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Subregular nilpotent orbits and explicit character formulas for modules over affine Lie algebras

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To Corrado De Concini, with admiration

Abstract: Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra and let $\widehat{\mathfrak{g}}$ be the corresponding affine Lie algebra. Kac and Wakimoto observed that in some cases the coefficients in the character formula for a simple highest weight $\widehat{\mathfrak{g}}$-module are either bounded or are given by a linear function of the weight. We explain and generalize this observation using Kazhdan-Lusztig theory, by computing values at $q = 1$ of certain (parabolic) affine inverse Kazhdan-Lusztig polynomials. In particular, we obtain explicit character formulas for some $\widehat{\mathfrak{g}}$-modules of negative integer level k when $\mathfrak{g}$ is of type D_n, E_6, E_7, E_8 and $k \geqslant -2, -3, -4, -6$ respectively, as conjectured by Kac and Wakimoto.

The calculation relies on the explicit description of the canonical basis in the cell quotient of the anti-spherical module over the affine Hecke algebra corresponding to the subregular cell. We also present an explicit description of the corresponding objects in the derived category of equivariant coherent sheaves on the Springer resolution, they correspond to irreducible objects in the heart of a certain t-structure related to the so called non-commutative Springer resolution.

1. Introduction

1.1.

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over $\mathbb{C}$ and let $\widehat{\mathfrak{g}} = \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$ be the corresponding affine Lie algebra.

Computing characters of highest weight $\widehat{\mathfrak{g}}$-modules is a classical problem of representation theory. For example, for integrable modules the answer is given by the Kac character formula which can be viewed as a direct generalization of the Weyl character formula for characters of finite dimensional

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representations of $\mathfrak{g}$. In the general case, the character formula involves the affine Kazhdan-Lusztig polynomials (or rather their evaluation at $q = 1$). While conceptually deep and algorithmically computable, this answer is much more complicated than the explicit expression appearing in the Kac character formula. It is unlikely that an essential simplification is possible in general, however, it is interesting to explore special cases when a simple character formula exists. In particular, in [22] the second author and Wakimoto identified (partly conjecturally) cases when the coefficients in the sum appearing in the character formula either take values $0, \pm 1$ (for $\mathfrak{g}$ of type A_n, $n \geqslant 2$) or depend *linearly* on the indexing weight (for $\mathfrak{g}$ of type D or E). In the present paper we partly prove their conjecture and extend their results, while connecting it to the Kazhdan-Lusztig theory. Instead of working with the (parabolic) Kazhdan-Lusztig polynomials directly, we analyze them using their relation to the Grothendieck group of equivariant coherent sheaves on the Springer resolution constructed from the Langlands dual group. Elementary geometric properties of the Springer resolution provide a transparent explanation for algebraic properties of Kazhdan-Lusztig polynomials and allow one to compute some of them effectively.

We should mention that most information about the canonical bases we use is already contained in Lusztig's work [34, 35]. The new result in this direction we provide is the realization of the basis elements as classes of explicit objects in the derived category of coherent sheaves, which arise as irreducible objects in the heart of a certain t-structure.

In order to present the content of this work in more detail we introduce further notation. Let $\mathfrak{g}^\vee$ be the Langlands dual Lie algebra. Let $\mathcal{N} \subset \mathfrak{g}^\vee$ be the nilpotent cone. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, $Q^\vee \subset \mathfrak{h}$ the coroot lattice, and $W \subset \mathrm{End}(\mathfrak{h})$ the Weyl group of $\mathfrak{g}$. Let $G^\vee$ be the adjoint group with Lie algebra $\mathfrak{g}^\vee$.

Let $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}}$ be the Cartan subalgebra of $\widehat{\mathfrak{g}}$, containing $\mathfrak{h}$. Let $\widehat{W} = W \ltimes Q^\vee$ be the affine Weyl group. For $\gamma \in Q^\vee$ we denote by t_γ the corresponding element of $\widehat{W}$ and by $w_\gamma \in \widehat{W}$ the shortest element of the coset $t_\gamma W \subset \widehat{W}$.

1.2. Characters of certain irreducible $\widehat{\mathfrak{g}}$-modules

Recall the notion of two-sided cells in $\widehat{W}$, which are certain subsets in $\widehat{W}$ (the main reference is [30], see also [16] for a short exposition). There exists a canonical bijection between the set of two-sided cells in $\widehat{W}$ and $G^\vee$-orbits on $\mathcal{N}$ (see [31]). We denote by $\mathbb{O}_c \subset \mathcal{N}$ the nilpotent orbit, corresponding to a cell $c \subset \widehat{W}$.

Let $c = c_{\mathrm{subreg}} \subset \widehat{W}$ be the cell, corresponding to the subregular nilpotent orbit. For $\nu \in Q^\vee$ let $c_\nu \subset \widehat{W}$ be the two-sided cell that contains w_ν.

We can now describe the main results of this paper. Pick $\Lambda \in \widehat{\mathfrak{h}}^*$. Let $L(\Lambda)$ be the irreducible $\widehat{\mathfrak{g}}$-module with highest weight Λ. Assume that the level of $L(\Lambda)$ is greater than $-h^\vee$ and that $\Lambda + \widehat{\rho}$ is integral quasi-dominant (these notions are defined in Section 2.1.5 and Definition 2.4). Let $w \in \widehat{W}$ be the longest element such that $w(\Lambda + \widehat{\rho})$ is dominant.

It follows from [19, Section 0.3] (see Theorem A.1 and Equation (3.7)) that the character of $L(\Lambda)$ can be expressed in terms of values at $q = 1$ of affine inverse Kazhdan-Lusztig polynomials $\mathbf{m}_w^{w_\gamma}(q)$ for $\gamma \in Q^\vee$ (see [25, Section 2] or Appendix A for the definitions). Using this observation, we derive explicit formulas for characters of $L(\Lambda)$ such that the corresponding w lies in c and is equal to w_ν for some $\nu \in Q^\vee$ (see Theorems 2.9, 2.16). We describe such Λ explicitly and compare formulas that we obtain with the results of [22], partly proving [22, Conjecture 3.2] (see Propositions 2.13, 2.15, 2.19, 2.21). Let us now describe the approach that we use to compute the values $\mathbf{m}_{w_\nu}^{w_\gamma}(1) = \mathbf{m}_{w_\nu}^{w_\gamma}$.

Consider the $\widehat{W}$-module $M := \mathbb{Z}\widehat{W} \otimes_{\mathbb{Z}W} \mathbb{Z}_{\mathrm{sign}}$, called the anti-spherical $\widehat{W}$-module. This module contains a *standard* basis $\{T_\gamma \mid \gamma \in Q^\vee\}$ and a *canonical* basis $\{C_\gamma \mid \gamma \in Q^\vee\}$ in the sense of Kazhdan-Lusztig (see Appendix A for details). By definition,

$$t_\gamma \cdot 1 = T_\gamma = \sum_{\nu \in Q^\vee} \mathbf{m}_{w_\nu}^{w_\gamma} C_\nu. \tag{1}$$

We explicitly describe a certain quotient of the module M and then consider the image of (1) in this quotient to determine the numbers $\mathbf{m}_{w_\nu}^{w_\gamma}$ for $w_\nu \in c$ (see the next section for more details).

1.3. Modules over $\widehat{W}$ via Springer theory

Let us recall the "coherent" realization of the module M and then describe the approach that we use to compute $\mathbf{m}_{w_\nu}^{w_\gamma}$ for $w_\nu \in c$.

Let $\pi\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ be the Springer resolution. For $\gamma \in Q^\vee$ we denote by $\mathcal{O}_{\mathcal{B}}(\gamma)$ the corresponding line bundle on the flag variety $\mathcal{B}$ of $G^\vee$ and by $\mathcal{O}_{\widetilde{\mathcal{N}}}(\gamma)$ its pull back to $\widetilde{\mathcal{N}}$. For every $G^\vee$-invariant locally closed subvariety $X \subset \mathcal{N}$ there is a natural action $\widehat{W} \curvearrowright K^{G^\vee}(\pi^{-1}(X))$ (see [33]), where by $K^{G^\vee}(\pi^{-1}(X))$ we denote the Grothendieck group of $G^\vee$-equivariant coherent sheaves on $\pi^{-1}(X)$.

Remark 1.1. This action comes from the identification $\mathbb{Z}\widehat{W} \xrightarrow{\sim} K^{G^\vee}(\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}})$ (see [33, Sections 7, 8] or [11, 26]), and the algebra structure on $K^{G^\vee}(\widetilde{\mathcal{N}} \times_{\mathcal{N}}$

$\widetilde{\mathcal{N}}$) is given by convolution. The algebra $K^{G^\vee}(\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}})$ acts naturally on $K^{G^\vee}(\pi^{-1}(X))$.

It is known (see, for example, [11]) that $K^{G^\vee}(\widetilde{\mathcal{N}})$ is isomorphic to the anti-spherical $\widehat{W}$-module M. The standard basis of $M \simeq K^{G^\vee}(\widetilde{\mathcal{N}})$ can be described explicitly: it consists of classes of line bundles $\mathcal{O}_{\widetilde{\mathcal{N}}}(\gamma)$, $\gamma \in Q^\vee$. The canonical basis does not have any explicit description. It can be shown that the canonical basis consists of classes of irreducible objects of the heart of the "exotic" t-structure on the derived category $D^b(\mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}}))$ (see [5] and [7] for details).

Let $U \subset \mathcal{N}$ be an open $G^\vee$-invariant subvariety. Set $\widetilde{U} := \pi^{-1}(U)$. It follows from [7, §11.3] that the kernel of the restriction homomorphism $K^{G^\vee}(\widetilde{\mathcal{N}}) \twoheadrightarrow K^{G^\vee}(\widetilde{U})$ is freely generated (as a module over $\mathbb{Z}$) by elements C_ν such that $\mathbb{O}_{c_\nu} \not\subset U$. In particular, $K^{G^\vee}(\widetilde{U})$ admits a canonical basis parametrized by $\{\nu \in Q^\vee \mid \mathbb{O}_{c_\nu} \subset U\}$.

Recall now that our goal is to compute the numbers $\mathbf{m}_{w_\nu}^{w_\gamma}$ for $w_\nu \in c$, where $c \subset \widehat{W}$ is the cell, corresponding to the subregular nilpotent. The numbers $\mathbf{m}_{w_\nu}^{w_\gamma}$ are determined by equation (1) as follows.

Consider $U = \mathbb{O}_e \cup \mathbb{O}^{\mathrm{reg}} \subset \mathcal{N}$, where $\mathbb{O}_e \subset \mathcal{N}$ is the $G^\vee$-orbit of a subregular nilpotent element $e \in \mathcal{N}$. It follows from the above that the canonical basis in $K^{G^\vee}(\widetilde{U})$ is parametrized by $\{1\} \cup \{\nu \in Q^\vee \mid w_\nu \in c\}$. For $\gamma, \nu \in Q^\vee$ such that $w_\nu \in c \cup \{1\}$ let $\bar{T}_\gamma$ and $\bar{C}_\nu$ be the images of T_γ and C_ν in $K^{G^\vee}(\widetilde{U})$. Taking the image of (1) in $K^{G^\vee}(\widetilde{U})$, we conclude that

$$t_\gamma \cdot \bar{1} = \bar{T}_\gamma = \sum_{\nu \in Q^\vee,\, w_\nu \in c \cup \{1\}} \mathbf{m}_{w_\nu}^{w_\gamma} \bar{C}_\nu.$$

So, to compute the coefficients $\mathbf{m}_{w_\nu}^{w_\gamma}$ as above, it is enough to describe the $\widehat{W}$-module structure on $K^{G^\vee}(\widetilde{U})$ and the canonical basis $\{\bar{C}_\nu \mid w_\nu \in c \cup \{1\}\}$ in it. Let us describe the answer.

The cell c has an explicit description (see [29, Proposition 3.8]): it consists of elements $w \in \widehat{W}$ with *unique* reduced decomposition. The subset $\{w \in c \mid w = w_\nu \text{ for some } \nu \in Q^\vee\}$ consists of elements $w \in c$ such that the reduced decomposition of w ends by s_0 (simple reflection, corresponding to the 0th vertex of the Dynkin diagram of $\widehat{\mathfrak{g}}$). This set can be described explicitly (see Corollary 5.2 and Lemma 6.1). It turns out that for $\mathfrak{g}$ of type D, E it is parametrized by the set $\widehat{I}$ of vertices of the Dynkin diagram of $\widehat{\mathfrak{g}}$, and for $\mathfrak{sl}_n$ ($n \geqslant 3$) it is parametrized by $\mathbb{Z}$ (that should be considered as the set of vertices of the Dynkin diagram A_∞).

We prove (see Proposition 5.16) that in types D, E the $\widehat{W}$-module $K^{G^\vee}(\widetilde{U}) \otimes \mathbb{Z}_{\mathrm{sign}}$ is isomorphic to the integral form $\widehat{\mathfrak{h}}_{\mathbb{Z}}$ of the Cartan subalgebra

$\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}}$, i.e. that

$$K^{G^\vee}(\widetilde{U}) \simeq (\mathbb{Z}Q^\vee \oplus \mathbb{Z}K \oplus \mathbb{Z}d) \otimes \mathbb{Z}_{\mathrm{sign}}$$

as $\widehat{W}$-modules. After this identification the canonical basis consists of d and minus simple coroots of $\widehat{\mathfrak{g}}$.

In type A we identify $K^{\mathrm{PGL}_n}(\mathcal{B}_e)$ with $(\mathfrak{h}_{\infty,\mathbb{Z}} \oplus \mathbb{Z}d) \otimes \mathbb{Z}_{\mathrm{sign}}$ and describe explicitly the $\widehat{W}$-action on the latter (see Section 6.4.2 and Proposition 6.17). After this identification the canonical basis consists of d and minus simple coroots of $\mathfrak{sl}_\infty$.

1.4. Structure of the paper

The paper is organized as follows. In Section 2 we recall the structure theory and representation theory of affine Lie algebras (see Section 2.1), we then give character formulas for certain $L(\Lambda)$ (see Theorem 2.5 for types D, E and Theorem 2.16 for type A) and rewrite them in more explicit terms (see Propositions 2.13, 2.15 for types D, E and Propositions 2.19, 2.21 for type A). In Section 3 we recall categories $\mathcal{O}$ for $\widehat{\mathfrak{g}}$ and describe characters of irreducible $\widehat{\mathfrak{g}}$-modules via values at $q = 1$ of (affine) inverse Kazhdan-Lusztig polynomials. In Section 4 we recall the Springer resolution and the geometric realization of the anti-spherical $\widehat{W}$-module M, we also recall some information about the canonical basis, in particular, we describe explicitly the canonical basis of $K^{G^\vee}(\widetilde{U})$. In Section 5 we describe $\widehat{W}$-module $K^{G^\vee}(\widetilde{U})$ explicitly for $\mathfrak{g}$ of type D, E (see Proposition 5.16). We then compute $\mathbf{m}_{w_\nu}^{w_\gamma}$ for $w_\nu \in c$ and derive Theorem 2.9 (see Section 5.3). In Section 6 we describe $\widehat{W}$-module $K^{\mathrm{PGL}_n}(\widetilde{U})$ explicitly (see Proposition 6.17). We then compute $\mathbf{m}_{w_\nu}^{w_\gamma}$ for $w_\nu \in c$ and derive Theorem 2.16 (see Section 6.7). In Section 7 we discuss possible generalizations. Appendix A contains the information about Kazhdan-Lusztig bases that we use.

2. Affine Lie algebras and their representation theory

2.1. Affine Lie algebras: structure theory and irreducible highest weight modules [21]

2.1.1. Simple Lie algebra $\mathfrak{g}$: notations Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over $\mathbb{C}$. We fix a Catan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and denote by Δ the set of roots of $(\mathfrak{h}, \mathfrak{g})$ and by W the Weyl group of $(\mathfrak{h}, \mathfrak{g})$. Let Q be the root lattice of $\mathfrak{g}$. Let $\alpha_1, \dots, \alpha_r \in \Delta$ be a set of simple roots, and $\theta \in \Delta$ be the highest root. We denote by $\Delta_+ \subset \Delta$ the set of positive roots and set

$\rho := \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$. We fix the nondegenerate invariant symmetric bilinear form $(\ ,\ )$ on $\mathfrak{g}$ normalized by $(\theta,\theta)=2$. Let $\alpha_1^\vee,\dots,\alpha_r^\vee\in\mathfrak{h}$ be the simple coroots defined by

$$\langle\alpha_j,\alpha_i^\vee\rangle = a_{ij},$$

where $A=(a_{ij})_{i,j=1,\dots,r}$ is the Cartan matrix of $\mathfrak{g}$. Let $\Delta^\vee$ be the W-orbit of $\{\alpha_1^\vee,\dots,\alpha_r^\vee\}$. We also denote by $\theta^\vee\in\mathfrak{h}$ the highest coroot of $\Delta^\vee$.

2.1.2. Affine Lie algebra $\widehat{\mathfrak{g}}$ We denote by $\widehat{\mathfrak{g}}$ the affine Lie algebra, corresponding to $\mathfrak{g}$. Recall that

$$\widehat{\mathfrak{g}} := \mathfrak{g}[t^{\pm1}]\oplus\mathbb{C}K\oplus\mathbb{C}d$$

with the bracket defined as follows ($a,b\in\mathfrak{g}$, $m,n\in\mathbb{Z}$):

$$[at^m,bt^n]=[a,b]t^{m+n}+m(a,b)\delta_{m,-n}K,\ [d,at^n]=nat^n,\ [K,\widehat{\mathfrak{g}}]=0.$$

The Lie algebra $\widehat{\mathfrak{g}}$ has a nondegenerate invariant symmetric bilinear form $(\ ,\ )$ defined by

$$(at^m,bt^n)=\delta_{m,-n}(a,b),\ (\mathbb{C}K\oplus\mathbb{C}d,\mathfrak{g}[t^{\pm1}])=0,$$
$$(K,K)=(d,d)=0,\ (K,d)=1.$$

This bilinear form restricts to a nondegenerate bilinear form on the Cartan subalgebra of $\widehat{\mathfrak{g}}$:

$$\widehat{\mathfrak{h}} := \mathfrak{h}\oplus\mathbb{C}K\oplus\mathbb{C}d.$$

We extend every $\gamma\in\mathfrak{h}^*$ to the linear function on $\widehat{\mathfrak{h}}$ by setting $\langle\gamma,\mathbb{C}K\oplus\mathbb{C}d\rangle=0$. Let $\delta\in\widehat{\mathfrak{h}}^*$ be the linear function given by $\langle\delta,\mathfrak{h}\oplus\mathbb{C}K\rangle=0$, $\langle\delta,d\rangle=1$. Set $\alpha_0:=\delta-\theta\in\mathfrak{h}^*$, $\alpha_0^\vee:=K-\theta^\vee\in\mathfrak{h}$. Then $\{\alpha_0,\alpha_1,\dots,\alpha_r\}$ are simple roots of $\widehat{\mathfrak{g}}$ and $\{\alpha_0^\vee,\alpha_1^\vee,\dots,\alpha_r^\vee\}$ are simple coroots. Define the fundamental weights $\Lambda_i\in\widehat{\mathfrak{h}}^*$ by

$$\langle\Lambda_i,\alpha_j^\vee\rangle := \delta_{i,j},\ i,j=0,1,\dots,r.$$

We denote by $\eta\colon\widehat{\mathfrak{h}}\xrightarrow{\sim}\widehat{\mathfrak{h}}^*$ the identification induced by the bilinear form $(\ ,\ )$.

From now on, we assume for simplicity that $\mathfrak{g}$ is of type A,D,E. So we have

$$\eta(\alpha_i^\vee)=\alpha_i,\ \eta(\theta^\vee)=\theta,\ \eta(\alpha_0^\vee)=\alpha_0,\ \eta(K)=\delta,\ \eta(d)=\Lambda_0,\ i=1,\dots,r.$$

2.1.3. Root system of $\widehat{\mathfrak{g}}$ Let $\widehat{\Delta}$ be the root system of $\widehat{\mathfrak{g}}$. Recall that $\widehat{\Delta}$ can be decomposed as a disjoint union of the sets of *real* and *imaginary* roots:

$$\widehat{\Delta} = \widehat{\Delta}^{\mathrm{re}} \cup \widehat{\Delta}^{\mathrm{im}},$$

where

$$\widehat{\Delta}^{\mathrm{re}} = \{\alpha + n\delta \mid \alpha \in \Delta,\, n \in \mathbb{Z}\},\ \widehat{\Delta}^{\mathrm{im}} = \{n\delta \mid n \in \mathbb{Z}_{\neq 0}\}.$$

A root $\alpha \in \widehat{\Delta}$ is called *positive* if it can be obtained as a *nonnegative* linear combination of simple roots $\alpha_i \in \widehat{\Delta}$, $i = 0, 1, \ldots, r$. The subset of $\widehat{\Delta}$, consisting of positive roots, will be denoted $\widehat{\Delta}_+ \subset \widehat{\Delta}$ and can be described as follows:

$$\widehat{\Delta}_+ = \widehat{\Delta}_+^{\mathrm{re}} \cup \widehat{\Delta}_+^{\mathrm{im}},$$

where

$$\widehat{\Delta}_+^{\mathrm{re}} = \Delta_+ \cup \{\alpha + n\delta \mid \alpha \in \Delta,\, n \in \mathbb{Z}_{\geqslant 1}\},\ \widehat{\Delta}_+^{\mathrm{im}} = \{n\delta \mid n \in \mathbb{Z}_{\geqslant 1}\}.$$

To every $\gamma \in \widehat{\Delta}^{\mathrm{re}}$ we associate the corresponding coroot $\gamma^\vee \in \widehat{\mathfrak{h}}$, defined by $\gamma^\vee := \alpha^\vee + nK \in \widehat{\mathfrak{h}}$ if $\gamma = \alpha + n\delta \in \widehat{\mathfrak{h}}^*$.

2.1.4. Weyl group of $\widehat{\mathfrak{g}}$ Let $\widehat{W}$ be the Weyl group of $\widehat{\mathfrak{g}}$. The group $\widehat{W}$ is the subgroup of $\mathrm{Aut}(\widehat{\mathfrak{h}})$ generated by the reflections s_i defined by

$$s_i(x) := x - \langle \alpha_i, x\rangle \alpha_i^\vee,\ x \in \widehat{\mathfrak{h}},\ i = 0, 1, \ldots, r. \tag{2}$$

For $\gamma \in Q^\vee$ define the operator $t_\gamma \in \mathrm{Aut}(\widehat{\mathfrak{h}})$ by

$$t_\gamma(x) = x + \langle \delta, x\rangle \gamma - ((x, \gamma) + \frac{1}{2}|\gamma|^2 \langle \delta, x\rangle) K,$$

where $|\gamma|^2 = (\gamma, \gamma)$.

The group $\widehat{W}$ is generated by s_i, $i = 1, \ldots, r$, and t_γ, $\gamma \in Q^\vee$, so that we obtain the identification

$$\widehat{W} = Q^\vee \rtimes W.$$

Remark 2.1. Note that $\widehat{\mathfrak{h}}$ contains $\mathbb{C}K$ as the trivial subrepresentation of $\widehat{W}$. Note also that $\mathbb{C}K \oplus \mathfrak{h} \subset \widehat{\mathfrak{h}}$ is a subrepresentation: for $x \in \mathbb{C}K \oplus \mathfrak{h}$ the action of t_γ is given by

$$t_\gamma(x) = x - (x, \gamma)K,\ \gamma \in Q^\vee.$$

The group $\widehat{W}$ is a Coxeter group generated by reflections $s_0, s_1, \dots, s_r$ and so is equipped with the *length* function

$$\ell\colon \widehat{W} \to \mathbb{Z}_{\geqslant 0},$$

where $\ell(w)$ is the length of a shortest expression of w in terms of the s_i.

