathsf{Mns}(B'', B''')$,
$$\mathsf{a}^{\mathcal{F}' \cup \mathcal{F}}_{\mathcal{F}''} \cdot \mathsf{a}^{\mathcal{F}}_{\mathcal{F}'} = \mathsf{a}^{\mathcal{F}}_{\mathcal{F}'' \cup \mathcal{F}'} \cdot \mathsf{a}^{\mathcal{F}'}_{\mathcal{F}''}$$
as isomorphisms $F_{\mathcal{F}''} \circ F_{\mathcal{F}'} \circ F_{\mathcal{F}} \Rightarrow F_{\mathcal{F}'' \cup \mathcal{F}' \cup \mathcal{F}}$.
- **Vertical factorisation.** For any $B'' \subseteq B' \subseteq B$, $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$ and $\mathcal{F}', \mathcal{G}' \in \mathsf{Mns}(B', B'')$,
$$\Upsilon_{(\mathcal{G}' \cup \mathcal{G})(\mathcal{F}' \cup \mathcal{F})} \circ \mathsf{a}^{\mathcal{F}}_{\mathcal{F}'} = \mathsf{a}^{\mathcal{G}}_{\mathcal{G}'} \circ \begin{pmatrix} \Upsilon_{\mathcal{G}\mathcal{F}} \\ \circ \\ \Upsilon_{\mathcal{G}'\mathcal{F}'} \end{pmatrix}$$
as isomorphisms $F_{\mathcal{F}'} \circ F_{\mathcal{F}} \Rightarrow F_{\mathcal{G}'} \circ F_{\mathcal{G}}$.
- **Generalised braid relations.** For any $B \subseteq \mathbb{D}$, $i \neq j \in B$ and maximal nested sets $\mathcal{K}[i], \mathcal{K}[j]$ on B such that $\{i\} \in \mathcal{K}[i]$, $\{j\} \in \mathcal{K}[j]$, the following holds in $\mathrm{Aut}\, F_{\mathcal{K}[i]}$
$$\underbrace{\mathsf{Ad}\,(\Upsilon_{ij})\,(S^{\mathsf{a}}_j) \cdot S^{\mathsf{a}}_i \cdot \mathsf{Ad}\,(\Upsilon_{ij})\,(S^{\mathsf{a}}_j) \cdots}_{m_{ij}} = \underbrace{S^{\mathsf{a}}_i \cdot \mathsf{Ad}\,(\Upsilon_{ij})\,(S^{\mathsf{a}}_j) \cdot S^{\mathsf{a}}_i \cdots}_{m_{ij}}$$
where $\Upsilon_{ij} = \Upsilon_{\mathcal{K}[i]\mathcal{K}[j]}$ and $S^{\mathsf{a}}_i = \mathrm{Ad}\, \mathsf{a}^{\mathcal{K}[i]_i}_{\mathcal{K}[i]^i}(S_i) \in \mathrm{Aut}\, F_{\mathcal{K}[i]}$.[8]
- **Coproduct identity.** For any $i \in D$, the following holds in $\mathrm{Aut}(F_{\{i\}} \otimes F_{\{i\}})$
$$J_i^{-1} \circ F_{\{i\}}(c_i) \circ \Delta(S_i) \circ J_i = c_\emptyset \circ (S_i \otimes S_i) \tag{6.1}$$

[7]Note that S_i is not assumed to be a tensor automorphism of $F_{\{i\}}$.
[8]$\mathcal{K}[i]_i$ and $\mathcal{K}[i]^i$ denote the truncations of $\mathcal{K}[i]$ at $\{i\}$.

where J_i is the tensor structure on $F_{\{i\}}$ and $c_i, c_\emptyset$ are the opposite braidings in $\mathcal{C}_i$ and $\mathcal{C}_\emptyset$, respectively.[9]

6.3. Representations of braid groups

Let $\mathcal{B}_{\mathbb{D}}^{\underline{m}}$ be the braid group with generators S_i, $i \in \mathbb{D}$, and relations (2.2) for the labelling $\underline{m}$. Let $\mathcal{B}_B^{\underline{m}} \leqslant \mathcal{B}_{\mathbb{D}}^{\underline{m}}$ be the subgroup generated by S_i with $i \in B$. Finally, let $\mathcal{B}_n$ be the braid group associated to the symmetric group $\mathfrak{S}_n$, with generators $T_1, \dots, T_{n-1}$, and br_n the set of complete bracketings on the non-commutative monomial $x_1 x_2 \cdots x_n$.

Let $\mathscr{C} = (\mathcal{C}_B, F_{\mathcal{F}}, \Upsilon_{\mathcal{F}\mathcal{G}}, \mathsf{a}_{\mathcal{F}'}^{\mathcal{F}}, S_i)$ be a braided Coxeter category. Then, there is a family of representations

$$\lambda_{\mathcal{F},b}^{\mathscr{C}} : \mathcal{B}_B^{\underline{m}} \times \mathcal{B}_n \to \mathrm{Aut}(F_{\mathcal{F}}^{\boxtimes n})$$

labelled by $B \subseteq \mathbb{D}$, $\mathcal{F} \in \mathsf{Mns}(B)$, and $b \in \mathsf{br}_n$, which is uniquely determined by the conditions

- $\lambda_{\mathcal{F},b}^{\mathscr{C}}(S_i) = \mathrm{Ad}(\mathsf{a}_{\mathcal{F}^i}^{\mathcal{F}_i})(S_i)_{1\dots n}$ if $\{i\} \in \mathcal{F}$ and $\lambda_{\mathcal{G},b}^{\mathscr{C}} = \mathrm{Ad}(\Upsilon_{\mathcal{G}\mathcal{F}})_{1\dots n} \circ \lambda_{\mathcal{F},b}^{\mathscr{C}}$.
- $\lambda_{\mathcal{F},b}^{\mathscr{C}}(T_i) = R_{B,i,i+1}^{\vee}$ if $b = x_1 \cdots (x_i x_{i+1}) \cdots x_n$ and $\lambda_{\mathcal{F},b'}^{\mathscr{C}} = \mathrm{Ad}(\Phi_{B,b'b}) \circ \lambda_{\mathcal{F},b}^{\mathscr{C}}$, where Φ_B and $R_B^{\vee}$ are the associativity and commutativity constraints of $\mathcal{C}_B$.

6.4. Equivalence of braided Coxeter categories

Let $\mathscr{C}$, $\mathscr{C}'$ be two braided Coxeter categories of type $(\mathbb{D}, \underline{m})$. An equivalence $\mathbf{H} : \mathscr{C} \to \mathscr{C}'$ is the data of

- For any $B \subseteq \mathbb{D}$, a braided tensor equivalence $H_B : \mathcal{C}_B \to \mathcal{C}'_B$
- For any $B' \subseteq B$ and $\mathcal{F} \in \mathsf{Mns}(B, B')$, an isomorphism $\gamma_{\mathcal{F}}$ of tensor functors

$$\begin{tikzcd} \mathcal{C}_B \arrow[r, "H_B"] \arrow[d, "F_{\mathcal{F}}"'] & \mathcal{C}'_B \arrow[d, "F'_{\mathcal{F}}"] \arrow[dl, Rightarrow, "\gamma_{\mathcal{F}}"] \\ \mathcal{C}_{B'} \arrow[r, "H_{B'}"'] & \mathcal{C}'_{B'} \end{tikzcd} \tag{6.2}$$

These are required to preserve the generalised associators, vertical joins, and local monodromies.

[9] Given a braided monoidal category with braiding β, we set $\beta_{X,Y}^{\mathrm{op}} := \beta_{Y,X}^{-1}$.

- For any $B' \subseteq B \subseteq \mathbb{D}$ and $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$,

$$\Upsilon_{\mathcal{G}\mathcal{F}} \circ \gamma_{\mathcal{F}} = \gamma_{\mathcal{G}} \circ \Upsilon'_{\mathcal{G}\mathcal{F}}$$

  as isomorphisms $F'_{\mathcal{F}} \circ H_B \Rightarrow H_{B'} \circ F_{\mathcal{G}}$.
- For any $B'' \subseteq B' \subseteq B \subseteq \mathbb{D}$, $\mathcal{F} \in \mathsf{Mns}(B, B')$, and $\mathcal{F}' \in \mathsf{Mns}(B', B'')$,

$$\gamma_{\mathcal{F}'\cup\mathcal{F}} \circ (\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'})' = \mathsf{a}^{\mathcal{F}}_{\mathcal{F}'} \circ \begin{pmatrix} \gamma_{\mathcal{F}} \\ \circ \\ \gamma_{\mathcal{F}'} \end{pmatrix}$$

  as isomorphisms $F'_{\mathcal{F}'} \circ F'_{\mathcal{F}} \circ H_B \Rightarrow H_{B'} \circ F_{\mathcal{F}\cup\mathcal{F}'}$.
- For any $i \in \mathbb{D}$, $S_i \circ \gamma_{\emptyset i} = \gamma_{\emptyset i} \circ S'_i$ as isomorphisms $F'_i \circ H_i \Rightarrow H_\emptyset \circ F_i$.

Let $\mathbf{H} : \mathscr{C} \to \mathscr{C}'$ be an equivalence of braided Coxeter categories. Then, the representations of the braid groups $\lambda^{\mathscr{C}}_{\mathcal{F},b}$ and $\lambda^{\mathscr{C}'}_{\mathcal{F},b}$ are equivalent through the natural isomorphism $\gamma_{\mathcal{F}} : F'_{\mathcal{F}} \circ H_B \Rightarrow F_{\mathcal{F}}$.

6.5. The braided Coxeter category $\mathscr{O}^{\mathrm{int}}_{\boldsymbol{U_\hbar\mathfrak{g}},\mathbf{R},\mathbf{S}}$

Let now A be a symmetrisable generalised Cartan matrix, $(\mathfrak{h}, \Pi, \Pi^\vee)$ a realisation of A, $\mathfrak{g}$ the corresponding Kac–Moody algebra and $\mathbb{D}$ its Dynkin diagram with the standard labelling (2.2), thus $\mathcal{B}^m_{\mathbb{D}} = \mathcal{B}_W$. To simplify the exposition, we assume that A is of finite or affine type, and $\mathfrak{h}$ is its minimal realisation.

For any proper subdiagram $B \subsetneq \mathbb{D}$, we denote by $\mathfrak{g}_B \subsetneq \mathfrak{g}$ the subalgebra generated by $\{e_i, f_i, h_i\}_{i\in B}$, and set $\mathfrak{g}_{\mathbb{D}} = \mathfrak{g}$.[10] Similarly, we denote by $U_\hbar\mathfrak{g}_B \subsetneq U_\hbar\mathfrak{g}$ the subalgebra topologically generated by $\{E_i, F_i, h_i\}_{i\in B}$, and set $U_\hbar\mathfrak{g}_{\mathbb{D}} = U_\hbar\mathfrak{g}$.

Then, the braided Coxeter category $\mathscr{O}^{\mathrm{int}}_{U_\hbar\mathfrak{g},\mathbf{R},\mathbf{S}}$ is given by the following data.

- The diagrammatic category corresponding to $B \subseteq \mathbb{D}$ is the monoidal category $\mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar\mathfrak{g}_B}$, with braiding induced by the universal R-matrix $\mathbf{R}_B$ of $U_\hbar\mathfrak{g}_B$.
- For any $B' \subseteq B$ and $\mathcal{F} \in \mathsf{Mns}(B, B')$, $F_{\mathcal{F}}$ is the restriction functor $\mathrm{Res}^\hbar_{B'B} : \mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar\mathfrak{g}_B} \to \mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar\mathfrak{g}_{B'}}$ with the trivial tensor structure.
- The generalised associators and vertical joins are trivial.

[10]Since A is assumed to be of finite or affine type, $\mathfrak{g}_B = \mathfrak{g}'_B$ is the Kac–Moody algebra corresponding to the Cartan submatrix A_B. For a general A, the definition of $\mathfrak{g}_B$ and $U_\hbar\mathfrak{g}_B$ requires a realisation which is *diagrammatic* in the sense of [ATL15, Sect. 2.4].

