Pure and Applied Mathematics Quarterly
Volume 20, Number 1, 307–426, 2024

A general framework for the analytic Langlands correspondence

Pavel Etingof, Edward Frenkel, and David Kazhdan

To Corrado De Concini with admiration

Abstract: We discuss a general framework for the analytic Langlands correspondence over an arbitrary local field F introduced and studied in our works [EFK1, EFK2, EFK3], in particular including non-split and twisted settings. Then we specialize to the archimedean cases ($F = \mathbb{C}$ and $F = \mathbb{R}$) and give a (mostly conjectural) description of the spectrum of the Hecke operators in various cases in terms of opers satisfying suitable reality conditions, as predicted in part in [EFK2, EFK3] and [GW]. We also describe an analogue of the Langlands functoriality principle in the analytic Langlands correspondence over $\mathbb{C}$ and show that it is compatible with the results and conjectures of [EFK2]. Finally, we apply the tools of the analytic Langlands correspondence over archimedean fields in genus zero to the Gaudin model and its generalizations, as well as their q-deformations.

1. Introduction

1.1. Overview

Let X be a smooth irreducible projective curve of genus $\mathrm{g} > 1$ and G a reductive algebraic group, both defined over a local field F. Let $\mathrm{Bun}_G^\circ(X)$ be the variety of regularly stable principal G-bundles on X (see e.g. [EFK1], Section 1.1). In [EFK1, EFK2, EFK3], motivated in part by the works [BK1, Ko, La, Te], we proposed the **analytic Langlands correspondence**, which is the study of the spectrum of **Hecke operators** acting on the space of complex-valued half-densities on $\mathrm{Bun}_G^\circ(X)(F)$. Justifying its name, this correspondence is a natural analytic analog of two previously known settings of Langlands correspondence – **arithmetic** (for curves over a finite field) and

arXiv: 2311.03743
Received October 27, 2023.

geometric (involving $\mathcal{D}$-modules on complex curves and on moduli stacks of G-bundles on curves), to both of which it is actually intimately related. We also proposed a ramified generalization of the analytic Langlands correspondence for G-bundles with level structure at an F-rational effective divisor $D \subset X$ (which allows one to also consider $\mathrm{g} = 0, 1$).

However, previously for simplicity we focused on the basic case when G and D are split, and in fact mostly assumed that $F = \mathbb{C}$. Yet the natural generality of the (arithmetic) Langlands correspondence is that of a flat reductive group scheme $\mathcal{G}$ over X, for example, one defined by an action of the étale fundamental group $\pi_1^{\mathrm{et}}(X)$ on G. Roughly speaking, one of the main goals of this paper is to discuss the (ramified) analytic Langlands correspondence in this more general setting, when F is arbitrary and G, D are not necessarily split; in particular, this means that we need focus on Galois-theoretic aspects of the theory. We also allow twists by $Z(G)$-gerbes on X and, in the ramified case, by unitary representations of the group of changes of the level structure. A detailed discussion of the general framework of the analytic Langlands correspondence with a focus on these additional features is the subject of the first half of the paper.

The second half of the paper is dedicated to the archimedean cases, $F = \mathbb{C}$ and $F = \mathbb{R}$. In these cases, as shown in [EFK2], the Hecke operators $H_{x,\lambda}$ commute with the **quantum Hitchin Hamiltonians** and also satisfy a certain differential equation with respect to $x \in X$ involving these Hamiltonians, called the **universal oper equation**. As a result, the joint spectrum of the Hecke and quantum Hitchin Hamiltonians is (conjecturally) labeled by $G^\vee$**-opers** L on X satisfying a certain topological condition called a **reality condition**. For $F = \mathbb{C}$, as explained in [EFK2], this is the condition that the monodromy of L can be conjugated into an inner (conjecturally, split) real form $G^\vee_{\mathbb{R}}$ of $G^\vee$. On the other hand, for $F = \mathbb{R}$ the theory depends on several additional pieces of data (an antiholomorphic involution of X, an inner class of G, a form G^σ of G in this class attached to each oval of $X(\mathbb{R})$, etc.), and the exact form of the reality condition depends on these details. We work out this condition in several examples for $G = GL_1$ and PGL_2, generalizing [EFK3], Subsection 4.7 and [GW], Section 6.

Finally, we explain how the **generalized Bethe Ansatz method for the Gaudin model** can be viewed (in several ways) as an instance of the tamely ramified analytic Langlands correspondence in genus 0 over $\mathbb{R}$ and $\mathbb{C}$. Interestingly, the role of Hecke operators in this setting is played by **Baxter's Q-operator of the Gaudin model**, the $q \to 1$ limit of the Q-operator of the XXZ quantum spin chain introduced by R. Baxter. Motivated by this, we discuss a q-deformation of the archimedean analytic Langlands correspondence in genus 0.

1.2. Summary of the main results

To summarize, in this paper we accomplish the following.

1. We formulate the problem of (ramified) analytic Langlands correspondence in a general setting when the group G and the ramification divisor $D \subset X$ are not necessarily split and ramification points carry unitary representations of the groups of changes of level structure, in presence of possible twists by a Z-gerbe on X and an action of $\pi_1(X)$ on the root datum of G. We state the conjecture on compactness of Hecke operators, which leads to discreteness of their spectrum.

2. For $G = PGL_2$ and genus 0 with split ramification divisor, in the tamely ramified case when ramification points carry principal series representations of $G(F)$, we compute explicitly the Hecke operators H_x (which are known to be compact in this case) and find their asymptotics near ramification points. This gives the asymptotics of eigenvalues of H_x.

3. In the cases $F = \mathbb{C}$ and $F = \mathbb{R}$, we conjecturally describe the spectrum of Hecke operators in the setting of (1) in terms of opers with monodromy representation satisfying suitable "reality conditions". We formulate reality conditions in various special cases, and show that spectral opers must satisfy these conditions. In particular, we work out reality conditions for $G = GL_1$, $F = \mathbb{R}$, and also $G = PGL_2$, $F = \mathbb{R}$ in the ramified and tamely ramified cases. We describe behavior of eigenvalues of H_x near real ovals of X (when they are present) in various situations.

4. In the case $F = \mathbb{C}$, we discuss in detail in Section 3.6 the Hecke operators corresponding to the *principal weights* of $G^\vee$ (such that the corresponding irreducible representation of $\mathfrak{g}^\vee$ remains irreducible under a principal $\mathfrak{sl}_2$ subalgebra), following the approach of [EFK2], Section 5. We then consider the general case. In Subsection 3.7, we prove that for a generic $G^\vee$-oper χ on a curve of genus $\mathrm{g} > 1$, the Zariski closure M_χ of the image of its monodromy representation is equal to $G^\vee$. We also show if $G^\vee$ is connected simple group of adjoint type, then for any $G^\vee$-oper χ, the group M_χ is a connected simple subgroup of $G^\vee$ that contains a principal PGL_2 subgroup of $G^\vee$. This allows us to elucidate the conjectural formula for the eigenvalues of the Hecke operators presented in [EFK2], Conjecture 5.1 (see Subsection 3.8). We also use these results in Subsection 3.9 to describe an analogue of the Langlands functoriality principle in the analytic Langlands correspondence over $\mathbb{C}$ and show that it is compatible with the results and conjectures of [EFK2].

5. We describe several settings of the Gaudin model in terms of the analytic Langlands correspondence, enabling us to describe the spectrum of the commuting Gaudin Hamiltonians. In particular, we reinterpret the known

description of the spectrum of the Gaudin Hamiltonians in the case of the tensor product of finite-dimensional representations in terms of monodromy-free opers [F2, R] as a special case of real analytic Langlands correspondence. Namely, we use our results in the "quaternionic" real case discussed in Subsection 4.7 to obtain this description of the spectrum. We also explain the connection to the Bethe Ansatz method of diagonalization of these Hamiltonians.

Our interpretation of the Gaudin model in terms of the analytic Langlands correspondence allows us to extend it to a more general setting in which the space of states is infinite-dimensional and the traditional Bethe Ansatz methods do not apply; for example, the tensor product of unitary principal series representations.

In all of these cases, the key new element is the existence of the Hecke operators commuting with the Gaudin Hamiltonians and the fact that they satisfy differential equations (the universal oper equations), which can be used to describe the analytic properties of the $G^\vee$-opers encoding the possible joint eigenvalues of the Gaudin Hamiltonians. We show that the Hecke operators in this setting are closely related to the analogue of the Baxter Q-operator in the Gaudin model. We use this relation to discuss a possible q-deformation of the analytic Langlands correspondence, where the role of Hecke operators is played by Baxter's Q-operators of the XXZ model (and its generalization from SL_2 to a general simple complex Lie group).

We note that some results on the spectra of the Gaudin Hamiltonians for SL_2 in the real case were obtained in [NRS] from the point of view of $N = 2$ SUSY 4d gauge theory (see also [JLN]). It would be interesting to see if there is a connection between their results and ours.

1.3. Structure of the paper

The structure of the paper is as follows.

In Section 2 we give an informal description of the general framework for the analytic Langlands correspondence over an arbitrary local field F. We start with reviewing the theory of forms of reductive groups, especially over non-archimedean local fields (the Kneser-Bruhat-Tits theory). Then we explain that in the unramified case the appropriate moduli space of F-rational G-bundles is determined by an **inner class** $C(s)$ of G over F, $s \in H^1(F, \operatorname{Out} G)$. We also explain that every such bundle P and a geometric point x of X defines an F-**form** G^σ of G over the field E_x of definition of x in $C(s)$. Thus in the tamely ramified case ($\ell = 1$) with ramification divisor $D \subset X$, the input data of the theory is a choice, for each $t \in D$, of a form G^σ of G over E_t in $C(s)$ and a unitary representation V of $G^\sigma(E_t)$. For such input data, we define Hecke operators on the L^2 space of the moduli space of F-rational G-bundles, pose the spectral problem for them, consider various examples and present several conjectures that we've proved in some interesting special cases.

In Section 3 we discuss the case $F = \mathbb{C}$. We start by reviewing the results and conjectures of [EFK1, EFK2, EFK3] on parametrizing the spectrum Σ of Hecke and quantum Hitchin Hamiltonians by real opers on the Riemann surface $X(\mathbb{C})$. Next, we consider the differential equations satisfied by the Hecke operators H_λ in various cases: $G = PGL_2$ and $X = \mathbb{P}^1$ with parabolic structures at finitely many points in Subsection 3.3 (recalling and generalizing the results of [EFK3]); $G = PGL_n$, X of genus $\mathrm{g} > 1$, and $\lambda = \omega_1$ in Subsection 3.4 (recalling the results of [EFK2]); a generalization of the latter case to an arbitrary principal λ in Subsection 3.6; and the general case in Subsections 3.5 and 3.8. In Subsection 3.7 we describe the Zariski closures of the monodromy representations of $G^\vee$-opers in the case that $G^\vee$ is a connected simple group of adjoint type and $\mathrm{g} > 1$. In particular, we show that the monodromy of a generic $G^\vee$-oper is dense in $G^\vee$. We use this in Subsection 3.8 to elucidate some results of [EFK2] and in Subsection 3.9 to describe an analogue of the Langlands functoriality principle in the analytic Langlands correspondence. In Subsection 3.10 we discuss the twists by $Z(G)$-gerbes and by $\mathrm{Aut}\, G$-torsors on X. We also consider the twists by unitary representations at ramification points for $G = PGL_2$. For all these twists, we describe (conjecturally) the set of opers parametrizing Σ.

In Section 4, we consider the case $F = \mathbb{R}$. In this case the curve X is a Riemann surface with an antiholomorphic involution τ. We first assume that τ has no fixed points on X (i.e., $X(\mathbb{R}) = \emptyset$) and review the conjectures from [GW], Section 6 on the opers that are expected to label the eigenspaces of Hecke operators. We also show following [W] that these conjectures hold for $G = GL_1$. Then we proceed to the general case, when $X(\mathbb{R})$ may be nonempty, discussed in [GW], Subsection 6.3. This case is more complicated since it involves boundary conditions for oper solutions on the ovals of $X(\mathbb{R})$. We describe what happens for $G = GL_1$, and then propose a conjectural reality condition for spectral opers for $G = SL_2$. In this case, we have two forms of G (both inner) – the split form and the compact form, giving rise to two types of ovals – real and quaternionic, respectively. We propose the boundary conditions for spectral opers on both real and quaternionic ovals. We also explain what happens in presence of ramification points, and in the case $X = \mathbb{P}^1$ with the usual real structure and all ramification points real, we recover the description of spectral opers from [EFK3], Subsection 4.7 ("balanced" opers).

Finally, in Section 5 we interpret the Gaudin model and its generalization in terms of the analytic Langlands correspondence. In particular, for $G = SL_2$, we derive the description of the spectrum of the Gaudin Hamiltonians in terms of monodromy-free PGL_2-opers [F2, R] from a special case of the analytic Langlands correspondence over $\mathbb{R}$ (namely, for the compact form of $SL_2(\mathbb{C})$). We give a description of the spectrum of the Gaudin Hamiltonians on the tensor product of representations of the unitary principal series of $SL_2(\mathbb{R})$ in terms of balanced PGL_2-opers. We also discuss a q-deformation of the analytic Langlands correspondence in genus 0 and its connection with Bethe Ansatz method for the XXZ model (a q-deformation of the Gaudin model).

2. Analytic Langlands correspondence over a general local field

2.1. Varieties over arbitrary fields

Let L be a separably closed field and X be an algebraic variety defined over L. Since X is reduced, $X(L) = X(\overline{L})$, where $\overline{L} \supset L$ is an algebraic closure of L; in particular, $X(L) \neq \emptyset$. For $\gamma \in \operatorname{Aut} L$, let ${}^\gamma X$ be the twist of X by γ. Thus $X(L) = {}^\gamma X(L)$ and the structure sheaf of ${}^\gamma X$ is obtained from the one of X by twisting the scalar multiplication by γ.

