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On the Drinfeld coproduct

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Al mio maestro Corrado De Concini, con stima, gratitudine e affetto

Abstract: This paper provides a construction of the Drinfeld coproduct Δ_v on an affine quantum Kac-Moody algebra or on a quantum affinization $\mathcal{U}$ through the exponentials of some locally nilpotent derivations, thus proving that this "coproduct" with values in a suitable completion of $\mathcal{U}\otimes\mathcal{U}$ is well defined.

For the affine quantum algebras, Δ_v is also obtained as "t-equivariant limit" of the Drinfeld-Jimbo coproduct Δ.

0. Introduction

In this paper $\mathcal{U}$ is either the quantum affinization of a generalized symmetrizable Cartan matrix $A=(a_{ij})_{i,j\in I}$ (see [22, 29]) or an affine quantum algebra (see [11, 21, 12]). The untwisted affine quantum algebras are the affinizations relative to finite Cartan matrices; on the other hand twisted affine quantum algebras are not affinizations, and affinizations relative to non-finite Cartan matrices are not affine quantum algebras.

The Drinfeld realization of the affine quantum algebras (see [10]) makes evident the "translation" automorphisms t_i ($i\in I$), through which the weight lattice $P\cong\mathbb{Z}^I$ acts on $\mathcal{U}$: indeed the set of the relations among the generators $X_{i,r}^{\pm}$ is invariant with respect to suitable "translations" $(i,r)\mapsto(i,r\pm\tilde{d}_i)$.

The definition of the quantum affinizations generalizes the passage from the finite quantum algebras to (the Drinfeld realization of) the untwisted affine quantum algebras, including the action of $\mathbb{Z}^I$.

The quantum algebras (and in particular the affine quantum algebras) are endowed with a coproduct Δ (the Drinfeld-Jimbo coproduct): it is defined in terms of the Drinfeld-Jimbo generators, and its expression in terms of the Drinfeld generators (the generators of the Drinfeld realization of the affine quantum algebras) is not trivial at all; in particular the expression of $\Delta(X_{i,r}^{\pm})$ is not obtained translating the indices in the formula for $\Delta(X_{i,0}^{\pm})$, which shows that Δ is not $\mathbb{Z}^I$-equivariant.

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The literature (see [8, 9, 19]) mentions an unpublished note where Drinfeld defined a $\mathbb{Z}^I$-equivariant "coproduct" Δ_v on the quantum affinizations and on the affine quantum algebras, giving its expression on the generators. The study of Δ_v and of its properties plays a non-trivial role in the representation theory of different environments, and in the study of the tensor properties of suitable categories of representations: not just for the affine quantum algebras, the quantum affinizations and the toroidal algebras arising from the generalized symmetrizable Cartan matrices (which are the proper object of this study), but also for other versions of the toroidal algebras, current algebras, vertex algebras, Yangians, quantum shuffles (see for example [13, 14, 17, 19, 22, 15]).

In this paper we prove that Δ_v is (defines) a well defined algebra homomorphism.

It is worth recalling that Δ_v is not properly a coproduct, because it takes values in $\mathcal{U}\otimes\mathcal{U}((v))$ rather than in $\mathcal{U}\otimes\mathcal{U}$ itself: in this paper a particular care is reserved to the definition of a smaller algebra $\mathcal{U}\hat{\otimes}\mathcal{U}\subseteq\mathcal{U}\otimes\mathcal{U}((v))$ consisting of the limits of some "convergent" sequences defined in $\mathcal{U}\otimes\mathcal{U}$, and of its subspaces and subalgebras where all the constructions that we use make sense.

In this setting of (convenient) completions of the tensor powers of $\mathcal{U}$, Δ_v satisfies some properties of the coproducts (coassociativity and existence of the counit).

As for the proof that Δ_v is a well defined algebra homomorphism, the difficulty arises from the Serre relations: that these relations are preserved by Δ_v has been proven in [9] for the simply laced case and in [13] (Section 4) and in [17] for the untwisted affine quantum algebras; but for the general (generalized) symmetrizable Cartan matrix, the expression of the Drinfeld coproduct applied to the Serre relations is very complicated, and hard to approach by direct computations.

In this work a different strategy is presented: the bracket by the Drinfeld generators is deformed so as to get a locally nilpotent derivation D on a suitable subalgebra $\mathcal{V}$ of $\mathcal{U}\hat{\otimes}\mathcal{U}$; the comparison of Δ_v with $\exp(D)$, which is an algebra automorphism of $\mathcal{V}$, provides a proof that the Drinfeld coproduct is well defined.

The construction of the locally nilpotent derivation D, which is at the base of our strategy for avoiding to check that Δ_v preserves the Serre relations, rely on a new structure of Q_0-graded vector space (not Q_0-graded algebra!) on $\mathcal{U}$ and on a projection π of $\mathcal{U}$ on its positive part: both this Q_0-grading and the projection π depend on the triangular decomposition of $\mathcal{U}$. So it is in the triangular decomposition (result that is highly non-trivial) that the difficulties are hidden, and it is thanks to this result (already proven in [19] and [5]) that we can now skip the obstacle.

This proof has been presented in two talks that I gave at the Departments of Mathematics of the Université Paris VII - Denis Diderot (May 30th, 2014) and of the Università degli Studi di Roma “Tor Vergata” (January 12th, 2018), but it was never written or published.

In the last two sections of the paper we restrict to the case when $\mathcal{U}$ is an affine quantum algebra. As recalled above, in this case $\mathcal{U}$ is a Hopf algebra, with coproduct $\Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$ (see [11, 21, 12]): we are interested in understanding whether Δ_v is somehow related to Δ.

Comparing the expressions of Δ and Δ_v on $\{X^{\pm}_{i,0} | i \in I\}$ (set of generators of the finite subalgebra of $\mathcal{U}$) one notices that $\Delta_v(X^{\pm}_{i,0})$ can be seen as $\Delta(X^{\pm}_{i,0})$ plus some “v-queues”, which in this case can be described, roughly speaking, as terms that vanish at $v = 0$: this suggests that there could be a connection between Δ and Δ_v; on the other hand, as already remarked, Δ_v is P-equivariant while Δ is not; moreover the construction of $\mathcal{U} \hat{\otimes} \mathcal{U}$ as “component-wise filtered” completion of $\mathcal{U} \otimes \mathcal{U}$ with respect to a suitable topology provides a notion of convergence.

Thanks to these three observations, the question about the connection between Δ and Δ_v can be formulated more precisely: can Δ_v be seen as a P-equivariant deformation (limit) of Δ?

To answer this question we recall the approach to the P-action from the point of view of the Drinfeld-Jimbo presentation of the affine quantum algebras. $\mathcal{U}$ is endowed with a braid group action (see [27, 26]), whose restriction to P provides a P-action (which, up to signs, is the “evident” P-action on the affinization, see [2, 1, 4, 5]). Δ is not equivariant with respect to the braid group action, but the conjugation of Δ by the braid group, though non-trivial, can be described through the R-matrix of $\mathcal{U}$ and is studied in details in [24, 25, 2, 6, 7]. This fact allows us to recognize that Δ and Δ_v applied to the Drinfeld generators (that are related among themselves through the P-action) differ by some terms that we can manage to control, finally ending up with a description of Δ_v as what we called “t-equivariant limit” of Δ.

Also this proof has been presented (but not written or published) in a talk that I gave at the Department of Mathematics of the Università degli Studi di Roma “Tor Vergata” (April 14th, 2023).

It is interesting that the connection between Δ, the braid group action and the R-matrix has already produced the description of a relation between Δ and Δ_v, in terms of conjugation by an invertible element related to the universal R-matrix (see [30]): we conclude the paper sketching some very fast observations about the comparison between the P-equivariant limit (our result) and the R-conjugation (see [30]), which would however require a deeper insight.

1. Preliminaries

In this section we recall the preliminary notions and fix the main notations used in the paper (see [3, 23]).

Notation 1.1. $A = (a_{ij})_{i,j\in I}$ denotes a generalized indecomposable symmetrizable Cartan matrix.

The set of indices I is $\{1,\dots,n\}$.

$D = diag(d_i|i \in I)$ is the diagonal matrix with relatively prime positive integral diagonal entries d_i such that DA is symmetric; $d = max\{d_i|i \in I\}$.

$\mathfrak{g} = \mathfrak{g}(A)$ is the Kac-Moody algebra associated to A.

A and $\mathfrak{g}$ are said to be finite if DA is positive definite.

A and $\mathfrak{g}$ are said to be affine if DA is positive semidefinite of rank $n-1$.

It is well known that for every affine Kac-Moody algebra $\mathfrak{g}$ there exist a finite Kac-Moody algebra $\mathfrak{g}_0$ and a finite order automorphism χ of $\mathfrak{g}_0$ such that $\mathfrak{g}$ is the universal central extension of $(\mathfrak{g}_0 \otimes \mathbb{C}[t^{\pm}])^{\chi}$ (χ acts on t by multiplication by a primitive $o(\chi)^{th}$ root of 1); since also $\mathfrak{g}_0^{\chi}$ is a finite Kac-Moody algebra, this description of the affine Kac-Moody algebras provides a map $A \mapsto A_f$ from the set of the affine Cartan matrices to the set of the finite Cartan matrices such that $\mathfrak{g}(A_f) = \mathfrak{g}_0^{\chi} \subseteq \mathfrak{g}(A)$.

The (finite) order of χ is denoted by k; k can be 1, 2, or 3.

The Cartan matrix A is said to be untwisted if $\chi = id$ (that is $k = 1$), twisted if $\chi \neq id$ (that is $k = 2$ or 3).

Let us recall the well known classification of the finite and affine Cartan matrices:

A is finite $\Leftrightarrow A = A_{n\geq 1},\ B_{n\geq 2},\ C_{n\geq 3}, D_{n\geq 4}, E_6,\ E_7,\ E_8,\ F_4,\ G_2$.

(C_2 and D_3 are also defined, and we have $C_2 = B_2$, $D_3 = A_3$).

A is affine $\Leftrightarrow A = X_{\tilde{n}}^{(k)}$ with $X_{\tilde{n}}$ finite ($\mathfrak{g}_0 = \mathfrak{g}(X_{\tilde{n}})$) and $k = o(\chi)$.

More precisely:

A is untwisted $\Leftrightarrow$ $A = X_n^{(1)}$ with X_n finite (thus in this case $\tilde{n} = n$, $\mathfrak{g}_0 = \mathfrak{g}_0^{\chi}$).

A is twisted $\Leftrightarrow A = A_{2n}^{(2)}\ (n \geq 1),\ A_{2n-1}^{(2)}\ (n \geq 3),\ D_{n+1}^{(2)}\ (n \geq 2), D_4^{(3)}, E_6^{(2)}$.

The map $A \mapsto A_f$ is the following:

$X_n^{(1)} \mapsto X_n$,

$$A_{2n}^{(2)} \mapsto \begin{cases} A_1 & \text{if } n = 1 \\ B_n & \text{if } n > 1, \end{cases}$$

$A_{2n-1}^{(2)} \mapsto C_n$, $D_{n+1}^{(2)} \mapsto B_n$, $D_4^{(3)} \mapsto G_2$, $E_6^{(2)} \mapsto F_4$.

Remark 1.2. Of course the correspondence $A \mapsto (A_f, k) = (A_f, 1)$ where A is untwisted affine classifies the untwisted affine Cartan matrices; but remark that the map $A \mapsto (A_f, k)$ where A is affine (not necessarily untwisted) does not classify the affine Cartan matrices, because $(A_{2n}^{(2)})_f = (D_{n+1}^{(2)})_f = B_n$ if $n \geq 2$.

Hence, in order to distinguish $A_{2n}^{(2)}$ from $D_{n+1}^{(2)}$ and obtain a classification, we shall replace the pairs (A_f, k) by the triples $(A_f, k, \tilde{d})$ where the parameter $\tilde{d}$ can assume two values if $(A_f, k) = (B_n, 2)$ and is uniquely determined by (A_f, k) in the other cases.

The direct dependence of the definition of the affine quantum algebra associated to A on $(A_f, k, \tilde{d})$ is described in Definition 2.2, and the parameter $\tilde{d}$ is chosen to this aim.

Remark 1.3. As we shall recall in Section 2 and in particular in Remark 2.13, for each generalized symmetrizable Cartan matrix we consider a (unique) quantum affinization, whose construction generalizes the passage from a finite quantum algebra to the Drinfeld realization of its untwisted affine quantum algebra.

In order to deal at the same time with the affine quantum algebras and the quantum affinizations, we are going to introduce, following Remarks 1.2 and 1.3, a set $\mathcal{D}$ classifying the algebras which are either affine quantum algebras or quantum affinizations.

Definition 1.4. We denote by $\mathcal{D}$ the set $\mathcal{D} = \{(A, k, \tilde{d})\}$ where:

- A is a generalized symmetrizable Cartan matrix
- $k = \begin{cases} 1 & \text{if } A \text{ is not finite} \\ 1 \text{ or } 2 & \text{if } A = A_1 \\ 1 \text{ or } d & \text{otherwise} \end{cases}$
- $\tilde{d} = \begin{cases} 1 & \text{if } A \text{ is not finite or } A = A_1 \\ \text{a positive divisor of } k & \text{if } A = B_n \\ k & \text{otherwise} \end{cases}$

Remark 1.5. 1. Let A, k be as in Definition 1.4. Remark that if $(A, k) \neq (B_n, 2)$ then there exists a unique $\tilde{d}$ such that $(A, k, \tilde{d}) \in \mathcal{D}$; if $(A, k) = (B_n, 2)$ then $\tilde{d} = 1$ or 2; if A is not finite then $k = \tilde{d} = 1$.

2. As discussed in Remark 1.2, $\{(A, k, \tilde{d}) \in \mathcal{D} | A \text{ is finite}\}$ is in 1-1 correspondence with $\{A | A \text{ is affine}\}$: more precisely the map $A \mapsto (A_f, k)$ (defined on the set of the affine Cartan matrices) can be refined to a $\mathcal{D}$-valued injection

$A \mapsto (A_f, k, \tilde{d})$ by setting $A_{2n}^{(2)} \mapsto (B_n, 2, 1)$, $D_{n+1}^{(2)} \mapsto (B_n, 2, 2)$; the image of this injection is $\{(A, k, \tilde{d}) \in \mathcal{D} | A \text{ is finite}\}$.

3. It follows from Remarks 1.2 and 1.3 and from points 1. and 2. that $\mathcal{D}$ is the classifying set that we need (see Definition 2.2).

4. If A is finite, consider the 1-1 correspondence of point 2.: then the condition $k = 1$ (hence $k = \tilde{d} = 1$) identifies the untwisted affine matrices; the condition $k \neq \tilde{d}$ (or equivalently $k = 2$, $\tilde{d} = 1$) identifies the affine matrices of type $A_{2n}^{(2)}$; the condition $k = \tilde{d} \neq 1$ identifies the twisted affine matrices of type different from $A_{2n}^{(2)}$.

5. The set $\{(A, k, \tilde{d}) \in \mathcal{D} | k = 1\}$ is in 1-1 correspondence with the quantum affinizations.

Notation 1.6. Let $(A, k, \tilde{d}) \in \mathcal{D}$, $D = diag(d_i | i \in I)$ and I as in Notation 1.1 and Definition 1.4. We denote by $\tilde{d}_i$ and $\tilde{d}_{ij}$ $(i, j \in I)$ the following positive integers:

$$\tilde{d}_i = \begin{cases} 1 & \text{if } \tilde{d} = 1 \\ d_i & \text{otherwise,} \end{cases} \qquad \tilde{d}_{ij} = max\{\tilde{d}_i, \tilde{d}_j\}$$

Before introducing the algebra $\mathcal{U}$ object of this study, let's quickly review and adapt the definition of the root lattice, the "finite" root lattice and the "weight" lattice.

Definition 1.7. The root lattice Q is the free abelian group on $\{\alpha_i | i \in I\} \cup \{\delta\}$, that is

$$Q = (\oplus_{i \in I} \mathbb{Z}\alpha_i) \oplus \mathbb{Z}\delta;$$

the α_i's are called simple roots, δ is called the "imaginary" root.

The "finite" root lattice Q_0 is the subgroup of Q generated by the simple roots, that is

$$Q_0 = \oplus_{i \in I} \mathbb{Z}\alpha_i.$$

The "weight" lattice P is the abelian group

$$P = \oplus_{i \in I} \mathbb{Z}\omega_i \subseteq Hom(Q_0, \mathbb{Z})$$

where the ω_i's are defined by $<\omega_i, \alpha_j> = \delta_{ij}\tilde{d}_i$ for all $i, j \in I$ and are called fundamental weights.

A weight $\omega = \sum_{i \in I} m_i \omega_i$ is said to be dominant if $m_i \geq 0$ for all $i \in I$; the set of the dominant weights is denoted by P_+.

P embeds into $Hom(Q, \mathbb{Z})$ by $<\omega, \delta> = 0$ $\forall \omega \in P$ and acts on Q by $\omega(\beta) = \beta - <\omega, \beta>\delta$ $\forall \omega \in P$, $\beta \in Q$.

2. The algebra $\mathcal{U}$ and its main structures

We shall attach a $\mathbb{C}(q)$-associative algebra $\mathcal{U} = \mathcal{U}(Y)$ to each $Y \in \mathcal{D}$ (see [10, 2, 22, 29, 19, 4]); to this aim we fix the following (more or less usual) notation.

Notation 2.1. For all $i \in I$ denote by q_i the element $q_i = q^{d_i} \in \mathbb{Z}[q] \subseteq \mathbb{C}(q)$.

Definition 2.2. Let $Y = (A, k, \tilde{d}) \in \mathcal{D}$; $\mathcal{U} = \mathcal{U}(Y)$ is the $\mathbb{C}(q)$-algebra defined by generators and relations in the following way:
generators:

$$C^{\pm 1},\ k_i^{\pm 1}\ (i \in I),\quad H_{i,r}\ ((i,r) \in I \times (\mathbb{Z} \setminus \{0\})),\quad X_{i,r}^{\pm}\ ((i,r) \in I \times \mathbb{Z});$$

relations:

$$(C) \qquad CC^{-1} = 1 \text{ and } C \text{ is central}$$

$$(K) \qquad k_i k_i^{-1} = 1 = k_i^{-1} k_i,\ \ k_i k_j = k_j k_i$$

$$(X) \qquad X_{i,r}^{\pm} = 0 \ \text{ if } \tilde{d}_i \not| r$$

$$(KX) \qquad k_i X_{j,r}^{\pm} = q_i^{\pm a_{ij}} X_{j,r}^{\pm} k_i$$

$$(HX) \qquad [H_{i,r}, X_{j,s}^{\pm}] = \pm b_{ijr} C^{\frac{r \mp |r|}{2}} X_{j,r+s}^{\pm}$$

$$(X\pm) \qquad [X_{i,r}^{+}, X_{j,s}^{-}] = \begin{cases} \delta_{ij} \frac{k_i C^{-s} \tilde{H}_{i,r+s}^{+} - k_i^{-1} C^{-r} \tilde{H}_{i,r+s}^{-}}{q_i - q_i^{-1}} & \text{if } \tilde{d}_j | s \\ 0 & \text{otherwise} \end{cases}$$

(XX)

$$\begin{cases} \sum_{\sigma \in \mathcal{S}_2} \sigma.\left([X_{i,r_1 \pm 2}^{\pm}, X_{i,r_2}^{\pm}]_{q^2} - q^4 [X_{i,r_1 \pm 1}^{\pm}, X_{i,r_2 \pm 1}^{\pm}]_{q^{-6}}\right) = 0 & \text{if } k \neq \tilde{d} = d_i = d_j \\ [X_{i,r \pm \tilde{d}_{ij}}^{\pm}, X_{j,s}^{\pm}]_{q_i^{a_{ij}}} + [X_{j,s \pm \tilde{d}_{ij}}^{\pm}, X_{i,r}^{\pm}]_{q_j^{a_{ji}}} = 0 & \text{otherwise} \end{cases}$$

$$(XXX) \qquad \sum_{\sigma \in \mathcal{S}_3} \sigma.[[X_{i,r_1 \pm 1}^{\pm}, X_{i,r_2}^{\pm}]_{q^2}, X_{i,r_3}^{\pm}]_{q^4} = 0 \ \text{ if } k \neq \tilde{d} = d_i = d_j$$

(S)

$$\sum_{\sigma \in \mathcal{S}_{1-a_{ij}}} \sigma. \sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1 - a_{ij} \\ m \end{bmatrix}_{q_i} X_{i,r_1}^{\pm} \cdot \ldots \cdot X_{i,r_m}^{\pm} X_{j,s}^{\pm} X_{i,r_{m+1}}^{\pm} \cdot \ldots \cdot X_{i,r_{1-a_{ij}}}^{\pm} = 0.$$

where

$$b_{ijr} = \begin{cases} 0 & \text{if } \tilde{d}_{ij} \not| r \\ \frac{[2r]_q (q^{2r} + (-1)^{r-1} + q^{-2r})}{r} & \text{if } k \neq \tilde{d} = d_i = d_j \\ \frac{[\tilde{r} a_{ij}]_{q_i}}{\tilde{r}} & \text{otherwise, with } \tilde{r} = \frac{r}{\tilde{d}_{ij}} \end{cases}$$

$$\sum_{r\in\mathbb{Z}} \tilde{H}^{\pm}_{i,r} u^{-r} = \exp\left(\pm(q_i - q_i^{-1}) \sum_{r>0} H_{i,\pm r} u^{\mp r}\right)$$

and $\mathcal{S}_a$ acts on $\mathbb{Z}^a$, that is $\sigma.f(r_1,\dots,r_a) = f(r_{\sigma(1)},\dots,r_{\sigma(a)})$; for example if $\sigma\in\mathcal{S}_3$ then

$$\sigma.[[X^+_{i,r_1+1}, X^+_{i,r_2}]_{q^2}, X^+_{i,r_3}]_{q^4} = [[X^+_{i,r_{\sigma^{-1}(1)}+1}, X^+_{i,r_{\sigma^{-1}(2)}}]_{q^2}, X^+_{i,r_{\sigma^{-1}(3)}}]_{q^4}.$$

For each $\beta = r\delta + \sum_{i\in I} r_i\alpha_i \in Q$ we set $k_\beta = C^r \prod_{i\in I} k_i^{r_i}$.

Remark 2.3. Other useful relations can be deduced from the defining relations of $\mathcal{U}$ (see Remark 2.11 and, for further details, [4]).

Notice that for all $i\in I$ the sets $\{H_{i,r}|r>0\}$ and $\{H_{i,r}|r<0\}$ generate the same subalgebras of $\mathcal{U}$ respectively as $\{\tilde{H}^+_{i,r}|r\in\mathbb{Z}\}$ and $\{\tilde{H}^-_{i,r}|r\in\mathbb{Z}\}$.

Recall that $\tilde{H}^{\pm}_{i,0} = 1$ and $\tilde{H}^{\pm}_{i,r} = 0$ if $\pm r < 0$.

The relations involving the $H_{i,r}$'s can be given an equivalent formulation in terms of the $\tilde{H}^{\pm}_{i,r}$'s.

It is useful to express both the generators and the relations defining $\mathcal{U}$ in a more compact way, by generating series of families of elements and relations.

To this aim let us introduce some notations.

Notation 2.4. Let $\mathcal{A}$ be an algebra and let

$$a_i(u_1,\dots,u_m) = \sum_{r_1,\dots,r_m\in\mathbb{Z}} a_{i,r_1,\dots,r_m} u_1^{-r_1}\cdot\ldots\cdot u_m^{-r_m} \in \mathcal{A}[[u_1^{\pm1},\dots,u_m^{\pm1}]]$$

be the generating series of the elements $a_{i,r_1,\dots,r_m}\in\mathcal{A}$.

The subalgebra/ideal of $\mathcal{A}$ generated by $\{a_{i,r_1,\dots,r_m}|i\in I; r_1,\dots,r_m\in\mathbb{Z}\}$ is also said to be the subalgebra/ideal of $\mathcal{A}$ generated by the $a_i(u)$'s.

Given $a(u_1,\dots,u_m)\in\mathcal{A}[[u_1^{\pm1},\dots,u_m^{\pm1}]]$, $h\in\{1,\dots,m\}$, $s\in\mathbb{Z}$ we denote by

$$a(u_1,\dots,u_m)\rfloor_{u_h^s} \in \mathcal{A}[[u_1^{\pm1},\dots,u_{h-1}^{\pm1},u_{h+1}^{\pm1},\dots,u_m^{\pm1}]]$$

the coefficient of u_h^s in $a(u_1,\dots,u_m)$.

Notation 2.5. We denote by $\delta(u)\in\mathbb{Z}[[u^{\pm1}]]$ the formal power series

$$\delta(u) = \sum_{r\in\mathbb{Z}} u^{-r}.$$

Remark 2.6. $\delta(u)$ has the following properties:

$\delta(u) = \delta(u^{-1})$;

$u\delta(u) = \delta(u)$;

For every abelian group M and for all $P \in M[u^{\pm 1}]$, the identity $P(u)\delta(u) = P(1)\delta(u)$ holds in $M[[u^{\pm 1}]]$.

Remark 2.7. Let $\mathcal{A}$ be a $\mathbb{Q}$-algebra and denote by $y \leftrightarrow \tilde{y}$ the 1-1 correspondence

$$u\mathcal{A}[[u]] \leftrightarrow 1 + u\mathcal{A}[[u]]$$

defined by $\tilde{y} = \exp(y)$ or equivalently $y = \ln(\tilde{y})$; denote by c a central element of $\mathcal{A}[[u]]$ and by x an element of $\mathcal{A}[[u]]$.

i) If $[h_1, h_2] = c$, then $[h_1, \tilde{h}_2] = c\tilde{h}_2$, and vice versa.
ii) If $[h, x] = cx$, then $\tilde{h}x = \tilde{c}x\tilde{h}$, and vice versa.

Definition 2.8. For $i, j \in I$ define $\mathcal{P}_{ij}(x, y) \in \mathbb{Z}[x^{\pm 1}, y]$ and $\mathcal{B}_{ij}(x, y) \in \mathbb{Q}(x, y) \cap \mathbb{Z}[x^{\pm 1}][[y]]$ as follows:

$$\mathcal{P}_{ij}(x, y) = \begin{cases} (1 - x^4 y)(1 + x^{-2} y) & \text{if } k \neq \tilde{d} = d_i = d_j \\ 1 - x^{d_i a_{ij}} y^{\tilde{d}_{ij}} & \text{otherwise} \end{cases}$$

and

$$\mathcal{B}_{ij}(x, y) = \frac{\mathcal{P}_{ij}(x^{-1}, y)}{\mathcal{P}_{ij}(x, y)}.$$

Remark 2.9. For all $i, j \in I$ we have that:

i) $\mathcal{B}_{ij}(q, u) = \mathcal{B}_{ji}(q, u)$;
ii) $\mathcal{B}_{ij}(q^{-1}, u) = \mathcal{B}_{ij}(q, u)^{-1}$;
iii) $\ln \mathcal{B}_{ij}(q, u) = (q_i - q_i^{-1}) \sum_{r>0} b_{ijr} u^r$.

In particular

$$[h, x] = \pm(q_i - q_i^{-1}) \sum_{r>0} b_{ijr} u^r x \Leftrightarrow \exp(h)x = \mathcal{B}_{ij}(q, u)^{\pm 1} x \exp(h).$$

We can now re-write Definition 2.2 in the following, equivalent way.