- The local monodromy corresponding to $i \in \mathbb{D}$ is the quantum Weyl group operator $\mathbf{S}_i \in \mathrm{Aut}(\mathsf{f}^{\mathrm{int}}_{\hbar,i})$.

Remarks.

(1) The braided Coxeter structure on $\mathscr{O}^{\mathrm{int}}_{U_\hbar\mathfrak{g},\mathbf{R},\mathbf{S}}$ is particularly simple in that the restriction functors, the generalised associators, and the vertical join do not depend upon the choice of a maximal nested set $\mathcal{F} \in \mathsf{Mns}(B, B')$, but only on the subdiagrams $B' \subseteq B$.

(2) The category $\mathscr{O}^{\mathrm{int}}_{U_\hbar\mathfrak{g},\mathbf{R},\mathbf{S}}$ gives rise to a single representation of the braid group $\mathcal{B}_W$ (independent of $\mathcal{F}$) which is the quantum Weyl group action $\rho : \mathcal{B}_W \to \mathrm{Aut}(\mathsf{f}^{\mathrm{int}}_\hbar)$ from Section 4.2.

(3) Strictly speaking, for the coproduct identity (6.1) to hold, it is necessary to consider a Cartan correction of the quantum Weyl group operator $\mathbf{S}_i$ (cf. [ATL15, Sec. 17.3]). For simplicity, we shall gloss over this technical detail and refer the reader to [ATL15].

6.6. The braided Coxeter category $\mathscr{O}^{\hbar,\mathrm{int}}_{\mathfrak{g},\nabla}$

In [ATL15, Sec. 16], we defined a braided Coxeter category $\mathscr{O}^{\mathrm{int}}_{\mathfrak{g},\nabla}$ which underlies the equivariant monodromy of the Casimir connection, together with that of the KZ equations for all the subalgebras $\mathfrak{g}_B \subseteq \mathfrak{g}$. In outline, $\mathscr{O}^{\mathrm{int}}_{\mathfrak{g},\nabla}$ is described as follows.

- The diagrammatic category corresponding to $B \subseteq \mathbb{D}$ is the braided monoidal category $\mathcal{O}^{\hbar,\mathrm{int}}_{\infty,\mathfrak{g}_B}$, with associativity and commutativity constraints given by the KZ associator Φ^{∇}_B and R-matrix $R^{\nabla}_B = \exp(\hbar\Omega_B/2)$, where $\Omega_B \in \mathfrak{g}_B\widehat{\otimes}\mathfrak{g}_B$ is the Casimir tensor of $\mathfrak{g}_B$, cf. [Dri90].
- For any $B' \subseteq B$ and $\mathcal{F} \in \mathsf{Mns}(B, B')$, $F_\mathcal{F}$ is the standard restriction functor $\mathsf{f}_{B'B} : \mathcal{O}^{\hbar,\mathrm{int}}_{\infty,\mathfrak{g}_B} \to \mathcal{O}^{\hbar,\mathrm{int}}_{\infty,\mathfrak{g}_{B'}}$, with tensor structure given by the relative twists $J^{\nabla}_\mathcal{F}$ constructed in [TL16], see also [ATL15, Sec. 13].
- For any $B' \subseteq B$ and $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$, the natural isomorphism of tensor functors $F_\mathcal{G} \Rightarrow F_\mathcal{F}$ is given by the De Concini–Procesi (relative) associator $\Upsilon^{\nabla}_{\mathcal{F}\mathcal{G}}$ constructed in [DCP95], see also [ATL15, Sec. 8].
- The vertical joins are trivial.
- The local monodromy corresponding to any $i \in \mathbb{D}$ is the operator (cf. (5.2))

$$S^{\nabla}_i = \widetilde{s}_i \cdot \exp(\hbar\mathcal{K}_{\alpha_i}/4) \tag{6.3}$$

Remark. Contrary to the local monodromies S^{∇}_i, the data $(\Phi^{\nabla}_B, R^{\nabla}_B, J^{\nabla}_\mathcal{F}, \Upsilon^{\nabla}_{\mathcal{F}\mathcal{G}})$ acts on category $\mathcal{O}_\infty$ modules. By replacing the diagrammatic categories

$\mathcal{O}^{\hbar,\mathrm{int}}_{\infty,\mathfrak{g}_B}$ with $\mathcal{O}^{\hbar}_{\infty,\mathfrak{g}_B}$ and excluding the S_i^{∇}, one obtains a braided *pre*-Coxeter category $\mathscr{O}^{\hbar}_{\mathfrak{g},\nabla}$ [ATL15, Sec. 15].

In 6.7–6.9, we briefly outline the construction of the relative De Concini–Procesi associators $\Upsilon^{\nabla}_{\mathcal{F}\mathcal{G}}$ and the relative twists $J^{\nabla}_{\mathcal{F}}$.

6.7. Monodromy data of the Casimir connection

Following Cherednik [Che89, Che91] and De Concini–Procesi [DCP95] (see also [ATL15, Sec. 8]), for any $\mathcal{F} \in \mathsf{Mns}(\mathbb{D})$, there is a canonical universal solution $G_{\mathcal{F}}$ of $\nabla_{\mathcal{K}}$ valued in $\mathrm{Aut}(\mathsf{f})$. It is uniquely determined by its prescribed asymptotics on a point at infinity $\mathsf{p}_{\mathcal{F}}$ corresponding to a choice of blow-up coordinates on X associated to $\mathcal{F}$.

For any $\mathcal{F},\mathcal{G} \in \mathsf{Mns}(\mathbb{D})$, the *De Concini–Procesi associator* $\Upsilon^{\nabla}_{\mathcal{F}\mathcal{G}}$ is the element of $\mathrm{Aut}(\mathsf{f})$ defined by

$$G_{\mathcal{G}}(x) = G_{\mathcal{F}}(x) \cdot \Upsilon^{\nabla}_{\mathcal{F}\mathcal{G}}$$

where x lies in the fundamental Weyl chamber. The datum of the De Concini–Procesi associators yields a combinatorial description of the equivariant monodromy of $\nabla_{\mathcal{K}}$ as follows (cf. [ATL15, Thm. 9.3]). Let S_i^{∇} be given by (6.3). Then, there is a family of representations

$$\mu_{\mathcal{F}} : \mathcal{B}_W \to \mathrm{Aut}(\mathsf{f}^{\mathrm{int}})$$

labelled by $\mathcal{F} \in \mathsf{Mns}(\mathbb{D})$, which is uniquely determined by the conditions

- $\mu_{\mathcal{F}}(S_i) = S_i^{\nabla}$ if $\{i\} \in \mathcal{F}$
- $\mu_{\mathcal{G}} = \mathrm{Ad}(\Upsilon_{\mathcal{G}\mathcal{F}}) \circ \mu_{\mathcal{F}}$

The representation $\mu_{\mathcal{F}}$ is the equivariant monodromy of $\nabla_{\mathcal{K}}$ computed with respect to the fundamental solution $G_{\mathcal{F}}$.

6.8. Generalised associators

For any $B \subseteq \mathbb{D}$, one similarly obtains the associators $\Upsilon^{\nabla}_{\mathcal{F}\mathcal{G}} \in \mathrm{Aut}(\mathsf{f}_B)$ with $\mathcal{F},\mathcal{G} \in \mathsf{Mns}(B)$ which, together with the local monodromies $\{S_i^{\nabla}\}_{i\in B}$, describe the equivariant monodromy of the Casimir connection of $\mathfrak{g}_B$. These associators are related to those for $\mathfrak{g}$ as follows. Let $\mathcal{H} \in \mathsf{Mns}(\mathbb{D}, B)$ and $\mathcal{F},\mathcal{G} \in \mathsf{Mns}(B,\emptyset)$. Then, [DCP95, Thm. 3.6] implies that

$$\Upsilon^{\nabla}_{\mathcal{H}\cup\mathcal{G}\,\mathcal{H}\cup\mathcal{F}} = \iota_{\mathbb{D}B}(\Upsilon^{\nabla}_{\mathcal{G}\mathcal{F}}) \tag{6.4}$$

where $\iota_{\mathbb{D}B} : \mathrm{End}(\mathsf{f}_B) \to \mathrm{End}(\mathsf{f}_{\mathbb{D}})$ is induced by the equality $\mathsf{f}_{\mathbb{D}} = \mathsf{f}_{\mathbb{D}B} \circ \mathsf{f}_B$.

The relative associators corresponding to an inclusion $B' \subseteq B$ are constructed as follows. Let $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$, choose $\mathcal{H} \in \mathsf{Mns}(B', \emptyset)$, and set

$$\Upsilon^{\nabla}_{\mathcal{G}\mathcal{F}} = \Upsilon^{\nabla}_{\mathcal{G}\cup\mathcal{H}\,\mathcal{F}\cup\mathcal{H}}$$

One then proves that the definition is independent of the choice of $\mathcal{H}$, and that $\Upsilon^{\nabla}_{\mathcal{G}\mathcal{F}}$ centralises $\mathfrak{g}_{B'}$ [DCP95, Thm. 3.6], and therefore can be thought of as an automorphism of the restriction functor $\mathsf{f}_{B'B} : \mathcal{O}^{\hbar}_{\infty,\mathfrak{g}_B} \to \mathcal{O}^{\hbar}_{\infty,\mathfrak{g}_{B'}}$.

These associators satisfy the vertical factorisation since if $B'' \subseteq B' \subseteq B$, $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$, $\mathcal{F}', \mathcal{G}' \in \mathsf{Mns}(B', B'')$,

$$\Upsilon^{\nabla}_{\mathcal{G}\cup\mathcal{G}'\,\mathcal{F}\cup\mathcal{F}'} = \Upsilon^{\nabla}_{\mathcal{G}\cup\mathcal{G}'\,\mathcal{G}\cup\mathcal{F}'} \cdot \Upsilon^{\nabla}_{\mathcal{G}\cup\mathcal{F}'\,\mathcal{F}\cup\mathcal{F}'} = \iota_{BB'}(\Upsilon^{\nabla}_{\mathcal{G}'\mathcal{F}'}) \cdot \Upsilon^{\nabla}_{\mathcal{G}\mathcal{F}}$$

where the second equality follows from (6.4) and the definition of $\Upsilon^{\nabla}_{\mathcal{G}\mathcal{F}}$.

6.9. Monodromy data of the joint KZ-Casimir system

The tensor structures $\{J^{\nabla}_{\mathcal{F}}\}_{\mathcal{F}\in\mathsf{Mns}(\mathbb{D})}$ on the forgetful functor $\mathsf{f} = \mathsf{f}_{\mathbb{D}}$ are obtained from the *dynamical* KZ equations in $n = 2$ points

$$d - \left(\mathsf{h}\frac{\Omega}{z} + \mu^{(1)}\right) dz \tag{6.5}$$

where $z = z_1 - z_2$, $\mu \in \mathfrak{h}$ and $\mu^{(1)} = \mu \otimes 1$ as follows.

These admit a canonical solution G_0 which is asymptotic to $z^{\mathsf{h}\Omega}$ near $z = 0$. If μ is regular and real, they also admit two canonical solutions $G_{\pm}$ which are asymptotic to $z^{\mathsf{h}\Omega_0} \cdot \exp(z\mu^{(1)})$ as $z \to \infty$ with $\mathrm{Im}\, z \gtrless 0$, where Ω_0 is the projection of Ω onto $\mathfrak{h} \otimes \mathfrak{h}$ [TL16, Sect. 6]. Define the *differential twist* $J_{\pm}(\mu)$ by

$$J_{\pm}(\mu) = G_0^{-1}(z) \cdot G_{\pm}(z)$$

where $\mathrm{Im}\, z \gtrless 0$.