Now let F be any field. Let F_{sep} be a separable closure of F and for a field extension $F \subset E \subset F_{\mathrm{sep}}$, $\Gamma_E := \mathrm{Gal}(F_{\mathrm{sep}}/E)$ be the absolute Galois group of E.

Let X be a variety defined over F. Let $\tau(\gamma) : {}^\gamma X \to X$, $\gamma \in \Gamma_F$, be the collection of isomorphisms defining the F-structure on X. This defines an action

$$\gamma \mapsto \tau(\gamma) : X(F_{\mathrm{sep}}) \to X(F_{\mathrm{sep}})$$

of Γ_F on $X(F_{\mathrm{sep}})$, and the set $X(E)$ of E-points of X is the fixed point set of $\Gamma_E \subset \Gamma_F$ on $X(F_{\mathrm{sep}})$.

Let $x \in X(F_{\mathrm{sep}})$ be a point with stabilizer $\Gamma_x \subset \Gamma_F$. We will call the field $E := F_{\mathrm{sep}}^{\Gamma_x}$ the **field of definition of** x; thus $\Gamma_x = \Gamma_E$.[1] Let $X_E \subset X(F_{\mathrm{sep}})$ be the subset of points with field of definition E. Then $X(E)$ is the disjoint union of X_K over all $F \subset K \subset E$.

[1]Note that the field of definition E of a point x is not just an abstract field extension of F, but rather a subfield of F_{sep} containing F; so if fields of definition of two points are not Galois then they may be distinct but nevertheless isomorphic.

Example 2.1. Let $F = \mathbb{R}$, then $F_{\mathrm{sep}} = \overline{F} = \mathbb{C}$ and $\Gamma_F = \mathbb{Z}/2$. An $\mathbb{R}$-structure on X is given by an antiholomorphic involution $\tau : X(\mathbb{C}) \to X(\mathbb{C})$. Furthermore, the real locus $X_{\mathbb{R}} = X(\mathbb{R}) = X(\mathbb{C})^{\tau}$ is just the set of fixed points of τ on $X(\mathbb{C})$, and $X_{\mathbb{C}} = X(\mathbb{C}) \setminus X(\mathbb{R})$.

2.2. Forms of reductive groups

Let G be a split connected reductive algebraic group over $\mathbb{Z}$ corresponding to a polarized root datum Δ_G. In particular, this means that we fix a positive Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Let G_{ad} be the corresponding adjoint group. For any field E let $\operatorname{Aut} G(E) := \operatorname{Aut} \Delta_G \ltimes G_{\mathrm{ad}}(E)$; here $\operatorname{Aut} \Delta_G = \operatorname{Out} G$ is the group of outer automorphisms of G.[2] This group acts on $G(K)$ for any field extension K of E.

Let F be a field whose characteristic is either zero or coprime to the determinant of the Cartan matrix of G.[3]

The classification of forms of G over F is as described in Subsection 2.1, except that unlike the case of general varieties, we already have a distinguished F-form of G (the **split form**) defining an action $g \mapsto \gamma(g)$ of Γ_F on $G(F_{\mathrm{sep}})$, so we can describe all F-forms of G by counting from this form, in terms of Galois cohomology. Namely, forms of G over F are parametrized by the (continuous) Galois cohomology $H^1(\Gamma_F, \operatorname{Aut} G(F_{\mathrm{sep}}))$ ([S]). Specifically, let $\theta : \Gamma_F \to \operatorname{Aut} G(F_{\mathrm{sep}})$ be a 1-cocycle, i.e.,

$$\theta(\gamma_1\gamma_2) = \theta(\gamma_1) \circ \gamma_1(\theta(\gamma_2)).$$

Then we have an action $\sigma = \sigma_\theta$ of Γ_F on $G(F_{\mathrm{sep}})$ given by

$$\sigma(\gamma) = \theta(\gamma) \circ \gamma,$$

and conversely, an action σ of Γ_F such that for $\gamma \in \Gamma_F$, $\theta_\sigma(\gamma) := \sigma(\gamma) \circ \gamma^{-1} \in \operatorname{Aut} G(F_{\mathrm{sep}})$ gives rise to a 1-cocycle $\theta = \theta_\sigma$. The form G^σ of G corresponding to σ (or θ) is defined by its functor of points

$$G^\sigma(A) := G(A \otimes_F F_{\mathrm{sep}})^{\Gamma_F}$$

[2] The group $\operatorname{Aut} \Delta_G$ may be infinite, for example if G is an n-dimensional torus then $\operatorname{Aut} \Delta_G = GL_n(\mathbb{Z})$. Thus $\operatorname{Aut} G$ is not an algebraic group, in general. More specifically, it is an algebraic group iff the center of G has dimension ≤ 1 (e.g., for $G = GL_n$ or G semisimple). But this is not important for our considerations.

[3] This assumption is not essential and is made to simplify the exposition.

for any commutative F-algebra A. In particular, for any field extension $F \subset E \subset F_{\mathrm{sep}}$, $G^\sigma(E)$ is the subgroup of fixed points of $\sigma(\Gamma_E)$. In other words, $G^\sigma(E)$ is the group of elements of $G(F_{\mathrm{sep}})$ satisfying the equation

$$\gamma(g) = \theta(\gamma)^{-1}(g)$$

for all $\gamma \in \Gamma_E$.

For example, if $\sigma = 1$ then $G^\sigma = G_{\mathrm{spl}}$, the split form of G.

Moreover, if an element $a \in \operatorname{Aut} G(F_{\mathrm{sep}})$ transforms a 1-cocycle θ_{σ_1} into a 1-cocycle θ_{σ_2} then it canonically defines an isomorphism $\chi(a) : G^{\sigma_1} \to G^{\sigma_2}$ such that for any b transforming θ_{σ_2} into θ_{σ_3} we have $\chi(ba) = \chi(b) \circ \chi(a)$. In other words, denoting by $\mathbf{A} = \mathbf{A}(F, G)$ the groupoid whose objects are 1-cocycles $\theta : \Gamma_F \to \operatorname{Aut} G(F_{\mathrm{sep}})$ and morphisms from θ to θ' are elements $a \in \operatorname{Aut} G(F_{\mathrm{sep}})$ such that $\theta'(\gamma) = a\theta(\gamma)\gamma(a)^{-1}$, we obtain a functor $\theta_\sigma \mapsto G^\sigma$ from $\mathbf{A}$ to the category of algebraic F-groups. In particular, G^σ depends only on the cohomology class $[\theta_\sigma]$ up to isomorphism.

Since the sequence

$$1 \to G_{\mathrm{ad}}(F_{\mathrm{sep}}) \to \operatorname{Aut} G(F_{\mathrm{sep}}) \to \operatorname{Aut} \Delta_G \to 1 \tag{2.1}$$

splits, it defines a short exact sequence of pointed sets

$$1 \to H^1(\Gamma_F, G_{\mathrm{ad}}(F_{\mathrm{sep}})) \to H^1(\Gamma_F, \operatorname{Aut} G(F_{\mathrm{sep}})) \to H^1(\Gamma_F, \operatorname{Aut} \Delta_G) \to 1, \tag{2.2}$$

where $H^1(\Gamma_F, \operatorname{Aut} \Delta_G) = \operatorname{Hom}(\Gamma_F, \operatorname{Aut} \Delta_G)/\text{conjugation}$ (note that Γ_F acts trivially on Δ_G since we start with the split form). In other words, the second map in (2.2) is injective and its image coincides with the kernel (i.e., the preimage of 1) of the third map, which is surjective.

Recall that a form G^σ is called **inner** if $[\theta_\sigma] \in H^1(\Gamma_F, G_{\mathrm{ad}}(F_{\mathrm{sep}}))$, i.e., if it projects to $1 \in H^1(\Gamma_F, \operatorname{Aut} \Delta_G)$. The **inner class** of G^σ is the collection $C(\sigma)$ of all forms G^η of G such that $[\theta_\sigma]$ and $[\theta_\eta]$ map to the same element of $H^1(\Gamma_F, \operatorname{Aut} \Delta_G)$. Thus inner classes are labeled by conjugacy classes $[s]$ of homomorphisms $s : \Gamma_F \to \operatorname{Aut} \Delta_G$. For example, $C(1)$ (the inner class of the split form) consists of all the inner forms of G.

Note that since the sequence (2.1) is canonically split, so is the sequence (2.2). Thus for every $s \in \operatorname{Hom}(\Gamma_F, \operatorname{Aut} \Delta_G)$ the inner class $C(s)$ has a canonical representative called the **quasi-split form** of G in $C(s)$ and denoted by G^s; this is the only form of G in $C(s)$ which has an F-rational Borel subgroup. For example, the quasi-split inner form is the split form. By (2.2), this implies that each inner class $C(s)$ can be canonically identified with $H^1(\Gamma_F, G^s_{\mathrm{ad}}(F_{\mathrm{sep}}))$ (with the action of Γ_F defined by s).

2.3. Forms of reductive groups over a local field

Now let F be a non-archimedean local field. We recall the theory of forms of reductive groups over F. We start with two classical theorems.

Theorem 2.2 (Kneser, Bruhat-Tits, [BT], 4.7)**.** *For a simply connected semisimple group G over F, one has $H^1(\Gamma_F, G(F_{\mathrm{sep}})) = 1$.*

Theorem 2.3 (Tate duality for Galois cohomology, [S], II.5.2, Theorem 2)**.** *Let A be a finite Γ_F-module of order prime to $\mathrm{char}(F)$ and $A^* := \mathrm{Hom}(A, \mathbb{G}_m)$. Then for $0 \le i \le 2$, $H^i(\Gamma_F, A)$ is finite and we have a canonical isomorphism*

$$H^i(\Gamma_F, A) \cong \mathrm{Hom}(H^{2-i}(\Gamma_F, A^*), \mathbb{Q}/\mathbb{Z}).$$

In particular, taking $i = 2$, we get $H^2(\Gamma_F, A) \cong \mathrm{Hom}((A^*)^{\Gamma_F}, \mathbb{Q}/\mathbb{Z}) = A(-1)_{\Gamma_F}$, the coinvariants of Γ_F in the negative Tate twist $A(-1)$ of A.

Corollary 2.4. *Let G^s be a simply connected quasi-split semisimple group over F, G^s_{ad} the corresponding adjoint group, and $Z^s(F_{\mathrm{sep}})$ the center of $G^s(F_{\mathrm{sep}})$ regarded as a Γ_F-module (with action defined by s). Then there is a natural inclusion*

$$H^1(\Gamma_F, G^s_{\mathrm{ad}}(F_{\mathrm{sep}})) \hookrightarrow Z^s(F_{\mathrm{sep}})(-1)_{\Gamma_F}.$$

Proof. By Theorem 2.2 and the “long” exact sequence of Galois cohomology, we have an embedding $\xi : H^1(\Gamma_F, G^s_{\mathrm{ad}}(F_{\mathrm{sep}})) \hookrightarrow H^2(\Gamma_F, Z^s(F_{\mathrm{sep}}))$, so the result follows from Theorem 2.3, using that p does not divide the determinant of the Cartan matrix of G. ☐

In fact, there is an even stronger result:

Corollary 2.5. *The map of Corollary 2.4 is an isomorphism.*

Proof. It is known ([K], [T]) that the map of Corollary 2.4 is surjective, so the result follows. ☐

Thus we obtain the following corollary. Let G^s_{sc} be the universal cover of G^s_{ad}.

Corollary 2.6. *Let G be a split connected reductive group and suppose $s \in \mathrm{Hom}(\Gamma_F, \mathrm{Aut}\,\Delta_G)$. Then the inner class $C(s)$ is naturally identified with the group $Z^s(F_{\mathrm{sep}})(-1)_{\Gamma_F}$, where Z^s is the center of G^s_{sc}.*

Thus we get

Proposition 2.7. *Over any local field, every inner class of a split connected reductive group G is finite.*[4]

Proof. Recall that if $F = \mathbb{R}$ then we have the well known Cartan classification of forms of G over F. This implies the proposition in the archimedean case. In the non-archimedean case the proposition follows from Corollary 2.6. $\square$

Example 2.8. Let $G = SL_n$ and $\mathrm{char}(F)$ be coprime to n.

(1) *The split inner class.* In this case, $Z^s = \mu_n$. Thus by Corollary 2.6, forms in this class are parametrized by $\mathbb{Z}/n$. Namely, we identify $\mathbb{Z}/n$ with $\frac{1}{n}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z} = \mathrm{Br}(F)$. Thus every $m \in \mathbb{Z}/n$ (represented by an integer in $[1, n]$) gives rise to a central division algebra D_m over F of dimension $(n/m)^2$, and the corresponding form G^σ is $SL_{n/m,D_m}$ (i.e., $G^\sigma(E) = SL_{n/m}(E \otimes_F D_m)$ for a field extension $E \supset F$).

(2) *The non-split inner class C_L attached to a separable quadratic field extension L of F ($n \geq 3$).* The quadratic extension defines a character $\chi_L : \Gamma_F \to \pm 1$, which gives rise to an action of Γ_F on $\mathbb{Z}$; we denote this module by $\mathbb{Z}_L$. Then $Z^s = \mu_n \otimes_{\mathbb{Z}} \mathbb{Z}_L$, so the inner class C_L is parametrized by $(\mathbb{Z}_L/n)_{\Gamma_F}$, which is trivial if n is odd and $\mathbb{Z}/2$ if n is even. Thus we should expect the quasi-split form and also an additional form for even n. And indeed, this is the case: these forms are the corresponding **special unitary groups**. Namely, recall that if $N : L^\times \to F^\times$ is the norm map then $|F^\times/N(L^\times)| = 2$. Thus we have two equivalence classes of nondegenerate Hermitian forms on L^n up to isomorphism – B_+ whose determinant is a norm and B_- whose determinant is not. So we have the corresponding special unitary groups $SU^+_{n,L}$ and $SU^-_{n,L}$ (namely, $SU^\pm_{n,L}(E) = SU^\pm_n(E \otimes_F L)$ for a field extension $E \supset F$). However, if n is odd then for any non-norm $a \in F^\times$, the forms aB_- and B_+ are equivalent, so these two groups are isomorphic.