Recall that

$$\varepsilon(w) := (-1)^{\ell(w)} = \det_{\widehat{\mathfrak{h}}} w \text{ and } \varepsilon(t_\gamma) = 1 \text{ for all } \gamma \in Q^\vee. \tag{3}$$

2.1.5. Irreducible highest weight representations of $\widehat{\mathfrak{g}}$ and their characters Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+ \subset \mathfrak{g}$ be the Borel subalgebra, corresponding to our choice of simple roots $\alpha_1, \dots, \alpha_r$. One defines the corresponding Borel subalgebra of $\widehat{\mathfrak{g}}$:

$$\widehat{\mathfrak{b}} := \widehat{\mathfrak{h}} \oplus \mathfrak{n}_+ \oplus \bigoplus_{n>0} \mathfrak{g}t^n.$$

Given $\Lambda \in \widehat{\mathfrak{h}}^*$ one extends it to the character of $\widehat{\mathfrak{b}}$ by zero on all other summands. Then there exists a unique irreducible $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ with highest weight Λ. Let us recall the construction of $L(\Lambda)$. Consider the Verma module

$$M(\Lambda) := U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{b}})} \mathbb{C}_\Lambda,$$

where $\mathbb{C}_\Lambda$ is the one dimensional representation of $\widehat{\mathfrak{b}}$ given by the character Λ. Then $L(\Lambda)$ is the unique (nonzero) irreducible quotient of the module $M(\Lambda)$. Let $\kappa = \Lambda(K)$ be the scalar by which $K \in \widehat{\mathfrak{g}}$ acts on $L(\Lambda)$ (and $M(\Lambda)$). This scalar is called the *level* of $L(\Lambda)$ (and the level of Λ), and is denoted by $\kappa(\Lambda)$.

For $\mu \in \widehat{\mathfrak{h}}^*$ and a $\widehat{\mathfrak{g}}$-module M let $M_\mu \subset M$ be the (generalized) weight space of M with weight μ. The characters of $M = L(\Lambda)$ or $M(\Lambda)$ are defined as the following (formal) series:

$$\operatorname{ch} M := \sum_{\mu \in \widehat{\mathfrak{h}}^*} (\dim M_\mu) \cdot e^\mu,$$

here e^μ are formal exponentials such that $e^{\mu_1} \cdot e^{\mu_2} = e^{\mu_1+\mu_2}$ (note that $\dim M_\mu < \infty$).

Remark 2.2. Note that one can consider e^μ as a function on $\widehat{\mathfrak{h}}$ whose value on $h \in \widehat{\mathfrak{h}}$ is equal to $e^{\mu(h)}$. Then $\operatorname{ch} M$ can be considered as the series $(\operatorname{ch} M)(h) = \operatorname{tr}_M e^h$. This series is convergent in the domain $\{h \in \widehat{\mathfrak{h}} \mid \operatorname{Re}(\alpha_i(h)) > 0, i = 0, 1, \dots, r\}$.

We set

$$\widehat{\rho} := \rho + h^\vee \Lambda_0,$$

where $h^\vee$ is the dual Coxeter number ($= \frac{1}{2}$ the eigenvalue on $\mathfrak{g}$ of the Casimir element). Recall that $\langle \widehat{\rho}, \alpha_i^\vee \rangle = 1$ for every $i = 0, 1, \dots, r$, and $\widehat{\rho} = \sum_{i=0}^r \Lambda_i$.

Set

$$\widehat{R} := e^{\widehat{\rho}} \prod_{\alpha \in \widehat{\Delta}_+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}.$$

Then

$$\widehat{R} \operatorname{ch} M(\Lambda) = e^{\Lambda + \widehat{\rho}}.$$

The main result of this note are explicit formulas for characters of modules $L(\Lambda)$ for certain Λ of integer level $\kappa(\Lambda) > -h^\vee$.

Remark 2.3. Note that the condition that the level of Λ is greater than $-h^\vee$ is equivalent to the fact that the level of $\Lambda + \widehat{\rho}$ is *positive*.

2.2. Motivation and main result

2.2.1. Motivation and Kac-Wakimoto conjecture We start with the following definition.

Definition 2.4. An element $\Lambda \in \widehat{\mathfrak{h}}^*$ is called *regular* if it has trivial stabilizer w.r.t. $\widehat{W} \curvearrowright \widehat{\mathfrak{h}}^*$; this condition is equivalent to $\langle \Lambda, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$. Element Λ is called *singular* if it is not regular (i.e. has nontrivial stabilizer in $\widehat{W}$). An element $\Lambda \in \widehat{\mathfrak{h}}^*$ is called *integral* if $\langle \Lambda, \alpha_i^\vee \rangle \in \mathbb{Z}$, $i = 0, 1, \dots, r$; Λ is called *dominant* (resp. *quasi-dominant*) if $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geqslant 0}$ for $i = 0, 1, \dots, r$ (resp. for $i = 1, \dots, r$).

Consider the following (shifted) action of $\widehat{W}$ on $\widehat{\mathfrak{h}}^*$:

$$w \circ \Lambda := w(\Lambda + \widehat{\rho}) - \widehat{\rho}.$$

For $\lambda \in \widehat{\mathfrak{h}}^*$ we denote by $\widehat{W}_\lambda \subset \widehat{W}$ the stabilizer of λ w.r.t. the shifted action.

Recall that by [23] for any $\Lambda \in \widehat{\mathfrak{h}}^*$ of level $\kappa(\Lambda) > -h^\vee$ one has

$$\widehat{R} \operatorname{ch} L(\Lambda) = \sum_{w \in \widehat{W}} c(w) e^{w(\Lambda + \widehat{\rho})} \text{ for some } c(w) \in \mathbb{Z}. \tag{4}$$

In addition, if Λ is quasi-dominant integral, one has (use W-invariance of $\operatorname{ch} L(\Lambda)$)

$$c(ut_\gamma) = \varepsilon(u) c(t_\gamma), \ u \in W, \ \gamma \in Q^\vee.$$

The motivation for this work is the following theorem proven in [22, Section 3].

Theorem 2.5. *Let $\mathfrak{g}$ be a simple Lie algebra of type D_n $(n \geqslant 4)$ or E_6, E_7, E_8 and Λ be a weight of $\widehat{\mathfrak{g}}$ of negative integral level κ such that the following conditions hold:*

(*i*) *Λ is quasi-dominant integral,*
(*ii*) *there exists a root $\alpha \in \Delta_+$, such that $(\Lambda + \widehat{\rho}, \delta - \alpha) = 0$, and if $\beta \in \widehat{\Delta}_+$ is orthogonal to $\Lambda + \widehat{\rho}$, then $\beta = \delta - \alpha$,*
(*iii*) *(extra hypothesis) in (4) one has: $c(t_\gamma)$ is a linear function in $\gamma \in Q^\vee$ plus constant.*

Then

$$\widehat{R}\,\mathrm{ch}\,L(\Lambda) = \frac{1}{2}\sum_{u\in W}\varepsilon(u)\Big(\sum_{\gamma\in Q^\vee}(\langle\alpha,\gamma\rangle+1)e^{ut_\gamma(\Lambda+\widehat{\rho})}\Big). \tag{5}$$

Example 2.6. Let us give examples of Λ, satisfying conditions (i), (ii) of Theorem 2.5 for $\mathfrak{g}$ of type D_4. To $\Lambda \in \widehat{\mathfrak{h}}^*$ we associate the element $w \in \widehat{W}$ that is the longest element such that $\lambda := w \circ \Lambda$ is dominant. Then the following is a *partial* list of Λ of level -1, satisfying conditions (i) and (ii) of Theorem 2.5 (we label the branching node of the Dynkin diagram of D_4 by 2), and the corresponding $\alpha \in \Delta_+$ and $w \in \widehat{W}$, $\lambda \in \widehat{\mathfrak{h}}^*$:

1) $\Lambda = -\Lambda_0$, $\alpha = \theta = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$, $w = s_0$, $\lambda = -\Lambda_0$,
2) $\Lambda = -2\Lambda_0 + \Lambda_k$, $\alpha = \theta - \alpha_2$, $w = s_2 s_0$, $\lambda = \Lambda_k - \Lambda_2$,
3) $\Lambda = -3\Lambda_0 + \Lambda_2$, $\alpha = \theta - \alpha_2$, $w = s_2 s_0$, $\lambda = \Lambda_0 - \Lambda_2$,
4) $\Lambda = -3\Lambda_0 + \Lambda_k + \Lambda_l$, $\alpha = \theta - \alpha_2 - \alpha_p$, $w = s_p s_2 s_0$, $\lambda = -\Lambda_p$,
5) $\Lambda = -4\Lambda_0 + 2\Lambda_k + \Lambda_l$, $\alpha = \theta - \alpha_2 - \alpha_l$, $w = s_l s_p s_2 s_0$, $\lambda = -\Lambda_l$,
6) $\Lambda = -5\Lambda_0 + 2\Lambda_k + \Lambda_l + \Lambda_p$, $\alpha = \theta - \alpha_2 - \alpha_k$, $w = s_k s_p s_l s_2 s_0$, $\lambda = -\Lambda_k$,
7) $\Lambda = -4\Lambda_0 + 3\Lambda_k$, $\alpha = \theta - \alpha_2 - \alpha_p - \alpha_l$, $w = s_2 s_l s_p s_2 s_0$, $\lambda = \Lambda_k - \Lambda_2$, where k, l are distinct elements of the set $\{1, 3, 4\}$ and $p \in \{1, 3, 4\} \setminus \{k, l\}$.

Examples 1)–4) are given in [22, Example 3.5].

Example 2.7. Let $\mathfrak{g}$ be of type D_n $(n \geqslant 4)$, E_6, E_7, E_8 and let b be the maximal among the coefficients a_i in $\theta = \sum_{i=1}^r a_i\alpha_i$. Then there is no Λ, satisfying conditions (i) and (ii) of Theorem 2.5 if $\kappa(\Lambda) < -b$, and there is only one, $\Lambda = -b\Lambda_0$, if $\kappa(\Lambda) = -b$. Furthermore, $\Lambda = -\kappa\Lambda_0$ for $\kappa \in \mathbb{Z}$, $1 \leqslant \kappa \leqslant b$ always satisfies conditions (i) and (ii) of Theorem 2.5, and then $\alpha = \theta - \sum_{i=1}^{\kappa-1}\alpha_i$, $w = s_{\kappa-1}\ldots s_0$, where we label vertices of the Dynkin

diagram of $\widehat{\mathfrak{g}}$ in such a way that the affine vertex has label 0 and i, $i+1$ are adjacent for $i \in \{0, 1, \dots, b-2\}$ (so, in particular, the label of the branching point is $b-1$).

It is conjectured in [22, Conjecture 3.2] that if $\mathfrak{g}$ is of type D_4, E_6, E_7, E_8, then the extra hypothesis (iii) holds (so we obtain the character formula (5) for $L(\Lambda)$ such that Λ satisfies conditions $(i), (ii)$ of Theorem 2.5). The main motivation of this work was to prove this conjecture in some cases. For example, in the case of $\mathfrak{g} = D_4$ we prove the conjecture in Examples 2.6 1)–4) (but not for 5)–7)). We don't know how to prove the conjecture in general.

We will actually compute (under some conditions) the numbers $c(t_\gamma)$ explicitly that will allow us to compute characters of $L(\Lambda)$ for certain Λ that appear in Theorem 2.5 and also of certain other Λ (see Theorem 2.9 and Propositions 2.13, 2.15 below).

2.2.2. Main result: types *D* and *E* Let $\mathfrak{g}$ be of type D_n $(n \geqslant 4)$, E_6, E_7, E_8. Before formulating the main result for types D and E we need to introduce some notation.

Let I be the set of vertices of the Dynkin diagram of $\mathfrak{g}$. We fix a labeling of I by the numbers $1, \dots, r$. Let $\widehat{I} = I \cup \{0\}$ be the set of vertices of the Dynkin diagram of $\widehat{\mathfrak{g}}$. For $i \in \widehat{I}$ consider the unique segment in $\widehat{I}$, connecting i and 0. Let l be the length of this segment (i.e. this segment consists of $l+1$ vertices). Let

$$0 = j_0, j_1, \dots, j_{l-1}, j_l = i$$

be the set of vertices that form the segment above. We set

$$w_i := s_i s_{j_{l-1}} \dots s_{j_1} s_0.$$

Remark 2.8. Note that for $i = 0$ we have $w_0 = s_0$.

We are now ready to describe the main result. The following theorem holds (see Section 5.3 for the proof).

Theorem 2.9. *Let $\mathfrak{g}$ be of type D_n $(n \geqslant 4)$ or E_6, E_7, E_8. Pick $i \in \{0, 1, \dots, r\}$ and let $\lambda \in \widehat{\mathfrak{h}}^*$ be an integral weight, such that $\lambda + \widehat{\rho}$ is dominant, $\Lambda = w_i^{-1} \circ \lambda$ is quasi-dominant and w_i is the longest element in the coset $\widehat{W}_\lambda w_i$. Then*

$$\widehat{R} \operatorname{ch} L(\Lambda) = \sum_{u \in W} \varepsilon(u w_i) \sum_{\gamma \in Q^\vee} \left\langle \Lambda_i, \gamma + \frac{|\gamma|^2}{2} K \right\rangle e^{u t_\gamma w_i (\Lambda + \widehat{\rho})}. \tag{6}$$

Remark 2.10. Note that we are not assuming in Theorem 2.9 that the level of Λ is negative. Actually, Lemma 2.11 below implies that $\kappa(\Lambda) \geqslant 0$ in case (a) and $\kappa(\Lambda) \geqslant -b$ in case (b) of this lemma (here b is as in Example 2.7 above).

The following lemma describes explicitly all the pairs λ, i, satisfying the conditions of Theorem 2.9.

Lemma 2.11. *Let $\mathfrak{g}$ be of type D_n $(n \geqslant 4)$ or E_6, E_7, E_8. Elements $\lambda \in \widehat{\mathfrak{h}}^*$, $i \in \{0, 1, \dots, r\}$ as in Theorem 2.9 are described as follows. There are two possibilities:*

(a) *$\lambda + \widehat{\rho}$ is regular dominant integral and i is an arbitrary element of $\widehat{I}$,*
(b) *$\lambda = -\Lambda_i + \sum_{k \neq i} m_k \Lambda_k + x\delta$ for some $i \in \{0, 1, \dots, r\}$, $x \in \mathbb{C}$, and $m_k \in \mathbb{Z}_{\geqslant 0}$.*

Proof. Let us first of all show that if λ, i are as in Theorem 2.9, then either $\lambda + \widehat{\rho}$ is regular or $\lambda = -\Lambda_i + \sum_{k \neq i} m_k \Lambda_k + x\delta$.

Indeed, assume that $\lambda + \widehat{\rho}$ is not regular. Recall that $\lambda + \widehat{\rho}$ is dominant, so its stabilizer in $\widehat{W}$ (with respect to the standard non shifted action of $\widehat{W}$) is generated by some simple reflections $s_{i_1}, \dots, s_{i_t}$. Our goal is to show that $t = 1$ and $i_1 = i$. Otherwise there exists $d \in \{1, \dots, t\}$ such that $i_d \neq i$. Consider the element

$$w' := s_{i_d} w_i = s_{i_d} s_i s_{j_{l-1}} \dots s_{j_1} s_0$$

and note that $w'^{-1} \circ \lambda = w_i^{-1} \circ \lambda$ is quasi-dominant (by our assumptions). We claim that $\ell(w') = \ell(w_i) + 1$ or equivalently $\ell(w'^{-1}) = \ell(w_i^{-1}) + 1$. To see this we need to show that $w_i^{-1}(\alpha_{i_d})$ is a positive root. If i_d does not lie in the set $\{0, j_1, \dots, j_{l-1}\}$ and i_d is not adjacent to i, then $w_i^{-1}(\alpha_{i_d}) = \alpha_{i_d}$ is clearly positive. If α_{i_d} is adjacent to $i = j_l$ and is not equal to j_{l-1}, then $w_i^{-1}(\alpha_{i_d}) = \alpha_{i_d} + \alpha_i$ is positive. Finally, if $i_d = j_p$ for some $p \in \{0, 1, \dots, l-1\}$, then we have $w_i^{-1}(\alpha_{i_d}) = \alpha_{j_{p+1}}$ is positive.

So we have shown that $\ell(w') = \ell(w_i) + 1$, $w_i^{-1} \circ \lambda$ is quasi-dominant and $w' = s_{i_d} w_i \in \widehat{W}_\lambda w_i$. This contradicts to our assumptions (we assumed that w_i is the longest element in $\widehat{W}_\lambda w_i$). So we conclude that $\widehat{W}_\lambda = \{1, s_i\}$ and λ must be of the form $-\Lambda_i + \sum_{k \neq i} m_k \Lambda_k$ with m_k being nonnegative.

It remains to show that if λ, i satisfy assumption (a) or (b) of Lemma 2.11, then they satisfy the assumptions of Theorem 2.9. The only nontrivial part is to check that $w_i^{-1} \circ \lambda = w_i^{-1}(\lambda + \widehat{\rho}) - \widehat{\rho}$ is quasi-dominant.

Let us decompose $\lambda = \sum_k m_k \Lambda_k + x\delta$. We have $\lambda + \widehat{\rho} = \sum_k (m_k + 1)\Lambda_k + x\delta$. We need to apply $w_i^{-1} = s_0 s_{j_1} \dots s_{j_{l-1}} s_i$ to $\lambda + \widehat{\rho}$ and show that after substituting $\widehat{\rho}$ we get quasi-dominant element. Indeed, recall that if $j \in \widehat{I}$,

then the action of s_j on some Λ_k is equal to Λ_k if $k \neq j$, the action of s_j on Λ_j is equal to $-\Lambda_j$ plus the sum of $\Lambda_{j'}$, where $j' \in \widehat{I}$ runs through all vertices that are adjacent to j. We easily conclude that the coefficient of $w_i^{-1}(\lambda+\widehat{\rho})$ in front of some Λ_{j_p} ($p \in \{1,\dots,l\}$) is equal to $m_{j_{p-1}}+1>0$, the coefficient in front of Λ_0 is equal to $-(m_0+m_{j_1}+\cdots+m_{j_{l-1}}+m_i+l+1)$ and coefficients in front of other Λ_k are at least m_k+1, so are positive. We have shown that $w_i^{-1}\circ\lambda$ is quasi-dominant. □

Remark 2.12. The last paragraph of the proof of Lemma 2.11 can be omitted since it follows from Lemma 3.11 below.

The following proposition shows how to deduce formulas for characters of (certain) modules appearing in [22, Conjecture 3.2] from Theorem 2.9 (note that we get the same character formula as in [22, Theorem 3.1]).

Proposition 2.13. *Let $\mathfrak{g}$ be of type D_n ($n \geqslant 4$) or E_6, E_7, E_8. Assume that $\lambda = -\Lambda_i + \sum_{k\neq i} m_k\Lambda_k + x\delta$ for some $i \in \{0,1,\dots,r\}$, $x \in \mathbb{C}$ and $m_k \in \mathbb{Z}_{\geqslant 0}$ for $k \neq i$. Let $\Lambda = w_i^{-1}\circ\lambda$ and $\alpha = w_i^{-1}(\alpha_i)+\delta$. Then we have*

$$\widehat{R}\,\mathrm{ch}\,L(\Lambda) = \frac{1}{2}\sum_{u\in W}\varepsilon(u)\Big(\sum_{\gamma\in Q^\vee}(\langle\alpha,\gamma\rangle+1)e^{ut_\gamma(\Lambda+\widehat{\rho})}\Big). \tag{7}$$

Proof. It follows from Lemma 2.11 that λ, i satisfy conditions of Theorem 2.9. So the character of $L(\Lambda)$ is computed using the formula (6). Pick $u \in W$, $\gamma \in Q^\vee$ and let us compute the coefficient of (6) in front of $e^{ut_\gamma(\lambda+\widehat{\rho})}$. Assume first that $i=0$. It follows that $\lambda=\Lambda$, $i=0$ and $w_i = s_0$. Since $s_0 = s_\theta t_{-\theta}$ and $s_0(\Lambda+\widehat{\rho}) = \Lambda+\widehat{\rho}$, we conclude that $us_\theta t_{s_\theta(\gamma)-\theta}(\Lambda+\widehat{\rho}) = ut_\gamma(\Lambda+\widehat{\rho})$, so the coefficient in front of $e^{ut_\gamma(\Lambda+\widehat{\rho})}$ in (6) is equal to

$$\begin{aligned}&-\varepsilon(u)\Big\langle\Lambda_0,\gamma+\frac{|\gamma|^2}{2}K\Big\rangle+\varepsilon(u)\Big\langle\Lambda_0,s_\theta(\gamma)-\theta+\frac{|s_\theta(\gamma)-\theta|^2}{2}K\Big\rangle\\&=\varepsilon(u)\Big(\frac{|s_\theta(\gamma)-\theta|^2}{2}-\frac{|\gamma|^2}{2}\Big)=\varepsilon(u)(\langle\theta,\gamma\rangle+1).\end{aligned}$$

Assume now that $i\neq 0$. Note that $\lambda+\widehat{\rho} = s_i(\lambda+\widehat{\rho})$, hence, $us_it_{s_i(\gamma)}(\lambda+\widehat{\rho}) = ut_\gamma(\lambda+\widehat{\rho})$, so the coefficient in front of $e^{ut_\gamma(\lambda+\widehat{\rho})}$ is equal to

$$\begin{aligned}&\varepsilon(uw_i)\Big\langle\Lambda_i,\gamma+\frac{|\gamma|^2}{2}K\Big\rangle-\varepsilon(uw_i)\Big\langle\Lambda_i,s_i(\gamma)+\frac{|\gamma|^2}{2}K\Big\rangle\\&=\varepsilon(uw_i)\langle\Lambda_i,\alpha_i^\vee\rangle\langle\alpha_i,\gamma\rangle=\varepsilon(uw_i)\langle w_i(\alpha-\delta),\gamma\rangle=\varepsilon(uw_i)\langle\alpha,w_i^{-1}\gamma\rangle.\end{aligned}$$

Consider the element $w_i' := w_i t_\theta = s_i s_{j_{l-1}} \dots s_{j_1} s_\theta$. We have shown that the coefficient in front of

$$e^{ut_\gamma(\lambda+\widehat{\rho})} = e^{ut_\gamma w_i(\Lambda+\widehat{\rho})} = e^{uw_i' t_{w_i'^{-1}(\gamma)-\theta}(\Lambda+\widehat{\rho})}$$

is equal to

$$\begin{aligned}\varepsilon(uw_i)\langle\alpha, w_i^{-1}\gamma\rangle &= \varepsilon(uw_i')\langle\alpha, t_\theta(w_i'^{-1}\gamma)\rangle \\ &= \varepsilon(uw_i')\langle\alpha, w_i'^{-1}\gamma - (\theta, w_i'^{-1}\gamma)K\rangle = \varepsilon(uw_i')\langle\alpha, w_i'^{-1}\gamma\rangle \\ &= \varepsilon(uw_i')(\langle\alpha, w_i'^{-1}\gamma - \theta\rangle + \langle\alpha, \theta\rangle),\end{aligned}$$

so it remains to check that $(\alpha, \theta) = 1$. Indeed, recall that $\alpha = w_i^{-1}(\alpha_i) + \delta$, so

$$\begin{aligned}(\alpha, \theta) &= (w_i^{-1}(\alpha_i), \theta) = (\alpha_i, w_i(\theta)) = -(\alpha_i, w_i(\delta - \theta)) \\ &= -(\alpha_i, w_i(\alpha_0)) = (\alpha_i, s_i s_{j_{l-1}} \dots s_{j_1}\alpha_0) \\ &= (\alpha_i, \alpha_0 + \alpha_{j_1} + \dots + \alpha_{j_{l-1}} + \alpha_i) = 1.\end{aligned}$$

□

Remark 2.14. In order to identify formulas (5) and (7), we need to show that if $\alpha = w_i^{-1}(\alpha_i) + \delta$, then $(\delta - \alpha, \Lambda + \widehat{\rho}) = 0$ and $\alpha \in \Delta_+$. Note that $(\alpha_i, \lambda + \widehat{\rho}) = 0$ and $\lambda + \widehat{\rho} = w_i(\Lambda + \widehat{\rho})$. It follows that $(w_i^{-1}(\alpha_i), \Lambda + \widehat{\rho}) = 0$, so indeed $(\delta - \alpha, \Lambda + \widehat{\rho}) = 0$. Note also that

$$\begin{aligned}w_i^{-1}(\alpha_i) &= s_0 s_{j_1} \dots s_{j_{l-1}} s_i(\alpha_i) \\ &= -\alpha_i - \alpha_{j_{l-1}} - \dots - \alpha_{j_1} - \alpha_0 = -\delta + \theta - \alpha_{j_1} - \dots - \alpha_{j_{l-1}} - \alpha_i,\end{aligned}$$

so $\alpha = \theta - \alpha_{j_1} - \dots - \alpha_{j_{l-1}} - \alpha_i \in \Delta_+$.