Definition 2.10. Let $Y = (A, k, \tilde{d}) \in \mathcal{D}$ and let ζ be a primitive k^{th} root of 1; $\mathcal{U} = \mathcal{U}(Y)$ is the $\mathbb{C}(q)$-algebra generated by

$$C^{\pm 1},\ k_i^{\pm 1},\ \tilde{H}_i^{\pm}(u) = \sum_{r \in \mathbb{Z}} \tilde{H}_{i,r}^{\pm} u^{-r},\ X_i^{\pm}(u) = \sum_{r \in \mathbb{Z}} X_{i,r}^{\pm} u^{-r} \quad (i \in I)$$

with relations:

(C) $$CC^{-1} = 1 \text{ and } C \text{ is central}$$

$$(K) \qquad k_ik_i^{-1} = 1 = k_i^{-1}k_i, \ k_ik_j = k_jk_i$$

$$(\tilde{H}) \qquad \tilde{H}_i^{\pm}(u) \in 1 + u^{\mp 1}\mathcal{U}[[u^{\mp 1}]]$$

$$(X) \qquad X_i^{\pm}(\zeta^{\frac{k}{d_i}}u) = X_i^{\pm}(u)$$

$$(KX) \qquad k_iX_j^{\pm}(u) = q_i^{\pm a_{ij}}X_j^{\pm}(u)k_i$$

$$(HX+) \qquad \tilde{H}_i^{\pm}(u_1)X_j^{\pm}(u_2) = X_j^{\pm}(u_2)\tilde{H}_i^{\pm}(u_1)\mathcal{B}_{ij}(q; u_1^{\mp 1}u_2^{\pm 1})$$

$$(HX-) \qquad \tilde{H}_i^{\pm}(u_1)X_j^{\mp}(u_2) = X_j^{\mp}(u_2)\tilde{H}_i^{\pm}(u_1)\mathcal{B}_{ij}(q; C^{\pm 1}u_1^{\mp 1}u_2^{\pm 1})^{-1}$$

$$(X\pm) \quad [X_i^+(u_1), X_j^-(u_2)] = \delta_{ij}\frac{k_i\tilde{H}_i^+(u_1)\delta(Cu_1^{-1}u_2) - k_i^{-1}\tilde{H}_i^-(u_2)\delta(Cu_1u_2^{-1})}{q_i - q_i^{-1}}$$

$$(XX) \quad \mathcal{P}_{ij}(q, u_1^{\mp 1}u_2^{\pm 1})X_i^{\pm}(u_1)X_j^{\pm}(u_2) = q_i^{a_{ij}}\mathcal{P}_{ij}(q^{-1}, u_1^{\mp 1}u_2^{\pm 1})X_j^{\pm}(u_2)X_i^{\pm}(u_1)$$

$$(XXX) \qquad \sum_{\sigma\in\mathcal{S}_3}\sigma.(u_1^{\pm 1} - (q^2 + q^4)u_2^{\pm 1} + q^6u_3^{\pm 1})X_i^{\pm}(u_1)X_i^{\pm}(u_2)X_i^{\pm}(u_3) = 0$$

and the Serre relations (S)

$$\sum_{\sigma\in\mathcal{S}_{1-a_{ij}}}\sigma.[\dots[[X_j^{\pm}(u), X_i^{\pm}(u_1)]_{q_i^{-a_{ij}}}, X_i^{\pm}(u_2)]_{q_i^{-a_{ij}-2}}, \dots, X_i^{\pm}(u_{1-a_{ij}})]_{q_i^{a_{ij}}} = 0$$

Remark 2.11. The following relations also hold in $\mathcal{U}$:

$$(H) \qquad \tilde{H}_i^{\pm}(\zeta^{\frac{k}{d_i}}u) = \tilde{H}_i^{\pm}(u)$$

$$(KH) \qquad k_i\tilde{H}_j^{\pm}(u) = \tilde{H}_j^{\pm}(u)k_i$$

$$(HH+) \qquad \tilde{H}_i^{\pm}(u_1)\tilde{H}_j^{\pm}(u_2) = \tilde{H}_j^{\pm}(u_2)\tilde{H}_i^{\pm}(u_1)$$

$$(HH-) \qquad \tilde{H}_i^{+}(u_1)\tilde{H}_j^{-}(u_2) = \tilde{H}_j^{-}(u_2)\tilde{H}_i^{+}(u_1)\frac{\mathcal{B}_{ij}(q; C^{-1}u_1^{-1}u_2)}{\mathcal{B}_{ij}(q; Cu_1^{-1}u_2)}.$$

Moreover the $H_i^{\pm}(u)$'s are defined by the following relation:

$$(H\tilde{H}) \qquad \tilde{H}_i^{\pm}(u) = \exp(\pm(q_i - q_i^{-1})H_i^{\pm}(u)).$$

On the other hand the relations (XX) and (XXX) are redundant except for the rank 1 case (that is in the affine cases $A_1^{(1)}$ and $A_2^{(2)}$, see [4]); conversely they are necessary in the description of the positive and negative part of $\mathcal{U}$.

Define also $\tilde{K}_{i,r}^{\pm} = k_i^{\pm 1}\tilde{H}_{ir}^{\pm}$ for all $i \in I$, $r \in \mathbb{Z}$ (in particular $\tilde{K}_{i,0}^{\pm} = k_i^{\pm 1}$) or equivalently

$$(\tilde{K}) \qquad \tilde{K}_i^{\pm}(u) = k_i^{\pm 1}\tilde{H}_i^{\pm}(u),$$

which is both a definition and a relation between the elements involved.

Remark 2.12. $\{C^{\pm 1}, \tilde{K}_i^{\pm}(u), X_i^{\pm}(u) | i \in I\}$ is a set of generators of $\mathcal{U}$.

Remark 2.13. If $k = 1$, then $\mathcal{U}((A, k, \tilde{d}))$ is the quantum affinization $\hat{\mathcal{U}}_q(\mathfrak{g}(A))$.

If A is finite, then $\mathcal{U}((A, k, \tilde{d}))$ is the Drinfeld realization $\mathcal{U}_q^{Dr}(\mathfrak{g}(\hat{A}))$, where $\hat{A}$ is the affine Cartan matrix corresponding to $(A, k, \tilde{d})$.

If A is finite and $k = 1$, then $\hat{\mathcal{U}}_q(\mathfrak{g}(A)) = \mathcal{U}_q^{Dr}(\mathfrak{g}(\hat{A}))$ (untwisted affine quantum algebras are quantum affinizations).

If A is finite and $k \neq 1$, then $\mathcal{U}((A, k, \tilde{d})) = \mathcal{U}_q^{Dr}(\mathfrak{g}(\hat{A}))$ is not a quantum affinization.

If A is affine, then $\mathcal{U}((A, k, \tilde{d}))$ is called quantum toroidal algebra (see [16, 29] and the review paper [18]): remark that there are various versions of the quantum toroidal algebras (for example attached to $\mathfrak{gl}_1$ or $\mathfrak{gl}_n$, see [14]): in principle we can expect that the argument of this paper, which relies on the triangular decomposition, should work also in these cases; but the details should be verified, and eventually adapted with care.

Definition 2.14. The algebra $\mathcal{U}$ is a Q-graded algebra: $\mathcal{U} = \oplus_{\beta \in Q} \mathcal{U}_\beta$, where

$$C^{\pm 1}, k_i^{\pm 1} \in \mathcal{U}_0, \;\; H_{i,r} \in \mathcal{U}_{r\delta}, X_{i,r}^{\pm} \in \mathcal{U}_{r\delta \pm \alpha_i}.$$

The Q-gradation induces a Q_0-gradation by $\mathbb{Z}$-graded vector spaces:

$$\mathcal{U} = \bigoplus_{\gamma \in Q_0} \mathcal{U}_{[\gamma]}, \quad \text{with} \;\; \mathcal{U}_{[\gamma]} = \bigoplus_{r \in \mathbb{Z}} \mathcal{U}_{\gamma + r\delta}.$$

For all $m \geq 1$, $\mathcal{U}^{\otimes m}$ inherits from the $Q^{\oplus m}$-gradation induced by the Q-gradation of $\mathcal{U}$:

i) a Q-gradation: $\mathcal{U}^{\otimes m} = \oplus_{\beta \in Q} (\mathcal{U}^{\otimes m})_\beta$ where

$$(\mathcal{U}^{\otimes m})_\beta = \bigoplus_{\substack{(\beta_1, \ldots, \beta_m) \in Q^{\oplus m} \\ \beta_1 + \cdots + \beta_m = \beta}} \mathcal{U}_{\beta_1} \otimes \cdots \otimes \mathcal{U}_{\beta_m};$$

ii) a $Q_0^{\oplus m}$-gradation with $\mathbb{Z}^m$-graded homogeneous components:

$$\mathcal{U}^{\otimes m} = \bigoplus_{(\gamma_1, \ldots, \gamma_m) \in Q_0^{\oplus m}} (\mathcal{U}^{\otimes m})_{[\gamma_1, \ldots, \gamma_m]}$$

where

$$(\mathcal{U}^{\otimes m})_{[\gamma_1, \ldots, \gamma_m]} = \mathcal{U}_{[\gamma_1]} \otimes \cdots \otimes \mathcal{U}_{[\gamma_m]} = \bigoplus_{(r_1, \ldots, r_m) \in \mathbb{Z}^m} \mathcal{U}_{\gamma_1 + r_1\delta} \otimes \cdots \otimes \mathcal{U}_{\gamma_m + r_m\delta}.$$

Notation 2.15. $\mathcal{U}^+, \mathcal{U}^-$ and $\mathcal{U}^0$ denote the $\mathbb{C}(q)$-subalgebras of $\mathcal{U}$ generated respectively by $\{X_i^+(u)|i \in I\}$, $\{X_i^-(u)|i \in I\}$ and $\{C^{\pm 1}, \tilde{H}_i^+(u), \tilde{H}_i^-(u)|i \in I\}$.

$\mathcal{U}^{0,+}$, $\mathcal{U}^{0,-}$ and $\mathcal{U}^{0,0}$ denote the $\mathbb{C}(q)$-subalgebras of $\mathcal{U}$ generated respectively by $\{\tilde{H}_i^+(u)|i \in I\}$, $\{\tilde{H}_i^-(u)|i \in I\}$ and $\{C^{\pm 1}\}$.

$\mathcal{U}^K$ denotes the $\mathbb{C}(q)$-subalgebra of $\mathcal{U}$ generated by $\{k_i^{\pm 1}|i \in I\}$ and finally $\mathcal{U}^{0,K}$ denotes the $\mathbb{C}(q)$-subalgebra of $\mathcal{U}$ generated by $\mathcal{U}^0$ and $\mathcal{U}^K$, that is the subalgebra generated by $\{C^{\pm 1}, \tilde{K}_i^{\pm}(u)|i \in I\}$.

Remark 2.16. Of course $\mathcal{U}^{\pm}$, $\mathcal{U}^0$, $\mathcal{U}^{0,\pm}$ are Q-graded subalgebras of $\mathcal{U}$; $\mathcal{U}^{0,0}$, $\mathcal{U}^K \subseteq \mathcal{U}_0$.

Remark 2.17. It is well known that the multiplication of $\mathcal{U}$, which is a Q-homogeneous map, induces the following isomorphisms of Q-graded vector spaces (refinement of the so called "triangular decomposition"):

$$\mathcal{U} \cong \mathcal{U}^+ \otimes \mathcal{U}^{0,K} \otimes \mathcal{U}^-,$$
$$\mathcal{U}^{0,K} \cong \mathcal{U}^0 \otimes \mathcal{U}^K \quad \text{(isomorphism of algebras)},$$
$$\mathcal{U}^0 \cong \mathcal{U}^{0,+} \otimes \mathcal{U}^{0,0} \otimes \mathcal{U}^{0,-}.$$

See [19] for the quantum affinizations and [5] for the affine quantum algebras.

As already underlined in the Introduction, the triangular decomposition is crucial for the argument of this paper. It shall be given a slightly different formulation in Remark 6.2 using the following description (Remark 2.18) of the subalgebras of $\mathcal{U}$ introduced in Notation 2.15, which is an immediate and well known consequence of Remark 2.17.

Remark 2.18. 1) $\mathcal{U}^{0,\pm}$, $\mathcal{U}^{0,0}$ and $\mathcal{U}^K$ are algebras of polynomials:

$$\mathcal{U}^{0,+} \cong \mathbb{C}(q)[\tilde{H}_{i,r}^+|i \in I, \tilde{d}_i|r, r > 0] = \mathbb{C}(q)[H_{i,r}|i \in I, \tilde{d}_i|r, r > 0],$$
$$\mathcal{U}^{0,-} \cong \mathbb{C}(q)[\tilde{H}_{i,r}^-|i \in I, \tilde{d}_i|r, r < 0] = \mathbb{C}(q)[H_{i,r}|i \in I, \tilde{d}_i|r, r < 0],$$
$$\mathcal{U}^{0,0} \cong \mathbb{C}(q)[C^{\pm 1}],$$
$$\mathcal{U}^K \cong \mathbb{C}(q)[k_i^{\pm 1}|i \in I];$$

2) $\mathcal{U}^0$ is the $\mathbb{C}(q)$-algebra generated by

$$\{C^{\pm 1}, \tilde{H}_i^+(u), \tilde{H}_i^-(u)|i \in I\}$$

with relations (C), $(\tilde{H})$, (H), $(HH+)$ and $(HH-)$.

Definition 2.19. 1. $\Omega : \mathcal{U} \to \mathcal{U}$ is the $\mathbb{C}$-antilinear anti-involution defined on the generators by:

$$\Omega(q) = q^{-1},\ \Omega(C) = C^{-1},\ \Omega(k_i) = k_i^{-1},$$
$$\Omega(\tilde{H}_i^{\pm}(u)) = \tilde{H}_i^{\mp}(u^{-1}),\ \Omega(X_i^{\pm}(u)) = X_i^{\mp}(u^{-1});$$

(remark that $\Omega(H_i^{\pm}(u)) = H_i^{\mp}(u^{-1})$).

2. For all $\omega \in P$, $t_\omega : \mathcal{U} \to \mathcal{U}$ is the $\mathbb{C}(q)$-algebra automorphism defined on the generators by:

$$t_\omega(k_\beta) = k_{\beta - <\omega,\beta>\delta},\ t_\omega(\tilde{H}_i^{\pm}(u)) = \tilde{H}_i^{\pm}(u),\ t_\omega(X_i^{\pm}(u)) = u^{\mp<\omega,\alpha_i>} X_i^{\pm}(u).$$

For all $i \in I$, t_i denotes the automorphism t_{ω_i}.

3. Completion of graded vector spaces

In order to define on $\mathcal{U}$ the Drinfeld coproduct Δ_v with values in a completion of $\mathcal{U} \otimes \mathcal{U}$, we have to choose this completion. In the literature it is usually remarked that Δ_v takes values in $\mathcal{U} \otimes \mathcal{U}((v))$; but this choice has some drawbacks that we want to avoid.

On one hand remark that the elements of $\mathcal{U} \otimes \mathcal{U}((v))$ are limits of sequences in $\mathcal{U} \otimes \mathcal{U}[v^{\pm 1}]$, not in $\mathcal{U} \otimes \mathcal{U}$:

$$a(v) = \sum_{r \geq R} a_r v^r = \lim_{N \to \infty} \left(\sum_{r=R}^{N} a_r v^r \big(= a(v)_N \big) \right) \quad (a_r \in \mathcal{U} \otimes \mathcal{U}\ \forall r). \tag{3.1}$$

The idea is to identify $a(v)_N$ with $\sum_{r=R}^{N} a_r \in \mathcal{U} \otimes \mathcal{U}$ by choosing a convenient subalgebra of $\mathcal{U} \otimes \mathcal{U}((v))$ so that its intersection with $\mathcal{U} \otimes \mathcal{U}[v^{\pm 1}]$ is isomorphic to $\mathcal{U} \otimes \mathcal{U}$ via the evaluation of v at 1. This is done by providing $\mathcal{U} \otimes \mathcal{U}$ with a structure of $\mathbb{Z}$-graded algebra $\mathcal{U} \otimes \mathcal{U} = \oplus_{d \in \mathbb{Z}} (\mathcal{U} \otimes \mathcal{U})^{(d)}$ so that

$$\mathcal{U} \otimes \mathcal{U} \cong \oplus_{d \in \mathbb{Z}} (\mathcal{U} \otimes \mathcal{U})^{(-d)} v^d \subseteq \bigcup_{R \in \mathbb{Z}} \prod_{d \geq R} (\mathcal{U} \otimes \mathcal{U})^{(-d)} v^d \subseteq (\mathcal{U} \otimes \mathcal{U})((v)).$$

Thus v has just the role of underlining the $\mathbb{Z}$-grading of the elements a_r in (3.1) and to control that the infinite sums involved in our definitions and arguments make sense in our completion.

This construction is to be extended to a $\mathbb{Z}^2$-graded algebra structure on $\mathcal{U}^{\otimes 3}$ (and more generally to a $\mathbb{Z}^m$-graded algebra structure on $\mathcal{U}^{\otimes (m+1)}$) in order to deal with the coassociativity of Δ_v.

On the other hand $\mathcal{U}\otimes\mathcal{U}((v))$ has no gradation, while $\mathcal{U}\otimes\mathcal{U}$ is $(Q\oplus Q)$-graded: in particular we are interested in preserving the Q-gradation arising from the projection $Q\oplus Q\ni(\beta,\beta')\mapsto\beta+\beta'\in Q$ (see Definition 2.14, i)), because we want Δ_v to be a Q-homogeneous homomorphism. This could be solved by restricting to a Q-graded version of our construction: we can first define the completion of each homogeneous component $(\mathcal{U}\otimes\mathcal{U})_\beta$ of $\mathcal{U}\otimes\mathcal{U}$ for $\beta\in Q$ (more precisely we shall define a $\mathbb{Z}$-graded completion of $\oplus_{r_1,r_2\in\mathbb{Z}}(\mathcal{U}_{\gamma_1+r_1\delta}\otimes\mathcal{U}_{\gamma_2+r_2\delta})$ for all $(\gamma_1,\gamma_2)\in Q_0\oplus Q_0$) and then define the whole completion $\mathcal{U}\hat{\otimes}\mathcal{U}$ as direct sum of these partial completions.

Finally, we need to make sure that with this smaller completion (smaller than $\mathcal{U}\otimes\mathcal{U}((v))$) the morphisms that we want to induce from $\mathcal{U}^{\otimes 2}$ (mainly $t_i\otimes t_i$ for the P-equivariance of Δ_v, $\Delta_v\otimes id$ and $id\otimes\Delta_v$ for its coassociativity, $\sigma\circ\Omega^{\otimes 2}$ for the symmetry between positive and negative parts preserved by Δ_v and useful in avoiding repetitive computations) are well defined (see Definition 2.19 and Remark 3.21): it is with this need in mind that the $\mathbb{Z}^m$-gradation of $\mathcal{U}^{\otimes(m+1)}$ mentioned above will be chosen.

1. *The filtered completion of a $\mathbb{Z}^m$-graded vector space.*

Let $V=\oplus_{\mathbf{r}=(r_1,\dots,r_m)\in\mathbb{Z}^m}V^{(\mathbf{r})}$ be a $\mathbb{Z}^m$-graded vector space; then, if the base field is endowed with the discrete topology, V is a topological vector space with the topology induced by

$$\Big\{\bigoplus_{r_1+\cdots+r_m\le N}V^{(\mathbf{r})}|N\in\mathbb{Z}\Big\}$$

as a fundamental system of neighborhoods of 0.

The $\mathbb{Z}^m$-gradation induces a filtration of V: for all $\mathbf{R}=(R_1,\dots,R_m)\in\mathbb{Z}^m$ let ${}_{\mathbf{R}}V=\oplus_{\mathbf{r}\le\mathbf{R}}V^{(\mathbf{r})}\subseteq V$, where $\mathbf{r}\le\mathbf{R}\Leftrightarrow r_h\le R_h\ \forall h=1,\dots,m$.

Then ${}_{\mathbf{R}}V\subseteq{}_{\mathbf{R}'}V$ for all $\mathbf{R}\le\mathbf{R}'$, $V=\sum_{\mathbf{R}\in\mathbb{Z}^m}{}_{\mathbf{R}}V=\bigcup_{\mathbf{R}\in\mathbb{Z}^m}{}_{\mathbf{R}}V$, and each ${}_{\mathbf{R}}V$ is a topological vector space with the topology induced by V.

The completion of the topological vector space ${}_{\mathbf{R}}V$ is $\prod_{\mathbf{r}\le\mathbf{R}}V^{(\mathbf{r})}$ and for all $\mathbf{R}\le\mathbf{R}'$ the embedding ${}_{\mathbf{R}}V\subseteq{}_{\mathbf{R}'}V$ induces an embedding $\prod_{\mathbf{r}\le\mathbf{R}}V^{(\mathbf{r})}\subseteq\prod_{\mathbf{r}\le\mathbf{R}'}V^{(\mathbf{r})}$.

Definition 3.2. Let $V=\oplus_{\mathbf{r}\in\mathbb{Z}^m}V^{(\mathbf{r})}$ be a $\mathbb{Z}^m$-graded vector space. The filtered $\mathbb{Z}^m$-graded completion $\bar V$ of V is

$$\bar V=\varinjlim_{\mathbf{R}}\prod_{\mathbf{r}\le\mathbf{R}}V^{(\mathbf{r})}=\bigcup_{\mathbf{R}\in\mathbb{Z}^m}\prod_{\mathbf{r}\le\mathbf{R}}V^{(\mathbf{r})}\Big(\subseteq\prod_{\mathbf{r}\in\mathbb{Z}^m}V^{(\mathbf{r})}\Big).$$

$\bar V$ can also be described as follows.

Notation 3.3. Given a vector space V and a positive integer m, let us denote by $V[\mathbf{v}^{\pm 1}] \subseteq V((\mathbf{v})) \subseteq V[[\mathbf{v}^{\pm 1}]]$ the following vector spaces:

$$V[\mathbf{v}^{\pm 1}] = V[v_1^{\pm 1}, \dots, v_m^{\pm 1}], \quad V[[\mathbf{v}^{\pm 1}]] = V[[v_1^{\pm 1}, \dots, v_m^{\pm 1}]],$$
$$V((\mathbf{v})) = V((v_1, \dots, v_m)) = \Big\{ \sum_{\mathbf{r} \le \mathbf{R}} x_{\mathbf{r}} v^{-\mathbf{r}} \in V[[v^{\pm 1}]] \Big| \mathbf{R} \in \mathbb{Z}^m \Big\}$$

where for all $\mathbf{r} = (r_1, \dots, r_m) \in \mathbb{Z}^m$ we set $\mathbf{v}^{\mathbf{r}} = v_1^{r_1} \cdot \dots \cdot v_m^{r_m}$.

Remark that if V is a $\mathbb{Z}^m$-graded vector space then $V[\mathbf{v}^{\pm 1}]$ is $\mathbb{Z}^m$-graded by $V[\mathbf{v}^{\pm 1}]^{(\mathbf{r})} = \oplus_{\mathbf{r}_1 + \mathbf{r}_2 = r} V^{(\mathbf{r}_1)} v^{\mathbf{r}_2}$, and $V \cong V[\mathbf{v}^{\pm 1}]^{(\mathbf{0})}$.

$V[[\mathbf{v}^{\pm 1}]]$ and $V((\mathbf{v}))$ are not $\mathbb{Z}^m$-graded.

If $V = \oplus_{\mathbf{r} \in \mathbb{Z}^m} V^{(\mathbf{r})}$ is $\mathbb{Z}^m$-graded, denote by $V[[\mathbf{v}^{\pm 1}]]^{(0)}$ and $V((\mathbf{v}))^{(0)}$ the following vector spaces:

$$V[[\mathbf{v}^{\pm 1}]]^{(0)} = \Big\{ \sum_{\mathbf{r} \in \mathbb{Z}^m} x_{\mathbf{r}} v^{-\mathbf{r}} \in V[[\mathbf{v}^{\pm 1}]] \Big| x_{\mathbf{r}} \in V^{(\mathbf{r})} \ \forall \mathbf{r} \Big\},$$
$$V((\mathbf{v}))^{(0)} = V[[\mathbf{v}^{\pm 1}]]^{(0)} \cap V((\mathbf{v})).$$

Of course $V[\mathbf{v}^{\pm 1}]^{(\mathbf{0})} = V[[\mathbf{v}^{\pm 1}]]^{(0)} \cap V[\mathbf{v}^{\pm 1}] \subseteq V((\mathbf{v}))^{(0)} \subseteq V[[\mathbf{v}^{\pm 1}]]^{(0)}$.

Remark 3.4. Let $V = \oplus_{\mathbf{r} \in \mathbb{Z}^m} V^{(\mathbf{r})}$ be a $\mathbb{Z}^m$-graded vector space. Then

$$\bar{V} \cong V((\mathbf{v}))^{(0)},$$

the isomorphism from $V((\mathbf{v}))^{(0)}$ to $\bar{V}$ being the evaluation of $v_1, \dots, v_m$ at 1.

Remark that if $m = 0$ then $\bar{V} = V$.

Remark 3.5. Let $V = \oplus_{\mathbf{r} \in \mathbb{Z}^m} V^{(\mathbf{r})}, W = \oplus_{\mathbf{r} \in \mathbb{Z}^m} W^{(\mathbf{r})}$ be $\mathbb{Z}^m$-graded vector spaces; then $V \otimes W$ is $\mathbb{Z}^m$-graded by

$$(V \otimes W)^{(\mathbf{r})} = \oplus_{\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{r}} V^{(\mathbf{r}_1)} \otimes W^{(\mathbf{r}_2)}$$

and $\bar{V} \otimes \bar{W}$ naturally embeds into $\overline{V \otimes W}$:

$$\sum_{\mathbf{r}_1 \in \mathbb{Z}^m} x_{\mathbf{r}_1} \mathbf{v}^{-\mathbf{r}_1} \otimes \sum_{\mathbf{r}_2 \in \mathbb{Z}^m} y_{\mathbf{r}_2} \mathbf{v}^{-\mathbf{r}_2} \mapsto \sum_{\mathbf{r} \in \mathbb{Z}^m} \sum_{\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{r}} x_{\mathbf{r}_1} \otimes y_{\mathbf{r}_2} \mathbf{v}^{-\mathbf{r}}.$$

Remark 3.6. Let $V = \oplus_{\mathbf{r} \in \mathbb{Z}^m} V^{(\mathbf{r})}, W = \oplus_{\mathbf{r} \in \mathbb{Z}^m} W^{(\mathbf{r})}$ be $\mathbb{Z}^m$-graded vector spaces and $f : V \to W$ be a continuous linear map. If for all $\mathbf{R} \in \mathbb{Z}^m$ there exists $\mathbf{S} \in \mathbb{Z}^m$ such that $f({}_{\mathbf{R}}V) \subseteq {}_{\mathbf{S}}W$ (this condition is fulfilled if there exists an ordering preserving injective map $\varphi : \mathbb{Z}^m \to \mathbb{Z}^m$ such that $f(V^{(\mathbf{r})}) \subseteq W^{(\varphi(\mathbf{r}))}$ for all $\mathbf{r} \in \mathbb{Z}^m$) then f induces $\bar{f} : \bar{V} \to \bar{W}$.

In particular if f is homogeneous of degree $\mathbf{d}(\in \mathbb{Z}^m)$ ($f(V^{(\mathbf{r})}) \subseteq W^{(\mathbf{r}+\mathbf{d})}$ $\forall \mathbf{r} \in \mathbb{Z}^m$) or if it permutes the degrees ($f(V^{(\mathbf{r})}) \subseteq W^{(\sigma.\mathbf{r})}$ for some $\sigma \in \mathcal{S}_m$) then it satisfies the above conditions, hence it induces $\bar{f} : \bar{V} \to \bar{W}$.

On the other hand, if f "reverses" the grading (for instance if $f(V^{(\mathbf{r})}) \subseteq W^{(-\sigma.\mathbf{r})}$ for all $\mathbf{r} \in \mathbb{Z}^m$ and a fixed $\sigma \in \mathcal{S}_m$) then f is not continuous and does not induce any $\bar{f}$ from $\bar{V}$ to $\bar{W}$.

We need a completion that, despite the loss of some good properties as the one just described with respect to the permutations of the degrees, allows us to extend f when $f(V^{(r_1,\dots,r_m)}) \subseteq W^{(-r_m,\dots,-r_1)}$.