We note that $SU^+_{2,L} = SL_{1,D_2}$ and $SU^-_{2,L} = SL_2$ (for any L).

Remark 2.9. If $\mathrm{char}(F) = 0$ then F has finitely many extensions of every fixed degree. Thus for any finite group Γ, there are finitely many homomorphisms $\Gamma_F \to \Gamma$. Also by a theorem of Jordan and Zassenhaus, for each n there are finitely many finite subgroups of $GL_n(\mathbb{Z})$ up to isomorphism, so the number of homomorphisms $\Gamma_F \to GL_n(\mathbb{Z})$ (i.e., of F-forms of the n-dimensional torus) is finite. It follows that the number of inner classes over F of any split connected reductive group is finite as well.

[4] We have shown this when $\mathrm{char}(F)$ does not divide the determinant of the Cartan matrix of F, but this assumption is, in fact, unnecessary.

This is, however, false in positive characteristic $p > 0$, as in this case $|\mathrm{Hom}(\Gamma_F, \mathbb{Z}/p)| = \infty$. For example, for $p = 2$ there are infinitely many separable quadratic extensions of F, so there are infinitely many inner classes of SL_n for $n \geq 3$. But this is not going to matter for us here.

Finally, we have

Proposition 2.10. *Let T be an abelian reductive group over F. Then*

(i) there exists a finite subgroup $R \subset H^1(\Gamma_F, T(F_{\mathrm{sep}}))$ such that the quotient $H^1(\Gamma_F, T(F_{\mathrm{sep}}))/R$ embeds into $H^1(\Gamma_F, (T/T_0)(F_{\mathrm{sep}}))$, where T_0 is the identity component of T;
(ii) If $\mathrm{char}(F)$ is coprime to the order of T/T_0 then $H^1(\Gamma_F, T(F_{\mathrm{sep}}))$ is finite.

Proof. By Theorem 2.3, (ii) follows from (i), so it remains to prove (i).
We have an exact sequence

$$H^1(\Gamma_F, T_0(F_{\mathrm{sep}})) \to H^1(\Gamma_F, T(F_{\mathrm{sep}})) \to H^1(\Gamma_F, (T/T_0)(F_{\mathrm{sep}})).$$

Thus it suffices to prove (i) when $T = T_0$ is connected (a torus), i.e., to show that in this case $H^1(\Gamma_F, T(F_{\mathrm{sep}}))$ is finite. Let $\Gamma_E \subset \Gamma_F$ be the kernel of the action of Γ_F on T. Then we have the inflation-restriction exact sequence

$$0 \to H^1(\Gamma_F/\Gamma_E, T(E)) \to H^1(\Gamma_F, T(F_{\mathrm{sep}})) \to H^1(\Gamma_E, T(F_{\mathrm{sep}}))^{\Gamma_F/\Gamma_E}.$$

By Hilbert Theorem 90, the last term vanishes, so

$$H^1(\Gamma_F, T(F_{\mathrm{sep}})) \cong H^1(\Gamma_F/\Gamma_E, T(E)).$$

But the group $H^1(\Gamma_F/\Gamma_E, T(E))$ has exponent dividing $N = [E : F]$, so it is finite. This implies the result. □

2.4. Principal G-bundles

Let $F, F_{\mathrm{sep}}, \Gamma_F$ be as above, X be an algebraic F-variety, and G an algebraic F-group. We denote by $\mathrm{Bun}_G(X)$ the F-stack of principal G-bundles on X. Then Γ_F acts on the set $\mathrm{Bun}_G(X)(F_{\mathrm{sep}})$. If a G-bundle $P \in \mathrm{Bun}_G(X)(F_{\mathrm{sep}})$ is defined by transition functions $g_{ij} : U_i \cap U_j \to G$ where $\{U_i\}$ is an open cover of X, and $\gamma \in \Gamma_F$, then we can define the G-bundle P^γ by the transition functions $g_{ij}^\gamma : U_i^\gamma \cap U_j^\gamma \to G$. It is clear that this definition is independent on choices and gives rise to an action of Γ_F on $\mathrm{Bun}_G(X)(F_{\mathrm{sep}})$.

Now let us write this definition slightly more explicitly. Let $\theta : \Gamma_F \to \operatorname{Aut} G(F_{\mathrm{sep}})$ be a 1-cocycle, $\sigma(\gamma) := \theta(\gamma) \circ \gamma$ be the corresponding action of Γ_F, and G^σ be the form of G over F defined by σ. Also let τ be the natural action of Γ_F on $X(F_{\mathrm{sep}})$ (the F-structure on X). Then given a principal G^σ-bundle P on X defined over F_{sep}, we have the G^σ-bundle $P^\gamma = (\sigma, \tau)(\gamma)P$ with transition functions $g_{ij}^\gamma = g_{ij}^{(\sigma,\tau,\gamma)}$ given by

$$g_{ij}^{(\sigma,\tau,\gamma)} = \sigma(\gamma) \circ g_{ij} \circ \tau(\gamma)^{-1}.$$

Let $\operatorname{Bun}_{G,\sigma}(X,\tau) := \operatorname{Bun}_G(X)(F_{\mathrm{sep}})^{\Gamma_F}$ be the set of fixed points of this action of Γ_F.

Now suppose that G is a split connected reductive group over $\mathbb{Z}$. If $h : \Gamma_F \to G^\sigma_{\mathrm{ad}}(F_{\mathrm{sep}})$ is a 1-cocycle then

$$g_{ij}^{(h\sigma,\tau,\gamma)} = h(\gamma) \circ g_{ij}^{(\sigma,\tau,\gamma)}.$$

Hence $h(\gamma)$ defines a canonical isomorphism $(h\sigma,\tau)(\gamma)P \cong (\sigma,\tau)(\gamma)P$. In other words, $(\sigma,\tau)(\gamma)$ depends only on the inner class $C(\sigma)$ of σ, i.e. on $s \in \operatorname{Hom}(\Gamma_F, \operatorname{Aut}\Delta_G)$ such that $C(\sigma) = C(s)$. Thus $(\sigma,\tau)(\gamma) = (s,\tau)(\gamma)$ for all γ and $\operatorname{Bun}_{G,\sigma}(X,\tau) = \operatorname{Bun}_{G,s}(X,\tau)$.

2.5. Moduli of *G*-bundles on a smooth projective curve

Let X be a smooth irreducible projective curve of genus $\mathrm{g} \geq 2$ over F. In this case we have a notion of a **regularly stable** principal G-bundle on X, which is a stable bundle whose group of automorphisms is the minimal possible, i.e., reduces to the center Z of G.

The set $\operatorname{Bun}^\circ_G(X)(F_{\mathrm{sep}}) \subset \operatorname{Bun}_G(X)(F_{\mathrm{sep}})$ of regularly stable bundles is the set of F_{sep}-points of a smooth algebraic variety $\operatorname{Bun}^\circ_G(X)$ of dimension $(\mathrm{g}-1)\dim G$ defined over F (this is, in fact, the underlying variety of a stack which is the quotient of a variety by the trivial action of Z). Moreover, every pair (s,τ) defines a form $\operatorname{Bun}^\circ_G(X)_{s,\tau}$ of this variety.

Let

$$\operatorname{Bun}^\circ_{G,s}(X,\tau) := \operatorname{Bun}^\circ_G(X)_{s,\tau}(F) \subset \operatorname{Bun}_{G,s}(X,\tau)$$

be the subset of isomorphism classes of regularly stable bundles.

Define a **pseudo-F-structure** on $P \in \operatorname{Bun}_{G,s}(X,\tau)$ to be a collection of isomorphisms

$$A(\gamma) : (s,\tau)(\gamma)P \to P,\ \gamma \in \Gamma_F.$$

Such a data defines a 2-cocycle $a = a_A$ on Γ_F with coefficients in $Z^s(F_{\mathrm{sep}})$ such that

$$A(\gamma_1\gamma_2) = A(\gamma_1) \circ (s,\tau)(\gamma_1)(A(\gamma_2)) \circ a(\gamma_1,\gamma_2). \tag{2.3}$$

We will say that a pseudo-F-structure A is an F-**structure** if $a_A = 1$.

Any two pseudo-F-structures A, A' on P differ by a 1-cochain $c : \Gamma_F \to Z^s(F_{\mathrm{sep}})$, and $a_A/a_{A'} = dc$. Thus the class $[a_A] \in H^2(\Gamma_F, Z^s(F_{\mathrm{sep}}))$ does not depend on A and only depends on P, so we'll denote it by α_P. So we obtain

Lemma 2.11. *We have a decomposition*

$$\mathrm{Bun}^\circ_{G,s}(X,\tau) = \sqcup_{\alpha\in H^2(\Gamma_F, Z^s(F_{\mathrm{sep}}))} \mathrm{Bun}^\circ_{G,s,\alpha}(X,\tau),$$

where $\mathrm{Bun}^\circ_{G,s,\alpha}(X,\tau)$ *is the subset of* P *with* $\alpha_P = \alpha$.

In fact, it is more natural to consider a Galois covering of $\mathrm{Bun}^\circ_{G,s,\alpha}(X,\tau)$ which keeps track of the isomorphism A. To this end, fix a 2-cocycle a representing α. Then for a principal bundle $P \in \mathrm{Bun}^\circ_{G,s,\alpha}(X,\tau)$ a solution A of (2.3) is unique up to multiplication by a 1-cocycle $c : \Gamma_F \to Z^s(F_{\mathrm{sep}})$. On the other hand, if c is a coboundary then A and cA are equivalent by an element of $Z(F_{\mathrm{sep}})$. Thus the set of solutions A of (2.3) up to isomorphism is a torsor over $H^1(\Gamma_F, Z^s(F_{\mathrm{sep}}))$. In other words, the set $\mathrm{Bun}^\circ_{G,s,a}(X,\tau)$ of isomorphism classes of pseudo-F-structures (P, A) satisfying (2.3) with $a_A = a$ is a $H^1(\Gamma_F, Z^s(F_{\mathrm{sep}}))$-torsor over $\mathrm{Bun}^\circ_{G,s,\alpha}(X,\tau)$.

Furthermore, if $[a] = [a'] = \alpha$ and $a/a' = dc$ then multiplication by the 1-cochain c defines a bijection $\nu_c : \mathrm{Bun}^\circ_{G,s,a}(X,\tau) \cong \mathrm{Bun}^\circ_{G,s,a'}(X,\tau)$ which depends on c only up to coboundaries, and if c is a 1-cocycle (so $a = a'$), this recovers the action of $H^1(\Gamma_F, Z^s(F_{\mathrm{sep}}))$ on the fibers of the projection

$$\mathrm{Bun}^\circ_{G,s,a}(X,\tau) \to \mathrm{Bun}^\circ_{G,s,\alpha}(X,\tau).$$

In other words, for any α we may consider the groupoid $\mathbf{H}_\alpha$ whose objects are 2-cocycles a with $[a] = \alpha$ and morphisms are $\mathrm{Hom}(a,a') = \{c : a/a' = dc\}$ with composition defined by addition. Then canonically we have an $\mathbf{H}_\alpha$-set $\mathrm{Bun}^\circ_{G,s,\alpha}(X,\tau)$ (a functor $\mathbf{H}_\alpha \to \mathbf{Sets}$), which is the collection of sets $\mathrm{Bun}^\circ_{G,s,a}(X,\tau)$, $[a] = \alpha$ with an action of the groupoid $\mathbf{H}_\alpha$.

2.6. The form of G attached to a principal bundle and a point

Now let $x \in X(F_{\mathrm{sep}})$ be a point with field of definition E and stabilizer $\Gamma_x = \Gamma_E \subset \Gamma_F$. Let $P \in \mathrm{Bun}^\circ_{G,s}(X,\tau)$ and P_x be the fiber of P at x. Fix a pseudo-F-structure A on P. Then for $\gamma \in \Gamma_E$ we have an isomorphism $A(\gamma) : P_x \to P_x$. By (2.3), if we identify P_x with G, we obtain a 1-cocycle $\theta : \Gamma_E \to G^s_{\mathrm{ad}}(F_{\mathrm{sep}})$; indeed, the a_A-factor goes away upon projection to the adjoint group. Moreover, this cocycle is independent on the choice of A, since two different choices differ by a central element of G which maps to 1 in G_{ad}.

When we change the identification $P_x \cong G$ by $g \in G(F_{\mathrm{sep}})$, the cocycle θ changes by the coboundary dg, so we obtain a well defined cohomology class $[\theta] \in H^1(\Gamma_E, G^s_{\mathrm{ad}}(F_{\mathrm{sep}}))$. This class defines an action $\sigma = \sigma_\theta$ of Γ_E on $G(F_{\mathrm{sep}})$. Thus we get

Proposition 2.12. *The above procedure assigns to a bundle $P \in \mathrm{Bun}^\circ_{G,s}(X,\tau)$ and a point $x \in X(F_{\mathrm{sep}})$ with field of definition E, an E-form G^σ of G in the inner class $C_E(s)$.*

Moreover, the connecting homomorphism

$$\xi : H^1(\Gamma_E, G^s_{\mathrm{ad}}(F_{\mathrm{sep}})) \to H^2(\Gamma_E, Z^s(F_{\mathrm{sep}}))$$

maps $[\theta]$ to $\alpha_P|_{\Gamma_E}$. It follows that if $X(E) \neq \emptyset$ then $\alpha_P|_{\Gamma_E}$ comes from an element of $H^2(\Gamma_E, Z^s_0(F_{\mathrm{sep}}))$, where $Z_0 := Z \cap [G,G]$.