Then, $J_{\pm}(\mu)$ kills the KZ associator for $\mathfrak{g}$. As a function of $\mu \in \mathcal{C}$, where $\mathcal{C}$ is the fundamental Weyl chamber, $J_{\pm}(\mu)$ is real analytic and varies according to the Casimir equations [TL16, Sect. 7]

$$d_{\mathfrak{h}} J_{\pm} = \frac{\mathsf{h}}{2} \sum_{\alpha\in\Delta_+} \frac{d\alpha}{\alpha} \left(\Delta(\mathcal{K}^+_\alpha) J_{\pm} - J_{\pm}\left(\mathcal{K}^+_\alpha \otimes 1 + 1 \otimes \mathcal{K}^+_\alpha\right)\right)$$

It follows that, for any maximal nested set $\mathcal{F} \in \mathsf{Mns}(\mathbb{D})$, the twist

$$J^{\nabla}_{\mathcal{F}} = \Delta(G_{\mathcal{F}}(\mu))^{-1} \cdot J_{\pm}(\mu) \cdot G_{\mathcal{F}}(\mu)^{\otimes 2}$$

where $G_{\mathcal{F}}(\mu)$ is the fundamental solution of the Casimir connection corresponding to $\mathcal{F}$ (see 6.7), is independent of $\mu \in \mathcal{C}$, and a tensor structure on $\mathsf{f}_{\mathbb{D}}$.

The relative twists $J_{\mathcal{F}}^{\nabla}$ corresponding to any $B' \subseteq B$ and $\mathcal{F} \in \mathsf{Mns}(B, B')$ are obtained by relying on vertical factorisation as follows. Fix $H \in \mathsf{Mns}(B', \emptyset)$, let $F_{\mathcal{F}\cup\mathcal{H}}^{\nabla}$ and $F_{\mathcal{H}}^{\nabla}$ be the tensor structures on $\mathsf{f}_B, \mathsf{f}_{B'}$ corresponding to $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{H}$ respectively. Then, define $J_{\mathcal{F}}^{\nabla}$ by

$$\mathsf{f}_{B'}(J_{\mathcal{F}}^{\nabla}) = J_{\mathcal{F}\cup\mathcal{H}}^{\nabla} \cdot (J_{\mathcal{H}}^{\nabla})^{-1}$$

More precisely, the right-hand side is a collection of natural isomorphisms

$$\mathsf{f}_{B'}\left(\mathsf{f}_{B'B}(U) \otimes \mathsf{f}_{B'B}(V)\right) \to \mathsf{f}_B(U \otimes V) = \mathsf{f}_{B'}\left(\mathsf{f}_{B'B}(U \otimes V)\right)$$

defined for any $U, V \in \mathcal{O}_{\infty,\mathfrak{g}_B}^{\hbar}$. One can prove that it satisfies the centraliser property, *i.e.*, commutes with the action of $\mathfrak{g}_{B'}$ [TL16, Sect. 8]. Since $\mathsf{f}_{B'}$ is faithful, it follows that it is of the form $\mathsf{f}_{B'}(J_{\mathcal{F}}^{\nabla})$ for a unique $J_{\mathcal{F}}^{\nabla}$. Moreover, the latter is independent of the choice of $\mathcal{H}$.

7. The equivariant monodromy theorem

We review in this section the main result of [ATL15], which extends that of [TL08, TL16] to the case of an arbitrary symmetrisable Kac–Moody algebra, and yields an equivalence of braided Coxeter categories $\mathscr{O}_{\mathfrak{g},\nabla}^{\hbar,\mathrm{int}} \to \mathscr{O}_{U_\hbar\mathfrak{g},\mathbf{R},\mathbf{S}}^{\mathrm{int}}$. Its proof relies on the Etingof–Kazhdan equivalence, which is briefly reviewed in 7.1–7.2.

7.1. The Etingof–Kazhdan equivalence

In [EK08, Thm. 4.2], Etingof and Kazhdan construct an equivalence of categories $\mathsf{F} : \mathcal{O}_{\infty,\mathfrak{g}}^{\hbar} \to \mathcal{O}_{\infty,U_\hbar\mathfrak{g}}$, together with an isomorphism α of functors

$$\begin{tikzcd} \mathcal{O}_{\infty,\mathfrak{g}}^{\hbar} \arrow[rr, "\mathsf{F}"] \arrow[dr, "\mathsf{f}"'] & \arrow[d, phantom, "\Leftarrow \alpha"] & \mathcal{O}_{\infty,U_\hbar\mathfrak{g}} \arrow[dl, "\mathsf{f}_\hbar"] \\ & \mathrm{Vect}_\hbar & \end{tikzcd}$$

where f and $\mathsf{f}_\hbar$ are the forgetful functors.[11] The equivalence F is the identity on $\mathfrak{h}$-modules and preserves integrability [ATL15, Lemma 22.9]. It therefore

[11] More precisely, in [EK08] Etingof–Kazhdan construct an equivalence F between the larger categories of deformation Drinfeld–Yetter modules over the negative Borel

gives rise to a diagram of functors in which every face commutes

(7.1)
$$\begin{tikzcd}
\mathcal{O}^{\hbar,\mathrm{int}}_{\infty,\mathfrak{g}} \arrow[r, "\mathsf{F}^{\mathrm{int}}"] \arrow[d] \arrow[dd, bend right=50, "{\mathsf{f}_0}^{\mathrm{int}}"'] \arrow[ddd, bend right=70, "\mathsf{f}^{\mathrm{int}}"'] & \mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar\mathfrak{g}} \arrow[d] \arrow[dd, bend left=50, "{\mathsf{f}_{\hbar,0}}^{\mathrm{int}}"] \arrow[ddd, bend left=70, "{\mathsf{f}_\hbar}^{\mathrm{int}}"] \\
\mathcal{O}^{\hbar}_{\infty,\mathfrak{g}} \arrow[r, "\mathsf{F}"] \arrow[d, "\mathsf{f}_0"] \arrow[dd, bend right=50, "\mathsf{f}"'] & \mathcal{O}_{\infty,U_\hbar\mathfrak{g}} \arrow[d, "\mathsf{f}_{\hbar,0}"'] \arrow[dd, bend left=50, "\mathsf{f}_\hbar"] \\
\mathrm{Mod}^{\hbar}_{\mathfrak{h}} \arrow[r, equal] \arrow[d, "\mathsf{f}_{\mathfrak{h}}"] & \mathrm{Mod}^{\hbar}_{\mathfrak{h}} \arrow[d, "\mathsf{f}_{\mathfrak{h}}"'] \\
\mathrm{Vect}_\hbar \arrow[r, equal] & \mathrm{Vect}_\hbar
\end{tikzcd}$$

where the vertical arrows are restriction functors, and the natural isomorphisms are either trivial or induced from α.[12]

7.2. The Etingof–Kazhdan isomorphism

In terms of completions, the Etingof–Kazhdan equivalence (F,α) gives rise to an isomorphism $\Psi:\mathrm{End}(\mathsf{f}_\hbar)\to\mathrm{End}(\mathsf{f})$ via the composition

(7.2)
$$\mathrm{End}(\mathsf{f}_\hbar)\longrightarrow\mathrm{End}(\mathsf{f}_\hbar\circ\mathsf{F})\to\mathrm{End}(\mathsf{f})$$

where the first isomorphism is induced by F, and the second is given by $\mathrm{Ad}(\alpha)$. By (7.1), Ψ restricts to an isomorphism $\Psi^{\mathrm{int}}:\mathrm{End}(\mathsf{f}_\hbar^{\mathrm{int}})\to\mathrm{End}(\mathsf{f}^{\mathrm{int}})$ such that

$$\begin{tikzcd}
\mathrm{End}(\mathsf{f}_\hbar^{\mathrm{int}}) \arrow[r, "\Psi^{\mathrm{int}}"] & \mathrm{End}(\mathsf{f}^{\mathrm{int}}) \\
\mathrm{End}(\mathsf{f}_\hbar) \arrow[u] \arrow[r, "\Psi"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{End}(\mathsf{f}) \arrow[u] \\
\mathrm{End}(\mathsf{f}_{\mathfrak{h}}) \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{End}(\mathsf{f}_{\mathfrak{h}}) \arrow[u]
\end{tikzcd}$$

subalgebra $\mathfrak{b}^-$, and admissible Drinfeld–Yetter modules over $U_\hbar\mathfrak{b}^-$ (see also [ATL18, 6.13]). It easily follows that F restricts to an equivalence $\mathcal{O}^{\hbar}_{\infty,\mathfrak{g}}\to\mathcal{O}_{\infty,U_\hbar\mathfrak{g}}$ since it is the identity on Drinfeld–Yetter $\mathfrak{h}$-modules, see [ATL15, Lemma 22.11]. By the same argument, it also restricts to an equivalence $\mathcal{O}^{\hbar}_{\mathfrak{g}}\to\mathcal{O}_{U_\hbar\mathfrak{g}}$.

[12]The categories $\mathcal{W}_\hbar$, $\mathcal{O}^{\hbar}_{\infty,\mathfrak{g}}$ and $\mathcal{O}_{\infty,U_\hbar\mathfrak{g}}$ naturally fit within the diagram (7.1), but are omitted for simplicity.

where the vertical arrows are restriction to category $\mathcal{O}_\infty$ and integrable modules.

7.3. The classical Drinfeld algebra

Let $\mathcal{D}$ be the analogue of the Drinfeld algebra $\mathcal{D}_\hbar$ for $U\mathfrak{g}[\![\hbar]\!]$ (cf. Section 3.1). Namely, for any $\beta \in \mathsf{Q}_+$, let $\mathcal{B}_\beta = \{X_{\beta,p}\}$ be a basis of $U\mathfrak{n}_\beta^+$ and $\mathcal{B} = \bigsqcup_{\beta\in\mathsf{Q}_+} \mathcal{B}_\beta$. Set

$$\mathcal{D}_0 = \left\{ \sum_{X\in\mathcal{B}} c_X X \,:\, c_X \in U\mathfrak{b}^- \right\} = \prod_{\beta\in\mathsf{Q}_+} U\mathfrak{b}^- \otimes U\mathfrak{n}_\beta^+ \supset U\mathfrak{g}$$

and $\mathcal{D} = \mathcal{D}_0[\![\hbar]\!]$. The algebra structure of $U\mathfrak{g}[\![\hbar]\!]$ extends to one on $\mathcal{D}$ and yields a chain of morphisms $U\mathfrak{g}[\![\hbar]\!] \subset \mathcal{D} \to \mathrm{End}(\mathsf{f})$. Proceeding as in Section 3 one shows that $\mathcal{D}$ embeds into $\mathrm{End}(\mathsf{f})$ and $\mathrm{End}(\mathsf{f}^{\mathrm{int}})$.

7.4. The monodromy theorem

In [ATL15, Thm. 22.1] we prove the following.

Theorem.

(1) There is a canonical equivalence of braided pre-Coxeter categories (cf. Remark 6.6)

$$\mathbf{H}_\mathfrak{g} = (H_B, \gamma_\mathcal{F}) : \mathscr{O}^\hbar_{\mathfrak{g},\nabla} \to \mathscr{O}_{U_\hbar\mathfrak{g},\mathbf{R},\mathbf{S}}$$

such that

– *for any $B \subseteq \mathbb{D}$, the equivalence H_B is the Etingof–Kazhdan functor*

$$\mathsf{F}_{\mathfrak{g}_B} : \mathcal{O}^\hbar_{\infty,\mathfrak{g}_B} \to \mathcal{O}_{\infty,U_\hbar\mathfrak{g}_B}$$

– *for any $B' \subseteq B$ and $\mathcal{F} \in \mathsf{Mns}(B,B')$, the natural isomorphism $\gamma_\mathcal{F}$ is induced by the action of an invertible weight zero element $\mathsf{g}_\mathcal{F}$ in the Drinfeld algebra of $\mathfrak{g}_B$,* i.e., *there is a commutative diagram of functors*

$$\begin{tikzcd}
 & & & \mathcal{O}_{\infty,U_\hbar\mathfrak{g}_B} \arrow[ddr, "\mathsf{f}_{\hbar,B'B}"] \arrow[dddd, dotted, "\mathsf{f}_{\hbar,B}"'] \arrow[dddll, Rightarrow, "\gamma_{\mathcal{F}}"'] & \\
\mathcal{O}^\hbar_{\infty,\mathfrak{g}_B} \arrow[urrr, "\mathsf{F}_{\mathfrak{g}_B}"] \arrow[ddr, "\mathsf{f}_{B'B}"] \arrow[dddd, bend right, "\mathsf{f}_B"'] & & & & \\
 & & & & \mathcal{O}_{\infty,U_\hbar\mathfrak{g}_{B'}} \arrow[ddl, "\mathsf{f}_{\hbar,B'}"] \\
 & \mathcal{O}^\hbar_{\infty,\mathfrak{g}_{B'}} \arrow[urrr, "\mathsf{F}_{\mathfrak{g}_{B'}}"] \arrow[ddl, "\mathsf{f}_{B'}"] \arrow[l, Rightarrow, "\mathsf{g}_{\mathcal{F}}"] & & & \\
 & & & \mathrm{Vect}_\hbar \arrow[dlll, equal] & \\
\mathrm{Vect}_\hbar & & & &
\end{tikzcd} \tag{7.3}$$

where the unmarked back face is the identity and the two unmarked lateral faces are the isomorphisms α for $\mathfrak{g}_B$ and $\mathfrak{g}_{B'}$.

