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A 1-dimensional formal group over the prismatization of $\operatorname{Spf} \mathbb{Z}_p$

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Dedicated to Corrado De Concini

Abstract: Let Σ denote the prismatization of $\operatorname{Spf} \mathbb{Z}_p$. The multiplicative group over Σ maps to the prismatization of $\mathbb{G}_m \times \operatorname{Spf} \mathbb{Z}_p$. We prove that the kernel of this map is the Cartier dual of some 1-dimensional formal group over Σ. We obtain some results about this formal group (e.g., we describe its Lie algebra). We give a very explicit description of the pullback of the formal group to the quotient stack $Q/\mathbb{Z}_p^\times$, where Q is the q-de Rham prism.

Keywords: Prismatic cohomology, prismatization, q-de Rham prism, formal group, Breuil-Kisin twist.

1. Introduction

Let p be a prime.

1.1. Subject of this article

In their remarkable work [BS] B. Bhatt and P. Scholze introduced the theory of *prismatic cohomology* of p-adic formal schemes. B. Bhatt and J. Lurie realized that the theory of [BS] has a stacky reformulation; it is based on a certain *prismatization functor,* which we denote[1] by $X \mapsto X^{\Delta}$. This is a functor from the category of bounded p-adic formal schemes to that of stacks.[2]

Following [D3], we write $\Sigma := (\operatorname{Spf} \mathbb{Z}_p)^{\Delta}$. The stack Σ plays a fundamental role in the theory of prismatic cohomology.

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[1] Bhatt and Lurie [BL, BL2] write WCart_X instead of X^{Δ} and WCart instead of $\Sigma := (\operatorname{Spf} \mathbb{Z}_p)^{\Delta}$.

[2] Bhatt and Lurie also define a derived version of the prismatization functor. The difference between derived and non-derived prismatization is irrelevant for our article.

In general, there is no canonical map $X \times \Sigma \to X^{\Delta}$. However, such a map exists if $X = \mathbb{G}_m \times \operatorname{Spf} \mathbb{Z}_p$. Moreover, this map is a faithfully flat group homomorphism (more precisely, a homomorphism from a commutative group scheme over Σ to a Picard stack over Σ). Let G_Σ be its kernel; it is a flat affine commutative group scheme over Σ.

Our first main result (Theorem 2.7.5) says that G_Σ is the Cartier dual of some 1-dimensional formal group over Σ, which we denote by H_Σ. Then $\operatorname{Lie}(H_\Sigma) = \underline{\operatorname{Hom}}(G_\Sigma, (\mathbb{G}_a)_\Sigma)$ is a line bundle on Σ. It turns out to be inverse to the Breuil-Kisin-Tate module $\mathcal{O}_\Sigma\{1\}$ (see Theorem 2.7.10). The corresponding homomorphism $G_\Sigma \to \mathcal{O}_\Sigma\{1\}$ is explicitly constructed in [BL, BL2] and called the *prismatic logarithm*; it is used in [BL] to define the *prismatic first Chern class*.

We obtain some results about the formal group H_Σ (see §2.9), but we are unable to describe it explicitly. However, in §2.10–2.11 we give a very explicit description of the pullback of H_Σ to the quotient stack $Q/\mathbb{Z}_p^\times$, where Q is the q-de Rham prism.

The author's study of G_Σ and H_Σ was motivated by the desire to understand certain aspects of [BS] and [BL, BL2] (see Remark 2.7.4 and Appendix A for more details). On the other hand, H_Σ could be interesting from the topologist's point of view.

Let us note that the group scheme G_Σ is also introduced in [BL2] (under the name of G_{WCart}).

1.2. Organization

The main results are formulated in §2. We also formulate there a question about G_Σ and a conjecture about H_Σ (see §2.8 and Conjecture 2.12.4).

In §3 we discuss some general results and constructions related to formal groups. In §4 we prove the results formulated in §2.

In §5 we describe and compare several "realizations" of the group scheme $G_Q := G_\Sigma \times_\Sigma Q$; the first one immediately follows from the definition of G_Q, and the others come from the description of its Cartier dual. A key role is played by the expressions $(1+(q-1)z)^{\frac{t}{q-1}}$ and $q^{\frac{pt}{q-1}}$; the second expression is closely related to the *q-logarithm* in the sense of [ALB, §4].

In Appendix A we explain how to compute the prismatic cohomology of the punctured affine line over $\operatorname{Spf} \mathbb{Z}_p$ using some results formulated in §2.

In Appendix B we discuss the Cartier dual of the divided powers version of $\mathbb{G}_m$. As explained in §4.6.1, the end of Appendix B is related to §4. Appendix B is closely related to the material from [BL] about the "Sen operator".

In Appendices C and D we describe the Cartier dual of $\hat{\mathbb{G}}_m$ and of its "rescaled" version. This material is used in §5. As noted by the reviewer, a substantial part of Appendices C and D is contained in [MRT].

2. Formulations of the main results

We fix a prime p. Let W denote the scheme of p-typical Witt vectors; this is a ring scheme over $\mathbb{Z}$.

2.1. Some conventions

A ring in which p is nilpotent is said to be *p-nilpotent.* A scheme S is said to be *p-nilpotent* if $p \in H^0(S, \mathcal{O}_S)$ is locally nilpotent.

Unless specified otherwise, the word "stack" will mean a stack of groupoids on the category of schemes equipped with the fpqc topology.

Schemes and formal schemes are particular classes of stacks. E.g., $\operatorname{Spf} \mathbb{Z}_p$ is the functor that associates to a scheme S the set with one element if S is p-nilpotent and the empty set otherwise.

For us, $\mathbb{A}^1 := \operatorname{Spec} \mathbb{Z}[x]$. Given a stack $\mathscr{X}$, we write $\mathbb{A}^1_{\mathscr{X}} := \mathbb{A}^1 \times \mathscr{X}$. E.g., $\mathbb{A}^1_{\operatorname{Spf} \mathbb{Z}_p}$ is the Spf of the p-adic completion of $\mathbb{Z}_p[x]$.

Similarly, $\mathbb{G}_a$, $\mathbb{G}_m$, W are group schemes over $\mathbb{Z}$, from which $(\mathbb{G}_a)_{\mathscr{X}}$, $(\mathbb{G}_m)_{\mathscr{X}}$, $W_{\mathscr{X}}$ are obtained by base change to $\mathscr{X}$.

2.2. δ-schemes and δ-stacks

2.2.1. Definitions A *Frobenius lift* for a stack $\mathscr{X}$ is a morphism $F : \mathscr{X} \to \mathscr{X}$ equipped with a 2-isomorphism between the endomorphism of $\mathscr{X} \otimes \mathbb{F}_p$ induced by F and the Frobenius endomorphism of $\mathscr{X} \otimes \mathbb{F}_p$. A *$\delta$-stack* is a stack $\mathscr{X}$ equipped with a Frobenius lift.

We say "δ-structure" instead of "δ-stack structure". We say "δ-morphism" instead of "morphism of δ-stacks".

A δ-stack which is a scheme (resp. formal scheme) is called a *δ-scheme* (resp. *formal δ-scheme*).

2.2.2. Comparison with δ-rings According to [BS, Def. 2.1], a δ-ring is a ring A equipped with a map $\delta : A \to A$ satisfying certain identities. These identities ensure that the map $\phi : A \to A$ given by $\phi(a) = a^p + p\delta(a)$ is a ring homomorphism (and therefore a Frobenius lift). If A is p-torsion-free then a δ-ring structure on A is *the same* as a Frobenius lift for A or equivalently, a δ-structure on $\operatorname{Spec} A$ in the sense of §2.2.1. If A is not p-torsion-free then the two notions are different, so the definitions of §2.2.1 are not so good. However, they are convenient enough for this article (because the rings that appear in it are p-torsion-free).

2.2.3. Group δ-schemes and ring δ-schemes By a *group δ-scheme* over a δ-stack $\mathscr{X}$ we mean a group object in the category of δ-stacks equipped with a schematic[3] δ-morphism to $\mathscr{X}$. The definition of *ring δ-scheme* is similar.

2.2.4. Examples (i) The endomorphism $F : \mathbb{G}_m \to \mathbb{G}_m$ defined by $F(x) = x^p$ makes $\mathbb{G}_m$ into a group δ-scheme over $\mathbb{Z}$.

(ii) The *Witt vector Frobenius* $F : W \to W$ makes W into a ring δ-scheme over $\mathbb{Z}$.

2.3. The formal δ-scheme W_{prim}

Let us recall the material from [D3, §4.1]. The same material is contained in [BL], but the notation in [BL] is different: our W_{prim} is denoted there by WCart_0.

2.3.1. A locally closed subscheme of W Let $A \subset W \otimes \mathbb{F}_p$ be the locally closed subscheme obtained by removing $\mathrm{Ker}(W \twoheadrightarrow W_2) \otimes \mathbb{F}_p$ from $\mathrm{Ker}(W \twoheadrightarrow W_1) \otimes \mathbb{F}_p$. In terms of the usual coordinates $x_0, x_1, \dots$ on the scheme W, the subscheme $A \subset W$ is defined by the equations $p = x_0 = 0$ and the inequality $x_1 \neq 0$.

2.3.2. Definition of W_{prim} Define W_{prim} to be the formal completion of W along the locally closed subscheme A from §2.3.1. In other words, for any scheme S, an S-point of W_{prim} is a morphism $S \to W$ which maps S_{red} to A. If S is p-nilpotent and if we think of a morphism $S \to W$ as a sequence of functions $x_n \in H^0(S, \mathcal{O}_S)$ then the condition is that x_0 is locally nilpotent and x_1 is invertible. If S is not p-nilpotent then $W_{\mathrm{prim}}(S) = \emptyset$.

W_{prim} is a formal affine δ-scheme (the δ-structure is induced by the one on W, see §2.2.4). In terms of the usual coordinates $x_0, x_1, \dots$ on W, the coordinate ring of W_{prim} is the completion of $\mathbb{Z}_p[x_0, x_1, \dots][x_1^{-1}]$ with respect to the ideal (p, x_0) or equivalently, the p-adic completion of the ring $\mathbb{Z}_p[x_1, x_1^{-1}, x_2, x_3, \dots][[x_0]]$.

2.4. The δ-stack Σ

Let us recall the material from [D3, §4.2]. The same material is contained in [BL], but the notation in [BL] is different: our Σ is denoted there by WCart and called the *Cartier-Witt stack*.

[3] A morphism of stacks $\mathscr{Y} \to \mathscr{X}$ is said to be *schematic* if $\mathscr{Y} \times_{\mathscr{X}} S$ is a scheme for any scheme S equipped with a morphism to $s \to \mathscr{X}$.

2.4.1. Action of $W^\times$ on W_{prim} The morphism

$$W^\times \times W_{\mathrm{prim}} \to W_{\mathrm{prim}}, \quad (u, \xi) \mapsto u^{-1}\xi \tag{2.1}$$

defines an action of $W^\times$ on W_{prim} ("action by division"). The reason why we prefer it to the action by multiplication is explained in [D3, §4.2.6]. The difference between the two actions is irrelevant for most purposes. Note that $W^\times$ is a δ-scheme, W_{prim} is a formal δ-scheme, and (2.1) is a δ-morphism.

2.4.2. Σ as a quotient stack The δ-stack Σ is defined as follows:

$$\Sigma := W_{\mathrm{prim}}/W^\times. \tag{2.2}$$

In other words, Σ is the fpqc-sheafification of the presheaf of groupoids

$$R \mapsto \ W_{\mathrm{prim}}(R)/W(R)^\times.$$

It is also the Zariski sheafification of this presheaf (see [BL] or [D3, §4.2.2]).

2.4.3. The S-points of Σ Instead of using the definition from §2.4.2, one can use a direct description of the groupoid of S-points of Σ, where S is any scheme (see [BL] or [D3, §4.2.2]).

2.5. The group δ-scheme G'_Σ over Σ

2.5.1. The group scheme $G'_{W_{\mathrm{prim}}}$ We are going to define a flat affine commutative group δ-scheme $G'_{W_{\mathrm{prim}}}$ over W_{prim} equipped with a homomorphism

$$G'_{W_{\mathrm{prim}}} \to W^\times_{W_{\mathrm{prim}}} := W_{\mathrm{prim}} \times W^\times \tag{2.3}$$

of group δ-schemes over W_{prim}.

As a formal δ-scheme, $G'_{W_{\mathrm{prim}}} := W_{\mathrm{prim}} \times W$. The map

$$G'_{W_{\mathrm{prim}}} \times_{W_{\mathrm{prim}}} G'_{W_{\mathrm{prim}}} \to G'_{W_{\mathrm{prim}}}, \quad (\xi, x_1, x_2) \mapsto (\xi, x_1 + x_2 + \xi x_1 x_2)$$

is a group operation (to check this, use that ξ is topologically nilpotent).

The homomorphism (2.3) is given by

$$(\xi, x) \mapsto (\xi, 1 + \xi x).$$

2.5.2. The group scheme G'_Σ over Σ Recall that $\Sigma = W_{\mathrm{prim}}/W^\times$. The δ-morphism

$$W^\times \times (W_{\mathrm{prim}} \times W) \to W_{\mathrm{prim}} \times W; \quad (u, \xi, x) \mapsto (u^{-1}\xi, ux)$$

defines an action of $W^\times$ on $G'_{W_{\mathrm{prim}}} := W_{\mathrm{prim}} \times W$, which lifts the action (2.1) on W_{prim} and preserves the group structure on $G'_{W_{\mathrm{prim}}}$ and the map (2.3).

So $G'_{W_{\mathrm{prim}}}$ descends to a commutative group δ-scheme G'_Σ over Σ equipped with a δ-homomorphism

$$G'_\Sigma \to W^\times_\Sigma := W^\times \times \Sigma. \tag{2.4}$$

G'_Σ is affine and flat over Σ because $G'_{W_{\mathrm{prim}}}$ is affine and flat over W_{prim}.

2.5.3. Relation to the prismatization of $\mathbb{G}_m$ The Bhatt-Lurie approach to prismatic cohomology is based on the *prismatization functor* $X \mapsto X^{\Delta}$ from the category of p-adic formal schemes[4] to that of δ-stacks algebraic over Σ, see [D3, §1.4]. If X is a scheme over $\mathbb{Z}$ we set $X^{\Delta} := (X \hat{\otimes} \mathbb{Z}_p)^{\Delta}$, where $X \hat{\otimes} \mathbb{Z}_p$ is the p-adic completion of X.

In particular, one can apply the prismatization functor to $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$. It is easy to check that $\mathbb{G}_m^{\Delta}$ has a natural structure of strictly commutative Picard stack over Σ, and one has a canonical isomorphism of of strictly commutative Picard stacks

$$\mathbb{G}_m^{\Delta} \xrightarrow{\sim} \mathrm{Cone}(G'_\Sigma \to W^\times_\Sigma), \tag{2.5}$$

where the meaning of "Cone" is explained in [D3, §1.3.1–1.3.2]; moreover, the isomorphism (2.5) is compatible with the δ-structures. We skip the details because the isomorphism (2.5) will be used only to *motivate* the study of G'_Σ and its subgroup G_Σ introduced below.

2.6. The group δ-scheme G_Σ over Σ

2.6.1. Teichmüller embedding We have the Teichmüller embedding

$$\mathbb{G}_m \hookrightarrow W^\times$$

and the retraction $W^\times \twoheadrightarrow \mathbb{G}_m$ (to a Witt vector one assigns its 0-th component). Both $\mathbb{G}_m$ and $W^\times$ are group δ-schemes over $\mathbb{Z}$ (see §2.2.4). The Teichmüller embedding is a δ-homomorphism. The retraction $W^\times \to \mathbb{G}_m$ is a homomorphism but *not* a δ-homomorphism.

[4] A *p-adic formal scheme* is a stack X equipped with a schematic morphism $X \to \mathrm{Spf}\, \mathbb{Z}_p$.

2.6.2. Definition G_Σ is the preimage of the subgroup $(\mathbb{G}_m)_\Sigma \subset W_\Sigma^\times$ under the homomorphism (2.4). Equivalently, G_Σ is the kernel of the homomorphism

$$G'_\Sigma \to (W^\times/\mathbb{G}_m)_\Sigma \tag{2.6}$$

that comes from (2.4).

2.6.3. Pieces of structure on G_Σ Clearly, G_Σ is a commutative affine group δ-scheme over Σ equipped with a δ-homomorphism

$$G_\Sigma \to (\mathbb{G}_m)_\Sigma. \tag{2.7}$$

2.6.4. Notation For a stack $\mathscr{X}$ over Σ, we write $G_\mathscr{X}$ (resp. $G'_\mathscr{X}$) for the pullback of G_Σ (resp. G'_Σ) to $\mathscr{X}$.

2.7. Results about G_Σ

Proposition 2.7.1. *The homomorphism (2.6) is faithfully flat.*

The proof is given in §4.2.

Corollary 2.7.2. *G_Σ is flat over Σ.*

Proof. Follows from Proposition 2.7.1 because G_Σ is the kernel of (2.6). □

Combining Proposition 2.7.1 with (2.5), one gets the following

Corollary 2.7.3. *One has a canonical isomorphism of strictly commutative Picard stacks*

$$\mathbb{G}_m^{\Delta} \xrightarrow{\sim} \operatorname{Cone}(G_\Sigma \to (\mathbb{G}_m)_\Sigma), \tag{2.8}$$

compatible with the δ-structures. □

Remark 2.7.4. Combining Corollary 2.7.3 with our results on G_Σ and its Cartier dual H_Σ, one can compute the derived direct images of the structure sheaf under the morphism

$$(\mathbb{A}^1 \setminus \{0\})^{\Delta} = \mathbb{G}_m^{\Delta} \to (\operatorname{Spf} \mathbb{Z}_p)^{\Delta} = \Sigma,$$

see Appendix A. This is not really a new result but rather a new point of view[5] on a key result of [BS].

[5] The prismatization functor and the groups G_Σ, H_Σ do not appear in [BS].

Corollary 2.7.2 can be strengthened as follows.

Theorem 2.7.5. *G_Σ is the Cartier dual of some 1-dimensional commutative formal group H_Σ over Σ.*

The proof is given in §4.3.4. The precise definition of a formal group over a scheme $\mathscr{X}$ is given in §3.2; in the case that $\mathscr{X}$ is a stack see §3.3.3(ii). (According to these definitions, a formal group is locally on $\mathscr{X}$ defined by a formal group law.)

Corollary 2.7.6. *$\underline{\mathrm{Hom}}(G_\Sigma, (\mathbb{G}_a)_\Sigma)$ is a line bundle over Σ.*

Proof. By Theorem 2.7.5, $\underline{\mathrm{Hom}}(G_\Sigma, (\mathbb{G}_a)_\Sigma) = \mathrm{Lie}(H_\Sigma)$. □

Our next goal is to formulate Theorem 2.7.10, which says that the line bundle $\underline{\mathrm{Hom}}(G_\Sigma, (\mathbb{G}_a)_\Sigma)$ is canonically isomorphic to $\mathcal{O}_\Sigma\{-1\}$, i.e., the inverse of the Breuil-Kisin-Tate module[6] $\mathcal{O}_\Sigma\{1\}$. To explain the word "canonically", we have to discuss $\rho_{\mathrm{dR}}^* G_\Sigma$, where $\rho_{\mathrm{dR}} : \mathrm{Spf}\, \mathbb{Z}_p \to \Sigma$ is the "de Rham point" of Σ.

2.7.7. The "de Rham pullback" of G_Σ

The element $p \in W(\mathbb{Z}_p)$ defines a morphism

$$\mathrm{Spf}\, \mathbb{Z}_p \to W_{\mathrm{prim}}.$$

The corresponding morphism $\mathrm{Spf}\, \mathbb{Z}_p \to \Sigma$ is called the *de Rham point* of Σ and denoted by $\rho_{\mathrm{dR}} : \mathrm{Spf}\, \mathbb{Z}_p \to \Sigma$.

Let $G_{\mathrm{dR}} := \rho_{\mathrm{dR}}^* G_\Sigma$. By the definition of G_Σ, for any p-nilpotent ring A we have

$$G_{\mathrm{dR}}(A) := \{x \in W(A) \mid 1 + px \in A^\times \subset W(A)^\times\}, \tag{2.9}$$

where $A^\times \subset W(A)^\times$ is the image of the Teichmüller embedding, and the group operation on $G_{\mathrm{dR}}(A)$ is given by $(x_1, x_2) \mapsto x_1 + x_2 + px_1x_2$.

We have a canonical homomorphism

$$G_{\mathrm{dR}} \to (\mathbb{G}_a)_{\mathrm{Spf}\, \mathbb{Z}_p}, \quad x \mapsto p^{-1} \log(1 + px_0) := \sum_{n=1}^{\infty} \frac{(-p)^{n-1}}{n} x_0^n, \tag{2.10}$$

where x_0 is the 0-th component of the Witt vector x (the formula makes sense because the numbers $\frac{(-p)^{n-1}}{n}$ are in $\mathbb{Z}_p$ and converge to 0).

[6]For the definition of $\mathcal{O}_\Sigma\{1\}$, see [D3, §4.9] or [BL]; one of the equivalent definitions is essentially recalled in §2.9.6. Let us note that in [BL] our $\mathcal{O}_\Sigma\{1\}$ is called the Breuil-Kisin line bundle and denoted by $\mathcal{O}_{\mathrm{WCart}}\{1\}$.

Proposition 2.7.8. *The homomorphism (2.10) induces an isomorphism*

$$G_{\mathrm{dR}} \xrightarrow{\sim} (\mathbb{G}_a^\sharp)_{\mathrm{Spf}\,\mathbb{Z}_p},$$

where $\mathbb{G}_a^\sharp$ is the divided powers version of $\mathbb{G}_a$.

The proposition is proved in §4.4.

Corollary 2.7.9. *The homomorphism (2.10) induces an isomorphism*

$$(\mathbb{G}_a)_{\mathrm{Spf}\,\mathbb{Z}_p} \xrightarrow{\sim} \underline{\mathrm{Hom}}(G_{\mathrm{dR}}, (\mathbb{G}_a)_{\mathrm{Spf}\,\mathbb{Z}_p}). \tag{2.11}$$

Theorem 2.7.10. *There is a unique isomorphism*

$$\mathcal{O}_\Sigma\{-1\} \xrightarrow{\sim} \underline{\mathrm{Hom}}(G_\Sigma, (\mathbb{G}_a)_\Sigma) \tag{2.12}$$

whose ρ_{dR}-pullback is the isomorphism (2.11).

In the theorem and the next corollary we tacitly use that $\rho_{\mathrm{dR}}^*\mathcal{O}_\Sigma\{1\}$ is canonically trivial, see [D3, §4.9]. The existence of (2.12) is proved in §2.9.6; uniqueness follows from the equality

$$H^0(\Sigma, \mathcal{O}_\Sigma) = \mathbb{Z}_p, \tag{2.13}$$

which is proved in [D3, Cor. 4.7.2]. Combining Theorem 2.7.10 with (2.13), we get

Corollary 2.7.11. *There is a unique homomorphism $G_\Sigma \to \mathcal{O}_\Sigma\{1\}$, whose ρ_{dR}-pullback is the homomorphism (2.10).* □

A homomorphism $G_\Sigma \to \mathcal{O}_\Sigma\{1\}$ with this property is explicitly constructed in [BL] (see also [BL2, §4]); it is denoted there by $\log_{\Delta}$ and called the *prismatic logarithm*. The prismatic logarithm is used in [BL] to define the *prismatic first Chern class*.

2.7.12. Pullback of G_Σ to the Hodge-Tate divisor We have a homomorphism $W \to W_1 = \mathbb{A}^1$ (to a Witt vector it associates its 0-th coordinate). It induces a morphism

$$\Sigma = W_{\mathrm{prim}}/W^\times \to \mathbb{A}^1/\mathbb{G}_m.$$

Let $\Delta_0 \subset \Sigma$ be the preimage of $\{0\}/\mathbb{G}_m \subset \mathbb{A}^1/\mathbb{G}_m$. Then Δ_0 is an effective Cartier divisor on Σ (in the sense of [D3, §2.10–2.11]). It is called the *Hodge-Tate divisor*. Let us note that in [BL] this divisor is denoted by $\mathrm{WCart}^{\mathrm{HT}}$.

Let G_{Δ_0} be the pullback of G_Σ to Δ_0. Let $\mathscr{M}$ be the conormal line bundle of $\Delta_0 \subset \Sigma$. Let $\mathscr{M}^\sharp$ be the divided powers version of $\mathscr{M}$ (so $\mathscr{M}^\sharp$ and $\mathscr{M}$ are obtained from $\mathbb{G}_a^\sharp$ and $\mathbb{G}_a$ by twisting them with the same $\mathbb{G}_m$-torsor on Δ_0).

Proposition 2.7.13. *G_{Δ_0} is isomorphic to $\mathscr{M}^\sharp$. Accordingly, the Cartier dual of G_{Δ_0} is isomorphic to the formal completion of the line bundle $\mathscr{M}^*$ along its zero section.*

The proposition will be proved in §4.3.5.

Let us note that Proposition 2.7.13 agrees with Theorem 2.7.10 because the pullback of $\mathscr{O}_\Sigma\{1\}$ to Δ_0 is known to be canonically isomorphic to $\mathscr{M}$ (e.g., see [D3, Lemma 4.9.7(ii)] and [D3, §4.9.1]).

2.7.14. Warning Let $\bar\rho_{\mathrm{dR}} : \operatorname{Spec} \mathbb{F}_p \to \Sigma$ be the restriction of

$$\rho_{\mathrm{dR}} : \operatorname{Spf} \mathbb{Z}_p \to \Sigma.$$

Then $\bar\rho_{\mathrm{dR}}$ lands into $\Delta_0 \subset \Sigma$. So one can compute $\rho_{\mathrm{dR}}^* G_\Sigma$ using either Theorem 2.7.10 or Proposition 2.7.13. Thus we get two isomorphisms

$$\rho_{\mathrm{dR}}^* G_\Sigma \xrightarrow{\sim} (\mathbb{G}_a^\sharp)_{\mathbb{F}_p}.$$

In §4.4.6 we will see that they differ by a *non-linear* automorphism of $(\mathbb{G}_a^\sharp)_{\mathbb{F}_p}$ (there are plenty of such automorphisms because the Cartier dual of $(\mathbb{G}_a^\sharp)_{\mathbb{F}_p}$ is $(\hat{\mathbb{G}}_a)_{\mathbb{F}_p}$).

2.8. A question about G_Σ

2.8.1. By §3.5, any formal group has a canonical "degeneration" into its Lie algebra. In particular, we have a canonical formal group over $\Sigma \times \mathbb{A}^1$ whose restriction to $\Sigma \times \{1\}$ is H_Σ and whose restriction to $\Sigma \times \{0\}$ is $\mathrm{Lie}(H_\Sigma)$. By Theorem 2.7.10, $\mathrm{Lie}(H_\Sigma) = \mathscr{O}_\Sigma\{-1\}$.

2.8.2. Passing to the Cartier dual, we get a canonical affine group scheme over $\Sigma \times \mathbb{A}^1$ whose restriction to $\Sigma \times \{1\}$ is G_Σ and whose restriction to $\Sigma \times \{0\}$ is $(\mathscr{O}_\Sigma\{1\})^\sharp$ (i.e., the divided powers version of the line bundle $\mathscr{O}_\Sigma\{1\}$).