Example 2.13. Let $G = \mathbb{G}_m$, $s = 1$ (the split 1-dimensional torus). Then

$$H^2(\Gamma_F, Z^s(F_{\mathrm{sep}})) = H^2(\Gamma_F, F^\times_{\mathrm{sep}}) = \mathrm{Br}(F),$$

the Brauer group of F. However, if $X(E) \neq \emptyset$ and $\alpha \in \mathrm{Br}(F)$ is such that $\mathrm{Bun}^\circ_{G,s,\alpha}(X,\tau) \neq \emptyset$ then the image of α in $\mathrm{Br}(E)$ is trivial, i.e., the central division F-algebra D_α splits over E. Thus, assuming that E is a Galois extension of F, we get that α belongs to $\mathrm{Br}(E/F) := H^2(\mathrm{Gal}(E/F), E^\times_{\mathrm{sep}})$, the relative Brauer group which is a finite subgroup of $\mathrm{Br}(F)$.

Thus all components $\mathrm{Bun}^\circ_{G,s,\alpha}(X,\tau)$ are empty except finitely many. It is not hard to show that the same is true in general.

2.7. Principal bundles on curves over a local field

Now let F be a local field. Then $\mathrm{Bun}^\circ_{G,s,\alpha}(X,\tau)$ is an analytic F-manifold of dimension $(\mathrm{g}-1)\dim G$. Thus by Proposition 2.10, for any a with $[a] = \alpha$, $\mathrm{Bun}^\circ_{G,s,a}(X,\tau)$ is also an analytic manifold of this dimension (as it is a finite covering of $\mathrm{Bun}^\circ_{G,s,\alpha}(X,\tau)$). Note that these manifolds are non-empty for $\alpha = 1$ even if $X(F) = \emptyset$ (although they might be empty for some $\alpha \neq 1$).

Let E/F be a finite extension and $X_E \subset X(F_{\mathrm{sep}})$ be the subset of points with field of definition equal to E. Then X_E is an open subset of $X(E)$, hence a 1-dimensional analytic E-manifold.

Corollary 2.14. *If F is non-archimedean and $P \in \mathrm{Bun}^{\circ}_{G,s,\alpha}(X,\tau)$ then the E-form G^{σ} of G in the class $C_E(s)$ attached to P and $x \in X_E$ in Subsection 2.6 is independent on x.*

Proof. By Theorem 2.2 the map ξ is injective. Thus there exists a unique θ up to coboundaries such that $\xi([\theta]) = \alpha$, and we have $\sigma = \sigma_{\theta}$. □

However, for $F = \mathbb{R}$, Corollary 2.14 is not true (even for $G = SL_2$). In this case $X(\mathbb{R}) = X_{\mathbb{R}}$ is a union of ovals, and the form G^{σ} attached to a given bundle P and $x \in X(\mathbb{R})$ is only **locally** constant in x, i.e., may depend on the oval to which x belongs. This leads to interesting topological phenomena described in Subsection 4.4 below.

2.8. Hecke operators

As before, let F be a local field and a a 2-cocycle such that $[a] = \alpha$. Consider the Hilbert space

$$\mathcal{H}(s,\tau,a) := L^2(\mathrm{Bun}^{\circ}_{G,s,a}(X,\tau))$$

of square-integrable half-densities on the analytic F-manifold $\mathrm{Bun}^{\circ}_{G,s,a}(X,\tau)$. The collection of Hilbert spaces $\mathcal{H}(s,\tau,a)$, $[a] = \alpha$ is an $\mathbf{H}_{\alpha}$-Hilbert space (unitary representation of the groupoid $\mathbf{H}_{\alpha}$) which we will denote by $\mathcal{H}(s,\tau,\alpha)$. We have a decomposition

$$\mathcal{H}(s,\tau,\alpha) = \oplus_{\chi}\mathcal{H}(s,\tau,\alpha,\chi),$$

where $\mathcal{H}(s,\tau,\alpha,\chi)$ is the isotypic component of the character

$$\chi : H^1(\Gamma_F, Z^s(F_{\mathrm{sep}})) \to \mathbb{C}^{\times}.$$

Note that $\mathcal{H}(s,\tau,\alpha,\chi)$ is a well defined Hilbert space up to scaling by a phase factor. Namely, it is canonically isomorphic up to a phase factor to the space $L^2(\mathrm{Bun}^{\circ}_{G,s,\alpha}(X,\tau),\mathcal{L}_{\chi})$ of half-densities with values in $\mathcal{L}_{\chi}$, where $\mathcal{L}_{\chi}$ is the complex line bundle over $\mathrm{Bun}^{\circ}_{G,s,\alpha}(X,\tau)$ associated to the principal bundle $\mathrm{Bun}^{\circ}_{G,s,a}(X,\tau) \to \mathrm{Bun}^{\circ}_{G,s,\alpha}(X,\tau)$ via the character χ (namely, the line bundle $\mathcal{L}_{\chi}$ is independent of a up to scaling by a phase factor).

We would like to define the action of commuting **Hecke operators** on $\mathcal{H}(s,\tau,\alpha)$ and to find the joint spectral decomposition of the algebra generated by these operators, see [EFK2]. In more down-to-earth terms, these should be operators on the Hilbert space $\mathcal{H}(s,\tau,a)$ for any fixed choice of the representative a of α which commute with the action of $H^1(\Gamma_F, Z^s(F_{\mathrm{sep}}))$, i.e., preserve the spaces $\mathcal{H}(s,\tau,\alpha,\chi)$. We view the eigenfunctions of these operators as the automorphic forms in the setting of the analytic Langlands correspondence.

The Hecke operators are defined as follows. Let Λ be the coweight lattice of G and $\lambda \in \Lambda^+$ be a dominant coweight. Then we define a variety $\mathcal{Z}_\lambda$ called the (regularly stable) **Hecke correspondence** equipped with a map

$$(p_1, p_2, q) : \mathcal{Z}_\lambda \to \mathrm{Bun}_G^\circ(X)^2 \times X,$$

see [BD1]. Namely, $\mathcal{Z}_\lambda$ consists of triples $T = (P_1, P_2, x)$ where $P_1, P_2 \in \mathrm{Bun}_G^\circ(X)$ are principal G-bundles on X which are identified outside x and are in relative position λ at x (see [EFK2], p.4), and $p_i(T) = P_i$, $q(T) = x$. For $x \in X(F)$ let $\mathcal{Z}_{\lambda,s,\tau,x}$ be the set of pairs (P_1, P_2) such that $(P_1, P_2, x) \in \mathcal{Z}_\lambda(F_{\mathrm{sep}})$ and P_1, P_2 are F-rational with respect to (s, τ), i.e., belong to $\mathrm{Bun}_{G,s}(X, \tau)$.[5] It is easy to see that if $(P_1, P_2) \in \mathcal{Z}_{\lambda,s,\tau,x}$ then $\alpha_{P_1} = \alpha_{P_2}$. Thus we may "define" the Hecke operator $H_{x,\lambda}$ on $\mathcal{H}(s, \tau, \alpha)$ by the formula

$$(H_{x,\lambda}\psi)(P) = \int_{\mathcal{Z}_{\lambda,s,\tau,x}(P)} \psi(Q)\, \|dQ\|\,,$$

where $\mathcal{Z}_{\lambda,s,\tau,x}(P) := \{Q : (P, Q) \in \mathcal{Z}_{\lambda,s,\tau,x}\}$ and dQ is an appropriate canonically defined algebraic volume element introduced by Beilinson and Drinfeld in [BD1], see also [EFK2], Theorem 1.1. If $(\lambda, \rho_G) \in \mathbb{Z} + \frac{1}{2}$, where ρ_G is the half-sum of positive roots of G, then dQ depends on a choice of a spin structure on X, but in any case $\|dQ\|$ is independent of choices. Thus the domain of integration $\mathcal{Z}_{\lambda,s,\tau,x}(P)$ is (non-canonically) isomorphic to the set $\mathrm{Gr}^\lambda_{G,\sigma}(F)$ of fixed points of the cell Gr^λ_G in the affine Grassmannian Gr_G over F_{sep} under the Γ_F-action corresponding to the F-form G^σ of G defined by (P, x) ([EFK2], Introduction and Section 5). The element dQ is a $-(\lambda, \rho_G)$-form on X, thus the family of operators $\{H_{x,\lambda}, x \in X(F)\}$ is an operator-valued $-(\lambda, \rho_G)$-density on $X(F)$.

The word "define" is in quotation marks because we can prove the convergence of these integrals only in some special cases. However, we expect that if ψ is a smooth function (locally constant in the non-archimedean case) compactly supported in the locus of sufficiently generic bundles then the integral $H_{x,\lambda}\psi$ is convergent and defines a function whose restriction to the open dense subset of **very stable**[6] bundles is smooth. This is indeed easy to see when

[5] For $\mathcal{Z}_{\lambda,s,\tau,x}$ to be non-empty, λ must be invariant under the action of Γ_F on Δ_G via s.

[6] A G-bundle P on X is said to be very stable if it does not admit a nonzero nilpotent Higgs field, i.e. a section of $\Omega^1(X, \mathrm{ad}\, P)$ taking values in the nilpotent cone of $\mathrm{Lie}\, G$. It is known that very stable bundles are stable and form a dense open set in the variety of stable bundles, see [Z] and references therein.

the genus g is big with respect to λ, so that the codimension of the unstable locus in the moduli stack of G-bundles is bigger than $\dim \mathrm{Gr}_G^\lambda = 2(\lambda, \rho_G)$.

This makes the operator $H_{x,\lambda}$ at least densely defined on $\mathcal{H}(s,\tau,\alpha)$, although it is not obvious that it lands in $\mathcal{H}(s,\tau,\alpha)$. We have conjectured in [EFK2, EFK3] (see also [BK2]) that in fact it does, and moreover it extends to a bounded operator which is norm-continuous in x (see Conjecture 2.16 below).

If so, then one can show that the operators $H_{x,\lambda}$ are normal (with $H_{x,\lambda}^\dagger = H_{x,\lambda^*}$), commute for different x and λ (as Hecke modifications at different points are done independently), and

$$H_{x,\lambda_1} H_{x,\lambda_2} = H_{x,\lambda_1+\lambda_2}.$$

2.9. Hecke operators for effective divisors with coefficients in the coweight lattice

It is possible that $X(F) = \emptyset$, then there are no $x \in X(F)$, so we cannot define the operators $H_{x,\lambda}$. But we can make a generalization.[7] Namely, let $M(X,G)$ be the set of Γ_F-invariant (for the action defined by s) functions $\mu : X(F_{\mathrm{sep}}) \to \Lambda^+$ with finite support (i.e., effective divisors with coefficients in Λ). The set $M(X,G)$ is a commutative monoid graded by Λ^+: the degree $|\mu|$ is the sum of all values of μ. Let $M(X,G)[\lambda]$ be the part of $M(X,G)$ of degree $\lambda \in \Lambda^+$.

For $\mu \in M(X,G)$, let $\mathcal{Z}_{\mu,s,\tau}$ be the set of pairs (P_1,P_2) of F-rational bundles which are identified outside $\operatorname{supp}\mu$ and are in relative position $\mu(x)$ at each point $x \in \operatorname{supp}\mu$. Then we can define the Hecke operator

$$(H_\mu \psi)(P) = \int_{\mathcal{Z}_{\mu,s,\tau}(P)} \psi(Q) \left\| dQ \right\|,$$

where $\mathcal{Z}_{\mu,s,\tau}(P) := \{Q : (P,Q) \in \mathcal{Z}_{\mu,s,\tau}\}$. For example, if $x \in X(F)$ and $\mu = \lambda\delta_x$ then $H_\mu = H_{x,\lambda}$.

The set $M(X,G)[\lambda]$ is the direct limit of subsets

$$M(X,G,E)[\lambda] = \{\mu \in M(X,G)[\lambda] |\operatorname{supp}\mu \subset X(E)\}$$

over finite Galois extensions $F \subset E \subset F_{\mathrm{sep}}$. Note that $M(X,G,E)[\lambda]$ has a natural topology induced by the topology of F, so we have the topology of

[7] In the case of $X = \mathbb{P}^1$ with ramification points and $G = PGL_2$, this generalization is discussed in [EFK3], Subsection 3.12.

direct limit on $M(X,G)[\lambda]$ for each λ, hence on $M(X,G)$. We expect that the operator H_μ is norm-continuous with respect to μ in this topology.

Moreover, let $x \in X_E$ for a finite extension $F \subset E \subset F_{\rm sep}$ and $\mu_x := \sum_{\gamma\in\Gamma_F/\Gamma_E}\gamma(\lambda\delta_x)$ for some Γ_E-invariant coweight $\lambda \in \Lambda^+$. Then H_{μ_x} is an operator-valued $-(\lambda,\rho_G)$-density on the 1-dimensional analytic E-manifold X_E.

These operators are expected to have similar properties to $H_{x,\lambda}$ for $x \in X(F)$ (bounded, normal with $H_\mu^\dagger = H_{\mu^*}$, $H_{\mu_1}H_{\mu_2} = H_{\mu_1+\mu_2}$), except that they always exist, as $X(F_{\rm sep}) \neq \emptyset$.

Remark 2.15. Let C be a finite central subgroup of G defined over F (for example, we can take $C = Z$ if G is semisimple). Then in genus zero the Hecke operators H_μ make sense more generally, when $\mu \in M(X,G/C)[\lambda]$ with $\lambda \in \Lambda_+$.

2.10. The spectral decomposition

As we have mentioned, the main problem of the analytic Langlands correspondence is to describe the joint spectral decomposition of the Hecke operators H_μ.

The following conjecture was made in [EFK2] (see Conjecture 1.2 and Corollary 1.3).