The following proposition gives formulas for characters of certain $L(\Lambda)$ for regular $\Lambda + \widehat{\rho}$ of nonnegative level.

Proposition 2.15. *Let $\mathfrak{g}$ be of type D_n ($n \geqslant 4$) or E_6, E_7, E_8. Pick $i \in \{0, 1, \dots, r\}$ and let $\lambda \in \widehat{\mathfrak{h}}^*$ be integral such that $\lambda + \widehat{\rho}$ is regular dominant. Set $\Lambda = w_i^{-1} \circ \lambda$, then*

$$\widehat{R}\,\mathrm{ch}\,L(\Lambda) = \sum_{u\in W} \varepsilon(uw_i) \sum_{\gamma\in Q^\vee} \left\langle \Lambda_i, \gamma + \frac{|\gamma|^2}{2}K \right\rangle e^{ut_\gamma w_i(\Lambda+\widehat{\rho})}.$$

Proof. Follows from Theorem 2.9 together with Lemma 2.11. □

2.2.3. Main result: type A Assume that $\mathfrak{g} = \mathfrak{sl}_n$. Recall that $Q^\vee$ is the sublattice of $\mathbb{Z}^{\oplus n}$, consisting of $(a_1, \dots, a_n)$ such that $\sum_{k=1}^n a_k = 0$. Let $\epsilon_1, \dots, \epsilon_n$ be the standard basis of $\mathbb{Z}^{\oplus n}$. For $i \in \mathbb{Z}$ let $[i] \in \{0, 1, \dots, n-1\}$ be the class of i modulo n. We set

$$w_i = \begin{cases} s_{[i]} s_{[i-1]} \dots s_1 s_0 & \text{for } i > 0, \\ s_0 & \text{for } i = 0, \\ s_{[-i]} s_{[-i+1]} \dots s_{[-1]} s_0 & \text{for } i < 0, \end{cases}$$

and for $i \in \mathbb{Z}$, $k = 1, \dots, n$, and $a \in \mathbb{Z}$ we set

$$z_i(a\epsilon_k) := \begin{cases} |\mathbb{Z}_{\leqslant i} \cap [k, k + (a-1)n] \cap (k + n\mathbb{Z})| & \text{for } a \in \mathbb{Z}_{\geqslant 0}, \\ -|\mathbb{Z}_{\leqslant i} \cap [k + an, k - n] \cap (k + n\mathbb{Z})| & \text{for } a \in \mathbb{Z}_{\leqslant 0}. \end{cases}$$

We are now ready to describe the main result for type A. The following theorem holds (see Section 6.7 for the proof).

Theorem 2.16. *Let $\mathfrak{g}$ be $\mathfrak{sl}_n$ ($n \geqslant 3$). Pick $i \in \mathbb{Z}$ and let $\lambda \in \widehat{\mathfrak{h}}^*$ be an integral weight, such that $\lambda + \widehat{\rho}$ is dominant, $\Lambda = w_i^{-1} \circ \lambda$ is quasi-dominant and w_i is the longest element in the coset $\widehat{W}_\lambda w_i$. Then*

$$\widehat{R} \operatorname{ch} L(\Lambda) = -\sum_{u \in W} \varepsilon(u w_i) \sum_{\gamma \in Q^\vee} \Big(\sum_{k=1}^n z_i(-\langle \epsilon_k, \gamma \rangle \epsilon_k) \Big) e^{u t_\gamma w_i (\Lambda + \widehat{\rho})}.$$

Similarly to the D, E case (see Lemma 2.11) the pairs λ, i, satisfying the conditions of Theorem 2.16 can be described explicitly.

Lemma 2.17. *Let $\mathfrak{g}$ be $\mathfrak{sl}_n$ ($n \geqslant 3$). Elements $\lambda \in \widehat{\mathfrak{h}}^*$, $i \in \mathbb{Z}$ as in Theorem 2.16 are described as follows. There are two possibilities:*

(a) $\lambda + \widehat{\rho}$ is regular dominant and i is an arbitrary element of $\mathbb{Z}$,

(b) $\lambda = -\Lambda_{[i]} + \sum_{k \neq [i]} m_k \Lambda_k + x\delta$ for some $i \in \mathbb{Z}$, $x \in \mathbb{C}$, and $m_k \in \mathbb{Z}_{\geqslant 0}$.

Proof. Same as the proof of Lemma 2.11. □

The following lemma will be useful.

Lemma 2.18. *For $i \in \mathbb{Z} \setminus n\mathbb{Z}$, $a \in \mathbb{Z}$ we have*

$$z_i(-a\epsilon_{[i]}) - z_i(-a\epsilon_{[i]+1}) = \begin{cases} 0 & \text{for } a \geqslant 0,\ i \notin [[i] - an, [i] - n] \\ -1 & \text{for } a \geqslant 0,\ i \in [[i] - an, [i] - n] \\ 0 & \text{for } a \leqslant 0,\ i \notin [[i], [i] + (-a-1)n] \\ 1 & \text{for } a \leqslant 0,\ i \in [[i], [i] + (-a-1)n] \end{cases}$$

$$= \begin{cases} 1 & \text{for } i \in [[i], [i] + (-a-1)n] \\ 0 & \text{for } i \notin [[i] - an, [i] + (-a-1)n] \\ -1 & \text{for } i \in [[i] - an, [i] - n]. \end{cases}$$

Proof. Straightforward. □

Proposition 2.19. *Let $\mathfrak{g} = \mathfrak{sl}_n$, $n \geqslant 3$. Pick $i \in \mathbb{Z}$ and assume that $\lambda = -\Lambda_{[i]} + \sum_{k \neq [i]} m_k \Lambda_k + x\delta$ for some $m_k \in \mathbb{Z}_{\geqslant 0}$, $x \in \mathbb{C}$. Set $\Lambda = w_i^{-1} \circ \lambda$. Then we have*

$$\widehat{R}\,\mathrm{ch}\, L(\Lambda) = \sum_{u \in W} \varepsilon(u) \sum_{\gamma \in Q^\vee,\, \langle \overline{\Lambda}_{n-1}, \gamma \rangle \geqslant 0} e^{u t_\gamma(\Lambda)} \text{ for } i \geqslant 0, \tag{8}$$

$$\widehat{R}\,\mathrm{ch}\, L(\Lambda) = \sum_{u \in W} \varepsilon(u) \sum_{\gamma \in Q^\vee,\, \langle \overline{\Lambda}_{1}, \gamma \rangle \geqslant 0} e^{u t_\gamma(\Lambda)} \text{ for } i \leqslant 0, \tag{9}$$

where $\overline{\Lambda}_1, \overline{\Lambda}_{n-1} \in \mathfrak{h}^*$ are the corresponding fundamental weights of $\mathfrak{sl}_n$.

Proof. We have

$$\widehat{R}\,\mathrm{ch}\, L(\Lambda) = \sum_{u \in W} -\varepsilon(uw_i) \sum_{\gamma \in Q^\vee} \Big(\sum_{k=1}^{n} z_i(-\langle \epsilon_k, \gamma \rangle \epsilon_k) \Big) e^{u t_\gamma w_i(\Lambda + \widehat{\rho})}. \tag{10}$$

Recall that $\lambda + \widehat{\rho} = w_i(\Lambda + \widehat{\rho})$ and $s_{[i]}(\lambda + \widehat{\rho}) = \lambda + \widehat{\rho}$. Assume first that $i \notin n\mathbb{Z}$. Decompose $\gamma = \sum_{k=1}^{n} a_k \epsilon_k$. We have $e^{u t_\gamma(\lambda + \widehat{\rho})} = e^{u s_{[i]} t_{s_{[i]}(\gamma)}(\lambda + \widehat{\rho})}$, so the coefficient in front of $e^{u t_\gamma(\lambda + \widehat{\rho})}$ is equal to the sum

$$\begin{aligned} &-\varepsilon(uw_i)\Big(\sum_{k=1}^{n} z_i(-\langle \epsilon_k, \gamma \rangle \epsilon_k) \Big) + \varepsilon(uw_i)\Big(\sum_{k=1}^{n} z_i(-\langle \epsilon_k, s_{[i]}(\gamma) \rangle \epsilon_k) \Big) \\ &\quad = -\varepsilon(uw_i)\Big(z_i(-a_{[i]}\epsilon_{[i]}) + z_i(-a_{[i]+1}\epsilon_{[i]+1}) - z_i(-a_{[i]+1}\epsilon_{[i]}) - z_i(-a_{[i]}\epsilon_{[i]+1}) \Big) \\ &\quad = \varepsilon(uw_i)\Big((z_i(-a_{[i]+1}\epsilon_{[i]}) - z_i(-a_{[i]+1}\epsilon_{[i]+1})) - (z_i(-a_{[i]}\epsilon_{[i]}) - z_i(-a_{[i]}\epsilon_{[i]+1})) \Big). \end{aligned}$$

Decompose $i = kn + [i]$, $k \in \mathbb{Z}_{\geqslant 0}$. Using Lemma 2.18, we conclude that

$$z_i(-a_{[i]}\epsilon_{[i]}) - z_i(-a_{[i]}\epsilon_{[i]+1}) = \begin{cases} 1 & \text{for } i \in [[i], [i] + (-a_{[i]} - 1)n] \\ 0 & \text{for } i \notin [[i], [i] + (-a_{[i]} - 1)n] \end{cases} \tag{11}$$

$$= \begin{cases} 1 & \text{for } k \in [0, -a_{[i]} - 1] \\ 0 & \text{for } k \notin [0, -a_{[i]} - 1] \end{cases}$$

$$= \begin{cases} 1 & \text{for } -a_{[i]} - k - 1 \geqslant 0 \\ 0 & \text{for } -a_{[i]} - k - 1 < 0, \end{cases}$$

(12)

$$z_i(-a_{[i]+1}\epsilon_{[i]}) - z_i(-a_{[i]+1}\epsilon_{[i]+1}) = \begin{cases} 1 & \text{for } i \in [[i], [i] + (-a_{[i]+1} - 1)n] \\ 0 & \text{for } i \notin [[i], [i] + (-a_{[i]+1} - 1)n] \end{cases}$$

$$= \begin{cases} 1 & \text{for } -a_{[i]+1} - k - 1 \geqslant 0 \\ 0 & \text{for } -a_{[i]+1} - k - 1 < 0. \end{cases}$$

It follows that the coefficient in front of $e^{ut_\gamma(\lambda+\widehat{\rho})}$ is equal to $\varepsilon(uw_i)$ times the difference of (12) and (11).

Let us now compute the coefficient in front of $e^{ut_\gamma(\lambda+\widehat{\rho})}$ in (8). We first should write $ut_\gamma w_i = u't_{\gamma'}$, $u' \in S_n$, $\gamma' \in Q^\vee$. Let us compute u', γ'. Recall that $w_i = s_{[i]}s_{[i-1]} \dots s_1 s_0$ and let w_i' be the element, obtained from w_i by replacing s_0 with s_θ in the decomposition above. It is clear that $u' = uw_i'$. Let us now compute γ'. Recall that $i = kn + [i]$, $k \in \mathbb{Z}_{\geqslant 0}$ and decompose $k + 1 = l(n-1) + p$, $l \in \mathbb{Z}_{\geqslant 0}$, $1 \leqslant p \leqslant n-1$.

We have

$$\gamma' = w_i'^{-1}(\gamma) + p\epsilon_n - \sum_{j=1}^{p} \epsilon_j + l\Big((n-1)\epsilon_n - \sum_{j=1}^{n-1} \epsilon_j\Big),$$

so

$$\langle \overline{\Lambda}_{n-1}, \gamma' \rangle = \langle \overline{\Lambda}_{n-1}, w_i'^{-1}(\gamma) \rangle - k - 1 = \langle w_i'(\overline{\Lambda}_{n-1}), \gamma \rangle - k - 1 = -a_{[i]+1} - k - 1.$$

Now $e^{u't_{\gamma'}(\Lambda+\widehat{\rho})}$ gives

$$\varepsilon(uw_i) \cdot \begin{cases} 1 & \text{for } -a_{[i]+1} - k - 1 \geqslant 0 \\ 0 & \text{for } -a_{[i]+1} - k - 1 < 0 \end{cases} \tag{13}$$

to the coefficient in front of $e^{u't_{\gamma'}(\Lambda+\widehat{\rho})}$ in (8). Another part comes from $e^{us_{[i]}t_{s_{[i]}(\gamma)}w_i(\Lambda+\widehat{\rho})}$, it is easy to see that the corresponding coefficient is equal to

$$-\varepsilon(uw_i) \cdot \begin{cases} 1 & \text{for } -a_{[i]} - k - 1 \geqslant 0 \\ 0 & \text{for } -a_{[i]} - k - 1 < 0, \end{cases} \tag{14}$$

so the total coefficient of (8) in front of $e^{u't_{\gamma'}(\Lambda+\widehat{\rho})}$ is equal to the sum of (13) and (14). This sum is clearly equal to the difference of (12) and (11) times $\varepsilon(uw_i)$.

Assume now that $i \in n\mathbb{Z}$. Let us identify (10) and (8) in this case. We have $e^{ut_\gamma(\lambda+\widehat{\rho})} = e^{us_\theta t_{s_\theta(\gamma)-\theta}(\lambda+\widehat{\rho})}$, so the coefficient in front of $e^{ut_\gamma(\lambda+\widehat{\rho})}$ is equal to

$$\begin{aligned}
&-\varepsilon(uw_i)\Big(\sum_{k=1}^{n} z_i(-\langle\epsilon_k,\gamma\rangle\epsilon_k)\Big) + \varepsilon(uw_i)\Big(\sum_{k=1}^{n} z_i(-\langle\epsilon_k, s_\theta(\gamma)-\theta\rangle\epsilon_k)\Big)\\
&\quad = -\varepsilon(uw_i)\Big(z_i(-a_1\epsilon_1) + z_i(-a_n\epsilon_n) - z_i(-(a_n-1)\epsilon_1) - z_i(-(a_1+1)\epsilon_n)\Big)\\
&\quad = \varepsilon(uw_i)\Big((z_i(-(a_n-1)\epsilon_1) - z_i(-a_n\epsilon_n)) - (z_i(-a_1\epsilon_1) - z_i(-(a_1+1)\epsilon_n))\Big).
\end{aligned}$$

Decompose $i = kn$, $k \in \mathbb{Z}_{\geqslant 0}$. We have

$$z_i(-(a_n-1)\epsilon_1) - z_i(-a_n\epsilon_n) = \begin{cases} 1, & \text{if } a_n + k - 1 \geqslant 0\\ 0, & \text{if } a_n + k - 1 < 0, \end{cases}$$

$$z_i(-a_1\epsilon_1) - z_i(-(a_1+1)\epsilon_n) = \begin{cases} 1, & \text{if } k + a_1 \geqslant 0\\ 0, & \text{if } k + a_1 < 0. \end{cases}$$

Let us now compute the (total) coefficient in front of $e^{ut_\gamma(\lambda+\widehat{\rho})}$ in (8). Decompose $k+1 = l(n-1)+p$, $l \in \mathbb{Z}_{\geqslant 0}$, $1 \leqslant p \leqslant n-1$. We have $ut_\gamma w_i = u't_{\gamma'}$, where $u' = uw_i'$,

$$\gamma' = w_i'^{-1}(\gamma) + p\epsilon_n - \sum_{j=1}^{p}\epsilon_j + l\Big((n-1)\epsilon_n - \sum_{j=1}^{n-1}\epsilon_j\Big).$$

It follows that $\langle\overline{\Lambda}_{n-1}, \gamma'\rangle = -a_1 - k - 1$, so $e^{u't_{\gamma'}(\Lambda+\widehat{\rho})}$ gives

$$\varepsilon(uw_i)\cdot\begin{cases} 1 & \text{for } -a_1 - k - 1 \geqslant 0\\ 0 & \text{for } -a_1 - k - 1 < 0. \end{cases}$$

Recall also that $e^{ut_\gamma(\lambda+\widehat{\rho})} = e^{us_\theta t_{s_\theta(\gamma)-\theta}(\lambda+\widehat{\rho})}$, so the corresponding coefficient is

$$-\varepsilon(uw_i)\cdot\begin{cases} 1 & \text{for } -a_n - k \geqslant 0\\ 0 & \text{for } -a_n - k < 0. \end{cases}$$

We see that the (total) coefficient of both (10) and (8) in front of $e^{ut_\gamma(\lambda+\widehat{\rho})}$ is equal to

$$\varepsilon(uw_i)\cdot\begin{cases}1 & \text{for } a_n+k>0>a_1+k\\ 0 & \text{for } a_n+k\leqslant 0,\ a_1+k<0 \text{ or } a_n+k>0,\ a_1+k\geqslant 0\\ -1 & \text{for } a_1+k\geqslant 0\geqslant a_n+k.\end{cases}$$

Formula (9) follows from (8), using the involution of the Dynkin diagram of $\widehat{\mathfrak{sl}}_n$, which keeps the 0th node fixed. □

Restricting attention to the negative level cases in Theorem 2.16 yields the following corollary, proven in [22, Theorem 1.1] by different methods.

Corollary 2.20. *For $\Lambda=-(1+i)\Lambda_0+i\Lambda_{n-1}$, $i\in\mathbb{Z}_{\geqslant 0}$, we have*

$$\widehat{R}\,\mathrm{ch}\,L(\Lambda)=\sum_{u\in W}\varepsilon(u)\sum_{\gamma\in Q^\vee,\,\langle\overline{\Lambda}_{n-1},\gamma\rangle\geqslant 0}e^{ut_\gamma(\Lambda)}.$$

For $\Lambda=-(1+i)\Lambda_0+i\Lambda_1$, $i\in\mathbb{Z}_{\geqslant 0}$ we have

$$\widehat{R}\,\mathrm{ch}\,L(\Lambda)=\sum_{u\in W}\varepsilon(u)\sum_{\gamma\in Q^\vee,\,\langle\overline{\Lambda}_{1},\gamma\rangle\geqslant 0}e^{ut_\gamma(\Lambda)}.$$

Proposition 2.21. *Let $\mathfrak{g}=\mathfrak{sl}_n$ ($n\geqslant 3$). Pick $i\in\mathbb{Z}$ and let $\lambda\in\widehat{\mathfrak{h}}^*$ be integral such that $\lambda+\widehat{\rho}$ is regular dominant. Set $\Lambda:=w_i^{-1}\circ\lambda$, then*

$$\widehat{R}\,\mathrm{ch}\,L(\Lambda)=-\sum_{u\in W}\varepsilon(uw_i)\sum_{\gamma\in Q^\vee}\sum_{k=1}^{n}z_i(-\langle\epsilon_k,\gamma\rangle\epsilon_k)e^{ut_\gamma w_i(\Lambda+\widehat{\rho})}.$$

Proof. Follows from Theorem 2.16 together with Lemma 2.17. □

2.2.4. Main steps of the proof of Theorem 2.9 Let us describe the main steps of the proof of Theorem 2.9. Proof of Theorem 2.16 is similar. We use notations of Theorem 2.9.

The first observation is that

$$\widehat{R}\,\mathrm{ch}\,L(\Lambda)=\sum_{w\in\widehat{W}}\varepsilon(ww_i^{-1})\mathbf{m}^w_{w_i}e^{w^{-1}(\lambda+\widehat{\rho})},\tag{15}$$

where $\mathbf{m}^w_{w_i}=\mathbf{m}^w_{w_i}(1)$ are values at $q=1$ of inverse Kazhdan-Lusztig polynomials $\mathbf{m}^w_{w_i}(q)$ for $\widehat{W}$ (see [19, Section 0.3] or Theorem A.4 below).

Using that Λ is integral quasi-dominant, we conclude from (15) that

$$\begin{aligned}\widehat{R}\,\mathrm{ch}\,L(\Lambda) &= \sum_{\gamma\in Q^\vee}\varepsilon(w_\gamma w_i^{-1})\mathbf{m}_{w_i}^{w_\gamma}\sum_{u\in W}\varepsilon(u)e^{uw_\gamma^{-1}(\lambda+\widehat{\rho})}\\ &= \varepsilon(w_i)\sum_{\gamma\in Q^\vee}\sum_{u\in W}\varepsilon(w_\gamma u)\mathbf{m}_{w_i}^{w_\gamma}e^{uw_\gamma^{-1}(\lambda+\widehat{\rho})}\\ &= \varepsilon(w_i)\sum_{\gamma\in Q^\vee}\sum_{u\in W}\varepsilon(u)\mathbf{m}_{w_i}^{w_\gamma}e^{ut_{-\gamma}(\lambda+\widehat{\rho})},\end{aligned}$$

where $w_\gamma\in t_\gamma W$ is the shortest element of the coset $t_\gamma W$.

By the definitions together with Proposition A.6 below, the values $\mathbf{m}_{w_i}^{w_\gamma}(1)$ are computed as follows. Consider the group algebra $\mathbb{Z}\widehat{W}$. This algebra admits two bases H_w and C_w, $w\in\widehat{W}$, called, respectively, the standard and the canonical basis (see Appendix A for the definitions). Recall the anti-spherical $\widehat{W}$-module

$$M=\mathbb{Z}\widehat{W}\otimes_{\mathbb{Z}W}\mathbb{Z}_{\mathrm{sign}}.$$

Taking images of H_w, C_w under the natural surjection $\widehat{W}\twoheadrightarrow M$, we obtain standard and canonical bases in M to be denoted H'_w, C'_w respectively.

Then we have

$$H'_{w_\gamma}=\sum_{\nu\in Q^\vee}\varepsilon(w_\gamma w_\nu^{-1})\mathbf{m}_{w_\nu}^{w_\gamma}C'_{w_\nu}, \tag{16}$$

and our goal is to compute the numbers $\mathbf{m}_{w_i}^{w_\gamma}$. Set

$$T_\gamma:=\varepsilon(w_\gamma)H'_{w_\gamma},\ C_\nu:=\varepsilon(w_\nu)C'_{w_\nu}.$$

We see that

$$T_\gamma=\sum_{\nu\in Q^\vee}\mathbf{m}_{w_\nu}^{w_\gamma}C_\nu. \tag{17}$$

The module M has a "coherent" realization as the equivariant K-theory $K^{G^\vee}(\widetilde{\mathcal{N}})$ of the Springer resolution for the Langlands dual group $G^\vee$ (see [11]). It follows from [3, 7] that after the identification $M\simeq K^{G^\vee}(\widetilde{\mathcal{N}})$ elements T_γ correspond to the classes of natural line bundles $\mathcal{O}_{\widetilde{\mathcal{N}}}(\gamma)$ on $\widetilde{\mathcal{N}}$. Elements of the canonical basis C_ν correspond to classes of irreducible objects in the "exotic" t-structure on $D^b(\mathrm{Coh}^G(\widetilde{\mathcal{N}}))$ defined and studied in [5, 7].

Recall that our goal is to compute the numbers $\mathbf{m}_{w_i}^{w_\gamma}$. It turns out that all these numbers are already "contained" in a certain quotient of the $\widehat{W}$-module $K^{G^\vee}(\widetilde{\mathcal{N}})$. Let us describe this quotient in our situation.