Question 2.8.3. *How to give a direct construction of the group scheme from §2.8.2?*

2.9. Results about H_Σ

2.9.1. Pieces of structure on H_Σ The δ-structure on G_Σ is a group homomorphism

$$G_\Sigma \to F^* G_\Sigma$$

whose restriction to $\Sigma \otimes \mathbb{F}_p$ is the geometric Frobenius. Dualizing this, we get a group homomorphism

$$\varphi : F^* H_\Sigma \to H_\Sigma \tag{2.14}$$

whose restriction to $\Sigma \otimes \mathbb{F}_p$ is the Verschiebung.

The homomorphism (2.7) yields a section

$$s : \Sigma \to H_\Sigma. \tag{2.15}$$

Since (2.7) is a δ-homomorphism, we have

$$\varphi(F^* s) = ps \tag{2.16}$$

(when writing ps we are using the additive notation for the group operation in H_Σ).

Theorem 2.9.2. *Let $s : \Sigma \to H_\Sigma$ and $\varphi : F^* H_\Sigma \to H_\Sigma$ be as in §2.9.1. Then*

(i) $s^{-1}(0_\Sigma) = \Delta_0$, where $0_\Sigma \subset H_\Sigma$ is the zero section and $\Delta_0 \subset \Sigma$ is the Hodge-Tate divisor (see §2.7.12);

(ii) $\varphi : F^* H_\Sigma \to H_\Sigma$ factors as $F^* H_\Sigma \xrightarrow{\sim} H_\Sigma(-\Delta_0) \to H_\Sigma$.

Here $H_\Sigma(-\Delta_0)$ is the formal group obtained from H_Σ by *rescaling* via the invertible subsheaf $\mathcal{O}_\Sigma(-\Delta_0) \subset \mathcal{O}_\Sigma$, see §3.4. If you wish, $H_\Sigma(-\Delta_0)$ is a formal group equipped with a homomorphism $H_\Sigma(-\Delta_0) \to H_\Sigma$ vanishing at Δ_0 and universal with this property (see Lemma 3.4.9 and Proposition 3.6.3).

A proof of Theorem 2.9.2 is given in §4.8.

Corollary 2.9.3. *The substack of zeros of the section $p^n s$ equals $\Delta_0 + \cdots + \Delta_n$, where $\Delta_i := (F^i)^{-1}(\Delta_0)$.*

Proof. Combine Theorem 2.9.2 with (2.16). ☐

2.9.4. The "de Rham pullback" of H_Σ Let $H_{\mathrm{dR}} := \rho_{\mathrm{dR}}^* H_\Sigma$, where $\rho_{\mathrm{dR}} : \operatorname{Spf} \mathbb{Z}_p \to \Sigma$ is as in §2.7.7. Then H_{dR} is a formal group over $\operatorname{Spf} \mathbb{Z}_p$ equipped with the following pieces of structure. First, (2.15) induces $s_{\mathrm{dR}} : \operatorname{Spf} \mathbb{Z}_p \to H_{\mathrm{dR}}$. Second, (2.14) induces a homomorphism $\varphi_{\mathrm{dR}} : H_{\mathrm{dR}} \to H_{\mathrm{dR}}$ (here we use that $F \circ \rho_{\mathrm{dR}} = \rho_{\mathrm{dR}}$).

Proposition 2.9.5. *(i) There exists a unique isomorphism*

$$(H_{\mathrm{dR}}, s_{\mathrm{dR}}) \xrightarrow{\sim} ((\hat{\mathbb{G}}_a)_{\operatorname{Spf}\mathbb{Z}_p}, p : \operatorname{Spf}\mathbb{Z}_p \to (\hat{\mathbb{G}}_a)_{\operatorname{Spf}\mathbb{Z}_p}),$$

where $(\hat{\mathbb{G}}_a)_{\operatorname{Spf}\mathbb{Z}_p}$ *is the formal additive group over* $\operatorname{Spf}\mathbb{Z}_p$*.*

(ii) φ_{dR} *equals* $p \in \operatorname{End} H_{\mathrm{dR}}$*.*

Proof. Uniqueness in (i) is obvious. Existence in (i) follows from Proposition 2.7.8 because $\hat{\mathbb{G}}_a$ is Cartier dual to $\mathbb{G}_a^\sharp$ via the pairing

$$\hat{\mathbb{G}}_a \times \mathbb{G}_a^\sharp \to \mathbb{G}_m, \quad (u, v) \mapsto \exp(uv).$$

Statement (ii) follows from (i) because $\varphi(s_{\mathrm{dR}}) = p s_{\mathrm{dR}}$ by (2.16). □

2.9.6. Proof of Theorem 2.7.10 $\underline{\operatorname{Hom}}(G_\Sigma, (\mathbb{G}_a)_\Sigma) = \operatorname{Lie}(H_\Sigma)$ because H_Σ is dual to G_Σ. By Theorem 2.9.2(ii) and Proposition 2.9.5(i), $\operatorname{Lie}(H_\Sigma)$ is a line bundle on Σ equipped with an isomorphism

$$F^* \operatorname{Lie}(H_\Sigma) \xrightarrow{\sim} \operatorname{Lie}(H_\Sigma)(-\Delta_0)$$

and a trivialization of $\rho_{\mathrm{dR}}^* \operatorname{Lie}(H_\Sigma)$. So one has a canonical isomorphism $\operatorname{Lie}(H_\Sigma) \xrightarrow{\sim} \mathcal{O}_\Sigma\{-1\}$, see [D3, §4.9]. The corresponding isomorphism

$$\underline{\operatorname{Hom}}(G_\Sigma, (\mathbb{G}_a)_\Sigma) \xrightarrow{\sim} \mathcal{O}_\Sigma\{-1\}$$

has the desired property. □

2.10. The pullback of H_Σ to the q-de Rham prism Q

2.10.1. Recollections on Q Let $Q := \operatorname{Spf}\mathbb{Z}_p[[q-1]]$, where $\mathbb{Z}_p[[q-1]]$ is equipped with the $(p, q-1)$-adic topology. Define $F : Q \to Q$ by $q \mapsto q^p$. Then (Q, F) is a formal δ-scheme. More abstractly, Q is the formal δ-scheme underlying the formal group δ-scheme $(\hat{\mathbb{G}}_m)_{\operatorname{Spf}\mathbb{Z}_p}$ over $\operatorname{Spf}\mathbb{Z}_p$, and the δ-structure on Q comes from the δ-structure on $\mathbb{G}_m$ introduced in §2.2.4.

Let Φ_p denote the cyclotomic polynomial. The element

$$\Phi_p([q]) = 1 + [q] + \cdots [q^{p-1}] \in W(\mathbb{Z}_p[[q-1]])$$

defines a morphism $Q \to W$ and, in fact, a morphism $Q \to W_{\mathrm{prim}}$. This is a δ-morphism because $F(\Phi_p([q]) = \Phi_p(q^p)$. Let $\pi : Q \to \Sigma$ be the composite morphism $Q \to W_{\mathrm{prim}} \to \Sigma$. It is known that π is faithfully flat. For us, the *q-de Rham prism* is the pair (Q, π).

Set $(\Delta_0)_Q := \Delta_0 \times_\Sigma Q \subset Q$; by the definition of π, we have

$$(\Delta_0)_Q = \operatorname{Spf} \mathbb{Z}_p[[q-1]]/(\Phi_p(q)) \subset \operatorname{Spf} \mathbb{Z}_p[[q-1]] = Q. \tag{2.17}$$

More details about (Q, π) can be found in [BL] and [D3, Appendix B].

2.10.2. Pieces of structure on H_Q Let $G_Q := \pi^* G_\Sigma$, $H_Q := \pi^* H_\Sigma$. By definition, a section $Q \to G_Q$ is the same as an element $x \in W(\mathbb{Z}_p[[q-1]])$ such that $1 + x\Phi_p([q])$ is Teichmüller. We will use the section $\sigma : Q \to G_Q$ corresponding to $x = [q] - 1$ (then $1 + x\Phi_p([q]) = [q^p]$). It is easy to see that $\sigma : Q \to G_Q$ is a δ-morphism. The section σ is a key advantage of Q over Σ.

Since G_Q is dual to H_Q, the section $\sigma : Q \to G_Q$ defines a homomorphism

$$\sigma^* : H_Q \to (\hat{\mathbb{G}}_m)_Q. \tag{2.18}$$

On the other hand, base-changing the pieces of structure on H_Σ described in §2.9.1, we get similar pieces of structure on H_Q. Namely, we get a group homomorphism

$$\varphi_Q : F^* H_Q \to H_Q \tag{2.19}$$

whose restriction to $Q \otimes \mathbb{F}_p$ is the Verschiebung and a section

$$s_Q : Q \to H_Q. \tag{2.20}$$

such that

$$\varphi_Q(F^* s_Q) = p s_Q \tag{2.21}$$

(when writing ps_Q we are using the additive notation for the group operation in H_Q).

(2.18) interacts with (2.19)–(2.20) as follows.

Lemma 2.10.3. *(i) The following diagram commutes:*

$$\begin{array}{ccc} F^* H_Q & \xrightarrow{\varphi_Q} & H_Q \\ \downarrow{\scriptstyle F^*(\sigma_Q^*)} & & \downarrow{\scriptstyle \sigma_Q^*} \\ (\hat{\mathbb{G}}_m)_Q & \xrightarrow{\mathrm{id}} & (\hat{\mathbb{G}}_m)_Q \end{array}$$

(ii) $\sigma^* \circ s_Q = q^p \in \hat{\mathbb{G}}_m(Q)$.

Proof. Statement (i) follows from σ being a δ-morphism.

Composing $\sigma_Q : Q \to G_Q$ with the homomorphism $G_Q \to (\mathbb{G}_m)_Q$ that comes from (2.7), we get $1 + (q-1) \cdot \Phi_p(q) = q^p \in \mathbb{G}_m(Q)$. Statement (ii) follows. ☐

Theorem 2.10.4. *The homomorphism $\sigma^* : H_Q \to (\hat{\mathbb{G}}_m)_Q$ induces an isomorphism*

$$H_Q \xrightarrow{\sim} (\hat{\mathbb{G}}_m)_Q(-D),$$

where $D \subset Q$ is the divisor $q = 1$.

The proof is given in §4.7.3.

2.10.5. H_Q as a formal scheme By Theorem 2.10.4, H_Q identifies with $(\hat{\mathbb{G}}_m)_Q(-D)$. So the formal scheme H_Q can be obtained as follows: first, blow up the formal scheme

$$(\hat{\mathbb{G}}_m)_Q = Q \times \hat{\mathbb{G}}_m = \operatorname{Spf} \mathbb{Z}_p[[q-1, q'-1]]$$

along the subscheme $q = q' = 1$, then H_Q is the formal completion of the blow-up along the strict preimage of the unit section of $(\hat{\mathbb{G}}_m)_Q$. In other words,

$$H_Q = \operatorname{Spf} \mathbb{Z}_p[[q-1, z]], \quad \text{where } z = \frac{q'-1}{q-1}. \tag{2.22}$$

2.10.6. The formal group H_Q in explicit terms In terms of the coordinate z from (2.22), H_Q corresponds to the following formal group law over $\mathbb{Z}_p[[q-1]]$:

$$z_1 * z_2 = \frac{(1+(q-1)z_1)(1+(q-1)z_2) - 1}{q-1} = z_1 + z_2 + (q-1)z_1 z_2. \tag{2.23}$$

Let us describe in these terms the pieces of structure on H_Q defined in §2.10.2. The homomorphism $\sigma^* : H_Q \to (\hat{\mathbb{G}}_m)_Q$ is just the map $(q,z) \mapsto (q, 1+(q-1)z)$. By Lemma 2.10.3(ii), the section $s_Q : Q \to H_Q$ is given by $z = \frac{q^p-1}{q-1} = \Phi_p(q)$. It remains to describe the homomorphism $\varphi_Q : F^* H_Q \to H_Q$. The formal group $F^* H_Q$ corresponds to the group law

$$y_1 * y_2 = y_1 + y_2 + (q^p - 1) y_1 y_2, \tag{2.24}$$

which is the F-pullback of (2.23). By Lemma 2.10.3(i), the homomorphism φ_Q is the homomorphism from (2.24) to (2.23) given by $z = \Phi_p(q) \cdot y$.

2.10.7. The group scheme $\boldsymbol{H_Q^{\mathrm{alg}}}$ Let $H_Q^{\mathrm{alg}} := \operatorname{Spf} A$, where A is the completion of $\mathbb{Z}_p[q, z]$ for the $(p, q-1)$-adic topology. The morphism $H_Q^{\mathrm{alg}} \to \operatorname{Spf} \mathbb{Z}_p[[q-1]] = Q$ is affine (in particular, schematic). The r.h.s. of (2.23) is a polynomial, so it gives a morphism

$$H_Q^{\mathrm{alg}} \times_Q H_Q^{\mathrm{alg}} \to H_Q^{\mathrm{alg}}.$$

This morphism makes H_Q^{alg} into a smooth affine group scheme over Q. The formal completion of H_Q^{alg} along its unit identifies with H_Q (if you wish, H_Q^{alg} is an *algebraization* of the formal group H_Q in the sense of §2.12.1). The homomorphism $\varphi_Q : F^* H_Q \to H_Q$ comes from a homomorphism $F^* H_Q^{\mathrm{alg}} \to H_Q^{\mathrm{alg}}$. The group H_Q^{alg} has a remarkable section $\tilde{s}_Q : Q \to H_Q^{\mathrm{alg}}$ given by $z = 1$; one has a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{\tilde{s}_Q} & H_Q^{\mathrm{alg}} \\ {\scriptstyle s_Q}\downarrow & & \downarrow{\scriptstyle p} \\ H_Q & \hookrightarrow & H_Q^{\mathrm{alg}} \end{array}$$

(s_Q was defined in §2.20 and described in §2.10.6).

2.10.8. Restriction of $\boldsymbol{H_Q^{\mathrm{alg}}}$ to $\boldsymbol{(\Delta_0)_Q}$ To get a feel of H_Q^{alg} let us discuss its restriction to $(\Delta_0)_Q$.

Recall that $(\Delta_0)_Q := \Delta_0 \times_\Sigma Q$, where $\Delta_0 \subset \Sigma$ is the Hodge-Tate divisor; explicitly, $(\Delta_0)_Q = \operatorname{Spf} \mathbb{Z}_p[[q-1]]/(\Phi_p(q)) \subset \operatorname{Spf} \mathbb{Z}_p[[q-1]] = Q$. Let $\zeta \in \mathbb{Z}_p[[q-1]]/(\Phi_p(q))$ be the image of q; then ζ is a primitive p-th root of 1.

Let $H_{(\Delta_0)_Q}^{\mathrm{alg}}$ be the restriction of H_Q^{alg} to $(\Delta_0)_Q$. It is easy to check that one has an exact sequence

$$0 \to (\mathbb{Z}/p\mathbb{Z})_{(\Delta_0)_Q} \xrightarrow{i} H_{(\Delta_0)_Q}^{\mathrm{alg}} \xrightarrow{\lambda} (\mathbb{G}_a)_{(\Delta_0)_Q} \to 0; \tag{2.25}$$

here i takes $1 \in \mathbb{Z}/p\mathbb{Z}$ to the section $\tilde{s}_{(\Delta_0)_Q} : (\Delta_0)_Q \to H_{(\Delta_0)_Q}$ given by $z = 1$ (then $1 + (\zeta - 1)z = \zeta$ is a p-th root of unity), and λ is given by $(\zeta - 1)^{-1} \cdot \log(1 + (\zeta - 1)z)$ (which is a power series in z whose coefficients are in $\mathbb{Z}_p[\zeta] = \mathbb{Z}_p[[q-1]]/(\Phi_p(q))$ and converge to 0).

The exact sequence (2.25) shows that $H_{(\Delta_0)_Q}$ is isomorphic to $(\hat{\mathbb{G}}_a)_{(\Delta_0)_Q}$. But $H_{(\Delta_0)_Q}^{\mathrm{alg}}$ is not isomorphic to $(\mathbb{G}_a)_{(\Delta_0)_Q}$ because

$$\operatorname{Hom}((\mathbb{Z}/p\mathbb{Z})_{(\Delta_0)_Q}, (\mathbb{G}_a)_{(\Delta_0)_Q}) = 0.$$

2.11. The action of $\mathbb{Z}_p^\times$ on H_Q

We keep the notation of §2.10.

2.11.1. The action of $\mathbb{Z}_p^\times$ on Q The pro-finite (and therefore pro-algebraic) group $\mathbb{Z}_p^\times$ acts on the formal δ-scheme $Q = \operatorname{Spf} \mathbb{Z}_p[[q-1]]$: the automorphism of Q corresponding to $n \in \mathbb{Z}_p^\times$ is

$$q \mapsto q^n = \sum_{i=0}^{\infty} \frac{n(n-1)\dots(n-i+1)}{i!}(q-1)^i.$$

In terms of the identification $Q = (\hat{\mathbb{G}}_m)_{\operatorname{Spf} \mathbb{Z}_p}$, this action comes from the isomorphism

$$\mathbb{Z}_p \xrightarrow{\sim} \operatorname{End}((\hat{\mathbb{G}}_m)_{\operatorname{Spf} \mathbb{Z}_p}).$$

2.11.2. The action of $\mathbb{Z}_p^\times$ on H_Q It is easy to show that the morphism $\pi : Q \to \Sigma$ from §2.10.1 factors through the quotient stack $Q/\mathbb{Z}_p^\times$ (see [BL] or [D3, Appendix B]). Therefore the formal group scheme H_Q is $\mathbb{Z}_p^\times$-equivariant.

The morphisms $\varphi_Q : F^* H_Q \to H_Q$ and $s_Q : Q \to H_Q$ are $\mathbb{Z}_p^\times$-equivariant because they are π-pullbacks of $\varphi : F^* H_\Sigma \to H_\Sigma$ and $s : \Sigma \to H_\Sigma$.

Proposition 2.11.3. *The morphism $\sigma^* : H_Q \to (\hat{\mathbb{G}}_m)_Q := (\hat{\mathbb{G}}_m)_{\operatorname{Spf} \mathbb{Z}_p} \times Q$ is $\mathbb{Z}_p^\times$-equivariant assuming that $(\hat{\mathbb{G}}_m)_{\operatorname{Spf} \mathbb{Z}_p}$ is equipped with the following $\mathbb{Z}_p^\times$-action:*[7] *$n \in \mathbb{Z}_p^\times$ acts as raising to the power of n.*

The proof is given in §4.9. Proposition 2.11.3 means that if we think of H_Q as an affine blow-up of $(\hat{\mathbb{G}}_m)_{\operatorname{Spf} \mathbb{Z}_p} \times Q$ (see §2.10.5) then the action of $\mathbb{Z}_p^\times$ on H_Q is the most natural one.

Corollary 2.11.4. *In terms of §2.10.6, the action of $n \in \mathbb{Z}_p^\times$ on H_Q is given by*

$$(q, z) \mapsto \left(q^n, \frac{h_n(z,q)}{h_n(1,q)}\right),$$

where

$$h_n(z,q) = \frac{(1+(q-1)z)^n - 1}{q-1} = \sum_{i=1}^{\infty} \frac{n(n-1)\dots(n-i+1)}{i!} z^i (q-1)^{i-1}$$

(so $h_n(1,q) = \frac{q^n-1}{q-1}$). □

[7] This $\mathbb{Z}_p^\times$-action is the same as the one from §2.11.1 (recall that Q is just the formal scheme underlying the formal group $(\hat{\mathbb{G}}_m)_{\operatorname{Spf} \mathbb{Z}_p}$).

Remark 2.11.5. Corollary 2.11.4 combined with §2.10.6 gives a complete description of the image of the formal group H_Σ under the pullback functor

$$\{\text{Formal groups over } \Sigma\} \to \{\mathbb{Z}_p^\times\text{-equivariant formal groups over } Q\}. \tag{2.26}$$

If $p > 2$ this functor is fully faithful by [BL, Thm. 3.8.3], so our description of the image of H_Σ under (2.26) could be considered as a (not very good) description of H_Σ itself.

Remark 2.11.6. Let H_Q^{alg} be as in §2.10.7. The action of $\mathbb{Z}_p^\times$ on H_Q comes from an action of $\mathbb{Z}_p^\times$ on H_Q^{alg}; the latter is given by the formula from Corollary 2.11.4 (this formula makes sense in the context of H_Q^{alg} because the reduction of $h_n(z, q)$ modulo any power of $q - 1$ is a polynomial in z).

2.12. A conjectural algebraization of H_Σ

2.12.1. Algebraizations of formal groups Let H be a formal group over a stack $\mathscr{X}$. By an *algebraization* of H we mean an isomorphism class of pairs consisting of a smooth affine group scheme G over $\mathscr{X}$ with connected fibers and an isomorphism $H \xrightarrow{\sim} \hat{G}$, where $\hat{G}$ is the formal completion of G along its unit. Let $\mathrm{Alg}(H)$ denote the set of algebraizations of H.

2.12.2. The sheaf property of Alg Suppose that in addition to $\mathscr{X}$ and H, we are given a morphism of stacks $\mathscr{X}' \to \mathscr{X}$ such that the corresponding morphism of fpqc-sheaves of sets is surjective (in other words, for every scheme S and every morphism $S \to \mathscr{X}$, the morphism $\mathscr{X}' \times_{\mathscr{X}} S \to S$ has a section fpqc-locally on S). Then we have an exact sequence of sets

$$\mathrm{Alg}(H) \to \mathrm{Alg}(H') \rightrightarrows \mathrm{Alg}(H''),$$

where H' and H'' are the pullbacks of H to $\mathscr{X}'$ and $\mathscr{X}' \times_{\mathscr{X}} \mathscr{X}'$, respectively. In particular, the map $\mathrm{Alg}(H) \to \mathrm{Alg}(H')$ is injective.

2.12.3. Good news (i) By Theorem 2.9.2(ii), $F^* H_\Sigma = H_\Sigma(-\Delta_0)$. On the other hand, $H_\Sigma(-\Delta_0)$ has a canonical algebraization constructed in §3.6.4 "by pure thought". Thus we get a canonical element $\alpha \in \mathrm{Alg}(F^* H_\Sigma)$.

(ii) The morphism $F : \Sigma \to \Sigma$ satisfies the condition of §2.12.2 (because $F : W \to W$ is faithfully flat). So the canonical map $\mathrm{Alg}(H_\Sigma) \to \mathrm{Alg}(F^* H_\Sigma)$ is injective. Thus α comes from at most one algebraization of H_Σ.

Conjecture 2.12.4. *Such an algebraization of H_Σ exists.*

The conjectural algebraization of H_Σ will be denoted by H_Σ^{alg}.

2.12.5. Evidence in favor of Conjecture 2.12.4 Even though H_Σ^{alg} is conjectural, the corresponding algebraizations of H_Q and H_{Δ_0} are *unconditional,* as explained below.

(i) Let α be as in §2.12.3(i). Then the image of α in $\mathrm{Alg}(F^*H_Q)$ comes from a (unique) element $\beta \in \mathrm{Alg}(H_Q)$, namely the one described in §2.10.7.

(ii) Let $\beta_0 \in \mathrm{Alg}(H_{(\Delta_0)_Q})$ be the image of β. Using the explicit description of β_0 from §2.10.8, one can check that β_0 comes from a (unique) algebraization $H_{\Delta_0}^{\mathrm{alg}}$ of H_{Δ_0}. Namely, while H_{Δ_0} is the formal completion of a certain line bundle $\mathscr{M}^*$ over Δ_0 (see Proposition 2.7.13), $H_{\Delta_0}^{\mathrm{alg}}$ is a $(\mathbb{Z}/p\mathbb{Z})$-covering of $\mathscr{M}^*$.

3. Generalities on formal groups and their Cartier duals

3.1. The notion of based formal S-polydisk

3.1.1. Notation If S is a scheme then the formal completion of $\mathbb{A}_S^n := \mathbb{A}^n \times S$ along its zero section will be denoted by $\hat{\mathbb{A}}_S^n$.

3.1.2. Definition Let S be a scheme. Let X be a formal S-scheme and $\sigma : S \to X$ a section. We say that (X, σ) is a *based formal S-polydisk* if Zariski-locally on S there exists an S-isomorphism $(X, \sigma) \xrightarrow{\sim} (\hat{\mathbb{A}}_S^n, 0)$ for some $n \in \mathbb{Z}_+$; here $0 : S \to \hat{\mathbb{A}}_S^n$ is the zero section.

3.1.3. Notation The category of based formal S-polydisks will be denoted by $\mathrm{Polyd}(S)$. For fixed $n \in \mathbb{Z}_+$, let $\mathrm{Polyd}_n(S) \subset \mathrm{Polyd}(S)$ be the full subcategory of based formal S-polydisks of dimension n (i.e., locally isomorphic to $(\hat{\mathbb{A}}_S^n, 0)$).

3.1.4. Automorphisms of $(\hat{\mathbb{A}}_S^n, 0)$ The functor that to a scheme S associates the group of S-automorphisms of $(\hat{\mathbb{A}}_S^n, 0)$ is representable by an affine group scheme $\mathscr{D}_n$ over $\mathbb{Z}$.

Lemma 3.1.5. *The underlying groupoid of* $\mathrm{Polyd}_n(S)$ *is canonically equivalent to that of $\mathscr{D}_n$-torsors on S.*

Proof. It suffices to show that any $\mathscr{D}_n$-torsor on S is Zariski-locally trivial. Indeed, $\mathscr{D}_n$ can be represented as a projective limit of a diagram of group schemes

$$\ldots \to G_2 \to G_1 \to G_0 = GL(n)$$

in which all morphisms are faithfully flat and for each n the group scheme $\mathrm{Ker}(G_{n+1} \to G_n)$ is isomorphic to a power of $\mathbb{G}_a$. □

Corollary 3.1.6. *The assignment $S \mapsto$* $\mathrm{Polyd}(S)$ *is a stack for the fpqc topology (not merely the Zariski topology).* □

3.2. Formal groups

Let $\mathfrak{F}(S)$ (resp. $\mathfrak{F}_n(S)$) be the category of group objects in $\operatorname{Polyd}(S)$ (resp. in $\operatorname{Polyd}_n(S)$). Objects of $\mathfrak{F}(S)$ will be called *formal groups* over S; in other words, by a formal group over S we mean a group object H in the category of formal S-schemes such that the pair $(H, e: S \to H)$ is a based formal S-polydisk. Objects of $\mathfrak{F}_n(S)$ are called n-dimensional formal groups.

3.3. Cartier duals of commutative formal groups

Lemma 3.3.1. *Let S be a scheme and $H \in \mathfrak{F}^{\mathrm{com}}(S)$. Then the Cartier dual H^* exists as a flat affine group scheme over S. Moreover, $H^* = \operatorname{Spec} \mathcal{A}$, where the quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{A}$ is locally free as an $\mathcal{O}_S$-module.*

Proof. We can assume that S is affine and that the based formal polydisk $(H, e: S \to H)$ is isomorphic to $(\hat{\mathbb{A}}^n_S, 0)$. Let $A := H^0(S, \mathcal{O}_S)$. Then the coordinate ring of H (viewed as a topological A-module) is the dual of a free A-module. The lemma follows. □

3.3.2. Notation Let $\mathfrak{F}^*(S)$ be the full subcategory of the category of group S-schemes formed by Cartier duals of commutative formal groups over S.

3.3.3. Remarks (i) The assignments $S \mapsto \mathfrak{F}(S)$ and $S \mapsto \mathfrak{F}^*(S)$ are stacks for the fpqc topology (not merely the Zariski topology). This follows from Corollary 3.1.6.

(ii) By Corollary 3.1.6 and the previous remark, if S is a fpqc-stack rather than a scheme one can still talk about $\operatorname{Polyd}(S)$, $\mathfrak{F}(S)$, and $\mathfrak{F}^*(S)$.