Conjecture 2.16. If G is semisimple and μ is nonzero on every simple factor then the operator H_μ is compact, and the intersection of the kernels of H_μ (for various μ) is zero. Thus the spectrum of $\{H_\mu\}$ is discrete, i.e., we have an orthogonal decomposition

$$\mathcal{H}(s,\tau,\alpha) = \oplus_k \mathcal{H}(s,\tau,\alpha)_k,$$

where $\mathcal{H}(s,\tau,\alpha)_k$ are finite dimensional joint eigenspaces.

2.11. The abelian case

Consider the abelian case, i.e., $G = \mathbb{G}_m^n$ is a torus. Then an inner class (or, equivalently, an F-form) of G is defined by a homomorphism with finite image $s : \Gamma_F \to GL_n(\mathbb{Z})$, and all bundles are stable with the automorphism group $G = Z$. Thus $\mathrm{Bun}^\circ_{G,s}(X,\tau)$ is the group $\mathrm{Pic}(X)^n(F_{\rm sep})^{\Gamma_F}$ where Γ_F acts via (s,τ). We have a homomorphism

$$\varphi : \mathrm{Pic}(X)^n(F_{\rm sep})^{\Gamma_F} \to H^2(\Gamma_F, Z^s(F_{\rm sep})),$$

so we may consider the fibers

$$\mathrm{Pic}(X)^n(F_{\mathrm{sep}})^{\Gamma_F}_{\alpha} := \varphi^{-1}(\alpha),$$

and

$$\mathcal{H}(s,\tau,\alpha,\chi) = L^2(\mathrm{Pic}(X)^n(F_{\mathrm{sep}})^{\Gamma_F}_{\alpha}, \mathcal{L}_\chi).$$

The Hecke operators act by translations. Thus the spectral decomposition in this case is just the decomposition of $L^2(\mathrm{Pic}(X)^n(F_{\mathrm{sep}})^{\Gamma_F}_{\alpha}, \mathcal{L}_\chi)$ into the characters of some finite index subgroup of $\mathrm{Pic}(X)^n(F_{\mathrm{sep}})^{\Gamma_F}_0 = \mathrm{Ker}\,\varphi$.

The group $\mathrm{Pic}(X)^n(F_{\mathrm{sep}})^{\Gamma_F}$ is isomorphic (albeit non-canonically) to the product of the compact group $\mathrm{Pic}_0(X)^n(F_{\mathrm{sep}})^{\Gamma_F}$ and a lattice. So the problem boils down to finding the character group of $\mathrm{Pic}_0(X)^n(F_{\mathrm{sep}})^{\Gamma_F}$. In the archimedean case, this reduces to classical Fourier analysis, so let us consider the non-archimedean case.

Example 2.17. Suppose $n = 1, s = 1$ and $\mathrm{char}(F) = 0$, so that F is an extension of $\mathbb{Q}_p$ of some degree m. Then we need to describe the character group of $J(F)$ where $J := \mathrm{Pic}_0(X)$. Let $\mathcal{O}_F$ be the ring of integers in F, $\mathfrak{m} \subset \mathcal{O}_F$ the maximal ideal, and $\mathbb{F}_q = \mathcal{O}_F/\mathfrak{m}$ the residue field of characteristic p. Suppose that X has a good reduction X_0 over $\mathbb{F}_q$. In this case by Hensel's lemma, $J(F) = J(\mathcal{O}_F)$ and thus we have a short exact sequence

$$0 \to J(\mathfrak{m}) \to J(F) \to J(\mathbb{F}_q) \to 0,$$

which implies that we have a short exact sequence of character groups

$$0 \to J(\mathbb{F}_q)^\vee \to J(F)^\vee \to J(\mathfrak{m})^\vee \to 0.$$

The group $J(\mathfrak{m})$ is the formal Lie group attached to J over $\mathcal{O}_F$, which is a finitely generated $\mathbb{Z}_p$-module of rank $m\mathrm{g}$. So it has a (non-canonical) decomposition

$$J(\mathfrak{m}) \cong \mathrm{Tors}(J(\mathfrak{m})) \oplus \mathbb{Z}_p^{m\mathrm{g}},$$

where $\mathrm{Tors}(J(\mathfrak{m}))$ is a finite abelian p-group whose exact structure depends on the arithmetic of F and X. Thus

$$J(\mathfrak{m})^\vee \cong \mathrm{Tors}(J(\mathfrak{m}))^\vee \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{m\mathrm{g}}.$$

On the other hand, the group $J(\mathbb{F}_q)^\vee$ can be described as usual through global class field theory of the function field $\mathbb{F}_q(X_0)$.

Remark 2.18. In general, recall that G is isogenous to a product of a torus and a semisimple group. Thus the problem of computing the spectrum of Hecke operators for a general G effectively reduces to the case of semisimple G, once the abelian case has been understood.

2.12. The archimedean case

Consider now the case when $F = \mathbb{R}$ or $F = \mathbb{C}$. In this case we have the **quantum Hitchin system** of Beilinson and Drinfeld ([BD1]). This is a commutative algebra $\mathcal{D}$ consisting of global differential operators acting on a square root of the canonical bundle on $\mathrm{Bun}_G(X)$, which is naturally isomorphic to the algebra of polynomial functions on the affine space $\mathrm{Op}_{\mathfrak{g}^\vee}$ of **Beilinson-Drinfeld opers** for the Langlands dual Lie algebra $\mathfrak{g}^\vee$ of $\mathfrak{g} = \mathrm{Lie}\, G$. The algebra $\mathcal{D}$ acts by unbounded (i.e., densely defined) operators on $\mathcal{H}(s, \tau, \alpha)$, namely, on the subspace of smooth half-densities with compact support. Moreover, for $F = \mathbb{C}$ there is a similar action of the complex conjugate algebra $\overline{\mathcal{D}}$ which commutes with $\mathcal{D}$, so we get an action of $\mathcal{A} := \mathcal{D} \otimes \overline{\mathcal{D}}$.

Furthermore, these algebras commute in an algebraic sense with Hecke operators, i.e., the Schwartz kernels of the Hecke operators satisfy appropriate differential equations, see [EFK2].[8] It is moreover expected that they commute in a stronger, analytic sense. Namely, we conjecture that the operators from $\mathcal{D}$ (and in the complex case, $\overline{\mathcal{D}}$) have canonical normal extensions which strongly commute with each other and with Hecke operators, thereby having a common spectral decomposition.

2.13. The Schwartz space

Let G be a semisimple group.

1. Assume first that F is a non-archimedean local field. Let ϕ be a locally constant $\mathcal{L}_\chi$-valued half-density on $\mathrm{Bun}^\circ_{G,s,\alpha}(X, \tau)$ supported at sufficiently generic bundles. We conjecture that the space $\mathcal{S}(\phi)$ generated by ϕ under the action of Hecke operators is finite dimensional. Then $\mathcal{S}(\phi)$ a direct sum of eigenspaces of Hecke operators.[9] So we may consider the **Schwartz space**

[8]More precisely, these differential equations are proved in [EFK2] for $F = \mathbb{C}$, but the same argument applies to $F = \mathbb{R}$, except that we get just a single equation rather than two complex conjugate equations.

[9]Since the action of Hecke operators on these finite dimensional spaces is given by matrices with algebraic entries, this would imply that all eigenvalues of Hecke operators are algebraic numbers, as conjectured by Kontsevich in [Ko], p. 19.

$\mathcal{S}(s,\tau,\alpha,\chi) = \sum_\phi \mathcal{S}(\phi)$. Equivalently, $\mathcal{S}(s,\tau,\alpha,\chi)$ is the space of smooth (=finite) vectors in $\mathcal{H}(s,\tau,\alpha,\chi)$, or the algebraic direct sum of joint eigenspaces of Hecke operators.

2. Now assume that F is archimedean. For $\lambda \in (\Lambda^+)^{\Gamma_F}$, let $\mathcal{H}_\lambda \subset \mathcal{H}$ be the (algebraic) sum of the images of H_μ with $|\mu| = \lambda$. Then we define the **Schwartz space** $\mathcal{S}(s,\tau,\alpha,\chi) \subset \mathcal{H}(s,\tau,\alpha,\chi)$ to be the intersection of $\mathcal{H}_\lambda$ over all λ; thus unlike the non-archimedean case, the Schwartz space is no longer countably dimensional, but it has a natural Fréchet topology defined by a collection of seminorms in which it is complete. It is clear that all eigenfunctions of Hecke operators belong to $\mathcal{S}(s,\tau,\alpha,\chi)$.

Moreover, we expect that the Schwartz space $\mathcal{S}(s,\tau,\alpha)$ is the intersection of domains of the operators from $\mathcal{D}$. In other words, it is the space of square integrable half-densities whose Fourier coefficients c_Λ with respect to an orthonormal eigenbasis ψ_Λ of the algebra $\mathcal{D}$ decay rapidly (faster than any polynomial) as a function of the eigenvalues Λ_i of generators $D_i \in \mathcal{D}$.

Remark 2.19. Another, more geometric definition of the Schwartz space based on the geometry of the stack of G-bundles on X, which is conjecturally equivalent to the above, is proposed in the non-archimedean case in [BK2], see also [BKP1], [BKP2]. A similar definition is expected to exist in the archimedean case.

2.14. The ramified case

The above picture has a generalization to the ramified case, i.e., the case of bundles with level structure along an F-rational effective divisor on X. This is, in particular, required to extend the analytic Langlands correspondence to curves of genus 0 and 1 (as such curves admit stable bundles only in the ramified setting). Before considering the ramified case we remind the following general construction.

Let F be a local field, Y be an analytic F-manifold, and $\mathbf{G}$ an analytic F-group acting properly and freely on Y. Let π be a unitary representation of $\mathbf{G}$. Let $V_\pi := (Y \times \pi)/\mathbf{G}$ be the associated Hilbert space bundle over the F-manifold $Y/\mathbf{G}$ (the action of $\mathbf{G}$ is given by $g(y,v) = (gy, gv)$). So we have the projection $V_\pi \to Y/\mathbf{G}$ with fiber π. Let ψ be a measurable half-density on $Y/\mathbf{G}$ with values in V_π. Then $|\psi|^2$ is a density on $Y/\mathbf{G}$. Let $\mathcal{H}(Y,\mathbf{G},\pi)$ be the Hilbert space of all ψ with $\int_{Y/\mathbf{G}} |\psi|^2 < \infty$.

More generally, if $\mathcal{L}$ is a $\mathbf{G}$-equivariant hermitian line bundle on Y (i.e., a hermitian line bundle on $Y/\mathbf{G}$), the we can define the Hilbert space $\mathcal{H}(Y,\mathbf{G},\pi,\mathcal{L})$ of $\mathcal{L}$-valued sections ψ with $\int_{Y/\mathbf{G}} |\psi|^2 < \infty$.

If $\mathbf{P} \subset \mathbf{G}$ is a closed subgroup and π a unitary representation of $\mathbf{P}$ then we have a canonical isomorphism

$$\mathcal{H}(Y, \mathbf{P}, \pi, \mathcal{L}) \cong \mathcal{H}(Y, \mathbf{G}, \mathrm{ind}_{\mathbf{P}}^{\mathbf{G}} \pi, \mathcal{L}),$$

where ind denotes unitary induction. Moreover, if π is a representation of a closed subgroup $\mathbf{P} \subset \mathbf{N} \subset N_{\mathbf{G}}(\mathbf{P})$ then this space carries a unitary action of $\mathbf{N}/\mathbf{P}$.

We are now ready to discuss the ramified case. The most general setting is as follows.

Let $t_1, \ldots, t_N$ be distinct points in $X(F_{\mathrm{sep}})$ and ζ_i be local parameters on X near t_i. Let $\ell \geq 1$. Denote by $\mathrm{Bun}_G^{\ell}(X, t_1, \ldots, t_N)$ the stack of principal G-bundles on X trivialized on the ℓ-th nilpotent neighborhood $D_\ell(t_i)$ of each t_i (i.e., modulo ζ_i^ℓ). We have a torsor $\mathrm{Bun}_G^{\ell}(X, t_1, \ldots, t_N) \to Bun_G(X)$ for the algebraic group $\prod_{i=1}^N \mathrm{Map}(D_\ell(t_i), G) \cong \prod_{i=1}^N G_\ell$, where for a commutative ring R, $G_\ell(R) := G(R[\zeta]/\zeta^\ell)$ (the identification is made using the local parameters ζ_i).

Now assume that t_i are permuted by Γ_F acting via τ. Let $\{t_i, i \in S\}, S \subset [1, n]$ be a set of representatives of orbits of this action. Then given $s : \Gamma_F \to \mathrm{Aut}\, \Delta_G$, we have an action of Γ_F on $\mathrm{Bun}_G^{\ell}(X, t_1, \ldots, t_N)(F_{\mathrm{sep}})$. Denote by $\mathrm{Bun}_{G,s}^{\ell\circ}(X, \tau, t_1, \ldots, t_N)$ the subset of regularly stable bundles fixed by this action. Let $F \subset E_i \subset F_{\mathrm{sep}}$ be the field of definition of t_i, i.e., $E_i = F_{\mathrm{sep}}^{\Gamma_{t_i}}$. Recall that in Subsection 2.6, to every $P \in \mathrm{Bun}_{G,s}^{\ell\circ}(X, \tau, t_1, \ldots, t_N)$ and each i we have attached an E_i-form G^{σ_i} of G in the inner class s. This form, in turn, defines an E_i-form $G_\ell^{\sigma_i}$ of G_ℓ. Let $\mathrm{Bun}_{G,s}^{\ell\circ}(X, \tau, t_1, \ldots, t_N, \sigma_1, \ldots, \sigma_N)$ be the subset of bundles defining the fixed forms $\sigma_1, \ldots, \sigma_N$ (clearly, if $\gamma t_i = t_j$ then $\gamma E_i = E_j$ and $\gamma \sigma_i = \sigma_j$). This is an analytic F-manifold with an action of $\mathbf{G} := \prod_{i \in S} G_\ell^{\sigma_i}(E_i)$, and, as in the unramified case, it is the disjoint union of open submanifolds $\mathrm{Bun}_{G,s,\alpha}^{\ell\circ}(X, \tau, t_1, \ldots, t_N, \sigma_1, \ldots, \sigma_N)$.