This goal will be achieved by modifying the grading, or better by changing a $\mathbb{Z}^m$-graded vector space into a $\mathbb{Z}$-graded vector space with $\mathbb{Z}^{m-1}$-graded components, through a construction which has the further advantage to produce a $\mathbb{Z}$-graded completion.

2. *The $\mathbb{Z}$-graded completion of a $\mathbb{Z}^m$-graded vector space ($m \geq 1$).*

Definition 3.7. Let $V = \oplus_{(r_1,\dots,r_m)\in\mathbb{Z}^m} V^{(r_1,\dots,r_m)}$ be a $\mathbb{Z}^m$-graded vector space with $m \geq 1$. Then V is a $\mathbb{Z}$-graded vector space with $\mathbb{Z}^{m-1}$-graded homogeneous components:

$$V = \bigoplus_{r\in\mathbb{Z}} V_r \quad \text{where} \quad V_r = \bigoplus_{r_1+\cdots+r_m=r} V^{(r_1,\dots,r_m)}$$

and

$$V_r = \bigoplus_{(s_2,\dots,s_m)\in\mathbb{Z}^{m-1}} (V_r)^{(s_2,\dots,s_m)}$$

with

$$(V_r)^{(s_2,\dots,s_m)} = V^{(r-s_2,s_2-s_3,\dots,s_k-s_{k+1},\dots,s_{m-1}-s_m,s_m)};$$

equivalently $V^{(r_1,\dots,r_m)} = V_{r_1+\cdots+r_m}^{(r_2+\cdots+r_m,r_3+\cdots+r_m,\dots,r_{m-1}+r_m,r_m)}$.

Then the $\mathbb{Z}$-graded completion of V is defined to be the $\mathbb{Z}$-graded vector space $\hat{V}$ whose homogeneous components are the filtered $\mathbb{Z}^{m-1}$-completions $\overline{V_r}$ of the V_r's.

Remark 3.8. Let $f : V \to W$ be a homogeneous linear map of $\mathbb{Z}^m$-graded vector spaces of degree $(d_1,\dots,d_m)$; then f induces a linear map $\hat{f} : \hat{V} \to \hat{W}$ homogeneous of degree $d_1 + \cdots + d_m$: more precisely

$$f(V_r^{(s_2,\dots,s_m)}) \subseteq W_{r+d_1+\cdots+d_m}^{(s_2+d_2+\cdots+d_m,\dots,s_{m-1}+d_{m-1}+d_m,s_m+d_m)},$$

that is $f|_{V_r} : V_r \to W_{r+d_1+\cdots+d_m}$ is homogeneous of degree $(d_2+\cdots+d_m,\dots,d_m)$, hence it induces $\overline{f|_{V_r}} : \overline{V_r} \to \overline{W_{r+d_1+\cdots+d_m}}$ and, by direct sum, $\hat{f} : \hat{V} \to \hat{W}$.

On the other hand if f maps $V^{(r_1,\dots,r_m)}$ to $W^{(-r_m,\dots,-r_1)}$ then $f(V_r)\subseteq W_{-r}$ and $f(V_r^{(s_2,\dots,s_m)})\subseteq W_{-r}^{(s_m-r,\dots,s_2-r)}$, inducing $\overline{f|_{V_r}}:\overline{V_r}\to\overline{W_{-r}}$ (see Remark 3.6) and, by direct sum, $\hat f:\hat V\to\hat W$.

Notation 3.9. Let $V_1,\dots,V_m$ be $\mathbb{Z}$-graded vector spaces. Then $V_1\otimes\cdots\otimes V_m$ is a $\mathbb{Z}^m$-graded vector space: if $V_i=\oplus_{r\in\mathbb{Z}}V_i^{(r)}$,

$$V_1\otimes\cdots\otimes V_m=\bigoplus_{(r_1,\dots,r_m)\in\mathbb{Z}^m}V_1^{(r_1)}\otimes\cdots\otimes V_m^{(r_m)}.$$

Its $\mathbb{Z}$-graded completion is denoted by $V_1\hat\otimes\cdots\hat\otimes V_m$.

Remark 3.10. Let V_1, V_2, V_3 be $\mathbb{Z}$-graded vector spaces. Remark that in general $V_1\hat\otimes V_2\hat\otimes V_3$, $(V_1\hat\otimes V_2)\hat\otimes V_3$ and $V_1\hat\otimes(V_2\hat\otimes V_3)$ are different. More precisely there are natural embeddings

$$V_1\hat\otimes V_2\hat\otimes V_3\hookrightarrow(V_1\hat\otimes V_2)\hat\otimes V_3\quad\text{and}\quad V_1\hat\otimes V_2\hat\otimes V_3\hookrightarrow V_1\hat\otimes(V_2\hat\otimes V_3)$$

and we have

$$V_1\hat\otimes V_2\hat\otimes V_3=(V_1\hat\otimes V_2)\hat\otimes V_3\cap V_1\hat\otimes(V_2\hat\otimes V_3).$$

Indeed these three $\mathbb{Z}$-graded vector spaces are generated by elements (homogeneous of degree r, as r varies in $\mathbb{Z}$) of the form

$$\sum_{\substack{(r_1,r_2,r_3)\\ r_1+r_2+r_3=r}}x_1\otimes x_2\otimes x_2\quad\text{with}\quad x_i\in V_i^{(r_i)}$$

where $(r_1,r_2,r_3)=(r-r_2-r_3,r_2,r_3)$ are subject respectively to the following conditions:

i) $\exists R,R'$ such that $r_3\le R$, $r_2+r_3\le R'$;
ii) $\exists R,R'_s$ $(\forall s\in\mathbb{Z})$ such that $r_3\le R$, $r_2\le R'_{r_3}$;
iii) $\exists R',R_s$ $(\forall s\in\mathbb{Z})$ such that $r_3\le R_{r_2+r_3}$, $r_2+r_3\le R'$.

Of course: i) $\Rightarrow$ ii); i) $\Rightarrow$ iii); ii) and iii) $\Rightarrow$ i).

The same argument shows that for all $m\ge 2$

$$V_1\hat\otimes\cdots\hat\otimes V_m=\bigcap_{k=1}^{m-1}(V_1\hat\otimes\cdots\hat\otimes V_k)\hat\otimes(V_{k+1}\hat\otimes\cdots\hat\otimes V_m).$$

Then $\hat\otimes$ is not associative, but $V_1\hat\otimes\cdots\hat\otimes V_m$ is contained in all the $\hat\otimes$-products of $V_1,\dots,V_m$ (in this order) however associated.

Notation 3.11. Let $m > 0$: we denote by $\sigma_m \in \mathcal{S}_m$ the permutation reversing the ordering, that is the permutation defined by $\sigma_m(i) = m+1-i$; given vector spaces $V_1, \dots, V_m$, we denote again by σ_m the homomorphism (involution)

$$\sigma_m : V_1 \otimes \cdots \otimes V_m \to V_m \otimes \cdots \otimes V_1$$

defined by the "action" of σ_m: $\sigma_m(x_1 \otimes \cdots \otimes x_m) = x_m \otimes \cdots \otimes x_1$.

Remark that if the V_i's are $\mathbb{Z}$-graded then σ_m maps $(V_1 \otimes \cdots \otimes V_m)^{(\mathbf{r})}$ to $(V_m \otimes \cdots \otimes V_1)^{(\sigma_m.\mathbf{r})}$.

Lemma 3.12. *Let $V_1, \dots, V_m$, $W_1, \dots, W_m$ be $\mathbb{Z}$-graded vector spaces, $f_i : V_i \to W_i$ linear maps such that $f_i(V_i^{(r)}) \subseteq W_i^{(-r)}$ for all $r \in \mathbb{Z}$ and $m_1, \dots, m_h$ positive integers such that $m_1 + \cdots + m_h = m$. Then:*

i) the f_i's induce $\hat{f}^{(m)} = \widehat{\sigma_m \circ (\otimes f_i)} : V_1 \hat{\otimes} \cdots \hat{\otimes} V_m \to W_m \hat{\otimes} \cdots \hat{\otimes} W_1$, which again maps elements of degree r in elements of degree $-r$;

ii) if for all $j = 1, \dots, h$ we denote by $\mathbf{V}_j$ and $\mathbf{W}_j$ the vector spaces

$$\mathbf{V}_j = V_{m_1+\cdots+m_{j-1}+1} \hat{\otimes} \cdots \hat{\otimes} V_{m_1+\cdots+m_{j-1}+m_j},$$
$$\mathbf{W}_j = W_{m_1+\cdots+m_{j-1}+m_j} \hat{\otimes} \cdots \hat{\otimes} W_{m_1+\cdots+m_{j-1}+1}$$

and by F_j the map induced, as in point i), by $f_{m_1+\cdots+m_{j-1}+1}, \dots, f_{m_1+\cdots+m_j}$, then

$$\hat{F}^{(h)} : \mathbf{V}_1 \hat{\otimes} \cdots \hat{\otimes} \mathbf{V}_h \to \mathbf{W}_h \hat{\otimes} \cdots \hat{\otimes} \mathbf{W}_1$$

is such that $\hat{F}^{(h)}|_{V_1 \hat{\otimes} \cdots \hat{\otimes} V_m} = \hat{f}^{(m)}$.

Proof. i) depends on the fact that for $r_1, \dots, r_m \in \mathbb{Z}$ the conditions

$$r_1 + \cdots + r_m = r, \quad r_l + \cdots + r_m \le R_l \quad \forall l = 2, \dots, m \quad (r, R_2, \dots, R_l \in \mathbb{Z} \text{ fixed})$$

are equivalent to (imply) the conditions

$$-r_m - \cdots - r_1 = -r, \quad -r_l - \cdots - r_1 \le -r + R_{l+1} \quad \forall l = 1, \dots, m-1.$$

ii) is obvious. □

Notation 3.13. With the notations of Lemma 3.12, by extension we shall denote by $\hat{f}^{(m)}$ all the maps induced by $f_1, \dots, f_m$ on the $\hat{\otimes}$-products of $V_1, \dots, V_m$ however associated (in this order). For example also $\hat{F}^{(h)}$ is denoted by $\hat{f}^{(m)}$.

Remark 3.14. Let U, V be $\mathbb{Z}$-graded vector spaces. Remark that in general $U \hat{\otimes} V \ncong V \hat{\otimes} U$.

But suppose that U has just a finite number of non-zero homogeneous components. Then it is immediate to see that $U \hat{\otimes} V \cong U \otimes V \cong V \hat{\otimes} U$.

In particular the 1-dimensional vector space of degree zero is the unit for $\hat{\otimes}$.

Remark 3.15. Let V_i, U_i $(i = 1, \dots, m)$ be $\mathbb{Z}$-graded vector spaces and $f_i : V_i \to U_i$ homogeneous linear maps of degree respectively d_i. Then Remark 3.8 implies that the f_i's induce $\hat{f} : V_1 \hat{\otimes} \cdots \hat{\otimes} V_m \to U_1 \hat{\otimes} \cdots \hat{\otimes} U_m$ of degree $d_1 + \cdots + d_m$.

In particular, if there is $h \in \{1, \dots, m\}$ such that $V_h = V_h' \hat{\otimes} V_h''$, then

$$\hat{f} : V_1 \hat{\otimes} \cdots \hat{\otimes} V_{h-1} \hat{\otimes} (V_h' \hat{\otimes} V_h'') \hat{\otimes} V_{h+1} \hat{\otimes} \cdots \hat{\otimes} V_m \to U_1 \hat{\otimes} \cdots \hat{\otimes} U_m$$

restricts to a map

$$V_1 \hat{\otimes} \cdots \hat{\otimes} V_{h-1} \hat{\otimes} V_h' \hat{\otimes} V_h'' \hat{\otimes} V_{h+1} \hat{\otimes} \cdots \hat{\otimes} V_m \to U_1 \hat{\otimes} \cdots \hat{\otimes} U_m$$

But conversely, if there is $h \in \{1, \dots, m\}$ such that $U_h = U_h' \hat{\otimes} U_h''$, it is not necessarily true that

$$\hat{f} : V_1 \hat{\otimes} \cdots \hat{\otimes} V_m \to U_1 \hat{\otimes} \cdots \hat{\otimes} U_{h-1} \hat{\otimes} (U_h' \hat{\otimes} U_h'') \hat{\otimes} U_{h+1} \hat{\otimes} \cdots \hat{\otimes} U_m$$

takes values in $U_1 \hat{\otimes} \cdots \hat{\otimes} U_{h-1} \hat{\otimes} U_h' \hat{\otimes} U_h'' \hat{\otimes} U_{h+1} \hat{\otimes} \cdots \hat{\otimes} U_m$ (consider the example $V_1 = U_1 = U_1' \hat{\otimes} U_1''$, $V_2 = U_2$, $f_i = id_{V_i}$).

This obvious remark (that the identity maps $id_{U_1' \hat{\otimes} U_1''}$ and id_{U_2} do not induce a map $(U_1' \hat{\otimes} U_1'') \hat{\otimes} U_2 \to U_1' \hat{\otimes} U_1'' \hat{\otimes} U_2$, see Remark 3.10) suggests the problem of understanding under which conditions a homogeneous map $f : V \to V' \hat{\otimes} V''$ and the identities id_{V_i} induce a map

$$\begin{aligned} &V_1 \hat{\otimes} \cdots \hat{\otimes} V_h \hat{\otimes} V \hat{\otimes} V_{h+1} \hat{\otimes} \cdots \otimes V_N \\ &\quad \to V_1 \hat{\otimes} \cdots \hat{\otimes} V_h \hat{\otimes} V' \hat{\otimes} V'' \hat{\otimes} V_{h+1} \hat{\otimes} \cdots \hat{\otimes} V_N. \end{aligned}$$

The problem comes from the fact that if f has degree d and $x \in V$ has degree r then

$$f(x) = \sum_{s \leq S} x_{r+d-s,s} v^{-s} \quad \text{with} \quad x_{r_1,r_2} \in (V')^{(r_1)} \otimes (V'')^{(r_2)}$$

and in general there is no relation between S and r.

In order to explain the control that we shall require on S, let us consider a notation encoding also the total degree, or equivalently the degrees in both factors:

$$f(x) = \sum_{s \le S} x_{r+d-s,s} \tilde{v}^{-r-d} v^{-s} = \sum_{s \le S} x_{r+d-s,s} \tilde{v}^{-(r+d-s)} (\tilde{v}v)^{-s}$$

where $(\tilde{v}, v)$ encodes the total degree (in this case $r+d$) and the degree s in the $(r+d)$-component, $(\tilde{v}, \tilde{v}v)$ encodes the degrees in the first and second factors of $\mathcal{U} \otimes \mathcal{U}$ and $(v\tilde{v}, v^{-1})$ encodes the total degree (again $r+d$) and the degree $(r+d-s)$ of the first component; remark that

$$f(x) \in (V' \otimes V'')((v))[\tilde{v}^{\pm 1}] = (V' \otimes V'')((v))[(v\tilde{v})^{\pm 1}].$$

We can require a "right control":

$$f(x) \in (V' \otimes V'')[[v]][\tilde{v}^{\pm 1}], \text{ which corresponds to } S = 0;$$

or a "left control":

$$f(x) \in (V' \otimes V'')[[v]][(v\tilde{v})^{\pm 1}], \text{ which corresponds to } S = r + d.$$

Definition 3.16. Let V, V', V'' be $\mathbb{Z}$-graded vector spaces and $f : V \to V' \hat{\otimes} V''$ be a degree d homogeneous map; an element $x \in V$ is said to be f-bounded if its homogeneous components x_r are such that

$$f(x_r) = \sum_{s \le max\{0, r+d\}} y_{r+d-s,s} = \sum_{s \le max\{0, r+d\}} y_{r+d-s,s} v^{-s}.$$

Of course the set $V[f^b]$ of the f-bounded elements of V is a $\mathbb{Z}$-graded vector subspace of V. A subset of V is said to be f-bounded if all of its elements are f-bounded, that is if it is contained in $V[f^b]$.

Lemma 3.17. *Let $V_1, \dots, V_m, V', V''$ be $\mathbb{Z}$-graded vector spaces, $1 \le h \le m$ (h fixed), $f : V_h \to V' \hat{\otimes} V''$ a homogeneous map of degree d,*

$$\begin{aligned} F = id^{\hat{\otimes}(h-1)} \hat{\otimes} f \hat{\otimes} id^{\hat{\otimes}(m-h)} &: V_1 \hat{\otimes} \cdots \hat{\otimes} V_m \\ &\to V_1 \hat{\otimes} \cdots \hat{\otimes} V_{h-1} \hat{\otimes} (V' \hat{\otimes} V'') \hat{\otimes} \cdots \hat{\otimes} V_m. \end{aligned}$$

If

$$x \in V_1 \hat{\otimes} \cdots \hat{\otimes} V_{h-1} \hat{\otimes} V_h[f^b] \hat{\otimes} V_{h+1} \hat{\otimes} \cdots \hat{\otimes} V_m,$$

then

$$F(x) \in V_1 \hat{\otimes} \cdots \hat{\otimes} V_{h-1} \hat{\otimes} V' \hat{\otimes} V'' \hat{\otimes} V_{h+1} \hat{\otimes} \cdots \hat{\otimes} V_m.$$

Proof. Let $x = \sum x_{r_1,\dots,r_m}$ $(x_{r_1,\dots,r_m} \in V_1^{(r_1)} \otimes \cdots \otimes V_m^{(r_m)})$: with no loss of generality we can suppose that x is homogeneous of degree r; thus the conditions for $x_{r_1,\dots,r_m}$ to be different from zero is that

$$r_l + \cdots + r_m \le R_l \quad \text{for all } l > 1.$$

On the other hand the hypothesis on x implies that

$$id^{\otimes(h-1)} \otimes f \otimes id^{\otimes(m-h)}(x_{r_1,\dots,r_m}) = \sum_{s \le \max\{0, r_h+d\}} y_{r_1,\dots,r_{h-1},r_h+d-s,s,r_{h+1},\dots,r_m}$$

so that

$$F(x) = \sum y_{r_1,\dots,r_{h-1},r',r'',r_{h+1},\dots,r_m}$$

with the conditions (see Definition 3.16)

$r_l + \cdots + r_m \le R_l$ for all $l > h$,

$r'' + r_{h+1} + \cdots + r_m \le R_{h+1} + max\{0, r_h + d\} \le max\{R_{h+1}, R_h + d\}$,

$r_l + \cdots + r_{h-1} + r' + r'' + r_{h+1} + \cdots + r_m \le R_l + d$ for all $l = 2, \dots, h$,

which imply that $F(x) \in V_1 \hat{\otimes} \cdots \hat{\otimes} V_{h-1} \hat{\otimes} V' \hat{\otimes} V'' \hat{\otimes} V_{h+1} \hat{\otimes} \cdots \hat{\otimes} V_m$. □

3. *$\mathcal{U}$ and its tensor powers.*

We are now ready to introduce the completion of $\mathcal{U}^{\otimes m}$ that we are interested in. This completion preserves both the Q-gradation and the $Q_0^{\oplus m}$-gradation that $\mathcal{U}^{\otimes m}$ inherits from the $Q^{\oplus m}$-gradation, but not the $Q^{\oplus m}$-gradation itself.

Definition 3.18. The completion $\mathcal{U}^{\hat{\otimes} m}$ of $\mathcal{U}^{\otimes m}$ is the $Q_0^{\oplus m}$-graded vector space whose homogeneous component of degree $(\gamma_1, \dots, \gamma_m) \in Q_0^{\oplus m}$ is the $\mathbb{Z}$-graded completion of the $\mathbb{Z}^m$-graded vector space $(\mathcal{U}^{\otimes m})_{[\gamma_1,\dots,\gamma_m]}$:

Denoting by $(\mathcal{U}^{\otimes m})_{[\gamma_1,\dots,\gamma_m;r]}$ the r-component of $(\mathcal{U}^{\otimes m})_{[\gamma_1,\dots,\gamma_m]}$, we have

$$(\mathcal{U}^{\hat{\otimes} m})_{[\gamma_1,\dots,\gamma_m]} = \mathcal{U}_{[\gamma_1]} \hat{\otimes} \cdots \hat{\otimes} \mathcal{U}_{[\gamma_m]}, \quad \mathcal{U}^{\hat{\otimes} m} = \bigoplus_{(\gamma_1,\dots,\gamma_m) \in Q_0^{\oplus m}} (\mathcal{U}^{\hat{\otimes} m})_{[\gamma_1,\dots,\gamma_m]},$$

that is

$$\mathcal{U}^{\hat{\otimes} m} = \bigoplus_{\substack{(\gamma_1,\dots,\gamma_m) \in Q_0^{\oplus m} \\ r \in \mathbb{Z}}} \overline{(\mathcal{U}^{\otimes m})_{[\gamma_1,\dots,\gamma_m;r]}}.$$

Remark that the $\mathbb{Z}$-graded completion of $\mathcal{U}^{\otimes m}$ (considering its $\mathbb{Z}^m$-gradation regardless of its $Q_0^{\oplus m}$-gradation) is bigger than the $\mathcal{U}^{\hat{\otimes} m}$ just defined (and is not $Q_0^{\oplus m}$-graded), thus it is by a small abuse of notation that we denote by $\mathcal{U}^{\hat{\otimes} m}$ our <u>component-wise</u> $\mathbb{Z}$-graded completion of $\mathcal{U}^{\otimes m}$.

Remark 3.19. The combination of the $\mathbb{Z}$-gradation of $\mathcal{U}^{\hat{\otimes} m}$ resulting from the $\mathbb{Z}$-graded completion, with its Q_0-gradation induced by the $Q_0^{\oplus m}$-gradation, provides $\mathcal{U}^{\hat{\otimes} m}$ with a Q-gradation: given $\beta = \gamma + r\delta \in Q$ (with $\gamma \in Q_0$ and $r \in \mathbb{Z}$)

$$(\mathcal{U}^{\hat{\otimes} m})_\beta = \bigoplus_{\substack{(\gamma_1,\dots,\gamma_m)\in Q_0^{\oplus m}\\ \sum_{i=1}^m \gamma_i=\gamma}} \overline{(\mathcal{U}^{\otimes m})_{[\gamma_1,\dots,\gamma_m;r]}}.$$

Since $\overline{(\mathcal{U}^{\otimes m})_{[\gamma_1,\dots,\gamma_m;r]}} \supseteq \bigoplus_{\substack{(r_1,\dots,r_m)\in\mathbb{Z}^m\\ \sum_{i=1}^m r_i=r}} \mathcal{U}_{\gamma_1+r_1\delta}\otimes\cdots\otimes\mathcal{U}_{\gamma_m+r_m\delta}$, it follows that

$$(\mathcal{U}^{\hat{\otimes} m})_\beta \supseteq \bigoplus_{\substack{(\beta_1,\dots,\beta_m)\in Q^{\oplus m}\\ \beta_1+\dots+\beta_m=\beta}} \mathcal{U}_{\beta_1\otimes\cdots\otimes\mathcal{U}_{\beta_m}} = (\mathcal{U}^{\otimes m})_\beta,$$

which means that the Q-gradations of $\mathcal{U}^{\otimes m}$ and $\mathcal{U}^{\hat{\otimes} m}$ are compatible, or that the $\mathbb{Z}$-graded completion preserves the Q-gradation.

Remark 3.20. Let $m = 2$. Then the discussion of the present section implies that

$$\mathcal{U}\hat{\otimes}\mathcal{U} = \oplus_{\beta\in Q}(\mathcal{U}\hat{\otimes}\mathcal{U})_\beta = \bigoplus_{\substack{(\gamma_1,\gamma_2)\in Q_0\oplus Q_0\\ r\in\mathbb{Z}}} \overline{(\mathcal{U}\otimes\mathcal{U})_{[\gamma_1,\gamma_2;r]}}$$

can be described as a subspace of $(\mathcal{U}\otimes\mathcal{U})((v))$.

Indeed

$$\overline{(\mathcal{U}\otimes\mathcal{U})_{[\gamma_1,\gamma_2;r]}} = (\mathcal{U}\otimes\mathcal{U})_{[\gamma_1,\gamma_2;r]}((v))^{(0)} =$$

$$= \left\{ \sum_{r_2\le R_2} x_{r_2} v^{-r_2} \middle| R_2\in\mathbb{Z},\ x_{r_2}\in\mathcal{U}_{\gamma_1+(r-r_2)\delta}\otimes\mathcal{U}_{\gamma_2+r_2\delta} \right\} \subseteq (\mathcal{U}\otimes\mathcal{U})((v)).$$

$\mathcal{U}\hat{\otimes}\mathcal{U}$ is the subspace of $(\mathcal{U}\otimes\mathcal{U})((v))$ generated by the $\overline{(\mathcal{U}\otimes\mathcal{U})_{[\gamma_1,\gamma_2;r]}}$'s.

Remark 3.21. Remark that:

-) $\mathcal{U}^{\hat{\otimes} m}$ is a Q-graded algebra: indeed the multiplication of $\mathcal{U}$ induces

$$(\mathcal{U}^{\otimes m})_{[\gamma_1,\dots,\gamma_m;r]} \otimes (\mathcal{U}^{\otimes m})_{[\gamma_1',\dots,\gamma_m';r']} \to (\mathcal{U}^{\otimes m})_{[\gamma_1+\gamma_1',\dots,\gamma_m+\gamma_m';r+r']}$$

(degree zero homogeneous map of $\mathbb{Z}^{m-1}$-graded vector spaces), thus the claim follows from Remarks 3.5 and 3.19.

-) for all $i \in I$ and for all $h = 1, \ldots, m$ the automorphism $t_{i,h} = id^{\otimes(h-1)} \otimes t_i \otimes id^{\otimes(m-h)}$ of $\mathcal{U}^{\otimes m}$ induces an automorphism

$$\hat{t}_{i,h} : \mathcal{U}^{\hat{\otimes}\, m} \to \mathcal{U}^{\hat{\otimes}\, m}$$

mapping

$$(\mathcal{U}^{\hat{\otimes}\, m})_{[\gamma_1,\ldots,\gamma_m;r]} \ \text{ to } \ (\mathcal{U}^{\hat{\otimes}\, m})_{[\gamma_1,\ldots,\gamma_m;r-<\omega_i,\gamma_h>]} :$$

indeed t_i maps $\mathcal{U}_{[\gamma;r]} = \mathcal{U}_{\gamma+r\delta}$ to $\mathcal{U}_{\gamma+(r-<\omega_i,\gamma>)\delta} = \mathcal{U}_{[\gamma;r-<\omega_i,\gamma>]}$, hence the claim follows from Remark 3.15.
In particular for all $\omega \in P$, the automorphism $t_\omega^{\otimes m}$ of $\mathcal{U}^{\otimes m}$ induces an algebra automorphism $t_\omega^{\hat{\otimes}\, m}$ of $\mathcal{U}^{\hat{\otimes}\, m}$ such that for all $\beta \in Q$ $(t_\omega^{\hat{\otimes}\, m})((\mathcal{U}^{\hat{\otimes}\, m})_\beta) = (\mathcal{U}^{\hat{\otimes}\, m})_{\beta-<\omega,\beta>\delta}$.

-) $\Omega^{\otimes m}$ does not define an (anti)automorphism of $\mathcal{U}^{\hat{\otimes}\, m}$; but if we define σ_m by

$$\sigma_m : \mathcal{U}^{\otimes m} \ni x_1 \otimes \cdots \otimes x_m \mapsto x_m \otimes \cdots \otimes x_1 \in \mathcal{U}^{\otimes m}$$

then $\sigma_m \circ \Omega^{\otimes m}|_{(\mathcal{U}^{\otimes m})_{[\gamma_1,\ldots,\gamma_m;r]}}$ maps

$$(\mathcal{U}^{\otimes m})^{(s_2,\ldots,s_m)}_{[\gamma_1,\ldots,\gamma_m;r]} \ \text{ to } \ (\mathcal{U}^{\otimes m})^{(s_m-r,\ldots,s_2-r)}_{[-\gamma_m,\ldots,-\gamma_1;-r]},$$

hence it satisfies the conditions of Remark 3.6 and induces a $\mathbb{C}(q)$-antilinear ($\mathbb{R}$-linear) antiinvolution

$$\hat{\Omega}^{(m)} : \mathcal{U}^{\hat{\otimes}\, m} \to \mathcal{U}^{\hat{\otimes}\, m}$$

such that $\hat{\Omega}^{(m)}((\mathcal{U}^{\hat{\otimes}\, m})_\beta) = (\mathcal{U}^{\hat{\otimes}\, m})_{-\beta}$ for all $\beta \in Q$.