Let $e \in \mathcal{N}$ be a subregular nilpotent element of $\mathfrak{g}^\vee$ and recall that $\mathbb{O}_e \subset \mathcal{N}$ is the corresponding nilpotent orbit. Recall also that $\mathbb{O}^{\mathrm{reg}} \subset \mathcal{N}$ is the regular nilpotent orbit, so that $U = \mathbb{O}^{\mathrm{reg}} \cup \mathbb{O}_e$ is an open $G^\vee$-invariant subvariety of $\mathcal{N}$. Recall that $\widetilde{U} \subset \widetilde{\mathcal{N}}$ is the preimage of U. We have the natural surjection of $\widehat{W}$-modules $K^{G^\vee}(\widetilde{\mathcal{N}}) \twoheadrightarrow K^{G^\vee}(\widetilde{U})$. It follows from the results of the first author (see [7, §11.3]) that the kernel of this surjection is spanned over $\mathbb{Z}$ by C_ν for $w_\nu \notin \{1, w_0, w_1, \dots, w_r\}$.

Taking the image of the equality (17) in $K^{G^\vee}(\widetilde{U})$, we conclude that

$$\bar{T}_\gamma = \bar{C}_0 + \sum_{i=0,1,\dots,r} \mathbf{m}_{w_i}^{w_\gamma} \bar{C}_{\nu_i}, \tag{18}$$

where by $\bar{x}$ we mean the image of the element $x \in M \simeq K^{G^\vee}(\widetilde{\mathcal{N}})$ in $K^{G^\vee}(\widetilde{U})$ and ν_i is the image of w_i in $Q^\vee \simeq \widehat{W}/W$.

It turns out (see Proposition 5.16) that the $\widehat{W}$-module $K^{G^\vee}(\widetilde{U})$ is isomorphic to $\widehat{\mathfrak{h}}_{\mathbb{Z}} \otimes \mathbb{Z}_{\mathrm{sign}}$ (with the $\widehat{W}$-action on $\widehat{\mathfrak{h}}_{\mathbb{Z}}$ given by (2)). The canonical basis elements $\bar{C}_0$ and $\bar{C}_{\nu_i}$ are $d \otimes 1$ and $-\alpha_i^\vee \otimes 1$. Using the equation (18), this allows us to compute the numbers $\mathbf{m}_{w_i}^{w_\gamma}$ explicitly.

3. Categories $\mathcal{O}$ for $\hat{\mathfrak{g}}$ and characters of irreducible modules via Kazhdan-Lusztig polymomials

3.1. Category $\mathcal{O}_\kappa$ for $\hat{\mathfrak{g}}$ and its decomposition into blocks

Consider a module $L(\Lambda)$ of integer level $\kappa(\Lambda) > -h^\vee$, $\Lambda \in \widehat{\mathfrak{h}}^*$. The module $L(\Lambda)$ is an object of the (affine) category $\mathcal{O}$, denoted by $\mathcal{O}_\kappa$ and defined as follows (see, for example, [18, Section 3]). Set $\widehat{Q}_+ := \sum_{i \in \hat{I}} \mathbb{Z}_{\geqslant 0} \alpha_i$.

Definition 3.1. The category $\mathcal{O}_\kappa$ is the full subcategory of the category of $\hat{\mathfrak{g}}$-modules of level κ, consisting of $\hat{\mathfrak{g}}$-modules N such that

(a) $N = \bigoplus_{\mu \in \widehat{\mathfrak{h}}^*} N_\mu$, where N_μ is a generalized μ-weight space for $\widehat{\mathfrak{h}}$,
(b) $\dim N_\mu < \infty$,
(c) for any $\mu \in \widehat{\mathfrak{h}}^*$ there exists only finitely many $\beta \in \mu + \widehat{Q}^+$ such that $N_\beta \neq 0$.

Let us recall the block decomposition of the category $\mathcal{O}_\kappa$. Irreducible objects of this category are precisely $L(\Lambda)$ such that $\kappa = \kappa(\Lambda)$. The category $\mathcal{O}_\kappa$ can be decomposed into blocks $\mathcal{O}_{\kappa,\xi}$ as follows. Let $\widehat{\mathfrak{h}}^*/_\circ \widehat{W}$ be the set of equivalence classes with respect to the shifted $\widehat{W}$-action. For $\Lambda \in \widehat{\mathfrak{h}}^*$ let $\overline{\Lambda} \in \widehat{\mathfrak{h}}^*/_\circ \widehat{W}$ be the class of Λ.

Definition 3.2. For $\xi \in \widehat{\mathfrak{h}}^*/_\circ\widehat{W}$ let $\mathcal{O}_{\kappa,\xi} \subset \mathcal{O}_\kappa$ be the full subcategory of $\mathcal{O}_\kappa$, consisting of modules, all of whose composition factors are $L(\Lambda)$ with $\overline{\Lambda} = \xi$.

Proposition 3.3. *The category $\mathcal{O}_\kappa$ decomposes as a direct sum as follows:*

$$\mathcal{O}_\kappa = \bigoplus_{\xi \in \widehat{\mathfrak{h}}^*/_\circ\widehat{W}} \mathcal{O}_{\kappa,\xi}.$$

Proof. Follows from [13, Theorem 5.7]. □

The subcategory $\mathcal{O}_{\kappa,\overline{\Lambda}}$, corresponding to an integral Λ, is called an *integral block*. Let $\mathcal{O}_{\kappa,\xi}$ be an integral block; then there exists the unique $\lambda \in \xi$ such that $\lambda + \widehat{\rho}$ is dominant integral (since $\kappa(\Lambda) > -h^\vee$).

Recall that $\widehat{W}_\lambda \subset \widehat{W}$ is the stabilizer of λ (w.r.t. the $\circ$ action). Fixing an integral $\lambda \in \widehat{\mathfrak{h}}^*$, such that $\lambda + \widehat{\rho}$ is dominant, we obtain the identification

$$\widehat{W}_\lambda\backslash\widehat{W} \xrightarrow{\sim} \mathrm{Irr}(\mathcal{O}_{\kappa,\xi}),\ \overline{w} \mapsto L(w^{-1} \circ \lambda).$$

For $\overline{w} \in \widehat{W}_\lambda\backslash\widehat{W}$ let $L_{\overline{w}}$ and $M_{\overline{w}} \in \mathcal{O}_{\kappa,\xi}$ be simple and Verma modules, corresponding to the coset $\overline{w}$ i.e.:

$$L_{\overline{w}} := L(w^{-1} \circ \lambda),\ M_{\overline{w}} := M(w^{-1} \circ \lambda).$$

Definition 3.4. For an integral $\lambda \in \widehat{\mathfrak{h}}^*$ such that $\lambda + \widehat{\rho}$ is dominant let ${}^\lambda\widehat{W}$ be the subset of $\widehat{W}$, consisting of *maximal* length representatives of *left* $\widehat{W}_\lambda$-cosets $\widehat{W}_\lambda\backslash\widehat{W}$.

We have the bijection ${}^\lambda\widehat{W} \xrightarrow{\sim} \widehat{W}_\lambda\backslash\widehat{W}$ that sends w to $\widehat{W}_\lambda w$. For $w \in {}^\lambda\widehat{W}$ let L_w, $M_w \in \mathcal{O}_{\kappa,\xi}$ be simple and Verma modules, corresponding to the coset $\widehat{W}_\lambda w$:

$$L_w := L_{\overline{w}},\ M_w := M_{\overline{w}}.$$

3.2. Classes of irreducible objects of a regular integral block $\mathcal{O}_{\kappa,\xi}$

Let $\mathcal{O}_{\kappa,\xi}$ be a regular integral block. Consider the Grothendieck K-group $K_0(\mathcal{O}_{\kappa,\xi})$ of the category $\mathcal{O}_{\kappa,\xi}$. For an object $N \in \mathcal{O}_{\kappa,\xi}$ we denote by $[N] \in K_0(\mathcal{O}_{\kappa,\xi})$ the corresponding class. For $v \in \widehat{W}$ and $w \in \widehat{W}$ let $\mathbf{m}_v^w$ be determined by:

$$[L_v] = \sum_{w \in \widehat{W}} \varepsilon(wv^{-1})\mathbf{m}_v^w[M_w]. \tag{19}$$

Remark 3.5. The numbers $\mathbf{m}_v^w$ are given by the affine inverse Kazhdan-Lusztig polynomials $\mathbf{m}_v^w(q)$ evaluated at $q = 1$ (see Theorem A.4 or [19, Section 0.3]).

3.3. Classes of irreducible objects of integral blocks $\mathcal{O}_{\kappa,\xi}$

Let $\mathcal{O}_{\kappa,\xi}$ be an integral block of level $\kappa > -h^\vee$ (we are not assuming that ξ is regular).

Proposition 3.6. *Pick $v \in {}^\lambda\widehat{W}$. We have*

$$[L_v] = \sum_{w\in\widehat{W}} \varepsilon(wv^{-1})\mathbf{m}_v^w[M_{\overline{w}}] = \sum_{w\in{}^\lambda\widehat{W}} \varepsilon(wv^{-1})\Big(\sum_{u\in\widehat{W}_\lambda} \varepsilon(u)\mathbf{m}_v^{uw}\Big)[M_w].$$

Proof. The claim can be deduced from [18, Theorem 1.1]. It also follows from the equality (19) by using translation functors (see [13, Theorem 5.13] or [18, Section 3]). □

3.4. Subcategory $\mathcal{R}_\kappa \subset \mathcal{O}_\kappa$ and irreducible objects in integral blocks $\mathcal{R}_{\kappa,\xi}$

We will be interested in integral quasi-dominant Λ (see Definition 2.4). The last condition corresponds to the fact that $L(\Lambda)$ lies in the subcategory $\mathcal{R}_\kappa \subset \mathcal{O}_\kappa$, defined as follows.

Definition 3.7. The category $\mathcal{R}_\kappa$ is the full subcategory of $\mathcal{O}_\kappa$, consisting of modules $N \in \mathcal{O}_\kappa$ such that the action of $\mathfrak{g}[t]$ on N is locally finite. For a block $\mathcal{O}_{\kappa,\xi}$ we denote by $\mathcal{R}_{\kappa,\xi} \subset \mathcal{O}_{\kappa,\xi}$ the full subcategory of $\mathcal{O}_{\kappa,\xi}$ by objects in $\mathcal{R}_\kappa$.

Let us now describe irreducible objects of the category $\mathcal{R}_{\kappa,\xi}$. We start from the case when $\xi \in \widehat{\mathfrak{h}}^*/_\circ\widehat{W}$ is regular (see Definition 2.4).

Definition 3.8. Let $\widehat{W}^f$ be the set of *minimal* length representatives of *right* W-cosets in $\widehat{W}$. We have $\widehat{W}^f \xrightarrow{\sim} \widehat{W}/W \simeq Q^\vee$ and let $\nu \mapsto w_\nu$ be the inverse bijection.

Lemma 3.9. *For integral regular ξ the category $\mathcal{R}_{\kappa,\xi}$ is the Serre subcategory of $\mathcal{O}_{\kappa,\xi}$ whose irreducible objects are L_v, $v \in \widehat{W}^f$.*

Proof. It is clear that $\mathcal{R}_{\kappa,\xi} \subset \mathcal{O}_{\kappa,\xi}$ is the Serre subcategory. It remains to show that $L_v \in \mathcal{R}_{\kappa,\xi}$ iff $v \in \widehat{W}^f$. Indeed, let us first of all note that $v \in \widehat{W}^f$ iff $v^{-1}(\lambda+\widehat{\rho})$ is quasi-dominant (indeed, if $v \in \widehat{W}^f$ and $v^{-1}(\lambda+\widehat{\rho})$ is not quasi-dominant then there exists $i \in \{1,\dots,r\}$ such that $\langle v^{-1}(\lambda+\widehat{\rho}), \alpha_i^\vee\rangle < 0$, so $\langle \lambda+\widehat{\rho}, v(\alpha_i^\vee)\rangle < 0$ i.e. $v(\alpha_i^\vee)$ is negative, hence, $\ell(vs_i) = \ell(v) - 1$ that contradicts to $v \in \widehat{W}^f$, similarly if $v^{-1}(\lambda+\widehat{\rho})$ is quasi-dominant but $v \notin \widehat{W}^f$ then there exists $i \in \{1,\dots,r\}$ such that $v(\alpha_i^\vee)$ is negative that contradicts to $\langle v^{-1}(\lambda+\widehat{\rho}), \alpha_i^\vee\rangle > 0$).

It remains to show that $L(\Lambda) \in \mathcal{R}_\kappa$ iff Λ is quasi-dominant. Assume that $L(\Lambda) \in \mathcal{R}_\kappa$ and consider the $\mathfrak{g}$-submodule of $L(\Lambda)$ generated by the highest weight vector of $L(\Lambda)$. This is a finite dimensional module with highest weight $\Lambda|_{\mathfrak{h}}$. It follows that $\Lambda|_{\mathfrak{h}}$ is dominant i.e. Λ is quasi-dominant.

Assume now that Λ is quasi-dominant. Let $V(\Lambda)$ be the irreducible (finite dimensional) representation of $\mathfrak{g}$ with highest weight $\Lambda|_{\mathfrak{h}}$. Consider $V(\Lambda)$ as a module over $\widehat{\mathfrak{g}}_+ := \mathfrak{g}[t] \oplus \mathbb{C}K \oplus \mathbb{C}d$, letting $t\mathfrak{g}[t]$ act via zero, K act via the multiplication by $\Lambda(K)$, and d act via the multiplication by $\Lambda(d)$. Consider the induced module $\mathrm{Ind}_{\widehat{\mathfrak{g}}_+}^{\widehat{\mathfrak{g}}} V(\Lambda) := U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}}_+)} V(\Lambda)$. It is easy to see that $\mathrm{Ind}_{\widehat{\mathfrak{g}}_+}^{\widehat{\mathfrak{g}}} V(\Lambda) \in \mathcal{R}_\kappa$. Since $L(\Lambda)$ is a quotient of $\mathrm{Ind}_{\widehat{\mathfrak{g}}_+}^{\widehat{\mathfrak{g}}} V(\Lambda)$, we conclude that $L(\Lambda) \in \mathcal{R}_\kappa$. □

In general (for singular ξ) irreducible objects of $\mathcal{R}_{\kappa,\xi} \subset \mathcal{O}_{\kappa,\xi}$ can be described as follows. Recall that irreducible objects of $\mathcal{O}_{\kappa,\xi}$ are in bijection with ${}^\lambda\widehat{W}$.

Definition 3.10. Let ${}^\lambda\widehat{W}^f$ be the intersection ${}^\lambda\widehat{W} \cap \widehat{W}^f \subset \widehat{W}$. Using the identification $\widehat{W}^f \xrightarrow{\sim} Q^\vee$, we can identify ${}^\lambda\widehat{W} \cap \widehat{W}^f$ with the subset of $Q^\vee$ to be denoted ${}^\lambda Q^\vee$.

Lemma 3.11. *For integral ξ the category $\mathcal{R}_{\kappa,\xi}$ is the Serre subcategory of $\mathcal{O}_{\kappa,\xi}$ whose irreducible objects are L_v, $v \in {}^\lambda\widehat{W}^f$, where $\lambda \in \xi$ is such that $\lambda + \widehat{\rho}$ is dominant integral.*

Proof. Clearly $\mathcal{R}_{\kappa,\xi}$ is a Serre subcategory of $\mathcal{O}_{\kappa,\xi}$. It follows from the proof of Lemma 3.9 that an irreducible object $L_v \in \mathcal{O}_{\kappa,\xi}$ ($v \in {}^\lambda\widehat{W}$) lies in $\mathcal{R}_{\kappa,\xi}$ iff the element $v^{-1} \circ \lambda$ is quasi-dominant.

It remains to show that $v \in {}^\lambda\widehat{W}^f$ iff $v^{-1} \circ \lambda$ is quasi-dominant. Indeed, assume that $v \in {}^\lambda\widehat{W}$ is such that $\Lambda := v^{-1} \circ \lambda$ is quasi-dominant. It follows that $\Lambda + \widehat{\rho} = v^{-1}(\lambda + \widehat{\rho})$ pairs with all $\alpha_i^\vee$, $i = 1, 2, \ldots, r$, by *positive* numbers. If $v \notin W^f$, then there exists $i \in \{1, 2, \ldots, r\}$ such that $\ell(vs_i) = \ell(v) - 1$. This is equivalent to $v(\alpha_i) \in \widehat{\Delta}_-$. On the other hand we have

$$\langle \lambda + \widehat{\rho}, v(\alpha_i^\vee) \rangle = \langle v^{-1}(\lambda + \widehat{\rho}), \alpha_i^\vee \rangle = \langle \Lambda + \widehat{\rho}, \alpha_i^\vee \rangle > 0. \tag{20}$$

Recall now that $\lambda + \widehat{\rho}$ is quasi-dominant, so the pairing $\langle \lambda + \widehat{\rho}, v(\alpha_i^\vee) \rangle$ must be nonpositive, contradicting to (20).

Assume now that $v \in {}^\lambda\widehat{W}^f$. Then $v \in \widehat{W}^f$, and since $\lambda + \widehat{\rho}$ is dominant, it is clear that $\Lambda + \widehat{\rho} = v^{-1}(\lambda + \widehat{\rho})$ is quasi-dominant (same argument as in the proof of Lemma 3.9). It remains to show that there is no $i \in \{1, \ldots, r\}$ such that $\langle \Lambda + \widehat{\rho}, \alpha_i^\vee \rangle = 0$. Assume that such i exists. Then $s_i(\Lambda + \widehat{\rho}) = \Lambda + \widehat{\rho}$, i.e.

$(vs_i)^{-1}\circ\lambda = v^{-1}\circ\lambda$, hence, $vs_i \in \widehat{W}_\lambda v$. Recall that v is the longest element of $\widehat{W}_\lambda v$, so $\ell(vs_i) = \ell(v) - 1$, and that contradicts the fact that v is the shortest element of vW (since $vs_i \in vW$ and vs_i is shorter than v). □

3.5. Classes of irreducibles of regular integral block $\mathcal{R}_{\kappa,\xi}$

Assume that $\xi \in \widehat{\mathfrak{h}}^*/_\circ\widehat{W}$ is regular integral and consider the corresponding category $\mathcal{R}_{\kappa,\xi}$.

Proposition 3.12. *Suppose that $v \in \widehat{W}^f$. Then we have*

$$[L_v] = \sum_{w\in\widehat{W}^f} \varepsilon(wv^{-1}) \sum_{u\in W} \varepsilon(u)\mathbf{m}_v^w[M_{wu}]. \tag{21}$$

Proof. This follows from W-invariance of $\operatorname{ch} L_v$. □

Recall now that we have the bijection $\widehat{W}^f \xrightarrow{\sim} Q^\vee$ and the inverse bijection sends ν to w_ν. Then the equality (21) can be rewritten as follows: for $v = w_\nu$ we have

$$[L_v] = \sum_{\gamma\in Q^\vee} \varepsilon(w_\gamma v^{-1}) \sum_{u\in W} \varepsilon(u)\mathbf{m}_v^{w_\gamma}[M_{w_\gamma u}].$$

3.6. Classes of irreducibles of integral block $\mathcal{R}_{\kappa,\xi}$

Assume now that $\mathcal{R}_{\kappa,\xi}$ is an arbitrary (possibly singular) integral block.

Theorem 3.13. *For $v \in {}^\lambda\widehat{W}^f$ we have*

$$\begin{aligned}[L_v] &= \sum_{w\in\widehat{W}^f} \varepsilon(wv^{-1}) \sum_{u\in W} \varepsilon(u)\mathbf{m}_v^w[M_{\overline{wu}}] \\ &= \sum_{w\in{}^\lambda\widehat{W}^f} \varepsilon(wv^{-1}) \sum_{u\in W} \varepsilon(u)\Big(\sum_{\sigma\in\widehat{W}_\lambda} \varepsilon(\sigma)\mathbf{m}_v^{\sigma w}\Big)[M_{wu}].\end{aligned} \tag{22}$$

Proof. Follows from Proposition 3.6 and the W-invariance of $\operatorname{ch} L_v$. □

Equivalently the equality (22) can be rewritten as follows. Pick $v \in {}^\lambda\widehat{W}^f$, then

$$\begin{aligned}[L_v] &= \sum_{\gamma\in Q^\vee} \varepsilon(w_\gamma v^{-1}) \sum_{u\in W} \varepsilon(u)\mathbf{m}_v^{w_\gamma}[M_{w_\gamma u}] \\ &= \sum_{\gamma\in{}^\lambda Q^\vee} \varepsilon(w_\gamma v^{-1}) \sum_{u\in W} \varepsilon(u)\Big(\sum_{\sigma\in\widehat{W}_\lambda} \varepsilon(\sigma)\mathbf{m}_v^{\sigma w_\gamma}\Big)[M_{w_\gamma u}].\end{aligned} \tag{23}$$

Remark 3.14. Note that if $w \in {}^\lambda\widehat{W}^f$ and $\sigma \in \widehat{W}_\lambda$, then $\sigma w \in \widehat{W}^f$.

3.7. Characters of $L(\Lambda) \in \mathcal{R}_{\kappa,\xi}$

Recall that we are assuming that Λ is integral and quasi-dominant. This corresponds to the fact that $L(\Lambda) \in \mathcal{R}_{\kappa,\xi}$, where $\xi = \widehat{W} \circ \Lambda$. Let $\lambda \in \xi$ be the (unique) element such that $\lambda + \widehat{\rho}$ is dominant. Let $v \in {}^{\lambda}\widehat{W}^f$ be the element such that $\Lambda = v \circ \lambda$.

The equality (23) can be obviously rewritten in the following way:

$$[L(\Lambda)] = \sum_{\gamma \in Q^\vee} \varepsilon(w_\gamma v^{-1}) \sum_{u \in W} \varepsilon(u) \mathbf{m}_v^{w_\gamma} [M((u^{-1} w_\gamma^{-1}) \circ \lambda)].$$

Hence the formula for the character of $L(\Lambda)$ is

$$\widehat{R} \operatorname{ch} L(\Lambda) = \sum_{\gamma \in Q^\vee} \varepsilon(w_\gamma v^{-1}) \sum_{u \in W} \varepsilon(u) \mathbf{m}_v^{w_\gamma} e^{u w_\gamma^{-1}(\lambda + \widehat{\rho})}.$$

Let $\nu \in Q^\vee$ be such that $v = w_\nu$. We conclude that

$$\begin{aligned} \widehat{R} \operatorname{ch} L(\Lambda) &= \sum_{\gamma \in Q^\vee} \sum_{u \in W} \varepsilon(u w_\gamma w_\nu) \mathbf{m}_{w_\nu}^{w_\gamma} e^{u(w_\gamma^{-1} t_\gamma) t_{-\gamma}(\lambda + \widehat{\rho})} \\ &= \sum_{\gamma \in Q^\vee} \sum_{u \in W} \varepsilon(u w_\nu) \mathbf{m}_{w_\nu}^{w_\gamma} e^{u t_{-\gamma}(\lambda + \widehat{\rho})}. \end{aligned} \tag{24}$$

So all the information about the character of $L(\Lambda)$ is contained in the numbers $\mathbf{m}_{w_\nu}^{w_\gamma}$ for $\gamma \in Q^\vee$.

3.8. Description of $\mathbf{m}_{w_\nu}^{w_\gamma}$ via anti-spherical module M

In this section we recall some results of Appendix A. Consider the group algebra $\mathbb{Z}\widehat{W}$. Then $\mathbb{Z}\widehat{W}$ admits two bases H_w and C_w, indexed by $\widehat{W}$, and called, respectively, the standard and the canonical basis (whose definition involves deformation of $\mathbb{Z}\widehat{W}$ to the Hecke algebra of $\widehat{W}$, see Appendix A for details). Recall the anti-spherical module $M = \mathbb{Z}\widehat{W} \otimes_{\mathbb{Z}W} \mathbb{Z}_{\mathrm{sign}}$. Module M admits the standard basis H'_w and canonical basis C'_w indexed by $w \in \widehat{W}^f$ and defined as the image of H_w and C_w under the natural surjection $\mathbb{Z}\widehat{W} \twoheadrightarrow M$.