Fix a closed subgroup $\mathbf{P} \subset \mathbf{G}$ acting freely and properly on

$$\mathrm{Bun}_{G,s}^{\ell\circ}(X, \tau, t_1, \ldots, t_N, \sigma_1, \ldots, \sigma_N)$$

and let π be an irreducible unitary representation of $\mathbf{P}$. Analogously to the unramified case, define the Hilbert space

$$\mathcal{H} = \mathcal{H}(s, \tau, \alpha, \chi, t_1, \ldots, t_N, \pi) := \mathcal{H}(\mathrm{Bun}_{G,s,\alpha}^{\ell\circ}(X, \tau, t_1, \ldots, t_N, \sigma_1, \ldots, \sigma_N), \mathbf{P}, \pi, \mathcal{L}_\chi),$$

where $\mathcal{L}_\chi$ is the line bundle defined in Subsection 2.8 (it is clear that this space does not change if we replace the set of stable points by its $\mathbf{P}$-invariant dense open subset).

The space $\mathcal{H}$ (conjecturally) carries an action of Hecke operators H_μ where $\mu(t_i) = 0$ for all i, and the main question of the (ramified) analytic Langlands correspondence is to describe the joint spectrum of H_μ (which in general won't be discrete).[10] Note that if π is a unitary representation of a closed subgroup $\mathbf{P} \subset \mathbf{N} \subset N_{\mathbf{G}}(\mathbf{P})$ (for example, $\pi = \mathbb{C}$, $\mathbf{N} = N_{\mathbf{G}}(\mathbf{P})$), then this space carries a commuting action of $\mathbf{N}/\mathbf{P}$.

Moreover, in the archimedean case the space $\mathcal{H}$ carries a (densely defined) action of the quantum Hitchin system (extended from the unramified case to produce a quantum integrable system), which conjecturally commutes with the Hecke operators and thus has compatible spectral decomposition.[11]

Consider first the case $\mathrm{g} \geq 2$. In this case we may define stable bundles as above, to be the bundles stable in the usual sense with automorphism group Z. Then we may take $\mathbf{P}$ to be the entire group $\mathbf{G}$, so that $\pi = \bigotimes_{i \in S} \pi_i$, where π_i are irreducible unitary representations of $G_\ell^{\sigma_i}(E_i)$. For instance, for $\ell = 1$ (tame ramification) π_i are irreducible unitary representations of the groups of E_i-points of the reductive groups G^{σ_i}.

In the (generally simpler) case $\mathrm{g} \leq 1$ the situation is a bit more tricky since $\mathbf{G}$ does not act freely any more, so one has to take proper subgroups $\mathbf{P} \subset \mathbf{G}$ which act freely. For simplicity consider the tamely ramified case $\ell = 1$. For $i \in S$ let $\mathbf{P}_i$ be cocompact closed subgroups of $\mathbf{G}_i := G^{\sigma_i}(E_i)$ and π_i be finite dimensional unitary representations of $\mathbf{P}_i$. Let $\mathbf{P} := \prod_{i \in S} \mathbf{P}_i$ and

[10]The definition of Hecke operators is the same as in [EFK2] in the unramified case, using the Beilinson-Drinfeld isomorphism a from [EFK2], Theorem 1.1. Ramification data does not alter this definition. However, the definition is still conjectural because of the analytic issues (landing in L^2, boundedness) which are present already in the unramified case and are discussed in [EFK2].

[11]Recall that the moduli stack $\mathrm{Bun}_G^\ell(X, t_1, \dots, t_N)$ can be represented as a double quotient

$$\mathrm{Bun}_G^\ell(X, t_1, \dots, t_N) = \mathrm{Ker}(G[[t]] \to G_\ell)^N \backslash G((t))^N / G(\mathcal{O}(X)),$$

and the quantum Hitchin system is obtained by reduction of two-sided invariant differential operators on the Kac-Moody central extension of $G((t))^N$ at the critical level (the Feigin-Frenkel center of the Kac-Moody algebra) down to the double quotient. Because of the critical level, we get commuting differential operators acting on half-forms on $\mathrm{Bun}_G^\ell(X, t_1, \dots, t_N)$, and Beilinson and Drinfeld showed that they form a quantum integrable system (i.e., the number of algebraically independent commuting operators is the maximal possible, namely $\dim \mathrm{Bun}_G^\ell(X, t_1, \dots, t_N)$).

$\pi := \bigotimes_{i\in S}\pi_i$. If $\mathbf{P}$ acts freely then we may define the space $\mathcal{H}$ as above and we expect the spectrum of Hecke operators on this space to be discrete.

Example 2.20. 1. $\mathbf{P}_i = P_i(E_i)$ for parabolic subgroups $P_i \subset G^{\sigma_i}$ defined over E_i; this corresponds to doing harmonic analysis on the moduli space of bundles with parabolic structures. For instance, if $P_i = B_i$ are Borel subgroups then we must have $\sigma_i = s$ for all i.

Another extreme is $P_i = G^{\sigma_i}$ and $\pi_i = \mathbb{C}$. Then we recover the unramified case discussed above.

2. $\mathbf{G}_i$ are compact. In this case we may take $\mathbf{P}_i = 1$, so $\pi = \mathbb{C}$, and the space $\mathcal{H}$ carries a commuting action of the compact group $\mathbf{G}$, so we have

$$\mathcal{H} = \oplus_{\rho\in\widehat{\mathbf{G}}}\mathcal{H}_\rho \otimes \rho^*$$

where $\rho = \bigotimes_{i\in S}\rho_i$ for some irreducibles ρ_i of $\mathbf{G}_i$, and $\mathcal{H}_\rho := (\mathcal{H}\otimes\rho)^{\mathbf{G}}$. In this case the Hecke operators act on $\mathcal{H}_\rho$ for each ρ.

Of course, these two examples can also be combined, with (1) occurring at some of the points t_i and (2) at other ones.

2.15. The ramified genus 0 case

In the case $\mathrm{g} = 0$, i.e., $X = \mathbb{P}^1$, the space $\mathcal{H}$ can be described entirely in terms of Lie theory. Indeed, suppose that the group G is semisimple and simply connected and $\ell = 1$ (the tamely ramified case). In this case, it suffices to consider only trivial G-bundles (with trivializations at t_i). Thus the moduli space is non-empty only if $\sigma_i = \sigma|_{\Gamma_{E_i}}$ for all i up to conjugation. In this case, we have a natural inclusion $G^\sigma(F) \hookrightarrow G^{\sigma_i}(E_i)$, and the moduli space looks like $(\prod_{i\in S} G^{\sigma_i}(E_i))/G^\sigma(F)$. Thus

$$\mathcal{H} = \mathcal{H}\left(\left(\prod_{i\in S} G^{\sigma_i}(E_i)\right)/G^\sigma(F), \prod_{i\in S}\mathbf{P}_i, \bigotimes_{i\in S}\pi_i\right).$$

This picture extends in an obvious way to the case of wild ramification.[12]

For instance, in Example 2.20(2), we have

$$\mathcal{H}_\rho = \left(\bigotimes_{i\in S}\rho_i\right)^{G^\sigma(F)},$$

[12] The parameters α and χ do not appear here, since the automorphism group of the trivial G-bundle trivialized on a a non-empty subset of X is trivial.

a finite dimensional space. In this case, all the analytic issues disappear and the spectral problem is automatically well defined.

Example 2.21. Let $\ell = 1$, $\mathrm{g} = 0$, $N \geq 2$, $t_i \in X(F)$, $\sigma_i = \sigma$ for all $1 \leq i \leq N$. Let $\mathbf{P}_i = 1$, $\pi_i = \mathbb{C}$. Then

$$\mathcal{H} = L^2(G^\sigma(F)^N / G^\sigma(F)_{\text{diagonal}}).$$

So the spectral decomposition of $\mathcal{H}$ is an extension of the Plancherel formula for $G^\sigma(F)$. Namely, we have

(2.4)

$$\mathcal{H} = \int^{\oplus}_{\widehat{G^\sigma(F)}^N} \mathrm{Mult}_{G^\sigma(F)}(V_N^*, V_1 \otimes \cdots \otimes V_{N-1}) \otimes (V_1^* \otimes \cdots \otimes V_N^*) d\mu(V_1) \cdots d\mu(V_N),$$

where $d\mu(V)$ is the Plancherel measure of $G^\sigma(F)$ and $\mathrm{Mult}_{G^\sigma(F)}(L, M)$ is the multiplicity space[13] of an irreducible tempered representation L of $G^\sigma(F)$ in a tempered representation M, defined using Plancherel theory (see [D], Subsection 18.8).[14] For example, for $N = 2$ by Schur's lemma the multiplicity space vanishes unless $V_1 \cong V_2^*$, in which case it is 1-dimensional, so formula (2.4) takes the form

$$\mathcal{H} = \int^{\oplus}_{\widehat{G^\sigma(F)}} (V^* \otimes V) d\mu(V),$$

which is the usual Plancherel formula for $L^2(G^\sigma(F))$.

Each multiplicity space

$$\mathcal{H}_{V_1,\ldots,V_N} := \mathrm{Mult}_{G^\sigma(F)}(V_N^*, V_1 \otimes \cdots \otimes V_{N-1}) \tag{2.5}$$

carries an action of commuting Hecke operators. For example, as follows from [EFK3], Section 3, if $G = PGL_2$, G^σ is the split form, and $x \in \mathbb{P}^1(F)$, then the Hecke operator has the form

$$H_x \psi = \int_F \left(\begin{pmatrix} 0 & x - t_1 \\ 1 & -s \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0 & x - t_{N-1} \\ 1 & -s \end{pmatrix} \right)^{-1} \psi \, \|ds\| \tag{2.6}$$

[13] The multiplicity space $\mathrm{Mult}_{G^\sigma(F)}(L, M)$ is a generalization of $\mathrm{Hom}_{G^\sigma(F)}(L, M)$ in presence of continuous spectrum. Namely, it coincides with the latter when M is a Hilbert direct sum (finite or infinite) of irreducible unitary representations.

[14] Note that the representation $V_1 \otimes \cdots \otimes V_N$ is tempered, so does not contain the trivial representation in its spectrum. Hence in the L^2 context we cannot talk about $\mathrm{Mult}_{G^\sigma(F)}(\mathbb{C}, V_1 \otimes \cdots \otimes V_N)$. This is why we break the symmetry and dualize one of the factors. However, it can be shown that the result does not depend on the choice of this factor.

where the tensor product acts in $V_1 \otimes \cdots \otimes V_{N-1}$.

Remark 2.22. One may take V_i to be **projective** representations of G^σ (i.e., replace G^σ with a central extension), as long as the product of Schur multipliers attached to them equals 1. For instance, in Example 2.21 we may replace $SL_2(\mathbb{R})$ by its universal cover $\widetilde{SL_2(\mathbb{R})}$. For an example of this see Remark 5.14 below.

2.16. The ramified genus 0 case for PGL_2

2.16.1. Hypergeometric integrals Let F be a local field and $a, b \in i\mathbb{R}$. For $\alpha \in \mathbb{C}$, $\mathrm{Re}\,\alpha > 0$ let

$$K(\alpha) := \int_{v \in F : \|v\| \le 1} \|v\|^{\alpha-1} \|dv\|.$$

Under suitable normalization of the Lebesgue measure on F we have $K(\alpha) = \frac{1}{\alpha}$ if F is archimedean and $K(\alpha) = \frac{\log q}{q^\alpha - 1}$ if not, where q is the order of the residue field of F.

For $\mathrm{Re}\,\alpha = 0$ this integral does not converge (even conditionally). But we can regularize it if $\alpha \neq 0$ by using an ϵ-deformation:

$$\int_{v \in F : \|v\| \le 1} \|v\|^{\alpha-1} \|dv\| := \lim_{\epsilon \to 0+} \int_{v \in F : \|v\| \le 1} \|v\|^{\alpha+\epsilon-1} \|dv\|. \tag{2.7}$$

Denote by Γ^F the **Γ-function** of F, which is the meromorphic function on $\mathbb{C}$ defined by the condition that

$$\mathcal{F} \|u\|^{a-1} = \Gamma^F(a) \|u\|^{-a},$$

where $\mathcal{F}$ is the Fourier transform on F (so $\Gamma^F(a)\Gamma^F(1-a) = 1$).[15]

Recall the **beta integral**

$$\int_F \|s\|^{\alpha-1} \|s-1\|^{\beta-1} \|ds\| = B^F(\alpha, \beta) := \frac{\Gamma^F(\alpha)\Gamma^F(\beta)}{\Gamma^F(\alpha+\beta)}.$$

Thus

$$B^F(\alpha, \beta) = B^F(\beta, \alpha) = B^F(\alpha, 1-\alpha-\beta).$$

[15] Here for simplicity we consider the Γ-function for unramified characters, but all the formulas extend straightforwardly to general multiplicative characters of F.

Now consider the integral

$$I(x) := \int_{s\in F:\|s\|\geq 1} \|s\|^{\alpha-1}\|s-x\|^{\beta-1}\|ds\|$$

where $\operatorname{Re}\alpha=0$, $0<\operatorname{Re}\beta<1$. Setting $s=xv^{-1}$, we get

$$I(x)=\|x\|^{\alpha+\beta-1}\int_{\|v\|\leq\|x\|}\|v\|^{-\alpha-\beta}\|v-1\|^{\beta-1}\|dv\|.$$

Thus $I(x)$ depends only on $\|x\|$.