(2) $\mathbf{H}_{\mathfrak{g}}$ *restricts to an equivalence of braided Coxeter categories*

$$\mathbf{H}_{\mathfrak{g}}^{\mathrm{int}} = (H_B^{\mathrm{int}}, \gamma_{\mathcal{F}}) : \mathscr{O}^{\hbar,\mathrm{int}}_{\mathfrak{g},\nabla} \to \mathscr{O}^{\mathrm{int}}_{U_\hbar\mathfrak{g},\mathbf{R},\mathbf{S}}$$

where $H_B^{\mathrm{int}} = \mathsf{F}^{\mathrm{int}}_{\mathfrak{g}_B}$.

(3) *For any* $\mathcal{F} \in \mathsf{Mns}(\mathbb{D})$, *the isomorphism*

$$\Psi_{\mathcal{F}}^{\mathrm{int}} = \mathrm{Ad}(\mathsf{g}_{\mathcal{F}}) \circ \Psi^{\mathrm{int}} : \mathrm{End}(\mathsf{f}_\hbar^{\mathrm{int}}) \to \mathrm{End}(\mathsf{f}^{\mathrm{int}})$$

intertwines the quantum Weyl group and the monodromy actions of $\mathcal{B}_W$, i.e.,

$$\begin{tikzcd}
 & \mathcal{B}_W \arrow[dl, "\lambda"'] \arrow[dr, "\mu_{\mathcal{F}}"] \arrow[d, phantom, "\circlearrowleft"] & \\
\mathrm{End}(\mathsf{f}_\hbar^{\mathrm{int}}) \arrow[rr, "\Psi_{\mathcal{F}}^{\mathrm{int}}"'] & {} & \mathrm{End}(\mathsf{f}^{\mathrm{int}})
\end{tikzcd} \tag{7.4}$$

where $\mu_{\mathcal{F}} = \mathscr{P}^{\mathcal{F}}_{\tau,\mathscr{B}}$ *denotes the monodromy action of* $\mathcal{B}_W$ *around the point at infinity in the De Concini–Procesi compactification of* X *corresponding to* $\mathcal{F}$.

Since the diagrammatic equivalences are fixed, the proof amounts to constructing suitable isomorphisms (6.2). The construction is in two steps. First, we prove that $\mathscr{O}^{\mathrm{int}}_{U_\hbar\mathfrak{g},\mathbf{R},\mathbf{S}}$ is equivalent to a braided Coxeter category $\mathscr{O}^{\hbar,\mathrm{int}}_{\mathfrak{g},\mathbf{R},\mathbf{S}}$ with diagrammatic categories $\mathcal{O}^{\hbar,\mathrm{int}}_{\infty,\mathfrak{g}_B}$, $B \subseteq \mathbb{D}$, and standard restriction functors with non-trivial tensor structures. The equivalence is given by the diagrammatic Etingof–Kazhdan functors, equipped with natural isomorphisms $\widetilde{\gamma}_{\mathcal{F}}$ whose construction is carried out in [ATL18, ATL19a]. Then, relying on the rigidity result from [ATL19b], we prove that $\mathscr{O}^{\hbar,\mathrm{int}}_{\mathfrak{g},\mathbf{R},\mathbf{S}}$ is equivalent to $\mathscr{O}^{\hbar,\mathrm{int}}_{\mathfrak{g},\nabla}$ with diagrammatic equivalences given by the identity functors. Finally, we observe that, by [ATL19a, Thm. 10.7], the resulting isomorphisms $\gamma_{\mathcal{F}}$ satisfy (7.3) for weight zero elements $\mathsf{g}_{\mathcal{F}}$ in the Drinfeld algebra.

8. The monodromy theorem in category $\mathcal{O}_\infty$

In this section, we show the equivalence of the actions of $\mathcal{P}_W$ constructed in Sections 4 and 5. The proof relies on the equivariant monodromy Theorem 7.4, the explicit description of the actions of $\mathcal{P}_W$ from Sections 4 and 5, and the following auxiliary result.

8.1. Isomorphism between Drinfeld algebras

We show below that the isomorphism $\Psi : \mathrm{End}(\mathsf{f}_\hbar) \to \mathrm{End}(\mathsf{f})$ (7.2) restricts to an isomorphism $\Psi^{\mathcal{D}} : \mathcal{D}_\hbar \to \mathcal{D}$. Our proof closely follows Etingof and Kazhdan's argument [EK08, Rem. p. 535] for the analogous algebra $\mathcal{Q}_\hbar = \lim_\beta U_\hbar\mathfrak{g}/I_\beta$ (cf. Remark 3.1), and completes their affirmative answer to a question raised by Drinfeld [Dri92, Question 8.2].[13]

For any $\beta = \sum_i k_i\alpha_i \in \mathsf{Q}_+$, define the height of β by $\mathrm{ht}\,\beta = \sum_i k_i$. For any $n \geqslant 0$, let $J_n \subseteq U_\hbar\mathfrak{g}$ be the left ideal generated by $(U_\hbar\mathfrak{n}^+)_\beta$ with $\mathrm{ht}(\beta) > n$. Set $U_\hbar^{(n)} = U_\hbar\mathfrak{g}/J_n$, and denote by $\iota^\hbar_{mn} : U_\hbar^{(n)} \to U_\hbar^{(m)}$ $(m \leqslant n)$ the natural morphisms. Their classical analogues $U^{(n)}$ and $\iota_{mn} : U^{(n)} \to U^{(m)}$ $(m \leqslant n)$ are defined similarly for $U\mathfrak{g}[\![\hbar]\!]$.

[13] The argument in [EK08] is not complete since the modules $U\mathfrak{g}/I_\beta$ are not equicontinuous for an arbitrary Kac–Moody algebra $\mathfrak{g}$, so that the Etingof–Kazhdan equivalence F cannot be applied to them. In particular, the existence of an isomorphism between $\mathcal{Q}_\hbar$ and its classical counterpart raised in [Dri92, Question 8.2] is not settled by [EK08]. Theorem 8.1 yields such an isomorphism for the algebra $\mathcal{D}_\hbar$.

Theorem.

1. *There is a canonical isomorphism of $U_\hbar\mathfrak{g}$-modules $\mathcal{D}_\hbar \simeq \lim_n U_\hbar^{(n)}$.*
2. *There is a canonical isomorphism of $U\mathfrak{g}[\![\hbar]\!]$-modules $\mathcal{D} \simeq \lim_n U^{(n)}$.*
3. *Ψ restricts to an isomorphism of algebras $\Psi^{\mathcal{D}} : \mathcal{D}_\hbar \to \mathcal{D}$.*

Proof. (1)–(2) The action of $\mathcal{D}_\hbar$ on the cyclic vector yields surjective morphisms $\phi_n : \mathcal{D}_\hbar \to U_\hbar^{(n)}$ of $U_\hbar\mathfrak{g}$-modules such that $\iota^\hbar_{mn} \circ \phi_n = \phi_m$. The corresponding morphism $\phi : \mathcal{D}_\hbar \to \lim_n U_\hbar^{(n)}$ is easily seen to be an isomorphism.

(3) The algebra structure of $\mathcal{D}_\hbar$ is encoded by the morphisms between the modules $U_\hbar^{(n)}$. Namely, we have a natural isomorphism

$$\mathcal{D}_\hbar^{\mathrm{op}} \simeq \mathrm{End}_{U_\hbar\mathfrak{g}}\left(\lim_n U_\hbar^{(n)}\right) \simeq \lim_m \operatorname*{colim}_n \mathrm{Hom}_{U_\hbar\mathfrak{g}}(U_\hbar^{(n)}, U_\hbar^{(m)})$$

(see also [App13, Appendix A.1]). A similar results holds for $\mathcal{D}$.

The module $U^{(n)}$ (resp. $U_\hbar^{(n)}$) does not lie in $\mathcal{O}^\hbar_{\infty,\mathfrak{g}}$ (resp. $\mathcal{O}_{\infty,U_\hbar\mathfrak{g}}$) since it is free over $U\mathfrak{h}[\![\hbar]\!]$. However, the fact that $U\mathfrak{n}^+_\beta v = 0$ (resp. $U_\hbar\mathfrak{n}^+_\beta v = 0$) for all but finitely many $\beta \in \mathsf{Q}_+$ for any weight vector $v \in U^{(n)}$ (resp. $v \in U_\hbar^{(n)}$) implies that $U^{(n)}$ is an equicontinuous $\mathfrak{g}$-module and therefore a Drinfeld–Yetter module over $\mathfrak{b}^-$, and that $U_\hbar^{(n)}$ is an admissible Drinfeld–Yetter module over $U_\hbar\mathfrak{b}^-$. One can therefore apply the equivalence F to $U^{(n)}$, and finds that $\mathsf{F}(U^{(n)}) = U_\hbar^{(n)}$ and $\mathsf{F}(\iota_{mn}) = \iota^\hbar_{mn}$ [EK08, Thms. 4.1–4.2]. This yields a collection of natural isomorphisms

$$\mathrm{Hom}_{U\mathfrak{g}[\![\hbar]\!]}(U^{(n)}, U^{(m)}) \simeq \mathrm{Hom}_{U_\hbar\mathfrak{g}}(U_\hbar^{(n)}, U_\hbar^{(m)})$$

and the desired isomorphism $\Psi^{\mathcal{D}} : \mathcal{D}_\hbar \to \mathcal{D}$. □

8.2. The monodromy theorem

Theorem. *The monodromy of the normally ordered Casimir connection on a $\mathfrak{g}$-module $\mathcal{V} \in \mathcal{O}^\hbar_{\infty,\mathfrak{g}}$ is canonically equivalent to the normally ordered quantum Weyl group action of the pure braid group $\mathcal{P}_W$ on the Etingof–Kazhdan quantisation $\mathsf{F}(\mathcal{V}) \in \mathcal{O}_{\infty,U_\hbar\mathfrak{g}}$.*

Proof. Let $\mathcal{F} \in \mathsf{Mns}(\mathbb{D})$. By Theorem 7.4 (3), there is a weight zero element $\mathsf{g}_\mathcal{F} \in \mathcal{D}^\times \subset \mathrm{Aut}(\mathsf{f})$ such that $\Psi^{\mathrm{int}}_\mathcal{F} = \mathrm{Ad}(\mathsf{g}_\mathcal{F}) \circ \Psi^{\mathrm{int}}$ intertwines the quantum Weyl group and the monodromy actions of $\mathcal{B}_W$, cf. (7.4). We claim that this yields a commutative diagram

$$\begin{tikzcd}
 & & \mathcal{P}_W \arrow[dll, "\lambda_{\varepsilon,\mathscr{B}}"'] \arrow[drr, "\mathscr{P}^{\mathcal{F}}"] \arrow[ddl, "\lambda_{\varepsilon,\mathscr{B}}"'] \arrow[ddr, "\mathscr{P}^{\mathcal{F}}"] & & \\
\mathrm{End}(\mathsf{f}_\hbar^{\mathrm{int}}) \arrow[rrrr, "\Psi_{\mathcal{F}}^{\mathrm{int}}"] & & & & \mathrm{End}(\mathsf{f}^{\mathrm{int}}) \\
 & \mathcal{D}_\hbar \arrow[rr, "\Psi_{\mathcal{F}}^{\mathcal{D}}"'] \arrow[dl, hook] & & \mathcal{D} \arrow[dr, hook] & \\
\mathrm{End}(\mathsf{f}_\hbar) \arrow[uu] \arrow[rrrr, "\Psi_{\mathcal{F}}"'] & & & & \mathrm{End}(\mathsf{f}) \arrow[uu]
\end{tikzcd}$$

where $\Psi_{\mathcal{F}}^{\mathcal{D}} = \mathrm{Ad}(\mathsf{g}_{\mathcal{F}}) \circ \Psi^{\mathcal{D}}$, $\mathscr{P}_{\mathcal{F}}$ denotes the normally ordered monodromy action of $\mathcal{P}_W$ around the point at infinity corresponding to $\mathcal{F}$, and every face is commutative. Then, the result follows from the commutativity of the back face.