4. The Drinfeld "coproduct"

We shall now introduce the Drinfeld "coproduct" Δ_v of $\mathcal{U}$: it is a function with values in the completion $\mathcal{U} \hat{\otimes} \mathcal{U} \subseteq (\mathcal{U} \otimes \mathcal{U})((v))$ of $\mathcal{U} \otimes \mathcal{U}$, which explains the v-notation for this map. Since it doesn't take values in $\mathcal{U} \otimes \mathcal{U}$, this map is not properly a coproduct; but it satisfies properties similar to those defining the coproducts (it is "coassociative" and admits a "counit", see Section 7), which explains the term "coproduct" and the Δ-notation.

Hence Δ_v will define a tensor structure on a convenient category of representations of $\mathcal{U}$: $x.(y_1 \otimes y_2) = \Delta_v(x)(y_1 \otimes y_2)$, where $x \in \mathcal{U}$ and the y_i's are elements of $\mathcal{U}$-modules V_i; of course, since $\Delta_v(x) \in \mathcal{U} \hat{\otimes} \mathcal{U}$ will involve infinite sums, we have to make sure that $\Delta_v(x)(y_1 \otimes y_2)$ makes sense by restricting to the representations with suitable properties.

In particular consider representations V of $\mathcal{U}$ provided with some $\mathbb{Z}$-gradation $V = \oplus_{r\in\mathbb{Z}} V^{(r)}$ such that $\mathcal{U}_{\gamma+r\delta}(V^{(s)}) \subseteq V^{(s+r)}$ if $\gamma \in Q_0$. If we require some condition assuring that for all $y \in V^{(s)}$ there exists $N \in \mathbb{Z}$ such that

$$\mathcal{U}_{\gamma+r\delta}(y) = 0 \quad \text{for all} \quad r > N, \tag{4.1}$$

then $\mathcal{U} \hat{\otimes} \mathcal{U}$ acts on $\tilde{V} \otimes V$, which will become via Δ_v a $\mathcal{U}$-module satisfying (4.1): this will provide the category of such modules with a tensor product. The same is true in the symmetric condition that $\mathcal{U}_{\gamma+r\delta}(y) = 0$ for all $r < N$.

We will be able to construct also another "tensor" structure on some category of $\mathcal{U}$-modules, thanks to $\hat{\otimes}$. Indeed if we choose a category of $\mathcal{U}$-modules preserving the Q-gradation: $\mathcal{U}_\beta(V_\lambda) \subseteq V_{\lambda+\beta}$; we can apply the $\hat{\otimes}$ to define this "tensor" structure: such modules V_1 and V_2 are direct sums of $\mathbb{Z}$-graded subspaces of the form $\oplus_{r\in\mathbb{Z}}(V_i)_{\lambda+r\delta}$, and $V_1 \hat{\otimes} V_2$ is defined similarly to the way in which we defined $\mathcal{U} \hat{\otimes} \mathcal{U}$ (by component-wise $\mathbb{Z}$-graded completion, see Definition 3.18). Then $V_1 \hat{\otimes} V_2$ is a $\mathcal{U} \hat{\otimes} \mathcal{U}$-module, and will inherit a $\mathcal{U}$-module structure via Δ_v.

We can also put together these two constructions, considering the $\mathcal{U}$-modules V such that, for convenient λ's:

$$V = \bigoplus_{r\in\mathbb{Z},\lambda} V_{\lambda+r\delta},$$
$$\mathcal{U}_{\gamma+r\delta}(V_{\lambda+s\delta}) \subseteq V_{\lambda+\gamma+(r+s)\delta},$$
$$\mathcal{U}_{\gamma+r\delta}(V_{\lambda+s\delta}) = 0 \quad \text{for all} \quad r > N_V + s.$$

The coproduct Δ_v will endow the category of these modules with both $\otimes$ and $\hat{\otimes}$.

The claim that Δ_v is a $\mathbb{C}(q)$-algebra homomorphism from $\mathcal{U}$ to $\mathcal{U} \hat{\otimes} \mathcal{U}$ is the main concern of this paper: Δ_v is defined on the generators of $\mathcal{U}$ and the aim of this paper is to prove that the relations defining $\mathcal{U}$ are preserved by Δ_v. Some of the relations are not difficult to verify, but it becomes very hard to deal with the Serre relations when $a_{ij} < -1$. In this paper we propose a strategy to overcome this problem.

Recall 4.2. Let us recall the identification of $\mathcal{U} \otimes \mathcal{U}$ with $(\mathcal{U} \otimes \mathcal{U})[v^{\pm 1}]^{(0)}$ (here $\mathbf{v} = v$):

$$\mathcal{U} \otimes \mathcal{U} = \oplus_{\gamma\in Q_0, r\in\mathbb{Z}} \mathcal{U} \otimes \mathcal{U}_{\gamma+r\delta} \cong \bigoplus_{\gamma\in Q_0, r\in\mathbb{Z}} \mathcal{U} \otimes \mathcal{U}_{\gamma+r\delta} v^{-r}.$$

More precisely we described the $(Q\oplus Q)$-gradation of $\mathcal{U}\otimes\mathcal{U}$ as a $(Q_0\oplus Q_0\oplus\mathbb{Z})$-gradation by $\mathbb{Z}$-graded vector spaces:

$$\mathcal{U}\otimes\mathcal{U}=\bigoplus_{\gamma_1,\gamma_2\in Q_0,r\in\mathbb{Z}}(\mathcal{U}\otimes\mathcal{U})_{[\gamma_1,\gamma_2;r]}$$

where

$$(\mathcal{U}\otimes\mathcal{U})_{[\gamma_1,\gamma_2;r]}=\bigoplus_{s\in\mathbb{Z}}\mathcal{U}_{\gamma_1+(r-s)\delta}\otimes\mathcal{U}_{\gamma_2+s\delta}\cong\bigoplus_{s\in\mathbb{Z}}\mathcal{U}_{\gamma_1+(r-s)\delta}\otimes\mathcal{U}_{\gamma_2+s\delta}v^{-s}$$

and its completion is

$$\overline{(\mathcal{U}\otimes\mathcal{U})_{[\gamma_1,\gamma_2;r]}}=\Big\{\sum_{s\leq S}x_sv^{-s}|S\in\mathbb{Z},\ x_s\in\mathcal{U}_{\gamma_1+(r-s)\delta}\otimes\mathcal{U}_{\gamma_2+s\delta}\Big\}$$

(see Remark 3.20).

In order to show that this construction is symmetric in the two "factors" of $\mathcal{U}\hat{\otimes}\mathcal{U}$, remark also that for all $\gamma_1,\gamma_2\in Q_0$ the elements of the $\mathbb{Z}$-graded vector space $\bigoplus_{r\in\mathbb{Z}}\overline{(\mathcal{U}\otimes\mathcal{U})_{[\gamma_1,\gamma_2;r]}}$ have the form

$$\sum_{r=m}^{M}\sum_{s\leq S_r}x_{r,s}\quad(x_{r,s}\in\mathcal{U}_{\gamma_1+(r-s)\delta}\otimes\mathcal{U}_{\gamma_2+s\delta})$$

(where S_r can be replaced by $max\{S_r|m\leq r\leq M\}$) or equivalently, with a notation that reflects the $\mathbb{Z}$-grading of each factor of $\mathcal{U}\otimes\mathcal{U}$ (and reveals the symmetry of this construction in the two factors),

$$\sum_{\substack{m\leq r_1+r_2\leq M\\ r_2\leq R_2}}y_{r_1,r_2}=\sum_{\substack{m\leq r_1+r_2\leq M\\ r_1\geq R_1}}y_{r_1,r_2}$$

where $y_{r_1,r_2}\in\mathcal{U}_{\gamma_1+r_1\delta}\otimes\mathcal{U}_{\gamma_2+r_2\delta}$. Going back to our v-notation (that we need only to control the infinite sums), we underline that this element is denoted by

$$\sum_{\substack{m\leq r_1+r_2\leq M\\ r_2\leq R_2}}y_{r_1,r_2}v^{-r_2}\in\mathcal{U}\otimes\mathcal{U}((v)).$$

The elements of $\mathcal{U}\hat{\otimes}\mathcal{U}$ are finite sums for $(\gamma_1,\gamma_2)\in Q_0\oplus Q_0$ of elements of this form.

Notation 4.3. Let $Y(u)=\sum_{r\in\mathbb{Z}}Y_ru^{-r}\in\mathcal{U}[[u^{\pm1}]]$. Then for all central invertible element $z\in\mathcal{U}$ of degree zero we use the notations

$$Y(u\otimes z)=\sum_{r\in\mathbb{Z}}Y_r\otimes z^{-r}u^{-r}\in(\mathcal{U}\otimes\mathcal{U})[[u^{\pm1}]]$$

for the family of elements $Y_r\otimes z^{-r}\in\mathcal{U}\otimes\mathcal{U}$, and

$$Y(z\otimes uv)=\sum_{r\in\mathbb{Z}}z^{-r}\otimes Y_rv^{-r}u^{-r}\in(\mathcal{U}\otimes\mathcal{U})[[(uv)^{\pm1}]]\subseteq(\mathcal{U}\otimes\mathcal{U})[v^{\pm1}][[u^{\pm1}]],$$

for the family of elements $z^{-r}\otimes Y_rv^{-r}\in\mathcal{U}\otimes\mathcal{U}[v^{\pm1}]$.

Remark 4.4. Consider the identification of $\mathcal{U}\otimes\mathcal{U}$ with $(\mathcal{U}\otimes\mathcal{U})[v^{\pm1}]^{(0)}$ and more generally of $\mathcal{U}\hat{\otimes}\mathcal{U}$ with a subalgebra of $(\mathcal{U}\otimes\mathcal{U})((v))^{(0)}\subseteq(\mathcal{U}\otimes\mathcal{U})((v))$ (see Recall 4.2).

1. $Y(u\otimes z)\in\mathcal{U}\hat{\otimes}\mathcal{U}[[u^{\pm1}]]$ because z has degree zero; if we have no other conditions on the Y_r's, in general $Y(z\otimes uv)\notin\mathcal{U}\hat{\otimes}\mathcal{U}[[u^{\pm1}]]$.
2. If there exists $\gamma\in Q_0$ such that $Y_r\in\mathcal{U}_{\gamma+r\delta}$ for all $r\in\mathbb{Z}$, then

$$Y(z\otimes uv)\in(\mathcal{U}\otimes\mathcal{U})[v^{\pm1}]^{(0)}[[u^{\pm1}]],$$

that is it represents an element of $(\mathcal{U}\otimes\mathcal{U})[[u^{\pm1}]]\hookrightarrow(\mathcal{U}\hat{\otimes}\mathcal{U})[[u^{\pm1}]]$.
3. Given $Y(u),Y'(u)\in\mathcal{U}[[u^{\pm1}]]$ and z,z' central invertible elements of $\mathcal{U}$ of degree zero, we have that $Y(u\otimes z)Y'(z'\otimes uv)=Y'(z'\otimes uv)Y(u\otimes z)$ is a well defined element

$$\sum_{r,s\in\mathbb{Z}}Y_r(z')^{-s}\otimes Y'_sz^{-r}v^{-s}u^{-r-s}=\sum_{m,s\in\mathbb{Z}}Y_{m-s}(z')^{-s}\otimes Y'_sz^{s-m}v^{-s}u^{-m}$$

of

$$(\mathcal{U}\otimes\mathcal{U})[[u^{\pm1},(uv)^{\pm1}]]=(\mathcal{U}\otimes\mathcal{U})[[v^{\pm1}]][[u^{\pm1}]].$$

4. If $Y(u)\in\mathcal{U}((u^{-1}))$ or $Y'(u)\in\mathcal{U}((u))$ then

$$Y(u\otimes z)Y'(z'\otimes uv)\in(\mathcal{U}\otimes\mathcal{U})((v))[[u^{\pm1}]]$$

because for all $m\in\mathbb{Z}$ the coefficient $Y_{m-s}(z')^{-s}\otimes Y'_sz^{s-m}$ of $v^{-s}u^{-m}$ is zero if $m-s<<0$ or $s>>0$, that is if $s>>0$.

If moreover there exist $\gamma,\gamma'\in Q_0$ such that $Y_r\in\mathcal{U}_{\gamma+r\delta}$ and $Y'_r\in\mathcal{U}_{\gamma'+r\delta}$ for all $r\in\mathbb{Z}$, then

$$Y(u\otimes z)Y'(z'\otimes uv)\in\sum_{m\in\mathbb{Z}}\overline{(\mathcal{U}\otimes\mathcal{U})_{[\gamma,\gamma';m]}}u^{-m}\subseteq(\mathcal{U}\hat{\otimes}\mathcal{U})[[u^{\pm1}]].$$

5. If $Y(u)$ and $Y'(u)$ are both in $\mathcal{U}((u^{-1}))$ or both in $\mathcal{U}((u))$ then

$$Y(u \otimes z)Y'(z' \otimes uv) \in (\mathcal{U} \otimes \mathcal{U})[v^{\pm 1}][[u^{\pm 1}]].$$

If moreover there exist $\gamma, \gamma' \in Q_0$ such that $Y_r \in \mathcal{U}_{\gamma + r\delta}$ and $Y'_r \in \mathcal{U}_{\gamma' + r\delta}$ for all $r \in \mathbb{Z}$, then

$$Y(u \otimes z)Y'(z' \otimes uv) \in (\mathcal{U} \otimes \mathcal{U})[v^{\pm 1}]^{(0)}[[u^{\pm 1}]]$$

that is it represents a family of elements of $(\mathcal{U} \otimes \mathcal{U}) \hookrightarrow \mathcal{U} \hat{\otimes} \mathcal{U}$.

Definition 4.5. Let us denote by $\mathcal{X}$ the following subset (set of generators) of $\mathcal{U}$:

$$\begin{aligned}\mathcal{X} = \{k_\beta,\ H_{i,r_0},\ \tilde{H}^{\pm}_{i,\pm r},\ C^s\tilde{K}^{\pm}_{i,\pm r},\ X^{\pm}_{i,r} | \beta \in Q, i \in I, r_0 \neq 0, r, s \in \mathbb{Z}\} = \\ = \{k_\beta,\ H^{\pm}_i(u),\ \tilde{H}^{\pm}_i(u),\ C^s\tilde{K}^{\pm}_i(u),\ X^{\pm}_i(u) | \beta \in Q, i \in I, s \in \mathbb{Z}\}\end{aligned}$$

and let us define the function $\Delta_v : \mathcal{X} \to \mathcal{U} \hat{\otimes} \mathcal{U}$ as follows:

$$\Delta_v(k_\beta) = k_\beta \otimes k_\beta,$$
$$\Delta_v(H_{i,r}) = \begin{cases} H_{i,r} \otimes 1 + C^r \otimes H_{i,r} v^{-r} & \text{if } r > 0 \\ H_{i,r} \otimes C^r + 1 \otimes H_{i,r} v^{-r} & \text{if } r < 0 \end{cases}$$
$$\Delta_v(\tilde{H}^+_{i,r}) = \sum_{r_1 + r_2 = r} C^{r_2}\tilde{H}^+_{i,r_1} \otimes \tilde{H}^+_{i,r_2} v^{-r_2}$$
$$\Delta_v(\tilde{H}^-_{i,r}) = \sum_{r_1 + r_2 = r} \tilde{H}^-_{i,r_1} \otimes C^{r_1}\tilde{H}^-_{i,r_2} v^{-r_2}$$
$$\Delta_v(C^s\tilde{K}^+_{i,r}) = \sum_{r_1 + r_2 = r} C^{s+r_2}\tilde{K}^+_{i,r_1} \otimes C^s\tilde{K}^+_{i,r_2} v^{-r_2}$$
$$\Delta_v(C^s\tilde{K}^-_{i,r}) = \sum_{r_1 + r_2 = r} C^s\tilde{K}^-_{i,r_1} \otimes C^{s+r_1}\tilde{K}^-_{i,r_2} v^{-r_2}$$
$$\Delta_v(X^+_{i,r}) = X^+_{i,r} \otimes 1 + \sum_{r_1 + r_2 = r} k_i C^{r_2}\tilde{H}^+_{i,r_1} \otimes X^+_{i,r_2} v^{-r_2}$$
$$\Delta_v(X^-_{i,r}) = 1 \otimes X^-_{i,r} v^{-r} + \sum_{r_1 + r_2 = r} X^-_{i,r_1} \otimes k_i^{-1} C^{r_1}\tilde{H}^-_{i,r_2} v^{-r_2},$$

which can be written more compactly as:

$$\Delta_v(k_\beta) = k_\beta \otimes k_\beta,$$
$$\Delta_v(H^+_i(u)) = H^+_i(u \otimes 1) + H^+_i(C^{-1} \otimes uv),$$

$$\Delta_v(H_i^-(u)) = H_i^-(u\otimes C^{-1}) + H_i^-(1\otimes uv),$$
$$\Delta_v(\tilde{H}_i^+(u)) = \tilde{H}_i^+(u\otimes 1)\tilde{H}_i^+(C^{-1}\otimes uv),$$
$$\Delta_v(C^s\tilde{K}_i^-(u)) = (C^s\otimes C^s)\tilde{K}_i^-(u\otimes C^{-1})\tilde{K}_i^-(1\otimes uv),$$
$$\Delta_v(C^s\tilde{K}_i^+(u)) = (C^s\otimes C^s)\tilde{K}_i^+(u\otimes 1)\tilde{K}_i^+(C^{-1}\otimes uv),$$
$$\Delta_v(\tilde{H}_i^-(u)) = \tilde{H}_i^-(u\otimes C^{-1})\tilde{H}_i^-(1\otimes uv),$$
$$\Delta_v(X_i^+(u)) = X_i^+(u\otimes 1) + \tilde{K}_i^+(u\otimes 1)X_i^+(C^{-1}\otimes uv)$$
$$\Delta_v(X_i^-(u)) = X_i^-(1\otimes uv) + X_i^-(u\otimes C^{-1})\tilde{K}_i^-(1\otimes uv).$$

Remark that k_β, $H_{i,r}$, $\tilde{H}_{i,r}^{\pm}$ are mapped in $\mathcal{U}\otimes\mathcal{U}$ by Δ_v. On the other hand the elements $\Delta_v(X_{i,r}^{\pm})$ belong to $\mathcal{U}\hat{\otimes}\mathcal{U}$ but not to $\mathcal{U}\otimes\mathcal{U}$.

Remark 4.6. Remark that $\mathcal{X}$ is Ω-stable and that $\Delta_v\circ\Omega = \hat{\Omega}^{(2)}\circ\Delta_v$.
Moreover $\mathcal{X}$ is t_ω-stable and $\Delta_v\circ t_\omega = t_\omega^{\hat{\otimes}2}\circ\Delta_v$ (for all $\omega\in P$).

Proposition 4.7. *The relations*

$$(C),\ (K),\ (\tilde{H}),\ (X),\ (KX),\ (H),\ (H\tilde{H}),\ (HX+),\ (HX-),\ (X\pm),\ (\tilde{K})$$

are preserved by Δ_v *(see also [19]).*

Proof. The proof that the relations (C), (K), $(\tilde{H})$, (X), (KX), (H), $(H\tilde{H})$ and $(\tilde{K})$ are preserved by Δ_v is immediate, and left to the reader.

$(HX+)$: $\Delta_v(\tilde{H}_i^+(u_1))\Delta_v(X_j^+(u_2)) =$

$$\begin{aligned}
&= \tilde{H}_i^+(u_1\otimes 1)\tilde{H}_i^+(C^{-1}\otimes u_1v)X_j^+(u_2\otimes 1) +\\
&+ \tilde{H}_i^+(u_1\otimes 1)\tilde{H}_i^+(C^{-1}\otimes u_1v)\tilde{K}_j^+(u_2\otimes 1)X_j^+(C^{-1}\otimes u_2v) =\\
&= \tilde{H}_i^+(u_1\otimes 1)X_j^+(u_2\otimes 1)\tilde{H}_i^+(C^{-1}\otimes u_1v) +\\
&+ \tilde{H}_i^+(u_1\otimes 1)\tilde{K}_j^+(u_2\otimes 1)\tilde{H}_i^+(C^{-1}\otimes u_1v)X_j^+(C^{-1}\otimes u_2v) =\\
&= X_j^+(u_2\otimes 1)\tilde{H}_i^+(u_1\otimes 1)\mathcal{B}_{ij}(q;u_1^{-1}u_2\otimes 1)\tilde{H}_i^+(C^{-1}\otimes u_1v) +\\
&+ \tilde{K}_j^+(u_2\otimes 1)\tilde{H}_i^+(u_1\otimes 1)X_j^+(C^{-1}\otimes u_2v)\tilde{H}_i^+(C^{-1}\otimes u_1v)\mathcal{B}_{ij}(q;1\otimes u_1^{-1}u_2) =\\
&= \Delta_v(X_j^+(u_2))\Delta_v(\tilde{H}_i^+(u_1))\mathcal{B}_{ij}(q,u_1^{-1}u_2);
\end{aligned}$$

together with the Ω-equivariance of Δ_v, this proves that $(HX+)$ is preserved.

$(HX-)$: $\Delta_v(\tilde{H}_i^-(u_1))\Delta_v(X_j^+(u_2)) =$

$$\begin{aligned}
&= \tilde{H}_i^-(u_1\otimes C^{-1})\tilde{H}_i^-(1\otimes u_1v)X_j^+(u_2\otimes 1) +\\
&+ \tilde{H}_i^-(u_1\otimes C^{-1})\tilde{H}_i^-(1\otimes u_1v)\tilde{K}_j^+(u_2\otimes 1)X_j^+(C^{-1}\otimes u_2v) =
\end{aligned}$$

$$
\begin{aligned}
&= \tilde{H}_i^-(u_1 \otimes C^{-1}) X_j^+(u_2 \otimes 1) \tilde{H}_i^-(1 \otimes u_1 v) + \\
&+ \tilde{H}_i^-(u_1 \otimes C^{-1}) \tilde{K}_j^+(u_2 \otimes 1) \tilde{H}_i^-(1 \otimes u_1 v) X_j^+(C^{-1} \otimes u_2 v) = \\
&= X_j^+(u_2 \otimes 1) \tilde{H}_i^-(u_1 \otimes C^{-1}) \mathcal{B}_{ij}(q, C^{-1} u_1 u_2^{-1} \otimes C^{-1})^{-1} \tilde{H}_i^-(1 \otimes u_1 v) + \\
&+ \tilde{K}_j^+(u_2 \otimes 1) \tilde{H}_i^-(u_1 \otimes C^{-1}) \frac{\mathcal{B}_{ij}(q; C u_1 u_2^{-1} \otimes C^{-1})}{\mathcal{B}_{ij}(q; C^{-1} u_1 u_2^{-1} \otimes C^{-1})} \cdot \\
&\cdot X_j^+(C^{-1} \otimes u_2 v) \tilde{H}_i^-(1 \otimes u_1 v) \mathcal{B}_{ij}(q, C \otimes C^{-1} u_1 u_2^{-1})^{-1} = \\
&= \Delta_v(X_j^+(u_2)) \Delta_v(\tilde{H}_i^-(u_1)) \Delta_v(\mathcal{B}_{ij}(q, C^{-1} u_1 u_2^{-1})^{-1});
\end{aligned}
$$

together with the Ω-equivariance of Δ_v, this proves that $(HX-)$ is preserved.

$(X\pm)$: $[\Delta_v(X_i^+(u_1)), \Delta_v(X_j^-(u_2))] =$

$$
\begin{aligned}
&= [X_i^+(u_1 \otimes 1) + \tilde{K}_i^+(u_1 \otimes 1) X_i^+(C^{-1} \otimes u_1 v), \\
&, X_j^-(1 \otimes u_2 v) + X_j^-(u_2 \otimes C^{-1}) \tilde{K}_j^-(1 \otimes u_2 v)] = \\
&= [X_i^+(u_1 \otimes 1), X_j^-(u_2 \otimes C^{-1})] \tilde{K}_j^-(1 \otimes u_2 v) + \\
&+ \tilde{K}_i^+(u_1 \otimes 1) [X_i^+(C^{-1} \otimes u_1 v), X_j^-(1 \otimes u_2 v)] + \\
&+ \tilde{K}_i^+(u_1 \otimes 1) X_j^-(u_2 \otimes C^{-1}) X_i^+(C^{-1} \otimes u_1 v) \tilde{K}_j^-(1 \otimes u_2 v) + \\
&- X_j^-(u_2 \otimes C^{-1}) \tilde{K}_i^+(u_1 \otimes 1) \tilde{K}_j^-(1 \otimes u_2 v) X_i^+(C^{-1} \otimes u_1 v) = \\
&= \frac{\delta_{ij}}{q_i - q_I^{-1}} \Big(\tilde{K}_i^+(u_1 \otimes 1) \delta(C u_1^{-1} u_2 \otimes C^{-1}) \tilde{K}_j^-(1 \otimes u_2 v) + \\
&- \tilde{K}_i^-(u_2 \otimes C^{-1}) \delta(C u_1 u_2^{-1} \otimes C) \tilde{K}_j^-(1 \otimes u_2 v) + \\
&+ \tilde{K}_i^+(u_1 \otimes 1) \tilde{K}_i^+(C^{-1} \otimes u_1 v) \delta(C \otimes C u_1^{-1} u_2) + \\
&- \tilde{K}_i^+(u_1 \otimes 1) \tilde{K}_i^-(1 \otimes u_2 v) \delta(C^{-1} \otimes C u_1 u_2^{-1}) \Big) + \\
&+ q_i^{-a_{ij}} X_j^-(u_2 \otimes C^{-1}) \tilde{K}_i^+(u_1 \otimes 1) \mathcal{B}_{ij}(q, C u_1^{-1} u_2 \otimes C^{-1})^{-1} \cdot \\
&\cdot X_i^+(C^{-1} \otimes u_1 v) \tilde{K}_j^-(1 \otimes u_2 v) + \\
&- q_i^{-a_{ij}} X_j^-(u_2 \otimes C^{-1}) \tilde{K}_i^+(u_1 \otimes 1) \cdot \\
&\cdot X_i^+(C^{-1} \otimes u_1 v) \tilde{K}_j^-(1 \otimes u_2 v) \mathcal{B}_{ji}(q, C \otimes C^{-1} u_1^{-1} u_2)^{-1} = \\
&= \frac{\delta_{ij}}{q_i - q_I^{-1}} \Big((k_i \otimes k_i^{-1}) \tilde{H}_i^+(u_1 \otimes 1) \tilde{H}_i^-(1 \otimes u_2 v) \cdot \\
&\cdot \big(\delta((C \otimes C^{-1}) u_1^{-1} u_2) - \delta((C^{-1} \otimes C) u_1 u_2^{-1})\big) + \\
&- (k_i^{-1} \otimes k_i^{-1}) \tilde{H}_i^-(u_2 \otimes C^{-1}) \tilde{H}_i^-(1 \otimes u_2 v) \delta((C \otimes C) u_1 u_2^{-1}) + \\
&+ (k_i \otimes k_i) \tilde{H}_i^+(u_1 \otimes 1) \tilde{H}_i^+(C^{-1} \otimes u_1 v) \delta((C \otimes C) u_1^{-1} u_2) \Big) =
\end{aligned}
$$

$$= \frac{\delta_{ij}}{q_i - q_I^{-1}}\Big(\Delta_v(k_i)\Delta_v(\tilde{H}_i^+(u_1))\Delta_v(\delta(Cu_1^{-1}u_2)) + \\ - \Delta_v(k_i)^{-1}\Delta_v(\tilde{H}_i^-(u_2))\Delta_v(\delta(Cu_1u_2^{-1}))\Big);$$

this proves that $(X\pm)$ is preserved. □

In order to prove that Δ_v defines a $\mathbb{C}(q)$-algebra homomorphism, we are left to show that the relations (XX), (XXX) and (S) are preserved. (XX) is easily checked (see [19]) and (S) has been proven when $a_{ij}a_{ji} \leq 3$ (see [9] and [13]), but in general the expression for the coproduct applied to the Serre relations is extremely complicated.