The following lemma is a generalization of [EFK3], Lemma 8.1.

Lemma 2.23. *If $\|\cdot\|^\alpha\neq 1$ then we have*

$$I(x)=K(-\alpha)\|x\|^{\beta-1}+B^F(\alpha,\beta)\|x\|^{\alpha+\beta-1}+o(\|x\|^{\operatorname{Re}\beta-1}),\ \ x\to\infty.$$

Proof. We write

$$\|x\|^{1-\alpha-\beta}I(x)=\int_{\|v\|\leq R}\|v\|^{-\alpha-\beta}\|v-1\|^{\beta-1}\|dv\|$$
$$+\int_{R<\|v\|\leq\|x\|}\|v\|^{-\alpha-\beta}\|v-1\|^{\beta-1}\|dv\|=\int_{\|v\|\leq R}\|v\|^{-\alpha-\beta}\|v-1\|^{\beta-1}\|dv\|$$
$$+\int_{R<\|v\|\leq\|x\|}\left(\left\|1-v^{-1}\right\|^{\beta-1}-1\right)\|v\|^{-\alpha-1}\|dv\|+\int_{R<\|v\|\leq\|x\|}\|v\|^{-\alpha-1}\|dv\|.$$

The first two integrals have a limit as $x\to\infty$, namely

$$C_R(\alpha,\beta):=\int_{\|v\|\leq R}\|v\|^{-\alpha-\beta}\|v-1\|^{\beta-1}\|dv\|$$
$$+\int_{R<\|v\|}\left(\left\|1-v^{-1}\right\|^{\beta-1}-1\right)\|v\|^{-\alpha-1}\|dv\|,$$

while the last integral equals

$$K(-\alpha)(\|x\|^{-\alpha}-R^{-\alpha}).$$

So we get

$$I(x)=K(-\alpha)\|x\|^{\beta-1}+(C_R(\alpha,\beta)-K(-\alpha)R^{-\alpha})\|x\|^{\alpha+\beta-1}+o(\|x\|^{\operatorname{Re}\beta-1})$$

as $x\to\infty$. Thus $C_R(\alpha,\beta)-K(-\alpha)R^{-\alpha}$ is independent of R. In fact, it is easy to show that for any R,

$$C_R(\alpha,\beta)-K(-\alpha)R^{-\alpha}=B^F(\alpha,\beta).$$

This implies the result. ☐

The following lemma is a generalization of [EFK3], Lemma 8.2.

Lemma 2.24. *Let α, β be as above and φ be a locally integrable function on $\mathbb{P}^1(F)$ which is smooth near ∞ and has power decay at* 0. *Let*

$$I(x, \varphi) := \int_F \varphi(s) \|s\|^{\alpha-1} \|s-x\|^{\beta-1} \|ds\|, \ \|x\| \gg 1.$$

Then if $\|\cdot\|^\alpha \neq 1$ then

$$I(x, \varphi) = \|x\|^{\beta-1} \int_F \varphi(s) \|s\|^{\alpha-1} \|ds\| + \varphi(\infty) B^F(\alpha, \beta) \|x\|^{\alpha+\beta-1} + o(\|x\|^{\mathrm{Re}\beta-1})$$

as $x \to \infty$, where the integral is understood in the sense of ϵ-deformation.

Proof. If $\varphi(\infty) = 0$, this follows by taking the limit directly, and if $\varphi = \mathbf{1}_{\|s\| \geq 1}$ is the indicator function then this is Lemma 2.23. So the result follows from the decomposition

$$\varphi(s) = (\varphi(s) - \varphi(\infty)\mathbf{1}_{\|s\|\geq 1}) + \varphi(\infty)\mathbf{1}_{\|s\|\geq 1}.$$ ☐

Now for $x \in F$, $x \neq 0, 1$, $0 < \mathrm{Re}\,\gamma < 1$, and consider the hypergeometric integral

$$\Phi^F(\alpha, \beta, \gamma; x) := \int_F \|s\|^{\alpha-\gamma} \|s-1\|^{\gamma-1} \|s-x\|^{\beta-1} \|ds\|.$$

Taking in Lemma 2.24 $\varphi(s) = \|1 - s^{-1}\|^{\gamma-1}$, where $0 < \mathrm{Re}\,\gamma < 1$, we get

Proposition 2.25.

$$\Phi^F(\alpha, \beta, \gamma; x) = B^F(-\alpha, \gamma) \|x\|^{\beta-1} + B^F(\alpha, \beta) \|x\|^{\alpha+\beta-1} + o(\|x\|^{\mathrm{Re}\beta-1})$$

as $x \to \infty$.

2.16.2. Principal series representations Let G be a split connected reductive algebraic group over a local field F, $B \subset G$ a Borel subgroup defined over F, and let $\mathbf{G} := G(F), \mathbf{B} := B(F)$. Then any integral weight μ of G defines a character $\mathbf{B} \to F^\times$, denoted $b \mapsto b^\mu$. Let $\rho := \rho_G$ be the sum of fundamental weights of G. Then densities on $\mathbf{G}/\mathbf{B}$ can be realized as functions $h : \mathbf{G} \to \mathbb{C}$ such that $h(gb) = \|b^{-2\rho}\| h(g)$, $g \in \mathbf{G}$, $b \in \mathbf{B}$.

Now let χ be a unitary character $\mathbf{B} \to \mathbb{C}^\times$ (or, equivalently, $\mathbf{T} \to \mathbb{C}^\times$, where $\mathbf{T} := \mathbf{B}/[\mathbf{B}, \mathbf{B}]$). Then we can define the **principal series representation** $M(\chi)$ of $\mathbf{G}$, which is the space of functions $f : \mathbf{G} \to \mathbb{C}$ such that

$$f(gb) = \chi(b) \left\| b^{-\rho} \right\| f(g), \ \ g \in \mathbf{G}, b \in \mathbf{B},$$

and $|f|^2$ is an L^1-density on $\mathbf{G}/\mathbf{B}$, with the action of $\mathbf{G}$ by left multiplication $((gf)(x) := f(g^{-1}x))$. Then $M(\chi)$ is a unitary representation of $\mathbf{G}$ with inner product

$$(f_1, f_2) = \int_{\mathbf{G}/\mathbf{B}} f_1 \overline{f}_2.$$

It is well known (see e.g. [ABV] for the archimedean cases and [Ca] for non-archimedean ones) that the representations $M(\chi)$ are irreducible and $M(\chi_1) \cong M(\chi_2)$ iff $\chi_1 = w\chi_2$ for some element w of the Weyl group W of G. These isomorphisms are nontrivial and are called **intertwining operators**; we discuss them below for $G = PGL_2$.

2.16.3. Principal series representations for $\boldsymbol{PGL_2}$ and intertwining operators Let us now consider the case $G = PGL_2$. Then $\mathbf{T} = F^\times$, so for $c \in i\mathbb{R}$ we may take $\chi_c(t) = \|t\|^c, t \in \mathbf{T}$, and define the **unramified (or spherical) principal series representation** $M(\chi_c)$. We will set $\lambda := -1+c$ and denote $M(\chi_c)$ by M_λ. Thus $M_\lambda = L^2(\mathbb{P}^1(F), \|K\|^{-\frac{\lambda}{2}})$, where K is the canonical bundle (as $c \in i\mathbb{R}$, this is naturally a Hilbert space with inner product $(f_1, f_2) = \int_{\mathbb{P}^1(F)} f_1 \overline{f}_2$).

It is well known that we have an isomorphism of unitary representations $\iota : M_\lambda \cong M_{-\lambda-2}$ such that $\iota^2 = \mathrm{Id}$, namely the **intertwining operator**

$$\iota : L^2(\mathbb{P}^1(F), \|K\|^{-\frac{\lambda}{2}}) \to L^2(\mathbb{P}^1(F), \|K\|^{\frac{\lambda+2}{2}})$$

given by

$$\iota = \mathcal{F} \circ \|\cdot\|^{\lambda+1} \circ \mathcal{F}^{-1},$$

where $\|\cdot\|^{\lambda+1}$ denotes the operator of multiplication by the function $\|\cdot\|^{\lambda+1}$. In other words, for smooth f vanishing near ∞ we have

$$(\iota f)(z) = \lim_{\epsilon \to 0+} \frac{1}{\Gamma^F(-\lambda - 1 + \epsilon)} \int_F f(w) \|z - w\|^{-\lambda-2+\epsilon} \|dzdw\|^{\frac{\lambda+2}{2}}. \tag{2.8}$$

Observe that this integral is absolutely convergent near the diagonal $z = w$

and the right hand side tends to the identity operator when $\lambda \to -1$.[16] Thus for $\lambda = -1$ we have $\iota = 1$.

The existence of ι implies that M_λ depends only on the Casimir eigenvalue $\frac{1}{2}(\lambda+1)^2 = \frac{c^2}{2}$.

Let $\lambda_j \in -1 + i\mathbb{R}$, $j = 0, \dots, m+1$, $V_j := M_{\lambda_j}$, $j \in [0, m+1]$ and $\boldsymbol{\lambda} := (\lambda_0, \dots, \lambda_m, \lambda_{m+1})$. Consider the Hilbert space

$$\mathcal{H}(\boldsymbol{\lambda}) := \mathcal{H}_{V_0,\dots,V_{m+1}} = \mathrm{Mult}_{PGL_2(F)}(M^*_{\lambda_{m+1}}, M_{\lambda_0} \otimes \cdots \otimes M_{\lambda_m}). \tag{2.9}$$

Similarly to [EFK3], Section 3.3, the space $\mathcal{H}(\boldsymbol{\lambda})$ may be realized as the space of functions $\psi(y_0, \dots, y_m)$ on F^{m+1} invariant under simultaneous translations $y_j \mapsto y_j + C$, homogeneous of degree $\frac{1}{2}(\sum_{j=0}^m \lambda_j - \lambda_{m+1})$, and specializing at $y_0 = 0, y_m = 1$ to square integrable functions on F^{m-1} (see e.g. [P, Re] for $F = \mathbb{R}$, [Na, Wi] for $F = \mathbb{C}$, [Ma] for a general local field). The inner product on $\mathcal{H}(\boldsymbol{\lambda})$ is given by

$$(\psi, \eta) = \int_{F^{m-1}} \psi(0, y_1, \dots, y_{m-1}, 1)\overline{\eta(0, y_1, \dots, y_{m-1}, 1)}\, \|dy_1 \cdots dy_{m-1}\|\,.$$

Note that if $\varepsilon = (\varepsilon_0, \dots, \varepsilon_m, \varepsilon_{m+1}) \in (\mathbb{Z}/2)^{m+2}$ is a collection of signs, and $(\varepsilon \circ \boldsymbol{\lambda})_j := \varepsilon_j(\lambda_j + 1) - 1$, then we have isomorphisms[17] $R_\varepsilon = R_\varepsilon^{\boldsymbol{\lambda}} : \mathcal{H}(\boldsymbol{\lambda}) \cong \mathcal{H}(\varepsilon \circ \boldsymbol{\lambda})$ given by composing the isomorophisms ι in all variables y_j for which $\varepsilon_j = -1$, and $R_{\varepsilon\varepsilon'} = R_\varepsilon R_{\varepsilon'}$. Thus we can think of the collection of spaces $\mathcal{H}(\boldsymbol{\lambda})$ attached to an orbit $\mathcal{O}$ of $(\mathbb{Z}/2)^{m+2}$ as a single Hilbert space $\mathcal{H}(\mathcal{O})$ with multiple realizations $\mathcal{H}(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \mathcal{O}$ connected by a consistent family of isomorphisms.

Furthermore, we can write an explicit formula for R_ε. For example, let us write an explicit formula for $R := R_{1,\dots,1,-1} : \mathcal{H}(\boldsymbol{\lambda}) \to \mathcal{H}(\boldsymbol{\lambda}')$, where $\boldsymbol{\lambda}' := (\lambda_0, \dots, \lambda_m, -\lambda_{m+1} - 2)$. In fact, we can write a more general formula for the operator

$$R : \mathrm{Mult}_{PGL_2(F)}(M^*_\lambda, V) \to \mathrm{Mult}_{PGL_2(F)}(M^*_{-\lambda-2}, V)$$

induced by ι for any tempered representation V of $PGL_2(F)$ (note that $R^2 = 1$, so $R^\dagger = R$). To this end, assume first that V is a direct sum of irreducible representations and realize elements of $\mathrm{Mult}_{PGL_2(F)}(M^*_\lambda, V)$ as

[16] Here we use the ϵ-deformation as in (2.7) to make sense of the divergent integral in (2.8).

[17] Here and below we will consider various operators with source $\mathcal{H}(\boldsymbol{\lambda})$. While these operators by definition depend on $\boldsymbol{\lambda}$, we will drop it from the notation when no confusion is possible.