We first prove the commutativity of the top face. Since $\mathsf{g}_{\mathcal{F}} \in \mathcal{D}$ is weight zero and $\mathsf{F}^{\mathrm{int}} : \mathcal{O}_{\infty,\mathfrak{g}}^{\hbar,\mathrm{int}} \to \mathcal{O}_{\infty,U_\hbar\mathfrak{g}}^{\mathrm{int}}$ is the identity at the level of $\mathfrak{h}$-modules in $\mathrm{Vect}_\hbar$, $\Psi_{\mathcal{F}}^{\mathrm{int}} = \mathrm{Ad}(\mathsf{g}_{\mathcal{F}}) \circ \Psi^{\mathrm{int}}$ intertwines the characters of $\mathcal{P}_W$ given by $\varepsilon(p_\alpha) = \exp(\iota\pi h_\alpha)$, and $\mathscr{B}(p_\alpha) = \exp(\hbar t_\alpha/2)$. Therefore, by Theorem 4.5 (1) and Proposition 5.6 (3), we can remove ε and $\mathscr{B}$, and obtain the result.

The commutativity of the lateral faces follows from Sections 4 and 5. Namely, by Theorem 4.5 (2) and Section 4.6, the normally ordered quantum Weyl group action of the pure braid group $\mathcal{P}_W \subset \mathcal{B}_W$ factors through the Drinfeld algebra $\mathcal{D}_\hbar \subset \mathrm{End}(\mathsf{f}_\hbar)$. Moreover, by definition, $\mathscr{P}$ is the normally ordered monodromy action of $\mathcal{P}_W$, which readily factors through the classical Drinfeld algebra $\mathcal{D} \subset \mathrm{End}(\mathsf{f})$.

The commutativity of the bottom and front faces follows from Section 8.1. Namely, by Theorem 3.1 (and its analogue for $U\mathfrak{g}[\![\hbar]\!]$), the restriction to integrable category $\mathcal{O}_\infty$ modules yields the embeddings $\mathcal{D}_\hbar \hookrightarrow \mathrm{End}(\mathsf{f}_\hbar^{\mathrm{int}})$ and $\mathcal{D} \hookrightarrow \mathrm{End}(\mathsf{f}^{\mathrm{int}})$. Since $\mathsf{g}_{\mathcal{F}} \in \mathcal{D}$, it follows from Theorem 8.1 that $\Psi_{\mathcal{F}}^{\mathrm{int}}$ also restricts to an isomorphism $\Psi_{\mathcal{F}}^{\mathcal{D}} = \mathrm{Ad}(\mathsf{g}_{\mathcal{F}}) \circ \Psi^{\mathcal{D}} \colon \mathcal{D}_\hbar \to \mathcal{D}$.

Finally, since $\mathcal{D}$ embeds in $\mathrm{End}(\mathsf{f}^{\mathrm{int}})$, the commutativity of the top, lateral, bottom, and front faces yields that of the diagram

$$\begin{tikzcd}
 & \mathcal{P}_W \arrow[dl, "\lambda_{\varepsilon,\mathscr{B}}"'] \arrow[dr, "\mathscr{P}^{\mathcal{F}}"] \arrow[d, phantom, "\circlearrowleft"] & \\
\mathcal{D}_\hbar \arrow[rr, "\Psi_{\mathcal{F}}^{\mathcal{D}}"'] & {} & \mathcal{D}
\end{tikzcd}$$

and the result follows. $\square$

8.3. The equivariant monodromy theorem

The following is a direct consequence of Theorem 8.2.

Theorem. *Let $\mathcal{V}$ be a $\mathfrak{g}$-module in category $\mathcal{O}^{\hbar}_{\infty}$, $\mathsf{F}(\mathcal{V})$ its Etingof–Kazhdan quantisation,*

$$\mathscr{P}_{\varepsilon,\mathscr{B}} : \mathcal{P}_W \to GL(\mathcal{V}) \qquad \text{and} \qquad \lambda : \mathcal{P}_W \to GL(\mathsf{F}(\mathcal{V}))$$

the equivariant monodromy of the Casimir connection given by Proposition 5.6, and quantum Weyl group action given by Theorem 4.5.

Then, $\mathscr{P}_{\varepsilon,\mathscr{B}}$ and λ are canonically equivalent. Specifically, for any $\mathcal{F} \in \mathsf{Mns}(\mathbb{D})$ the following diagram is commutative

$$\begin{tikzcd} & \mathcal{P}_W \arrow[dl, "\lambda"'] \arrow[dr, "{\mathscr{P}_{\varepsilon,\mathscr{B}}}^{\mathcal{F}}"] \arrow[d, phantom, "\circlearrowleft"] & \\ \mathrm{End}(\mathsf{f}_\hbar) \arrow[rr, "\Psi_{\mathcal{F}}"'] & {} & \mathrm{End}(\mathsf{f}) \end{tikzcd}$$

8.4. Extension to other Lie associators

Although Theorem 8.2 and Corollary 8.3 are formulated in terms of the tensor equivalence $\mathsf{F} : \mathcal{O}_\infty \to \mathcal{O}_{\infty, U_\hbar\mathfrak{g}}$ corresponding to the KZ associator, they hold true for the Etingof–Kazhdan equivalence corresponding to an arbitrary Lie associator Φ.

Indeed, by [ATL18, ATL19a] the braided Coxeter category $\mathscr{O}^{\mathrm{int}}_{U_\hbar\mathfrak{g},\mathbf{R},\mathbf{S}}$ underlying the R-matrix and quantum Weyl group of $U_\hbar\mathfrak{g}$ (see 6.5) is equivalent to a braided Coxeter category $\mathscr{O}^{\hbar,\mathrm{int},\Phi}_{\mathfrak{g},\mathbf{R},\mathbf{S}}$ with diagrammatic categories $\{\mathcal{O}^{\hbar,\mathrm{int}}_{\infty,\mathfrak{g}_B}\}_{B\subseteq\mathbb{D}}$, and standard restriction functors, with the corresponding horizontal equivalences $\mathcal{O}^{\hbar}_{\infty,\mathfrak{g}_B} \to \mathcal{O}_{\infty,U_\hbar\mathfrak{g}_B}$ given by the Etingof–Kazhdan tensor equivalence $\mathsf{F}^{\Phi}_{\mathfrak{g}_B}$ corresponding to $\mathfrak{g}_B$ and the choice of Φ.

By the rigidity result of [ATL19b], $\mathscr{O}^{\hbar,\mathrm{int},\Phi}_{\mathfrak{g},\mathbf{R},\mathbf{S}}$ is equivalent to $\mathscr{O}^{\hbar,\mathrm{int}}_{\mathfrak{g},\nabla}$, with diagrammatic equivalences given by the identity functors endowed with a non-trivial tensor structure.

Composing yields an equivalence $\mathscr{O}^{\hbar,\mathrm{int}}_{\mathfrak{g},\nabla} \to \mathscr{O}^{\mathrm{int}}_{U_\hbar\mathfrak{g},\mathbf{R},\mathbf{S}}$ whose diagrammatic equivalences are the Etingof–Kazhdan functors corresponding to Φ, which then yields Theorem 8.2 and Corollary 8.3 for F^{Φ}.

9. Parabolic pure braid group actions

In this section, we extend the results of Sections 4 and 8 to parabolic pure braid groups.

9.1. The group $\mathcal{PB}_{\mathbf{J}}$

For any subset $\mathbf{J} \subseteq \mathbf{I}$, let $\mathcal{PB}_{\mathbf{J}} \subseteq \mathcal{B}_W$ be the preimage of $W_{\mathbf{J}} = \langle s_j \rangle_{j\in\mathbf{J}}$ under the projection $\mathcal{B}_W \to W$. Thus, $\mathcal{PB}_\emptyset = \mathcal{P}_W$ and $\mathcal{PB}_{\mathbf{I}} = \mathcal{B}_W$. The *parabolic pure braid group* $\mathcal{PB}_{\mathbf{J}}$ is generated by the braid group $\mathcal{B}_{W_{\mathbf{J}}}$ and the pure braid group $\mathcal{P}_W$. Moreover, as an abstract group,

$$\mathcal{PB}_{\mathbf{J}} \simeq (\mathcal{P}_W \rtimes \mathcal{B}_{W_{\mathbf{J}}})/\widetilde{\mathcal{P}}_{W_{\mathbf{J}}} \quad \text{where} \quad \widetilde{\mathcal{P}}_{W_{\mathbf{J}}} = \{(p, p^{-1}) \,|\, p \in \mathcal{P}_{W_{\mathbf{J}}}\} \subset \mathcal{P}_W \rtimes \mathcal{B}_{W_{\mathbf{J}}}$$

9.2. Quantum Weyl group action of $\mathcal{PB}_{\mathbf{J}}$

Let $U_\hbar\mathfrak{g}_{\mathbf{J}} \subseteq U_\hbar\mathfrak{g}$ be the Hopf subalgebra generated by $\{E_j, F_j, h_j\}_{j\in\mathbf{J}}$, and $\mathcal{O}^{\mathbf{J}\text{-int}}_{\infty,U_\hbar\mathfrak{g}} \subseteq \mathcal{O}_{\infty,U_\hbar\mathfrak{g}}$ the full subcategory of modules whose restriction to $U_\hbar\mathfrak{g}_{\mathbf{J}}$ is integrable. We have the inclusions

$$\mathcal{O}^{\text{int}}_{\infty,U_\hbar\mathfrak{g}} \subset \mathcal{O}^{\mathbf{J}\text{-int}}_{\infty,U_\hbar\mathfrak{g}} \subset \mathcal{O}_{\infty,U_\hbar\mathfrak{g}}$$

together with the equalities $\mathcal{O}^{\emptyset\text{-int}}_{\infty,U_\hbar\mathfrak{g}} = \mathcal{O}_{\infty,U_\hbar\mathfrak{g}}$ and $\mathcal{O}^{\mathbf{I}\text{-int}}_{\infty,U_\hbar\mathfrak{g}} = \mathcal{O}^{\text{int}}_{\infty,U_\hbar\mathfrak{g}}$.

Let $\mathsf{f}_\hbar^{\mathbf{J}\text{–int}} : \mathcal{O}^{\mathbf{J}\text{-int}}_{\infty,U_\hbar\mathfrak{g}} \to \mathrm{Vect}_\hbar$ be the forgetful functor. We define below and in 9.3 two actions

$$\lambda, \lambda_{\varepsilon[\mathbf{J}],\mathscr{B}[\mathbf{J}]} : \mathcal{PB}_{\mathbf{J}} \to \mathrm{Aut}(\mathsf{f}_\hbar^{\mathbf{J}\text{–int}})$$

such that

- for $\mathbf{J} = \emptyset$, they recover the quantum Weyl group action $\lambda : \mathcal{P}_W \to \mathrm{Aut}(\mathsf{f}_\hbar)$ from Theorem 4.5 (3) and the normally ordered quantum Weyl group action $\lambda_{\varepsilon,\mathscr{B}} : \mathcal{P}_W \to U_\hbar\mathfrak{g}$ from Section 4.6, respectively.
- for $\mathbf{J} = \mathbf{I}$, both give the quantum Weyl group action $\lambda : \mathcal{B}_W \to \mathrm{Aut}(\mathsf{f}_\hbar^{\text{int}})$.