By Theorem A.4 we have

$$H'_{w_\gamma} = \sum_{\nu \in Q^\vee} \varepsilon(w_\gamma w_\nu^{-1}) \mathbf{m}_{w_\nu}^{w_\gamma} C'_{w_\nu}. \tag{25}$$

We set

$$T_\gamma := \varepsilon(w_\gamma) H'_{w_\gamma}, \; C_\nu = \varepsilon(w_\nu) C'_{w_\nu}.$$

For $\nu, \gamma \in Q^\vee$ we see that

$$T_\gamma = \sum_{\nu \in Q^\vee} \mathbf{m}_{w_\nu}^{w_\gamma} C_\nu. \tag{26}$$

Remark 3.15. Note that $\mathbf{m}_{w_\nu}^{w_\nu} = 1$.

4. Geometry of Springer resolution and realization of M

We will analyze $\mathbf{m}_{w_\nu}^{w_\gamma}$ based on the "coherent" realization of M (as the equivariant K-theory of the Springer resolution for the Langlands dual group $G^\vee$). Let us first of all recall basic things about the Springer resolution.

4.1. Springer resolution

Recall that $G^\vee$ is the adjoint group with Lie algebra $\mathfrak{g}^\vee$, $\mathcal{B}$ is the flag variety of $\mathfrak{g}^\vee$ and $\mathcal{N} \subset \mathfrak{g}^\vee$ is the variety of nilpotent elements. Recall also that $\widetilde{\mathcal{N}} = T^*\mathcal{B}$ and $\pi\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ is the projection (Springer) map.

The lattice $Q^\vee$ is the root lattice of $G^\vee$ that identifies with the weight lattice $\mathrm{Hom}(T, \mathbb{C}^\times)$ of characters of a maximal torus $T \subset G^\vee$ (here we use that $G^\vee$ is adjoint). For $\gamma \in Q^\vee$ we denote by $\mathcal{O}_{\mathcal{B}}(\gamma) := G^\vee \times^B \mathbb{C}_{-\gamma}$ the corresponding $G^\vee$-equivariant line bundle on $\mathcal{B}$, and $\mathcal{O}_{\widetilde{\mathcal{N}}}(\gamma)$ is the pull back of $\mathcal{O}_{\mathcal{B}}(\gamma)$ to $\widetilde{\mathcal{N}}$.

The variety $\widetilde{\mathcal{N}}$ contains an open $G^\vee$-orbit $\mathbb{O}^{\mathrm{reg}}$ (we identify $\mathbb{O}^{\mathrm{reg}} \subset \mathcal{N}$ with its preimage in $\widetilde{\mathcal{N}}$). The complement $\widetilde{\mathcal{N}} \setminus \mathbb{O}^{\mathrm{reg}}$ is the divisor in $\widetilde{\mathcal{N}}$. It is a standard fact that the irreducible components of this divisor are parametrized by simple coroots $\alpha^\vee = \alpha_i^\vee$, $i = 1, \dots, r$, as follows. Let $B \in \mathcal{B}$ be a Borel subgroup and let $P_{\alpha^\vee} \supset B$ be the minimal parabolic, corresponding to $\alpha^\vee$. Let $\pi_{\alpha^\vee}\colon \mathcal{B} = G^\vee/B \twoheadrightarrow G/P_{\alpha^\vee}$ be the projection. Set $\widetilde{\mathcal{N}}_{\alpha^\vee} := T^*(G^\vee/P_{\alpha^\vee}) \times_{G^\vee/P_{\alpha^\vee}} \mathcal{B}$. The differential of $\pi_{\alpha^\vee}$ provides the closed embedding $i_{\alpha^\vee}\colon \widetilde{\mathcal{N}}_{\alpha^\vee} \subset \widetilde{\mathcal{N}}$. Subvarieties $\widetilde{\mathcal{N}}_{\alpha^\vee} \subset \widetilde{\mathcal{N}}$ are precisely the irreducible components of $\widetilde{\mathcal{N}} \setminus \mathbb{O}^{\mathrm{reg}}$.

The following exact sequence is standard (see, for example, [5, Equation (13)] or [1, Lemma 5.3]).

Lemma 4.1. *There is a canonical (in particular, $G^\vee$-equivariant) exact sequence of (coherent) sheaves on $\widetilde{\mathcal{N}}$:*

$$0 \to \mathcal{O}_{\widetilde{\mathcal{N}}}(\alpha^\vee) \to \mathcal{O}_{\widetilde{\mathcal{N}}} \to i_{\alpha^\vee *}\mathcal{O}_{\widetilde{\mathcal{N}}_{\alpha^\vee}} \to 0. \tag{27}$$

4.2. Realization of M via the equivariant K-theory of $\widetilde{\mathcal{N}}$ and canonical basis

For a group H, acting on an algebraic variety X, we let $K^H(X)$ denote the Grothendieck group of H-equivariant coherent sheaves on X.

Theorem 4.2 (See e.g. [11]). *We have a canonical isomorphism $M \simeq K^{G^\vee}(\widetilde{\mathcal{N}})$, such that the element T_γ is sent to $[\mathcal{O}_{\widetilde{\mathcal{N}}}(\gamma)]$, the action of $\gamma \in Q^\vee \subset \widehat{W}$ corresponds to the automorphism induced by the functor $\mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_{\widetilde{\mathcal{N}}}} \mathcal{O}_{\widetilde{\mathcal{N}}}(\gamma)$.*

We will need some information about the image of C_ν in $K^{G^\vee}(\widetilde{\mathcal{N}})$. Recall the notion of a two-sided cell in $\widehat{W}$ (see [30]). These are certain subsets in $\widehat{W}$, the set of two-sided cells comes equipped with a partial order. There exists a canonical bijection between the set of two-sided cells and $G^\vee$-orbits on $\mathcal{N}$ (see [31]). The order on two-sided sets corresponds to the adjunction order on nilpotent orbits (see [6, Theorem 4(b)]).

We will write $\mathbb{O}_c$ for the orbit, corresponding to the two sided cell of c, and $\mathbb{O}_{\leqslant c}$ for its closure. Let $\widetilde{\mathbb{O}}_c$, $\widetilde{\mathbb{O}}_{\leqslant c}$ be the (reduced) preimages of $\mathbb{O}_c$, $\mathbb{O}_{\leqslant c}$ in $\widetilde{\mathcal{N}}$.

Let $K^{G^\vee}_{\leqslant c}(\widetilde{\mathcal{N}}) \subset K^{G^\vee}(\widetilde{\mathcal{N}})$ denote the subgroup generated by classes of sheaves supported on $\widetilde{\mathbb{O}}_{\leqslant c}$. Let $K^{G^\vee}_{<c}(\widetilde{\mathcal{N}}) \subset K^{G^\vee}(\widetilde{\mathcal{N}})$ denote the subgroup generated by classes of sheaves supported on $\widetilde{\mathbb{O}}_{\leqslant c} \setminus \widetilde{\mathbb{O}}_c$.

Theorem 4.3 ([7, §11.3]). *The $\mathbb{Z}$-module $K^{G^\vee}_{\leqslant c}(\widetilde{\mathcal{N}})$ is spanned by the elements of the canonical basis C_ν such that $w_\nu \in c' \leqslant c$. The quotient $K^{G^\vee}_{\leqslant c}(\widetilde{\mathcal{N}})/K^{G^\vee}_{<c}(\widetilde{\mathcal{N}})$ has a $\mathbb{Z}$-basis, consisting of classes of C_ν, $w_\nu \in c$.*

Corollary 4.4. *For every open $G^\vee$-invariant locally closed subvariety $U \subset \mathcal{N}$ and $\widetilde{U} = \pi^{-1}(U)$, the kernel of the surjection $K^{G^\vee}(\widetilde{\mathcal{N}}) \twoheadrightarrow K^{G^\vee}(\widetilde{U})$ is spanned over $\mathbb{Z}$ by $\{C_\nu \mid \mathbb{O}_{w_\nu} \not\subset U\}$.*

Proof. Follows from Theorem 4.3. □

In fact, [3, 7] provide more information on the images of C_ν in $K^{G^\vee}(\widetilde{\mathcal{N}})$: these are exactly the classes of irreducible objects in the heart of a certain t-structure on $D^b(\mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}}))$, the so called exotic t-structure, introduced in [5] (and called "perversely exotic" in [8]). Recall their explicit description.

Theorem 4.5. *There exist a $G^\vee$-equivariant vector bundle $\mathcal{E}$ on $\widetilde{\mathcal{N}}$ with the following properties.*

0) The structure sheaf $\mathcal{O}$ is a direct summand in $\mathcal{E}$. Also, $\mathcal{E}^*$ is globally generated.

1) Let $\mathbf{A} = \mathrm{End}(\mathcal{E})^{op}$. Then the functor $\mathcal{F} \mapsto RHom(\mathcal{E}, \mathcal{F})$ provides equivalences

$$D^b(\mathrm{Coh}(\widetilde{\mathcal{N}})) \cong D^b(\mathbf{A} - mod),$$
$$D^b(\mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}})) \cong D^b(\mathbf{A} - mod^{G^\vee}),$$

where $\mathbf{A} - mod$ is the category of finitely generated $\mathbf{A}$-modules.

2) Irreducible objects in the heart of the exotic t-structure are in bijection with pairs $(\mathbb{O}, M)$ where $\mathbb{O}$ is a $G^\vee$-orbit in $\mathcal{N}$ and M is an irreducible equivariant module for $\mathbf{A}|_{\mathbb{O}}$.
More precisely, given $(\mathbb{O}, M)$ as above there exists an object $\mathbf{L}_{\mathbb{O},M} \in D^b(\mathbf{A}-mod^{G^\vee})$ uniquely characterized by the following properties: $\mathbf{L}_{\mathbb{O},M}$ is supported on the closure of $\mathbb{O}$, its restriction to the open subset $\mathbb{O}$ of its support is isomorphic to $M\Big[-\frac{\mathrm{codim}(\mathbb{O})}{2}\Big]$; for every orbit $\mathbb{O}' \neq \mathbb{O}$ the object $i^*_{\mathbb{O}'}(\mathbf{L}_{\mathbb{O},M})$ is concentrated in cohomological degrees less than $\frac{\mathrm{codim}(\mathbb{O}')}{2}$ and the object $i^!_{\mathbb{O}'}(\mathbf{L}_{\mathbb{O},M})$ is concentrated in cohomological degrees greater than $\frac{\mathrm{codim}(\mathbb{O}')}{2}$. The object $\mathbf{L}_{\mathbb{O},M}$ is irreducible in the heart of the exotic t-structure and every such irreducible is isomorphic to $\mathbf{L}_{\mathbb{O},M}$ for some $(\mathbb{O}, M)$.

3) The classes of $\mathbf{L}_{\mathbb{O},M}$ form the canonical basis in M, where we identified $K^{G^\vee}(\widetilde{\mathcal{N}}) \simeq M$ using the map $[\mathcal{O}(\gamma)] \mapsto T_{-\gamma}$.

Proof. The vector bundle $\mathcal{E}$ is introduced in [8, Theorem 1.5.1], which asserts that $\mathcal{E}$ is a tilting generator, i.e. statement 1) holds. It contains $\mathcal{O}$ as a direct summand by [8, Theorem 1.8.2 (a,1)]. Statements 2), 3) follow from [8, Theorem 6.2.1].

It remains to show that $\mathcal{E}^*$ is globally generated. Recall from [8, §1.8] a collection of tilting vector bundles on $\widetilde{\mathcal{N}}$ parametrized by alcoves, here $\mathcal{E}$ corresponds to the fundamental alcove and $\mathcal{E}^*$ is the tilting bundle corresponding to the anti-fundamental one. A vector bundle $\mathcal{V}$ is globally generated iff for every morphism from $f\colon \mathcal{V} \to k_x$, where k_x is a skyscraper sheaf, there exists a morphism $\phi\colon \mathcal{O} \to \mathcal{V}$ such that $f \circ \phi \neq 0$. It is easy to see that when $\mathcal{V}$ is a dilation equivariant vector bundle on $\widetilde{\mathcal{N}}$, it suffices to consider x in the zero section $G^\vee/B^\vee \subset \widetilde{\mathcal{N}}$. Moreover, it is enough to prove a similar statement over a field k of a large positive characteristic. In that case, we can apply localization functor corresponding to the point -2ρ in the anti-fundamental alcove to translate this statement into one in the representation theory of the Lie

algebra $\mathfrak{g}^\vee$ over k. The localization equivalence relates the functor of global sections to translation to the singular central character $-\rho$, a skyscraper sheaf is identified with a module M_0^* where M_0 is a baby Verma module with highest weight zero, while $\mathcal{E}^*$ (pulled back to the formal neighborhood of the zero section) is identified with a projective generator in the corresponding category of $\mathfrak{g}^\vee$-modules. Thus the statement reduces to showing that translation functor $T_{-2\rho\to-\rho}$ does not kill any nonzero submodule of M_0^*. This follows from the standard fact that the adjunction arrow $M_0^* \to T_{-\rho\to-2\rho}T_{-2\rho\to\rho}M_0^*$ is injective. □

Remark 4.6. Notice that the isomorphisms between $K^{G^\vee}(\widetilde{\mathcal{N}})$ and the anti-spherical module M in Theorem 4.2 and in Theorem 4.5 are different: one sends $[\mathcal{O}(\gamma)]$ to T_γ while the other sends it to $T_{-\gamma}$. Both are natural from some perspective and both appear in the literature. A related issue is the choice of the isomorphism between the weight lattice and the Picard group of the flag variety: we use the one sending a dominant weight to a semi-ample line bundle, thus the weights of the action of a Borel subgroup on the nilpotent radical of its Lie algebra correspond to negative roots, while some authors prefer the opposite convention (see [11, Section 6.1.11]). We will work with the isomorphism of Theorem 4.2, see below.

For technical reasons we prefer to work with the globally generated tilting bundle $\mathcal{E}^*$ rather than with $\mathcal{E}$. Thus we set $A = End(\mathcal{E}^*)^{op} = \mathbf{A}^{op}$ and consider the equivalence $D^b(Coh(\widetilde{\mathcal{N}})) \cong D^b(A-mod)$, $\mathcal{F} \mapsto RHom(\mathcal{E}^*, \mathcal{F})$. In this approach it is more natural to use the isomorphism between $K^{G^\vee}(\widetilde{\mathcal{N}})$ and the anti-spherical module M sending $[\mathcal{O}(\gamma)]$ to T_γ, see Theorem 4.2. With this identification, elements of the canonical basis correspond to classes $\mathcal{L}_{\mathbb{O},M}$ where $\mathbb{O}$ is a $G^\vee$-orbit in $\mathcal{N}$ and M is an irreducible equivariant module for $A|_{\mathbb{O}}$ characterized as in Theorem 4.5 (2).

For $e \in \mathcal{N}$ let A_e denote the corresponding specialization of A.

We recall an explicit description of (complexes of) coherent sheaves corresponding to some irreducible A-modules. Let $\mathcal{A} = A-mod$, the category of finitely generated A-modules. We identify $\mathcal{A}$ with the corresponding full subcategory in $D^b(\mathrm{Coh}(\widetilde{\mathcal{N}}))$. For $e \in \mathcal{N}$ let $\mathcal{A}_e$ be the full subcategory in $\mathcal{A}$, consisting of objects set-theoretically supported on $\pi^{-1}(e)$.

Recall that e is a subregular nilpotent.

Irreducible components of $\mathcal{B}_e$ are parametrized by $\tilde{I}$; here $\tilde{I} = I$ if the Dynkin diagram of $\mathfrak{g}$ is simply laced (see Sections 5.1 and 6.3 below) and $\tilde{I}$ is the set of vertices of the unfolding of the Dynkin graph in general (see [37]). Each irreducible component Π_i, $i \in \tilde{I}$, is isomorphic to $\mathbb{P}^1$.

Lemma 4.7 (cf. [9, Example 5.3.3]). *Let $e \in \mathcal{N}$ be a subregular nilpotent. The irreducible objects in $\mathcal{A}_e$ are: $\mathcal{O}_{\Pi_i}(-1)[1]$, $\mathcal{O}_{\pi^{-1}(e)}$ where $\pi^{-1}(e)$ is the schematic fiber of the Springer map π.*

Proof. Property 1) in Theorem 4.5 shows that $\mathcal{E}$ is tilting, i.e. $Ext^{>0}(\mathcal{E},\mathcal{E}) = 0$. Since $\mathcal{O}$ is a direct summand of $\mathcal{E}$, it follows that $H^1(\mathcal{E}) = H^1(\mathcal{E}^*) = 0$. Using that $\mathcal{E}$ is globally generated we conclude that $\mathcal{E}|_{\Pi_i}$ is a sum of copies of $\mathcal{O}_{\Pi_i}$ and $\mathcal{O}_{\Pi_i}(1)$ (note that $R\Gamma$ has homological dimension 1 so H^1 is right exact, thus $H^1(\mathcal{E}) = 0$ implies $H^1(E|_{\Pi_i}) = 0$).

It is then clear that $\mathcal{O}_{\Pi_i}(-1)[1]$ lies in the heart.

Since $\mathcal{O}$ is a direct summand in $\mathcal{E}$ it follows that the objects $\mathcal{F}$ such that $R\Gamma(\mathcal{F}) = 0$ form a Serre subcategory $\mathcal{A}_e^0$ in $\mathcal{A}_e$.

For an object $\mathcal{F}$ supported on $\pi^{-1}(e)$ with $R\Gamma(\mathcal{F}) = 0$, each cohomology sheaf of $\mathcal{F}$ is an extension of sheaves of the form $\mathcal{O}_{\Pi_i}(-1)$, see, for example, [27, Theorem 2.3]. It follows that $\mathcal{A}_e^0$ consists of objects of the form $\mathcal{F}[1]$ where $\mathcal{F}$ is an extension of $\mathcal{O}_{\Pi_i}(-1)$ and that $\mathcal{O}_{\Pi_i}(-1)[1]$ are irreducible.

We know that the classes of irreducible objects form a basis in the Grothendieck group $K(\mathrm{Coh}(\pi^{-1}(e))$ which is isomorphic to homology of $\pi^{-1}(e)$ and has dimension $|\tilde{I}|+1$. Thus there exists a unique irreducible L_0 not isomorphic to $\mathcal{O}_{\Pi_i}(-1)[1]$. It remains to show that $L_0 \cong \mathcal{O}_{\pi^{-1}(e)}$. In the usual t-structure there exists a filtration of $\mathcal{O}_{\pi^{-1}(e)}$ starting with $\Bbbk_p$ for some point $p \in \mathcal{B}_e$ with the other terms being $\mathcal{O}_{\Pi_i}(-1)$. Since $\Bbbk_p$ clearly lies in the heart of our t-structure, we see that $\mathcal{O}_{\pi^{-1}(e)}$ is also in the heart, and is a subobject of $\Bbbk_p$.

It is clear that $\mathrm{Hom}(\mathcal{O}_{\Pi_i}(-1)[1], \mathcal{O}_{\pi^{-1}}(e))$ vanishes. We check that Hom vanishing in the other direction also holds. To this end, consider the Slodowy variety $\widetilde{S}_e$, resolving the Slodowy slice S_e to $e \in \mathcal{N}$ (see [37]). We have an exact sequence

$$0 \to \mathcal{O}_{\widetilde{S}_e^{(1)}}(-\pi^{-1}(e)) \to \mathcal{O}_{\widetilde{S}_e^{(1)}} \to \mathcal{O}_{\pi^{-1}(e)} \to 0. \tag{28}$$

Since S_e is affine, $\mathcal{O}_{\widetilde{S}_e^{(1)}}(-\pi^{-1}(e))$ is globally generated, which yields

$$\mathrm{Hom}(\mathcal{O}_{\pi^{-1}(e)}, \mathcal{O}_{\Pi_i}(-1)[1]) = 0.$$

It follows that both socle and cosocle of $\mathcal{O}_{\pi^{-1}(e)}$ is the sum of copies of L_0. If $\mathcal{O}_{\pi^{-1}(e)} \neq L_0$, then we get a nonconstant endomorphism of $\mathcal{O}_{\pi^{-1}(e)}$ but again using the exact sequence (28) we see that $\Gamma(\mathcal{O}_{\pi^{-1}(e)})$ is one dimensional. □

Remark 4.8. According to [8], base change of A to a slice to a nilpotent orbit is derived equivalent to the resolution of the slice, this yields a t-structure on the resolution of the slice. When e is subregular, the resolution of the slice

coincides with the minimal resolution of a rational (Kleinian) singularity. Comparing Lemma 4.7 with [27, Theorem 2.3] one sees that in this case the above t-structure coincides with one arising from the MacKay equivalence between the derived category of the resolution and the derived category of the orbifold.

Assume now $\mathfrak{g}$ is of type D or E. Then the centralizer $Z_{G^\vee}(e)$ is unipotent. It is clear each of the above irreducibles carries a unique equivariant $Z_{G^\vee}(e)$ structure. For $\mathfrak{g} = \mathfrak{sl}_n$ $(n \geqslant 3)$ we have $Z_e \simeq \mathbb{C}^\times$ (recall that $Z_e \subset Z_{G^\vee}(e)$ is the reductive part); for a $\mathbb{C}^\times$-equivariant sheaf $\mathcal{F}$ and $k \in \mathbb{Z}$ we denote by $\mathcal{F}\langle k\rangle$ the same sheaf but with the $\mathbb{C}^\times$-equivariant structure twisted by the character $t \mapsto t^k$ of $\mathbb{C}^\times$. It is clear that each of the above irreducibles carries unique (up to a shifting by $k \in \mathbb{Z}$) $Z_{G^\vee}(e)$-equivariant structure.

We will use the same notation for the resulting $Z_{G^\vee}(e)$-equivariant sheaves, as well as for the corresponding $G^\vee$-equivariant sheaves on the schematic preimage of the orbit $\pi^{-1}(\mathbb{O}_e) \xrightarrow{\iota} \widetilde{\mathcal{N}}$.

Consider $U = \mathbb{O}_e \cup \mathbb{O}^{\mathrm{reg}}$. We record a description of the canonical basis in $K^{G^\vee}(\widetilde{U})$ stemming from Theorem 4.5 and Lemma 4.7.

Proposition 4.9. *For $\mathfrak{g}$ of type D_n $(n \geqslant 4)$ or E_6, E_7, E_8 the canonical basis of $K^{G^\vee}(\widetilde{U})$ consists of classes of*

$$\mathcal{O}_{\widetilde{U}},\ \iota_*\mathcal{O}_{\pi^{-1}(\mathbb{O}_e)}[-1],\ \text{and } \iota_*\mathcal{O}_{\Pi_i}(-1),\ i = 1, \dots, r.$$

For $\mathfrak{g} = \mathfrak{sl}_n$ $(n \geqslant 3)$ the canonical basis of $K^{G^\vee}(\widetilde{U})$ consists of classes of

$$\mathcal{O}_{\widetilde{U}},\ \iota_*\mathcal{O}_{\pi^{-1}(\mathbb{O}_e)}[-1]\langle k\rangle,\ \text{and } \iota_*\mathcal{O}_{\Pi_i}(-1)\langle k\rangle,\ i = 1, \dots, n-1,\ k \in \mathbb{Z}.$$

Proof. We use the description of the canonical basis provided by the Theorem 4.5. It is easy to see that $\mathcal{O}_{\widetilde{\mathcal{N}}}$ satisfies the properties in Theorem 4.5 (2), so it corresponds to an element of the canonical basis (this is just the image of the unit element in the affine Hecke algebra).

Also, Theorem 4.5 together with Lemma 4.7 show the existence of irreducible objects in the heart of the exotic t-structure whose restriction to $\widetilde{U}$ coincides with the other objects listed in the Proposition (note that the objects that appear in Lemma 4.7 should be shifted by $[-\frac{\operatorname{codim}\mathbb{O}_e}{2}] = [-1]$). Thus the statement follows from Theorem 4.5 (3). □

Remark 4.10. The canonical basis in the corresponding module over the affine Hecke algebra was computed by other methods in [35] for type A and in [34] for types D and E. It is not hard to check that the basis in Proposition 4.9 after applying Grothendieck-Serre duality to it (see [33, Sections 6.10, 6.11, 6.12]) agrees with those earlier results.