In the following we propose a strategy to bypass this problem, which provides a proof that Δ_v is well defined on $\mathcal{U}^+$ so that in particular it preserves all the relations holding in $\mathcal{U}^+$ (and in $\mathcal{U}^-$).

5. Strategy

Let $\mathcal{A}$ be an associative algebra with 1 over a field of characteristic zero and let $D : \mathcal{A} \to \mathcal{A}$ be a locally nilpotent derivation. It is well known that $\exp(D) : \mathcal{A} \to \mathcal{A} = \sum_{n\geq 0} \frac{D^n}{n!}$ is a well defined algebra automorphism of $\mathcal{A}$. It is also well known that if D, D' are two commuting locally nilpotent derivations then $D + D'$ is a locally nilpotent derivation and $\exp(D + D') = \exp(D)\exp(D')$.

Recall that we want to prove that Δ_v defines an algebra homomorphism $\mathcal{U} \to \mathcal{U}\hat{\otimes}\mathcal{U}$, and that to this aim thanks to Proposition 4.7 it is enough to prove that Δ_v defines an algebra homomorphism $\mathcal{U}^+ \to \mathcal{U}\hat{\otimes}\mathcal{U}$.

This goal will be achieved by constructing a $\mathbb{C}(q)$-subalgebra $\mathcal{V}$ of $\mathcal{U}\hat{\otimes}\mathcal{U}$ containing $\mathcal{U}^+\otimes\mathbb{C}(q)$, and a locally nilpotent derivation $D : \mathcal{V} \to \mathcal{V}$ such that

$$\exp(D)(X_i^+(u)\otimes 1) = \Delta_v(X_i^+(u));$$

since the composition

$$\mathcal{U}^+ \cong \mathcal{U}^+\otimes\mathbb{C}(q) \subseteq \mathcal{V} \stackrel{\exp(D)}{\longrightarrow} \mathcal{V} \subseteq \mathcal{U}\hat{\otimes}\mathcal{U}$$

is a well defined algebra homomorphism, this implies at once that Δ_v preserves all the relations involving only the $X_{i,r}^+$'s.

Of course if $a \in \mathcal{A}$ is such that $D^2(a) = 0$ then $\exp(D)(a) = a + D(a)$. Hence, regarding the expression

$$\Delta_v(X_i^+(u)) = X_i^+(u\otimes 1) + \tilde{K}_i^+(u\otimes 1)X_i^+(C^{-1}\otimes uv)$$

it turns out that if the locally nilpotent derivation D is such that

$$D(X_i^+(u\otimes 1))=\tilde{K}_i^+(u\otimes 1)X_i^+(C^{-1}\otimes uv)\ \ \forall i\in I$$

(so that in particular $\tilde{K}_i^+(u\otimes 1)X_i^+(C^{-1}\otimes uv)\in\mathcal{V}[[u^{\pm 1}]]$) and

$$D(\tilde{K}_i^+(u\otimes 1)X_i^+(C^{-1}\otimes uv))=0,$$

then

$$\exp(D)(X_i^+(u\otimes 1))=\Delta_v(X_i^+(u)).$$

We make the idea of the proof more precise by requiring the derivation D to be the sum of n pairwise commuting locally nilpotent derivations D_j $(j\in I)$ such that for all $i\in I$

$$D_j(X_i^+(u\otimes 1))=\delta_{ij}\tilde{K}_i^+(u\otimes 1)X_i^+(C^{-1}\otimes uv), \tag{5.1}$$

$$D_j(\tilde{K}_i^+(u\otimes 1)X_i^+(C^{-1}\otimes uv))=0. \tag{5.2}$$

Remark 5.3. Let $\tilde{\mathcal{V}}\subseteq\mathcal{U}\hat{\otimes}\mathcal{U}$ be a subalgebra containing $\mathcal{U}^+\otimes\mathbb{C}(q)$, let $D_j:\tilde{\mathcal{V}}\to\tilde{\mathcal{V}}$ $(j\in I)$ be derivations such that (5.1) and (5.2) hold, and let $\mathcal{V}$ be the $\mathbb{C}(q)$-subalgebra of $\mathcal{U}\hat{\otimes}\mathcal{U}$ generated by

$$\{X_i^+(u\otimes 1),\tilde{K}_i^+(u\otimes 1)X_i^+(C^{-1}\otimes uv)|i\in I\}.$$

Then it is trivial to see that:

i) $\mathcal{U}^+\otimes\mathbb{C}(q)\subseteq\mathcal{V}\subseteq\tilde{\mathcal{V}}$;
ii) $\mathcal{V}$ is D_j-stable for all $j\in I$;
iii) $D_j|_{\mathcal{V}}:\mathcal{V}\to\mathcal{V}$ is a locally nilpotent derivation for all $j\in I$ (since it is locally nilpotent on the generators of $\mathcal{V}$);
iv) the $D_j|_{\mathcal{V}}$'s are mutually commuting derivations, as can be seen immediately by evaluating $[D_j,D_{j'}]$ on the generators of $\mathcal{V}$.

Hence the first step of our strategy (finding a suitable $\mathcal{V}$) is done:

Definition 5.4. $\mathcal{V}$ is the $\mathbb{C}(q)$-subalgebra of $\mathcal{U}\hat{\otimes}\mathcal{U}$ generated by

$$\{X_i^+(u\otimes 1),\tilde{K}_i^+(u\otimes 1)X_i^+(C^{-1}\otimes uv)|i\in I\}.$$

Remark 5.5. Notice that $\mathcal{V}$ is a Q-graded subalgebra of $\mathcal{U}\hat{\otimes}\mathcal{U}$ because

$$X_{i,r}^+\otimes 1,\quad \tilde{K}_i^+(u\otimes 1)X_i^+(C^{-1}\otimes uv)\rfloor_{u^{-r}}\in(\mathcal{U}\hat{\otimes}\mathcal{U})_{\alpha_i+r\delta}$$

and a Q_0-graded algebra with

$$X_i^+(u\otimes 1),\quad \tilde{K}_i^+(u\otimes 1)X_i^+(C^{-1}\otimes uv)\in(\mathcal{U}\hat{\otimes}\mathcal{U})_{[\alpha_i]}[[u^{\pm 1}]].$$

The second step of our strategy is to provide for all $j\in I$ the derivation $D_j:\mathcal{V}\to\mathcal{V}$ satisfying (5.1) and (5.2); the goal will be achieved constructing D_j as a deformation of a bracket.

Remark 5.6. Let us compare the term $\delta_{ij}\tilde{K}_i^+(u\otimes 1)$ appearing in the equation (5.1) with the commutator by $X_{j,-r}^-$:

$$[X_{j,-r}^-,X_i^+(u)]=-\frac{\delta_{ij}}{q_1-q_1^{-1}}\left(C^r u^{-r}\tilde{\mathbf{K}}_i^+(\mathbf{u})-C^{-r}u^{-r}\tilde{K}_i^-(Cu)\right),\tag{5.7}$$

so that (5.1) can be written as

$$D_j(X_i^+(u\otimes 1))=\delta_{ij}\sum_{r\in\mathbb{Z}}C^r u^{-r}\tilde{K}_i^+(u\otimes 1)(1\otimes X_{i,r}^+v^{-r})=$$

$$=\sum_{r\in\mathbb{Z}}\left(-(q_i-q_i^{-1})[X_{j,-r}^-,X_i^+(u)]+\delta_{ij}C^{-r}u^{-r}\tilde{K}_i^-(Cu)\right)\otimes X_{j,r}^+v^{-r}=$$

$$=-(q_i-q_i^{-1})\sum_{r\in\mathbb{Z}}[X_{j,-r}^-\otimes X_{j,r}^+v^{-r},X_i^+(u\otimes 1)]+\delta_{ij}\tilde{\mathbf{K}}_{\mathbf{i}}^-(\mathbf{Cu}\otimes\mathbf{1})\mathbf{X}_{\mathbf{i}}^+(\mathbf{C}\otimes\mathbf{uv}).$$

Remark that for all $r\in\mathbb{Z}$ the commutator $[X_{j,-r}^-\otimes X_{j,r}^+,\cdot]$ (which, using the v-notation for $\mathcal{U}\otimes\mathcal{U}\subseteq\mathcal{U}\hat{\otimes}\mathcal{U}\subseteq\mathcal{U}\otimes\mathcal{U}((v))$, is equal to $[X_{j,-r}^-\otimes X_{j,r}^+v^{-r},\cdot]$) is obviously a derivation of $\mathcal{U}\otimes\mathcal{U}$ and of $\mathcal{U}\hat{\otimes}\mathcal{U}$.

On the other hand remark that $\sum_{r\in\mathbb{Z}}[X_{j,r}^-\otimes X_{j,-r}^+v^r,\cdot]$ maps $\mathcal{U}\otimes\mathcal{U}$ (and even $\mathcal{U}^+\otimes\mathbb{C}(q)$) to $\mathcal{U}\otimes\mathcal{U}[[v^{\pm 1}]]$ and not to $\mathcal{U}\otimes\mathcal{U}((v))$.

Remark 5.8. The element $\sum_{r\in\mathbb{Z}}X_{j,r}^-\otimes X_{j,-r}^+v^r\in\mathcal{U}\otimes\mathcal{U}[[v^{\pm 1}]]$ is the coefficient of w^0 in $X_j^-(w)\otimes X_j^+(wv)\in\mathcal{U}\otimes\mathcal{U}[[v^{\pm 1},w^{\pm 1}]]$.

Remark that $[X_j^-(w)\otimes X_j^+(wv),\cdot]:\mathcal{U}\otimes\mathcal{U}\to\mathcal{U}\otimes\mathcal{U}[[v^{\pm 1},w^{\pm 1}]]$. It is not defined on $\mathcal{U}\hat{\otimes}\mathcal{U}\subseteq\mathcal{U}\otimes\mathcal{U}((v))$. We shall avoid this problem by considering the element $X_j^-(w)\otimes X_j^+(wxv)$ instead of $X_j^-(w)\otimes X_j^+(wv)$; in this way we get a well defined map

$$[X_j^-(w)\otimes X_j^+(wxv),\cdot]:\mathcal{U}\otimes\mathcal{U}[[v^{\pm 1}]]\to\mathcal{U}\otimes\mathcal{U}[[x^{\pm 1},v^{\pm 1},w^{\pm 1}]].$$

Remark also that $\mathcal{U}\otimes\mathcal{U}[[v^{\pm 1}]]$ and $\mathcal{U}\otimes\mathcal{U}[[x^{\pm 1},v^{\pm 1},w^{\pm 1}]]$ are left and right $\mathcal{U}\otimes\mathcal{U}$-modules but do not have a structure of $\mathcal{U}\otimes\mathcal{U}((v))$-modules (and $\mathcal{U}\otimes\mathcal{U}[[v^{\pm 1}]]$ is not a $\mathbb{C}(q)$-algebra), while $\mathcal{U}\otimes\mathcal{U}[x^{\pm 1}]((v))[[w^{\pm 1}]]$ is a left and right $\mathcal{U}\otimes\mathcal{U}((v))$-module (in particular it is a $\mathcal{V}$-module).

The idea is to deform the bracket $[X_j^-(w)\otimes X_j^+(wxv),\cdot]$ to a map $D_j(w,x)$ whose restriction to $\mathcal{V}$ takes values in $\mathcal{U}\otimes\mathcal{U}[x^{\pm1}]((v))[[w^{\pm1}]]$ and can thus be composed with the evaluation of x at 1, providing a map with values in $\mathcal{U}\otimes\mathcal{U}((v))[[w^{\pm1}]]$.

Then we shall prove that the map $D_j = ev_{x=1}\circ(D_j(w,x)|_{\mathcal{V}})\rfloor_{w^0}$ (see Notation 2.4) is a derivation satisfying (5.1) and (5.2).

6. The deformation $D_j(w,x)$ of $[X_j^-(w)\otimes X_j^+(wxv),\cdot]$

As suggested in Remark 5.6, we want to get rid of the terms $\tilde{K}_{i,r}^-$ arising from the commutation of $X_j^-(w)$ with $X_i^+(u)$.

In this section we construct some projections of $\mathcal{U}$ in itself that allow us to ignore these terms.

It is worth repeating that these structures and projections, fundamental for our argument, rely on the triangular decomposition of $\mathcal{U}$ (see Section 0 and Remarks 2.17 and 2.18).

Definition 6.1. Let us define the following subspaces of $\mathcal{U}$:

$\mathcal{U}^>$ is the $\mathbb{C}(q)$-subalgebra generated by $\mathcal{U}^+$ and $\mathcal{U}^0$;

$\mathcal{U}^{>,K}$ is the $\mathbb{C}(q)$-subalgebra generated by $\mathcal{U}^>$ and $\mathcal{U}^K$;

$\mathcal{U}^< = \Omega(\mathcal{U}^>)$ and $\mathcal{U}^{<,K} = \Omega(\mathcal{U}^{>,K})$;

$\mathcal{M}^- = \sum_{i\in I,r\in\mathbb{Z}}\mathcal{U}X_{i,r}^-$ is the $\mathcal{U}$-submodule (left ideal) generated by $\{X_{i,r}^-|i\in I, r\in\mathbb{Z}\}$.

Remark 6.2. $\mathcal{U}^>$, $\mathcal{U}^{>,K}$, $\mathcal{U}^<$, $\mathcal{U}^{<,K}$ and $\mathcal{M}^-$ are Q-graded subspaces of $\mathcal{U}$.

Moreover the relations defining $\mathcal{U}$ and its triangular decomposition imply that

$$\mathcal{U}^>\cong\mathcal{U}^+\otimes\mathcal{U}^0;$$
$$\mathcal{U}^{>,K}\cong\mathcal{U}^+\otimes\mathcal{U}^0\otimes\mathcal{U}^K;$$
$$\mathcal{U}\cong\mathcal{U}^>\otimes\mathbb{C}(q)[k_i^{\pm1}|i\in I]\otimes\mathcal{U}^-\cong\mathcal{U}^+\otimes\mathbb{C}(q)[k_i^{\pm1}|i\in I]\otimes\mathcal{U}^<;$$
$$\mathcal{U}\cong\mathcal{U}^{>,K}\oplus\mathcal{M}^-.$$

Definition 6.3. For all $\gamma\in Q_0$ define $\mathcal{U}{<}\gamma{>}$ to be the following subspace of $\mathcal{U}$:

$$\mathcal{U}{<}\gamma{>} = \mathcal{U}^>k_\gamma\mathcal{U}^- = \mathcal{U}^+k_\gamma\mathcal{U}^<.$$

This defines a structure of Q_0-graded vector space on $\mathcal{U}$ (that we shall refer to as "the new Q_0-gradation" of $\mathcal{U}$):

$$\mathcal{U} = \bigoplus_{\gamma\in Q_0}\mathcal{U}{<}\gamma{>}.$$

Define $p_\gamma : \mathcal{U} \to \mathcal{U}{<}\gamma{>} \subseteq \mathcal{U}$ to be the projection on the γ-component of $\mathcal{U}$.

Moreover define $\pi : \mathcal{U} \to \mathcal{U}$ to be the composition of morphisms of left $\mathcal{U}^{>,K}$-modules $\mathcal{U} \to \mathcal{U}/\mathcal{M}^- \cong \mathcal{U}^{>,K} \subseteq \mathcal{U}$.

Remark 6.4. The new Q_0-gradation of $\mathcal{U}$ has the properties that, for all $\gamma, \gamma' \in Q_0$:

i) $k_{\gamma'}\mathcal{U}{<}\gamma{>} = \mathcal{U}{<}\gamma+\gamma'{>} = \mathcal{U}{<}\gamma{>}k_{\gamma'}$.
ii) $\mathcal{U}{<}\gamma{>}$ is a left $\mathcal{U}^{>}$-submodule and a right $\mathcal{U}^{<}$-submodule of $\mathcal{U}$.

Equivalently, $\forall \gamma, \gamma', \gamma'' \in Q_0$ we have that:

iii) $p_\gamma(k_{\gamma'} a k_{\gamma''}) = k_{\gamma'} p_{\gamma-\gamma'-\gamma''}(a) k_{\gamma''}$.
iv) p_γ is a morphism of left $\mathcal{U}^{>}$-modules and a morphism of right $\mathcal{U}^{<}$-modules.

Moreover:

v) $\mathcal{U}^{>,K}$ and $\mathcal{M}^-$ are Q_0-graded subspaces of $\mathcal{U}$.

Equivalently:

vi) π commutes with all the p_γ's.

Finally:

vii) for all $\beta \in Q$, $\mathcal{U}_\beta$ is Q_0-graded with respect to the new Q_0-gradation:

$$\mathcal{U}_\beta = \oplus_{\gamma \in Q_0}(\mathcal{U}_\beta \cap \mathcal{U}{<}\gamma{>}),$$

that is the Q-gradation and the new Q_0-gradation of $\mathcal{U}$ are compatible and define a $(Q \times Q_0)$-gradation of $\mathcal{U}$.

Remark 6.5. 1) With the new Q_0-gradation just defined, $\mathcal{U}$ is not a Q_0-graded algebra: for example

$$X_i^{\pm}(u) \in \mathcal{U}{<}0{>}$$

but

$$0 \neq [X_i^+(u_1), X_i^-(u_2)] \in \mathcal{U}{<}\alpha_i{>} \oplus \mathcal{U}{<}-\alpha_i{>}.$$

2) Viceversa $\mathcal{U}^{>,K}$ is a Q_0-graded algebras (and so is $\mathcal{U}^{<,K}$). In particular

$$\forall \gamma, \gamma', \gamma'' \in Q_0, \quad \forall a, a', a'' \in \mathcal{U}^{>,K} \text{ with } a' \in \mathcal{U}{<}\gamma'{>}, \quad a'' \in \mathcal{U}{<}\gamma''{>},$$

we have

$$p_\gamma(a'aa'') = a' p_{\gamma-\gamma'-\gamma''}(a) a''.$$

3) Since $\mathcal{V} \subseteq (\mathcal{U}^{>,K} \otimes \mathcal{U}^+)((v)) \cap \mathcal{U} \hat{\otimes} \mathcal{U}$, the new Q_0-gradation induces on $\mathcal{V}$ a new structure of Q_0-graded algebra:

$$\mathcal{V}{<}\gamma{>} = \mathcal{V} \cap (\mathcal{U}{<}\gamma{>} \otimes \mathcal{U}^+)((v)).$$

Notice that $\mathcal{V}{<}\gamma{>} \subseteq \mathcal{U}{<}\gamma{>} \otimes \mathcal{U}^+_{[\gamma]}((v))$.

Definition 6.6. For all $j \in I$ let $D_j^0(w,x) : \mathcal{U} \otimes \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}[[(xv)^{\pm 1}, w^{\pm 1}]]$ be the map defined by

$$D_j^0(w,x)\Big|_{\mathcal{U}<\gamma>\otimes\mathcal{U}} = -(q_j - q_j^{-1})((\pi \circ p_{\gamma+\alpha_j}) \otimes id_{\mathcal{U}}) \circ [X_j^-(w) \otimes X_j^+(wxv), \cdot];$$
$$D_j(w,x) : \mathcal{U} \otimes \mathcal{U}[[v^{\pm 1}]] \to \mathcal{U} \otimes \mathcal{U}[[(xv)^{\pm 1}, w^{\pm 1}, v^{\pm 1}]] = \mathcal{U} \otimes \mathcal{U}[[x^{\pm 1}, v^{\pm 1}, w^{\pm 1}]]$$

is the map defined by

$$D_j(w,x)\left(\sum_{r\in\mathbb{Z}} a_r v^{-r}\right) = \sum_{r\in\mathbb{Z}} D_j^0(w,x)(a_r)v^{-r}.$$

Remark 6.7. Since $\pi(\mathcal{U}) \subseteq \mathcal{U}^{>,K}$, $\pi(\mathcal{M}^-) = 0$ and $\mathcal{M}^-\mathcal{U}^{0,K} \subseteq \mathcal{M}^-$, we have that

$$\pi(X_j^-(w)(\mathcal{M}^- + \mathcal{U}^{0,K})) = 0,$$

hence

$$D_j(w,x)(\mathcal{U} \otimes \mathcal{U}[[v^{\pm 1}]]) \subseteq \mathcal{U}^{>,K} \otimes \mathcal{U}[[x^{\pm 1}, v^{\pm 1}, w^{\pm 1}]],$$
$$D_j(w,x)((\mathcal{M}^- + \mathcal{U}^{0,K}) \otimes \mathcal{U}[[v^{\pm 1}]]) = 0$$

and for all $a \otimes a' \in \mathcal{U}{<}\gamma{>} \otimes \mathcal{U}$,

$$D_j^0(w,x)(a \otimes a') = -(q_j - q_j^{-1})(\pi \circ p_{\gamma+\alpha_j})(X_j^-(w)a) \otimes X_j^+(wxv)a'.$$

Lemma 6.8. *Let $\gamma \in Q_0$, $a \in \mathcal{U}{<}\gamma{>}$. There exists $R \in \mathbb{Z}$ such that*

$$\pi p_{\gamma+\alpha_j}(X_{j,r}^- a) = 0 \quad \forall r < R.$$

Proof. If $a \in \mathcal{M}^- + \mathcal{U}^{0,K}$ the claim is obvious (see Remark 6.7), so let $a = a^+a^0$ with $a^+ \in \mathcal{U}^+ \subseteq \mathcal{U}{<}0{>}$, $a^0 \in \mathcal{U}^{0,K}{<}\gamma{>}$.

Since by Remark 6.5, 2)

$$\pi p_{\gamma+\alpha_j}(X_{j,r}^- a^+ a^0) = \pi p_{\gamma+\alpha_j}([X_{j,r}^-, a^+]a^0 + a^+[X_{j,r}^-, a^0]) =$$
$$= \pi p_{\gamma+\alpha_j}([X_{j,r}^-, a^+]a^0) + a^+\pi p_{\gamma+\alpha_j}([X_{j,r}^-, a^0]) = \pi p_{\alpha_j}([X_{j,r}^-, a^+])a^0,$$

it is enough to prove the claim for $a \in \mathcal{U}^+$.

Now let $a = X^+_{i_1,r_1} \cdot \ldots \cdot X^+_{i_N,r_N}$ and $R = -max\{r_h | i_h = j\}$. Then for all $r < R$ and $h = 1, \ldots, N$ we have either

$$i_h \neq j \ \text{ and } \ [X^-_{j,r}, X^+_{i_h,r_h}] = 0$$

or

$$r + r_h < 0 \ \text{ and } \ p_{\alpha_j}([X^-_{j,r}, X^+_{i_h,r_h}]) = 0.$$

The claim follows. $\square$

Corollary 6.9. $D_j^0(w,x)(\mathcal{U}\otimes\mathcal{U}) \subseteq \mathcal{U}\otimes\mathcal{U}((xv))[[w^{\pm 1}]] \subseteq \mathcal{U}\otimes\mathcal{U}[x^{\pm 1}]((v))[[w^{\pm 1}]]$.

$D_j(w,x)(\mathcal{U}\otimes\mathcal{U}[v^{\pm 1}]^{(0)}) \subseteq \mathcal{U}\otimes\mathcal{U}[x^{\pm 1}]((v))[[w^{\pm 1}]]$.

Proof. It is enough to prove that

$$D_j^0(w,x)(a'\otimes a'') \in \mathcal{U}\otimes\mathcal{U}((xv))[[w^{\pm 1}]] \ \ \forall a' \in \mathcal{U}_{<\gamma>}, a'' \in \mathcal{U}$$

and to recall that $\mathcal{U}\otimes\mathcal{U}[x^{\pm 1}]((v))[[w^{\pm 1}]]$ is a $\mathbb{C}(q)[v^{\pm 1}]$-module.

Let $R \in \mathbb{Z}$ be such that $\pi p_{\gamma+\alpha_j}(X^-_{j,r}a') = 0$ for all $r < R$. Then

$$-\frac{1}{q_j - q_j^{-1}} D_j^0(w,x)(a'\otimes a'') =$$
$$= \sum_{r,s\in\mathbb{Z}} \pi p_{\gamma+\alpha_j}(X^-_{j,r}a') \otimes X^+_{j,s}a'' x^{-s}v^{-s}w^{-r-s} =$$
$$= \sum_{r\geq R, s\in\mathbb{Z}} \pi p_{\gamma+\alpha_j}(X^-_{j,r}a') \otimes X^+_{j,s}a'' x^{-s}v^{-s}w^{-r-s} =$$
$$= \sum_{r\geq R, s\in\mathbb{Z}} \pi p_{\gamma+\alpha_j}(X^-_{j,r}a') \otimes X^+_{j,s-r}a'' x^{-s+r}v^{-s+r}w^{-s}$$

which belongs to $\mathcal{U}\otimes\mathcal{U}((xv))[[w^{\pm 1}]]$. $\square$

Remark 6.10. Notice that

$$D_j(w,x)(\mathcal{U}\hat{\otimes}\mathcal{U}) \not\subseteq \mathcal{U}\otimes\mathcal{U}[x^{\pm 1}]((v))[[w^{\pm 1}]].$$

For example

$$\sum_{r\geq 0} X^+_{i,2r}\tilde{H}^-_{i,-r} \otimes \tilde{H}^-_{i,-r} v^r$$

is an element of $(\mathcal{U}\hat{\otimes}\mathcal{U})_{[\alpha_i,0;0]}$ but

$$D_i(w,x)\big|_{w^0}\left(\sum_{r\geq 0} X^+_{i,2r}\tilde{H}^-_{i,-r} \otimes \tilde{H}^-_{i,-r} v^r\right) =$$

$$= D_i(w,x)\big(X_i^+(u_1u_2)\tilde H_i^-(u_1u_2^2)\otimes \tilde H_i^-(u_1v)\big)\big|_{w^0,u_1^0,u_2^0} =$$
$$= \sum_{\substack{r,s\in\mathbb{Z}:\\ s\geq max(r,2r)}} k_i\tilde H^+_{i,s-2r}C^s\tilde H^-_{i,r-s}\otimes X^+_{i,s}\tilde H^-_{i,r-s}x^{-s}v^{-r}$$

which does not even belong to $\mathcal{U}\otimes\mathcal{U}[x^{\pm1}][[v^{\pm1}]]$ (it cannot be evaluated at $x=1$).

Remark 6.11. Let $a,b\in\mathcal{U}\otimes\mathcal{U}((v))$ (or even $a,b\in\mathcal{U}\otimes\mathcal{U}$) be such that

$$D_j(w,x)(a), D_j(w,x)(b)\in\mathcal{U}\otimes\mathcal{U}[x^{\pm1}]((v))[[w^{\pm1}]].$$

In general it is **not** true that $D_j(w,x)(ab)=D_j(w,x)(a)b+aD_j(w,x)(b)$.

For example let $a=X_j^-(u\otimes 1)$, $b=X_j^+(\tilde u\otimes 1)$. Then

$$D_j(w,x)(a)=D_j(w,x)(ab)=0\ \text{ but }\ aD_j(w,x)(b)\neq 0.$$

Lemma 6.12. *Let $a\in\mathcal{U}^{>,K}{<}\gamma'{>}\otimes\mathcal{U}((v))$ and $b\in\mathcal{U}^{>,K}{<}\gamma''{>}\otimes\mathcal{U}((v))$ be such that*

$$D_j(w,x)(a), D_j(w,x)(b)\in\mathcal{U}\otimes\mathcal{U}[x^{\pm1}]((v))[[w^{\pm1}]].$$

Then

$$D_j(w,x)(ab)=D_j(w,x)(a)b+aD_j(w,x)(b).$$

In particular the subspace $\bar{\mathcal{V}}=\oplus_{\gamma\in Q_0}\bar{\mathcal{V}}{<}\gamma{>}$ of $\mathcal{U}\otimes\mathcal{U}((v))$ defined by

$$\bar{\mathcal{V}}{<}\gamma{>}=\{a\in\mathcal{U}^{<,K}{<}\gamma{>}\otimes\mathcal{U}((v))|D_j(w,x)(a)\in\mathcal{U}\otimes\mathcal{U}[x^{\pm1}]((v))[[w^{\pm1}]]\}$$

is a subalgebra of $\mathcal{U}\otimes\mathcal{U}((v))$ and $D_j(w,x)|_{\bar{\mathcal{V}}}:\bar{\mathcal{V}}\to\mathcal{U}\otimes\mathcal{U}((v))$ is a derivation.