Let $\mathsf{f}^{\text{int}}_{\mathbf{J},\hbar} : \mathcal{O}^{\text{int}}_{\infty,U_\hbar\mathfrak{g}_{\mathbf{J}}} \to \mathrm{Vect}_\hbar$ be the forgetful functor and $\lambda_{\mathbf{J}} : \mathcal{B}_{W_{\mathbf{J}}} \to \mathrm{Aut}(\mathsf{f}^{\text{int}}_{\mathbf{J},\hbar})$ the quantum Weyl group action of $\mathcal{B}_{W_{\mathbf{J}}}$. Let $\lambda^{\mathbf{J}\text{–int}} : \mathcal{B}_{W_{\mathbf{J}}} \to \mathrm{Aut}(\mathsf{f}_\hbar^{\mathbf{J}\text{–int}})$ be its lift through the restriction functor $\mathcal{O}^{\mathbf{J}\text{-int}}_{\infty,U_\hbar\mathfrak{g}} \to \mathcal{O}^{\text{int}}_{\infty,U_\hbar\mathfrak{g}_{\mathbf{J}}}$.

Theorem. *The following holds.*

1. *The quantum Weyl group action of $\mathcal{PB}_{\mathbf{J}}$ on integrable modules in category $\mathcal{O}_{\infty,U_\hbar\mathfrak{g}}$ has a unique extension to an action $\lambda : \mathcal{PB}_{\mathbf{J}} \to \mathrm{Aut}(\mathsf{f}_\hbar^{\mathbf{J}\text{–int}})$ such that $\lambda|_{\mathcal{B}_{W_{\mathbf{J}}}} = \lambda^{\mathbf{J}\text{–int}}$ and $\lambda|_{\mathcal{P}_W}$ is the restriction of the action (4.2) to $\mathcal{O}^{\mathbf{J}\text{-int}}_{\infty,U_\hbar\mathfrak{g}} \subset \mathcal{O}_{\infty,U_\hbar\mathfrak{g}}$.*
2. *The map λ intertwines the inner action of $\mathcal{PB}_{\mathbf{J}}$ on $U_\hbar\mathfrak{g}$,* i.e., *for any element $Y \in U_\hbar\mathfrak{g}$ and $b \in \mathcal{PB}_{\mathbf{J}}$*

$$\lambda(b) Y \lambda(b)^{-1} = b(Y)$$

in $\mathrm{End}(\mathsf{f}_\hbar^{\mathbf{J}\text{–int}})$.

Proof. The uniqueness of λ follows from the fact that $\mathcal{PB}_{\mathbf{J}}$ is generated by $\mathcal{P}_W$ and $\mathcal{B}_{W_{\mathbf{J}}}$. To prove the existence of λ, it is enough to observe that on the one hand there is a commutative diagram

$$\begin{tikzcd} \mathcal{B}_W \arrow[r, "\lambda"] & \mathrm{Aut}(\mathsf{f}_\hbar^{\mathrm{int}}) \\ \mathcal{B}_{W_{\mathbf{J}}} \arrow[u, hook] \arrow[r, "\lambda^{\mathbf{J}\text{–int}}"'] & \mathrm{Aut}(\mathsf{f}_\hbar^{\mathbf{J}\text{–int}}) \arrow[u] \end{tikzcd}$$

where the right vertical arrow is induced by the inclusion $\mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar\mathfrak{g}} \subset \mathcal{O}^{\mathbf{J}\text{-int}}_{\infty,U_\hbar\mathfrak{g}}$. On the other, by Theorem 4.5, the quantum Weyl group action of $\mathcal{P}_W$ on integrable modules extends canonically to $\mathcal{O}_{\infty,U_\hbar\mathfrak{g}}$ and therefore to $\mathcal{O}^{\mathbf{J}\text{-int}}_{\infty,U_\hbar\mathfrak{g}} \subseteq \mathcal{O}_{\infty,U_\hbar\mathfrak{g}}$, *i.e.*, there is a commutative diagram

$$\begin{tikzcd} \mathcal{B}_W \arrow[r, "\lambda"] & \mathrm{Aut}(\mathsf{f}_\hbar^{\mathrm{int}}) \\ \mathcal{P}_W \arrow[u, hook] \arrow[r, "\lambda"'] & \mathrm{Aut}(\mathsf{f}_\hbar^{\mathbf{J}\text{–int}}) \arrow[u] \end{tikzcd}$$

The actions of $\mathcal{B}_{W_{\mathbf{J}}}$ and $\mathcal{P}_W$ on $\mathsf{f}_\hbar^{\mathrm{int}}$ give rise to an action of $\mathcal{P}_W \rtimes \mathcal{B}_{W_{\mathbf{J}}}$, since, for any $p \in \mathcal{P}_W$ and $b \in \mathcal{B}_{W_{\mathbf{J}}}$, one has

$$\begin{aligned} \lambda^{\mathbf{J}\text{–int}}(b) \cdot \lambda(p) &= \lambda^{\mathbf{J}\text{–int}}(b) \cdot \varepsilon_\hbar(p) \cdot \mathscr{K}(p) \\ &= b(\varepsilon_\hbar(p)) \cdot b(\mathscr{K}(p)) \cdot \lambda^{\mathbf{J}\text{–int}}(b) \\ &= \varepsilon_\hbar(bpb^{-1}) \cdot \mathscr{K}(bpb^{-1}) \cdot \lambda^{\mathbf{J}\text{–int}}(b) \\ &= \lambda(bpb^{-1}) \cdot \lambda^{\mathbf{J}\text{–int}}(b) \end{aligned}$$

where the third equality follows the $\mathcal{B}_W$-equivariance of $\mathscr{K}$ (Theorem 4.5 (2)). Moreover, they coincide on $\mathcal{P}_{W_{\mathbf{J}}} = \mathcal{P}_W \cap \mathcal{B}_{W_{\mathbf{J}}}$, and therefore give rise to the desired action $\lambda : \mathcal{PB}_{\mathbf{J}} \to \mathrm{Aut}(\mathsf{f}_\hbar^{\mathbf{J}\text{–int}})$. □

9.3. Normally ordered quantum Weyl group action of $\mathcal{PB}_{\mathbf{J}}$

Let $\Delta_{\mathbf{J}} \subseteq \Delta$ be the root subsystem generated by $\{\alpha_j\}_{j\in\mathbf{J}}$, and let

$$\varepsilon_\hbar^{[\mathbf{J}]} : \mathcal{PB}_{\mathbf{J}} \to \mathrm{Aut}(\mathsf{f}_\hbar^{\mathbf{J}\text{–int}}) \qquad \text{and} \qquad \mathscr{B}^{[\mathbf{J}]} : \mathcal{PB}_{\mathbf{J}} \to \exp(\hbar\mathfrak{h}) \tag{9.1}$$

be the morphisms uniquely defined by the following conditions.

- For any $b \in \mathcal{B}_{W_{\mathbf{J}}}$, $\varepsilon_\hbar^{[\mathbf{J}]}(b) = 1 = \mathscr{B}^{[\mathbf{J}]}(b)$.

- For any $\alpha \in \Delta^{\mathrm{re}}_{\mathbf{J},+}$, $\varepsilon_\hbar^{[\mathbf{J}]}(p_\alpha) = 1 = \mathscr{B}^{[\mathbf{J}]}(p_\alpha)$.
- For any $\alpha \in \Delta^{\mathrm{re}}_{+} \backslash \Delta^{\mathrm{re}}_{\mathbf{J},+}$, $\varepsilon_\hbar^{[\mathbf{J}]}(p_\alpha) = \exp(\iota\pi h_\alpha)$ and $\mathscr{B}^{[\mathbf{J}]}(p_\alpha) = \exp(\hbar t_\alpha/2)$.

Note that $\varepsilon_\hbar^{[\mathbf{J}]}$ and $\mathscr{B}^{[\mathbf{J}]}$ are both $\mathcal{B}_{W_\mathbf{J}}$-equivariant. They therefore give rise to a morphism

$$\lambda_{\varepsilon^{[\mathbf{J}]},\mathscr{B}^{[\mathbf{J}]}} : \mathcal{PB}_\mathbf{J} \to \mathrm{Aut}(\mathsf{f}_\hbar^{\mathbf{J}\text{-int}}) \qquad \text{by} \qquad \lambda(b) = \varepsilon_\hbar^{[\mathbf{J}]}(b) \cdot \lambda_{\varepsilon^{[\mathbf{J}]},\mathscr{B}^{[\mathbf{J}]}}(b) \cdot \mathscr{B}^{[\mathbf{J}]}(b)$$

for any $b \in \mathcal{PB}_\mathbf{J}$, which we shall refer to as the normally ordered quantum Weyl group action of $\mathcal{PB}_\mathbf{J}$ on $\mathcal{O}^{\mathbf{J}\text{-int}}_{\infty,U_\hbar\mathfrak{g}}$. If $\mathbf{J} = \emptyset$, $\lambda_{\varepsilon^{[\mathbf{J}]},\mathscr{B}^{[\mathbf{J}]}}$ is the action of $\mathcal{P}_W$ constructed in 4.6 while, if $\mathbf{J} = \mathbf{I}$, $\lambda_{\varepsilon^{[\mathbf{J}]},\mathscr{B}^{[\mathbf{J}]}}$ is the quantum Weyl group action of $\mathcal{B}_W$ on $\mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar\mathfrak{g}}$.

9.4. Tits extension and $\mathcal{PB}_\mathbf{J}$

Let now $\mathfrak{g}_\mathbf{J} \subseteq \mathfrak{g}$ be the subalgebra generated by $\{e_j, f_j\}_{j\in\mathbf{J}}$, $\mathcal{O}^{\hbar,\mathbf{J}\text{-int}}_{\infty,\mathfrak{g}} \subseteq \mathcal{O}^{\hbar}_{\infty,\mathfrak{g}}$ the full subcategory of modules whose restriction to $\mathfrak{g}_\mathbf{J}$ is integrable, and $\mathsf{f}^{\mathbf{J}\text{-int}} : \mathcal{O}^{\hbar,\mathbf{J}\text{-int}}_{\infty,\mathfrak{g}} \to \mathrm{Vect}_\hbar$ the forgetful functor.

Let $\varepsilon^{[\mathbf{J}]} : \mathcal{PB}_\mathbf{J} \to \mathrm{Aut}(\mathsf{f}^{\mathbf{J}\text{-int}})$ be the sign character defined as in (9.1), and define $\varepsilon_{[\mathbf{J}]} : \mathcal{P}_W \to \mathrm{Aut}(\mathsf{f}^{\mathbf{J}\text{-int}})$ by the relation

$$\varepsilon(p) = \varepsilon_{[\mathbf{J}]}(p) \cdot \varepsilon^{[\mathbf{J}]}(p)$$

for any $p \in \mathcal{P}_W$, where ε is given by (5.3). Thus, $\varepsilon_{[\mathbf{J}]}(p_\alpha) = \exp(\iota\pi h_\alpha)$ if $\alpha \in \Delta^{\mathrm{re}}_{\mathbf{J},+}$, and $\varepsilon_{[\mathbf{J}]}(p_\alpha) = 1$ if $\alpha \notin \Delta^{\mathrm{re}}_{\mathbf{J},+}$.