Proposition 3.3.4. *Let S be a scheme and $S_0 \subset S$ a closed subscheme whose ideal is nilpotent. Let G be a flat commutative group scheme over S such that $G \times_S S_0 \in \mathfrak{F}^*(S_0)$. Then $G \in \mathfrak{F}^*(S)$.*

As pointed out by the reviewer, the above proposition appears as Lemma 1.1.21 in J. Lurie's work [Lu]. Moreover, the Cartier duals of commutative formal groups play an important role in [Lu]

Proof. We can assume that $S = \operatorname{Spec} A$ and $S_0 = \operatorname{Spec} A_0$, where $A_0 = A/I$ and $I^2 = 0$. We can also assume the existence of an isomorphism of based formal S_0-polydisks

$$(G_0^*, e) \xrightarrow{\sim} (\hat{\mathbb{A}}^n_S, 0),$$

where G_0^* is the Cartier dual of G_0. To simplify notation, we will assume that $n = 1$.

G_0 is affine because $G_0 \in \mathfrak{F}^*(S_0)$. So G is affine. Let B be the coordinate ring of G and $B_0 := B \otimes_A A_0$. Then $G = \operatorname{Spec} B$ and $G_0 = \operatorname{Spec} B_0$.

Let $B^* := \operatorname{Hom}_A(B, A)$ and $B_0^* := \operatorname{Hom}_{A_0}(B_0, A_0)$ be the dual modules. We equip them with the weak topology. The coproduct in B and B_0 yields a topological algebra structure on B^* and B_0^*.

By assumption, we have an isomorphism of based formal S_0-disks

$$(G_0^*, e) \xrightarrow{\sim} (\hat{\mathbb{A}}^1_S, 0).$$

It induces an isomorphism of topological algebras $f_0 : A_0[[x]] \xrightarrow{\sim} B_0^*$ such that $l_0(1) = 0$, where $l_0 := f_0(x) \in B_0^*$ and $1 \in B_0$ is the unit. We will lift it to an isomorphism $f : A[[x]] \xrightarrow{\sim} B$ such that $l(1) = 0$, where $l = f(x) \in B^*$. This will show that $\operatorname{Spf} B^*$ is a formal group over $S = \operatorname{Spec} A$, whose Cartier dual is G.

The A_0-module B_0 is free because f_0^* identifies B_0 with the topological dual $(A_0[[x]])^*$, which is a free A_0-module. By assumption, B is flat over $\operatorname{Spec} A$. So B is a free A-module. Therefore we can lift l_0 to an element $l \in B^*$. Moreover, adding to l a multiple of the counit of B, we can achieve the equality $l(1) = 0$.

Let us prove that $l^n \to 0$. The problem is to show that for every $b \in B$ we have $l^n(b) = 0$ for big enough n. Let $F \subset B$ be a finitely generated A-submodule such that the coproduct $\Delta : B \to B \otimes_A B$ takes b to $\operatorname{Im}(F \otimes_A F \to B \otimes_A B)$. Since $l_0^n \to 0$, there exists $m \in \mathbb{N}$ such that for $n \geq m$ one has $l^n(F) \subset I := \operatorname{Ker}(B \to B_0)$. Then for $n \geq 2m$ one has $l^n(b) = (l^m \otimes l^{n-m})(\Delta(b)) \in I^2 = 0$.

Since $l^n \to 0$, there is a homomorphism of topological A-algebras $f : A[[x]] \to B$ such that $f(x) = l$. The dual map $f^* : B^* \to (A[[x]])^*$ is a homomorphism of free A-modules inducing an isomorphism modulo I. Therefore f^* is an isomorphism, and so is f. □

3.4. Rescaling formal groups

3.4.1. The monoidal category $\mathscr{M}(S)$ Given a scheme S, let $\mathscr{M}(S)$ be the category of pairs $(\mathscr{L}, \alpha : \mathscr{L} \to \mathcal{O}_S)$, where $\mathscr{L}$ is an invertible $\mathcal{O}_S$-module; this is a monoidal category with respect to tensor product.

Let $\mathscr{M}_{\text{inj}}(S) \subset \mathscr{M}(S)$ be the full monoidal subcategory of pairs $(\mathscr{L}, \alpha)$ such that $\operatorname{Ker} \alpha = 0$. In fact, the category $\mathscr{M}_{\text{inj}}(S)$ is an ordered set, which identifies with the set $\operatorname{Div}_+(S)$ of effective Cartier divisors on S equipped with the ordering opposite to the usual one: the invertible subsheaf of $\mathscr{L} \subset \mathcal{O}_S$ corresponding to $\operatorname{Div}_+(S)$ is $\mathcal{O}_S(-D)$. Moreover, the tensor product in

$\mathscr{M}_{\mathrm{inj}}(S)$ corresponds to addition in $\mathrm{Div}_+(S)$. For this reason, objects of $\mathscr{M}(S)$ are called *generalized Cartier divisors* in [BL].

Let $\mathscr{M}_{\mathrm{nilp}}(S) \subset \mathscr{M}(S)$ be the full subcategory of pairs $(\mathscr{L}, \alpha)$ such that α vanishes on S_{red}. One has $\mathscr{M}_{\mathrm{nilp}}(S) \cap \mathscr{M}_{\mathrm{inj}}(S) = \emptyset$.

The assignment $S \mapsto \mathscr{M}(S)$ is an fpqc-stack of monoidal categories.[8] For any $f : S' \to S$ one has $f^*(\mathscr{M}_{\mathrm{nilp}}(S)) \subset \mathscr{M}_{\mathrm{nilp}}(S')$; if f is flat then $f^*(\mathscr{M}_{\mathrm{inj}}(S)) \subset \mathscr{M}_{\mathrm{inj}}(S')$.

3.4.2. Remark The unit object of $\mathscr{M}(S)$ is a final object.

3.4.3. Goal We have the stack of monoidal categories $\mathscr{M}$ from §3.4.1. In §3.4.6 we will define an action of $\mathscr{M}$ on Polyd and on $\mathfrak{F}$, where Polyd is the stack of based formal polydisks (see §3.1) and $\mathfrak{F}$ is the stack of formal groups (see §3.2).

3.4.4. The prestacks $\mathrm{Polyd}_{\mathrm{pre}}$ and $\mathscr{M}_{\mathrm{pre}}$ Let $\mathrm{Polyd}_{\mathrm{pre}}(S) \subset \mathrm{Polyd}(S)$ be the full subcategory formed by formal schemes $\hat{\mathbb{A}}^n_S$. Then $\mathrm{Polyd}_{\mathrm{pre}}$ is a prestack of categories such that the associated fpqc-stack is Polyd.

Let $\mathscr{M}_{\mathrm{pre}}(S) \subset \mathscr{M}(S)$ be the full subcategory of pairs $(\mathscr{L}, \alpha)$ with $\mathscr{L} = \mathcal{O}_S$. Then $\mathscr{M}_{\mathrm{pre}}$ is a prestack of monoidal categories such that the associated fpqc-stack is $\mathscr{M}$. Explicitly, $\mathrm{Ob}\,\mathscr{M}_{\mathrm{pre}}(S) = H^0(S, \mathcal{O}_S)$, a morphism from $\alpha \in H^0(S, \mathcal{O}_S)$ to $\alpha' \in H^0(S, \mathcal{O}_S)$ is a presentation of α as $\alpha'\alpha''$, one has $\alpha_1 \otimes \alpha_2 = \alpha_1\alpha_2$, and so on. In other words, $\mathscr{M}_{\mathrm{pre}}(S)$ is obtained as follows: start with the multiplicative monoid $H^0(S, \mathcal{O}_S)$ viewed as a discrete monoidal category, then add morphisms $\psi_\alpha : \alpha \to 1$, subject to the relations $\psi_{\alpha_1\alpha_2} = \psi_{\alpha_1} \otimes \psi_{\alpha_2}$.

3.4.5. Action of $\mathscr{M}_{\mathrm{pre}}$ on $\mathrm{Polyd}_{\mathrm{pre}}$ (i) First, let us define a strict action of the multiplicative monoid $H^0(S, \mathcal{O}_S)$ on the category $\mathrm{Polyd}_{\mathrm{pre}}(S)$, which is trivial at the level of objects of $\mathrm{Polyd}_{\mathrm{pre}}(S)$. To this end, note that a morphism $\hat{\mathbb{A}}^m_S \to \hat{\mathbb{A}}^n_S$ is just a collection

$$(f_1, \ldots, f_n), \quad f_i \in H^0(S, \mathcal{O}_S)[[x_1, \ldots x_m]], \quad f_i(0) = 0.$$

Definition: $\alpha \in H^0(S, \mathcal{O}_S)$ takes $(f_1, \ldots, f_n)$ to $(\tilde{f}_1, \ldots, \tilde{f}_n)$, where

$$\tilde{f}_i(x_1, \ldots x_m) := \alpha^{-1} f_i(\alpha x_1, \ldots \alpha x_m). \tag{3.1}$$

The r.h.s. of (3.1) makes sense (even though α^{-1} is not assumed to exist) because $f_i(0) = 0$.

[8] The same stack is introduced in [D3], where it is denoted by $(\mathbb{A}^1/\mathbb{G}_m)_-$.

(ii) Let $\Phi_\alpha : \mathrm{Polyd}_{\mathrm{pre}}(S) \to \mathrm{Polyd}_{\mathrm{pre}}(S)$ be the functor corresponding to $\alpha \in H^0(S, \mathcal{O}_S)$. The explicit description of $\mathscr{M}_{\mathrm{pre}}(S)$ (see §3.4.4) shows that extending the above action of $H^0(S, \mathcal{O}_S)$ on $\mathrm{Polyd}_{\mathrm{pre}}(S)$ to an action of $\mathscr{M}_{\mathrm{pre}}(S)$ amounts to specifying natural transformations $\psi_\alpha : \Phi_\alpha \to \mathrm{Id}$ so that $\psi_{\alpha_1\alpha_2} = \psi_{\alpha_1} \circ \Phi_{\alpha_1}(\psi_{\alpha_2})$. We define the morphism $\hat{\mathbb{A}}^n_S = \Phi_\alpha(\hat{\mathbb{A}}^n_S) \xrightarrow{\psi_\alpha} \hat{\mathbb{A}}^n_S$ to be multiplication by α.

3.4.6. Action of $\mathscr{M}$ on Polyd and $\mathfrak{F}$ (i) In §3.4.5 we defined an action of $\mathscr{M}_{\mathrm{pre}}$ on $\mathrm{Polyd}_{\mathrm{pre}}$. It induces an action of $\mathscr{M}$ on Polyd.

(ii) The endofunctor of $\mathrm{Polyd}(S)$ corresponding to each object of $\mathscr{M}(S)$ preserves finite products. So $\mathrm{Polyd}(S)$ acts on the category of group objects in $\mathrm{Polyd}(S)$, i.e., on $\mathfrak{F}(S)$.

Lemma 3.4.7. *Let $\mathscr{M}_{\mathrm{nilp}}(S)$ be as in §3.4.1 and $(\mathscr{L}, \alpha) \in \mathscr{M}_{\mathrm{nilp}}(S)$. Then the rescaling functor $\Phi_{\mathscr{L},\alpha} : \mathfrak{F}(S) \to \mathfrak{F}(S)$ canonically factors as*

$$\mathfrak{F}(S) \to \mathrm{Aff}(S) \to \mathfrak{F}(S), \tag{3.2}$$

where $\mathrm{Aff}(S)$ is the category of smooth affine group S-schemes with connected fibers and the second arrow in (3.2) is the functor of formal completion along the unit. Moreover, if G is in the essential image of the functor $\mathfrak{F}(S) \to \mathrm{Aff}(S)$ then Zariski-locally on S, the pointed S-scheme $(G, 0)$ is isomorphic to $(\mathbb{A}^m_S, 0)$ for some m.

Proof. If in the situation of §3.4.5 the function $\alpha \in H^0(S, \mathcal{O}_S)$ is nilpotent then the formal series (3.1) is a polynomial. □

3.4.8. Notation Recall that $\mathscr{M}(S) \supset \mathscr{M}_{\mathrm{inj}}(S) = \mathrm{Div}_+(S)$ (see §3.4.1). If $D \in \mathrm{Div}_+(S)$ then the action of D on $\mathrm{Polyd}(S)$ or $\mathfrak{F}(S)$ will be denoted by $\mathscr{X} \mapsto \mathscr{X}(-D)$. By §3.4.2, we have a canonical morphism $\mathscr{X}(-D) \to \mathscr{X}$.

Lemma 3.4.9. *Let S be a scheme and $D \overset{i}{\hookrightarrow} S$ an effective Cartier divisor.*

(i) For any $\mathscr{X}, \mathscr{X}' \in \mathrm{Polyd}(S)$, the map

$$\mathrm{Mor}(\mathscr{X}', \mathscr{X}(-D)) \to \mathrm{Mor}(\mathscr{X}', \mathscr{X})$$

is injective. Its image is equal to the preimage of the distinguished element[9] *of $\mathrm{Mor}(i^*\mathscr{X}', i^*\mathscr{X})$.*

(ii) The same is true if $\mathscr{X}, \mathscr{X}'$ are formal groups over S. □

[9]This element is due to the fact that we are dealing with *based* formal S-polydisks.

3.4.10. Remark Lemma 3.4.9(i) can be generalized as follows. Assume that $(\mathscr{L},\alpha)\in\mathscr{M}(S)$. Let $D\overset{i}{\hookrightarrow}S$ be the closed subscheme corresponding to the ideal $\operatorname{Im}\alpha\subset\mathcal{O}_S$; let $S'\overset{\nu}{\hookrightarrow}S$ be the closed subscheme corresponding to the ideal $\operatorname{Ker}(\alpha^*:\mathcal{O}_S\to\mathscr{L}^*)$. Let $\mathscr{X},\mathscr{X}'\in\operatorname{Polyd}(S)$, and let $\tilde{\mathscr{X}}\in\mathscr{M}(S)$ be obtained by acting on $\mathscr{X}$ by $(\mathscr{L},\alpha)$. By §3.4.2, we have a canonical morphism $\tilde{\mathscr{X}}\to\mathscr{X}$ and therefore a morphism $f:\underline{\operatorname{Mor}}(\mathscr{X}',\tilde{\mathscr{X}})\to\underline{\operatorname{Mor}}(\mathscr{X}',\mathscr{X})$, where $\underline{\operatorname{Mor}}$ denotes the sheaf on S formed by morphisms. Then the sequence

$$\underline{\operatorname{Mor}}(\mathscr{X}',\tilde{\mathscr{X}})\xrightarrow{f}\underline{\operatorname{Mor}}(\mathscr{X}',\mathscr{X})\to i_*\underline{\operatorname{Mor}}(i^*\mathscr{X}',i^*\mathscr{X})$$

is exact in the following sense: the sections of $\operatorname{Im}f$ are precisely those sections of $\underline{\operatorname{Mor}}(\mathscr{X}',\mathscr{X})$ which map to the distinguished section of $i_*\underline{\operatorname{Mor}}(i^*\mathscr{X}',i^*\mathscr{X})$. Moreover,

$$\operatorname{Im}f=\nu_*\underline{\operatorname{Mor}}(\nu^*\mathscr{X}',\nu^*\tilde{\mathscr{X}})$$

(the two sheaves are equal as quotients of $\underline{\operatorname{Mor}}(\mathscr{X}',\tilde{\mathscr{X}})$).

3.4.11. What if *S* is a stack? It is straightforward to generalize the material of §3.4.1–3.4.10 to the situation where S is an algebraic stack[10] of groupoids in the sense of [D3, §2.4]. But algebraic stacks are not enough for us: the stack Σ and the q-de Rham prism Q are *formal* stacks rather than algebraic ones.

If S is any fpqc-stack we still have the monoidal category $\mathscr{M}(S)$ and its action on $\operatorname{Polyd}(S)$ and $\mathfrak{F}(S)$. For a reasonable class of stacks S (which includes all formal stacks, e.g., Σ, Q, and $Q\times_\Sigma Q$) one also has a good notion of effective Cartier divisor on S and an analog of Lemma 3.4.9, see §3.6 below.

3.5. Deformation of a formal group to the formal completion of its Lie algebra

In this subsection we briefly discuss a formal version of a particular case of the Fulton-MacPherson construction of *deformation to the normal cone*, see [F, Ch. 5], [Ve, §2], and also §10 of the article [R] (where some generalizations of the original construction are discussed).

[10] Let us note that the definition of algebraic stack from [D3, §2.4] involves no finiteness conditions.

3.5.1. Let $\mathscr{X} \in \mathrm{Polyd}(S)$, $\mathscr{X} = (X, \sigma : S \to X)$. Let $\mathscr{N}$ be the σ-pullback of the tangent bundle of X relative to S (or equivalently, the normal bundle of $\sigma(S) \subset X$). Let $\pi : \mathbb{A}^1_S \to S$ be the projection and $i_0 : S \to \mathbb{A}^1_S$ the zero section. Let $D := i_0(S) \subset \mathbb{A}^1_S$. Let

$$\tilde{\mathscr{X}} := (\pi^* \mathscr{X})(-D) \in \mathrm{Polyd}(\mathbb{A}^1_S).$$

One checks that $i_0^* \tilde{\mathscr{X}}$ canonically identifies with the formal completion of the vector bundle $\mathscr{N}$ along its zero section.

3.5.2. Now let $\mathscr{X} \in \mathscr{F}(S)$. Just as in §3.5.1, let $\tilde{\mathscr{X}} := (\pi^* \mathscr{X})(-D) \in \mathscr{F}(\mathbb{A}^1_S)$. Then the formal group $i_0^* \tilde{\mathscr{X}}$ canonically identifies with the formal completion of the vector bundle $\mathrm{Lie}(\mathscr{X})$ along its zero section.

3.6. An analog of Lemma 3.4.9 if S is a stack

3.6.1. A class of stacks Let S be an fpqc-stack of groupoids which can be represented as

$$S = \varinjlim (S_1 \hookrightarrow S_2 \hookrightarrow \dots), \tag{3.3}$$

where each S_i is an algebraic stack in the sense of [D3, §2.4] and the morphisms $S_i \to S_{i+1}$ are closed immersions. Such S is *pre-algebraic* in the sense of [D3, §2.3].

3.6.2. The notion of effective Cartier divisor We will use the notion of effective Cartier divisor on a pre-algebraic stack introduced in [D3, §2.10–2.11]. If S admits a presentation (3.3) the definition from [D3] is equivalent to the following one: an *effective Cartier divisor* on S is a closed substack $D \subset S$ such that

(i) the ideal $\mathcal{I}_n$ of the closed substack $D \cap S_n \subset S_n$ is an invertible sheaf on some closed substack $S'_n \subset S_n$;

(ii) the inductive limit[11] of the stacks S'_n equal S; equivalently, for every quasi-compact scheme $\tilde{S}$ every morphism $f : \tilde{S} \to S$ factors through some S'_n.

In this situation one can define the line bundle $\mathcal{O}_S(-D)$: its pullback to S'_n equals $\mathcal{I}_n$. Therefore we have $\mathscr{X}(-D)$ for $\mathscr{X} \in \mathrm{Polyd}(S)$ or for $\mathscr{X} \in \mathfrak{F}(S)$.

Proposition 3.6.3. *Lemma 3.4.9 remains valid for any stack S which admits a presentation* (3.3).

[11] It is easy to check that $S'_n \subset S'_{n+1}$, so the stacks S'_n form an inductive system.

Proof. It suffices to prove the analog of Lemma 3.4.9(i) for the stack S. To this end, for each n apply the analog of §3.4.10 for algebraic stacks to the pullback of $\mathcal{O}_S(-D)$ to S_n. □

The author expects that using §3.4.10 one can prove that Lemma 3.4.9 remains valid for any *pre-algebraic* stack in the sense of [D3, §2.3].

3.6.4. A corollary of Lemma 3.4.7 Let S be as in §3.6.1 and $H \in \mathfrak{F}(S)$. Let $D \subset S$ be an effective Cartier divisor such that for every scheme T and every morphism $T \to S$ one has $T \times_S D \supset T_{\mathrm{red}}$. Lemma 3.4.7 implies that in this situation the formal group $H(-D)$ can be canonically represented as a formal completion of a smooth affine group S-scheme with connected fibers. We denote this group scheme by $H(-D)^{\mathrm{alg}}$.

In particular, we have the group scheme $H_\Sigma(-\Delta_0)^{\mathrm{alg}}$ over Σ.

4. Proofs of the statements from §2

4.1. Recollections on the Hodge-Tate divisor $\Delta_0 \subset \Sigma$

By definition, $\Delta_0 \subset \Sigma := W_{\mathrm{prim}}/W^\times$ is the preimage of $\{0\}/\mathbb{G}_m \subset \mathbb{A}^1/\mathbb{G}_m$ under the morphism $W_{\mathrm{prim}}/W^\times \to \mathbb{A}^1/\mathbb{G}_m$.

The element $V(1) \in W(\mathbb{Z}_p)$ defines a morphism

$$\eta : \operatorname{Spf}\mathbb{Z}_p \to \Delta_0. \tag{4.1}$$

η is faithfully flat, and it identifies Δ_0 with the classifying stack

$$(\operatorname{Spf}\mathbb{Z}_p)/(W^\times)^{(F)},$$

where $(W^\times)^{(F)} := \operatorname{Ker}(F : W^\times \to W^\times)$; the proof of this fact is straightforward (see [BL] or Lemma 4.5.2 of [D3]).

4.2. Proof of Proposition 2.7.1

We have to show that the composite morphism

$$G'_\Sigma \to W_\Sigma^\times \to (W^\times/\mathbb{G}_m)_\Sigma \tag{4.2}$$

is faithfully flat.

4.2.1. Reductions Both G'_Σ and $(W^\times/\mathbb{G}_m)_\Sigma$ are flat over Σ. For any morphism from a quasi-compact scheme S to Σ, the ideal of the closed subscheme $S\times_\Sigma \Delta_0\subset S$ is nilpotent. So it suffices to check faithful flatness of (4.2) after base change to Δ_0 and even after further pullback via the faithfully flat morphism (4.1).

4.2.2. Pullback via $\boldsymbol{\eta}$: $\mathbf{Spf}\,\mathbb{Z}_{\boldsymbol{p}}\to\boldsymbol{\Sigma}$ Let G'_η be the pullback of G'_Σ via $\eta:\operatorname{Spf}\mathbb{Z}_p\to\Delta_0\subset\Sigma$. By §2.5.1, $G'_\eta = W_{\operatorname{Spf}\mathbb{Z}_p}$ (disregarding the group operation), and the η-pullback of (4.2) is the map

$$W_{\operatorname{Spf}\mathbb{Z}_p}\to (W^\times/\mathbb{G}_m)_{\operatorname{Spf}\mathbb{Z}_p}=\operatorname{Ker}(W^\times\twoheadrightarrow\mathbb{G}_m)_{\operatorname{Spf}\mathbb{Z}_p}$$

given by

$$x\mapsto 1+V(1)\cdot x = 1+V(Fx). \tag{4.3}$$

This map is faithfully flat because $F:W\to W$ is a Frobenius lift. □

4.3. The group schemes G_η, G_{Δ_0} and the proof of Theorem 2.7.5

Let G_{Δ_0} be the pullback of G_Σ to Δ_0. Let G_η be the pullback of G_Σ via $\eta:\operatorname{Spf}\mathbb{Z}_p\to\Delta_0\subset\Sigma$; this is a group scheme over $\operatorname{Spf}\mathbb{Z}_p$.

Proposition 4.3.1. *(i) There is a canonical isomorphism of group schemes*

$$G_\eta \xrightarrow{\sim} (W^{(F)})_{\operatorname{Spf}\mathbb{Z}_p}, \tag{4.4}$$

where $W^{(F)}:=\operatorname{Ker}(F:W\to W)$*.*

(ii) The homomorphism $G_\eta\to(\mathbb{G}_m)_{\operatorname{Spf}\mathbb{Z}_p}$ induced by (2.7) is trivial.

(iii) The η-pullback of the morphism $G_\Sigma\to F^*G_\Sigma$ from §2.9.1 is trivial.

Proof. Let us prove (i). By §2.5.1, G'_η is $W_{\operatorname{Spf}\mathbb{Z}_p}$ equipped with the group operation

$$(x_1,x_2)\mapsto x_1+x_2+V(1)\cdot x_1x_2.$$

By (4.3), the subgroup $G_\eta\subset G'_\eta$ is defined by the equation $V(1)\cdot x=0$ or equivalently, $Fx=0$.

Statement (ii) is clear because the homomorphism $G_\eta\to(\mathbb{G}_m)_{\operatorname{Spf}\mathbb{Z}_p}$ is the restriction of the map $G'_\eta=W_{\operatorname{Spf}\mathbb{Z}_p}\to W^\times_{\operatorname{Spf}\mathbb{Z}_p}$ given by $x\mapsto 1+V(1)\cdot x$.

To prove (iii), note that the morphism in question is $x\mapsto Fx$, but we already know that $G_\eta\subset G'_\eta$ is defined by the equation $Fx=0$. □

Lemma 4.3.2. *The canonical homomorphism* $W \twoheadrightarrow W_1 = \mathbb{G}_a$ *induces an isomorphism*

$$W^{(F)}_{\operatorname{Spec}\mathbb{Z}_{(p)}} \xrightarrow{\sim} (\mathbb{G}_a^\sharp)_{\operatorname{Spec}\mathbb{Z}_{(p)}},$$

where $\mathbb{G}_m^\sharp$ *is the divided powers additive group and* $\mathbb{Z}_{(p)}$ *is the localization of* $\mathbb{Z}$ *at* p*.*

For a proof of the lemma, see [BL] or [D3, Lemma 3.2.6].

Corollary 4.3.3. $G_\eta = (\mathbb{G}_a^\sharp)_{\operatorname{Spf}\mathbb{Z}_p}$*, so* G_η *is Cartier dual to the formal group* $(\hat{\mathbb{G}}_a)_{\operatorname{Spf}\mathbb{Z}_p}$*.*

Proof. Follows from Proposition 4.3.1 and Lemma 4.3.2. □

4.3.4. Proof of Theorem 2.7.5 Corollary 4.3.3 and Proposition 3.3.4 imply (similarly to §4.2.1) that G_Σ is the Cartier dual of some 1-dimensional formal group over Σ (which is denoted by H_Σ). □

4.3.5. Proof of Proposition 2.7.13 We have to construct an isomorphism $G_{\Delta_0} \xrightarrow{\sim} \mathscr{M}^\sharp$, where $\mathscr{M}$ is the conormal bundle of $\Delta_0 \subset \Sigma$. Corollary 4.3.3 provides an isomorphism $f : G_\eta \xrightarrow{\sim} \eta^*\mathscr{M}^\sharp$. By §4.1, $(W^\times)^{(F)}$ acts on G_η and $\eta^*\mathscr{M}^\sharp$, and the problem is to check that f is $(W^\times)^{(F)}$-equivariant. Indeed, $u \in (W^\times)^{(F)}$ acts on $G_\eta = W^{(F)}$ as multiplication by u, and it acts on $\eta^*\mathscr{M}^\sharp = \mathbb{G}_a^\sharp$ as multiplication by the 0-th component of the Witt vector u. □

We can now prove the following weaker version of Theorem 2.9.2(i).