5. The subregular type D, E case

In this section we assume that $\mathfrak{g}$ is of type D or E. Since $\mathfrak{g}$ is simply-laced, we have $\mathfrak{g} = \mathfrak{g}^\vee$. Recall that $e \in \mathfrak{g}$ is the subregular nilpotent element and $c \subset \widehat{W}$ is the corresponding two-sided cell. Recall the $\widehat{W}$-module $K^{G^\vee}(\widetilde{U})$, and that this module has a (canonical) basis $\bar{C}_\nu$ parametrized by ν such that the corresponding w_ν lies in $c \cup \{1\}$. Let us describe such ν, w_ν explicitly.

The following proposition follows from [29, Proposition 3.8], see also [40, Proposition 3.6].

Proposition 5.1. *The cell c, corresponding to the subregular nilpotent e, consists of all nonidentity elements $w \in \widehat{W}$ that have unique reduced decomposition.*

Corollary 5.2. *Let c be as in Proposition 5.1. The elements $\nu \in Q^\vee$ such that $w_\nu \in c$ can be described as follows. The set of possible ν is parametrized by $\widehat{I}$. For $i \in \widehat{I}$ let us connect i with $0 \in \widehat{I}$ by the segment:*

$$0 = j_0, j_1, \dots, j_{l-1}, j_l = i.$$

Then the element ν_i is equal to

$$\nu_i = s_i s_{j_{l-1}} \dots s_{j_1}(\theta) = \theta - \alpha_{j_1}^\vee - \dots - \alpha_{j_{l-1}}^\vee - \alpha_i^\vee \tag{29}$$

and the corresponding w_{ν_i} is

$$w_{\nu_i} = w_i = s_i s_{j_{l-1}} \dots s_{j_1} s_0.$$

Proof. It easily follows from Proposition 5.1 that the elements of c of the form w_ν are precisely the elements w_i, $i \in \widehat{I}$. Recall now that

$$w_i = s_i s_{j_{l-1}} \dots s_{j_1} s_0 = s_i s_{j_{l-1}} \dots s_{j_1} s_\theta t_{-\theta} = t_{s_i s_{j_{l-1}} \dots s_{j_1}(\theta)} s_\theta s_{j_1} \dots s_{j_{l-1}} s_i.$$

We conclude that ν that corresponds to w_i is equal to $s_i s_{j_{l-1}} \dots s_{j_1}(\theta)$. It remains to note that

$$\begin{aligned} s_i s_{j_{l-1}} \dots s_{j_1}(\theta) &= s_i s_{j_{l-1}} \dots s_{j_1}(\delta - \alpha_0) \\ &= \delta - s_i s_{j_{l-1}} \dots s_{j_1}(\alpha_0) = \delta - (\alpha_0 + \alpha_{j_1} + \dots + \alpha_{j_{l-1}} + \alpha_i) \\ &= \theta - \alpha_{j_1} - \dots - \alpha_{j_{l-1}} - \alpha_i. \end{aligned}$$

□

Remark 5.3. Note that $\nu_0 = \theta$, $w_0 = s_0$.

Our goal is to identify $\widehat{W}$-module $K^{G^\vee}(\widetilde{U}) \otimes \mathbb{Z}_{\text{sign}}$ with $\widehat{\mathfrak{h}}_\mathbb{Z} = \mathbb{Z}Q^\vee \oplus \mathbb{Z}K \oplus \mathbb{Z}d$.

5.1. Structure of $K(\mathcal{B}_e)$

Let us first of all describe the geometry of the variety $\mathcal{B}_e$. Recall that $\mathcal{B}_e$ is the fiber over e of the Springer resolution $\pi\colon \widetilde{\mathcal{N}} \to \mathcal{N}$. Directly from the definitions we have

$$\mathcal{B}_e = \{\mathfrak{b}' \in \mathcal{B} \mid e \in \mathfrak{n}_{\mathfrak{b}'}\},$$

where $\mathfrak{n}_{\mathfrak{b}'} \subset \mathfrak{b}$ is the unipotent radical of the Borel subalgebra $\mathfrak{b}'$.

The following lemma is standard (see [37]).

Lemma 5.4. *For every $i = 1, \dots, r$ there exists the unique parabolic subalgebra $\mathfrak{p}_{e,i}$ such that*

(1) *$\mathfrak{p}_{e,i}$ is conjugate to the standard minimal parabolic subalgebra, corresponding to $\alpha_i^\vee$,*
(2) *the nilradical of $\mathfrak{p}_{e,i}$ contains e.*

We denote the corresponding parabolic subgroup by $P_{e,i} \subset G^\vee$.

For $i \in 1, \dots, r$ let $\mathcal{P}_i = G^\vee/P_{e,i}$ be the variety of parabolic subalgebras of $\mathfrak{g}^\vee$ of type $\mathfrak{p}_{e,i}$. The following proposition is standard (see [37]).

Proposition 5.5. *The variety $\mathcal{B}_e$ has r irreducible components Π_i, $i = 1, \dots, r$. The component Π_i, corresponding to $i \in I$, is the fiber of the morphism $\mathcal{B} \to \mathcal{P}_i$ over the point $\mathfrak{p}_{e,i}$ (in particular, $\Pi_i \simeq \mathbb{P}^1$). For $i \neq j$ components Π_i, Π_j intersect iff $(\alpha_i, \alpha_j) = -1$. If this is the case, then Π_i, Π_j intersect transversally at one point.*

We pick any point $p \in \mathcal{B}_e$ and denote by $[\mathbb{C}_p]$ the class in $K(\mathcal{B}_e)$ of the skyscraper sheaf $\mathbb{C}_p$.

Remark 5.6. Note that $[\mathbb{C}_p]$ does not depend on the point p. Indeed, it is enough to check this for $\mathbb{P}^1$ where every skyscraper sheaf is equal to $[\mathcal{O}_{\mathbb{P}^1}] - [\mathcal{O}_{\mathbb{P}^1}(-1)]$.

For $m \in \mathbb{Z}$ we denote by O_i^m the class of $\mathcal{O}_{\Pi_i}(m)$ in $K(\Pi_i)$. We denote by $o_i^m \in K(\mathcal{B}_i)$ the direct image of O_i^m under the closed embedding $\Pi_i \subset \mathcal{B}_i$.

The following lemma holds by [34, Section 3.4].

Lemma 5.7. *The $\mathbb{Z}$-module $K(\mathcal{B}_e)$ is spanned by o_i^m, $i = 1, \dots, r$, $m \in \mathbb{Z}$, subject to relations*

$$o_i^m - o_i^{m-1} = o_j^m - o_j^{m-1},\ o_i^{m+1} - o_i^m = o_i^m - o_i^{m-1},\ \ i, j = 1, \dots, r.$$

Remark 5.8. The relations follow from the standard exact sequences on $\mathbb{P}^1$ (and their twistings by $\mathcal{O}_{\mathbb{P}^1}(m)$):

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to \mathbb{C}_p \to 0, \\ 0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0. \end{aligned} \tag{30}$$

Corollary 5.9 ([34, Section 3.4])**.** *The $\mathbb{Z}$-module $K(\mathcal{B}_e)$ has a basis o_i^{-1}, $i = 1, \dots, r$, $[\mathbb{C}_p]$.*

Proof. Easily follows from Lemma 5.7. □

Remark 5.10. *Note that for every $i \in I$ we have $[\mathbb{C}_p] = o_i^0 - o_i^{-1}$ and, more generally, $o_i^m - o_i^{m-1} = [\mathbb{C}_p]$ for every $m \in \mathbb{Z}$.*

The following lemma holds by [34, Lemma 3.6].

Lemma 5.11. *For $\gamma \in Q^\vee$ we have*

(a) $t_\gamma([\mathbb{C}_p]) = [\mathbb{C}_p]$,
(b) $t_\gamma(o_i^m) = o_i^{m+\langle \alpha_i, \gamma \rangle}$.

The following lemma holds by [34, Lemmas 3.7, 3.8].

Lemma 5.12. *For $i \in I$ we have*

(a) $s_i([\mathbb{C}_p]) = -[\mathbb{C}_p]$,
(b) $s_i(o_i^{-1}) = o_i^{-1}$.

The following lemma holds by [34, Lemma 3.12].

Lemma 5.13. *If $i, j \in I$ are such that $(\alpha_i, \alpha_j) = 0$, then $s_i(o_j^{-1}) = -o_j^{-1}$.*

The following lemma holds by [34, Lemma 3.11].

Lemma 5.14. *If $i, j \in I$ are such that $(\alpha_i, \alpha_j) = -1$, then $s_i(o_j^{-1}) = -o_j^{-1} - o_i^{-1}$.*

Recall that $\mathbb{Z}_{\text{sign}}$ is the one dimensional sign representation of $\widehat{W}$. We finally obtain the following proposition.

Proposition 5.15. *There is an isomorphism of $\widehat{W}$-modules*

$$K(\mathcal{B}_e) \simeq (\mathfrak{h}_{\mathbb{Z}} \oplus \mathbb{Z}K) \otimes \mathbb{Z}_{\text{sign}},$$

given by

$$K \otimes 1 \mapsto [\mathbb{C}_p],\ \alpha_i^\vee \otimes 1 \mapsto -o_i^{-1},\ i = 1, \dots, r.$$

Proof. The only nontrivial part is to compare the action of $t_{\alpha_j^\vee}$ on o_i^{-1} with the action of $t_{\alpha_j^\vee}$ on $\alpha_i^\vee$. If $(\alpha_i, \alpha_j) = 0$, then we have

$$t_{\alpha_j^\vee}(o_i^{-1}) = o_i^{-1}$$

as desired. If $(\alpha_i, \alpha_j) = -1$, then we have

$$t_{\alpha_j^\vee}(o_i^{-1}) = o_i^{-2} = o_i^{-1} - [\mathbb{C}_p].$$

Finally, if $i = j$ then we have

$$t_{\alpha_i^\vee}(o_i^{-1}) = o_i^1 = o_i^0 + \mathbb{C}_p = o_i^{-1} + 2[\mathbb{C}_p].$$ □

5.2. Structure of $K^{G^\vee}(\widetilde{U})$

Let us describe the $\widehat{W}$-module $K^{G^\vee}(\widetilde{U})$ (recall that $\mathfrak{g}$ is of type D, E). Recall that we have the closed embedding $\iota\colon \widetilde{\mathbb{O}}_e \subset \widetilde{U}$ (where $\widetilde{\mathbb{O}}_e$ is the schematic preimage of $\mathbb{O}_e \subset U$). It induces the homomorphism $\iota_*\colon K^{G^\vee}(\widetilde{\mathbb{O}}_e) \to K^{G^\vee}(\widetilde{U})$ of $\widehat{W}$-modules. Note also that we have the natural identification $K^{G^\vee}(\widetilde{\mathbb{O}}_e) = K(\mathcal{B}_e)$.

Recall that by Theorem 4.3 and Corollary 4.4 we have an exact sequence of $\widehat{W}$-modules

$$0 \to K(\mathcal{B}_e) \xrightarrow{\iota_*} K^{G^\vee}(\widetilde{U}) \to K^{G^\vee}(\mathbb{O}^{\mathrm{reg}}) \to 0.$$

Proposition 5.16. *There is an isomorphism of $\widehat{W}$-modules*

$$K^{G^\vee}(\widetilde{U}) \simeq (\mathfrak{h}_\mathbb{Z} \oplus \mathbb{Z}K \oplus \mathbb{Z}d) \otimes \mathbb{Z}_{\mathrm{sign}} = \widehat{\mathfrak{h}}_\mathbb{Z} \otimes \mathbb{Z}_{\mathrm{sign}},$$

given by ($i \in \{1, 2, \ldots, r\}$)

$$K \mapsto \iota_*[\mathbb{C}_p],\ \alpha_0^\vee \mapsto \iota_*[\mathcal{O}_{\pi^{-1}(e)}],\ \alpha_i^\vee \mapsto -\iota_* o_i^{-1},\ d \mapsto [\mathcal{O}_{\widetilde{U}}].$$

The images of the elements of the canonical basis under this isomorphism are as follows:

$$\bar{C}_0 = \bar{T}_0 \mapsto d,\ \bar{C}_{\nu_i} \mapsto -\alpha_i^\vee,\ i = 0, 1, \ldots, r,$$

where the ν_i are defined by (29).

Proof. Recall that $K^{G^\vee}(\widetilde{U})$ has a basis

$$\iota_*[\mathbb{C}_p], \iota_* o_1^{-1}, \ldots, \iota_* o_r^{-1}, [\mathcal{O}_{\widetilde{U}}].$$

Moreover, $\iota_*[\mathbb{C}_p], \iota_* o_1^{-1}, \dots, \iota_* o_r^{-1}$ form the submodule isomorphic to $K(\mathcal{B}_e) \simeq (\mathfrak{h}_\mathbb{Z} \oplus \mathbb{C}K) \otimes \mathbb{Z}_{\mathrm{sign}}$ (see Proposition 5.15).

It remains to compute the action of $\widehat{W}$ on $[\mathcal{O}_{\widetilde{U}}]$. The module $K^{G^\vee}(\widetilde{U})$ is the quotient of $K^{G^\vee}(\widetilde{\mathcal{N}})$, where the surjection $K^{G^\vee}(\widetilde{\mathcal{N}}) \twoheadrightarrow K^{G^\vee}(\widetilde{U})$ is induced by the restriction $\widetilde{\mathcal{N}} \supset \widetilde{U}$. Moreover, we have the identification of $\widehat{W}$-modules

$$K^{G^\vee}(\widetilde{\mathcal{N}}) \simeq \mathbb{Z}\widehat{W} \otimes_{\mathbb{Z}W} \mathbb{Z}_{\mathrm{sign}},\ [\mathcal{O}(\gamma)] \mapsto T_\gamma.$$

In particular, $[\mathcal{O}_{\widetilde{\mathcal{N}}}] \in K^{G^\vee}(\widetilde{\mathcal{N}})$ identifies with $1 \in \mathbb{Z}\widehat{W} \otimes_{\mathbb{Z}W} \mathbb{Z}_{\mathrm{sign}}$, so W acts on $[\mathcal{O}_{\widetilde{\mathcal{N}}}]$ via the sign representation. It remains to compute the action of $Q^\vee$ on $[\mathcal{O}_{\widetilde{U}}]$. In other words we need to compute the action of the elements $t_{\alpha_i^\vee}$, $i = 1, \dots, r$ on $[\mathcal{O}_{\widetilde{U}}]$. By the definitions and Lemma 4.1 we have

$$t_{\alpha_i^\vee} \cdot [\mathcal{O}_{\widetilde{U}}] = [\mathcal{O}_{\widetilde{U}}(\alpha_i^\vee)] = [\mathcal{O}_{\widetilde{U}}] - \iota_* o_i^0 = [\mathcal{O}_{\widetilde{U}}] - \iota_* o_i^{-1} - \iota_*[\mathbb{C}_p].$$

It remains to recall that the action of $t_{\alpha_i^\vee}$ on d is given by

$$t_{\alpha_i^\vee}(d) = d + \alpha_i^\vee - K.$$

The isomorphism of $\widehat{W}$-modules $K^{G^\vee}(\widetilde{U}) \simeq \widehat{\mathfrak{h}}_\mathbb{Z} \otimes \mathbb{Z}_{\mathrm{sign}}$ follows.

The claim about the canonical basis follows from Proposition 4.9. The fact that the element C_{ν_i} is equal to $-\alpha_i^\vee$ but not $-\alpha_j^\vee$ for some other j can be easily seen from the equality (use (29)):

$$t_{\nu_i}(d) = d + \nu_i - K = d - \alpha_0^\vee - \alpha_{j_1}^\vee - \dots - \alpha_i^\vee$$

together with Remark 3.15. □

Remark 5.17. One can avoid the use of Proposition 4.9 in the proof of Proposition 5.16 and instead use results of [34] on the canonical basis in $K^{G^\vee}(\mathcal{B}_e)$ together with [8, Theorem 5.3.5]. Recall that the canonical basis from [34] differs from the one in Proposition 4.9 by applying Grothendieck-Serre duality to it.

5.3. Computation of $\mathbf{m}_{w_i}^{w\gamma}$ and the proof of Theorem 2.9

Recall now that by Equation (17) we have

$$T_\gamma = \sum \mathbf{m}_{w_\nu}^{w_\gamma} C_\nu.$$

Taking the image of this equality in $K^{G^\vee}(\widetilde{U}) \simeq \widehat{\mathfrak{h}}_{\mathbb{Z}} \otimes \mathbb{Z}_{\mathrm{sign}}$ and using Proposition 5.16, we see that

$$t_\gamma(d) = d - \sum_{i=0,1,\dots,r} \mathbf{m}_{w_i}^{w_\gamma} \alpha_i^\vee.$$

So we can compute $\mathbf{m}_{w_i}^{w_\gamma}$. Indeed, this is the coefficient in front of $-\alpha_i^\vee$ in

$$t_\gamma(d) - d = \gamma - \frac{1}{2}|\gamma|^2 K.$$

We conclude that

$$\mathbf{m}_{w_i}^{w_\gamma} = \left\langle -\Lambda_i, \gamma - \frac{1}{2}|\gamma|^2 K \right\rangle = \left\langle \Lambda_i, -\gamma + \frac{|\gamma|^2}{2} K \right\rangle. \tag{31}$$

We are now ready to prove Theorem 2.9.

Proof of Theorem 2.9. Let i, λ, Λ be as in Theorem 2.9. Combining (24) and (31), we conclude that

$$\begin{aligned} \widehat{R}\,\mathrm{ch}\, L(\Lambda) &= \sum_{\gamma \in Q^\vee} \sum_{u \in W} \varepsilon(uw_i) \mathbf{m}_{w_i}^{w_{-\gamma}} e^{ut_\gamma(\lambda + \widehat{\rho})} \\ &= \sum_{\gamma \in Q^\vee} \sum_{u \in W} \varepsilon(uw_i) \left\langle \Lambda_i, \gamma + \frac{|\gamma|^2}{2} K \right\rangle e^{ut_\gamma w_i(\Lambda + \widehat{\rho})}, \end{aligned}$$

which is precisely the statement of Theorem 2.9. □

6. The subregular type A case

In this section we assume that $\mathfrak{g} = \mathfrak{sl}_n$ for some $n \in \mathbb{Z}_{\geqslant 3}$. Recall that our goal is to describe the $\widehat{W}$-module $K^{\mathrm{PGL}_n}(\widetilde{U})$ and the canonical basis in it.

6.1. Structure of $\widehat{W}$, its extended version $\widehat{W}^{\mathrm{ext}}$

Recall that $Q^\vee$ is the cocharacter lattice of SL_n that is equal to the character lattice of PGL_n. We can identify $Q^\vee$ with the following sublattice of $\mathbb{Z}^n$:

$$Q^\vee = \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid a_1 + \dots + a_n = 0\}.$$

Recall that $\epsilon_1, \dots, \epsilon_n$ is the standard basis of $\mathbb{Z}^n$. Simple coroots $\alpha_i^\vee \in Q^\vee$ are $\alpha_i^\vee = \epsilon_i - \epsilon_{i+1}$, $i = 1, \dots, n-1$, and the Weyl group of $\mathfrak{sl}_n$ is S_n. The action

of S_n on $Q^\vee$ is the standard action via permutations. Recall that

$$\widehat{W} = Q^\vee \rtimes S_n.$$

The group $\widehat{W}$ can be described as follows. It is the group of all permutations $\sigma\colon \mathbb{Z} \to \mathbb{Z}$ such that $\sigma(i+n) = \sigma(i) + n$ and $\sum_{i=1}^{n}(\sigma(i) - i) = 0$. The group of all permutations $\sigma\colon \mathbb{Z} \to \mathbb{Z}$ such that $\sigma(i+n) = \sigma(i) + n$ is isomorphic to $S_n \ltimes \mathbb{Z}^n$.

Let us recall the description of the elements $s_0, s_1, \ldots, s_{n-1}$, t_γ in these terms. Element s_i is given by

$$s_i(j) = \begin{cases} j+1 & \text{for } j \equiv i \pmod{n}, \\ j-1 & \text{for } j \equiv i+1 \pmod{n}, \\ j & \text{otherwise.} \end{cases} \tag{32}$$

For a lattice element $\gamma = (a_0, \ldots, a_{n-1}) \in \mathbb{Z}^n$ the corresponding element t_γ of $\mathbb{Z}^n \rtimes S_n$ is given by

$$t_\gamma(k) = k + a_{[k]}n,$$

where $[k] \in \mathbb{Z}/n\mathbb{Z} \simeq \{0, 1, \ldots, n-1\}$ is the class of k modulo n. So the element t_{ϵ_k} is given by

$$t_{\epsilon_k}(j) = \begin{cases} j+n & \text{for } j \equiv k \pmod{n}, \\ j & \text{otherwise.} \end{cases}$$

Let $P^\vee$ be the cocharacter lattice of PGL_n. We can identify $P^\vee$ with the following quotient of $\mathbb{Z}^n$:

$$P^\vee = \mathbb{Z}^{\oplus n}/\mathbb{Z}(\epsilon_1 + \cdots + \epsilon_n).$$

We have a natural embedding $Q^\vee \subset P^\vee$. It will be useful to consider the the *extended* affine Weyl group

$$\widehat{W}^{\mathrm{ext}} := P^\vee \rtimes S_n.$$

For $\gamma \in \mathbb{Z}^n$ we denote by $t_\gamma \in \widehat{W}^{\mathrm{ext}}$ the corresponding element of $\widehat{W}^{\mathrm{ext}}$. The group $\widehat{W}^{\mathrm{ext}}$ is generated by $t_{\epsilon_1}, \ldots, t_{\epsilon_n}$, $s_1, \ldots, s_{n-1}$.

6.2. Parametrization of the canonical basis of $K^{\mathrm{PGL}_n}(\widetilde{U})$

Recall that $e \in \mathfrak{sl}_n$ is the subregular nilpotent element and $c \subset \widehat{W}$ is the corresponding two-sided cell. Recall that the module $K^{\mathrm{PGL}_n}(\widetilde{U})$ has a canonical basis $\bar{C}_\nu$ parametrized by ν such that $w_\nu \in c \cup \{1\}$. It follows from Proposition 5.1 that the set of w_ν as above is given by

Lemma 6.1. *The elements $w_\nu \in c$ can be described as follows. The set of possible ν is parametrized by $\mathbb{Z}$. For $i \in \mathbb{Z}$, the corresponding element w_i is equal to*

$$w_i = \begin{cases} s_{[i]}s_{[i-1]}\dots s_1 s_0 & \text{for } i > 0 \\ s_0 & \text{for } i = 0, \\ s_{[i]}s_{[i+1]}\dots s_{[-1]}s_0 & \text{for } i < 0 \end{cases}$$

and ν_i is the image of w_i in $Q^\vee \simeq \widehat{W}/S_n$.

6.3. Description of $\mathcal{B}_e$ and $\mathbb{C}^\times$-equivariant line bundles on it

Let us now recall the explicit description of $\mathcal{B}_e$.

The element e can be described as follows. Let $V = \mathbb{C}^n$ be the standard representation of $\mathfrak{sl}_n$ and let $v_1, \dots, v_n$ be the standard basis of $\mathbb{C}^n$. Then the element e is:

$$e(v_{n-1}) = e(v_n) = 0,\ e(v_i) = v_{i+1},\ i = 1, \dots, n-2.$$

For every (ordered) basis $b_1, \dots, b_n$ of V let $\mathcal{F}(b_1, \dots, b_n)$ be the flag

$$\{0\} \subset \mathrm{Span}_{\mathbb{C}}(b_1) \subset \mathrm{Span}_{\mathbb{C}}(b_1, b_2) \subset \dots \subset \mathrm{Span}_{\mathbb{C}}(b_1, \dots, b_{n-1}) \subset V.$$

To every $(k, a) \in (\{1, \dots, n-1\} \times \mathbb{C}) \cup \{(0, 0)\}$ we associate the flag

$$\mathcal{F}_{k,a} = \mathcal{F}(v_{n-1}, v_{n-2}, \dots, v_{n-k+1}, v_{n-k} + a v_n, v_n, v_{n-k-1}, v_{n-k-2}, \dots, v_1).$$

The irreducible components of $\mathcal{B}_e$ are parametrized by $k \in \{1, 2, \dots, n-1\}$:

$$\Pi_k = \{\mathcal{F}_{k,a} \mid a \in \mathbb{C}\} \cup \{\mathcal{F}_{k+1,0}\}.$$

For $k = 1, 2, \dots, n$ we set

$$p_{k-1,k} := \mathcal{F}_{k-1,0} = \mathcal{F}(v_{n-1}, \dots, v_{n-k+1}, v_n, v_{n-k}, v_{n-k-1}, \dots, v_1).$$

For $k = 2, \dots, n-1$ we have $p_{k-1,k} = \mathcal{F}_{n-k,0} = \Pi_k \cap \Pi_{k-1}$.