Lemma. *Let $\mathcal{V}$ be a module in $\mathcal{O}^{\hbar,\mathbf{J}\text{-int}}_{\infty,\mathfrak{g}}$. Then, there is an action $\tau_{[\mathbf{J}]}$ of $\mathcal{PB}_\mathbf{J}$ on $\mathcal{V}$ uniquely determined by the following conditions.*

(1) The restriction of $\tau_{[\mathbf{J}]}$ to $\mathcal{B}_{W_\mathbf{J}}$ is given by the action $\tau_\mathbf{J}$ of the triple exponentials (2.3) indexed by $\mathbf{J}$.
(2) The restriction of $\tau_{[\mathbf{J}]}$ to $\mathcal{P}_W$ is given by the sign character $\varepsilon_{[\mathbf{J}]}$.

Proof. The result follows at once from Proposition 5.6 (1). □

Remark. Equivalently, $\tau_{[\mathbf{J}]}$ is given by a projection of $\mathcal{PB}_\mathbf{J}$ onto the Tits extension $\widetilde{W}_\mathbf{J}$. Note also that, for $\mathbf{J} = \emptyset$, $\tau_{[\mathbf{J}]}$ is trivial, while, for $\mathbf{J} = \mathbf{I}$, $\tau_{[\mathbf{J}]} = \tau$.

9.5. Monodromy action of $\mathcal{PB}_{\mathbf{J}}$ on category $\mathcal{O}_{\infty}^{\mathbf{J}\text{-int}}$

We construct below an action

$$\mathscr{P}_{\tau_{[\mathbf{J}]},\mathscr{B}_{[\mathbf{J}]}} : \mathcal{PB}_{\mathbf{J}} \to \mathrm{Aut}(\mathsf{f}^{\mathbf{J}\text{–int}})$$

by making the monodromy of the Casimir connection $\nabla_{\mathcal{K}}$ of $\mathfrak{g}$ equivariant, as described in 1.9 and 5.4–5.6, but only with respect to the parabolic subgroup $W_{\mathbf{J}}$. For $\mathbf{J} = \emptyset$, $\mathscr{P}_{\tau_{[\mathbf{J}]},\mathscr{B}_{[\mathbf{J}]}}$ is the monodromy $\mathscr{P} : \mathcal{P}_W \to \mathrm{Aut}(\mathsf{f})$ of $\nabla_{\mathcal{K}}$ (cf. Section 5.2) while, for $\mathbf{J} = \mathbf{I}$, $\mathscr{P}_{\tau_{[\mathbf{J}]},\mathscr{B}_{[\mathbf{J}]}}$ is the equivariant monodromy action $\mathscr{P}_{\tau,\mathscr{B}} : \mathcal{B}_W \to \mathrm{Aut}(\mathsf{f}^{\mathrm{int}})$ of Theorem 5.5.

Let $\mathscr{P} : \mathbf{\Pi}_1(\mathsf{X}; Wx_0) \to \mathcal{T}_{\mathfrak{g}}$ be the monodromy of $\nabla_{\mathcal{K}}$, where $\mathcal{T}_{\mathfrak{g}}$ is the image of the holonomy algebra (cf. 1.9), and consider its restriction to $\mathbf{\Pi}_1(\mathsf{X}; W_{\mathbf{J}}x_0)$. The lack of equivariance of $\mathscr{P}$ under $W_{\mathbf{J}}$ is controlled by the 1-cocycle

$$\mathscr{A}_{[\mathbf{J}]} = i_{\mathbf{J}}^* \mathscr{A}|_{W_{\mathbf{J}}} : W_{\mathbf{J}} \to \mathrm{Hom}(\mathbf{\Pi}_1(\mathsf{X}; W_{\mathbf{J}}x_0), \exp(\hbar\mathfrak{h}))$$

where $i_{\mathbf{J}} : \mathbf{\Pi}_1(\mathsf{X}; W_{\mathbf{J}}x_0) \to \mathbf{\Pi}_1(\mathsf{X}; Wx_0)$ is the inclusion.

The obstruction $\mathscr{A}_{[\mathbf{J}]}$ is related to the one for the Casimir connection of $\mathfrak{g}_{\mathbf{J}}$ as follows. Consider the quotient map

$$p_{\mathbf{J}} : \mathfrak{h}^{\mathrm{e}} \to \mathfrak{h}^{\mathrm{e}} / \bigcap_{\alpha\in\Delta_{\mathbf{J}}} \mathrm{Ker}(\alpha) \simeq \mathfrak{h}_{\mathbf{J}}^{\mathrm{e}}$$

$p_{\mathbf{J}}$ is equivariant under $W_{\mathbf{J}}$ and, by [Kac90, Prop. 3.12], restricts to a map $\mathsf{X} \to \mathsf{X}_{\mathbf{J}}$ of Tits cones. It therefore induces a morphism of groupoids $p_{\mathbf{J}} : \mathbf{\Pi}_1(\mathsf{X}; W_{\mathbf{J}}x_0) \to \mathbf{\Pi}_1(\mathsf{X}_{\mathbf{J}}; W_{\mathbf{J}}[x_0]_{\mathbf{J}})$, where $[x_0]_{\mathbf{J}} = p_{\mathbf{J}}(x_0)$, which we denote by the same symbol.

Lemma. *Let*

$$\mathscr{A}_{\mathbf{J}} : W_{\mathbf{J}} \to \mathrm{Hom}(\mathbf{\Pi}_1(\mathsf{X}_{\mathbf{J}}; W_{\mathbf{J}}[x_0]_{\mathbf{J}}), \exp(\hbar\mathfrak{h}_{\mathbf{J}}))$$

be the 1-cocycle measuring the lack of equivariance of the Casimir connection of $\mathfrak{g}_{\mathbf{J}}$ with respect to $W_{\mathbf{J}}$. Then, $\mathscr{A}_{[\mathbf{J}]} = p_{\mathbf{J}}^* \mathscr{A}_{\mathbf{J}}$.

Proof. Let $w \in W_{\mathbf{J}}$. By Remark 5.4, $\mathscr{A}_w$ is the monodromy of the connection $d - \mathsf{h}a_w$, where

$$\mathsf{h}a_w = \nabla_{\mathcal{K}} - w^*\nabla_{\mathcal{K}} = \mathsf{h} \sum_{\substack{\alpha\in\Delta_+:\\ w\alpha<0}} \frac{d\alpha}{\alpha} t_\alpha = \mathsf{h} \sum_{\substack{\alpha\in\Delta_{\mathbf{J},+}:\\ w\alpha<0}} \frac{d\alpha}{\alpha} t_\alpha = p_{\mathbf{J}}^*(\nabla_{\mathcal{K},\mathbf{J}} - w^*\nabla_{\mathcal{K},\mathbf{J}})$$

□

By Theorem 5.5 for $\mathfrak{g}_{\mathbf{J}}$, $\mathscr{A}_{\mathbf{J}} = d\mathscr{B}_{\mathbf{J}}$, where $\mathscr{B}_{\mathbf{J}} \in \mathrm{Hom}(\mathbf{\Pi}_1(\mathsf{X}_{\mathbf{J}}; W_{\mathbf{J}}[x_0]_{\mathbf{J}}), \exp(\hbar\mathfrak{h}_{\mathbf{J}}))$. Set $\mathscr{B}_{[\mathbf{J}]} = p_{\mathbf{J}}^*\mathscr{B}_{\mathbf{J}}$. Then,

$$\mathscr{A}_{[\mathbf{J}]} = p_{\mathbf{J}}^*\mathscr{A}_{\mathbf{J}} = p_{\mathbf{J}}^* d\mathscr{B}_{\mathbf{J}} = dp_{\mathbf{J}}^*\mathscr{B}_{\mathbf{J}} = d\mathscr{B}_{[\mathbf{J}]}$$

It follows that $\mathscr{B}_{[\mathbf{J}]}$ gives rise to a $W_{\mathbf{J}}$-equivariant morphism

$$\mathscr{P}_{\mathscr{B}_{[\mathbf{J}]}} : \mathbf{\Pi}_1(\mathsf{X}; W_{\mathbf{J}}x_0) \to \mathcal{T}_{\mathfrak{g}} \qquad\qquad \mathscr{P}_{\mathscr{B}_{[\mathbf{J}]}}(\gamma) = \mathscr{P}(\gamma)\cdot\mathscr{B}_{[\mathbf{J}]}(\gamma)$$

Consider next the equivalence of groupoids

$$P_{\mathbf{J}} : W_{\mathbf{J}} \ltimes \mathbf{\Pi}_1(\mathsf{X}; W_{\mathbf{J}}x_0) \to \mathbf{\Pi}_1(\mathsf{X}/W_{\mathbf{J}}; [x_0]) \cong \mathcal{PB}_{\mathbf{J}}$$

generalising (5.1). Composing with $P_{\mathbf{J}}^{-1}$ yields a morphism $\mathcal{PB}_{\mathbf{J}} \to W_{\mathbf{J}} \ltimes \mathcal{T}_{\mathfrak{g}}$ and its lift $\mathcal{PB}_{\mathbf{J}} \to \mathcal{PB}_{\mathbf{J}} \ltimes \mathcal{T}_{\mathfrak{g}}$. Combining this with the action $\tau_{[\mathbf{J}]}$ of $\mathcal{PB}_{\mathbf{J}}$ on $\mathsf{f}^{\mathbf{J}\text{–int}}$ defined in Lemma 9.4, yields the following generalisation of Theorem 5.5.

Theorem. *There is a morphism $\mathscr{P}_{\tau_{[\mathbf{J}]},\mathscr{B}_{[\mathbf{J}]}} : \mathcal{PB}_{\mathbf{J}} \to \mathrm{Aut}(\mathsf{f}^{\mathbf{J}\text{–int}})$ given by*

$$\mathscr{P}_{\tau_{[\mathbf{J}]},\mathscr{B}_{[\mathbf{J}]}}(b) = \tau_{[\mathbf{J}]}(b)\cdot\mathscr{P}(\widetilde{b})\cdot\mathscr{B}_{[\mathbf{J}]}(\widetilde{b})$$

where $\widetilde{b} \in \mathbf{\Pi}_1(\mathsf{X}; W_{\mathbf{J}}x_0)$ is the unique lift of b through x_0.

Remark. Note that, for any $j \in \mathbf{J}$, $\mathscr{B}_{[\mathbf{J}]}(\gamma_j) = \exp(\hbar t_{\alpha_j}/4)$, since $p_{\mathbf{J}}$ maps γ_j to the corresponding generator of $\gamma_{\mathbf{J},i}, \in \Pi_1(\mathsf{X}_{\mathbf{J}}; W_{\mathbf{J}}[x_0]_{\mathbf{J}})$ and $\mathscr{B}_{[\mathbf{J}]}(\gamma_j) = \exp(\hbar t_{\alpha_j}/4)$ by construction.

9.6. The monodromy theorem for $\mathcal{PB}_{\mathbf{J}}$

Theorem. *The $W_{\mathbf{J}}$-equivariant monodromy of the Casimir connection on a $\mathfrak{g}$-module $\mathcal{V} \in \mathcal{O}_{\infty,\mathfrak{g}}^{\hbar,\mathbf{J}\text{-int}}$ is canonically equivalent to the normally ordered quantum Weyl group action of the parabolic braid group $\mathcal{PB}_{\mathbf{J}}$ on the Etingof–Kazhdan quantisation $\mathsf{F}(\mathcal{V}) \in \mathcal{O}_{\infty,U_\hbar\mathfrak{g}}^{\mathbf{J}\text{-int}}$.*

Proof. The result follows from the combination of Theorem 7.4 for $\mathcal{B}_{W_{\mathbf{J}}}$ and Theorems 8.2–8.3 for $\mathcal{P}_W$.