Corollary 4.3.6. *The section* $s : \Sigma \to H_\Sigma$ *vanishes on* Δ_0*.*

Proof. As already mentioned in §4.1, $\eta : \operatorname{Spf}\mathbb{Z}_p \to \Delta_0$ is faithfully flat. So Proposition 4.3.1(ii) implies that the canonical homomorphism $G_\Sigma \to (\mathbb{G}_m)_\Sigma$ vanishes on Δ_0. By the definition of s (see §2.10.2), this means that $s : \Sigma \to H_\Sigma$ vanishes on Δ_0. □

4.4. The "de Rham pullback" of G_Σ and the proof of Proposition 2.7.8

4.4.1. Recollections Recall that $G_{\mathrm{dR}} := \rho_{\mathrm{dR}}^* G_\Sigma$, where $\rho_{\mathrm{dR}} : \operatorname{Spf}\mathbb{Z}_p \to \Sigma$ comes from the element $p \in W(\mathbb{Z}_p)$. For any p-nilpotent ring A we have

$$G_{\mathrm{dR}}(A) := \{x \in W(A) \mid 1 + px \in A^\times \subset W(A)^\times\}, \tag{4.5}$$

where $A^\times \subset W(A)^\times$ is the image of the Teichmüller embedding, and the group operation on $G_{\mathrm{dR}}(A)$ is given by

$$(x_1, x_2) \mapsto x_1 + x_2 + px_1x_2. \tag{4.6}$$

4.4.2. The homomorphisms $f : G_{\mathrm{dR}} \to W_{\operatorname{Spf}\mathbb{Z}_p}$ and $f_0 : G_{\mathrm{dR}} \to (\mathbb{G}_a)_{\operatorname{Spf}\mathbb{Z}_p}$ The coefficients of the formal series

$$f(x) := p^{-1}\log(1+px) = \sum_{n=1}^{\infty} \frac{(-p)^{n-1}}{n} x^n. \tag{4.7}$$

belong to $\mathbb{Z}_p$ and converge to 0. One has

$$f(x_1 + x_2 + px_1x_2) = f(x_1) + f(x_2).$$

So the series (4.7) defines a group homomorphism $f : G_{\mathrm{dR}} \to W_{\operatorname{Spf}\mathbb{Z}_p}$. Composing it with the canonical homomorphism $W_{\operatorname{Spf}\mathbb{Z}_p} \twoheadrightarrow (W_1)_{\operatorname{Spf}\mathbb{Z}_p} = (\mathbb{G}_a)_{\operatorname{Spf}\mathbb{Z}_p}$, we get a homomorphism

$$f_0 : G_{\mathrm{dR}} \to (\mathbb{G}_a)_{\operatorname{Spf}\mathbb{Z}_p}; \tag{4.8}$$

equivalently, $f_0(x) = p^{-1}\log(1+px_0)$, where x_0 is the 0-th component of the Witt vector x.

By Lemma 4.3.2, $(\mathbb{G}_a^\sharp)_{\operatorname{Spf}\mathbb{Z}_p} = W^{(F)}_{\operatorname{Spf}\mathbb{Z}_p}$, where $W^{(F)} := \operatorname{Ker}(F : W \to W)$. So Proposition 2.7.8 is equivalent to the following one.

Proposition 4.4.3. *There exists an isomorphism $G_{\mathrm{dR}} \xrightarrow{\sim} W^{(F)}_{\operatorname{Spf}\mathbb{Z}_p}$ whose composition with the canonical homomorphism $W^{(F)}_{\operatorname{Spf}\mathbb{Z}_p} \to (\mathbb{G}_a)_{\operatorname{Spf}\mathbb{Z}_p}$ equals* (4.8).

Note that the isomorphism in question is unique (to see this, identify $W^{(F)}_{\operatorname{Spf}\mathbb{Z}_p}$ with $(\mathbb{G}_a^\sharp)_{\operatorname{Spf}\mathbb{Z}_p}$). In §4.4.5 we will deduce Proposition 4.4.3 from the following lemma, which will be proved in §4.5.

Lemma 4.4.4. *The homomorphism $f : G_{\mathrm{dR}} \to W_{\operatorname{Spf}\mathbb{Z}_p}$ from §4.4.2 induces an isomorphism*

$$G_{\mathrm{dR}} \xrightarrow{\sim} W^{F=p}_{\operatorname{Spf}\mathbb{Z}_p}, \quad \text{where } W^{F=p}_{\operatorname{Spf}\mathbb{Z}_p} := \{y \in W_{\operatorname{Spf}\mathbb{Z}_p} \mid Fy = py\}. \tag{4.9}$$

The inverse isomorphism is given by $y \mapsto g(y)$, where g is the formal power series

$$g(y) := \frac{\exp(py) - 1}{p} = \sum_{n=1}^{\infty} \frac{p^{n-1}}{n!} y^n. \tag{4.10}$$

Note that if A is a p-nilpotent ring and $y \in W(A)$ satisfies $Fy = py$ then y is topologically nilpotent, so $h(y)$ makes sense for *any* formal power series h over $\mathbb{Z}_p$. In particular, this is true for the power series (4.10) (even though in the case $p = 2$ its coefficients do not converge to 0).

4.4.5. Deducing Proposition 4.4.3 from Lemma 4.4.4 The equation $Fy = py$ from (4.9) can be rewritten as $F(y - Vy) = 0$. The operator $\mathrm{id} - V$ is invertible because V is topologically nilpotent. So we get an isomorphism

$$\mathrm{id} - V : W^{F=p}_{\operatorname{Spf}\mathbb{Z}_p} \xrightarrow{\sim} W^{(F)}_{\operatorname{Spf}\mathbb{Z}_p}. \tag{4.11}$$

Composing it with (4.9), we get an isomorphism

$$G_{\mathrm{dR}} \xrightarrow{\sim} W^{(F)}_{\operatorname{Spf}\mathbb{Z}_p}, \tag{4.12}$$

which has the required property. □

4.4.6. Warning about the base change of (4.12) to $\operatorname{Spec}\mathbb{F}_p$ In $W(\mathbb{F}_p)$ we have $p = V(1)$. So by Proposition 4.3.1(i), the base change of G_{dR} to $\operatorname{Spec}\mathbb{F}_p$ identifies with $W^{(F)}_{\operatorname{Spec}\mathbb{F}_p}$. The group scheme $W^{F=p}_{\operatorname{Spec}\mathbb{F}_p} := \{y \in W_{\operatorname{Spec}\mathbb{F}_p} \mid Fy = py\}$ also equals $W^{(F)}_{\operatorname{Spec}\mathbb{F}_p}$: indeed, if A is an $\mathbb{F}_p$-algebra and $y \in W(A)$ then

$$Fy = py \Leftrightarrow (\mathrm{id} - V)Fy = 0 \Leftrightarrow Fy = 0.$$

We claim that the base change to $\operatorname{Spec}\mathbb{F}_p$ of the isomorphism (4.9) equals the identity (so the base change to $\operatorname{Spec}\mathbb{F}_p$ of (4.12) is $\mathrm{id} - V \neq \mathrm{id}$!!). This follows from the next

Lemma 4.4.7. *Let A be an $\mathbb{F}_p$-algebra and $x \in W^{(F)}(A)$. Then $px = x^p = 0$.*

Proof. We have $px = FVx = VFx = 0$. Write $x = [x_0] + Vy$, where x_0 is the 0-th coordinate of the Witt vector x. Then $x_0^p = 0$ and $Fy = 0$ (because in characteristic p the Witt vector Frobenius equals the usual one). So $py = VFy = 0$ and $(Vy)^2 = V(py^2) = 0$. Therefore $x^p = 0$. □

4.5. Proof of Lemma 4.4.4

4.5.1. By Corollary 2.7.2, G_{dR} is flat over $\operatorname{Spf}\mathbb{Z}_p$. By (4.11) and Lemma 4.3.2, $W^{F=p}_{\operatorname{Spf}\mathbb{Z}_p}$ is also flat over $\operatorname{Spf}\mathbb{Z}_p$.

4.5.2. Let us prove that the homomorphism $f : G_{\mathrm{dR}} \to W_{\operatorname{Spf}\mathbb{Z}_p}$ from Lemma 4.4.4 factors through $W^{F=p}_{\operatorname{Spf}\mathbb{Z}_p}$. Since $f \circ F = F \circ f$, it suffices to show that for $x \in G_{\mathrm{dR}}(A)$ one has

$$Fx = h(x), \tag{4.13}$$

where $h : G_{\mathrm{dR}} \to G_{\mathrm{dR}}$ is raising to the power of p in the sense of the operation (4.6); explicitly,

$$h(x) = \frac{(1+px)^p - 1}{p} := \sum_{i=1}^{p} \binom{p}{i} p^{i-1} x^i.$$

Since $1 + px \in A^\times \subset W(A)^\times$ we have $F(1+px) = (1+px)^p$, so

$$p(Fx - h(x)) = 0. \tag{4.14}$$

But the coordinate ring B of G_{dR} is flat over $\mathbb{Z}_p$ (see §4.5.1), so $W(B)$ is also flat over $\mathbb{Z}_p$. The elements $Fx - h(x)$ for all p-nilpotent rings A and all $x \in G_{\mathrm{dR}}(A)$ define an element $u \in W(B)$, and by (4.14) we have $pu = 0$. So $u = 0$, which proves (4.13).

4.5.3. The formal series (4.10) defines a homomorphism $g : W^{F=p}_{\mathrm{Spf}\,\mathbb{Z}_p} \to G'_{\mathrm{dR}} := \rho^*_{\mathrm{dR}} G'_\Sigma$, where G'_Σ is as in §2.5. Let us prove that this homomorphism factors through $G_{\mathrm{dR}} \subset G'_{\mathrm{dR}}$. The problem is to show that for any p-nilpotent ring A and any $x \in W^{F=p}_{\mathrm{Spf}\,\mathbb{Z}_p}(A)$ the Witt vector $1 + pg(x)$ is Teichmüller. It is clear that

$$F(1 + pg(x)) = (1 + pg(x))^p.$$

But the coordinate ring C of $W^{F=p}_{\mathrm{Spf}\,\mathbb{Z}_p}$ is flat over $\mathbb{Z}_p$ (see §4.5.1), so an element $u \in W(C)$ such that $Fu = u^p$ has to be Teichmüller.

4.5.4. By §4.5.1, G_{dR} and $W^{F=p}_{\mathrm{Spf}\,\mathbb{Z}_p}$ are flat over $\mathrm{Spf}\,\mathbb{Z}_p$. The morphism $f : G_{\mathrm{dR}} \to W^{F=p}_{\mathrm{Spf}\,\mathbb{Z}_p}$ becomes an isomorphism after base change to $\mathrm{Spec}\,\mathbb{F}_p$, see §4.4.6. So f itself is an isomorphism. Finally, it is easy to see that $f \circ g = \mathrm{id}$. □

4.5.5. Remark In §4.5.2–4.5.3 we used a flatness argument. Instead, one could use the canonical δ-ring structure on $W(A)$.

4.6. Remarks related to §4.4

Let $(W^\times)^{(F)} := \mathrm{Ker}(F : W^\times \to W^\times)$. In §4.6.1 we identify the group scheme G_{dR} from §4.4 with $((W^\times)^{(F)}/\mu_p)_{\mathrm{Spf}\,\mathbb{Z}_p}$. This allows us to think of (4.12) as an isomorphism

$$((W^\times)^{(F)}/\mu_p)_{\mathrm{Spf}\,\mathbb{Z}_p} \xrightarrow{\sim} W^{(F)}_{\mathrm{Spf}\,\mathbb{Z}_p}. \tag{4.15}$$

In §4.6.2 we show that the base change of (4.15) to $\operatorname{Spec}\mathbb{F}_p$ is somewhat unexpected. Related to this is Lemma 4.6.6, which says that the restriction of the formal group H_Q to the subscheme $\operatorname{Spf}\mathbb{Z}_p[[q-1]]/(q^p-1)\subset Q$ is somewhat unusual.

4.6.1. If $w\in(W^\times)^{(F)}(A)$ and $w_0\in A^\times$ is the 0-th component of the Witt vector w then $[w_0]/w=1+Vx$ for a unique $x\in W(A)$; moreover, $x\in G_{\mathrm{dR}}(A)$ because

$$1+px=F([w_0]/w)=[w_0^p].$$

It is easy to check that one thus gets an isomorphism

$$((W^\times)^{(F)}/\mu_p)_{\operatorname{Spf}\mathbb{Z}_p}\xrightarrow{\sim}G_{\mathrm{dR}}. \tag{4.16}$$

Composing (4.16) and (4.12), one gets an isomorphism (4.15).

On the other hand, one has canonical isomorphisms

$$(W^\times)^{(F)}_{\operatorname{Spf}\mathbb{Z}_p}\xrightarrow{\sim}(\mathbb{G}_m^\sharp)_{\operatorname{Spf}\mathbb{Z}_p},\quad W^{(F)}_{\operatorname{Spf}\mathbb{Z}_p}\xrightarrow{\sim}(\mathbb{G}_a^\sharp)_{\operatorname{Spf}\mathbb{Z}_p},$$

where $\mathbb{G}_m^\sharp$ and $\mathbb{G}_a^\sharp$ are the divided powers versions of $\mathbb{G}_m$ and $\mathbb{G}_a$ (see [BL] or Lemma 3.2.6 and §3.3.3 of [D3]). So one can think of (4.16) as an isomorphism $(\mathbb{G}_m^\sharp/\mu_p)_{\operatorname{Spf}\mathbb{Z}_p}\xrightarrow{\sim}G_{\mathrm{dR}}$, and one can think of (4.15) as an isomorphism

$$(\mathbb{G}_m^\sharp/\mu_p)_{\operatorname{Spf}\mathbb{Z}_p}\xrightarrow{\sim}(\mathbb{G}_a^\sharp)_{\operatorname{Spf}\mathbb{Z}_p}. \tag{4.17}$$

It is easy to check that the isomorphism (4.17) is equal to the isomorphism

$$\log:(\mathbb{G}_m^\sharp/\mu_p)_{\operatorname{Spf}\mathbb{Z}_p}\xrightarrow{\sim}(\mathbb{G}_a^\sharp)_{\operatorname{Spf}\mathbb{Z}_p}$$

from Proposition B.5.6(i) of Appendix B.

4.6.2. Warning In $W_{\mathbb{F}_p}$ one has $VF=FV$. Using this, it is easy to check that the map $W_{\mathbb{F}_p}\to W_{\mathbb{F}_p}^\times$ defined by $x\mapsto 1+Vx$ induces an isomorphism

$$f_{naive}:W_{\mathbb{F}_p}^{(F)}\xrightarrow{\sim}((W^\times)^{(F)}/\mu_p)_{\mathbb{F}_p}.$$

On the other hand, let $f:W_{\mathbb{F}_p}^{(F)}\xrightarrow{\sim}((W^\times)^{(F)}/\mu_p)_{\mathbb{F}_p}$ be the base change of the inverse of (4.15) to $\operatorname{Spec}\mathbb{F}_p$. It turns out that $f\neq f_{naive}$; more precisely, using §4.4.6 one gets

$$f(x)=f_{naive}(Vx-x). \tag{4.18}$$

The remaining part of §4.6 is closely related to formula (4.18).

4.6.3. Notation Let

$$T := \operatorname{Spf} B, \quad \text{where } B := \{(x,y) \in \mathbb{Z}_p \times \mathbb{Z}_p \mid x \equiv y \mod p\}.$$

The element $(p, V(1)) \in W(\mathbb{Z}_p) \times W(\mathbb{Z}_p) = W(\mathbb{Z}_p \times \mathbb{Z}_p)$ belongs to $W(B)$. It defines a morphism $T \to W_{\mathrm{prim}}$ and therefore a morphism $T \to \Sigma$. Let G_T and H_T be the pullbacks of G_Σ and H_Σ to T (so H_T is a formal group over T, and G_T is the Cartier dual affine group scheme over T).

Lemma 4.6.4. *(i) The pullback of G_T (resp. H_T) via each of the two closed immersions $i_1, i_2 : \operatorname{Spf} \mathbb{Z}_p \hookrightarrow T$ is isomorphic to $W^{(F)}_{\operatorname{Spf} \mathbb{Z}_p}$ (resp. to $(\hat{\mathbb{G}}_a)_{\operatorname{Spf} \mathbb{Z}_p}$).*

(ii) G_T is not isomorphic to $W^{(F)}_T$, and H_T is not isomorphic to $(\hat{\mathbb{G}}_a)_T$.

Proof. We have the isomorphisms $i_1^* G_T \xrightarrow{\sim} W^{(F)}_{\operatorname{Spf} \mathbb{Z}_p}$ and $i_2^* G_T \xrightarrow{\sim} W^{(F)}_{\operatorname{Spf} \mathbb{Z}_p}$ given by (4.12) and (4.4). Their pullbacks to $\operatorname{Spec} \mathbb{F}_p = i_1(\operatorname{Spf} \mathbb{Z}_p) \cap i_2(\operatorname{Spf} \mathbb{Z}_p)$ are different: by §4.4.6, they differ by $\mathrm{id} - V \in \operatorname{Aut} W^{(F)}_{\mathbb{F}_p}$.

It remains to show that the automorphism $\mathrm{id} - V \in \operatorname{Aut} W^{(F)}_{\mathbb{F}_p}$ is *not* in the image of $\operatorname{Aut} W^{(F)}_{\operatorname{Spf} \mathbb{Z}_p}$. The Cartier duals of $W^{(F)}_{\mathbb{F}_p}$ and $\mathrm{id} - V$ are $(\hat{\mathbb{G}}_a)_{\mathbb{F}_p}$ and $\mathrm{id} - \mathrm{Fr} \in \operatorname{Aut}(\hat{\mathbb{G}}_a)_{\mathbb{F}_p}$. It is clear that $\mathrm{id} - \mathrm{Fr}$ is not in the image of $\operatorname{Aut}(\hat{\mathbb{G}}_a)_{\operatorname{Spf} \mathbb{Z}_p} = \mathbb{Z}_p^\times$. □

4.6.5. A subscheme $T' \subset Q$ Let us formulate a variant of Lemma 4.6.4. As usual, let Q be the q-de Rham prism, i.e., $Q := \operatorname{Spf} \mathbb{Z}_p[[q-1]]$. Let $T' \subset Q$ be defined by the equation $q^p = 1$. Let T be as in §4.6.3. We have a commutative diagram

$$\begin{array}{ccc} T' & \longrightarrow & Q \\ \downarrow & & \downarrow \\ T & \longrightarrow & \Sigma \end{array}$$

in which the morphism $Q \to \Sigma$ is as in §2.10.1 and the morphism $T' \to T$ comes from the ring homomorphism

$$B \to \mathbb{Z}_p[q]/(q^p - 1), \quad (x,y) \mapsto y + \frac{x-y}{p} \cdot (1 + q + \cdots + q^{p-1}),$$

where B is as in §4.6.3.

Lemma 4.6.6. *As before, let $T' \subset Q$ be defined by the equation $q^p = 1$. Let $T'_1 \subset T'$ (resp. $T'_2 \subset T'$) be defined by the equation $q = 1$ (resp. by $1 + q + \cdots q^{p-1} = 0$). Let $H_{T'}$, $H_{T'_1}$, $H_{T'_2}$ be the pullbacks of H_Q to T', T'_1, T'_2. Then*

(i) $H_{T_1'} \simeq (\hat{\mathbb{G}}_a)_{T_1'}$ *and* $H_{T_2'} \simeq (\hat{\mathbb{G}}_a)_{T_2'}$*;*
(ii) $H_{T'}$ *is not isomorphic to* $(\hat{\mathbb{G}}_a)_{T'}$*.*

Proof. Statement (i) follows from Lemma 4.6.4(i). Statement (ii) is proved similarly to Lemma 4.6.4(ii). □

4.6.7. Remark In connection with Lemma 4.6.6, let us note that H_Q has a very explicit description, see (2.23). This description was deduced from Theorem 2.10.4, which will be proved in the next subsection.

4.7. Proof of Theorem 2.10.4

4.7.1. Recollections By (2.17), the effective divisor $(\Delta_0)_Q := \Delta_0 \times_\Sigma Q \subset Q$ is defined by the equation $\Phi_p(q) = 0$. Recall that $D \subset Q$ denotes the divisor $q = 1$. Since $q^p - 1 = (q-1) \cdot \Phi_p(q)$, we get

$$F^{-1}(D) = D + (\Delta_0)_Q. \tag{4.19}$$

We have a section $s_Q : Q \to H_Q$ and a homomorphism $\sigma^* : H_Q \to (\hat{\mathbb{G}}_m)_Q$. By Lemma 2.10.3(ii), $\sigma^* \circ s_Q$ is given by $q^p \in \hat{\mathbb{G}}_m(Q)$, so $s_Q^{-1}(\operatorname{Ker} \sigma^*)$ is the divisor $q^p = 1$. By (4.19), we get

$$s_Q^{-1}(\operatorname{Ker} \sigma^*) = D + (\Delta_0)_Q. \tag{4.20}$$

Lemma 4.7.2. *(i) The closed subscheme* $\operatorname{Ker} \sigma^* \subset H_Q$ *is equal to the divisor* $H_D + 0_Q$*, where* $H_D \subset H_Q$ *is the preimage of* D *and* $0_Q \subset H_Q$ *is the zero section.*

(ii) $s_Q^{-1}(0_Q) = (\Delta_0)_Q$*.*

Proof. By (4.20), $\operatorname{Ker} \sigma^* \neq H_Q$. Since $\operatorname{Ker} \sigma^* = (\sigma^*)^{-1}(0_Q)$ and 0_Q is a divisor in H_Q, we see that $\operatorname{Ker} \sigma^*$ is a divisor in H_Q.

From the definition of σ (see §2.10.2) it is clear that $\sigma : Q \to G_Q$ vanishes on D. So $\sigma^* : H_Q \to (\hat{\mathbb{G}}_m)_Q$ vanishes over D. Therefore $\operatorname{Ker} \sigma^* \geq H_D + 0_Q$. In other words,

$$\operatorname{Ker} \sigma^* = H_D + 0_Q + \mathfrak{D}, \qquad \text{where } \mathfrak{D} \geq 0.$$

Combining this with (4.20), we see that

$$s_Q^{-1}(0_Q) + s_Q^{-1}(\mathfrak{D}) = (\Delta_0)_Q.$$

But $s_Q^{-1}(0_Q) \geq (\Delta_0)_Q$ by Corollary 4.3.6. So $s_Q^{-1}(\mathfrak{D}) = 0$. Therefore $\mathfrak{D} = 0$. □

4.7.3. Proof of Theorem 2.10.4 We have to show that $\sigma^* : H_Q \to (\hat{\mathbb{G}}_m)_Q$ induces an isomorphism $H_Q \xrightarrow{\sim} (\hat{\mathbb{G}}_m)_Q(-D)$. Choose an isomorphism $H_Q \xrightarrow{\sim} \operatorname{Spf} \mathbb{Z}_p[[q-1,x]]$ of formal schemes over Q. Then σ^* is given by a formal series $f \in \mathbb{Z}_p[[q-1,x]]$ such that $f(x_1 \star x_2) = f(x_1)f(x_2)$, where $\star$ is the group operation in H_Q. By Lemma 4.7.2(i), $f = 1 + (q-1)g$, where

$$g \in x \cdot \mathbb{Z}_p[[q-1,x]]^\times. \tag{4.21}$$

Then $g(x_1 \star x_2) = g(x_1) + g(x_2) + (q-1)g(x_1)g(x_2) = g(x_1) * g(x_2)$, where $*$ is the group operation in $(\hat{\mathbb{G}}_m)_Q(-D)$. Combining this with (4.21), we see that g defines an isomorphism of formal groups $H_Q \xrightarrow{\sim} (\hat{\mathbb{G}}_m)_Q(-D)$. □

4.8. Proof of Theorem 2.9.2

As already mentioned in §2.10.1, the morphism $\pi : Q \to \Sigma$ is faithfully flat. So to prove Theorem 2.9.2, it suffices to check analogous statements about H_Q. The analog of Theorem 2.9.2(i) has already been proved, see Lemma 4.7.2(ii). It remains to show that the morphism $\varphi_Q : F^*H_Q \to H_Q$ factors as $F^*H_Q \xrightarrow{\sim} H_Q((-\Delta_0)_Q) \to H_\Sigma$. This follows from Lemma 2.10.3(i), Theorem 2.10.4 and formula (4.19). □

4.8.1. Remark The interested reader can prove Theorem 2.9.2 without using the q-de Rham prism. One can deduce it from Proposition 2.9.5 and the description of G_η given in the proof of Proposition 4.3.1(i). (Proposition 2.9.5 was deduced in §2 from Proposition 2.7.8, and the latter was proved in §4.4.)

4.9. Proof of Proposition 2.11.3

4.9.1. Recall that $Q = \operatorname{Spf} \mathbb{Z}_p[[q-1]]$, $H_Q = \operatorname{Spf} \mathbb{Z}_p[[q-1,z]]$, and the group operation on H_Q is given by $z_1 * z_2 = z_1 + z_2 + (q-1)z_1z_2$. We have a canonical section $s_Q : Q \to H_Q$; as explained in §2.10.6, it is given by $z = \frac{q^p-1}{q-1}$.

Lemma 4.9.2. *Let $\mathcal{K} \subset H_Q$ be a closed group subscheme such that $s_Q : Q \to H_Q$ factors through $\mathcal{K}$. Then $\mathcal{K} = H_Q$.*

Proof. Assume the contrary. Then there exists a nonzero regular function f on H_Q which vanishes on the image of each composite morphism

$$Q \xrightarrow{s_Q} H_Q \xrightarrow{n} H_Q, \quad n \in \mathbb{Z}. \tag{4.22}$$

Without loss of generality, we can assume that $f \in \mathbb{Z}_p[[q-1, z]]$ has the form

$$f = \sum_{i=0}^{\infty} a_i(z)(q-1)^i, \quad \text{where } a_i \in \mathbb{Z}_p[[z]],\ a_0 \neq 0.$$

By §4.9.1, the morphism (4.22) is given by $z = \frac{q^{pn}-1}{q-1}$. The value of $\frac{q^{pn}-1}{q-1}$ at $q = 1$ equals pn. So $a_0(pn) = 0$ for all $n \in \mathbb{Z}$, which contradicts the assumption $a_0 \neq 0$. □

4.9.3. Proof of Proposition 2.11.3 Recall that the formal groups H_Q and $(\hat{\mathbb{G}}_m)_Q$ are $\mathbb{Z}_p^\times$-equivariant (the action of $\mathbb{Z}_p^\times$ on $(\hat{\mathbb{G}}_m)_Q$ is as in the formulation of Proposition 2.11.3). So $\mathbb{Z}_p^\times$ acts on $\operatorname{Hom}(H_Q, (\hat{\mathbb{G}}_m)_Q)$. We have an element $\sigma^* \in \operatorname{Hom}(H_Q, (\hat{\mathbb{G}}_m)_Q)$, and the problem is to show that $\alpha(\sigma^*) = \sigma^*$ for all $\alpha \in \mathbb{Z}_p^\times$. By Lemma 4.9.2, it suffices to check that for every $\alpha \in \mathbb{Z}_p^\times$ one has

$$s_Q(Q) \subset \mathcal{K}_\alpha, \quad \text{where } \mathcal{K}_\alpha := \operatorname{Ker}(\alpha(\sigma^*) - \sigma^*) \subset H_Q. \tag{4.23}$$

The section $s_Q : Q \to H_Q$ is $\mathbb{Z}_p^\times$-equivariant because it comes from $s : \Sigma \to H_\Sigma$. By Lemma 2.10.3(ii), $\sigma^* \circ s_Q : Q \to (\hat{\mathbb{G}}_m)_Q$ is also $\mathbb{Z}_p^\times$-equivariant. So (4.23) holds. □

5. Several realizations of the group scheme G_Q

By definition, G_Q is the pullback of G_Σ to the q-de Rham prism Q. This immediately leads to the first realization of G_Q and its coordinate ring, see §5.1–5.2. In §5.3 we note that the coordinate ring of a certain extension of G_Q by $(\mu_p)_Q$ appears in the theory of q-logarithm from [ALB, BL].