Proof. Since $\mathcal{U}\otimes\mathcal{U}((v))$ is a $\mathbb{C}(q)$-algebra, ab, $D_j(w,x)(a)b$ and $aD_j(w,x)(b)$ are well defined. Remark 6.5, 2) implies that

$$-\frac{1}{q_j-q_j^{-1}}D_j(w,x)(ab)=\pi p_{\gamma'+\gamma''+\alpha_j}([X_j^-(w)\otimes X_j^+(wxv),ab])=$$
$$=\pi p_{\gamma'+\gamma''+\alpha_j}([X_j^-(w)\otimes X_j^+(wxv),a]b+a[X_j^-(w)\otimes X_j^+(wxv),b])=$$
$$=\pi p_{\gamma'+\alpha_j}([X_j^-(w)\otimes X_j^+(wxv),a])b+a\pi p_{\gamma''+\alpha_j}([X_j^-(w)\otimes X_j^+(wxv),b])=$$
$$=-\frac{1}{q_j-q_j^{-1}}(D_j(w,x)(a)b+aD_j(w,x)(b)).$$
□

Proposition 6.13. *Let $i, j \in I$. Then*

i) $D_j(w,x)(\tilde{K}_i(u\otimes 1)X_i^+(C^{-1}\otimes uv)) = 0$.
ii) $D_j(w,x)(X_i^+(u\otimes 1)) = \delta_{ij}\tilde{K}_i(u\otimes 1)X_i^+(C^{-1}\otimes uvx)\delta(Cu^{-1}w\otimes 1)$.
iii) $D_j(w,x)\big|_{\mathcal{V}}$ *is a derivation from* $\mathcal{V}$ *to the* $\mathcal{V}$*-module* $\mathcal{U}\otimes\mathcal{U}[x^{\pm 1}]((v))[[w^{\pm 1}]]$.

Proof. i) follows from Remark 6.7. ii) is an immediate computation:

$$\begin{gathered} D_j(w,x)(X_i^+(u\otimes 1)) = \\ = -(q_j - q_j^{-1})\pi p_{\alpha_j}([X_j^-(w\otimes 1), X_i^+(u\otimes 1)]X_j^+(1\otimes wxv)) = \\ = \delta_{ij}\tilde{K}_i(u\otimes 1)\delta(Cu^{-1}w\otimes 1)X_i^+(1\otimes wxv) \\ = \delta_{ij}\tilde{K}_i(u\otimes 1)X_i^+(C^{-1}\otimes uvx)\delta(Cu^{-1}w\otimes 1) \end{gathered}$$

Remark that

$$\begin{gathered} \tilde{K}_i(u\otimes 1)X_i^+(C^{-1}\otimes uvx)\delta(Cu^{-1}w\otimes 1) = \\ = \sum_{\substack{r,s,t:\\ r\le s+t}} k_i\tilde{H}^+_{i,s-r+t}C^{r-t}\otimes X^+_{i,r}x^{-r}v^{-r}w^{-t}u^{-s} \end{gathered}$$

is an element of $\mathcal{U}\otimes\mathcal{U}[x^{\pm 1}]((v))[[w^{\pm 1}, u^{\pm 1}]]$.

iii) follows from i), ii) and Lemma 6.12. □

Notation 6.14. For all $j \in I$ we set $D_j = ev_{x=1}\circ D_j(w,x)\rfloor_{w^0} : \mathcal{V} \to \mathcal{U}\otimes\mathcal{U}((v))$.

We can now conclude our argument.

Theorem 6.15. *Let $i, j \in I$. Then:*

$$\begin{gathered} D_j(X_i^+(u\otimes 1)) = \delta_{ij}\tilde{K}_i^+(u\otimes 1)X_i^+(C^{-1}\otimes uv); \\ D_j(\tilde{K}_i^+(u\otimes 1)X_i^+(C^{-1}\otimes uv)) = 0. \end{gathered}$$

Hence:

$D_j(\mathcal{V}) \subseteq \mathcal{V}$.

D_j *is a locally nilpotent derivation of* $\mathcal{V}$ *satisfying conditions (5.1) and (5.2).*

$D_jD_i = D_iD_j$.

Proof. Thanks to Remark 5.3 and Proposition 6.13, we only need to prove that

$$D_j(X_i^+(u\otimes 1)) = \delta_{ij}\tilde{K}_i^+(u\otimes 1)X_i^+(C^{-1}\otimes uv)$$

(which belongs to $\mathcal{V}$):

$$\begin{aligned}D_j(X_i^+(u\otimes 1)) = ev_{x=1}\circ D_j(w,x)(X_i^+(u\otimes 1))\rfloor_{w^0} =\\ = ev_{x=1}(\delta_{ij}\tilde{K}_i(u\otimes 1)X_i^+(C^{-1}\otimes uvx)\delta(Cu^{-1}w\otimes 1))\rfloor_{w^0} =\\ = \delta_{ij}\tilde{K}_i(u\otimes 1)X_i^+(C^{-1}\otimes uv).\end{aligned}$$ □

Theorem 6.16. *The map Δ_v extends (uniquely) to a homomorphism of $\mathbb{C}(q)$-algebras*

$$\Delta_v : \mathcal{U}\to\mathcal{U}\,\hat{\otimes}\,\mathcal{U}.$$

That is: the Drinfeld "coproduct" is well defined.

Proof. The claim is an immediate consequence of Proposition 4.7 and Theorem 6.16. Indeed the exponential $\exp(\sum_{j\in I}D_j)$ is a well defined algebra automorphism of $\mathcal{V}$ and the composition

$$\mathcal{U}^+\cong\mathcal{U}^+\otimes\mathbb{C}(q)\hookrightarrow\mathcal{V}\xrightarrow{\exp(\sum_{j\in I}D_j)}\mathcal{V}\hookrightarrow\mathcal{U}\,\hat{\otimes}\,\mathcal{U}$$

is an algebra homomorphism mapping $X_i^+(u)$ to $\Delta_v(X_i^+(u))$.

In particular Δ_v preserves the relations $(X)^+$, $(XX)^+$, $(XXX)^+$ and $(S)^+$ and, thanks to Remark 4.6, also the relations $(X)^-$, $(XX)^-$, $(XXX)^-$ and $(S)^-$. Together with Proposition 4.7, this implies the claim. □

Corollary 6.17. *$\Delta_v : \mathcal{U}\to\mathcal{U}^{\hat{\otimes}\,2}$ is homomorphism of Q-graded algebras.*

Δ_v commutes with Ω: $\Delta_v\circ\Omega = \hat{\Omega}^{(2)}\circ\Delta_v$.

Δ_v preserves the action of the weight lattice P: $\Delta_v\circ t_\omega = t_\omega^{\hat{\otimes}\,2}\circ\Delta_v$ for all $\omega\in P$.

7. Δ_v is a "coproduct"

Here we shall shortly show that Δ_v is "coassociative" and admits a "counit".

Since the study of the coassociativity involves $\mathcal{U}^{\otimes 3}$ and its $\mathbb{Z}$-graded completions, as we did in Recall 4.2 we start by recalling an explicit description of $\mathcal{U}^{\hat{\otimes}\,3}$ (here $\mathbf{v}=(v_1,v_2)$) and proving that this subalgebra of $\mathcal{U}^{\hat{\otimes}\,2}\,\hat{\otimes}\,\mathcal{U}$ and of $\mathcal{U}\,\hat{\otimes}\,\mathcal{U}^{\hat{\otimes}\,2}$ (and more generally $\mathcal{U}^{\hat{\otimes}\,m}$) is the correct setting where to investigate the coassociativity.

Recall 7.1. The identification of $\mathcal{U}\otimes\mathcal{U}\otimes\mathcal{U}$ with

$$\mathcal{U}\otimes\mathcal{U}\otimes\mathcal{U}[v_1^{\pm 1},v_2^{\pm 1}]^{(\mathbf{0})}\subseteq\mathcal{U}\otimes\mathcal{U}\otimes\mathcal{U}[v_1^{\pm 1},v_2^{\pm 1}]$$

can be described as follows:

$$\mathcal{U}\otimes\mathcal{U}\otimes\mathcal{U}=\bigoplus_{\substack{\gamma_2,\gamma_3\in Q_0\\ s_1,s_2\in\mathbb{Z}}}\mathcal{U}\otimes\mathcal{U}_{\gamma_2+(s_1-s_2)\delta}\otimes\mathcal{U}_{\gamma_3+s_2\delta}\cong$$
$$\cong\bigoplus_{\substack{\gamma_2,\gamma_3\in Q_0\\ s_1,s_2\in\mathbb{Z}}}\mathcal{U}\otimes\mathcal{U}_{\gamma_2+(s_1-s_2)\delta}\otimes\mathcal{U}_{\gamma_3+s_2\delta}v_1^{-s_1}v_2^{-s_2}$$

More precisely we described the $(Q^{\oplus 3})$-gradation of $\mathcal{U}^{\otimes 3}$ as a $(Q_0^{\oplus 3}\oplus\mathbb{Z})$-gradation by $\mathbb{Z}^2$-graded vector spaces:

$$\mathcal{U}^{\otimes 3}=\bigoplus_{\gamma_1,\gamma_2,\gamma_3\in Q_0,r\in\mathbb{Z}}(\mathcal{U}^{\otimes 3})_{[\gamma_1,\gamma_2,\gamma_3;r]}$$

where

$$(\mathcal{U}^{\otimes 3})_{[\gamma_1,\gamma_2,\gamma_3;r]}=\bigoplus_{(s_1,s_2)\in\mathbb{Z}^2}\mathcal{U}_{\gamma_1+(r-s_1)\delta}\otimes\mathcal{U}_{\gamma_2+(s_1-s_2)\delta}\otimes\mathcal{U}_{\gamma_3+s_2\delta}\cong$$
$$\cong\bigoplus_{(s_1,s_2)\in\mathbb{Z}^2}\mathcal{U}_{\gamma_1+(r-s_1)\delta}\otimes\mathcal{U}_{\gamma_2+(s_1-s_2)\delta}\otimes\mathcal{U}_{\gamma_3+s_2\delta}v_1^{-s_1}v_2^{-s_2}=$$
$$=\bigoplus_{r_1+r_2+r_3=r}\mathcal{U}_{\gamma_1+r_1\delta}\otimes\mathcal{U}_{\gamma_2+r_2\delta}\otimes\mathcal{U}_{\gamma_3+r_3\delta}v_1^{-r_2-r_3}v_2^{-r_3},$$

whose completion is

$$\overline{(\mathcal{U}^{\otimes 3})_{[\gamma_1,\gamma_2,\gamma_3;r]}}=\left\{\sum_{\substack{r_1+r_2+r_3=r\\ r_3\leq R,r_2+r_3\leq S}}x_{r_1,r_2,r_3}\middle|R,S\in\mathbb{Z}\right\}=$$
$$=\left\{\sum_{\substack{r_1+r_2+r_3=r\\ r_1\geq R,r_1+r_2\geq S}}x_{r_1,r_2,r_3}\middle|R,S\in\mathbb{Z}\right\}=$$
$$=\left\{\sum_{r_1+r_2+r_3=r}x_{r_1,r_2,r_3}v_1^{-r_2}(v_1v_2)^{-r_3}\in\mathcal{U}\otimes\mathcal{U}\otimes\mathcal{U}((v_1,v_2))\right\}$$

where the x_{r_1,r_2,r_3}'s are elements of $\mathcal{U}_{\gamma_1+r_1\delta}\otimes\mathcal{U}_{\gamma_2+r_2\delta}\otimes\mathcal{U}_{\gamma_3+r_3\delta}$.

The elements of $\mathcal{U}^{\hat{\otimes}\,3}$ are finite sums of elements of this form.

Lemma 7.2. *Let $\mathcal{U}[\Delta_v^b]$ be the subspace of the Δ_v-bounded elements of $\mathcal{U}$ (see Definition 3.16) and $\bar{\mathcal{X}}$ the (obviously $\mathbb{Z}$-graded) vector subspace of $\mathcal{U}$ generated by $\mathcal{X}$. Then:*

i) $\bar{\mathcal{X}}\subseteq\mathcal{U}[\Delta_v^b]$;

ii) $\Delta_v(\mathcal{X}) \subseteq \bar{\mathcal{X}} \hat{\otimes} \bar{\mathcal{X}} \subseteq \mathcal{U} \hat{\otimes} \mathcal{U}$;
iii) for all $m \geq h \geq 1$ *the homomorphism* $id^{\hat{\otimes}(h-1)} \hat{\otimes} \Delta_v \hat{\otimes} id^{\hat{\otimes}(m-h)}$ *maps* $\bar{\mathcal{X}}^{\hat{\otimes} m}$ *in* $\bar{\mathcal{X}}^{\hat{\otimes}(m+1)} (\subseteq \mathcal{U}^{\hat{\otimes}(m+1)})$.

Proof. It is a straightforward verification that the elements of $\mathcal{X}$ are Δ_v-bounded (which implies i)) and that ii) holds. iii) then follows at once from Lemma 3.17. □

Definition 7.3. Let $m \in \mathbb{N}$ and $h_1, h_2, \dots, h_m > 0$ be such that $h_i \leq i$ for all $i = 1, \dots, m$.

Define $\Delta^{(m)}_{v;h_1,\dots,h_m}$ as follows:

$$\Delta^{(0)}_v = id; \quad \Delta^{(m)}_{v;h_1,\dots,h_m} = (id^{\hat{\otimes}(h_m-1)} \hat{\otimes} \Delta_v \hat{\otimes} id^{\hat{\otimes}(m-h_m)} \circ \Delta^{(m-1)}_{v;h_1,\dots,h_{m-1}} \text{ if } m > 0.$$

Lemma 7.4. *With the notations of Definition 7.3 we have that* $\Delta^{(m)}_{v;h_1,\dots,h_m}$ *maps* $\mathcal{U}$ *in* $\mathcal{U}^{\hat{\otimes}(m+1)}$.

Proof. We prove the claim by induction on m, remarking that the claim is obvious for $m = 0$ and $m = 1$ and that Lemma 7.2, iii) implies that $\Delta^{(m)}_{v;h_1,\dots,h_m}(\bar{\mathcal{X}}) \subseteq \bar{\mathcal{X}}^{\hat{\otimes}(m+1)} \subseteq \mathcal{U}^{\hat{\otimes}(m+1)}$.

The inductive hypothesis implies that $\Delta^{(m-1)}_{v;h_1,\dots,h_{m-1}}(\mathcal{U}) \subseteq \mathcal{U}^{\hat{\otimes} m}$, so that

$$\Delta^{(m)}_{v;h_1,\dots,h_m}(\mathcal{U}) \subseteq \mathcal{U}^{\hat{\otimes}(h_m-1)} \hat{\otimes} (\mathcal{U} \hat{\otimes} \mathcal{U}) \hat{\otimes} \mathcal{U}^{\hat{\otimes}(m-h_m)}.$$

The claim now follows recalling that:
$\mathcal{X} (\subseteq \bar{\mathcal{X}})$ is a set of algebra generators of $\mathcal{U}$;
$id^{\hat{\otimes}(h_m-1)} \hat{\otimes} \Delta_v \hat{\otimes} id^{\hat{\otimes}(m-h_m)}$ is an algebra homomorphism;
$\mathcal{U}^{\hat{\otimes}(m+1)}$ is an algebra (a subalgebra of $\mathcal{U}^{\hat{\otimes}(h_m-1)} \hat{\otimes} (\mathcal{U} \hat{\otimes} \mathcal{U}) \hat{\otimes} \mathcal{U}^{\hat{\otimes}(m-h_m)}$). □

Lemma 7.5. *We have that*

$$(\Delta_v \hat{\otimes} id) \circ \Delta_v \circ \Omega = \hat{\Omega}^{(3)} \circ (id \hat{\otimes} \Delta_v) \circ \Delta_v$$

and

$$(id \hat{\otimes} \Delta_v) \circ \Delta_v \circ \Omega = \hat{\Omega}^{(3)} \circ (\Delta_v \hat{\otimes} id) \circ \Delta_v.$$

Proof. Consider the natural actions of the symmetric groups $\mathcal{S}_2$ and $\mathcal{S}_3$ respectively on $\mathcal{U}^{\otimes 2}$ and $\mathcal{U}^{\otimes 3}$ (so that $\sigma_2 = (1,2)$ and $\sigma_3 = (1,3)$, see Notation 3.11, Lemma 3.12 and Remark 3.21) and notice that for all $f : \mathcal{U} \to \mathcal{U}^{\otimes 2}$ we have

$$(f \otimes id) \circ \sigma_2 = (3,2,1) \circ (id \otimes f), \quad (id \otimes f) \circ \sigma_2 = (1,2,3) \circ (f \otimes id),$$

hence, recalling that $\hat\Omega^{(2)} = \widehat{\sigma_2 \circ \Omega^{\otimes 2}}$, $id \otimes \sigma_2 = (2,3)$, $\sigma_2 \otimes id = (1,2)$ and $(3,2,1)(2,3) = (1,2,3)(1,2) = (1,3)$, we get

$$(\Delta_v \,\hat\otimes\, id) \circ \Delta_v \circ \Omega = (\Delta_v \,\hat\otimes\, id) \circ \hat\Omega^{(2)} \circ \Delta_v =$$
$$= \widehat{\sigma_3 \circ \Omega^{\otimes 3}} \circ (id \,\hat\otimes\, \Delta_v) \circ \Delta_v = \hat\Omega^{(3)} \circ (id \,\hat\otimes\, \Delta_v) \circ \Delta_v$$

and

$$(id \,\hat\otimes\, \Delta_v) \circ \Delta_v \circ \Omega = (id \,\hat\otimes\, \Delta_v) \circ \hat\Omega^{(2)} \circ \Delta_v =$$
$$= \widehat{\sigma_3 \circ \Omega^{\otimes 3}} \circ (\Delta_v \,\hat\otimes\, id) \circ \Delta_v = \hat\Omega^{(3)} \circ (\Delta_v \,\hat\otimes\, id) \circ \Delta_v. \qquad \square$$

Proposition 7.6. Δ_v *is "coassociative":*

$$(\Delta_v \,\hat\otimes\, id) \circ \Delta_v = (id \,\hat\otimes\, \Delta_v) \circ \Delta_v : \mathcal{U} \to \mathcal{U}^{\hat\otimes 3}.$$

Equivalently $\Delta^{(m)}_{v;h_1,\dots,h_m}$ *is independent of* $h_1, \dots, h_m$. *Denote it simply as* $\Delta^{(m)}_v$. *Then* $\Delta^{(0)}_v = id$, $\Delta^{(1)}_v = \Delta_v$ *and*

$$\Delta^{(m)}_v = (id^{\hat\otimes(h-1)} \,\hat\otimes\, \Delta_v \,\hat\otimes\, id^{\hat\otimes(m-h)}) \circ \Delta^{(m-1)}_v \quad \forall m \geq h \geq 1.$$

Proof. It is enough to compute $(\Delta_v \,\hat\otimes\, id) \circ \Delta_v$ and $(id \,\hat\otimes\, \Delta_v) \circ \Delta_v$ on a set of generators of $\mathcal{U}$ and even, thanks to Lemma 7.5, just on $C^{\pm 1}$ (which is trivial), $\tilde K^+_i(u)$ and $X^+_i(u)$. These easy computations (left to the reader) are based on the following observations:

$$(\Delta_v \,\hat\otimes\, id)(\tilde K^+_i(u \otimes 1)) = \tilde K^+_i(u \otimes 1 \otimes 1)\tilde K^+_i(C^{-1} \otimes uv_1 \otimes 1),$$
$$(\Delta_v \,\hat\otimes\, id)(\tilde K^+_i(C^{-1} \otimes uv)) = \tilde K^+_i(C^{-1} \otimes C^{-1} \otimes uv_1v_2),$$
$$(id \,\hat\otimes\, \Delta_v)(\tilde K^+_i(u \otimes 1)) = \tilde K^+_i(u \otimes 1 \otimes 1),$$
$$(id \,\hat\otimes\, \Delta_v)(\tilde K^+_i(C^{-1} \otimes uv)) = \tilde K^+_i(C^{-1} \otimes uv_1 \otimes 1)\tilde K^+_i(C^{-1} \otimes C^{-1} \otimes uv_1v_2),$$
$$(\Delta_v \,\hat\otimes\, id)(X^+_i(u \otimes 1)) = X^+_i(u \otimes 1 \otimes 1) + \tilde K^+_i(u \otimes 1 \otimes 1)X^+_i(C^{-1} \otimes uv_1 \otimes 1),$$
$$(\Delta_v \,\hat\otimes\, id)(X^+_i(C^{-1} \otimes uv)) = X^+_i(C^{-1} \otimes C^{-1} \otimes uv_1v_2),$$
$$(id \,\hat\otimes\, \Delta_v)(X^+_i(u \otimes 1)) = X^+_i(u \otimes 1 \otimes 1),$$
$$(id \,\hat\otimes\, \Delta_v)(X^+_i(C^{-1} \otimes uv)) =$$
$$= X^+_i(C^{-1} \otimes uv_1 \otimes 1) + \tilde K^+_i(C^{-1} \otimes uv_1 \otimes 1)X^+_i(C^{-1} \otimes C^{-1} \otimes uv_1v_2). \square$$

Remark 7.7. In [19] Δ_v is proven to be not coassociative, the coassociativity holding just for the "limit" at $v = 1$. The apparent difference of this claim from Proposition 7.6 depends on the different completion used in the two papers: indeed our completion $\mathcal{U}\hat{\otimes}\mathcal{U}$ is (can be seen as) the evaluation (the limit) at $v=1$ of a subalgebra of the completion $\mathcal{U}'_q\hat{\otimes}\mathcal{U}'_q$ used in [19].

More precisely: as already pointed out, "our" v or v_1 and v_2 are not parameters, but just symbols to underline the grading and to control it. This means that we could also choose another notation for the same algebra $\mathcal{U}\hat{\otimes}\mathcal{U}$ without writing the v_i's (keeping otherwise the control of the range allowed for the infinite sums): this can be described as interpreting these symbols as parameters and evaluating them at 1, or passing to their limit for $v\to 1$.

But which is the setting where the evaluation and the limit make sense? The construction of $\mathcal{U}\hat{\otimes}\mathcal{U}$ is a choice for such a setting.

This observation about the status of the v_i's has two consequences: on one hand, as we shall see in Section 9, it provides the setting for the description of Δ_v as limit of Δ in the case of the affine quantum algebras; on the other hand it turns out to be important when working in $\mathcal{U}^{\hat{\otimes} m}$ with $m>2$, and in particular when dealing with the coassociativity, as we are doing now.

On the other hand the algebra $\mathcal{U}'_q\hat{\otimes}\mathcal{U}'_q=\mathcal{U}\otimes\mathcal{U}((v))$ considered in [19] is defined as the topological completion not of $\mathcal{U}\otimes\mathcal{U}$ but of the bigger $\mathcal{U}'_q\otimes_{\mathbb{C}}(q)((v))\,\mathcal{U}'_q=\mathbb{C}(q)((v))\otimes\mathcal{U}\otimes\mathcal{U}$ (actually in [19] q is to be intended as a complex parameter, but this is a minor difference, involving no problem) and contains many copies of $\mathcal{U}\otimes\mathcal{U}$, mapped isomorphically onto $\mathcal{U}\otimes\mathcal{U}$ via the "evaluation" of v at 1 and different from each other.

Moreover, coherently with the choice where v is an element of the ground field $\mathbb{C}(q)((v))$, $\mathcal{U}'_q\hat{\otimes}\mathcal{U}'_q\hat{\otimes}\mathcal{U}'_q=\mathcal{U}\otimes\mathcal{U}\otimes\mathcal{U}((v))$ and v behaves as a scalar.

As a final remark let us observe that, as the author underlines, the limit at $v=1$ is not defined; so it is just mentioned to underline that there is some trace of coassociativity of Δ_v, despite to the lack of the coassociativity in the strict sense.

Let us consider a simple example: if $r>0$

$$\Delta_v(H_{i,r})=H_{i,r}\otimes 1+C^r\otimes H_{i,r}v^{-r}$$

means that $\Delta_v(H_{i,r})=H_{i,r}\otimes 1+C^r\otimes H_{i,r}$ (element of degree r), where the right hand factors 1 and $H_{i,r}$ have degrees respectively 0 and r, encoded in the exponent of v. Then

$$\begin{gathered}(\Delta_v\otimes id)\circ\Delta_v(H_{i,r})=(id\otimes\Delta_v)\circ\Delta_v(H_{i,r})=\\=H_{i,r}\otimes 1\otimes 1+C^r\otimes H_{i,r}\otimes 1+C^r\otimes C^r\otimes H_{i,r},\end{gathered}$$

which is written as

$$H_{i,r}\otimes 1\otimes 1+(C^r\otimes H_{i,r}\otimes 1)v_1^{-r}+C^r\otimes C^r\otimes H_{i,r}(v_1v_2)^{-r}.$$

On the other hand in the setting of [19] we have

$$\Delta_v(H_{i,r})=H_{i,r}\otimes 1+C^r\otimes H_{i,r}v^{-r}\neq\Delta_v(H_{i,r})=H_{i,r}\otimes 1+C^r\otimes H_{i,r},$$

so that

$$(\Delta_v\otimes id)\circ\Delta_v(H_{i,r})=H_{i,r}\otimes 1\otimes 1+(C^r\otimes H_{i,r}\otimes 1)v^{-r}+C^r\otimes C^r\otimes H_{i,r}v^{-r},$$

while

$$(id\otimes\Delta_v)\circ\Delta_v(H_{i,r})=H_{i,r}\otimes 1\otimes 1+(C^r\otimes H_{i,r}\otimes 1)v^{-r}+C^r\otimes C^r\otimes H_{i,r}v^{-2r}.$$

In this simple example we can evaluate v at 1, since these elements belong to $\mathcal{U}\otimes\mathcal{U}\otimes\mathcal{U}[v^{-1}]$, or because, with the notations of the present paper, $\Delta_v(H_{i,r})\in\mathcal{U}\otimes\mathcal{U}\subseteq\mathcal{U}\hat{\otimes}\mathcal{U}$.

The same computation applied to $X_{i,r}^+$ confirms this picture, enlightening the problem with the evaluation of v at 1.