Specifically, let $B \subseteq \mathbb{D}$ be the subdiagram corresponding to $\mathbf{J}$, $\mathcal{F}$ a maximal nested set containing $B \subseteq \mathbb{D}$ corresponding to $\mathbf{J}$, and $\mathcal{F}_{\mathbf{J}}$ the induced maximal nested set on B. Let

$$\mathsf{f}_{\mathbf{J}} : \mathcal{O}_{\infty,\mathfrak{g}_{\mathbf{J}}}^{\hbar} \to \mathrm{Vect}_\hbar \qquad \text{and} \qquad \mathsf{f}_{\mathbf{J},\hbar} : \mathcal{O}_{\infty,U_\hbar\mathfrak{g}_{\mathbf{J}}} \to \mathrm{Vect}_\hbar$$

be the forgetful functors. By Theorem 7.4 (1) and (7.3), the isomorphism $\Psi_{\mathcal{F}}$ restricts to $\Psi_{\mathcal{F}_{\mathbf{J}}}$, *i.e.*, there is a commutative diagram

$$\begin{tikzcd} \mathrm{End}(\mathsf{f}_{\mathbf{J},\hbar}) \arrow[r, "\Psi_{\mathcal{F}_{\mathbf{J}}}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{End}(\mathsf{f}_{\mathbf{J}}^{\mathrm{int}}) \arrow[d] \\ \mathrm{End}(\mathsf{f}_{\hbar}) \arrow[r, "\Psi_{\mathcal{F}}"'] & \mathrm{End}(\mathsf{f}) \end{tikzcd}$$

where the vertical arrows are induced by the restriction functors $\mathcal{O}^{\hbar}_{\infty,\mathfrak{g}} \to \mathcal{O}^{\hbar}_{\infty,\mathfrak{g}_{\mathbf{J}}}$ and $\mathcal{O}_{\infty,U_\hbar\mathfrak{g}} \to \mathcal{O}_{\infty,U_\hbar\mathfrak{g}_{\mathbf{J}}}$, respectively.

Further, since the Etingof–Kazhdan equivalence preserves the categories of $\mathcal{O}^{\mathbf{J}\text{–int}}_{\infty}$ modules, $\Psi_{\mathcal{F}}$ restricts to an isomorphism $\Psi^{\mathbf{J}\text{–int}}_{\mathcal{F}} : \mathrm{End}(\mathsf{f}^{\mathbf{J}\text{–int}}_{\hbar}) \to \mathrm{End}(\mathsf{f}^{\mathbf{J}\text{–int}})$ such that

(1) There is a commutative diagram

$$\begin{tikzcd} \mathrm{End}(\mathsf{f}^{\mathrm{int}}_{\mathbf{J},\hbar}) \arrow[r, "\Psi^{\mathrm{int}}_{\mathcal{F}_{\mathbf{J}}}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{End}(\mathsf{f}_{\mathbf{J}}^{\mathrm{int}}) \arrow[d] \\ \mathrm{End}(\mathsf{f}^{\mathbf{J}\text{–int}}_{\hbar}) \arrow[r, "\Psi^{\mathbf{J}\text{–int}}_{\mathcal{F}}"'] & \mathrm{End}(\mathsf{f}^{\mathbf{J}\text{–int}}) \end{tikzcd}$$

where the vertical arrows are induced by the restriction functors $\mathcal{O}^{\hbar,\mathbf{J}\text{-int}}_{\infty,\mathfrak{g}} \to \mathcal{O}^{\hbar,\mathrm{int}}_{\infty,\mathfrak{g}_{\mathbf{J}}}$ and $\mathcal{O}^{\mathbf{J}\text{-int}}_{\infty,U_\hbar\mathfrak{g}} \to \mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar\mathfrak{g}_{\mathbf{J}}}$, respectively.

(2) There is a commutative diagram

$$\begin{tikzcd} \mathrm{End}(\mathsf{f}_{\hbar}) \arrow[r, "\Psi_{\mathcal{F}}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{End}(\mathsf{f}) \arrow[d] \\ \mathrm{End}(\mathsf{f}^{\mathbf{J}\text{–int}}_{\hbar}) \arrow[r, "\Psi^{\mathbf{J}\text{–int}}_{\mathcal{F}}"'] & \mathrm{End}(\mathsf{f}^{\mathbf{J}\text{–int}}) \end{tikzcd}$$

where the vertical arrows are induced by the inclusions $\mathcal{O}^{\hbar,\mathbf{J}\text{-int}}_{\infty,\mathfrak{g}} \to \mathcal{O}^{\hbar}_{\infty,\mathfrak{g}}$ and $\mathcal{O}^{\mathbf{J}\text{-int}}_{\infty,U_\hbar\mathfrak{g}} \to \mathcal{O}_{\infty,U_\hbar\mathfrak{g}}$, respectively.

We claim that $\Psi^{\mathbf{J}\text{–int}}_{\mathcal{F}}$ intertwines the actions of $\mathcal{B}_{W_{\mathbf{J}}}$ and $\mathcal{P}_W$, and therefore that of $\mathcal{PB}_{\mathbf{J}}$. To this end, consider the diagram

$$\begin{tikzcd}
& \mathcal{B}_{W_{\mathbf{J}}} \arrow[dl, "\lambda_{\mathbf{J}}"'] \arrow[dr, "\mathscr{P}^{\mathcal{F}_{\mathbf{J}}}_{\tau_{\mathbf{J}},\mathscr{B}_{\mathbf{J}}}"] \arrow[dd, hook] & \\
\mathrm{End}(\mathsf{f}^{\mathrm{int}}_{\mathbf{J},\hbar}) \arrow[rr, "\Psi^{\mathrm{int}}_{\mathcal{F}_{\mathbf{J}}}"] \arrow[dd] & & \mathrm{End}(\mathsf{f}^{\mathrm{int}}_{\mathbf{J}}) \arrow[dd] \\
& \mathcal{PB}_{\mathbf{J}} \arrow[dl, "\lambda_{\varepsilon^{[\mathbf{J}]},\mathscr{B}^{[\mathbf{J}]}}"'] \arrow[dr, "{\mathscr{P}_{\tau_{[\mathbf{J}]},\mathscr{B}_{[\mathbf{J}]}}}^{\mathcal{F}}"] & \\
\mathrm{End}(\mathsf{f}^{\mathbf{J}-\mathrm{int}}_{\hbar}) \arrow[rr, "\Psi^{\mathbf{J}-\mathrm{int}}_{\mathcal{F}}"'] & & \mathrm{End}(\mathsf{f}^{\mathbf{J}-\mathrm{int}})
\end{tikzcd}$$

The front face commutes by (1); the top face by Theorem 7.4 (3) for $\mathfrak{g}_{\mathbf{J}}$; the left lateral face by Theorem 9.2 (1). For the right lateral face, recall that, for any $b \in \mathcal{PB}_{\mathbf{J}}$,

$$\mathscr{P}^{\mathcal{F}}_{\tau_{[\mathbf{J}]},\mathscr{B}_{[\mathbf{J}]}}(b) = \tau_{[\mathbf{J}]}(b) \cdot \mathscr{P}^{\mathcal{F}}(\widetilde{b}) \cdot \mathscr{B}_{[\mathbf{J}]}(\widetilde{b})$$

Let $b \in \mathcal{B}_{W_{\mathbf{J}}}$. By Lemma 9.4 (1), we have that $\tau_{[\mathbf{J}]}(b) = \tau_{\mathbf{J}}(b)$. Then, by Remark 9.5, $\mathscr{B}_{[\mathbf{J}]}(\widetilde{b}) = \mathscr{B}_{\mathbf{J}}(\widetilde{b}_{\mathbf{J}})$, where $\widetilde{b}_{\mathbf{J}} \in \mathbf{\Pi}_1(\mathsf{X}_{\mathbf{J}}, W_{\mathbf{J}}[x_0]_{\mathbf{J}})$ denotes the unique lift of b through $[x_0]_{\mathbf{J}}$. Finally, $\mathscr{P}^{\mathcal{F}}(\widetilde{b}) = \mathscr{P}^{\mathcal{F}_{\mathbf{J}}}(\widetilde{b}_{\mathbf{J}})$ since the monodromy in the De Concini–Procesi compactification is recursive in nature [DCP95, Thm. 3.6]. Thus, $\Psi^{\mathbf{J}-\mathrm{int}}_{\mathcal{F}}$ intertwines the actions of $\mathcal{B}_{W_{\mathbf{J}}}$ through $\mathcal{PB}_{\mathbf{J}}$.

Similarly, consider the diagram

$$\begin{tikzcd}
& \mathcal{P}_{W} \arrow[dl, "\lambda_{\varepsilon^{[\mathbf{J}]},\mathscr{B}^{[\mathbf{J}]}}"'] \arrow[dr, "\mathscr{P}^{\mathcal{F}}_{\varepsilon^{[\mathbf{J}]},\mathscr{B}^{[\mathbf{J}]}}"] \arrow[dd, hook] & \\
\mathrm{End}(\mathsf{f}_{\hbar}) \arrow[rr, "\Psi_{\mathcal{F}}"] \arrow[dd] & & \mathrm{End}(\mathsf{f}) \arrow[dd] \\
& \mathcal{PB}_{\mathbf{J}} \arrow[dl, "\lambda_{\varepsilon^{[\mathbf{J}]},\mathscr{B}^{[\mathbf{J}]}}"'] \arrow[dr, "{\mathscr{P}_{\tau_{[\mathbf{J}]},\mathscr{B}_{[\mathbf{J}]}}}^{\mathcal{F}}"] & \\
\mathrm{End}(\mathsf{f}^{\mathbf{J}-\mathrm{int}}_{\hbar}) \arrow[rr, "\Psi^{\mathbf{J}-\mathrm{int}}_{\mathcal{F}}"'] & & \mathrm{End}(\mathsf{f}^{\mathbf{J}-\mathrm{int}})
\end{tikzcd}$$

Let $p \in \mathcal{P}_W$ and recall the identities

$$\varepsilon(p) = \varepsilon_{[\mathbf{J}]}(p) \cdot \varepsilon^{[\mathbf{J}]}(p) \quad \text{and} \quad \mathscr{B}(p) = \mathscr{B}_{[\mathbf{J}]}(p) \cdot \mathscr{B}^{[\mathbf{J}]}(p)$$

from 9.4 and Remark 9.5. The commutativity of the top face then follows from Theorem 8.2 by correcting simultaneously $\lambda_{\varepsilon,\mathscr{B}}$ and $\mathscr{P}^{\mathcal{F}}$ by $\varepsilon_{[\mathbf{J}]}$ and $\mathscr{B}_{[\mathbf{J}]}$. The left lateral face commutes by Theorem 9.2 (1). The right lateral face commutes by Lemma 9.4 (2). Thus, $\Psi_{\mathcal{F}}^{\mathbf{J}\text{-int}}$ intertwines the actions of $\mathcal{P}_W$ through $\mathcal{PB}_{\mathbf{J}}$. □

The (non normally ordered) quantum Weyl group action of $\mathcal{PB}_{\mathbf{J}}$ admits a similar monodromic interpretation, in analogy with Theorem 8.3. Namely, define $\mathscr{P}_{\tau,\mathscr{B}} : \mathcal{PB}_{\mathbf{J}} \to \mathrm{Aut}(\mathsf{f}_{\hbar}^{\mathbf{J}\text{-int}})$ by

$$\mathscr{P}_{\tau,\mathscr{B}}(b) = \varepsilon^{[\mathbf{J}]}(b) \cdot \mathscr{P}_{\tau_{[\mathbf{J}]},\mathscr{B}_{[\mathbf{J}]}}(b) \cdot \mathscr{B}^{[\mathbf{J}]}(b) \tag{9.2}$$

for any $b \in \mathcal{PB}_{\mathbf{J}}$. Then, the following holds.

Corollary. *Let $\mathcal{V}$ be a $\mathfrak{g}$-module in category $\mathcal{O}_{\infty}^{\hbar,\mathbf{J}\text{-int}}$, $\mathsf{F}(\mathcal{V})$ its Etingof–Kazhdan quantisation,*

$$\mathscr{P}_{\tau,\mathscr{B}} : \mathcal{PB}_{\mathbf{J}} \to GL(\mathcal{V}) \qquad \text{and} \qquad \lambda : \mathcal{PB}_{\mathbf{J}} \to GL(\mathsf{F}(\mathcal{V}))$$

the corrected $W_{\mathbf{J}}$-equivariant monodromy of the Casimir connection (9.2), and the quantum Weyl group action given by Theorem 9.2 respectively. Then, $\mathscr{P}_{\tau,\mathscr{B}}$ and λ are canonically equivalent.

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