On the other hand, Theorem 2.10.4 identifies G_Q with the Cartier dual of a very explicit formal group $(\hat{\mathbb{G}}_m)_Q(-D)$. This Cartier dual is denoted by $G_Q^!$. We explicitly describe $G_Q^!$ (see §5.4–5.5) and the isomorphism $G_Q^! \xrightarrow{\sim} G_Q$ (see §5.6).

In §5.7 we define group schemes $G_Q^{!?}, G_Q^{!!}$ and isomorphisms between them and $G_Q^!$. Unlike $G_Q^!$ and similarly to G_Q, both $G_Q^{!?}$ and G_Q are defined in terms of Witt vectors.

Let us note that in §5.5–5.6 a key role is played by the expressions $(1+(q-1)z)^{\frac{t}{q-1}}$ and $q^{\frac{pt}{q-1}}$. The closely related q-logarithm (in the sense of [ALB, BL]) appears in formula (5.16).

5.1. Recollections

5.1.1. The formal δ-scheme Q Let $Q := \operatorname{Spf} \mathbb{Z}_p[[q-1]]$, where $\mathbb{Z}_p[[q-1]]$ is equipped with the $(p, q-1)$-adic topology. Define $F : Q \to Q$ by $q \mapsto q^p$. Then (Q, F) is a formal δ-scheme.

5.1.2. Pieces of structure on G_Q Recall that according to the definition from §2.10.2,

$$G_Q := G_\Sigma \times_\Sigma Q,$$

where G_Σ is as in §2.6. G_Q is a formal scheme over Q. The morphism $G_Q \to Q$ is schematic and affine; by Corollary 2.7.2, it is flat.

Let us recall the pieces of structure on G_Q. Most of them come from similar pieces of structure on G_Σ (the only exception is (iii).

(i′) G_Q is a formal δ-scheme over the formal δ-scheme Q; in other words, G_Q is equipped with a Frobenius lift $F : G_Q \to G_Q$, which is compatible with $F : Q \to Q$.

(i″) G_Q is a group scheme over Q. The group structure is compatible with $F : G_Q \to G_Q$, so G_Q is a group δ-scheme over Q.

(ii) One has a canonical map $G_Q \to (\mathbb{G}_m)_Q$, which is a homomorphism of group δ-schemes over Q. As usual, $(\mathbb{G}_m)_Q := \mathbb{G}_m \times Q$, and the δ-scheme structure on $\mathbb{G}_m$ is given by raising to the power of p.

(iii) In §2.10.2 we defined a canonical section $\sigma : Q \to G_Q$, which is a δ-morphism.

5.1.3. Who is who For any p-nilpotent ring A one has

$$Q(A) = \{q \in A^\times \mid q-1 \text{ is nilpotent}\},$$
$$G_Q(A) = \{(q,x) \in Q(A) \times W(A) \mid 1 + \Phi_p([q])x \in \mathbb{G}_m(A)\}, \tag{5.1}$$

where Φ_p is the cyclotomic polynomial and $\mathbb{G}_m$ is identified with a subgroup of $W^\times$ via the Teichmüller embedding. The morphism $F : G_Q \to G_Q$ is given by

$$F(q,x) = (q^p, Fx).$$

The group operation on G_Q and the homomorphism $G_Q \to (\mathbb{G}_m)_Q$ are given by the maps

$$G_Q \times_Q G_Q \to G_Q, \quad (q, x_1, x_2) \mapsto (q, x_1 + x_2 + \Phi_p([q])x_1x_2),$$
$$G_Q \to \mathbb{G}_m, \quad (q,x) \mapsto 1 + \Phi_p([q])x.$$

The section $\sigma : Q \to G_Q$ is given by

$$\sigma(q) := (q, [q] - 1). \tag{5.2}$$

5.2. The coordinate ring of G_Q

The coordinate ring $H^0(G_Q, \mathcal{O}_{G_Q})$ is a $(p, q-1)$-adically complete $\mathbb{Z}_p[[q-1]]$-algebra. Since G_Q is flat over Q, for any open ideal $I \subset \mathbb{Z}_p[[q-1]]$ the tensor product $H^0(G_Q, \mathcal{O}_{G_Q}) \otimes_{\mathbb{Z}_p[[q-1]]} (\mathbb{Z}_p[[q-1]]/I)$ is flat over $\mathbb{Z}_p[[q-1]]/I$. In particular, $H^0(G_Q, \mathcal{O}_{G_Q})$ is p-torsion-free, so $F : G_Q \to G_Q$ induces on $H^0(G_Q, \mathcal{O}_{G_Q})$ a δ-ring structure in the sense of [J85] and [BS, §2]. Let us describe $H^0(G_Q, \mathcal{O}_{G_Q})$ as a δ-algebra over $\mathbb{Z}_p[[q-1]]$, where $\mathbb{Z}_p[[q-1]]$ is considered as a δ-ring with $\delta(q) = 0$.

Proposition 5.2.1. *Let R_0 be the δ-algebra over $\mathbb{Z}[q]$ with a single generator x_0 and a single defining relation $\delta(1+\Phi_p(q)x_0) = 0$. Let R be the $(p, q-1)$-adic completion of R_0. Then there is a unique isomorphism $R_0 \xrightarrow{\sim} H^0(G_Q, \mathcal{O}_{G_Q})$ of δ-algebras over $\mathbb{Z}_p[[q-1]]$ such that $x_0 \in R$ goes to the following function on G_Q: the value of the function on a pair (q, x) as in (5.1) is the 0-th component of the Witt vector x.*

Proof. Let Y be the affine scheme over $\mathbb{Z}[q]$ such that for any $\mathbb{Z}[q]$-algebra A one has

$$Y(A) = \{x \in W(A) \mid 1 + \Phi_p([q])x \in \tau(A)\},$$

where $\tau : A \to W(A)$ is the Teichmüller embedding. Then G_Q is the $(p, q-1)$-adic formal completion of Y.

Let us construct an isomorphism $R_0 \xrightarrow{\sim} H^0(Y, \mathcal{O}_Y)$. By definition, Y is a closed subscheme of $W_{\mathbb{Z}[q]} := W \times \operatorname{Spec} \mathbb{Z}[q]$. By §C.3.7 of Appendix C, the coordinate ring of $W_{\mathbb{Z}[q]}$ is a free δ-algebra over $\mathbb{Z}[q]$ on a single generator x_0, where x_0 is the function that takes a Witt vector to its 0-th component. Since the Teichmüller embedding $\mathbb{A}^1 \to W$ is a δ-morphism, we see that the ideal of Y in $W_{\mathbb{Z}[q]}$ is generated by $\delta^n(1+\Phi_p(q)x_0)$, $n > 0$. So $H^0(Y, \mathcal{O}_Y) = R_0$. ☐

5.3. G_Q and the q-logarithm in the sense of [ALB, BL]

This subsection is a commentary on the notion of q-logarithm[12] from [ALB, §4] and [BL, §2.6]; the main point is that the q-logarithm is the unique group homomorphism $G_Q \to (\mathbb{G}_a)_Q$ with a certain property (see the last sentence of §5.3.2). This material will be used in formula (5.16) and nowhere else.

[12] As explained in [BL, Prop. 2.6.10], the q-logarithm is closely related to the prismatic logarithm (i.e., to the homomorphism $G_\Sigma \to \mathcal{O}_\Sigma\{1\}$ from our Corollary 2.7.11). We do not discuss this relation here.

5.3.1. An extension of G_Q by $(\mu_p)_Q$ The definition of q-logarithm given in [BL, §2.6] following [ALB, §4] secretly uses the coordinate ring of a slight modification of G_Q. Namely, for any p-nilpotent ring A let

$$\tilde{G}_Q(A) := \{(q, x, u) \in Q(A) \times W(A) \times A^\times \mid 1 + \Phi_p([q])x = [u^p]\} \tag{5.3}$$

(so $\tilde{G}_Q$ is an extension of G_Q by $(\mu_p)_Q$). The δ-ring constructed in [BL, Prop. 2.6.5] is just the coordinate ring of $\tilde{G}_Q$; this easily follows from Proposition 5.2.1.

We have a section

$$\tilde{\sigma} : Q \to \tilde{G}_Q, \quad \tilde{\sigma}(q) := (q, [q] - 1, q), \tag{5.4}$$

which lifts the section $\sigma : Q \to \tilde{G}_Q$ defined by (5.2).

5.3.2. The q-logarithm On $\tilde{G}_Q$ we have an invertible regular function u, see formula (5.3); note that u^p is a regular function on G_Q (unlike u). The authors of [ALB, BL] define another regular function on $\tilde{G}_Q$ denoted by $\log_q(u)$ and called the *q-logarithm*[13] of u. As explained below, $\log_q(u)$ is, in fact, a regular function on G_Q itself.

Very informally, $\log_q(u) = \frac{q-1}{\log q} \cdot \log u$ (so $\log_q(u)$ is $q - 1$ times the logarithm of u with base q). From this informal description we see that $\log_q(u_1 u_2) = \log_q(u_1) + \log_q(u_2)$ and $\log_q(q) = q - 1$.

The precise definition of $\log_q(u)$ from [ALB, BL] can be paraphrased as follows: $\log_q(u)$ is the unique group homomorphism $\tilde{G}_Q \to (\mathbb{G}_a)_Q$ that takes the section (5.4) to the section $q - 1 : Q \to (\mathbb{G}_a)_Q$ (the existence and uniqueness of such a homomorphism is proved in [ALB, §4]; see also [BL, Prop. 2.6.9]).

Note that the group $\mathrm{Ker}(\tilde{G}_Q \twoheadrightarrow G_Q) = (\mu_p)_Q$ is killed by $\log_q(u)$ because

$$\mathrm{Hom}((\mu_p)_Q, (\mathbb{G}_a)_Q) = 0.$$

So $\log_q(u)$ *is a group homomorphism* $G_Q \to (\mathbb{G}_a)_Q$*; it is the unique homomorphism that takes the section* $\sigma : Q \to \tilde{G}_Q$ *from* (5.2) *to the section* $q - 1 : Q \to (\mathbb{G}_a)_Q$.

[13] Warning: in the literature the word "q-logarithm" is used for many quite different functions, see the article [KVA], especially its last section.

5.4. The group scheme $G^!_Q$

5.4.1. Definition of $H^!_Q$ and $G^!_Q$ Let $D \subset Q$ be the effective divisor $q = 1$. Let

$$H^!_Q := (\hat{\mathbb{G}}_m)_Q(-D),$$

i.e., $H^!_Q$ is the formal group over Q obtained from $(\hat{\mathbb{G}}_m)_Q$ by rescaling via the invertible subsheaf $\mathcal{O}_Q(-D) \subset \mathcal{O}_Q$, see §3.4.

Now define $G^!_Q$ to be the Cartier dual of $H^!_Q$. Then $G^!_Q$ is a flat affine group scheme over Q.

Theorem 2.10.4 yields canonical isomorphisms $H_Q \xrightarrow{\sim} H^!_Q$, $G^!_Q \xrightarrow{\sim} G_Q$. But we will disregard these isomorphisms until §5.6.

5.4.2. $H^!_Q$ in explicit terms As a formal scheme, $H^!_Q = \operatorname{Spf}\mathbb{Z}_p[[q-1, z]]$, and the group operation is

$$z_1 * z_2 = \frac{(1+(q-1)z_1)(1+(q-1)z_2)-1}{q-1} = z_1 + z_2 + (q-1)z_1 z_2. \tag{5.5}$$

Let $H^!$ be the formal group over $\mathbb{A}^1 = \operatorname{Spec}\mathbb{Z}[q]$ defined by the group law (5.5); then $H^!_Q = H^! \times_{\mathbb{A}^1} Q$.

5.4.3. Pieces of structure on $H^!_Q$ $H^!_Q$ is a formal group over Q equipped with a homomorphism

$$H^!_Q \to (\hat{\mathbb{G}}_m)_Q. \tag{5.6}$$

In terms of the coordinate z from §5.4.2, it is given by the function $1+(q-1)z$.

Since $F^{-1}(D) \supset D$, there is a unique homomorphism $\varphi_Q : F^* H^!_Q \to H^!_Q$ such that the diagram

$$\begin{array}{ccc} F^* H^!_Q & \xrightarrow{\varphi_Q} & H^!_Q \\ \downarrow & & \downarrow \\ F^*(\hat{\mathbb{G}}_m)_Q & \xrightarrow{\sim} & (\hat{\mathbb{G}}_m)_Q \end{array} \tag{5.7}$$

commutes; here the lower horizontal arrow comes from the fact that $(\hat{\mathbb{G}}_m)_Q := \mathbb{G}_m \times Q$. Over $Q \otimes \mathbb{F}_p$ the upper horizontal arrow of (5.7) becomes the Verschiebung (because the lower horizontal arrow does).

Moreover, the map $Q \to \hat{\mathbb{G}}_m$, $q \mapsto q^p$, defines a section $Q \to (\hat{\mathbb{G}}_m)_Q$, which comes from a section

$$s_Q : Q \to H^!_Q. \tag{5.8}$$

In terms of §5.4.2, s_Q is given by $z = \Phi_p(q)$.

The following diagram commutes:

$$\begin{array}{ccc} F^*H^!_Q & \xrightarrow{\varphi_Q} & H^!_Q \\ {\scriptstyle F^*(s_Q)}\uparrow & & \uparrow{\scriptstyle p} \\ Q & \xrightarrow{s_Q} & H^!_Q \end{array}$$

Note that (5.6) and $\varphi_Q : F^*H^!_Q \to H^!_Q$ come from similar pieces of structure on the formal group $H^!$ from §5.4.2; on the other hand, (5.8) does not have an analog for $H^!$.

5.4.4. Pieces of structure on $G^!_Q$ Dualizing §5.4.3, we get the following pieces of structure on $G^!_Q$, which are parallel to those from §5.1.2.

(i) The homomorphism $\varphi_Q : F^*H^!_Q \to H^!_Q$ yields a map $F : G^!_Q \to G^!_Q$, which makes $G^!_Q$ into a group δ-scheme over Q.

(ii) The section (5.8) yields a canonical map $G^!_Q \to (\mathbb{G}_m)_Q$, which is a homomorphism of group δ-schemes over Q.

(iii) The homomorphism (5.6) yields a canonical section $\sigma^! : Q \to G^!_Q$, which is a δ-morphism.

An explicit description of $G^!_Q$ (together with the above pieces of structure on it) will be given in Proposition 5.5.2.

5.4.5. The group scheme $G^!$ Let $G^!$ be the Cartier dual of the formal group $H^!$ from §5.4.2. Then $G^!_Q = G^! \times_{\mathbb{A}^1} Q$.

The pieces of structure from §5.4.4(i,iii) are pullbacks of similar pieces of structure on $G^!$. On the other hand, the piece of structure from §5.4.4(ii) does not have an analog for $G^!$.

The affine group scheme $G^!$ and its coordinate ring are described in Appendix D. We will use these results below.

5.5. Explicit description of $G^!_Q$

5.5.1. The ring B Let B_0 be the Hopf algebra over $\mathbb{Z}[h]$ from Proposition D.2.2 (see also §D.3.6 for a description of $B_0 \otimes \mathbb{Z}_{(p)}$). Let B be the (p, h)-adic completion of B_0. Then B is a topological Hopf algebra over $\mathbb{Z}_p[[q-1]]$, where $q = 1 + h$. Elements of B are infinite sums

(5.9)
$$\sum_{n=0}^{\infty} a_n \cdot \frac{t(t-q+1)\dots(t-(n-1)(q-1))}{n!}, \qquad \text{where } a_n \in \mathbb{Z}_p[[q-1]],\ a_n \to 0.$$

An element (5.9) is in B_0 if and only if $a_n \in \mathbb{Z}[q]$ for all n and $a_n = 0$ for $n \gg 0$. Note that B is torsion-free as a $\mathbb{Z}_p[[q-1]]$-module.

Proposition 5.5.2. *(a) The group scheme $G_Q^!$ identifies with* $\operatorname{Spf} B$ *so that in terms of this identification the pairing $G_Q^! \times H_Q^! \to (\mathbb{G}_m)_Q$ is given by the formal series*

(5.10)
$$(1+(q-1)z)^{\frac{t}{q-1}} := \sum_{n=0}^{\infty} \frac{t(t-q+1)\dots(t-(n-1)(q-1))}{n!} \cdot z^n \in B[[z]]^{\times},$$

where z is the coordinate on $H_Q^!$ from §5.4.2.

(a′) The regular function on $G_Q^!$ corresponding to $t \in B$ defines a group homomorphism

$$G_Q^! \to (\mathbb{G}_a)_Q. \tag{5.11}$$

(b) The homomorphism $\phi : B \to B$ corresponding to the morphism $F : G_Q^! \to G_Q^!$ from §5.4.4(i) is given by

$$\phi(q) = q^p,\ \phi(t) = \Phi_p(q)t.$$

Moreover, ϕ makes B into a δ-ring.

(c) The homomorphism $G_Q^! \to (\mathbb{G}_m)_Q$ from §5.4.4(ii) is given by the element

$$q^{\frac{pt}{q-1}} := \sum_{n=0}^{\infty} \frac{t(t-q+1)\dots(t-(n-1)(q-1))}{n!} \cdot \Phi_p(q)^n \in B^{\times}, \tag{5.12}$$

which is obtained from (5.10) by setting $z = \Phi_p(q)$ (the sum converges because $\Phi_p(1) = p$).

(c′) One has

$$q^{\frac{pt}{q-1}} = \sum_{n=0}^{\infty} \alpha_n, \qquad \text{where } \alpha_n := \frac{pt(pt-q+1)\dots(pt-(n-1)(q-1))}{n!};$$

more precisely, $\alpha_n \in B_0$ *and the series* $\sum_{n=0}^{\infty} \alpha_n$ *converges in* B *to the element* $q^{\frac{pt}{q-1}}$ *defined by* (5.12).

(d) The section $\sigma^! : Q \to G^!_Q$ *from §5.4.4(iii) corresponds to the algebra homomorphism* $B \to \mathbb{Z}_p[[q-1]]$ *such that* $t \mapsto q-1$.

The "true meaning" of the homomorphism (5.11) will be explained later, see formula (5.16).

Proof. Statement (a) follows from Proposition D.2.2 and formula (D.4).

Statement (a′) is clear from §D.2 or Proposition D.2.2(iii). It also follows from (5.10) combined with the formula

$$(1+(q-1)z)^{\frac{t_1+t_2}{q-1}} = (1+(q-1)z)^{\frac{t_1}{q-1}} \cdot (1+(q-1)z)^{\frac{t_2}{q-1}}.$$

By Lemma D.3.5(ii), $F : G^!_Q \to G^!_Q$ is the base change of the morphism $\Psi_p : G^! \to G^!$ from §D.3.4, so $\phi : B \to B$ is the base change of the homomorphism $\psi^p : B_0 \to B_0$ from Lemma D.3.3. Since B is p-torsion-free, ϕ makes B into a δ-ring. This proves (b).

Statement (c) is clear because the homomorphism $G^!_Q \to (\mathbb{G}_m)_Q$ comes from the section (5.8), which is given by $z = \Phi_p(q)$.

To prove (d), recall that $\sigma^!$ comes from the homomorphism (5.6), which is given by the function $1+(q-1)z$; this function is the result of substituting $t = q-1$ into (5.10).

Let us prove (c′). By Lemma D.2.3, $\alpha_n \in B_0$ and in the ring $B_0[[z]]$ one has

$$\sum_{n=0}^{\infty} \alpha_n z^n = (1+(q-1)vz)^{\frac{t}{q-1}} := \sum_{n=0}^{\infty} \beta_n v^n z^n, \tag{5.13}$$

where $\beta_n := \frac{t(t-q+1)\dots(t-(n-1)(q-1))}{n!}$ and

$$v = v(q,z) := \frac{(1+(q-1)z)^p - 1}{(q-1)z} = \sum_{i=1}^{p} \binom{p}{i} (q-1)^{i-1} z^{i-1} \in \mathbb{Z}[q,z].$$

Note that $v \in I[z]$, where $I \subset \mathbb{Z}[q]$ is the ideal $(p, q-1)$. So the r.h.s of (5.13) belongs to the subring $\varprojlim_m (B/I^m B)[z] \subset B[[z]]$. Therefore we can set $z = 1$ and get

$$\sum_n \alpha_n = \sum_n \beta_n \cdot v(q,1)^n = \sum_n \beta_n \cdot \Phi_p(q)^n;$$

in other words, $\sum_n \alpha_n$ equals the r.h.s of (5.12). □

5.6. The isomorphism $G_Q^! \xrightarrow{\sim} G_Q$ in explicit terms

5.6.1. The isomorphisms $H_Q \xrightarrow{\sim} H_Q^!$ and $G_Q^! \xrightarrow{\sim} G_Q$ Theorem 2.10.4 yields a canonical isomorphism $H_Q \xrightarrow{\sim} H_Q^!$. It is compatible with the pieces of structure on H_Q and $H_Q^!$ introduced in §2.10.2 and §5.4.3. So the Cartier dual isomorphism $G_Q^! \xrightarrow{\sim} G_Q$ transforms the pieces of structure on $G_Q^!$ from §5.4.4 into the corresponding pieces of structure on G_Q (see §5.1.2–5.1.3).

5.6.2. The isomorphism between the coordinate rings of $G_Q^!$ and G_Q Recall that $G_Q = \operatorname{Spf} R$, $G_Q^! = \operatorname{Spf} B$, where $R := \hat{R}_0$ and $B := \hat{B}_0$ are the $(p, q-1)$-adic completions of the $\mathbb{Z}[q]$-algebras R_0 and B_0 from Propositions 5.2.1 and D.2.2. So the canonical isomorphism $G_Q^! \xrightarrow{\sim} G_Q$ induces an isomorphism $R \xrightarrow{\sim} B$; using it, we identify R and B. Then the element $x_0 \in R_0$ from Proposition 5.2.1 and the element $t \in B_0$ from §5.5.1 live in the same ring $R = B$. Let us discuss the relation between them.

By Proposition 5.5.2(c), we have

$$1 + \Phi_p(q)x_0 = q^{\frac{pt}{q-1}}, \tag{5.14}$$

$$x_0 = \frac{q^{\frac{pt}{q-1}} - 1}{\Phi_p(q)} = \sum_{n=1}^{\infty} \frac{t(t-q+1)\dots(t-(n-1)(q-1))}{n!} \cdot \Phi_p(q)^{n-1}. \tag{5.15}$$

We claim that in terms of the q-logarithm (see §5.3.2) one has

$$t = \log_q(u), \qquad \text{where } u^p = 1 + \Phi_p(q)x_0, \tag{5.16}$$

which implies that $pt = \log_q(1 + \Phi_p(q)x_0)$. This follows from parts (a′), (d) of Proposition 5.5.2 and the definition of $\log_q(u)$ at the end of §5.3.2.

5.6.3. Remark Using (5.15), it is easy to show that R_0 and B_0 are *different* as subrings of $R = B$; moreover, $R_0/(q-1)R_0$ and $B_0/(q-1)B_0$ are different as subrings of the ring $R/(q-1)R = B/(q-1)B$.

5.6.4. Plan of what follows By Proposition 5.5.2, $G_Q^! = \operatorname{Spf} B$. By (5.1), the isomorphism

$$\operatorname{Spf} B = G_Q^! \xrightarrow{\sim} G_Q \tag{5.17}$$

is given by an element $x \in W(B)$ such that $1 + \Phi_p([q])x \in B^\times$, where $B^\times \subset W(B)^\times$ is the subgroup of Teichmüller elements. In Proposition 5.6.6 we will write a formula for x.

5.6.5. The homomorphism $\psi : B \to W(B)$ According to A. Joyal [J85], the forgetful functor from the category of δ-rings to that of rings has a right adjoint, which is nothing but the functor W. Our B is a δ-ring, so the unit of Joyal's adjunction yields a homomorphism of δ-rings $\psi : B \to W(B)$. It is the unique homomorphism of δ-rings $B \to W(B)$ whose composition with the canonical epimorphism $W(B) \twoheadrightarrow W_1(B) = B$ equals id_B. For any $b \in B$ the n-th Buium-Joyal component (see §C.3.7) of the Witt vector $\psi(b)$ equals $\delta^n(b)$.

Proposition 5.6.6. *One has*

$$x = \psi\Big(\frac{q^{\frac{pt}{q-1}} - 1}{\Phi_p(q)}\Big), \tag{5.18}$$

$$1 + \Phi_p([q])x = [q^{\frac{pt}{q-1}}], \tag{5.19}$$

where $\psi : B \to W(B)$ is as in §5.6.5 and $q^{\frac{pt}{q-1}} \in B$ is defined by (5.12) (so $q^{\frac{pt}{q-1}} - 1$ is divisible by $\Phi_p(q)$).

Proof. By §5.6.1, the morphism $\operatorname{Spf} B = G_Q^! \xrightarrow{\sim} G_Q$ is a δ-morphism. So $x : \operatorname{Spf} B \to W$ is a δ-morphism. Therefore the corresponding map $H^0(W, \mathcal{O}_W) \to B$ is a δ-homomorphism. So the description of $H^0(W, \mathcal{O}_W)$ from §C.3.7 shows that $x = \psi(x_0)$, where x_0 is the 0-th component of the Witt vector x. Combining this with (5.15), we get (5.18).

Formula (5.19) follows from (5.14) because $1 + \Phi_p([q])x$ is a Teichmüller element. ☐

5.7. The group schemes $G_Q^{!?}$ and $G_Q^{!!}$

Using Witt vectors, we will define group δ-schemes $G_Q^{!?}$ and $G_Q^{!!}$ over Q; each of them is canonically isomorphic to $G_Q^!$ and therefore to G_Q. The author is not sure that $G_Q^{!?}$ is really useful; this explains the question mark in the notation.

5.7.1. Definition of $G_Q^{!?}$ For any p-nilpotent ring A let

$$G_Q^{!?}(A) = \{(q, y) \in Q(A) \times W(A) \mid Fy = [q-1]^{p-1} \cdot y\}. \tag{5.20}$$

Then $G_Q^{!?} \subset W_Q$ is a group subscheme. The map

$$G_Q^{!?} \to G_Q^{!?}, \quad (q, y) \mapsto (q^p, [\Phi_p(q)] \cdot y)$$

makes $G_Q^{!?}$ into a group δ-scheme over the formal δ-scheme Q.

Proposition 5.7.2. *One has a canonical isomorphism*

$$G_Q^! \xrightarrow{\sim} G_Q^{!?}; \tag{5.21}$$

of group δ-schemes over Q; it is induced by the map (D.20).

Proof. Follows from Proposition D.4.10 and (D.17). ☐

5.7.3. Definition of $G_Q^{!!}$ For any p-nilpotent ring A let

$$G_Q^{!!}(A) = \{(q, y) \in Q(A) \times W(A) \mid Fy = \Phi_p([q]) \cdot y\}. \tag{5.22}$$

Then $G_Q^{!!} \subset W_Q$ is a group subscheme. Moreover, $G_Q^{!!}$ is a δ-subscheme if $W_Q = W \times Q$ is equipped with the product of the standard δ-structures on W an Q.