Definition 7.8. $\varepsilon:\mathcal{U}\to\mathbb{C}(q)$ is the $\mathbb{C}(q)$-algebra homomorphism defined by

$$\varepsilon(C^{\pm 1})=\varepsilon(\tilde{K}_i^{\pm}(u))=1,\quad \varepsilon(X_i^{\pm}(u))=0.$$

Proposition 7.9. *ε is a “counit” for Δ_v: $(\varepsilon\hat{\otimes}id)\circ\Delta_v=(id\hat{\otimes}\varepsilon)\circ\Delta_v=id$.*

Proof. Notice that (see Remark 3.14)

$$(\varepsilon\hat{\otimes}id):\mathcal{U}^{\hat{\otimes}2}\to\mathbb{C}(q)\hat{\otimes}\mathcal{U}\cong\mathcal{U},\quad (id\hat{\otimes}\varepsilon):\mathcal{U}^{\hat{\otimes}2}\to\mathcal{U}\hat{\otimes}\mathbb{C}(q)\cong\mathcal{U},$$
$$\varepsilon\circ\Omega=\varepsilon,\quad (\varepsilon\hat{\otimes}id)\circ\hat{\Omega}^{(2)}=\Omega\circ(id\hat{\otimes}\varepsilon),\quad (id\hat{\otimes}\varepsilon)\circ\hat{\Omega}^{(2)}=\Omega\circ(\varepsilon\hat{\otimes}id)$$

and

$$(\varepsilon\hat{\otimes}id)(\tilde{K}_i^+(u\otimes 1))=(id\hat{\otimes}\varepsilon)(\tilde{K}_i^+(C^{-1}\otimes uv))=1,$$
$$(\varepsilon\hat{\otimes}id)(\tilde{K}_i^+(C^{-1}\otimes uv))=(id\hat{\otimes}\varepsilon)(\tilde{K}_i^+(u\otimes 1))=\tilde{K}_i^+(u),$$
$$(\varepsilon\hat{\otimes}id)(X_i^+(u\otimes 1))=(id\hat{\otimes}\varepsilon)(X_i^+(C^{-1}\otimes uv))=0,$$
$$(\varepsilon\hat{\otimes}id)(X_i^+(C^{-1}\otimes uv))=(id\hat{\otimes}\varepsilon)(X_i^+(u\otimes 1))=X_i^+(u).$$

The claim follows. ∎

Corollary 7.10. $\Delta_v : \mathcal{U} \to \mathcal{U}\hat{\otimes}\mathcal{U}$ *is injective.*

Proposition 7.11. *Let* $S : \mathcal{U}^{0,K} \to \mathcal{U}^{0,K}$ *be defined by*

$$S(k_\beta) = k_{-\beta}, \quad S(\tilde{H}_i^+(u)) = \tilde{H}_i^+(Cu)^{-1}, \quad S(\tilde{H}_i^-(u)) = \tilde{H}_i^-(Cu)^{-1}.$$

Then S *is a well defined* $\mathbb{C}(q)$*-antiautomorphism of* $\mathcal{U}^{0,K}$ *commuting with* Ω *and it is the antipode for* $\Delta_v|_{\mathcal{U}^{0,K}}$*.*

Proof. That the relations defining $\mathcal{U}^{0,K}$ are preserved by S is an easy verification, and so is the commutation with Ω. The condition for S to be an antipode for Δ_v (that is $m\circ(S\otimes id)\circ\Delta_v = \varepsilon = m\circ(id\otimes S)\circ\Delta_v$) is equivalent to

$$\begin{gathered} S(k_\beta)k_\beta = 1 = k_\beta S(k_\beta), \\ S(\tilde{H}_i^+(u))\tilde{H}_i^+(Cu) = 1, \\ S(\tilde{H}_i^-(u))\tilde{H}_i^-(Cu) = 1, \end{gathered}$$

which is the definition of S. □

Remark 7.12. There does not exist an "antipode" $S : \mathcal{U} \to \mathcal{U}$ for Δ_v.

Proof. If such an S existed its restriction to $\mathcal{U}^{0,K}$ would be as in Proposition 7.11, hence the condition of being an "antipode" would imply

$$S(X_i^+(u)) + \tilde{K}_i^+(Cu)^{-1}X_i^+(Cuv) = 0$$

or equivalently

$$S(X_i^+(u)) = -\tilde{K}_i^+(Cu)^{-1}X_i^+(Cuv),$$

which does not define an element of $\mathcal{U}[[u^{\pm 1}]]$. □

Remark 7.13. In [17] Δ_v is proven to admit an antipode. Here too, as in Remark 7.7, the apparent different results depend on the chosen completions: indeed S is well defined if one admits it to take values in a suitable completion of $\mathcal{U}$. See [17] for a detailed discussion of the conditions to obtain a Hopf algebra structure.

8. Affine quantum algebras: some more definitions and notations

In this and in the next sections we consider the particular case when $\mathcal{U}$ is an affine quantum algebra (that is $\mathcal{U} = \mathcal{U}(A, k, \tilde{d}) \cong \mathcal{U}_q(\hat{A})$ with $\hat{A}$ affine Cartan matrix and $A = \hat{A}_f$ finite Cartan matrix). Then it is well known that $\mathcal{U}$ is endowed with a Hopf algebra structure

$$(\mathcal{U}, \Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}, \varepsilon : \mathcal{U} \to \mathbb{C}(q), S : \mathcal{U} \to \mathcal{U}).$$

We want to describe the connection between Δ and Δ_v; this also provides another proof that Δ_v is well defined, in the particular case when, precisely, $\mathcal{U}$ is an affine quantum algebra.

This section is devoted to recall some preliminary definitions and results about affine quantum algebras that we shall need (see [3, 11, 21, 12, 27, 26, 24, 6, 7]).

In Section 9 we introduce the notion of t-equivariant limit and prove that Δ_v is the t-equivariant limit of Δ.

Definition 8.1. Let $\hat{A} = (a_{ij})_{i,j\in\hat{I}}$ and $A = (a_{ij})_{i,j\in I}$ be respectively an affine and a finite Cartan matrix with $A = \hat{A}_f$, $\hat{A} \simeq (A, k, \tilde{d})$ (see Remark 1.5). Then $\hat{I} = I \cup \{0\}$ and $\mathcal{U}_q = \mathcal{U}_q(\hat{A})$ is the $\mathbb{C}(q)$-algebra generated by $\{E_i, F_i, K_i^{\pm 1} | i \in \hat{I}\}$ with relations

$$K_iK_i^{-1} = 1 = K_i^{-1}K_i, \quad K_iK_j = K_jK_i \quad \forall i, j \in \hat{I};$$
$$K_iE_j = q_i^{a_{ij}} E_jK_i, \quad K_iF_j = q_i^{-a_{ij}} F_jK_i \quad \forall i, j \in \hat{I};$$
$$[E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \quad \forall i, j \in \hat{I};$$
$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} E_i^m E_j E_i^{1-a_{ij}-m} = 0 \quad \forall i \neq j \in \hat{I}.$$
$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} F_i^m F_j F_i^{1-a_{ij}-m} = 0 \quad \forall i \neq j \in \hat{I}.$$

Remark 8.2. Setting $\Omega(q) = q^{-1}$, $\Omega(K_i) = K_i^{-1}$, $\Omega(E_i) = F_i$, $\Omega(F_i) = E_i$ for all $i \in \hat{I}$ defines a $\mathbb{C}$-antilinear antiinvolution of $\mathcal{U}_q$.

Remark 8.3. The sets $\{\alpha_i | i \in \hat{I}\}$ and $\{\alpha_i | i \in I\} \cup \{\delta\}$ are different $\mathbb{Z}$-bases of the same root lattice Q.

$\mathcal{U}_q$ is Q-graded: for all $i \in \hat{I}$, K_i, E_i, F_i have degrees respectively 0, α_i and $-\alpha_i$.

Definition 8.4. The coproduct $\Delta : \mathcal{U}_q \to \mathcal{U}_q \otimes \mathcal{U}_q$ is the $\mathbb{C}(q)$-algebra homomorphism defined on the generators by

$$\Delta(K_i) = K_i \otimes K_i, \;\; \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \;\; \Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i^{-1} \;\; \forall i \in \hat{I}.$$

Remark 8.5. Δ preserves the Q-gradation and $\Delta \circ \Omega = \sigma_2 \circ \Omega^{\otimes 2} \circ \Delta$ (we say that Δ is Ω-equivariant).

Definition 8.6. The Weyl group W is the subgroup of $Aut(Q)$ generated by the reflections $s_i : Q \to Q$ ($i \in \hat{I}$) (which are defined by $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$ for all $j \in \hat{I}$).

W_0 is the subgroup of W generated by $\{s_i | i \in I\}$.

Recalling that the weight lattice P can be identified to a subgroup of $Aut(Q)$ acting on Q by "translations" ($P \ni \omega : \beta \mapsto \beta - {<}\omega, \beta{>}\delta$, see Definition 1.7) we can define the extended Weyl group $\hat{W} = P \rtimes W_0$ and the subgroup $\mathcal{T}$ of

$$\left\{\tau : \hat{I} \to \hat{I} \middle| a_{\tau(i)\tau(j)} = a_{ij} \quad \forall i, j \in \hat{I}\right\}$$

such that $\hat{W} = W \rtimes \mathcal{T}$ (see [28, 20]).

Remark that the condition $a_{\tau(i)\tau(j)} = a_{ij}$ for all $i, j \in \hat{I}$ implies the injectivity, hence the bijectivity, of τ.

To the Weyl group(s) there are associated the braid group(s) and the corresponding projection(s): $\mathcal{B}$ is generated by $\{T_i | i \in \hat{I}\}$ with the braid relations (see [3]); $\hat{\mathcal{B}} = \mathcal{B} \rtimes \mathcal{T}$; $\hat{\mathcal{B}} \ni T \mapsto w \in \hat{W}$ is the group homomorphism defined by $T_i \mapsto s_i$, $\tau \mapsto \tau$ (for all $i \in \hat{I}$, $\tau \in \mathcal{T}$).

$\hat{\mathcal{B}}$ acts on $\mathcal{U}_q$ by the following formulas (see [27] and [26]):

$$T(K_\beta) = K_{w(\beta)} \ \text{ if } \ T \mapsto w, \ \text{ where } \ K_{\tilde{\beta}} = \prod_{i \in \hat{I}} K_i^{m_i} \ \text{ if } \ \tilde{\beta} = \sum_{i \in \hat{I}} m_i \alpha_i,$$

$$\tau(E_i) = E_{\tau(i)}, \quad T_i(E_i) = -F_i K_i,$$

$$T_i(E_j) = \sum_{s=0}^{-a_{ij}} (-1)^{s - a_{ij}} q_i^{-s} \begin{bmatrix} -a_{ij} \\ s \end{bmatrix}_{q_i} E_i^{-a_{ij}-s} E_j E_i^s \ \text{ if } \ i \neq j,$$

$$T(F_i) = \Omega T(E_i) \ \ \forall T \in \hat{\mathcal{B}}, \ i \in \hat{I}.$$

Remark 8.7. $\hat{W}$ fixes the imaginary root δ.

The length function $l : W \to \mathbb{N}$ extends to $\hat{W}$ by $l(w\tau) = l(w)$ for all $w \in W, \tau \in \mathcal{T}$.

The restriction of l to the set of the dominant weights P_+ (see Definition 1.7) is additive.

The map $s_i \mapsto T_i$, $\tau \mapsto \tau$ ($i \in \hat{I}$, $\tau \in \mathcal{T}$) extends to a section $T : \hat{W} \ni w \mapsto T_w \in \hat{\mathcal{B}}$ such that $T_{ww'} = T_w T_{w'}$ if $l(ww') = l(w) + l(w')$.

In particular $T\big|_{P_+}$ extends uniquely to a group homomorphism $P \hookrightarrow \hat{\mathcal{B}}$, hence it defines an action of P on $\mathcal{U}_q$.

The action of $\hat{\mathcal{B}}$ on $\mathcal{U}_q$ has the property that if $\hat{\mathcal{B}}\ni T\mapsto w\in\hat{W}$, then T maps elements of any degree $\beta\in Q$ to elements of degree $w(\beta)$.

Definition 8.8. The set Φ^{re} of the real roots is

$$\Phi^{re}=\{\alpha\in Q|\exists i\in\hat{I},\ w\in\hat{W}\text{ such that }\alpha=w(\alpha_i)\}.$$

The set Φ^{re}_+ of the positive real roots is

$$\Phi^{re}_+=\Big\{\alpha=\sum_{i\in\hat{I}}m_i\alpha_i\in\Phi^{re}|m_i\geq 0\ \forall i\in\hat{I}\Big\}.$$

The sets Φ^0 of the finite roots and Φ^0_+ of the positive finite roots are

$$\Phi^0=\Phi^{re}\cap Q_0=\{\alpha\in Q|\exists i\in I,\ w\in W_0\text{ such that }\alpha=w(\alpha_i)\},$$
$$\Phi^0_+=\Phi^0\cap\Phi^{re}_+.$$

Recall 8.9. For all $\alpha\in\Phi^{re}_+$ root vectors E_α and $F_\alpha=\Omega(E_\alpha)$ of degrees respectively α and $-\alpha$ can be defined through the action of the braid group, depending on a sequence $\iota:\mathbb{Z}\to\hat{I}$ with suitable properties. Here we recall the main facts on which the construction of the E_α's (and symmetrically of the F_α's) is based:

i) if $i\neq j\in I$ then $T_{\omega_i}(E_j)=E_j$ and $T_{\omega_i}T_{\omega_j}=T_{\omega_j}T_{\omega_i}$;

ii) if $w=s_{i_1}\cdot\ldots\cdot s_{i_l}\in W$ ($i_r\in\hat{I}$ for all r) has length l, then

$$\{s_{i_1}\cdot\ldots\cdot s_{i_{h-1}}(\alpha_{i_h})|h=1,\ldots,l\}=\{\alpha\in\Phi^{re}_+|w^{-1}(\alpha)\in-\Phi^{re}_+\}$$

and this set has l elements;

iii) for all $\alpha=\gamma+m\delta\in\Phi^{re}_+$ ($\gamma\in Q_0$) we have that:

- -) if $\gamma>0$, then $\omega^{-1}(\alpha)\in\Phi^{re}_+$ for all dominant weights $\omega\in P_+$ and there exists a dominant weight $\omega\in P_+$ such that $\omega(\alpha)\in-\Phi^{re}_+$;
- -) if $\gamma<0$, then $\omega(\alpha)\in\Phi^{re}_+$ for all dominant weights $\omega\in P_+$ and there exists a dominant weight $\omega\in P_+$ such that $\omega^{-1}(\alpha)\in-\Phi^{re}_+$;

iv) to any surjective map $i:\{1,\ldots,M\}\to I$ such that $\sum_{h=1}^M\omega_{i_h}\in P\cap W$ we can attach a sequence $\iota:\mathbb{Z}\to\hat{I}$ as follows:

- -) if N_h is the length of $\omega_{i_1}+\cdots+\omega_{i_h}\in P\subseteq\hat{W}$ then

$$s_{\iota(1)}\cdot\ldots\cdot s_{\iota(N_h)}\tau_h=\omega_{i_1}+\cdots+\omega_{i_h}$$

for some Dynkin diagram automorphism τ_h (of course $\tau_M=id$);

-) ι is periodic of period N_M: $\iota(r) = \iota(r + N_M)$ for all $r \in \mathbb{Z}$;

v) given ι as in iv), the map $\mathbb{Z} \ni r \mapsto \beta_r \in \Phi^{re}$ defined by

$$\beta_r = \begin{cases} s_{\iota(1)} s_{\iota(2)} \cdot \ldots \cdot s_{\iota(r-1)}(\alpha_{\iota(r)}) & \text{if } r \geq 1 \\ s_{\iota(0)} s_{\iota(-1)} \cdot \ldots \cdot s_{\iota(r+1)}(\alpha_{\iota(r)}) & \text{if } r \leq 0 \end{cases}$$

establishes a bijection between $\mathbb{Z}$ and Φ^{re}_+.

For the details see [3, 2, 6, 7, 1].

We are now ready to introduce the root vectors.

Definition 8.10. Let $\iota : \mathbb{Z} \to \hat{I}$ be as in Recall 8.9, iv). For all $r \in \mathbb{Z}$ the real root vectors E_{β_r} and F_{β_r} are defined as follows:

$$E_{\beta_r} = \begin{cases} T_{\iota(1)} T_{\iota(2)} \cdot \ldots \cdot T_{\iota(r-1)}(E_{\iota(r)}) & \text{if } r \geq 1 \\ T^{-1}_{\iota(0)} T^{-1}_{\iota(-1)} \cdot \ldots \cdot T^{-1}_{\iota(r+1)}(E_{\iota(r)}) & \text{if } r \leq 0, \end{cases}$$
$$F_{\beta_r} = \Omega(E_{\beta_r}).$$

Remark that in general these root vectors depend on ι, but Recall 8.9 implies that $E_{r\tilde{d}_i\delta\pm\alpha_i}$ is independent of the choice of ι:

$$E_{r\tilde{d}_i\delta+\alpha_i} = T^{-r}_{\omega_i}(E_i), \quad E_{r\tilde{d}_i\delta-\alpha_i} = -T^{r}_{\omega_i}(K_i^{-1}F_i) = -K_{r\tilde{d}_i\delta-\alpha_i} T^{r}_{\omega_i}(F_i).$$

Finally, for all $r > 0$, $i \in I$ the root vectors $E_{(r\delta,i)}$ and $F_{(r\delta,i)}$ of degrees respectively $r\delta$ and $-r\delta$ are defined by

$$\exp\left(\sum_{r>0} E_{(r\delta,i)} u^{-r}\right) = 1 + (q_i - q_i^{-1}) \sum_{r>0} (E_{r\tilde{d}_i\delta-\alpha_i} E_i - q_i^{-2} E_i E_{r\tilde{d}_i\delta-\alpha_i}) u^{-r\tilde{d}_i},$$
$$F_{(r\delta,i)} = \Omega(E_{(r\delta,i)}).$$

Remark 8.11. For all $\epsilon : \hat{I} \to \{\pm 1\}$, ϵ induces a groups homomorphism $\epsilon_Q : Q \to \{\pm 1\}$ ($\epsilon_Q(\alpha_i) = \epsilon(i)$ for all $i \in \hat{I}$) and a $\mathbb{C}(q)$-algebra automorphism $\epsilon_{\mathcal{U}} : \mathcal{U}_q \to \mathcal{U}_q$ setting

$$\epsilon_{\mathcal{U}}(K_i) = K_i, \quad \epsilon_{\mathcal{U}}(E_i) = \epsilon(i) E_i, \quad \epsilon_{\mathcal{U}}(F_i) = \epsilon(i) F_i;$$

$\epsilon_{\mathcal{U}}$ is an involution of $\mathcal{U}_q$ acting as $\epsilon_Q(\beta)$ on the elements of degree β.

Remark that any $\epsilon : I \to \{\pm 1\}$ can be (uniquely) extended to $\hat{\epsilon} : \hat{I} \to \{\pm 1\}$ so that $\hat{\epsilon}_Q(\delta) = 1$.

The automorphisms of $\mathcal{U}_q$ that are determined by the maps $\epsilon : \hat{I} \to \{\pm 1\}$ such that $\epsilon_Q(\delta) = 1$ commute among themselves and with the T_{ω_i}'s.

Definition 8.12. Let $o: I \to \{\pm 1\}$ be such that

i) $o(i)o(j) = -1$ for all $i, j \in I$ such that $a_{ij} < 0$; and
ii) in the case $X_{\tilde{n}}^{(k)} \neq A_{2n}^{(2)}$, $o(i) = 1$ if there exists $j \in I$ such that $a_{ij} = -2$.

For all $i \in I$ consider the map $\epsilon_i : I \to \{\pm 1\}$ defined by

$$\epsilon_i(j) = \begin{cases} o(i) & \text{if } j = i \\ 1 & \text{otherwise,} \end{cases}$$

extend it to $\hat{\epsilon}_i : \hat{I} \to \{\pm 1\}$ in such a way that $\hat{\epsilon}_i(\delta) = 1$ and denote by $(\hat{\epsilon}_i)_{\mathcal{U}}$ the corresponding automorphism of $\mathcal{U}_q$ (see Remark 8.11).

We define $T'_{\omega_i} = (\hat{\epsilon}_i)_{\mathcal{U}} T_{\omega_i}$ and $T'_{\omega} = \prod_{i \in I} (T'_{\omega_i})^{m_i}$ for all $\omega = \sum_{i \in I} m_i \omega_i \in P$.

We can now recall the results concerning the isomorphism between $\mathcal{U}(A, k, \tilde{d})$ and $\mathcal{U}_q(\hat{A})$.

Recall 8.13. It is well known that $\mathcal{U}(A, k, \tilde{d}) \cong \mathcal{U}_q(\hat{A})$ (see [4] and [5]). More precisely if $o: I \to \{\pm 1\}$ is as in Definition 8.12, there exists an isomorphism $\psi : \mathcal{U}(A, k, \tilde{d}) \to \mathcal{U}_q(\hat{A})$ characterized by the identification

$$E_i \leftrightarrow X_{i,0}^+, \quad F_i \leftrightarrow X_{i,0}^-, \quad K_i \leftrightarrow k_i$$

and by the condition $\psi \circ t_i = T'_{\omega_i} \circ \psi$ (for all $i \in I$).

In particular $\psi \circ \Omega = \Omega \circ \psi$ and $E_{(r\tilde{d}_i\delta, i)} \leftrightarrow o(i)^r H_{i, r\tilde{d}_i}$ for all $r > 0, i \in I$.

From now on we consider the identification $\mathcal{U}(A, k, \tilde{d}) = \mathcal{U}_q(\hat{A}) = \mathcal{U}$.

Definition 8.14. 1. We shall denote by $Q_{++} \subseteq Q_+ \subseteq Q$ the sets

$$Q_+ = \Big\{ \beta = r\delta + \sum_{i \in I} m_i \alpha_i \in Q \big| m_i \geq 0 \ \forall i \in I \Big\},$$

$$Q_{++} = \Big\{ \beta = r\delta + \sum_{i \in I} m_i \alpha_i \in Q_+ \big| \exists i \in I \text{ such that } m_i > 0 \Big\} = Q_+ \setminus \mathbb{Z}\delta.$$

2. Given $\beta_1, \beta_2 \in Q$ we say that

$$\beta_1 \geq \beta_2 \text{ if } \beta_1 - \beta_2 \in Q_+ \text{ and } \beta_1 > \beta_2 \text{ if } \beta_1 - \beta_2 \in Q_{++}.$$

Remark 8.15. Remark that $\geq$ is not an ordering (but its restriction to Q_0 is).

Remark 8.16. In $\mathcal{U}$ we have that, independently of the sequence ι:

i) $\mathcal{U}^{0,+}$ is the $\mathbb{C}(q)$-subalgebra generated by $\{E_{(r\delta,i)}|r>0,\ i\in I\}$;
ii) $\mathcal{U}^+$ is the $\mathbb{C}(q)$-subalgebra generated by

$$\{E_\alpha, K_{\alpha'}F_{\alpha'}|\alpha,\alpha'\in\Phi_+^{re},\alpha'<0<\alpha\}.$$

iii) $\mathcal{U}^-$ is the $\mathbb{C}(q)$-subalgebra generated by

$$\{F_\alpha, K_{-\alpha'}E_{\alpha'}|\alpha,\alpha'\in\Phi_+^{re},\alpha'<0<\alpha\}.$$

iv) The $\mathbb{C}(q)$-subalgebra generated by $\{E_\alpha|0<\alpha\in\Phi^{re}\}$ is the intersection between $\mathcal{U}^+$ and the $\mathbb{C}(q)$-subalgebra generated by $\{E_i|i\in\hat{I}\}$.
v) The $\mathbb{C}(q)$-subalgebra generated by $\{E_\alpha|0>\alpha\in\Phi^{re}\}$ is the intersection between $\oplus_{\beta\le 0}k_\beta\mathcal{U}_\beta^-$ and the $\mathbb{C}(q)$-subalgebra generated by $\{E_i|i\in\hat{I}\}$.
vi) Finally, for all $r>0$, $i\in I$ the sequence ι (see Recall 8.9 and Definition 8.10) can be chosen so that if $l=l(\omega_i)$ then

$$r\tilde{d}_i\delta+\alpha_i=s_{\iota(0)}s_{\iota(-1)}\cdot\ldots\cdot s_{\iota(-rl+1)}(\alpha_{\iota(-rl)})$$

and

$$\begin{aligned}&\{s_{\iota(0)}s_{\iota(-1)}\cdot\ldots\cdot s_{\iota(h+1)}(\alpha_{\iota(h)})|-rl<h\le 0\}=\\&=\{\alpha\in\Phi_+^{re}|\omega_i^r(\alpha)\in-\Phi_+^{re}\}\subseteq\{\alpha\in\Phi_+^{re}|\alpha\ge\alpha_i\}\end{aligned}$$

(see [6, 7]).

Remark 8.17. We are now ready to describe, not completely but with some accuracy that will turn out to be sufficient for our aim, the (Drinfeld-Jimbo) coproduct Δ on the (Drinfeld) generators $H_{i,r}$ and $X_{i,r}^+$ with positive r. Notation 8.18 is introduced to help this description.

It is interesting to recall that the following results depend on a strong relation between Δ and the braid group action, which, avoiding too many details, can be summarized as follows (see [24, 6, 7]): for every dominant weight $\omega\in P_+$ there exists a "partial R-matrix" $\tilde{R}_\omega$ such that

$$((T_\omega\otimes T_\omega)\circ\Delta\circ T_\omega^{-1})(x)=\tilde{R}_\omega\cdot\Delta(x)\cdot\tilde{R}_\omega^{-1}\quad\forall x\in\mathcal{U}.$$

This property, that we shall use in Section 9 to describe Δ_v as P-equivariant limit of Δ, was already useful to find a relation of conjugation between Δ and Δ_v: for example in [30] the author proved that Δ and Δ_v are conjugate through an invertible element $R_<$ arising from a decomposition of the universal R-matrix; see also Remark 9.16.

Notation 8.18. Given $i \in I$ we denote by $\mathcal{U}_i^+$ the left and right $\mathcal{U}^+$-submodule of $\mathcal{U}$ (two-sided ideal of $\mathcal{U}^+$) generated by $\{X_{i,r}^+ | r \in \mathbb{Z}\}$; if moreover $r \geq 0$ define $\mathcal{U}_{i,r}^+$ and $\mathcal{U}_{i,r}^{++}$ by

$$\mathcal{U}_{i,r}^+ = \bigoplus_{\substack{0 \leq s < r, \\ \gamma \in Q_0}} (\mathcal{U}_i^+)_{\gamma + s\delta}, \quad \mathcal{U}_{i,r}^{++} = \bigoplus_{\substack{\gamma \in Q_0: \\ \gamma > \alpha_i}} (\mathcal{U}_{i,r}^+)_{[\gamma]}.$$

Recall 8.19. Δ has the following properties: for all $i \in I$

i) $\Delta(H_{i,r}) - (H_{i,r} \otimes 1 + C^r \otimes H_{i,r}) \in \bigoplus_{\substack{0 \leq s < r, \\ \gamma \in Q_0}} C^r \mathcal{U}_{(r-s)\delta - \gamma}^- \otimes \mathcal{U}_{s\delta + \gamma}^+ \quad \forall r > 0.$

ii) The homogeneous component of $\Delta(H_{i,\tilde{d}_i})$ in $\mathcal{U}_{[-\alpha_i]} \otimes \mathcal{U}_{[\alpha_i]}$ is

$$-(q_i - q_i^{-1}) b_{ii\tilde{d}_i} C^{\tilde{d}_i} X_{i,\tilde{d}_i}^- \otimes X_{i,0}^+.$$

iii) $\Delta(X_{i,r}^+) - (X_{i,r}^+ \otimes 1 + k_i C^r \otimes X_{i,r}^+) \in \mathcal{U} \otimes ((\mathcal{U}_{i,r}^+)_{[\alpha_i]} \oplus \mathcal{U}_{i,r}^{++}) \quad \forall r \geq 0.$

See [6] for the untwisted case and [7] for the general affine case (and Remark 8.16, vi)).

The next Lemma, which is a refinement of Recall 8.19, iii), is the main result used in Section 9 to compare Δ and Δ_v.

Lemma 8.20. *For all $i \in I, r \geq 0$*

$$\Delta(X_{i,r}^+) = X_{i,r}^+ \otimes 1 + \sum_{s=0}^{r} k_i C^{r-s} \tilde{H}_{i,s} \otimes X_{i,r-s}^+ + Y_r$$

with $Y_r \in \mathcal{U} \otimes \mathcal{U}_{i,r}^{++}$.