Our first goal is to describe the action of $\widehat{W}$ on $K^{\mathrm{PGL}_n}(\widetilde{\mathbb{O}}_e)$. It will be more convenient to describe the action of $\widehat{W}^{\mathrm{ext}}$ on $K^{\mathrm{SL}_n}(\widetilde{\mathbb{O}}_e)$ first. Recall that

$$K^{\mathrm{PGL}_n}(\widetilde{\mathbb{O}}_e) = K^{Z_{e,\mathrm{PGL}_n}}(\mathcal{B}_e),\ K^{\mathrm{SL}_n}(\widetilde{\mathbb{O}}_e) = K^{Z_{e,\mathrm{SL}_n}}(\mathcal{B}_e),$$

where $Z_{e,\mathrm{PGL}_n} \subset \mathrm{PGL}_n$, $Z_{e,\mathrm{SL}_n} \subset \mathrm{SL}_n$ are reductive parts of the centralizers of e.

We have the identifications (compare the second identification with [35, Section 5.1])

$$\mathbb{C}^\times \xrightarrow{\sim} Z_{e,\mathrm{PGL}_n} \subset \mathrm{PGL}_n,\ t \mapsto \mathrm{diag}(1,1,\dots,1,t),$$
$$\mathbb{C}^\times \xrightarrow{\sim} Z_{e,\mathrm{SL}_n} \subset \mathrm{SL}_n,\ t \mapsto \mathrm{diag}(t^{-1},t^{-1},\dots,t^{-1},t^{n-1}).$$

We obtain two actions of $\mathbb{C}^\times$ on $\mathcal{B}_e$. The first action sends a flag $\mathcal{F}_{k,a}$ to the flag $\mathcal{F}_{k,ta}$ and the second action sends $\mathcal{F}_{k,a}$ to the flag $\mathcal{F}_{k,t^n a}$.

Fix $1 \leqslant k \leqslant n-1$ and consider the following action of $\mathbb{C}^\times$ on $\mathbb{P}^1$.

$$t \cdot [x:y] = [t^n x : y]. \tag{33}$$

The following lemma is straightforward (compare with [35, Section 5.4], [15, Section 3.6]).

Lemma 6.2. *Let $\mathbb{C}^\times$ act on $\mathbb{P}^1$ via (33). For every collection of integers i, j, satisfying $i - j = nm$ for some $m \in \mathbb{Z}$, there exists the unique $\mathbb{C}^\times$-equivariant line bundle on $\mathbb{P}^1$ such that $t \in \mathbb{C}^\times$ acts via t^j at the fiber over $[1:0]$ and acts via t^i at the fiber over $[0:1]$. Every $\mathbb{C}^\times$-equivariant line bundle on $\mathbb{P}^1$ can be obtained in this way. We denote the line bundle above by $\mathcal{O}^{j,i}$.*

Remark 6.3. The Euler characteristic of $\mathcal{O}^{j,i}$ is equal to $m+1$, so $\mathcal{O}^{j,i}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(m)$ as a line bundle.

Recall again that we have the action of $\mathbb{C}^\times \simeq Z_{e,\mathrm{SL}_n}$ on $\mathcal{B}_e$, which acts on every Π_k via (33). We identify $K^{Z_{e,\mathrm{SL}_n}}(\mathrm{pt}) = \mathbb{Z}[\xi^{\pm 1}]$, $K^{Z_{e,\mathrm{PGL}_n}} = \mathbb{Z}[\xi^{\pm n}]$. For $k \in \{1,\dots,n-1\}$ and $a, b \in \mathbb{Z}$ such that $a - b \in n\mathbb{Z}$ we denote by $O_k^{b,a}$ the line bundle on Π_k whose fiber over $p_{k-1,k}$ is ξ^b and the fiber over $p_{k,k+1}$ is ξ^a.

Lemma 6.4. *For every $a, b \in \mathbb{Z}$ such that $a - b \in n\mathbb{Z}$ we have*

$$O_k^{b,a} + O_k^{a,b} = (\xi^a + \xi^b) O_k^{0,0}.$$

Proof. Use the Euler sequence for $\Pi_k \simeq \mathbb{P}^1$ (see (30)). □

For $k = 1, \dots, n-1$ we set

$$\mathbf{O}_k := [O_k^{0,-n}].$$

Remark 6.5. *Note that the line bundle $O_k^{0,-n}$ has degree $m = -1$, i.e. is isomorphic to $\mathcal{O}_{\Pi_k}(-1)$.*

Let $\mathbb{C}_{p_{0,1}}$ be the skyscraper sheaf of the point $p_{0,1} \in \mathcal{B}_e$.

Lemma 6.6. *We have*

$$[\mathcal{O}_{\mathcal{B}_e}] = [\mathbb{C}_{p_{01}}] + \sum_{k=1}^{n-1} \xi^n \mathbf{O}_k.$$

Proof. Recall that $O_k^{0,-n}$ is a line bundle whose fiber over $p_{k-1,k}$ is 1 and the fiber over $p_{k,k+1}$ is ξ^{-n}. Clearly we have an exact sequence

$$0 \to \xi^n O_{n-1}^{0,-n} \to \mathcal{O}_{\mathcal{B}_e} \to \mathcal{O}_{\cup_{k=1}^{n-2}\Pi_k} \to 0.$$

The claim follows by induction. □

We set

$$\mathbf{O}_0 := -[\mathcal{O}_{\mathcal{B}_e}] = -[\mathbb{C}_{p_{01}}] - \sum_{k=1}^{n-1} \xi^n \mathbf{O}_k.$$

We finally extend $\mathbf{O}_k$ to every $k \in \mathbb{Z}$ in such a way that $\mathbf{O}_k = \xi^n \mathbf{O}_{k+n}$ for every $k \in \mathbb{Z}$. The set $\{\mathbf{O}_k \mid k \in \mathbb{Z}\}$ forms a basis of the $\mathbb{Z}$-module $K^{Z_{e,\mathrm{PGL}_n}}(\mathcal{B}_e)$.

6.4. Modules $\mathfrak{h}_{\infty,\mathbb{Z}}$, $\widehat{\mathfrak{h}}_{\infty,\mathbb{Z}}$ over $S_n \ltimes \mathbb{Z}^n$, $\widehat{W}$

6.4.1. Module $\mathfrak{h}_{\infty,\mathbb{Z}}$ Using the identification of $\mathbb{Z}^n \rtimes S_n$ with the permutations $\mathbb{Z} \to \mathbb{Z}$, we obtain the action of $\mathbb{Z}^n \rtimes S_n \curvearrowright \mathbb{Z}^{\oplus\mathbb{Z}}$ that sends ϵ_i to $\epsilon_{\sigma(i)}$. Consider now the action of $\mathbb{Z}[\xi^{\pm n}]$ on $\mathbb{Z}^{\oplus\mathbb{Z}}$ given by $\xi^n \cdot \epsilon_i = \epsilon_{i-n}$.

Remark 6.7. Note that $\mathbb{Z}^n \rtimes S_n$-action on $\mathbb{Z}^{\oplus\mathbb{Z}}$ commutes with the $\mathbb{Z}[\xi^{\pm n}]$-action.

Let $\mathfrak{h}_{\infty,\mathbb{Z}} \subset \mathbb{Z}^{\oplus\mathbb{Z}}$ be the submodule, consisting of elements $(a_i)_{i\in\mathbb{Z}}$ such that $\sum_{i\in\mathbb{Z}} a_i = 0$. We obtain the action $\widehat{W} \curvearrowright \mathfrak{h}_{\infty,\mathbb{Z}}$. Module $\mathfrak{h}_{\infty,\mathbb{Z}}$ has a $\mathbb{Z}$-basis $\{\alpha_i^\vee,\ i \in \mathbb{Z}\}$, where $\alpha_i^\vee = \epsilon_i - \epsilon_{i+1}$.

Remark 6.8. Recall that $\mathfrak{h}_{\infty,\mathbb{Z}}$ is a $\widehat{W}$-module over $\mathbb{Z}[\xi^{\pm n}]$. We can consider the quotient $\mathfrak{h}_{\infty,\mathbb{Z}}/(\xi^n - 1)\mathfrak{h}_{\infty,\mathbb{Z}}$. It is easy to see that $\widehat{W}$-module $\mathfrak{h}_{\infty,\mathbb{Z}}/(\xi^n-1)\mathfrak{h}_{\infty,\mathbb{Z}}$ is isomorphic to $\mathfrak{h}_{\mathbb{Z}} \oplus \mathbb{Z}K$ via $[\alpha_i^\vee] \mapsto \alpha_i^\vee$, $i = 0, 1, \dots, n-1$.

Remark 6.9. Let us define the symmetric bilinear form $(\ ,\ )$ on $\mathfrak{h}_{\infty,\mathbb{Z}}$. Consider the symmetric bilinear form $(\ ,\ )$ on $\mathbb{Z}^{\oplus\mathbb{Z}}$ given by $(\epsilon_i, \epsilon_j) = \delta_{[i],[j]}$. We denote by $(\ ,\ )$ its restriction to $\mathfrak{h}_{\infty,\mathbb{Z}}$. It is clear that $(\ ,\ )$ is $\mathbb{Z}^n \rtimes S_n$-invariant. It is also clear that

$$(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 2 & \text{if } [i] = [j], \\ -1 & \text{if } [i], [j] \in \mathbb{Z}/n\mathbb{Z} \text{ are adjacent}, \\ 0 & \text{otherwise.} \end{cases}$$

6.4.2. Module $\widehat{\mathfrak{h}}_{\infty,\mathbb{Z}}$ Recall the $\widehat{W}$-module $\mathfrak{h}_{\infty,\mathbb{Z}}$. Set

$$\widehat{\mathfrak{h}}_{\infty,\mathbb{Z}} := \mathfrak{h}_{\infty,\mathbb{Z}} \oplus \mathbb{Z}d$$

and define the $\widehat{W}$-module structure on it by $(i = 1, \dots, n-1)$

$$t_{\alpha_i^\vee}(d) = d + \epsilon_i - \epsilon_{i+1-n},\ w(d) = d,\ w \in S_n.$$

Clearly, we have an exact sequence of $\widehat{W}$-modules

$$0 \to \mathfrak{h}_{\infty,\mathbb{Z}} \to \widehat{\mathfrak{h}}_{\infty,\mathbb{Z}} \to \mathbb{Z}_{\text{triv}} \to 0.$$

Remark 6.10. Recall the bilinear form $(\ ,\ )$ on $\mathfrak{h}_{\infty,\mathbb{Z}}$. It can be extended to the bilinear from on $\widehat{\mathfrak{h}}_{\infty,\mathbb{Z}}$ by $(d, \alpha_i) = 0$ for $i \notin n\mathbb{Z}$ and $(d, \alpha_i) = 1$ for $i \in n\mathbb{Z}$, $(d, d) = 0$. It is easy to see that this form is $\widehat{W}$-invariant.

We can extend the action of $\widehat{W}$ on $\widehat{\mathfrak{h}}_{\infty,\mathbb{Z}}$ to the action of $\mathbb{Z}^n \rtimes S_n$ on $\mathbb{Z}^{\oplus\mathbb{Z}} \oplus \mathbb{Z}d$ via

$$t_{\epsilon_i}(d) = d + \epsilon_i.$$

For $\gamma \in \mathbb{Z}^n$ let us describe explicitly the element $t_\gamma(d)$. For $k \in \mathbb{Z}$ we set

$$\overline{\Lambda}_k^\infty := \sum_{l \leqslant k} \epsilon_l^* \in \mathfrak{h}_\infty^*.$$

Clearly

$$\langle \overline{\Lambda}_i^\infty, \alpha_k^\vee \rangle = \delta_{i,k}.$$

For $a \in \mathbb{Z}_{>0}$ we have

$$t_{a\epsilon_k}(d) = d + \epsilon_k + \dots + \epsilon_{k+(a-1)n},$$

so

$$\langle \overline{\Lambda}_i^{\infty}, t_{a\epsilon_k}(d)\rangle = |\mathbb{Z}_{\leqslant i} \cap [k, k+(a-1)n] \cap (k+n\mathbb{Z})|.$$

For $a<0$ we have

$$t_{a\epsilon_k}(d) = d - \epsilon_{k-n} - \cdots - \epsilon_{k+an},$$

so

$$\langle \overline{\Lambda}_i^{\infty}, t_{a\epsilon_k}(d)\rangle = -|\mathbb{Z}_{\leqslant i} \cap [k+an, k-n] \cap (k+n\mathbb{Z})|.$$

For $i \in \mathbb{Z}$, $k = 1, \dots, n$, and $a \in \mathbb{Z}$ we set

$$z_i(a\epsilon_k) := \begin{cases} |\mathbb{Z}_{\leqslant i} \cap [k, k+(a-1)n] \cap (k+n\mathbb{Z})| & \text{for } a \in \mathbb{Z}_{\geqslant 0}, \\ -|\mathbb{Z}_{\leqslant i} \cap [k+an, k-n] \cap (k+n\mathbb{Z})| & \text{for } a \in \mathbb{Z}_{\leqslant 0}. \end{cases} \tag{34}$$

We conclude that for $\gamma = a_1\epsilon_1 + a_2\epsilon_2 + \cdots + a_n\epsilon_n \in Q^{\vee}$ we have

$$t_{\gamma}(d) = d + \sum_{i\in\mathbb{Z}} \Big(\sum_{k=1}^{n} z_i(a_k\epsilon_k) \Big) \alpha_i^{\vee}.$$

6.5. Structure of modules $K^{Z_{e,\mathrm{SL}_n}}(\mathcal{B}_e)$, $K^{Z_{e,\mathrm{PGL}_n}}(\mathcal{B}_e)$

The main reference for this section is [35]. The goal of this section is to construct an isomorphism of $\widehat{W}$-modules $\mathfrak{h}_{\infty,\mathbb{Z}} \simeq K^{Z_{e,\mathrm{PGL}_n}}(\mathcal{B}_e)$. We describe the action of $\widehat{W}^{\mathrm{ext}}$ on $K^{Z_{e,\mathrm{SL}_n}}(\mathcal{B}_e)$ first.

The set $\{\mathbf{O}_k\}_{k=1,\dots,n-1} \cup \{[\mathbb{C}_{p_{0,1}}]\}$ forms a basis of the $\mathbb{Z}[\xi^{\pm 1}]$-module $K^{Z_{\mathrm{SL}_n}(e)}(\mathcal{B}_e)$. The following lemma holds by [35, Section 5.11].

Lemma 6.11. *We have ($k = 1, 2, \dots, n-1$)*

$$\begin{gathered} s_l(\mathbf{O}_{l-1}) = -\mathbf{O}_{l-1} - \mathbf{O}_l \text{ for } l = 2, \dots, n-1, \\ s_{l-1}(\mathbf{O}_l) = -\mathbf{O}_l - \mathbf{O}_{l-1} \text{ for } l = 2, \dots, n-1, \\ s_k(\mathbf{O}_k) = \mathbf{O}_k, \\ s_l(\mathbf{O}_k) = -\mathbf{O}_k \text{ for } l \neq k-1, k, k+1, \\ s_l([\mathbb{C}_{p_{0,1}}]) = -[\mathbb{C}_{p_{0,1}}] \text{ for } l = 2, \dots, n-1, \\ s_1([\mathbb{C}_{p_{0,1}}]) = -[\mathbb{C}_{p_{0,1}}] + (1-\xi^n)\mathbf{O}_1. \end{gathered}$$

Let $L_{k-1,k}$ be the line bundle on $\mathcal{B}_e$ whose fiber over $(\{0\} \subset D_1 \subset \cdots \subset D_{n-1} \subset V)$ is D_k/D_{k-1} (see [35, Section 5.1]). By definition, the action of $t_{-\epsilon_k}$ on $K^{Z_{\mathrm{SL}_n}(e)}(\mathcal{B}_e)$ is given by the tensor product with the line bundle $L_{k-1,k}$.

The fiber of $L_{k-1,k}$ over $p_{l-1,l}$ is equal to ξ^{-1} if $k \neq l$ and ξ^{n-1} if $k = l$. We conclude that

$$t_{-\epsilon_k}([O_l^{b,a}]) = [O_l^{b-1,a-1}] = \xi^{-1}[O_l^{b,a}] \text{ if } l \neq k-1, k, \tag{35}$$

$$t_{-\epsilon_k}([O_{k-1}^{b,a}]) = [O_{k-1}^{b-1,a+n-1}], \tag{36}$$

$$t_{-\epsilon_k}([O_k^{b,a}]) = [O_k^{b+n-1,a-1}]. \tag{37}$$

Lemma 6.12. *We have*

$$[\mathbb{C}_{p_{k-1,k}}] = [O_k^{0,0}] - [O_k^{n,0}],\ [\mathbb{C}_{p_{k,k+1}}] = [O_k^{0,0}] - [O_k^{0,-n}],$$
$$[\mathbb{C}_{p_{k-1,k}}] = [\mathbb{C}_{p_{k,k+1}}] + (1-\xi^n)\mathbf{O}_k,$$
$$[\mathbb{C}_{p_{01}}] = [\mathbb{C}_{p_{k-1,k}}] + \sum_{l=1}^{k-1}(1-\xi^n)\mathbf{O}_l.$$

Proof. All of these equalities directly follow from the standard exact sequences on $\mathbb{P}^1$. □

Corollary 6.13. *We have*

$$[O_k^{0,0}] = [\mathbb{C}_{p_{01}}] + \mathbf{O}_k + \sum_{l=1}^{k}(\xi^n - 1)\mathbf{O}_l = -\sum_{l=0}^{k-1}\mathbf{O}_l - \xi^n \sum_{l=k+1}^{n-1}\mathbf{O}_l.$$

Proof. By Lemma 6.12

$$\begin{aligned}[O_k^{0,0}] &= [\mathbb{C}_{p_{k,k+1}}] + [O_k^{0,-n}] = [\mathbb{C}_{p_{k,k+1}}] + \mathbf{O}_k \\ &= [\mathbb{C}_{p_{01}}] + \mathbf{O}_k + \sum_{l=1}^{k}(\xi^n - 1)\mathbf{O}_l = -\sum_{l=0}^{k-1}\mathbf{O}_l - \xi^n \sum_{l=k+1}^{n-1}\mathbf{O}_l.\end{aligned}$$

□

Lemma 6.14. *We have*

$$t_{-\epsilon_k}([\mathbb{C}_{p_{0,1}}]) = \xi^{-1}[\mathbb{C}_{p_{0,1}}],\ k = 2, \dots, n-1,$$
$$t_{-\epsilon_1}([\mathbb{C}_{p_{0,1}}]) = \xi^{n-1}[\mathbb{C}_{p_{0,1}}],$$
$$t_{-\epsilon_k}(\mathbf{O}_l) = \xi^{-1}\mathbf{O}_l \text{ if } l \neq k-1, k,$$
$$t_{-\epsilon_k}(\mathbf{O}_{k-1}) = \xi^{-1}([\mathbb{C}_{p_{k-1,k}}] + \mathbf{O}_{k-1}) = -\xi^{-1}\Big(\sum_{l=0}^{k-2}\mathbf{O}_l + \xi^n \sum_{l=k}^{n-1}\mathbf{O}_l\Big),$$
$$t_{-\epsilon_k}(\mathbf{O}_k) = \xi^{-1}(-[\mathbb{C}_{p_{k-1,k}}] + \mathbf{O}_k) = \xi^{-1}\Big(\sum_{l=0}^{k}\mathbf{O}_l + \xi^n \sum_{l=k}^{n-1}\mathbf{O}_l\Big).$$

Proof. The first three equalities follow from the fact that the fiber of $L_{k-1,k}$ over $p_{l-1,l}$ is ξ^{-1} if $k \neq l$ and ξ^{n-1} if $k = l$.

Since $[O_{k-1}^{-1,-1}] = \xi^{-1}[O_{k-1}^{0,0}]$ and using Corollary 6.13, we conclude that

$$[O_{k-1}^{-1,-1}] = \xi^{-1}[O_{k-1}^{0,0}] = -\xi^{-1}\sum_{l=0}^{k-2}\mathbf{O}_l - \xi^{n-1}\sum_{l=k}^{n-1}\mathbf{O}_l.$$

We see that

$$t_{-\epsilon_k}([O_{k-1}^{0,-n}]) = [O_{k-1}^{-1,-1}] = -\xi^{-1}\Big(\sum_{l=0}^{k-2}\mathbf{O}_l + \xi^n\sum_{l=k}^{n-1}\mathbf{O}_l\Big).$$

By Lemma 6.4 $[O_k^{0,-n}] + [O_k^{-n,0}] = (1+\xi^{-n})[O_k^{0,0}]$, so

$$[O_k^{0,-n}] = -[O_k^{-n,0}] + (1+\xi^{-n})[O_k^{0,0}].$$

Using (37) we see that

$$\begin{aligned} t_{-\epsilon_k}([O_k^{0,-n}]) &= t_{-\epsilon_k}(-[O_k^{-n,0}] + (1+\xi^{-n})[O_k^{0,0}]) \\ &= \xi^{-1}(-[O_k^{0,0}] + (1+\xi^{-n})[O_k^{n,0}]) \\ &= \xi^{-1}(\mathbf{O}_k + \xi^n\mathbf{O}_k - [O_k^{0,0}]) = \xi^{-1}\Big(\sum_{l=0}^{k}\mathbf{O}_l + \xi^n\sum_{l=k}^{n-1}\mathbf{O}_l\Big). \end{aligned}$$

□

Combining all the relations, we get the $\widehat{W}^{\mathrm{ext}}$-representation $K^{Z_{e,\mathrm{SL}_n}}(\mathcal{B}_e)$ with $\mathbb{Z}[\xi^{\pm 1}]$-basis $\mathbf{O}_1, \dots, \mathbf{O}_{n-1}, [\mathbb{C}_{p_{0,1}}]$ and the following action of $\widehat{W}^{\mathrm{ext}}$ (here $k \in \{1, \dots, n-1\}$):

$$\begin{gathered} s_k(\mathbf{O}_{k-1}) = -\mathbf{O}_{k-1} - \mathbf{O}_k, \\ s_{k-1}\mathbf{O}_k = -\mathbf{O}_k - \mathbf{O}_{k-1}, \\ s_k(\mathbf{O}_k) = \mathbf{O}_k, \\ s_l(\mathbf{O}_k) = -\mathbf{O}_k \text{ for } l \neq k-1, k, k+1, \\ s_k([\mathbb{C}_{p_{0,1}}]) = -[\mathbb{C}_{p_{0,1}}],\ k = 2, \dots, n-1, \\ s_1([\mathbb{C}_{p_{0,1}}]) = -[\mathbb{C}_{p_{0,1}}] + (1-\xi^n)\mathbf{O}_1, \\ t_{-\epsilon_k}([\mathbb{C}_{p_{0,1}}]) = \xi^{-1}[\mathbb{C}_{p_{0,1}}],\ k = 2, \dots, n-1, \\ t_{-\epsilon_1}([\mathbb{C}_{p_{0,1}}]) = \xi^{n-1}[\mathbb{C}_{p_{0,1}}], \\ t_{-\epsilon_k}(\mathbf{O}_l) = \xi^{-1}\mathbf{O}_l \text{ if } l \neq k-1, k, \end{gathered}$$

$$t_{-\epsilon_k}(\mathbf{O}_{k-1}) = -\xi^{-1}\Big(\sum_{l=0}^{k-2}\mathbf{O}_l + \xi^n\sum_{l=k}^{n-1}\mathbf{O}_l\Big) = -\xi^{-1}\sum_{l=k-n}^{k-2}\mathbf{O}_l,$$
$$t_{-\epsilon_k}(\mathbf{O}_k) = \xi^{-1}\Big(\sum_{l=0}^{k}\mathbf{O}_l + \xi^n\sum_{l=k}^{n-1}\mathbf{O}_l\Big) = \xi^{-1}\sum_{l=k-n}^{k}\mathbf{O}_l.$$

Proposition 6.15. *There is an isomorphism of $\widehat{W}$-modules over $\mathbb{Z}[\xi^{\pm n}]$:*

$$\mathfrak{h}_{\infty,\mathbb{Z}}\otimes\mathbb{Z}_{\mathrm{sign}} \simeq K^{Z_{e,\mathrm{PGL}_n}}(\mathcal{B}_e),$$

given by

$$\alpha_k^\vee\otimes 1\mapsto -\mathbf{O}_k,\ k\in\mathbb{Z},\ [\mathbb{C}_{p_{0,1}}]\mapsto \alpha_{1-n}^\vee+\cdots+\alpha_{-1}^\vee+\alpha_0^\vee=\epsilon_{1-n}-\epsilon_1.$$

Proof. Directly follows from the formulas for the action of $\widehat{W}^{\mathrm{ext}}$ on $K^{Z_{e,\mathrm{SL}_n}}(\mathcal{B}_e)$ above together with the fact that $t_{\alpha_k^\vee}=t_{\epsilon_k}\circ t_{-\epsilon_{k+1}}$. □

6.6. Structure of $K^{\mathbf{PGL_n}}(\widetilde{U})$

Recall that by Theorem 4.3 we have an exact sequence of $\widehat{W}$-modules

$$0\to K^{Z_{e,\mathrm{PGL}_n}}(\mathcal{B}_e)\to K^{\mathrm{PGL}_n}(\widetilde{U})\to\mathbb{Z}_{\mathrm{sign}}\to 0.$$

We have already described $\widehat{W}$-module $K^{Z_{e,\mathrm{PGL}_n}}(\mathcal{B}_e)$ explicitly. So in order to describe the action of $\widehat{W}$ on $K^{\mathrm{PGL}_n}(\widetilde{U})$ we just need to compute the action of $\widehat{W}$ on $[\mathcal{O}_{\widetilde{U}}]$. Since $K^{\mathrm{PGL}_n}(\widetilde{U})$ is the quotient of $K^{\mathrm{PGL}_n}(\widetilde{\mathcal{N}})$, we have $w([\mathcal{O}_{\widetilde{U}}])=\varepsilon(w)[\mathcal{O}_{\widetilde{U}}]$ for $w\in S_n$. It remains to determine the action of $t_{\alpha_k^\vee}$, $k=1,\dots,n-1$, on $[\mathcal{O}_{\widetilde{U}}]$.