5.7.4. The isomorphism $G_Q^! \xrightarrow{\sim} G_Q^{!!}$ Let $t \in H^0(G_Q^{!!}, \mathcal{O}_{G_Q^{!!}})$ be the function that takes $(q, y) \in G_Q^{!!}(A)$ to the 0-th component of the Witt vector y. Similarly to the proof of Proposition 5.2.1, one shows that $H^0(G_Q^{!!}, \mathcal{O}_{G_Q^{!!}})$ is the $(p, q-1)$-adic completion of the δ-algebra over $\mathbb{Z}[q]$ with a single generator t and a single relation

$$t^p + p\delta(t) = \Phi_p(q) \cdot t.$$

Combining this with §D.3.6 and Proposition 5.5.2(a,b), we get an isomorphism of δ-rings $H^0(G_Q^{!!}, \mathcal{O}_{G_Q^!}) \xrightarrow{\sim} H^0(G_Q, \mathcal{O}_{G_Q^!})$. The corresponding isomorphism $G_Q^! \xrightarrow{\sim} G_Q^{!!}$ is a group isomorphism by Proposition 5.5.2(a′).

5.7.5. Remark Combining Proposition 5.7.2 and §5.7.4 with the isomorphism $G_Q^! \xrightarrow{\sim} G_Q$ from §5.6, we get canonical isomorphisms between the group δ-schemes G_Q, $G_Q^{!?}$, and $G_Q^{!!}$. These group δ-schemes are defined in terms of W, but I do not know an explicit description of these isomorphisms in terms of the standard Witt vector formalism. However, *after* the "de Rham" specialization $q = 1$ the isomorphisms in question specialize to the explicit isomorphisms from §4.4 (note that G_Q, $G_Q^{!?}$, and $G_Q^{!!}$ specialize to G_{dR}, $W^{(F)}_{\mathrm{Spf}\,\mathbb{Z}_p}$, and $W^{F=p}_{\mathrm{Spf}\,\mathbb{Z}_p}$, respectively).

Appendix A. On the prismatic cohomology of $(\mathbb{A}^1 \setminus \{0\})_{\operatorname{Spf} \mathbb{Z}_p}$

A.1. The result

Let $\mathbb{G}_m^{\Delta}$ be the prismatization of $(\mathbb{A}^1 \setminus \{0\})_{\operatorname{Spf} \mathbb{Z}_p} = (\mathbb{G}_m)_{\operatorname{Spf} \mathbb{Z}_p}$. The projection $(\mathbb{G}_m)_{\operatorname{Spf} \mathbb{Z}_p} \to \operatorname{Spf} \mathbb{Z}_p$ induces a morphism $\pi : \mathbb{G}_m^{\Delta} \to (\operatorname{Spf} \mathbb{Z}_p)^{\Delta} = \Sigma$. The goal of this Appendix is to compute the higher derived images $R^i\pi_*\mathcal{O}_{\mathbb{G}_m^{\Delta}}$ using some results of §2. Here is the answer; it is almost contained[14] in [BS, §16].

Theorem A.1.1. *(i)* $R^i\pi_*\mathcal{O}_{\mathbb{G}_m^{\Delta}} = 0$ *if* $i \neq 0, 1$.

(ii) $R^0\pi_*\mathcal{O}_{\mathbb{G}_m^{\Delta}} = \mathcal{O}_\Sigma$.

(iii) $R^1\pi_*\mathcal{O}_{\mathbb{G}_m^{\Delta}} = \bigoplus_{n\in\mathbb{Z}} \mathcal{M}_n$, *where* $\mathcal{M}_0 := \mathcal{O}_\Sigma\{-1\}$ *and if* $n \neq 0$ *and* m *is the biggest number such that* $p^m | n$ *then* $\mathcal{M}_n := \mathcal{O}_\Sigma(\Delta_0 + \cdots + \Delta_m)/\mathcal{O}_\Sigma$. *Here* $\Delta_0 \subset \Sigma$ *is the Hodge-Tate divisor and* $\Delta_i := (F^i)^{-1}(\Delta_0)$.

A.2. Proof of Theorem A.1.1

By Corollary 2.7.3, $\mathbb{G}_m^{\Delta} = \operatorname{Cone}(G_\Sigma \to (\mathbb{G}_m)_\Sigma)$. Thus

$$R^i\pi_*\mathcal{O}_{\mathbb{G}_m^{\Delta}} = H^i(G_\Sigma, \mathcal{O}_\Sigma \otimes A),$$

where A is the regular representation of $\mathbb{G}_m$ (so $\mathcal{O}_\Sigma \otimes A$ is a $(\mathbb{G}_m)_\Sigma$-module and therefore a G_Σ-module). Equivalently,

$$R^i\pi_*\mathcal{O}_{\mathbb{G}_m^{\Delta}} = \bigoplus_{n\in\mathbb{Z}} H^i(G_\Sigma, \mathcal{O}_\Sigma \otimes \chi_n), \tag{A.1}$$

where χ_n is the 1-dimensional $\mathbb{G}_m$-module corresponding to the character $z \mapsto z^n$.

By Theorem 2.7.5, G_Σ is the Cartier dual of a 1-dimensional formal group H_Σ. Let $s : \Sigma \to H_\Sigma$ be the section corresponding to the canonical homomorphism $G_\Sigma \to (\mathbb{G}_m)_\Sigma$. For $n \in \mathbb{Z}$, let $D_n \subset H_\Sigma$ be the image of the composite morphism $\Sigma \xrightarrow{s} H_\Sigma \xrightarrow{n} H_\Sigma$; in particular, $D_0 \subset H_\Sigma$ is the image of the zero section. Then

$$H^i(G_\Sigma, \mathcal{O}_\Sigma \otimes \chi_n) = R^i\mathbf{0}^!\mathcal{O}_{D_n}, \tag{A.2}$$

where $\mathbf{0} : \Sigma \to H_\Sigma$ is the zero section and $\mathcal{O}_{D_n}$ is viewed as an $\mathcal{O}_{H_\Sigma}$-module.

[14] In [BS, §16] the pullback of $R^i\pi_*\mathcal{O}_{\mathbb{G}_m^{\Delta}}$ to the q-de Rham prism Q was computed. Theorem A.1.1 can be easily deduced from this computation.

Lemma A.2.1. *$R^i\mathbf{0}^!\mathcal{O}_{D_0} = 0$ for $i \neq 0, 1$, $R^0\mathbf{0}^!\mathcal{O}_{D_0} = \mathcal{O}_\Sigma$, and $R^1\mathbf{0}^!\mathcal{O}_{D_0} = \mathcal{O}_\Sigma\{-1\}$.*

Proof. By Theorem 2.7.10, $\mathrm{Lie}(H_\Sigma) = \mathcal{O}_\Sigma\{-1\}$. □

The following lemma is a reformulation of Corollary 2.9.3.

Lemma A.2.2. *Let $n = p^m n'$, where $(n', p) = 1$. Then the projection $H_\Sigma \to \Sigma$ induces an isomorphism $D_0 \cap D_n \xrightarrow{\sim} \Delta_0 + \cdots \Delta_m$, where $\Delta_0 \subset \Sigma$ is the Hodge-Tate divisor and $\Delta_i := (F^i)^{-1}(\Delta_0)$.* □

Corollary A.2.3. *If $n \neq 0$ then $R^i\mathbf{0}^!\mathcal{O}_{D_n} = 0$ for $i \neq 1$ and $R^1\mathbf{0}^!\mathcal{O}_{D_n} = \mathcal{M}_n$, where $\mathcal{M}_n$ is as in Theorem A.1.1(iii).* □

Combining Lemma A.2.1, Corollary A.2.3, and (A.1)–(A.2), we get Theorem A.1.1.

Appendix B. The Cartier dual of the divided powers version of $\mathbb{G}_m$

B.1. Plan

As usual, let $\mathbb{G}_m = \operatorname{Spec}\mathbb{Z}[x, x^{-1}]$ be the multiplicative group over $\mathbb{Z}$. Let $\mathbb{M}_m$ denote the scheme $\mathbb{A}^1 = \operatorname{Spec}\mathbb{Z}[x]$ viewed as a multiplicative monoid over $\operatorname{Spec}\mathbb{Z}$. Let $\mathbb{G}_m^\sharp$ (resp. $\mathbb{M}_m^\sharp$) be the PD hull of the unit in $\mathbb{G}_m$ (resp. in $\mathbb{M}_m$); explicitly,

$$\mathbb{M}_m^\sharp = \operatorname{Spec} A, \qquad \text{where } A := \mathbb{Z}\left[x, \frac{(x-1)^2}{2!}, \frac{(x-1)^3}{3!}, \dots\right] \tag{B.1}$$

and $\mathbb{G}_m^\sharp = \operatorname{Spec} A[1/x]$. The monoid structure on $\mathbb{M}_m$ and $\mathbb{G}_m$ extends to a monoid structure on $\mathbb{M}_m^\sharp$ and $\mathbb{G}_m^\sharp$. Moreover, the monoid $\mathbb{G}_m^\sharp$ is a group.

Theorem B.2.3 below describes the Cartier duals[15] of $\mathbb{G}_m^\sharp$ and $\mathbb{M}_m^\sharp$ (this description is likely to be known, but I was unable to find a reference). The description becomes even simpler after base change to Spf $\mathbb{Z}_p$, see §B.4.

In §B.5 we construct an exact sequence (B.10) of group schemes over Spf $\mathbb{Z}_p$, which plays an important role in [BL]. In §B.6 we discuss a variant of (B.10) over $\operatorname{Spec}\mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p.

[15] By the Cartier dual of $\mathbb{M}_m^\sharp$ we mean $\underline{\mathrm{Hom}}(\mathbb{M}_m^\sharp, \mathbb{M}_m)$; equivalently, the bialgebras corresponding to $\mathbb{M}_m^\sharp$ and its Cartier dual are dual to each other.

B.2. Formulation of the theorem

We will define ind-schemes Γ and Γ_+ equipped with a monoid structure; moreover, Γ is a group. Then we will identify the Cartier duals of $\mathbb{G}_m^\sharp$ and $\mathbb{M}_m^\sharp$ with Γ and Γ_+, respectively.

B.2.1. Definition of Γ and Γ_+ Given integers $a \leq b$, define a polynomial $f_{a,b} \in \mathbb{Z}[u]$ by

$$f_{a,b}(u) := \prod_{i=a}^{b} (u-i). \tag{B.2}$$

Define a closed subscheme $\Gamma^{[a,b]} \subset \mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[u]$ by

$$\Gamma^{[a,b]} := \operatorname{Spec} \mathbb{Z}[u]/(f_{a,b}).$$

The addition map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ induces a morphism $\Gamma^{[a,b]} \times \Gamma^{[c,d]} \to \Gamma^{[a+c,b+d]}$. So the ind-schemes

$$\Gamma := \Gamma^{[-\infty,\infty]} := \varinjlim_N \Gamma^{[-N,N]}, \quad \Gamma^+ := \Gamma^{[0,\infty]} := \varinjlim_N \Gamma^{[0,N]}$$

are monoids; moreover, Γ is a group.

B.2.2. The pairings We have a pairing

$$\mathbb{M}_m^\sharp \times \Gamma^+ \to \mathbb{M}_m, \quad (x,u) \mapsto x^u := \sum_{n=0}^{\infty} f_{0,n}(u) \cdot \frac{(x-1)^n}{n!}, \tag{B.3}$$

where $f_{0,n}$ is defined by formula (B.2). Since $\mathbb{G}_m^\sharp$ is a group, the morphism (B.3) maps $\mathbb{G}_m^\sharp \times \Gamma^+$ to $\mathbb{G}_m$. Define a pairing

$$\mathbb{G}_m^\sharp \times \Gamma \to \mathbb{G}_m \tag{B.4}$$

as follows: for each integer $a \geq 0$ its restriction to $\mathbb{G}_m^\sharp \times \Gamma^{[-a,\infty]}$ is given by

$$(x,u) \mapsto x^{-a} \cdot x^{u+a} = x^{-a} \cdot \sum_{n=0}^{\infty} f_{0,n}(u+a) \cdot \frac{(x-1)^n}{n!}.$$

Theorem B.2.3. *(i) The pairings (B.4) and (B.3) induce isomorphisms*

$$\mathbb{G}_m^\sharp \xrightarrow{\sim} \underline{\operatorname{Hom}}(\Gamma, \mathbb{G}_m), \quad \mathbb{M}_m^\sharp \xrightarrow{\sim} \underline{\operatorname{Hom}}(\Gamma^+, \mathbb{M}_m).$$

(ii) The coordinate ring of $\mathbb{G}_m^\sharp$ is a free $\mathbb{Z}$-module.[16]

B.3. Proof of Theorem B.2.3

B.3.1. Distributions on Γ and Γ^+ Let $\mathrm{Distr}(\Gamma^{[a,b]})$ be the $\mathbb{Z}$-module dual to the coordinate ring of $\Gamma^{[a,b]}$; equivalently, $\mathrm{Distr}(\Gamma^{[a,b]})$ is the $\mathbb{Z}$-module of those linear functionals $\mathbb{Z}[u] \to \mathbb{Z}$ that are trivial on the ideal $(f_{a,b}) \subset \mathbb{Z}[u]$. We think of elements of $\mathrm{Distr}(\Gamma^{[a,b]})$ as *distributions* on $\Gamma^{[a,b]}$. Let

$$\mathrm{Distr}(\Gamma) := \varinjlim_N \mathrm{Distr}(\Gamma^{[-N,N]}), \quad \mathrm{Distr}(\Gamma^+) := \varinjlim_N \mathrm{Distr}(\Gamma^{[0,N]}).$$

Then $\mathrm{Distr}(\Gamma)$ and $\mathrm{Distr}(\Gamma^+)$ are rings with respect to convolution; moreover, they are bialgebras over $\mathbb{Z}$.

For each $n \in \mathbb{Z}$ we have the functional $\mathbb{Z}[u] \to \mathbb{Z}$ given by evaluation at n; it defines an element $\delta_n \in \mathrm{Distr}(\Gamma)$. If $n \geq 0$ then $\delta_n \in \mathrm{Distr}^+(\Gamma)$. It is clear that $\delta_m\delta_n = \delta_{m+n}$ and δ_0 is the unit of $\mathrm{Distr}(\Gamma)$. So $\delta_{-n} = \delta_n^{-1}$.

Lemma B.3.2. *(i) For every $n \geq 0$ one has $(\delta_1 - \delta_0)^n \in n! \cdot \mathrm{Distr}(\Gamma^{[0,n]})$;*

(ii) the distributions $\frac{(\delta_1-\delta_0)^n}{n!} = \frac{(\delta_1-1)^n}{n!}$, $n \geq 0$, form a basis in $\mathrm{Distr}(\Gamma^+)$;

(iii) $\mathrm{Distr}(\Gamma)$ is equal to the localization $\mathrm{Distr}(\Gamma)[\delta_1^{-1}] = \mathrm{Distr}(\Gamma)[\delta_{-1}]$.

Proof. $(\delta_1 - \delta_0)^n$ is the unique element of $\mathrm{Distr}(\Gamma^{[0,n]})$ such that the corresponding functional $\mathbb{Z}[u] \to \mathbb{Z}$ takes u^n to $n!$ and $u^{n-1}, \ldots, u, 1$ to 0. The value of this functional on the polynomial $f_{0,n-1}$ equals $n!$. This implies (i)-(ii). Statement (iii) follows from (ii). ☐

B.3.3. End of the proof The pairings (B.4) and (B.3) induce bialgebra homomorphisms

$$\mathrm{Distr}(\Gamma) \to \mathrm{Fun}(\mathbb{G}_m^\sharp) \quad \text{and} \quad \mathrm{Distr}(\Gamma^+) \to \mathrm{Fun}(\mathbb{M}_m^\sharp), \tag{B.5}$$

where Fun stands for the coordinate ring. The homomorphisms (B.5) take δ_n to x^n, where x is the coordinate on $\mathbb{G}_m$ or $\mathbb{M}_m$. Lemma B.3.2 implies that the maps (B.5) are isomorphisms. Theorem B.2.3(i) follows.

It is easy to see that the $\mathbb{Z}$-module $\mathrm{Distr}(\Gamma^{[-N-1,N+1]})/\,\mathrm{Distr}(\Gamma^{[-N,N]})$ is free. Therefore the $\mathbb{Z}$-module $\mathrm{Distr}(\Gamma)$ is free. Theorem B.2.3(ii) follows. ☐

[16] A similar statement for $\mathbb{M}_m^\sharp$ is obvious, see formula (B.1).

B.4. Base change to Spf $\mathbb{Z}_p$

Fix a prime p. Let Γ be as in §B.2.1. Let

$$\Gamma_{\mathbb{Z}/p^n\mathbb{Z}} := \Gamma \times \operatorname{Spec} \mathbb{Z}/p^n\mathbb{Z}, \quad \Gamma_{\operatorname{Spf} \mathbb{Z}_p} := \Gamma \times \operatorname{Spf} \mathbb{Z}_p.$$

$\Gamma_{\mathbb{Z}/p^n\mathbb{Z}}$ is a group ind-scheme over $\mathbb{Z}/p^n\mathbb{Z}$, and $\Gamma_{\operatorname{Spf} \mathbb{Z}_p}$ is a group ind-scheme over Spf $\mathbb{Z}_p$. The next lemma shows that in fact, these ind-schemes are *formal schemes.*

Lemma B.4.1. *$\Gamma_{\mathbb{Z}/p^n\mathbb{Z}}$ is the formal completion of $\mathbb{A}^1_{\mathbb{Z}/p^n\mathbb{Z}} = \operatorname{Spec}(\mathbb{Z}/p^n\mathbb{Z})[u]$ along the subscheme of $\mathbb{A}^1_{\mathbb{F}_p}$ defined by the equation $u(u-1)\dots(u-p+1) = 0$.* □

The lemma yields canonical exact sequences

$$0 \to (\hat{\mathbb{G}}_a)_{\mathbb{Z}/p^n\mathbb{Z}} \to \Gamma_{\mathbb{Z}/p^n\mathbb{Z}} \to \mathbb{Z}/p\mathbb{Z} \to 0, \tag{B.6}$$

$$0 \to (\hat{\mathbb{G}}_a)_{\operatorname{Spf} \mathbb{Z}_p} \to \Gamma_{\operatorname{Spf} \mathbb{Z}_p} \to \mathbb{Z}/p\mathbb{Z} \to 0. \tag{B.7}$$

Remark B.4.2. If $n = 1$ the exact sequence (B.6) has a unique splitting. If $n > 1$ then (B.6) has no splittings.

B.5. Dualizing the exact sequence (B.7)

B.5.1. The homomorphism log : $(\mathbb{G}_m^\sharp)_{\operatorname{Spf} \mathbb{Z}_p} \to (\mathbb{G}_a)_{\operatorname{Spf} \mathbb{Z}_p}$ We have the homomorphism $\log : (\mathbb{G}_m^\sharp)_{\operatorname{Spf} \mathbb{Z}_p} \to (\mathbb{G}_a)_{\operatorname{Spf} \mathbb{Z}_p}$ given by

$$\log x := \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{(x-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)! \cdot \frac{(x-1)^n}{n!}.$$

Let $\mathbb{G}_a^\sharp$ be the divided powers additive group, i.e., the PD hull of 0 in $\mathbb{G}_a$; as a scheme,

$$\mathbb{G}_a^\sharp = \operatorname{Spec} \mathbb{Z}\Big[y, \frac{y^2}{2!}, \frac{y^3}{3!}, \dots\Big].$$

Lemma B.5.2. *The homomorphism* $\log : (\mathbb{G}_m^\sharp)_{\operatorname{Spf} \mathbb{Z}_p} \to (\mathbb{G}_a)_{\operatorname{Spf} \mathbb{Z}_p}$ *factors through $(\mathbb{G}_a^\sharp)_{\operatorname{Spf} \mathbb{Z}_p}$, so we get a homomorphism*

$$\log : (\mathbb{G}_m^\sharp)_{\operatorname{Spf} \mathbb{Z}_p} \to (\mathbb{G}_a^\sharp)_{\operatorname{Spf} \mathbb{Z}_p}. \tag{B.8}$$

Remark B.5.3. In the lemma the factorization is unique because the map $\mathrm{Fun}(\mathbb{G}_a) \to \mathrm{Fun}(\mathbb{G}_a^\sharp)$ becomes an isomorphism after tensoring by $\mathbb{Q}$ (here Fun stands for the ring of regular functions).

Proof of Lemma B.5.2. We have to show that $(\log x)^k$ is divisible by $k!$ in the ring of regular functions on $(\mathbb{G}_m^\sharp)_{\mathrm{Spf}\,\mathbb{Z}_p}$ for any $k > 0$. Since $\frac{d}{dx}(\log x)^k = kx^{-1}(\log x)^{k-1}$, this follows by induction on k. □

Lemma B.5.4. *The embedding $(\mu_p)_{\mathrm{Spf}\,\mathbb{Z}_p} \hookrightarrow (\mathbb{G}_m)_{\mathrm{Spf}\,\mathbb{Z}_p}$ comes from a unique homomorphism*

$$(\mu_p)_{\mathrm{Spf}\,\mathbb{Z}_p} \hookrightarrow (\mathbb{G}_m^\sharp)_{\mathrm{Spf}\,\mathbb{Z}_p}. \tag{B.9}$$

Proof. It suffices to show that $(\mu_p)_{\mathrm{Spec}\,\mathbb{Z}_p}$ is a PD-thickening of the unit of $(\mu_p)_{\mathrm{Spec}\,\mathbb{Z}_p}$. We have $(\mu_p)_{\mathrm{Spec}\,\mathbb{Z}_p} = \mathrm{Spec}\,A$, where $A = \mathbb{Z}_p[x]/(x^p - 1)$, and the unit corresponds to the ideal $I := (x-1) \subset A$, so the problem is to show that $f^p \in pI$ for $f \in I$. Indeed, the image of $(x-1)^p$ in A/pA is zero, so for $f \in I$ one has $f^p \in pA \cap I = pI$. □

Remark B.5.5. The composition of (B.9) and (B.8) is zero because

$$\mathrm{Hom}((\mu_p)_{\mathrm{Spf}\,\mathbb{Z}_p}, (\mathbb{G}_a^\sharp)_{\mathrm{Spf}\,\mathbb{Z}_p}) = 0.$$

Proposition B.5.6. *(i) The sequence*

$$0 \to (\mu_p)_{\mathrm{Spf}\,\mathbb{Z}_p} \to (\mathbb{G}_m^\sharp)_{\mathrm{Spf}\,\mathbb{Z}_p} \xrightarrow{\log} (\mathbb{G}_a^\sharp)_{\mathrm{Spf}\,\mathbb{Z}_p} \to 0, \tag{B.10}$$

whose morphisms are (B.9) *and* (B.8)*, is exact.*

(ii) The exact sequence (B.10) *is Cartier dual to* (B.7)*; the pairing between $(\mathbb{G}_m^\sharp)_{\mathrm{Spf}\,\mathbb{Z}_p}$ and $\Gamma_{\mathrm{Spf}\,\mathbb{Z}_p}$ is given by* (B.4)*, and the pairing*

$$(\mathbb{G}_a^\sharp)_{\mathrm{Spf}\,\mathbb{Z}_p} \times (\hat{\mathbb{G}}_a)_{\mathrm{Spf}\,\mathbb{Z}_p} \to \mathbb{G}_m$$

is the exponent of the product.

Proof. It suffices to prove that the morphisms (B.9) and (B.8) are dual to the corresponding morphisms in the exact sequence (B.7). This follows from the equality $x^u = \exp(u \cdot \log x)$. □

In the next subsection we describe another approach to the exact sequence (B.10).

B.6. A variant of (B.10) over $\mathbb{Z}_{(p)}$

Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at p. Base-changing $\mathbb{G}_m$ and μ_p to $\mathbb{Z}_{(p)}$, one gets group schemes $(\mathbb{G}_m)_{\mathbb{Z}_{(p)}}$ and $(\mu_p)_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$. Similarly to Lemma B.5.4, one sees that the embedding $(\mu_p)_{\mathbb{Z}_{(p)}} \hookrightarrow (\mathbb{G}_m)_{\mathbb{Z}_{(p)}}$ comes from a unique homomorphism

$$(\mu_p)_{\mathbb{Z}_{(p)}} \hookrightarrow (\mathbb{G}_m^\sharp)_{\mathbb{Z}_{(p)}}. \tag{B.11}$$

We are going to describe the cokernel of (B.11), see Proposition B.6.3. Then we will deduce exactness of (B.10) from this description, see §B.6.5.

B.6.1. The group schemes G and $G^\sharp$ Let G be the group scheme over $\mathbb{Z}$ whose group of A-points is the set $\{z \in A \,|\, 1+pz \in A^\times\}$ equipped with the operation $z_1 * z_2 := z_1 + z_2 + pz_1z_2$; in other words, G is the p-rescaled version of $\mathbb{G}_m$. We have a canonical homomorphism

$$G \to \mathbb{G}_m, \quad z \mapsto 1+pz. \tag{B.12}$$

As usual, let $G^\sharp$ be the divided powers version of G (i.e., the PD hull of the unit in G).

Lemma B.6.2. *There is a unique homomorphism*

$$\mathbb{G}_m^\sharp \to G^\sharp \tag{B.13}$$

such that the diagram

$$\begin{array}{ccc} \mathbb{G}_m^\sharp & \longrightarrow & G^\sharp \\ & {\scriptstyle p}\searrow & \downarrow \\ & & \mathbb{G}_m^\sharp \end{array}$$

commutes; here the vertical arrow comes from (B.12)*.*

Proof. As above, let z be the coordinate on G. Let x be the usual coordinate on $\mathbb{G}_m$ and $t := x - 1$. The homomorphism $(\mathbb{G}_m^\sharp)_\mathbb{Q} \xrightarrow{p} (\mathbb{G}_m^\sharp)_\mathbb{Q} = G_\mathbb{Q}^\sharp$ is given by $z = \frac{(1+t)^p - 1}{p}$. The problem is to check that $\frac{(1+t)^p-1}{p} = \sum_{i=1}^p m_i\gamma_i(t)$ for some $m_i \in \mathbb{Z}$ (here γ_i is the i-th divided power). This is clear. $\square$

Proposition B.6.3. *The homomorphism $(\mathbb{G}_m^\sharp)_{\mathbb{Z}_{(p)}} \to (G^\sharp)_{\mathbb{Z}_{(p)}}$ corresponding to* (B.13) *induces an isomorphism $(\mathbb{G}_m^\sharp)_{\mathbb{Z}_{(p)}}/(\mu_p)_{\mathbb{Z}_{(p)}} \xrightarrow{\sim} (G^\sharp)_{\mathbb{Z}_{(p)}}$.*

For a proof, see §B.7.

B.6.4. Passing to formal completions (i) Let $\hat{G}, \hat{\mathbb{G}}_a$ be the formal completions of the group schemes $G, \mathbb{G}_a$ along their units; these are formal groups over $\mathbb{Z}$. Let $\hat{G}_{\mathbb{Z}_{(p)}}, (\hat{\mathbb{G}}_a)_{\mathbb{Z}_{(p)}}$ be the corresponding formal groups over $\mathbb{Z}_{(p)}$. One has an isomorphism

$$\hat{G}_{\mathbb{Z}_{(p)}} \xrightarrow{\sim} (\hat{\mathbb{G}}_a)_{\mathbb{Z}_{(p)}}, \quad z \mapsto \frac{\log(1+pz)}{p} := \sum_{n=1}^{\infty} \frac{(-p)^{n-1}}{n} \cdot z^n. \tag{B.14}$$

(ii) The isomorphism (B.14) induces an isomorphism

$$G^\sharp_{\operatorname{Spf}\mathbb{Z}_p} \xrightarrow{\sim} (\mathbb{G}_a^\sharp)_{\operatorname{Spf}\mathbb{Z}_p} \tag{B.15}$$

because one can think of $G^\sharp_{\operatorname{Spf}\mathbb{Z}_p}$ (resp. $(\mathbb{G}_a^\sharp)_{\operatorname{Spf}\mathbb{Z}_p}$) as the p-adically completed PD version of $\hat{G}_{\mathbb{Z}_{(p)}}$ (resp. $(\hat{\mathbb{G}}_a)_{\mathbb{Z}_{(p)}}$).