Proof. Thanks to Recall 8.19, iii) we already know that

$$\Delta(X_{i,r}^+) = X_{i,r}^+ \otimes 1 + Y_r' + Y_r$$

with $Y_r' - k_i C^r \otimes X_{i,r}^+ \in \mathcal{U} \otimes (\mathcal{U}_{i,r}^+)_{[\alpha_i]}$, $Y_r \in \mathcal{U} \otimes \mathcal{U}_{i,r}^{++}$, so we just need to prove that

$$Y_r' = \sum_{s=0}^{r} k_i C^{r-s} \tilde{H}_{i,s} \otimes X_{i,r-s}^+.$$

We proceed by induction on r, the cases $r < \tilde{d}_i$ being trivial ($Y_0 = 0$, $X_{i,r}^+ = 0$ if $0 < r < \tilde{d}_i$).

Let $r \geq \tilde{d}_i$. Since $[H_{i,\tilde{d}_i}, X^+_{i,s}] = b_{ii\tilde{d}_i} X^+_{i,s+\tilde{d}_i}$ ($b_{ii\tilde{d}_i} \neq 0$) for all s, Recall 8.19 implies that there exists $Y \in \oplus_{\substack{0\leq s<\tilde{d}_i \\ \gamma\neq\alpha_i}} C^{\tilde{d}_i}\mathcal{U}^-_{(\tilde{d}_i-s)\delta-\gamma} \otimes \mathcal{U}^+_{\gamma+s\delta}$ such that

$$b_{ii\tilde{d}_i}\Delta(X^+_{i,r}) =$$

$$= \Bigg[H_{i,\tilde{d}_i} \otimes 1 + C^{\tilde{d}_i} \otimes H_{i,\tilde{d}_i} - (q_i - q_i^{-1}) b_{ii\tilde{d}_i} C^{\tilde{d}_i} X^-_{i,\tilde{d}_i} \otimes X^+_{i,0},$$

$$X^+_{i,r-\tilde{d}_i} \otimes 1 + \sum_{s=0}^{r-\tilde{d}_i} k_i C^{r-\tilde{d}_i-s} \tilde{H}_{i,s} \otimes X^+_{i,r-\tilde{d}_i-s}\Bigg] +$$

$$+ [\Delta(H_{i,\tilde{d}_i}), Y_{r-\tilde{d}_i}] + \Bigg[Y, X^+_{i,r-\tilde{d}_i} \otimes 1 + \sum_{s=0}^{r-\tilde{d}_i} k_i C^{r-\tilde{d}_i-s} \tilde{H}_{i,s} \otimes X^+_{i,r-\tilde{d}_i-s}\Bigg],$$

so that

$$\Delta(X^+_{i,r}) =$$

$$= X^+_{i,r} \otimes 1 + \sum_{s=0}^{r-\tilde{d}_i} k_i C^{r-s} \tilde{H}_{i,s} \otimes X^+_{i,r-s} + k_i \tilde{H}_{i,r} \otimes X^+_{i,0} + Y_r$$

(which is the claim) because

$$[\Delta(H_{i,\tilde{d}_i}), Y_{r-\tilde{d}_i}] + \Bigg[Y, \sum_{s=0}^{r-\tilde{d}_i} k_i C^{r-\tilde{d}_i-s} \tilde{H}_{i,s} \otimes X^+_{i,r-\tilde{d}_i-s}\Bigg]$$

obviously belongs to $\mathcal{U} \otimes \mathcal{U}^{++}_{i,r}$ and

$$[Y, X^+_{i,r-\tilde{d}_i} \otimes 1]$$

also belongs to $\mathcal{U} \otimes \mathcal{U}^{++}_{i,r}$ because it belongs to $\mathcal{U} \otimes \mathcal{U}^+_i$ (see Remark 8.16, vi), which implies that the root vectors E_α involved in the right hand factors of Y are of the form $\alpha = \gamma + m\delta$ with $Q_0 \ni \gamma \geq \alpha_i$) and its component in $\mathcal{U} \otimes \mathcal{U}_{[\alpha_i]}$ is zero (because of the condition $\gamma \neq \alpha_i$ in the definition of Y).

One can also prove by direct computation that $[Y, X^+_{i,r-\tilde{d}_i} \otimes 1] \in \mathcal{U} \otimes \mathcal{U}^{++}_{i,r}$ remarking that $X^-_{j,s}$ commutes with $X^+_{i,r\tilde{d}_i}$ for all $j \neq i$, $s \in \mathbb{Z}$: indeed $Y = \bar{Y} + \bar{\bar{Y}}$ with $\bar{Y} \in \mathcal{U} \otimes \mathcal{U}^{++}_{i,r}$ (so that $[\bar{Y}, X^+_{i,r\tilde{d}_i} \otimes 1] \in \mathcal{U} \otimes \mathcal{U}^{++}_{i,r}$) and $\bar{\bar{Y}} = \sum \bar{\bar{Y}}'_h \otimes \bar{\bar{Y}}''_h$ where the $\bar{\bar{Y}}'_h$'s belong to the subalgebra of $\mathcal{U}$ generated by $\{X^-_{j,s} | j \neq i, s \in \mathbb{Z}\}$ (so that $[\bar{\bar{Y}}, X^+_{i,r\tilde{d}_i} \otimes 1] = 0$). $\square$

9. Affine quantum algebras: Δ_v as t-equivariant limit of Δ

We are now ready to concentrate on the connection between Δ and the action of the weight lattice P.

We shall prove that Δ_v is the "t-equivariant limit" of Δ (see Notation 9.7 and Definition 9.8): the first concern of this section is to discuss the notion of limit and convergence in $\mathcal{U}\hat{\otimes}\mathcal{U}$.

Recall that $\mathcal{U}\hat{\otimes}\mathcal{U}$ is a $Q_0\oplus Q_0\oplus\mathbb{Z}$-graded vector space whose components are the completions of $\mathbb{Z}$-graded vector spaces.

The notion of convergence and limit in the completion $\bar{V}$ of a $\mathbb{Z}$-graded vector space $V=\oplus_{r\in\mathbb{Z}}V^{(r)}$ is easy to describe: a sequence $\{x_m\}_{m\in\mathbb{N}}\subseteq V$ is convergent if

$$\forall N\in\mathbb{Z}\ \exists M\geq 0 \text{ such that } \ \forall m\geq M: x_m-x_M\in\oplus_{s\leq N}V^{(s)};$$

its limit is the element $\bar{x}\in\bar{V}$ whose n^{th} component for $n>N$ is the n^{th}-component of x_m for $m\geq M$ (which is independent of m thanks to the choice of M).

Now if we have the direct sum $V=\oplus_a V_a$ of a family of $\mathbb{Z}$-graded vector spaces and the direct sum $\hat{V}=\oplus\bar{V}_a$ of their completions, every element of V (respectively $\hat{V}$) is the (finite) sum of its own a-components (that is projections on the V_a's, respectively $\bar{V}_a$); and to each sequence σ with values in V there corresponds for all a a sequence ${}_a\sigma$ with values in V_a (the projection of σ).

Which are the conditions for σ to have limit in $\hat{V}$? The first condition is that every ${}_a\sigma$ has limit in $\bar{V}_a$. Then the limit of σ should be the sum of the limits of the ${}_a\sigma$'s: so the second condition is that just a finite number of ${}_a\sigma$'s have limit different from zero.

Notation 9.1. For all $\gamma_1,\gamma_2\in Q_0$, $r\in\mathbb{Z}$ let $\mathsf{p}_{[\gamma_1,\gamma_2;r]}$ be the projection

$$\mathsf{p}_{[\gamma_1,\gamma_2;r]}:\mathcal{U}\hat{\otimes}\mathcal{U}\to(\mathcal{U}\hat{\otimes}\mathcal{U})_{[\gamma_1,\gamma_2;r]}.$$

Of course $\mathsf{p}_{[\gamma_1,\gamma_2;r]}$ maps $\mathcal{U}\otimes\mathcal{U}$ onto $(\mathcal{U}\otimes\mathcal{U})_{[\gamma_1,\gamma_2;r]}$.

Definition 9.2. Let us consider a sequence $\{x_m\}_{m\in\mathbb{N}}\subseteq\mathcal{U}\otimes\mathcal{U}$. We say that $\{x_m\}_{m\in\mathbb{N}}$ is convergent, and converges to $\bar{x}\in\mathcal{U}\hat{\otimes}\mathcal{U}$, if the following conditions are satisfied:

i) for all $\gamma_1,\gamma_2\in Q_0$, $r\in\mathbb{Z}$ there exists

$$\lim_{m\to\infty}\mathsf{p}_{[\gamma_1,\gamma_2;r]}(x_m)\in(\mathcal{U}\hat{\otimes}\mathcal{U})_{[\gamma_1,\gamma_2;r]};$$

equivalently: for all $N \in \mathbb{Z}$ there exists $M \geq 0$ such that for all $m \geq M$

$$\mathsf{p}_{[\gamma_1,\gamma_2;r]}(x_m - x_M) \in \bigoplus_{s \leq N} \mathcal{U}_{\gamma_1+(r-s)\delta} \otimes \mathcal{U}_{\gamma_2+s\delta};$$

ii) $\#\{(\gamma_1,\gamma_2,r) \in Q_0 \times Q_0 \times \mathbb{Z} | \lim_{m\to\infty} \mathsf{p}_{[\gamma_1,\gamma_2;r]}(x_m) \neq 0\} < \infty$;
iii) $\bar{x} = \sum_{(\gamma_1,\gamma_2,r)\in Q_0\times Q_0\times\mathbb{Z}} \lim_{m\to\infty} \mathsf{p}_{[\gamma_1,\gamma_2;r]}(x_m)$.

If $\{x_m\}_{m\in\mathbb{N}}$ converges to $\bar{x} \in \mathcal{U} \hat{\otimes} \mathcal{U}$ we write $\bar{x} = \lim_{m\to\infty} x_m$.

Remark 9.3. The sequence $\{x_m\}_{m\in\mathbb{N}} \subseteq \mathcal{U} \otimes \mathcal{U}$ converges to $\bar{x} \in \mathcal{U} \hat{\otimes} \mathcal{U}$ if and only if for all $\gamma_1, \gamma_2 \in Q_0$, $r, N \in \mathbb{Z}$ there exists $M \geq 0$ such that for all $m \geq M$

$$\mathsf{p}_{[\gamma_1,\gamma_2;r]}(\bar{x} - x_m) \in \sum_{s\leq N} \mathcal{U}_{\gamma_1+(r-s)\delta} \otimes \mathcal{U}_{\gamma_2+s\delta} \subseteq (\mathcal{U} \hat{\otimes} \mathcal{U})_{[\gamma_1,\gamma_2;r]}.$$

From this remark it is clear that in the general definition of convergence M depends on γ_1, γ_2, r, N but we have no control on this dependence. The problem is that for our needs this notion of convergence is too weak, and it is useful to introduce a stronger notion of convergence requiring some condition on the dependence of M on γ_1, γ_2, r, N (we shall require that M be actually independent of γ_1, γ_2, r and depend "almost linearly" on N).

Definition 9.4. A sequence $\{x_m\}_{m\in\mathbb{N}} \subseteq \mathcal{U}\otimes\mathcal{U}$ is said to strongly converge to $\bar{x} \in \mathcal{U} \hat{\otimes} \mathcal{U}$ if there exist $R_x \in \mathbb{Z}$, $M_x \in \mathbb{N}$ such that for all $\gamma_1, \gamma_2 \in Q_0, r, N \in \mathbb{Z}$ and for all $m \geq max\{M_x, R_x - N\}$ we have

$$\mathsf{p}_{[\gamma_1,\gamma_2;r]}(\bar{x} - x_m) \in \sum_{s\leq N} \mathcal{U}_{\gamma_1+(r-s)\delta} \otimes \mathcal{U}_{\gamma_2+s\delta} \subseteq (\mathcal{U} \hat{\otimes} \mathcal{U})_{[\gamma_1,\gamma_2;r]}.$$

A sequence $\{x_m\}_{m\in\mathbb{N}} \subseteq \mathcal{U}\otimes\mathcal{U}$ is said to be strongly convergent if there exists $\bar{x} \in \mathcal{U} \hat{\otimes} \mathcal{U}$ such that $\{x_m\}_{m\in\mathbb{N}}$ strongly converges to $\bar{x}$.

A sequence that strongly converges to $\bar{x}$ is convergent and converges to $\bar{x}$.

Remark 9.5. A sequence $\{x_m\}_{m\in\mathbb{N}} \subseteq \mathcal{U}\otimes\mathcal{U}$ strongly converges to $\bar{x} \in \mathcal{U} \hat{\otimes} \mathcal{U}$ if and only if there exist $R_x \in \mathbb{Z}$, $M_x \in \mathbb{N}$ such that for all $N \in \mathbb{Z}$ and for all $m \geq max\{M_x, R_x - N\}$ we have

$$\bar{x} - x_m \in \sum_{\beta\in Q,\gamma\in Q_0,s\leq N} \mathcal{U}_\beta \otimes \mathcal{U}_{\gamma+s\delta} \subseteq v^{-N}\mathcal{U} \otimes \mathcal{U}[[v]].$$

Equivalently $\{x_m\}_{m\in\mathbb{N}} \subseteq \mathcal{U}\otimes\mathcal{U}$ strongly converges to $\bar{x} \in \mathcal{U}\hat{\otimes}\mathcal{U}$ if and only if there exist $R_x \in \mathbb{Z}$, $M_x \in \mathbb{N}$ such that for all $m \geq M_x$ we have

$$\bar{x} - x_m \in v^{m-R_x}\mathcal{U}\otimes\mathcal{U}[[v]].$$

Remark 9.6. If $\{x_m\}_{m\in\mathbb{N}}$ and $\{y_m\}_{m\in\mathbb{N}} \subseteq \mathcal{U}\otimes\mathcal{U}$ strongly converge respectively to $\bar{x} \in v^{-d_x}\mathcal{U}\otimes\mathcal{U}[[v]]$ and $\bar{y} \in v^{-d_y}\mathcal{U}\otimes\mathcal{U}[[v]]$, then $\{x_m + y_m\}_{m\in\mathbb{N}}$ and $\{x_m y_m\}_{m\in\mathbb{N}}$ strongly converge respectively to $\bar{x}+\bar{y}$ and $\bar{x}\bar{y}$.

Let us now turn to the equivariant side of the problem.

Notation 9.7. From now on we denote by t the automorphism of $\mathcal{U}$ defined by

$$t = \prod_{i\in I} t_{\omega_i}.$$

Equivalently $t = t_\omega = T'_\omega$ where $\omega = \sum_{i\in I}\omega_i$.

Definition 9.8. Let x be an element of $\mathcal{U}$. The t-equivariant limit $\bar{\Delta}(x)$ of $\Delta(x)$ in $\mathcal{U}^{\hat{\otimes}2}$ is, if it exists,

$$\bar{\Delta}(x) = \lim_{m\to\infty}(t\otimes t)^m\Delta t^{-m}(x).$$

Proposition 9.9. *The set*

$$\bar{\mathcal{U}} = \{x \in \mathcal{U} | \{(t\otimes t)^m\Delta t^{-m}(x)\}_{m\in\mathbb{N}} \text{ is strongly convergent}\}$$

is an Ω-stable $\mathbb{C}(q)$-subalgebra of $\mathcal{U}$ and $\bar{\Delta}: \bar{\mathcal{U}} \to \mathcal{U}\hat{\otimes}\mathcal{U}$ is an Ω-equivariant $\mathbb{C}(q)$-algebra homomorphism, that is $\bar{\Delta}$ is a $\mathbb{C}(q)$-algebra homomorphism such that $\bar{\Delta}\Omega = \hat{\Omega}^{(2)}\bar{\Delta}$.

Proof. Since Δ, t and $t\otimes t$ are algebra homomorphisms, that $\bar{\mathcal{U}}$ is a subalgebra of $\mathcal{U}$ and $\bar{\Delta}$ a $\mathbb{C}(q)$-algebra homomorphism follows from Remark 9.6.

The Ω-equivariance of $\bar{\Delta}$ follows from the fact that

$$(t\otimes t)^m\Delta t^{-m}(\Omega(x)) = \sigma_2 \circ \Omega^{\otimes 2}(t\otimes t)^m\Delta t^{-m}(x)$$

and from the fact that if $x \in \mathcal{U}_\beta$ with $\beta = \gamma + r\delta$ ($\gamma \in Q_0$, $r \in \mathbb{Z}$) then

$$(t\otimes t)^m\Delta t^{-m}(x) \in \bigoplus_{\substack{\gamma_1,\gamma_2\in Q_0\\ r_1+r_2=r}} \mathcal{U}_{\gamma_1+r_1\delta}\otimes\mathcal{U}_{\gamma_2+r_2\delta}$$

and

$$\sigma_2 \circ \Omega^{\otimes 2}(t\otimes t)^m \Delta t^{-m}(x) \in \bigoplus_{\substack{\gamma_1,\gamma_2\in Q_0\\ r_1+r_2=r}} \mathcal{U}_{-\gamma_2-r_2\delta}\otimes\mathcal{U}_{-\gamma_1+(r_2-r)\delta},$$

so that if $x \in \bar{\mathcal{U}} \subseteq \oplus_{\beta\in Q}\mathcal{U}_\beta$ there exists $r \in \mathbb{Z}$ such that $R_{\Omega(x)} = R_x - r$ satisfies the condition for the strong convergence of $\{(t\otimes t)^m\Delta t^{-m}(\Omega(x))\}_{m\in\mathbb{N}}$ to $\hat{\Omega}^{(2)}\bar{\Delta}(x)$. □

Remark 9.10. For all $\beta \in Q$ $(t\otimes t)^m\Delta t^{-m}(k_\beta) = k_\beta\otimes k_\beta$ is a constant sequence, hence

$$\bar{\Delta}(k_\beta) = \Delta(k_\beta) = k_\beta\otimes k_\beta.$$

Lemma 9.11. *1. Let $x \in \mathcal{U}_{\beta_1}\otimes\mathcal{U}_{\beta_2}$ with $\beta_1,\beta_2 \in Q$ such that $<\omega,\beta_2> > 0$. Then $\{(t\otimes t)^m(x)\}_{m\in\mathbb{N}}$ strongly converges to zero.*

2. Let $r \in \mathbb{Z}$ and $\{x_m\}_{m\in\mathbb{N}} \in \mathcal{U}\otimes\mathcal{U}$ be such that $x_m \in \mathcal{U}\otimes\mathcal{U}^{++}_{i,r+m\tilde{d}_i}$ for all $m >> 0$.

Then $\{(t\otimes t)^m(x_m)\}_{m\in\mathbb{N}}$ strongly converges to zero.

Proof. 1. Let $\beta_2 = \gamma_2 + r_2\delta$ with $\gamma_2 \in Q_0$, $r_2 \in \mathbb{Z}$. Then

$$(t\otimes t)^m(x) \in \mathcal{U}_{\beta_1-m<\omega,\beta_1>\delta}\otimes\mathcal{U}_{\gamma_2+(r_2-m<\omega,\gamma_2>)\delta},$$

which implies the claim because $r_2 - m<\omega,\gamma_2> \le r_2 - m$.

2. x_m is a (finite) sum of elements in $\mathcal{U}\otimes(\mathcal{U}^{++}_{i,r+m\tilde{d}_i})_{\gamma+s\delta}$ with $0 \le s < r+m\tilde{d}_i$ and $\alpha_i < \gamma \in Q_0$. Since

$$(t\otimes t)^m(\mathcal{U}\otimes\mathcal{U}_{\gamma+s\delta}) \subseteq \mathcal{U}\otimes\mathcal{U}_{\gamma+(s-m<\omega,\gamma>)\delta}$$

the claim follows because

$$<\omega,\gamma> > <\omega,\alpha_i> = \tilde{d}_i,\quad s < r+m\tilde{d}_i \Rightarrow$$
$$\Rightarrow s - m<\omega,\gamma> < r + m(\tilde{d}_i - <\omega,\gamma>) \le r-m,$$

that is $s - m<\omega,\gamma> < r-m$. □

Proposition 9.12. *Let $i \in I$, $r \neq 0$. Then $(t\otimes t)^m\Delta t^{-m}(H_{i,r})$ is strongly convergent and*

$$\bar{\Delta}(H_{i,r}) = \begin{cases} H_{i,r}\otimes 1 + C^r\otimes H_{i,r} & \text{if } r>0,\\ 1\otimes H_{i,r} + H_{i,r}\otimes C^r & \text{if } r<0.\end{cases}$$

In particular $\mathcal{U}^0 \subseteq \bar{\mathcal{U}}$ and $\bar{\Delta} : \mathcal{U}^0 \to \mathcal{U}^0\otimes\mathcal{U}^0$.

Proof. Since $H_{i,-r} = \Omega(H_{i,r})$ it is enough to prove the claim for $r > 0$, observing that $1 \otimes H_{i,-r} + H_{i,-r} \otimes C^{-r} = \sigma_2 \Omega^{\otimes 2}(H_{i,r} \otimes 1 + C^r \otimes H_{i,r})$.

Remark that $t^{-m}(H_{i,r}) = H_{i,r}$, so that we want to study the sequence $(t \otimes t)^m \Delta(H_{i,r})$.

By Recall 8.19, i) we have that if $r > 0$ then

$$\Delta(H_{i,r}) - (H_{i,r} \otimes 1 + C^r \otimes H_{i,r}) \in (C^r \mathcal{U}^- \otimes \mathcal{U}^+)_{r\delta},$$

which implies that it is a (finite) sum of elements satisfying the condition of Lemma 9.11, hence strongly converges to zero; since $H_{i,r} \otimes 1 + C^r \otimes H_{i,r}$ is $t \otimes t$-stable, this implies that $(t \otimes t)^m \Delta(H_{i,r})$ strongly converges to $H_{i,r} \otimes 1 + C^r \otimes H_{i,r}$, which is the claim. □

Corollary 9.13. *Remark 9.10 and Proposition 9.12 imply that also $\tilde{H}^{\pm}_{i,r}$ and $\tilde{K}^{\pm}_{i,r} = k_i^{\pm 1} \tilde{H}^{\pm}_{i,r}$ belong to $\bar{\mathcal{U}}$ and that on these elements $\bar{\Delta}$ coincides with Δ_v.*

Proposition 9.14. *Let $i \in I$, $r \in \mathbb{Z}$. Then $(t \otimes t)^m \Delta t^{-m}(X^{\pm}_{i,r})$ is strongly convergent and*

$$\bar{\Delta}(X^+_{i,r}) = X^+_{i,r} \otimes 1 + \sum_{r_1+r_2=r} k_i C^{r_2} \tilde{H}^+_{i,r_1} \otimes X^+_{i,r_2} v^{-r_2},$$
$$\bar{\Delta}(X^-_{i,r}) = 1 \otimes X^-_{i,r} v^{-r} + \sum_{r_1+r_2=r} X^-_{i,r_1} \otimes k_i^{-1} C^{r_1} \tilde{H}^-_{i,r_2} v^{-r_2}.$$

Proof. As in Proposition 9.12 it is enough to prove the claim for $X^+_{i,r}$, since $X^-_{i,r} = \Omega(X^+_{i,-r})$ and, as already remarked,

$$1 \otimes X^-_{i,r} v^{-r} + \sum_{r_1+r_2=r} X^-_{i,r_1} \otimes k_i^{-1} C^{r_1} \tilde{H}^-_{i,r_2} v^{-r_2} =$$
$$= \hat{\Omega}^{(2)} \left(X^+_{i,-r} \otimes 1 + \sum_{r_1+r_2=-r} k_i C^{r_2} \tilde{H}^+_{i,r_1} \otimes X^+_{i,r_2} v^{-r_2} \right).$$

Let us fix $r \in \mathbb{Z}$. Then with the notations of Lemma 8.20 we have that for all $m >> 0$ (more precisely if $m \geq 0$ is such that $r + m\tilde{d}_i \geq 0$)

$$\Delta t^{-m}(X^+_{i,r}) = X^+_{i,r+m\tilde{d}_i} \otimes 1 + \sum_{s=0}^{r+m\tilde{d}_i} k_i C^{r+m\tilde{d}_i-s} \tilde{H}_{i,s} \otimes X^+_{i,r+m\tilde{d}_i-s} + Y_{r+m\tilde{d}_i},$$

hence

$$(t \otimes t)^m \Delta t^{-m}(X^+_{i,r}) = X^+_{i,r} \otimes 1 + \sum_{s=0}^{r+m\tilde{d}_i} k_i C^{r-s} \tilde{H}_{i,s} \otimes X^+_{i,r-s} + (t \otimes t)^m (Y_{r+m\tilde{d}_i}).$$

Lemma 9.11,2. implies that $\{(t\otimes t)^m(Y_{r+m\tilde{d}_i})\}_{m\in\mathbb{N}}$ strongly converges to zero.

On the other hand

$$\left\{\sum_{s=0}^{r+m\tilde{d}_i} k_iC^{r-s}\tilde{H}_{i,s}\otimes X^+_{i,r-s}\right\}_{m\in\mathbb{N}}$$

obviously strongly converges to

$$\sum_{s\geq 0} k_iC^{r-s}\tilde{H}_{i,s}\otimes X^+_{i,r-s},$$

which proves the claim. □

Theorem 9.15. *The subalgebra $\bar{\mathcal{U}}$ of $\mathcal{U}$ (see Proposition 9.9) is $\mathcal{U}$, that is for all $x\in\mathcal{U}$ the sequence $\{(t\otimes t)^m\Delta t^{-m}(x)\}_{m\in\mathbb{N}}$ is strongly convergent.*

The t-equivariant limit $\bar{\Delta}:\mathcal{U}\to\mathcal{U}\hat{\otimes}\mathcal{U}$ of Δ is an Ω-equivariant $\mathbb{C}(q)$-algebra homomorphism.

The map $\bar{\Delta}|_{\mathcal{X}}$ is equal to Δ_v: in particular the map Δ_v defined in Definition 4.5 extends to a well defined $\mathbb{C}(q)$-algebra homomorphism, that is the Drinfeld "coproduct" is well defined and it is the t-equivariant limit of Δ.

Proof. Remark 9.10, Propositions 9.12 and 9.14 and Corollary 9.13 imply that $\bar{\mathcal{U}}$ contains the set $\mathcal{X}$, on which Δ_v is defined, and that $\bar{\Delta}$ and Δ_v coincide on this set of generators of $\mathcal{U}$. Then Proposition 9.9 implies the claim. □

Remark 9.16. We have thus proven that for all $x\in\mathcal{U}$

$$\Delta_v(x)=\lim_{m\to\infty}(t_{\otimes}t)^m\Delta t^{-m}(x)=\lim_{m\to\infty}(T'_\omega\otimes T'_\omega)^m\Delta T'^{-m}_\omega(x)$$

where $\omega=\sum_{\in I}\omega_i$.

Remark that $(T'_\omega\otimes T'_\omega)^m\Delta T'^{-m}_\omega=(T_\omega\otimes T_\omega)^m\Delta T^{-m}_\omega$ (see Definition 8.12 for the action of T'_ω on $\mathcal{U}_\beta$), so that

$$\Delta_v(x)=\lim_{m\to\infty}(T_\omega\otimes T_\omega)^m\Delta T^{-m}_\omega(x).$$

But as we have already remarked (see Remark 8.17) there exist "partial R-matrices" $\bar{R}_m=\tilde{R}_{m\omega}$ such that

$$(T_\omega\otimes T_\omega)^m\Delta T^{-m}_\omega(x)=\bar{R}_m\Delta(x)\bar{R}_m^{-1}.$$

On the other hand (see [30])

$$\Delta_v(x)=R_<\Delta(x)R_<^{-1},$$

so that

$$R_<\Delta(x)R_<^{-1} = \lim_{m\to\infty} \bar{R}_m\Delta(x)\bar{R}_m^{-1}.$$

Of course this observation suggests the problem, that it is not possible to study here, of understanding if the $R_<$ considered in [30] is defined in our setting (that is if it belongs to $\mathcal{U}\hat{\otimes}\mathcal{U}$), if the $\bar{R}_m$'s (which are element of $\mathcal{U}\otimes\mathcal{U}$) have limit in $\mathcal{U}\hat{\otimes}\mathcal{U}$, and if $R_<$ can be described also as limit of the $\bar{R}_m$'s, or which is its relation with them, if there is any.

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