Lemma 6.16. *We have*

$$t_{\alpha_k^\vee}([\mathcal{O}_{\widetilde{U}}]) = [\mathcal{O}_{\widetilde{U}}] + \sum_{l=k+1-n}^{k-1}\mathbf{O}_l.$$

Proof. Indeed, using Lemma 4.1 and Corollary 6.13 we obtain

$$\begin{aligned} t_{\alpha_k^\vee}([\mathcal{O}_{\widetilde{U}}]) &= [\mathcal{O}_{\widetilde{U}}(\alpha_k^\vee)] = [\mathcal{O}_{\widetilde{U}}] - [\iota_* O_k^{0,0}] \\ &= [\mathcal{O}_{\widetilde{U}}] - [\mathbb{C}_{p_{01}}] - \mathbf{O}_k + \sum_{l=1}^{k}(1-\xi^n)\mathbf{O}_l = [\mathcal{O}_{\widetilde{U}}] + \sum_{l=k+1-n}^{k-1}\mathbf{O}_l. \end{aligned}$$

□

Proposition 6.17. *We have an isomorphism of $\widehat{W}$-modules $K^{\mathrm{PGL}_n}(\widetilde{U}) \simeq \widehat{\mathfrak{h}}_{\infty,\mathbb{Z}} \otimes \mathbb{Z}_{\mathrm{sign}}$. This isomorphism is given by*

$$\bar{C}_0 \mapsto d \otimes 1,\ \bar{C}_{\nu_i} \mapsto -\alpha_i^\vee,\ i \in \mathbb{Z}.$$

Proof. Follows from Proposition 6.15, Lemma 6.16 and Proposition 4.9. □

Remark 6.18. One can avoid the use of Proposition 4.9 in the proof of Proposition 6.17 and instead use results of [35] on the canonical basis in $K^{G^\vee}(\mathcal{B}_e)$ together with [8, Theorem 5.3.5], see also [15].

6.7. Computation of $\mathbf{m}_{w_i}^{w_\gamma}$ and the proof of Theorem 2.16

Recall now that

$$t_\gamma \cdot 1 = T_\gamma = \sum_\nu \mathbf{m}_{w_\nu}^{w_\gamma} C_\nu.$$

Taking the image of this equality in $K^{\mathrm{PGL}_n}(\widetilde{U}) \simeq \widehat{\mathfrak{h}}_{\infty,\mathbb{Z}} \otimes \mathbb{Z}_{\mathrm{sign}}$ and using Proposition 6.17, we see that

$$t_\gamma(d) = d - \sum_{i\in\mathbb{Z}} \mathbf{m}_{w_i}^{w_\gamma} \alpha_i^\vee.$$

So we can compute $\mathbf{m}_{w_i}^{w_\gamma}$. Indeed, this is just the coefficient in front of $-\alpha_i^\vee$ in

$$t_\gamma(d) - d = \sum_{i\in\mathbb{Z}} \Big(\sum_{k=1}^n z_i(\langle \epsilon_k, \gamma\rangle \epsilon_k) \Big) \alpha_i^\vee.$$

We conclude that

$$\mathbf{m}_{w_i}^{w_\gamma} = -\sum_{k=1}^n z_i(\langle \epsilon_k, \gamma\rangle \epsilon_k). \tag{38}$$

We are now ready to prove Theorem 2.16.

Proof of Theorem 2.16. Let i, λ, Λ be as in Theorem 2.9. Combining (24) and (38), we conclude that

$$\begin{aligned} \widehat{R}\,\mathrm{ch}\, L(\Lambda) &= \sum_{\gamma\in Q^\vee} \sum_{u\in W} \varepsilon(uw_i) \mathbf{m}_{w_i}^{w_{-\gamma}} e^{ut_\gamma(\lambda+\widehat{\rho})} \\ &= -\sum_{\gamma\in Q^\vee} \sum_{u\in W} \varepsilon(uw_i) \Big(\sum_{k=1}^n z_i(-\langle \epsilon_k, \gamma\rangle \epsilon_k) \Big) e^{ut_\gamma w_i(\Lambda+\widehat{\rho})} \end{aligned}$$

that is precisely the statement of Theorem 2.16. □

7. Possible generalizations

7.1. Non-simply laced case

Recall that in this paper we restrict ourselves to the simply laced case. One can consider arbitrary simple Lie algebra $\mathfrak{g}$. Using an approach similar to the one in this paper it should be possible to obtain explicit formulas for characters of certain $\widehat{\mathfrak{g}}$-modules $L(\Lambda)$ ("corresponding" to the subregular cell in $\widehat{W}$). We plan to return to this in the future.[1] The relevant question here is the explicit description of the $\widehat{W}$-module $K^{Z_e}(\mathcal{B}_e)$ (e is a subregular nilpotent of $\mathfrak{g}^\vee$). Let us describe the conjectural answer.

Remark 7.1. Note that the $\widehat{W}$-module $K(\mathcal{B}_e)$ is described in [35, Section 6].

Consider the affine Lie algebra $\widehat{\mathfrak{g}}$ and the corresponding (affine) Weyl group $\widehat{W}$ (see Section 2.1.4 or [21], [32, Section 1.6]). Our goal is to describe the $\widehat{W}$-module structure on $K^{Z_e}(\mathcal{B}_e)$. The Dynkin diagram of $(\widehat{\mathfrak{g}})^\vee$ can be obtained from a simply laced affine Dynkin diagram by folding (see for example [36, Section 14.1.5]). We denote the simply laced affine Lie algebra above by $\mathfrak{k}$ and denote by $W(\mathfrak{k})$ its Weyl group.

Remark 7.2. Note that $(\widehat{\mathfrak{g}})^\vee$ is a *twisted* affine Lie algebra (see Example 7.4).

Assume for simplicity that $\mathfrak{g}$ is B_n or F_4. By [29, Corollary 3.3] there is an embedding $\widehat{W} \subset W(\mathfrak{k})$ that sends a simple reflection of $\widehat{W}$ to the product of simple reflections over the corresponding orbit of folding. Let $\widetilde{\mathfrak{t}}_{\mathbb{Z}} \subset \mathfrak{k}$ be the (integral form of the) "reflection" representation of $W(\mathfrak{k})$ and $\widehat{\mathfrak{t}}_{\mathbb{Z}} = \widetilde{\mathfrak{t}} \oplus \mathbb{Z}d$ be the "Cartan" representation. Using the embedding $\widehat{W} \subset W(\mathfrak{k})$ we obtain the action of $\widehat{W}$ on $\widetilde{\mathfrak{t}}_{\mathbb{Z}}$, $\widehat{\mathfrak{t}}_{\mathbb{Z}}$. The following conjecture will be proven (and generalized to other types) in [28].

Conjecture 7.3. *Assume that $\mathfrak{g}$ is B_n ($n \geqslant 3$), F_4. We have isomorphisms of $\widehat{W}$-modules*

$$K^{Z_e}(\mathcal{B}_e) \simeq \widetilde{\mathfrak{t}}_{\mathbb{Z}} \otimes \mathbb{Z}_{\mathrm{sign}},\ K^{G^\vee}(\widetilde{U}) \simeq \widehat{\mathfrak{t}}_{\mathbb{Z}} \otimes \mathbb{Z}_{\mathrm{sign}}.$$

Example 7.4. Assume for example that $\mathfrak{g} = B_n$. Then $\widehat{\mathfrak{g}} = \widetilde{B}_n$, hence, $(\widehat{\mathfrak{g}})^\vee = A^{(2)}_{2n+1}$. We conclude that $\mathfrak{k} = \widetilde{D}_{2n}$. For $\mathfrak{g} = F_4$ we have $(\widehat{\mathfrak{g}})^\vee = E_6^{(2)}$ so $\mathfrak{k} = \widetilde{E}_7$.

Remark 7.5. For $\mathfrak{g} = B_n, F_4$ it is easy to see that the rank of $K^{Z_e}(\mathcal{B}_e)$ (over $\mathbb{Z}$) is equal to the rank of $\widetilde{\mathfrak{t}}_{\mathbb{Z}}$. Indeed, recall that by [37, Section 6.2] variety $\mathcal{B}_e$ can be identified with the fiber of the Springer resolution over a subregular nilpotent element of the unfolding of $\mathfrak{g}$. It follows that $K(\mathcal{B}_e)$ has a basis,

[1] This will be done in the joint paper [28] of the third author and Kenta Suzuki.

consisting of $[\mathcal{O}_{\Pi_i}(-1)]$, $[\mathcal{O}_{\mathcal{B}_e}]$, where i runs through the set of simple roots of the unfolding of $\mathfrak{g}$ (recall that to every *long* simple root α_i of $\mathfrak{g}$ corresponds the unique simple root of the unfolding of $\mathfrak{g}$ and to every *short* root α_i correspond two simple roots $\alpha_{i^\pm}$ of the unfolding of $\mathfrak{g}$). Recall also that $Z_e \simeq S_2$. Using the same argument as in [34, Section 1.25], one can show that the basis of $K^{Z_e}(\mathcal{B}_e)$ can be described as follows: for every long simple root α_i of $\mathfrak{g}$ we consider $[\mathcal{O}_{\Pi_i}(-1)^+]$, $[\mathcal{O}_{\Pi_i}(-1)^-]$, where $\pm$ correspond to two different Z_e-equivariant structures on $\mathcal{O}_{\Pi_i}(-1)$, we also have $[\mathcal{O}^\pm_{\mathcal{B}_e}]$ and finally for every short simple root α_i (of $\mathfrak{g}$) we have $[\mathcal{O}_{\Pi_{i^+}}(-1) \sqcup \mathcal{O}_{\Pi_{i^-}}(-1)]$ (this is the class supported on $\Pi_{i^+} \sqcup \Pi_{i^-}$). So we see that to every *long* root of $\widehat{\mathfrak{g}}$ (in particular, to the affine root) we associate two basis elements of $K^{Z_e}(\mathcal{B}_e)$ and every *short* root corresponds to the unique basis element of $K^{Z_e}(\mathcal{B}_e)$. It follows that the rank of $K^{Z_e}(\mathcal{B}_e)$ is equal to the number of vertices of the unfolding of the Dynkin diagram of $(\widehat{\mathfrak{g}})^\vee$ i.e. the number of simple roots of $\mathfrak{k}$ (the rank of $\widetilde{\mathfrak{t}}_{\mathbb{Z}}$).

7.2. Case of arbitrary nilpotent e

One can try to use a similar approach to the one in this paper to compute characters of more general $\widehat{\mathfrak{g}}$-modules $L(\Lambda)$ such that the level of Λ is greater than $-h^\vee$ and $\Lambda + \widehat{\rho}$ is integral quasi-dominant. Let $w \in \widehat{W}$ be the longest element such that $w(\lambda + \widehat{\rho})$ is dominant. Let $c \subset \widehat{W}$ be the two-sided cell that contains w. Let $e \in \mathfrak{g}^\vee$ be the corresponding nilpotent element (not necessarily subregular). For $e' \in \mathcal{N}$ we say that e' *is over* e if e is contained in the closure of the orbit $\mathbb{O}_{e'} = G^\vee \cdot e'$. Let $U \subset \mathcal{N}$ be the union of the $\mathbb{O}_{e'}$ such that e' is over e; this is an open subset of $\mathcal{N}$. Set $\widetilde{U} := \pi^{-1}(U)$.

It follows from the above that the character of $L(\Lambda)$ can be extracted from the $\widehat{W}$-module $K^{G^\vee}(\widetilde{U})$ and the canonical basis in it. Recall that $\widetilde{U}$ was constructed starting from a nilpotent element $e \in \mathfrak{g}^\vee$.

Recall that an element $e \in \mathcal{N}$ is called *distinguished* if it is not contained in a proper Levi subalgebra. Apparently the simplest case to consider is the case when e is *very distinguished* i.e. if every element $e' \in \mathcal{N}$ over e is distinguished. If this is the case then the module $K^{G^\vee}(\widetilde{U})$ is clearly finite dimensional. It follows that the function $\gamma \mapsto \mathbf{m}_{w_\nu}^{w_\gamma}$ is a *quasi-polynomial* in this case. Degrees and periods of these quasi-polynomials will be estimated in the Appendix in [28] (written by first and third authors joint with Kenta Suzuki), the main technical tool is the localization theorem in equivariant algebraic K-theory.

Example 7.6. A regular element is always very distinguished. A subregular element is very distinguished except in types A_n and B_n, when it is not distinguished. There are also other examples: one in F_4, two in E_7, three in E_8, etc., see e.g. [2].

Remark 7.7. Recall (see Proposition 5.1) that the cell c such that $\mathbb{O}_c$ is subregular consists of elements $w \neq 1$ with a unique minimal decomposition (see Proposition 5.1 below). A similar (but more complicated) description of the next case, which includes most very distinguished examples, should follow from [14].

Another interesting case to consider is the case of $\mathfrak{g} = \mathfrak{sl}_n$ and e being the two-block nilpotent, see [4] for the parametrization and description of the irreducible objects in the heart of the exotic t-structure in this case.

Appendix A. Basic facts about Kazhdan-Lusztig polynomials

A.1. Canonical and standard bases in Hecke algebra $\mathcal{H}_q(\widehat{W})$

The Hecke algebra $\mathcal{H}_q = \mathcal{H}_q(\widehat{W})$ over $\mathbb{Z}[q^{\pm 1}]$ is an $\mathbb{Z}[q^{\pm 1}]$-algebra with free $\mathbb{Z}[q^{\pm 1}]$-basis $\{H_w\}_{w\in\widehat{W}}$ whose multiplication is determined by the following:

$$H_w H_v = H_{wv} \text{ if } \ell(wv) = \ell(w) + \ell(v),$$
$$(H_s + 1)(H_s - q) = 0 \text{ for } s \in \{s_0, s_1, \dots, s_r\}.$$

We can define the Hecke algebra $\mathcal{H}_q(W)$ similarly.

We define an involutive ring endomorphism $\mathcal{H}_q(\widehat{W}) \ni h \mapsto \overline{h} \in \mathcal{H}_q(\widehat{W})$ by

$$\overline{\sum_{w\in\widehat{W}} a_w H_w} = \sum_{w\in\widehat{W}} \overline{a}_w H_{w^{-1}}^{-1},$$

where $\overline{q} = q^{-1}$. Let $\preccurlyeq$ be the Bruhat order on $\widehat{W}$.

Proposition A.1 ([24]). *For any $v \in \widehat{W}$ there exists a unique $C_v \in \mathcal{H}_q(\widehat{W})$, satisfying the following conditions:*

$$C_v = \sum_{w\preccurlyeq v} P_{w,v}(q) H_w$$

with $P_{v,v}(q) = 1$ and $P_{w,v}(q) \in \mathbb{Z}[q]$ of degree $\leqslant (\ell(v) - \ell(w) - 1)/2$ for $w \prec v$,

$$\overline{C}_v = q^{-\ell(v)} C_v.$$

The polynomials $P_{w,v}(q)$ are called Kazhdan-Lusztig polynomials. Let us now introduce *inverse* Kazhdan-Lusztig polynomials $\mathbf{m}_v^w(q)$ (see [25, Section 2] where they are denoted by $Q_{v,w}(q)$). These polynomials are determined

by:

$$H_w = \sum_{v \preccurlyeq w} \varepsilon(wv^{-1}) \mathbf{m}_v^w(q) C_v.$$

Remark A.2. Note that in [25] the polynomials $\mathbf{m}_v^w(q)$ are denoted by $Q_{v,w}(q)$. We change this notation since we have already reserved "Q" for the coroot lattice $Q^\vee$.

Lemma A.3. *For $v, w \in \widehat{W}$ we have $\mathbf{m}_v^w(q) = \mathbf{m}_{v^{-1}}^{w^{-1}}(q)$.*

Proof. Consider the anti-automorphism i of $\mathcal{H}_q(\widehat{W})$ given by $i(H_x) = H_{x^{-1}}$, $i(q) = q$. Map i commutes with the involution $\overline{\bullet}$. The claim follows. □

The following Theorem holds by [19, Section 0.3] together with Lemma A.3:

Theorem A.4. *For a dominant integral weight $\lambda \in \widehat{\mathfrak{h}}^*$ we have*

$$\operatorname{ch} L(v^{-1} \circ \lambda) = \sum_{w \in \widehat{W}} \varepsilon(wv^{-1}) \mathbf{m}_v^w(1) \operatorname{ch} M(w^{-1} \circ \lambda).$$

A.2. Canonical and standard bases in the anti-spherical module

Define the algebra homomorphism $\chi\colon \mathcal{H}_q(W) \to \mathbb{Z}[q]$ by $\chi(H_w) = \varepsilon(w)$. We define the induced module $\mathcal{M}$ (anti-spherical module over $\mathcal{H}_q(\widehat{W})$) by

$$\mathcal{M} := \mathcal{H}_q(\widehat{W}) \otimes_{\mathcal{H}_q(W)} \mathbb{Z}[q]$$

and define $\varphi\colon \mathcal{H}_q \twoheadrightarrow \mathcal{M}$ by $\varphi(h) = h \otimes 1$.

It is easily checked that $\mathcal{M} \ni m \mapsto \overline{m} \in \mathcal{M}$ is well defined by

$$\overline{\varphi(m)} = \varphi(\overline{m}).$$

For $\gamma \in Q^\vee$ set $H'_{w_\gamma} := \varphi(H_{w_\gamma})$. It is easily seen that $\mathcal{M}$ is a free $\mathbb{Z}[q^{\pm 1}]$-module with basis $\{H'_{w_\gamma}\}_{\gamma \in Q^\vee}$.

Proposition A.5 ([12])**.** *For any $\nu \in Q^\vee$ there exists a unique $C'_{w_\nu} \in \mathcal{M}$, satisfying the following conditions.*

$$C'_{w_\nu} = \sum_{\gamma \in Q^\vee,\, w_\gamma \preccurlyeq w_\nu} \tilde{P}_{w_\gamma, w_\nu}(q) H'_{w_\gamma}$$

with $\tilde{P}_{w_\nu, w_\nu}(q) = 1$ and $\tilde{P}_{w_\gamma, w_\nu}(q) \in \mathbb{Z}[q]$ of degree $\leqslant (\ell(w_\nu) - \ell(w_\gamma) - 1)/2$ for $w_\gamma \prec w_\nu$.

$$\overline{C'_{w_\nu}} = q^{-\ell(w_\nu)} C'_{w_\nu}.$$

It easily follows from Proposition A.5 and Theorem A.1 that

$$C'_{w_\nu} = \varphi(C_{w_\nu}) \text{ and } \varphi(C_w) = 0 \text{ if } w \neq w_\nu \text{ for any } \nu \in Q^\vee, \tag{39}$$

so

$$\mathcal{M} = \mathcal{H}_q / \langle C_w \mid w \notin \{w_\nu \mid \nu \in Q^\vee\} \rangle.$$

Polynomials $\tilde{P}_{w_\gamma, w_\nu}(q)$ are called parabolic Kazhdan-Lusztig polynomials. Let us now define the parabolic inverse Kazhdan-Lusztig polynomials $\tilde{\mathbf{m}}^{w_\gamma}_{w_\nu}(q)$. Following [19, Equation (2.40)], we define them by

$$H'_{w_\gamma} = \sum_{\nu \in Q^\vee,\, w_\nu \preccurlyeq w_\gamma} \varepsilon(w_\gamma w_\nu^{-1}) \tilde{\mathbf{m}}^{w_\gamma}_{w_\nu}(q) C'_{w_\nu}.$$

The following proposition holds by [38] (see also [19, Proposition 2.7]).

Proposition A.6. *For $\nu, \gamma \in Q^\vee$ we have $\tilde{\mathbf{m}}^{w_\gamma}_{w_\nu}(q) = \mathbf{m}^{w_\gamma}_{w_\nu}(q)$.*

Proof. Easily follows from the fact that $\varphi(H_{w_\gamma u}) = \varepsilon(u) H'_{w_\gamma}$, $u \in W$ together with (39). □

So we conclude that

$$H'_{w_\gamma} = \sum_{\nu \in Q^\vee} \varepsilon(w_\gamma w_\nu^{-1}) \mathbf{m}^{w_\gamma}_{w_\nu}(q) C'_{w_\nu}.$$

Let us now modify bases H'_{w_γ}, C'_{w_ν} as follows:

$$T_\gamma := \varepsilon(w_\gamma) H'_{w_\gamma}, \; C_\nu := \varepsilon(w_\nu) C'_{w_\nu}.$$

We have

$$T_\gamma = \sum_{\nu \in Q^\vee} \mathbf{m}^{w_\gamma}_{w_\nu}(q) C_\nu.$$

Remark A.7. The numbers $\tilde{\mathbf{m}}^{w_\gamma}_{w_\nu}(1) = \mathbf{m}^{w_\gamma}_{w_\nu}(1) = \mathbf{m}^{w_\gamma}_{w_\nu}$ are matrix coefficients of the transition matrix from classes of standard sheaves on the affine Grassmannian of G to classes of irreducible objects (IC sheaves) (see [19, Corollary 5.5] for details). Set $M = \mathcal{M}/(q-1) = \mathbb{Z}\widehat{W} \otimes_{\mathbb{Z}W} \mathbb{Z}_{\text{sign}}$. After the identification $K^{G^\vee}(\widetilde{\mathcal{N}}) \simeq M$ elements T_γ become $[\mathcal{O}_{\widetilde{\mathcal{N}}}(\gamma)]$ and C_γ are classes of irreducible objects in the heart of the "exotic" t-structure on $D^b(\mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}}))$ (see [3, 5, 7] for details).

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