Note that (B.14) is an isomorphism of *formal* groups over the *scheme* $\operatorname{Spec}\mathbb{Z}_{(p)}$, while (B.15) is an isomorphism of group *schemes* over the *formal scheme* $\operatorname{Spf}\mathbb{Z}_p$.

B.6.5. A proof of exactness of (B.10) Using that $\log(x^p) = p \cdot \log x$, one checks that the homomorphism $\log : (\mathbb{G}_m^\sharp)_{\operatorname{Spf}\mathbb{Z}_p} \to (\mathbb{G}_a^\sharp)_{\operatorname{Spf}\mathbb{Z}_p}$ from (B.10) equals the composite map

$$(\mathbb{G}_m^\sharp)_{\operatorname{Spf}\mathbb{Z}_p} \to G^\sharp_{\operatorname{Spf}\mathbb{Z}_p} \xrightarrow{\sim} (\mathbb{G}_a^\sharp)_{\operatorname{Spf}\mathbb{Z}_p},$$

where the first arrow comes from (B.13) and the second one is (B.15). So exactness of (B.10) follows from Proposition B.6.3.

B.7. Proof of Proposition B.6.3

B.7.1. Straightforward proof The kernel of the homomorphism

$$(\mathbb{G}_m^\sharp)_{\mathbb{Z}_{(p)}} \to (G^\sharp)_{\mathbb{Z}_{(p)}}$$

equals $(\mu_p)_{\mathbb{Z}_{(p)}}$. The problem is to show that the homomorphism is faithfully flat.

We will use the coordinates z and t from the proof of Lemma B.6.2. We have

$$z = \frac{(1+t)^p - 1}{p} = \gamma(t) + \sum_{i=1}^{p-1} n_i t^i \quad \text{for some } n_i \in \mathbb{Z}, \tag{B.16}$$

where $\gamma(t) := \frac{t^p}{p}$.

The coordinate ring of $(G^\sharp)_{\mathbb{Z}_{(p)}}$ is

$$A\left[\frac{1}{1+pz}\right], \text{ where } A := \mathbb{Z}_{(p)}[z, \gamma(z), \gamma^2(z), \ldots] \subset \mathbb{Q}[z].$$

The coordinate ring of $(\mathbb{G}_m^\sharp)_{\mathbb{Z}_{(p)}}$ is

$$B\left[\frac{1}{1+t}\right] = B\left[\frac{1}{1+pz}\right], \text{ where } B := \mathbb{Z}_{(p)}[t, \gamma(t), \gamma^2(t), \ldots] \subset \mathbb{Q}[t].$$

It suffices to show that the homomorphism $A \to B$ given by (B.16) makes B into a free A-module with basis $1, t, \ldots, t^{p-1}$. These elements form a basis of $B \otimes \mathbb{Q} = \mathbb{Q}[t]$ over $A \otimes \mathbb{Q} = \mathbb{Q}[z]$, so we only have to check that $1, t, \ldots, t^{p-1}$ generate B as an A-module. Note that as a $\mathbb{Z}_{(p)}$-module, B is generated by elements

$$\prod_{i=0}^{\infty} (\gamma^i(t))^{m_i}, \qquad \text{where } 0 \le m_i < p \text{ and } m_i = 0 \text{ for } i \gg 0.$$

By (B.16), $\prod_{i=0}^{\infty} (\gamma^i(t))^{m_i} = t^{m_0} \cdot \prod_{i>0} (\gamma^{i-1}(z))^{m_i} + \{\text{lower terms}\}$, so we can proceed by induction. ☐

B.7.2. Proof via Cartier duality (sketch) One can also prove Proposition B.6.3 by passing to the Cartier duals. Similarly to Theorem B.2.3, the Cartier dual of $G^\sharp$ identifies with the group ind-scheme Γ_p whose definition is parallel to that of Γ (see §B.2.1) but with the polynomial $\prod_{i=a}^{b} (u-i)$ from formula (B.2) being replaced by $\prod_{i=a}^{b} (u - pi)$. Details are left to the reader.

Appendix C. The Cartier dual of $\hat{\mathbb{G}}_m$

Let $\hat{\mathbb{G}}_m$ denote the formal multiplicative group over $\mathbb{Z}$. For any ring A one has

$$\hat{\mathbb{G}}_m(A) = \{y \in A^\times \mid y - 1 \text{ is nilpotent}\}.$$

In this section we give two descriptions of the Cartier dual of $\hat{\mathbb{G}}_m$, see §C.1 and §C.3. They are probably well known: the description from §C.3 is contained in [MRT], and the one from §C.1 was known to T. Ekedahl (see Remark 4 on p. 197 of [Ek]).

C.1. The Cartier dual in terms of the ring of integer-valued polynomials

C.1.1. The ring scheme $\mathscr{R}$ Let $\mathscr{R} := \underline{\mathrm{Hom}}(\hat{\mathbb{G}}_m, \hat{\mathbb{G}}_m)$. This is a unital ring scheme over Spec $\mathbb{Z}$. The action of $\mathscr{R}$ on $\mathrm{Lie}(\hat{\mathbb{G}}_m)$ defines a homomorphism of ring schemes

$$\mathscr{R} \to \mathbb{G}_a \tag{C.1}$$

(the multiplication operation in $\mathbb{G}_a$ is the usual one). The coordinate ring of $\mathbb{G}_a$ equals $\mathbb{Z}[u]$, so (C.1) induces a ring homomorphism

$$\mathbb{Z}[u] \to H^0(\mathscr{R}, \mathcal{O}_{\mathscr{R}}). \tag{C.2}$$

As a group scheme, $\mathscr{R}$ equals $\underline{\mathrm{Hom}}(\hat{\mathbb{G}}_m, \mathbb{G}_m)$, i.e., the Cartier dual of $\hat{\mathbb{G}}_m$. So $\mathscr{R}$ is a flat affine scheme over Spec $\mathbb{Z}$.

By Lie theory, the homomorphism (C.1) induces an isomorphism

$$\mathscr{R} \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{G}_a \otimes \mathbb{Q}. \tag{C.3}$$

The action of $\operatorname{Spec} \mathbb{Q}[u] = \mathbb{G}_a \otimes \mathbb{Q} = \mathscr{R} \otimes \mathbb{Q}$ on $\hat{\mathbb{G}}_m \otimes \mathbb{Q}$ is given by Newton's binomial formula

$$y^u = \sum_{n=0}^{\infty} \binom{u}{n} (y-1)^u, \qquad \binom{u}{n} := \frac{u(u-1)\dots(u-n+1)}{n!} \in \mathbb{Q}[u]. \tag{C.4}$$

The ring scheme $\mathscr{R}$ is commutative by virtue of (C.3) and flatness of $\mathscr{R}$ over Spec $\mathbb{Z}$.

The homomorphism (C.2) becomes an isomorphism after tensoring by $\mathbb{Q}$. So

$$\mathbb{Z}[u] \subset H^0(\mathscr{R}, \mathcal{O}_{\mathscr{R}}) \subset \mathbb{Q}[u].$$

The homomorphism $H^0(\mathscr{R}, \mathcal{O}_{\mathscr{R}}) \to H^0(\mathscr{R}, \mathcal{O}_{\mathscr{R}}) \otimes H^0(\mathscr{R}, \mathcal{O}_{\mathscr{R}})$ corresponding to addition (resp. multiplication) in $\mathscr{R}$ takes u to $u \otimes 1 + 1 \otimes u$ (resp. to $u \otimes u$). To finish the explicit description of $\mathscr{R}$, it remains to describe the subring $H^0(\mathscr{R}, \mathcal{O}_{\mathscr{R}}) \subset \mathbb{Q}[u]$.

Proposition C.1.2. $H^0(\mathscr{R}, \mathcal{O}_{\mathscr{R}}) = \mathrm{Int}$, *where* $\mathrm{Int} \subset \mathbb{Q}[u]$ *is the subring generated by the polynomials* $\binom{u}{n}$, $n \geq 0$.

Proof. $H^0(\mathscr{R}, \mathcal{O}_{\mathscr{R}})$ is the smallest subring $A \subset \mathbb{Q}[u]$ such that the action of $\operatorname{Spec} \mathbb{Q}[u] = \mathscr{R} \otimes \mathbb{Q}$ on $\hat{\mathbb{G}}_m$ extends to an action of Spec A on $\hat{\mathbb{G}}_m$. So $H^0(\mathscr{R}, \mathcal{O}_{\mathscr{R}})$ is generated by the coefficients of the formal series (C.4). ☐

C.1.3. On the ring Int It is well known that

$$\mathrm{Int} = \{f \in \mathbb{Q}[u] \mid f(m) \in \mathbb{Z} \text{ for all } m \in \mathbb{Z}\};$$

for this reason, Int is known as the *ring of integer-valued polynomials*. It is also well known that

(i) the polynomials $\binom{u}{n}$ form a *basis* of the $\mathbb{Z}$-module Int;

(ii) one has

$$\mathrm{Int} = \{f \in \mathrm{Fun}(\mathbb{Z},\mathbb{Z}) \mid \Delta^m(f) = 0 \text{ for some } m\}, \tag{C.5}$$

where $\mathrm{Fun}(\mathbb{Z},\mathbb{Z})$ is the ring of all functions $\mathbb{Z} \to \mathbb{Z}$ and $\Delta : \mathrm{Fun}(\mathbb{Z},\mathbb{Z}) \to \mathrm{Fun}(\mathbb{Z},\mathbb{Z})$ is the *difference operator* $\Delta : \mathrm{Fun}(\mathbb{Z},\mathbb{Z}) \to \mathrm{Fun}(\mathbb{Z},\mathbb{Z})$ defined by

$$(\Delta f)(u) = f(u+1) - f(u).$$

More details about the ring Int and some references can be found in [CC, Ch, Ek, El].

C.1.4. Remark Here is an interpretation of (C.5) via Cartier duality between $\mathscr{R}$ and $\hat{\mathbb{G}}_m$.

The Cartier dual of the embedding $\hat{\mathbb{G}}_m \hookrightarrow \mathbb{G}_m$ is a morphism $\mathbb{Z} \times \operatorname{Spec}\mathbb{Z} \to \mathscr{R}$, and the embedding

$$H^0(\mathscr{R}, \mathcal{O}_{\mathscr{R}}) = \mathrm{Int} \hookrightarrow \mathrm{Fun}(\mathbb{Z},\mathbb{Z}) \tag{C.6}$$

is the corresponding homomorphism of coordinate rings. As a $\mathbb{Z}$-module, $H^0(\mathscr{R}, \mathcal{O}_{\mathscr{R}})$ is the topological dual $(\mathbb{Z}[[y-1]])^*$, and the map (C.6) is just the natural map

$$\varphi : (\mathbb{Z}[[y-1]])^* \to (\mathbb{Z}[y, y^{-1}])^*.$$

So (C.5) means that φ is injective, and $\operatorname{Im}\varphi$ consists of those linear functionals on $\mathbb{Z}[y, y^{-1}]$ that are trivial on $(y-1)^m\mathbb{Z}[y, y^{-1}]$ for some m. This is, of course, true because $\mathbb{Z}[[y-1]]$ is the $(y-1)$-adic completion of $\mathbb{Z}[y, y^{-1}]$.

C.2. The reduction of the scheme $\mathscr{R}$ modulo p^n and the λ-ring structure on Int

C.2.1. The reduction of $\mathscr{R}$ modulo p^n Let p be a prime. If A is a ring in which p is nilpotent then $\hat{\mathbb{G}}_m \otimes A$ is the inductive limit of $\mu_{p^n} \otimes A$. The Cartier dual of μ_{p^n} is $\mathbb{Z}/p^n\mathbb{Z}$. So

$$\mathscr{R} \otimes A = (\mathbb{Z}_p)_A := \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})_A, \tag{C.7}$$

where $(\mathbb{Z}/p^n\mathbb{Z})_A$ is the constant ring scheme over Spec A with fiber $\mathbb{Z}/p^n\mathbb{Z}$.

C.2.2. Mahler's theorem Let $A = \mathbb{Z}/p^n\mathbb{Z}$. Combining (C.7) with the equality $H^0(\mathscr{R}, \mathcal{O}_{\mathscr{R}}) = \mathrm{Int}$, we get an isomorphism

$$\mathrm{Int}/p^n\,\mathrm{Int} \xrightarrow{\sim} \{\text{Locally constant functions } \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}\}; \tag{C.8}$$

the map (C.8) is as follows: given a function $f \in \mathrm{Int} \subset \mathrm{Fun}(\mathbb{Z}, \mathbb{Z})$, we reduce it modulo p^n and then extend from $\mathbb{Z}$ to $\mathbb{Z}_p$ by continuity. The isomorphism (C.8) is due to K. Mahler [Ma]. It is discussed, e.g., in [La, Ch. 4].

Lemma C.2.3. *For every prime p, the Frobenius endomorphism of* $\mathrm{Int}/p\,\mathrm{Int}$ *equals the identity.*

This well known fact follows from (C.8) or from (C.5).

C.2.4. Wilkerson's theorem on λ-rings Any λ-ring R is equipped with an action of the multiplicative monoid $\mathbb{N}$; the endomorphism of R corresponding to $n \in \mathbb{N}$ is denoted by ψ^n and called the n-th *Adams operation*. So we get a functor from the category of λ-rings to that of rings equipped with $\mathbb{N}$-action. C. Wilkerson [W] proved that this functor identifies the category of torsion-free λ-rings with the category of torsion-free rings equipped with an action of $\mathbb{N}$ satisfying the following condition: $\psi^p(x)$ is congruent to x^p modulo p for every prime p and every $x \in R$.

C.2.5. The λ-ring structure on Int By §C.2.4, a torsion free ring R such that for every prime p the Frobenius endomorphism of R/pR equals the identity is the same as a torsion-free λ-ring such that $\psi^n = \mathrm{id}$ for all n. It is known (see [W, El]) that for such R one has

$$\lambda_n(x) = \frac{x(x-1)\dots(x-n+1)}{n!} \quad \text{for all } n \in \mathbb{N}, x \in R. \tag{C.9}$$

By Lemma C.2.3, this applies to the ring Int. On the other hand, in the case $R = \mathrm{Int}$ the λ-ring structure comes from the embedding $\mathrm{Int} \hookrightarrow \mathrm{Fun}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z} \times \dots$ and the λ-ring structure on $\mathbb{Z}$, so (C.9) is clear.

C.2.6. Generators of Int $\otimes\mathbb{Z}_{(p)}$ Fix a prime p. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at p and $\mathrm{Int}_{(p)} := \mathrm{Int} \otimes \mathbb{Z}_{(p)} \subset \mathbb{Q}[u]$. For $x \in \mathrm{Ind}_{(p)}$ set

$$\delta(x) := (x - x^p)/p;$$

then $\delta(x)\in \mathrm{Int}_{(p)}$ by Lemma C.2.3. The pair $(\mathrm{Int}_{(p)},\delta:\mathrm{Int}_{(p)}\to\mathrm{Int}_{(p)})$ is a δ-ring in the sense of [J85] and [BS]. The following lemma is well known (e.g., see [El, §3]).

Lemma C.2.7. *(i) The elements $\delta^n(u)$, $n\in\mathbb{Z}_+$, generate* $\mathrm{Int}_{(p)}$ *as an $\mathbb{Z}_{(p)}$-algebra.*

(ii) Elements of the form

$$\prod_i (\delta^i(u))^{d_i}, \quad \textit{where } 0\le d_i<p \textit{ for all } i \textit{ and } d_i=0 \textit{ for } i\gg 0$$

form a basis of the $\mathbb{Z}_{(p)}$-module $\mathrm{Int}_{(p)}$.

Proof. It suffices to proof (ii). Let $n\ge 0$ be an integer. Write $n=\sum_i d_ip^i$, where $0\le d_i<p$ for all i and $d_i=0$ for $i\gg 0$. There exists $c\in\mathbb{Q}$ such that the polynomial

$$\binom{u}{n}-c\prod_i(\delta^i(u))^{d_i}$$

has degree $<n$. It remains to check that $c\in\mathbb{Z}_{(p)}$. To do this, use that $n!\in p^m\cdot\mathbb{Z}_{(p)}^{\times}$, where $m=\sum_i d_i(p^{i-1}+\cdots+p+1)$. □

C.3. The ring scheme $\mathscr{R}$ via Witt vectors

C.3.1. The ring scheme W_{big} Let W_{big} be the ring scheme of "big" Witt vectors. Recall that for any ring A, the additive group of $W_{\mathrm{big}}(A)$ is the subgroup of $A[[z]]^{\times}$ that consists of all power series with constant them 1. For each $n\in\mathbb{Z}$ one has the *Witt vector Frobenius* map $F_n:W_{\mathrm{big}}\to W_{\mathrm{big}}$, which is a ring scheme endomorphism; one has $F_mF_n=F_{mn}$ and $F_1=\mathrm{id}$. Recall that the unit of $W_{\mathrm{big}}(A)$ corresponds to $1-z\in A[[z]]^{\times}$.

C.3.2. The map $\mathscr{R}\to W_{\mathrm{big}}$ By definition, an A-point of $\mathscr{R}$ is an element $f\in A[[y-1]]$ satisfying the functional equation

$$f(y_1y_2)=f(y_1)f(y_2). \tag{C.10}$$

Associating to such f the formal power series $f(1-z)\in A[[z]]^{\times}$, we get a group homomorphism $\mathscr{R}(A)\to W_{\mathrm{big}}(A)$ functorial in A, i.e., a homomorphism of group schemes

$$\mathscr{R}\to W_{\mathrm{big}}. \tag{C.11}$$

This morphism is a closed immersion because (C.10) is a closed condition. Note that the map (C.11) takes $1 \in \mathscr{R}(\mathbb{Z})$ to $1 \in W_{\mathrm{big}}(\mathbb{Z})$ (see the end of §C.3.1).

C.3.3. Remark Here is a slightly different way of thinking about (C.11). Consider the unique homomorphism of unital rings $f : \mathbb{Z} \to W_{\mathrm{big}}(\mathbb{Z})$. Then each component of the Witt vector $f(n)$ is an (integer-valued) polynomial in n, so we get an element of $W_{\mathrm{big}}(\mathrm{Int})$, i.e., a morphism $\mathrm{Spec}\,\mathrm{Int} \to W_{\mathrm{big}}$. This is (C.11).

Proposition C.3.4. *(i) The map* (C.11) *is a homomorphism of ring schemes.*

(ii) It induces an isomorphism $\mathscr{R} \xrightarrow{\sim} W_{\mathrm{big}}^F$, *where*

$$W_{\mathrm{big}}^F := \{w \in W_{\mathrm{big}} \mid F_n(w) = w \text{ for all } n \in \mathbb{N}\}. \tag{C.12}$$

Proof. We know that $\mathscr{R}$ is flat over $\mathrm{Spec}\,\mathbb{Z}$ and the morphism $\mathscr{R} \to W_{\mathrm{big}}$ is a closed immersion. It is straightforward to check (i) and (ii) after base change to $\mathrm{Spec}\,\mathbb{Q}$. It remains to show that W_{big}^F is flat over $\mathrm{Spec}\,\mathbb{Z}$. This follows from Lemmas C.3.5–C.3.6 below. ☐

Lemma C.3.5. *Let p be a prime and W the ring scheme of p-typical Witt vectors. Let $F : W \to W$ be the Witt vector Frobenius and*

$$W^F := \{w \in W \mid F(w) = w\}. \tag{C.13}$$

Then the natural ring scheme morphism $W_{\mathrm{big}} \to W$ induces an isomorphism

$$W_{\mathrm{big}}^F \otimes \mathbb{Z}_{(p)} \xrightarrow{\sim} W^F \otimes \mathbb{Z}_{(p)} \tag{C.14}$$

Proof. The proof is based on the identification of $W_{\mathrm{big}} \otimes \mathbb{Z}_{(p)}$ with the product of infinitely many copies of $W \otimes \mathbb{Z}_{(p)}$ (the copies are labeled by positive integers coprime to p) and the usual description of the morphisms $F_n : W_{\mathrm{big}} \otimes \mathbb{Z}_{(p)} \to W_{\mathrm{big}} \otimes \mathbb{Z}_{(p)}$ in terms of this identification. ☐

Lemma C.3.6. *The scheme W^F defined by* (C.13) *is flat over $\mathbb{Z}$.*

Before proving the lemma, let us briefly recall the approach to W developed by Joyal [J85] (a detailed exposition of this approach can be found in [B16] and [BG, §1]).

C.3.7. Joyal's approach to W Let C be the coordinate ring of W. Let $\phi : C \to C$ be the homomorphism corresponding to $F : W \to W$. The map $W \otimes \mathbb{F}_p \to W \otimes \mathbb{F}_p$ induced by F is the usual Frobenius, so there is a map $\delta : C \to C$ such that $\phi(c) = c^p + p\delta(c)$ for all $c \in C$ (of course, the map δ is neither additive nor multiplicative).

The pair (C, δ) is a δ-ring in the sense of [J85] and [BS, §2]. The main theorem of [J85] says that C is the *free δ-ring* on x_0, where $x_0 \in C$ corresponds to the canonical homomorphism $W \to W/VW = \mathbb{G}_a$. This means that as a ring, C is freely generated by the elements $x_n := \delta^n(x_0)$, $n \geq 0$. We have

$$\phi(x_n) = x_n^p + px_{n+1} \tag{C.15}$$

The elements x_n (which are regular functions on W) are called *Buium-Joyal coordinates* or *Buium-Joyal components* (this terminology is introduced in [BG]). For $n > 1$ they are different from Witt components (i.e., the usual ones).

C.3.8. Proof of Lemma C.3.6 Let C be as in §C.3.7. Formula (C.15) implies that the coordinate ring of W^F is the quotient of C by the ideal I generated by the elements

$$x_n^p + px_{n+1} - x_n, \quad n \in \mathbb{Z}_+. \tag{C.16}$$

This quotient is a free $\mathbb{Z}$-module whose basis is formed by elements $\prod_i x_i^{d_i}$, where $0 \leq d_i < p$ for all i and $d_i = 0$ for $i \gg 0$. Indeed, these elements clearly generate C/I, and they are linearly independent in $(C/I) \otimes \mathbb{Q} = \mathbb{Q}[x_0]$. □

Appendix D. The rescaled $\hat{\mathbb{G}}_m$ and its Cartier dual

As noted by the reviewer, a substantial part of this Appendix and the previous one is contained in [MRT].

D.1. Rescaling $\hat{\mathbb{G}}_m$

D.1.1. The formal group $H^!$ Let $H^!$ be the formal group scheme over $\mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[h]$ defined by the formal group law

$$z_1 * z_2 = z_1 + z_2 + hz_1z_2.$$

Note that $1 + h \cdot (z_1 * z_2) = (1 + hz_1)(1 + hz_2)$. So we have a homomorphism of formal groups over $\mathbb{A}^1$

$$H^! \to \hat{\mathbb{G}}_m \times \mathbb{A}^1, \quad z \mapsto 1 + hz, \tag{D.1}$$

which induces an isomorphism over the locus $h \neq 0$.

After specializing h to 1 and 0, the formal group $H^!$ becomes $\hat{\mathbb{G}}_m$ and $\hat{\mathbb{G}}_a$, respectively. If you wish, $H^!$ is a deformation of $\hat{\mathbb{G}}_m$ to $\hat{\mathbb{G}}_a$.

D.1.2. Remarks (i) The action of $\mathbb{G}_m$ on $\mathbb{A}^1$ by multiplication lifts to an action of $\mathbb{G}_m$ on $H^!$: namely, $\lambda \in \mathbb{G}_m$ takes (h, z) to $(\lambda h, \lambda^{-1} z)$. So $H^!$ descends from $\mathbb{A}^1$ to the quotient stack $\mathbb{A}^1/\mathbb{G}_m$.

(ii) $H^!$ is obtained from $\hat{\mathbb{G}}_m$ by rescaling depending on a parameter h. This is a particular case of the construction of §3.5.

D.1.3. Plan In §D.2 (which is parallel to §C.1) we give a description of the Cartier dual $G^!$ of $H^!$. In §D.4–D.5 we describe $G^!$ in terms of Witt vectors in two different ways; the description from §D.4 is quite parallel to §C.3. In §D.3 we discuss a certain λ-ring structure on the coordinate ring of $G^!$.

D.2. The first description of the Cartier dual of $H^!$

D.2.1. The group scheme $G^!$ Let $G^!$ be the Cartier dual of $H^!$; this is a flat affine scheme over $\mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[h]$. The group scheme $\mathscr{R}$ from §C.1 can be obtained from $G^!$ by specializing h to 1. In this subsection we describe $G^!$ in the spirit of §C.1. Later we will give two different descriptions of $G^!$ in terms of Witt vectors (see Propositions D.4.10 and D.5.5).

For any $\mathbb{Z}[h]$-algebra A, an A-point of $G^!$ is a formal series $f \in 1+zA[[z]] \subset (A[[z]])^\times$ such that $f(z_1 * z_2) = f(z_1)f(z_2)$. Associating $f'(0)$ to such f, we get a homomorphism

$$G^! \to \mathbb{G}_a \times \mathbb{A}^1 \tag{D.2}$$

of group schemes over $\mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[h]$. The coordinate ring of $\mathbb{G}_a$ equals $\mathbb{Z}[t]$, so (D.2) induces a homomorphism of $\mathbb{Z}[h]$-algebras

$$\mathbb{Z}[h, t] \to H^0(G^!, \mathcal{O}_{G^!}). \tag{D.3}$$

By Lie theory, the homomorphism (D.2) becomes an isomorphism after base change to $\mathbb{A}^1_{\mathbb{Q}} = \operatorname{Spec} \mathbb{Q}[h]$. So we have a pairing $\mathbb{G}_a \times H^!_{\mathbb{A}^1_{\mathbb{Q}}} \to \mathbb{G}_m \times \mathbb{A}^1_{\mathbb{Q}}$, where $H^!_{\mathbb{A}^1_{\mathbb{Q}}} := H^! \times_{\mathbb{A}^1} \mathbb{A}^1_{\mathbb{Q}}$. The corresponding map $\mathbb{G}_a \times H^!_{\mathbb{A}^1_{\mathbb{Q}}} \to \mathbb{G}_m$ is given by the formal series

$$(1+hz)^{t/h} = \sum_{n=0}^{\infty} \frac{t(t-h)\dots(t-h(n-1))}{n!} \cdot z^n \in (\mathbb{Q}[h,t][[z]])^\times, \tag{D.4}$$

where t is the coordinate on $\mathbb{G}_a$ and z is the coordinate on $H^!$. Note that after substituting $h = 0$ the formal series (D.4) becomes equal to $\exp(tz)$.

The homomorphism (D.3) becomes an isomorphism after tensoring by $\mathbb{Q}$. Since $G^!$ is flat over $\mathbb{Z}[h]$, we see that $\mathbb{Z}[h,t] \subset H^0(G^!, \mathcal{O}_{G^!}) \subset \mathbb{Q}[h,t]$.

Proposition D.2.2. *(i) $H^0(G^!, \mathcal{O}_{G^!}) = B_0$, where $B_0 \subset \mathbb{Q}[h,t]$ is the subring generated over $\mathbb{Z}[h]$ by the polynomials*

$$\frac{t(t-h)\dots(t-h(n-1))}{n!}, \quad n \geq 0. \tag{D.5}$$

(ii) The polynomials (D.5) *form a basis of the $\mathbb{Z}[h]$-module B_0.*

(iii) The Hopf algebra structure on B_0 corresponding to the group structure on $G^!$ is given by $t \mapsto t \otimes 1 + 1 \otimes t$.

Proof. The proof of (i) is parallel to that of Proposition C.1.2.

Let us prove (ii). The product of two polynomials of the form (D.5) can be represented as an $\mathbb{Z}[h]$-linear combination of such polynomials using the formula

$$(1+hz_1)^{t/h}(1+hz_2)^{t/h} = (1+h(z_1 * z_2))^{t/h}, \quad \text{where } z_1 * z_2 = z_1 + z_2 + hz_1z_2.$$

So polynomials of the form (D.5) generate B_0 as a $\mathbb{Z}[h]$-module. They are linearly independent over $\mathbb{Z}[h]$ because the polynomial (D.5) has degree n with respect to u.

Finally, (iii) is clear because t is the pullback via (D.2) of the natural coordinate on $\mathbb{G}_a$. ☐

The following simple lemma is used in the proof of Proposition 5.5.2(c′).

Lemma D.2.3. *Let $m \in \mathbb{N}$. Then*

(i) the homomorphism $g_m : B_0 \to B_0$ induced by the morphism $G^! \xrightarrow{m} G^!$ takes t to mt;

(ii) $\frac{mt(mt-h)\dots(mt-(n-1)h)}{n!} \in B_0$ for all n;

(iii) in $B_0[[z]]$ one has the equality

$$\sum_{n=0}^{\infty} \frac{mt(mt-h)\dots(mt-(n-1)h)}{n!} \cdot z^n = (1+hvz)^{\frac{t}{h}},$$

where $v := \frac{(1+hz)^m-1}{hz} \in \mathbb{Z}[h,z]$ *and* $(1+hvz)^{\frac{t}{h}} := \sum_{n=0}^{\infty} \frac{t(t-h)\dots(t-(n-1)h)}{n!} \cdot v^n z^n$.

Proof. Statement (i) follows from Proposition D.2.2(iii). The expression from (ii) is just $g_m(t)$, where g_m is as in (i); so (ii) is clear. In statement (iii) one can replace B_0 by $B_0 \otimes \mathbb{Q} = \mathbb{Q}[h,t]$, so (iii) is classical. □

D.2.4. The homomorphism $\mathscr{R} \times \mathbb{A}^1 \to G^!$ Recall that $\mathscr{R}$ is the Cartier dual of $\hat{\mathbb{G}}_m$. So the Cartier dual of (D.1) is a homomorphism

$$\mathscr{R} \times \mathbb{A}^1 \to G^! \tag{D.6}$$

of group schemes over $\mathbb{A}^1 = \operatorname{Spec}\mathbb{Z}[h]$, which induces an isomorphism over the locus $h \neq 0$.

D.2.5. Relation between B_0 and Int The map (D.6) induces a homomorphism of $\mathbb{Z}[h]$-algebras

$$B_0 \to \operatorname{Int}[h], \tag{D.7}$$

which becomes an isomorphism after base change to $\mathbb{Z}[h,h^{-1}]$. The homorphism (D.7) takes t to hu, so the polynomial (D.5) goes to $h^n\binom{u}{n}$.

Equip $\mathbb{Q}[h,t]$ with the grading such that $\deg h = \deg t = 1$, then B_0 is a graded subring of $\mathbb{Q}[h,t]$. Equip $\operatorname{Int}[h]$ with the grading such that $\deg h = 1$ and all elements of Int have degree 0. Then the homomorphism (D.7) is graded.

The subring $\operatorname{Int} \subset \mathbb{Q}[u]$ from Proposition C.1.2 is filtered by degree of polynomials. Let $\operatorname{Int}_{\leq n}$ be the n-th term of this filtration. It is easy to see that (D.7) induces an isomorphism

$$B_0 \xrightarrow{\sim} \bigoplus_n h^n \operatorname{Int}_{\leq n}. \tag{D.8}$$

Thus the graded $\mathbb{Z}[h]$-algebra B_0 is obtained from the filtered ring Int by a very familiar procedure.

D.2.6. Remarks (i) $B_0/hB_0 = \operatorname{gr}\operatorname{Int}$ is the ring of divided powers polynomials in u.

(ii) One can rewrite (D.8) as an isomorphism

$$B_0 \xrightarrow{\sim} \operatorname{Int}[h] \cap \mathbb{Q}[h,hu] \subset \mathbb{Q}[h,u]; \tag{D.9}$$

under this isomorphism $t \in B_0$ corresponds to $hu \in \mathbb{Q}[h,u]$. Note that (D.9) induces an isomorphism $B_0 \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}[h,hu] \subset \mathbb{Q}[h,u]$.

D.3. λ-ring structure on B_0

D.3.1. Notation Let $q := h + 1 \in \mathbb{Z}[h]$; then $\mathbb{Z}[h] = \mathbb{Z}[q]$.

D.3.2. A λ-ring structure on $\mathbb{Z}[h]$ By §C.2.4, there is a unique λ-ring structure on $\mathbb{Z}[h] = \mathbb{Z}[q]$ such that $\psi^n(q) = q^n$ for all $n \in \mathbb{N}$.

Another way to get this λ-ring structure is to chose a field k and to identify $\mathbb{Z}[q, q^{-1}]$ (resp. $\mathbb{Z}[q]$) with the Grothendieck ring of the category of finite-dimensional representations of $(\mathbb{G}_m)_k$ (resp. of the multiplicative monoid over k) so that q identifies with the class of the tautological 1-dimensional representation.

Let us note that the λ-ring $\mathbb{Z}[q]$ is studied in Pridham's article [Pri].

In the next lemma we define a λ-ring structure on B_0; the definition will be motivated by Lemma D.3.5(ii).

Lemma D.3.3. *Consider B_0 as a graded ring (see §D.2.5). For $n \in \mathbb{N}$ let ψ^n be the endomorphism of B_0 whose restriction to the m-th graded piece of B_0 is multiplication by $(\frac{q^n-1}{q-1})^m$. Then*

(i) the endomorphisms ψ^n define a λ-ring structure on B_0;

(ii) in B_0 one has $\psi^n(q) = q^n$, so the map $\mathbb{Z}[q] = \mathbb{Z}[h] \hookrightarrow B_0$ is a homomorphism of λ-rings;

(iii) the diagram

$$\begin{array}{ccc} B_0 & \xrightarrow{\Delta} & B_0 \otimes_{\mathbb{Z}[q]} B_0 \\ {\scriptstyle \psi^n}\downarrow & & \downarrow{\scriptstyle \psi^n \otimes \psi^n} \\ B_0 & \xrightarrow{\Delta} & B_0 \otimes_{\mathbb{Z}[q]} B_0 \end{array}$$

commutes, where Δ is the coproduct.

Proof. By the definition of $\psi^n : B_0 \to B_0$, in the ring B_0 we have $\psi^n(q-1) = q^n - 1$ (because $q - 1 = h$ is in the degree 1 graded piece) and therefore $\psi^n(q) = q^n$.

Let us prove (i). It is easy to check that $\psi^n \circ \psi^{n'} = \psi^{nn'}$. So by §C.2.4, it remains to check that for every prime p the endomorphism of B_0/pB_0 induced by ψ^p equals the Frobenius. This follows from (D.8), the fact that $\psi^p(h)$ is congruent to h^p modulo p, and Lemma C.2.3, which says that the Frobenius endomorphism of $\operatorname{Int}/p\operatorname{Int}$ equals the identity.

To prove (iii), recall that $B_0 \otimes \mathbb{Q} = \mathbb{Q}[h, t]$, $\Delta(t) = t \otimes 1 + 1 \otimes t$ (see Proposition D.2.2(iii)) and $\psi^n(t) = \frac{q^n-1}{q-1} \cdot t$. ∎

D.3.4. The morphisms $\boldsymbol{\Psi_n : G^! \to G^!}$ The endomorphisms $\psi^n \in \operatorname{End} \mathbb{Z}[h]$ and $\psi^n \in \operatorname{End} B_0$ induce maps $\Psi_n : \mathbb{A}^1 \to \mathbb{A}^1$ and $\Psi_n : G^! \to G^!$. By Lemma D.3.3(ii), the diagram

$$\begin{array}{ccc} G^! & \xrightarrow{\Psi_n} & G^! \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \xrightarrow{\Psi_n} & \mathbb{A}^1 \end{array}$$

commutes, so we get a morphism

$$G^! \to \Psi_n^* G^! \tag{D.10}$$

of schemes over $\mathbb{A}^1$, where $\Psi_n^* G^!$ is the pullback of $G^!$ via $\Psi_n : \mathbb{A}^1 \to \mathbb{A}^1$. Moreover, (D.10) is a group homomorphism by Lemma D.3.3(iii).

Lemma D.3.5. *(i) Let $n \in \mathbb{N}$. Let $\Psi_n^* H^!$ be the pullback of $H^!$ via $\Psi_n : \mathbb{A}^1 \to \mathbb{A}^1$. Then there is a unique group homomorphism*

$$\Psi_n^* H^! \to H^! \tag{D.11}$$

which makes the following diagram commute:

$$\begin{array}{ccc} \Psi_n^* H^! & \longrightarrow & H^! \\ & \searrow & \downarrow \\ & & \hat{\mathbb{G}}_m \times \mathbb{A}^1 \end{array} \tag{D.12}$$

Here the vertical arrow is the map (D.1) and the diagonal one is its pullback via $\Psi_n : \mathbb{A}^1 \to \mathbb{A}^1$.

(ii) The homomorphisms (D.10) and (D.11) are Cartier dual to each other.

Proof. (i) $H^!$ is the formal group over $\mathbb{A}^1$ given by the group law

$$z_1 * z_2 = z_1 + z_2 + (q-1) z_1 z_2.$$

So $\Psi_n^* H^!$ is given by the group law $y_1 * y_2 = y_1 + y_2 + (q^n - 1) y_1 y_2$. The homomorphism (D.11) is given by $z = \frac{q^n - 1}{q - 1} \cdot y$.

(ii) The Cartier dual of the vertical arrow of (D.12) is the homomorphism $f : \mathscr{R} \times \mathbb{A}^1 \to G^!$ from (D.6). So it suffices to check commutativity of the diagram

$$\begin{array}{ccc} \Psi_n^* G^! & \longleftarrow & G^! \\ & \nwarrow^{\Psi_n^*(f)} & \uparrow f \\ & & \mathscr{R} \times \mathbb{A}^1 \end{array}$$

whose horizontal arrow is (D.10). This is equivalent to commutativity of the diagram

(D.13)
$$\begin{CD} \mathscr{R} \times \mathbb{A}^1 @>f>> G^! \\ @V{\mathrm{id}_{\mathscr{R}} \times \Psi_n}VV @VV{\Psi_n}V \\ \mathscr{R} \times \mathbb{A}^1 @>f>> G^! \end{CD}$$

and then (after passing to coordinate rings) to commutativity of the diagram

$$\begin{CD} \mathrm{Int} \otimes \mathbb{Z}[h] @<<< B_0 \\ @V{\mathrm{id} \otimes \psi^n}VV @VV{\psi^n}V \\ \mathrm{Int} \otimes \mathbb{Z}[h] @<<< B_0 \end{CD}$$

in which each horizontal arrow is the homomorphism (D.7). The commutativity of the latter diagram is clear from the definition of $\psi^n : B_0 \to B_0$ from Lemma D.3.3. □

D.3.6. The δ-ring $B_0 \otimes \mathbb{Z}_{(p)}$ Fix a prime p. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at p. Let $\phi \in \mathrm{End}(B_0 \otimes \mathbb{Z}_{(p)})$ be induced by $\psi^p \in \mathrm{End}\, B_0$. For every $b \in B_0 \otimes \mathbb{Z}_{(p)}$, the element $\delta(b) := \frac{\phi(b) - b^p}{p}$ belongs to $B_0 \otimes \mathbb{Z}_{(p)}$ by Lemma D.3.3(i). The map $\delta : B_0 \otimes \mathbb{Z}_{(p)} \to B_0 \otimes \mathbb{Z}_{(p)}$ makes $B_0 \otimes \mathbb{Z}_{(p)}$ into a δ-ring in the sense of [J85] and [BS]. The subring $\mathbb{Z}_{(p)}[q] \subset B_0 \otimes \mathbb{Z}_{(p)}$ is a δ-subring.

By the definition of ψ^p (see Lemma D.3.3), the element $t \in B_0 \otimes \mathbb{Z}_{(p)}$ satisfies the relation $\psi^p(t) := \frac{q^p - 1}{q - 1} \cdot t$ or equivalently,

(D.14)
$$t^p + p\delta(t) = \Phi_p(q) \cdot t.$$

On the other hand, let C be the δ-algebra over $\mathbb{Z}_{(p)}[q]$ with a single generator (denoted by t) and the *defining* relation (D.14). We claim that *the canonical homomorphism $C \to B_0 \otimes \mathbb{Z}_{(p)}$ is an isomorphism.* Indeed, elements of the form

$$\prod_i (\delta^i(t))^{d_i}, \quad \text{where } 0 \le d_i < p \text{ for all } i \text{ and } d_i = 0 \text{ for } i \gg 0$$

generate[17] the $\mathbb{Z}_{(p)}[q]$-module C and form a basis of the $\mathbb{Z}_{(p)}[q]$-module $B_0 \otimes \mathbb{Z}_{(p)}$ (the latter is similar to Lemma C.2.7).

D.3.7. Some generalizations The generalizations discussed here are not used in the rest of the article.

(i) In §D.3.2 we set $\psi^n(h) := (1+h)^n - 1$. This choice of ψ^n is motivated by our interest in the q-de Rham prism. On the other hand, one could set $\psi^n(h) := h^n$ and define $\psi^n : B_0 \to B_0$ by setting $\psi^n(b) = h^{m(n-1)}$ for b in the m-th graded piece of B_0. Then we would still get a λ-ring structure on $\mathbb{Z}[h]$ and B_0; moreover, Lemmas D.3.3 and D.3.5 would remain valid.

(ii) In §D.3.6 we considered the δ-ring structure on $B_0 \otimes \mathbb{Z}_{(p)}$ corresponding to the endomorphism of $B_0 \otimes \mathbb{Z}_{(p)}$ that acts on the m-th graded piece as multiplication by $((1+h)^p - 1)/h)^m$. If we replace $((1+h)^p - 1)/h$ by any polynomial $f \in \mathbb{Z}_{(p)}[h]$ congruent to h^{p-1} modulo p we would still get a δ-ring structure on $B_0 \otimes \mathbb{Z}_{(p)}$ such that the elements $\delta^i(t)$, $i \geq 0$, generate $B_0 \otimes \mathbb{Z}_{(p)}$ over $\mathbb{Z}_{(p)}[h]$.

D.4. The group scheme $G^!$ in terms of Witt vectors. I

D.4.1. λ-schemes By a λ-scheme we mean a scheme X equipped with a collection of endomorphisms $\Psi_n : X \to X$, $n \in \mathbb{N}$, such that $\Psi_m \circ \Psi_n = \Psi_{mn}$, $\Psi_1 = \mathrm{id}$, and for every prime p the morphism $\Psi_p : X \otimes \mathbb{F}_p \to X \otimes \mathbb{F}_p$ equals $\mathrm{Fr}_{X \otimes \mathbb{F}_p}$. (This definition is good enough for us because we will be dealing with schemes flat over $\mathbb{Z}$.) Similarly to §2.2.3 we have the notion of group λ-scheme over a λ-scheme.

D.4.2. Plan of §D.4–D.5 $G^!$ is a group λ-scheme over the λ-scheme $\mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[q]$ (see §D.3.2–D.3.4). We will describe two realizations of this group λ-scheme in terms of Witt vectors, denoted by $G^{!?}$ and $G^{!!}$. The definitions of $G^{!?}$ and $G^{!!}$ are given in §D.4.3 and §D.5.2, respectively. According to Propositions D.4.10 and D.5.5, the group λ-schemes $G^!$, $G^{!?}$, and $G^{!!}$ are canonically isomorphic.

Probably $G^{!!}$ is better than $G^{!?}$ (this opinion is influenced, in part, by my correspondence with Lance Gurney). However, let us start with $G^{!?}$, which is obtained by rescaling §C.3 in a straightforward way.

[17] To see this, note that $\delta^(t)^p = \phi(\delta^i(t)) - p\delta^{i+1}(t) = \delta^i(\Phi_p(q) \cdot t) - p\delta^{i+1}(t)$.

D.4.3. Definition of $G^{!?}$ Let W_{big} be the ring scheme of "big" Witt vectors (so $W_{\mathrm{big}} \times \mathbb{A}^1$ is a ring scheme over $\mathbb{A}^1$). Define $G^{!?} \subset W_{\mathrm{big}} \times \mathbb{A}^1$ to be the following subgroup:

$$G^{!?} := \{(w,q) \in W_{\mathrm{big}} \times \mathbb{A}^1 \mid F_m(w) = [q-1]^{m-1} w \text{ for all } m \in \mathbb{N}\}. \tag{D.15}$$

For $n \in \mathbb{N}$ define $\Psi_n : G^{!?} \to G^{!?}$ by

$$\Psi_n(w,q) = \left(\left[\frac{q^n - 1}{q-1} \right] \cdot w, q^n \right). \tag{D.16}$$

It is easy to check that for each prime p the morphism $\Psi_p : G^{!?} \otimes \mathbb{F}_p \to G^{!?} \otimes \mathbb{F}_p$ is equal to the Frobenius. So $G^{!?}$ is a group λ-scheme over $\mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[q]$.

D.4.4. Remark Let p be a prime and W the ring scheme of p-typical Witt vectors. Similarly to the proof of Lemma C.3.5, one shows that the canonical epimorphism $W_{\mathrm{big}} \twoheadrightarrow W$ induces an isomorphism

$$G^{!?} \otimes \mathbb{Z}_{(p)} \xrightarrow{\sim} \{(w,q) \in W \times \mathbb{A}^1_{\mathbb{Z}_{(p)}} \mid F(w) = [q-1]^{p-1} \cdot w\}. \tag{D.17}$$

Lemma D.4.5. *$G^{!?}$ is flat over $\mathbb{Z}[q]$.*

Proof. It suffices to show that the r.h.s of (D.17) is flat over $\mathbb{Z}_{(p)}[q]$. The argument is parallel to that of §C.3.8, but the role of the elements

$$x_n^p + p x_{n+1} - x_n$$

from §C.3.8 is played by $x_n^p + p x_{n+1} - (q-1)^{p^n(p-1)} x_n$. □

D.4.6. A homomorphism $G^! \to W_{\mathrm{big}} \times \mathbb{A}^1$ For any $\mathbb{Z}[q]$-algebra A, an A-point of $G^!$ is an element $f \in 1 + zA[[z]] \subset (A[[z]])^\times$ satisfying the functional equation

$$f(z_1) f(z_2) = f(z_1 + z_2 + (q-1) z_1 z_2). \tag{D.18}$$

Associating to such f the formal power series

$$f(-z) \in 1 + zA[[z]] = W_{\mathrm{big}}(A), \tag{D.19}$$

we get a group homomorphism $G^!(A) \to 1 + zA[[z]] = W_{\mathrm{big}}(A)$ functorial in A, i.e., a homomorphism of group schemes over $\mathbb{A}^1$

$$i : G^! \hookrightarrow W_{\mathrm{big}} \times \mathbb{A}^1. \tag{D.20}$$

The morphism (D.20) is a closed immersion because (D.18) is a closed condition.

D.4.7. Relation to the homomorphism $\mathscr{R}\to \boldsymbol{W}_{\mathbf{big}}$ It is easy to check that after the specialization $q=2$ (i.e., $q-1=1$) the homomorphism (D.20) becomes the homomorphism

$$\mathscr{R} \xrightarrow{\sim} W_{\mathrm{big}}^F \hookrightarrow W_{\mathrm{big}} \tag{D.21}$$

from §C.3.2 (the minus sign in (D.19) was introduced to ensure this). Moreover, one has the following

Lemma D.4.8. *(i) The following diagram commutes:*

$$\begin{array}{ccc} \mathscr{R}\times\mathbb{A}^1 & \longrightarrow & G^! \\ \big\downarrow & & \big\downarrow{\scriptstyle i} \\ W_{\mathrm{big}}\times\mathbb{A}^1 & \xrightarrow{[q-1]} & W_{\mathrm{big}}\times\mathbb{A}^1 \end{array} \tag{D.22}$$

Here the upper horizontal arrow is (D.6)*, the lower horizontal arrow is multiplication by the Teichmüller representative* $[q-1]\in W_{\mathrm{big}}(\mathbb{Z}[q])$*, the right vertical arrow is* (D.20)*, and the left vertical arrow comes from* (D.21)*.*

(ii) After base change to the open subset $\operatorname{Spec}\mathbb{Z}[q,(q-1)^{-1}]\subset\operatorname{Spec}\mathbb{Z}[q]=\mathbb{A}^1$*, the horizontal arrows of* (D.22) *become isomorphisms.*

Proof. Recall that for any $\mathbb{Z}[q]$-algebra A, multiplication by $[q-1]$ in $W_{\mathrm{big}}(A)$ takes a formal series $g(x)\in 1+xA[[x]]=W_{\mathrm{big}}(A)$ to $g((q-1)x)$. The rest is straightforward. □

D.4.9. Remarks (i) By Lemma C.2.3, $\mathscr{R}$ is a λ-scheme with $\Psi_n=\mathrm{id}$ for all n. Moreover, commutativity of (D.13) means that the upper horizontal arrow of (D.22) is a morphism of λ-schemes.

(ii) The lower horizontal arrow of (D.22) induces a morphism $W_{\mathrm{big}}^F\times\mathbb{A}^1\to G^{!?}$ of λ-schemes over $\mathbb{A}^1$, which becomes an isomorphism over the locus $q\neq 1$.

Proposition D.4.10. *The homomorphism* (D.20) *induces an isomorphism*

$$G^! \xrightarrow{\sim} G^{!?} \tag{D.23}$$

of λ-schemes over $\mathbb{A}^1$.

Proof. The schemes $G^!$ and $G^{!?}$ are flat over $\mathbb{Z}[q]$ (for $G^{!?}$ this is Lemma D.4.5). The morphism $i : G^! \to W_{\mathrm{big}} \times \mathbb{A}^!$ is a closed immersion. So it remains to show that i induces an isomorphism of λ-schemes $G^!_{q\neq 1} \xrightarrow{\sim} G^{!?}_{q\neq 1}$, where $G^!_{q\neq 1}$ and $G^{!?}_{q\neq 1}$ are the restrictions of $G^!$ and $G^{!?}$ to the locus $q \neq 1$. This follows from Lemma D.4.8 and §D.4.9. □

D.5. The group scheme $G^!$ in terms of Witt vectors. II

This subsection is a non-p-typical version of §5.7.4. Part (iii) of Lemma D.5.4 is somewhat surprising.

D.5.1. Recollections on $G^!$ $G^!$ is the group scheme over $\mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[q]$ such that for any $\mathbb{Z}[q]$-algebra A, $G^!(A)$ is the group of elements $f \in 1 + zA[[z]] \subset (A[[z]])^\times$ satisfying the functional equation

$$f(z_1)f(z_2) = f(z_1 + z_2 + (q-1)z_1 z_2). \tag{D.24}$$

Recall that $H^0(G^!, \mathcal{O}_{G^!}) = B_0$, where B_0 is as in Proposition D.2.2. The λ-scheme structure on $G^!$ was defined in §D.3.2–D.3.4.

D.5.2. Definition of $G^{!!}$ Define $G^{!!} \subset W_{\mathrm{big}} \times \mathbb{A}^1$ to be the following subgroup:

$$G^{!!} := \left\{ (w,q) \in W_{\mathrm{big}} \times \mathbb{A}^1 \mid F_m(w) = \frac{[q]^m - 1}{[q]-1} \cdot w \text{ for all } m \in \mathbb{N} \right\}, \tag{D.25}$$

where $\frac{[q]^m-1}{[q]-1} := 1 + [q] + \cdots + [q]^{m-1}$. Define $\Psi_n : G^{!!} \to G^{!!}$ by the following very simple formula:

$$\Psi_n(w,q) = (F_n(w), q^n). \tag{D.26}$$

Then $G^{!?}$ is a group λ-scheme over $\mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[q]$.

D.5.3. Remark Let p be a prime and W the ring scheme of p-typical Witt vectors. Similarly to the proof of Lemma C.3.5, one shows that the canonical epimorphism $W_{\mathrm{big}} \twoheadrightarrow W$ induces an isomorphism

$$G^{!!} \otimes \mathbb{Z}_{(p)} \xrightarrow{\sim} \{(w,q) \in W \times \mathbb{A}^1_{\mathbb{Z}_{(p)}} \mid F(w) = \Phi_p([q]) \cdot w\}. \tag{D.27}$$

Lemma D.5.4. *Equip W_{big} with the λ-scheme structure given by the Frobenius endomorphisms $F_n : W_{\mathrm{big}} \to W_{\mathrm{big}}$, $n \in \mathbb{N}$. Equip $G^!$ and $\mathbb{A}^1 = \mathrm{Spec}\,\mathbb{Z}[q]$ with the λ-structure from §D.3.2–D.3.4. Let $\pi : W_{\mathrm{big}} \twoheadrightarrow \mathbb{G}_a$ be the morphism that takes a Witt vector to its first component. Then*

(i) there exists a unique morphism

$$G^! \to W_{\mathrm{big}} \times \mathbb{A}^1 \tag{D.28}$$

of λ-schemes over $\mathbb{A}^1$ whose composition with the projection $W_{\mathrm{big}} \times \mathbb{A}^1 \to W_{\mathrm{big}} \xrightarrow{\pi} \mathbb{G}_a$ is given by the element $t \in B_0 = H^0(G^!, \mathcal{O}_{G^!})$ from Proposition D.2.2;

(ii) the map (D.28) is a group homomorphism;

(iii) the morphism (D.28) has the following explicit description: for any $\mathbb{Z}[q]$-algebra A, it takes a formal series $f \in 1 + zA[[z]]$ satisfying (D.24) to the formal series $f(\frac{z}{z-1})$ viewed as an element of $W_{\mathrm{big}}(A)$.

Proof. The coordinate ring of the λ-scheme W_{big} is known to be the free λ-ring on a single generator π. This implies (i). Statement (ii) follows from (iii).

Let us prove (iii). Our map $G^! \to W_{\mathrm{big}}$ is given by the unique element of $W_{\mathrm{big}}(B_0)$ whose n-th ghost component equals $\psi^n(t) = \frac{q^n-1}{q-1} \cdot t \in B_0$ (see the definition of ψ^n in Lemma D.3.3). Recall that the universal solution to (D.24) is given by

$$f(z) = (1 + (q-1)z)^{t/(q-1)}.$$

So it remains to check that for this f one has

$$-z\frac{d}{dz} \log f\Big(\frac{z}{z-1}\Big) = t \cdot \sum_{n=1}^{\infty} \frac{q^n - 1}{q-1} \cdot z^n.$$

This is straightforward; one uses that $1+(q-1)z/(z-1) = (1-qz)/(1-z)$. ☐

Proposition D.5.5. *The homomorphism (D.28) induces an isomorphism*

$$G^! \xrightarrow{\sim} G^{!!}, \tag{D.29}$$

where $G^{!!} \subset W_{\mathrm{big}} \times \mathbb{A}^1$ is as in §D.5.2.

Proof. It suffices to show that (D.28) induces an isomorphism $G^! \otimes \mathbb{Z}_{(p)} \xrightarrow{\sim} G^{!!} \otimes \mathbb{Z}_{(p)}$ for each prime p. The description of $G^{!!} \otimes \mathbb{Z}_{(p)}$ from (D.27) allows one to prove this quite similarly to §5.7.4. ☐

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