V-valued (generalized) $-\frac{\lambda}{2}$-densities $v = v(y)\,\|dy\|^{-\frac{\lambda}{2}}$ on $\mathbb{P}^1(F)$ equivariant under $PGL_2(F)$. Then formula (2.8) implies that

$$Rv = \lim_{\epsilon\to 0+} \frac{1}{\Gamma^F(-\lambda-1+\epsilon)} \int_F v(s)\,\|y-s\|^{-\lambda-2+\epsilon}\,\|ds\|\,\|dy\|^{\frac{\lambda+2}{2}},$$

which we will write for brevity as

$$Rv = \frac{1}{\Gamma^F(-\lambda-1)} \int_F v(s)\,\|y-s\|^{-\lambda-2}\,\|ds\|\,\|dy\|^{\frac{\lambda+2}{2}}.$$

Let $\widehat{v} := v(u^{-1})\,\|u\|^{\lambda}\,\|du\|^{-\frac{\lambda}{2}}$ be the image of v under the change of coordinates $u = y^{-1}$, so $\widehat{v}(u) := v(u^{-1})\,\|u\|^{\lambda}$. Then

$$\widehat{Rv}(u) = \frac{1}{\Gamma^F(-\lambda-1)} \int_F v(s)\,\|1-su\|^{-\lambda-2}\,\|ds\|.$$

Specializing this at $u=0$, we have

$$\widehat{Rv}(0) = \frac{1}{\Gamma^F(-\lambda-1)} \int_F v(s)\,\|ds\| = \frac{1}{\Gamma^F(-\lambda-1)} \int_F \widehat{v}(s^{-1})\,\|s\|^{\lambda}\,\|ds\|.$$

But s^{-1} is obtained from 0 by the element $g(s) \in PGL_2(F)$ sending u to $u+s^{-1}$, i.e. y to $(y^{-1}+s^{-1})^{-1} = \frac{y}{s^{-1}y+1}$. Thus $g(s) = \left(\begin{smallmatrix} 1 & 0 \\ s^{-1} & 1 \end{smallmatrix}\right)$. So we get

$$\widehat{Rv}(0) = \frac{1}{\Gamma^F(-\lambda-1)} \int_F g(s)^{-1}\widehat{v}(0)\,\|s\|^{\lambda}\,\|ds\|.$$

This formula continues to hold if V is not necessarily a direct sum of irreducible representations but rather a direct integral. For example, in our situation $V = V_0 \otimes \cdots \otimes V_m$ is the space of translation invariant functions in $y_0, \ldots, y_m$ which are homogeneous of degree $\frac{1}{2}(\sum_{i=0}^m \lambda_i - \lambda_{m+1})$, so we get

$$R\psi = \frac{1}{\Gamma^F(-\lambda_{m+1}-1)} \int_F g(s)^{-1}\psi\,\|s\|^{\lambda_{m+1}}\,\|ds\|,$$

i.e.,

$$R\psi(y_0, \ldots, y_m) =$$
$$\tfrac{1}{\Gamma^F(-\lambda_{m+1}-1)} \int_F \psi\left(\tfrac{y_0 s}{s-y_0}, \ldots, \tfrac{y_0 s}{s-y_m}\right)\|s\|^{\lambda_{m+1}-\sum_{i=0}^m \lambda_i} \prod_{i=0}^m \|s-y_i\|^{\lambda_i}\,\|ds\| =$$

$$\frac{1}{\Gamma^F(-\lambda_{m+1}-1)}\int_F \psi\left(\frac{s^2}{s-y_0},\ldots,\frac{s^2}{s-y_m}\right)\|s\|^{\lambda_{m+1}-\sum_{i=0}^m \lambda_i}\prod_{i=0}^m \|s-y_i\|^{\lambda_i}\|ds\|.$$

We thus obtain the following lemma.

Lemma 2.26. *We have*[18]

$$(R\psi)(y_0,\ldots,y_m) = \frac{1}{\Gamma^F(-\lambda_{m+1}-1)}\int_F \psi\left(\frac{1}{s-y_0},\ldots,\frac{1}{s-y_m}\right)\prod_{i=0}^m \|s-y_i\|^{\lambda_i}\|ds\|.$$

Remark 2.27. Setting in Lemma 2.26 $t_0=y_0=0$ and $\varphi(y_1,\ldots,y_m):=\psi(0,y_1,\ldots,y_m)$, we obtain

$$(R\varphi)(y_1,\ldots,y_m) = \frac{1}{\Gamma^F(-\lambda_{m+1}-1)}\int_F \varphi\left(\frac{y_1}{1-y_1s^{-1}},\ldots,\frac{y_m}{1-y_ms^{-1}}\right) \times\prod_{j=1}^m \left\|1-y_js^{-1}\right\|^{\lambda_j}\|s\|^{\lambda_{m+1}}\|ds\|. \tag{2.10}$$

We will also need to consider the operator $Q=Q^{\boldsymbol{\lambda}}:\mathcal{H}(\boldsymbol{\lambda})\to\mathcal{H}(\boldsymbol{\lambda}')$ given by

$$(Q\psi)(y_0,\ldots,y_m) = \frac{1}{\Gamma^F(\lambda_{m+1}+1)}\int_F \psi(y_0-t_0s,\ldots,y_m-t_ms)\|s\|^{\lambda_{m+1}}\|ds\|.$$

Setting $t_0=y_0=0$ and $\varphi:=\psi|_{y_0=0}$, we have

$$(Q\varphi)(y_1,\ldots,y_m) = \frac{1}{\Gamma^F(\lambda_{m+1}+1)}\int_F \psi(y_1-t_1s,\ldots,y_m-t_ms)\|s\|^{\lambda_{m+1}}\|ds\|. \tag{2.11}$$

Recall that by [EFK3], Proposition 3.3,

$$S_0(y_1,\ldots,y_m)=\left(\tfrac{t_1}{y_1},\ldots,\tfrac{t_m}{y_m}\right).$$

Thus, setting $\boldsymbol{\lambda}^*:=(-\lambda_0-2,\lambda_1,\ldots,\lambda_m,-\lambda_{m+1}-2)$, we get the unitary involution

$$S_0:\mathcal{H}(\boldsymbol{\lambda})\to\mathcal{H}(\boldsymbol{\lambda}^*)$$

[18] Note that this integral is not convergent near $s=\infty$ and should be understood in the sense of ϵ-deformation in λ_{m+1}, as explained in footnote 16.

given by

$$(S_0\varphi)(y_1,\dots,y_m)=\prod_{j=1}^{m}\left\|\tfrac{t_j}{y_j^2}\right\|^{-\frac{\lambda_j}{2}}\varphi(\tfrac{t_1}{y_1},\dots,\tfrac{t_m}{y_m}),$$

and we see that $Q=S_0RS_0$. In fact, since for any $1\le i\le m$ the operator S_iS_0 commutes with R, we have $Q=S_iRS_i$ for any $0\le i\le m$. We also see that

$$Q^2=1,\ Q^\dagger=Q.$$

2.16.4. Formulas for Hecke operators Now let us study the Hecke operators on the Hilbert space $\mathcal{H}_{V_0,\dots,V_{m+1}}$. This is the setting of the **tamely ramified analytic Langlands correspondence with parameters** λ_j for $G=PGL_2$ and $X=\mathbb{P}^1$ with $N=m+2$ ramification points. It generalizes the setting of [EFK3] where $\lambda_j=-1$ for all j, so that

$$\mathcal{H}_{V_0,\dots,V_{m+1}}=L^2(\mathrm{Bun}_G^\circ(X,t_0,\dots,t_{m+1}),\|K\|^{\frac{1}{2}})$$

is the space of square-integrable half-densities. Similarly to [EFK3], Subsection 3.3, we'll denote the ramification points by $t_0,\dots,t_{m+1}$ to align notation with [EFK3], and assume that $t_j\in X(F)$ for all j, while $t_{m+1}=\infty$.

First we would like to write a formula for the (modified) Hecke operator $\mathbb{H}_x$ that generalizes the formula for $\mathbb{H}_x$ from [EFK3], Proposition 3.9. Recall that we have two components of the moduli of bundles, Bun_0 and Bun_1 (bundles of even and odd degree), which are identified by the Hecke modification S_{m+1} at infinity. Thus, as in [EFK3], Section 3, we can use S_{m+1} to identify the sectors $\mathcal{H}^0,\mathcal{H}^1$ of the Hilbert space corresponding to bundles of degree 0 and 1. Then the modified Hecke operator $\mathbb{H}_x:\mathcal{H}^0\to\mathcal{H}^1$ can be written as an endomorphism of $\mathcal{H}^0$. In fact, it is convenient to write this endomorphism as an operator $\mathcal{H}(\boldsymbol{\lambda})\to\mathcal{H}(\boldsymbol{\lambda}')$. In other words, it maps homogeneous functions of degree $\frac{1}{2}(\sum_{j=0}^m\lambda_j-\lambda_{m+1})$ to homogeneous functions of degree $1+\frac{1}{2}\sum_{j=0}^{m+1}\lambda_j$.

Proposition 2.28. *The modified Hecke operator* $\mathbb{H}_x=\mathbb{H}_x^{\boldsymbol{\lambda}}:\mathcal{H}(\boldsymbol{\lambda})\to\mathcal{H}(\boldsymbol{\lambda}')$ *is given by the formula*

$$(\mathbb{H}_x\psi)(y_0,\dots,y_m)=\int_F\psi\left(\frac{t_0-x}{s-y_0},\dots,\frac{t_m-x}{s-y_m}\right)\prod_{j=0}^{m}\|s-y_j\|^{\lambda_j}\,\|ds\|\,.$$

The proof of Proposition 2.28 is parallel to the proof of [EFK3], Proposition 3.9.

We also have the ordinary (unmodified) Hecke operator $H_x : \mathcal{H}(\boldsymbol{\lambda}) \to \mathcal{H}(\boldsymbol{\lambda}')$, which differs from $\mathbb{H}_x$ by normalization:

$$H_x = \prod_{j=0}^{m} \|x - t_j\|^{-\frac{\lambda_j}{2}} \, \mathbb{H}_x.$$

This is a special case of formula (2.6). It is not hard to check that

$$R_\varepsilon \circ H_x = H_x \circ R_\varepsilon$$

when $\varepsilon_{m+1} = 1$; however, this does not hold for $\varepsilon_{m+1} = -1$ since we used S_{m+1} acting at $t_{m+1} = \infty$ to identify Bun_0 and Bun_1.

Remark 2.29. By setting $t_0 = y_0 = 0$ and $\varphi(y_1, \dots, y_m) := \psi(0, y_1, \dots, y_m)$, we obtain the following formula for the action of $\mathbb{H}_x$ on homogeneous functions of degree $\frac{1}{2}(\sum_{j=0}^m \lambda_j - \lambda_{m+1})$:

(2.12)
$$(\mathbb{H}_x\varphi)(y_1, \dots, y_m) = \int_F \varphi\left(\frac{t_1 s - x y_1}{s(s - y_1)}, \dots, \frac{t_m s - x y_m}{s(s - y_m)}\right) \prod_{j=0}^{m} \|s - y_j\|^{\lambda_j} \|ds\| .$$

This formula generalizes the formula of Theorem 3.6 in [EFK3].

We can further set $t_m = y_m = 1$ and get the following analog of [EFK3], Proposition 3.7. Let $\phi(y_1, \dots, y_{m-1}) = \varphi(y_1, \dots, y_{m-1}, 1) \in L^2(F^{m-1})$. Then

$$\begin{gathered}(\mathbb{H}_x\phi)(y_1, \dots, y_{m-1}) = \\ \int_F \phi\left(\frac{(t_1 s - x y_1)(s-1)}{(s - y_1)(s - x)}, \dots, \frac{(t_{m-1} s - x y_{m-1})(s-1)}{(s - y_{m-1})(s - x)}\right) \times \\ \left\|\frac{s(s-1)}{s-x}\right\|^{-\frac{1}{2}(\sum_{j=0}^m \lambda_j - \lambda_{m+1})} \prod_{j=0}^{m} \|s - y_j\|^{\lambda_j} \|ds\| .\end{gathered}$$

Similarly to [EFK3], Subsection 3.3, define the unitary operator $U_{s,x}$ on $L^2(F^{m-1})$ by

$$\begin{gathered}(U_{s,x}\phi)(y_1, \dots, y_{m-1}) := \\ \phi\left(\frac{(t_1 s - y_1 x)(s-1)}{(s - y_1)(s - x)}, \dots, \frac{(t_{m-1} s - y_{m-1} x)(s-1)}{(s - y_{m-1})(s - x)}\right) \times \\ \left\|\frac{s(s-1)}{s-x}\right\|^{-\frac{1}{2}\sum_{j=1}^{m-1} \lambda_j} \prod_{j=1}^{m-1} \left\|\frac{t_j - x}{(s - y_j)^2}\right\|^{-\frac{\lambda_j}{2}} .\end{gathered}$$

Then we get

$$(2.13)\quad H_x\phi = \|x\|^{-\frac{\lambda_0}{2}}\,\|x-1\|^{-\frac{\lambda_m}{2}} \times$$
$$\int_F \|s\|^{\frac{1}{2}(\lambda_0-\lambda_m+\lambda_{m+1})}\,\|s-1\|^{\frac{1}{2}(-\lambda_0+\lambda_m+\lambda_{m+1})}\,\|s-x\|^{\frac{1}{2}(\lambda_0+\lambda_m-\lambda_{m+1})}\,U_{s,x}\phi\,\|ds\|\,,$$

which generalizes [EFK3], Proposition 3.7. From this formula it follows by the argument of [EFK3], Proposition 3.10 that $\mathbb{H}_x$ is a bounded operator which depends norm-continuously on x for $x \neq t_j, \infty$.

2.16.5. Properties of Hecke operators The properties of the operator H_x are analogous to those in the untwisted case ([EFK3], Section 3). To avoid confusion, from now on the operators $R, Q, H_x : \mathcal{H}(\boldsymbol{\lambda}) \to \mathcal{H}(\boldsymbol{\lambda}')$ will be denoted by R_+, Q_+, H_{x+} and the operators $R, Q, H_x : \mathcal{H}(\boldsymbol{\lambda}') \to \mathcal{H}(\boldsymbol{\lambda})$ by R_-, Q_-, H_{x-} (thus $Q_+^\dagger = Q_-, R_+^\dagger = R_-$).

Proposition 2.30.

(i) The operators $H_{x,\pm}$, $x \in \mathbb{P}^1(F), x \neq t_j$ are compact.
(ii) $H_{x-}H_{y+} = H_{y-}H_{x+}$, $x, y \in \mathbb{P}^1(F), x, y \neq t_j$.
(iii) $H_{x+}^\dagger = H_{x-}$.

Proof. (i) is proved analogously to [EFK3], Proposition 3.13. (ii), (iii) are proved analogously to [EFK3], Proposition 3.11. (ii) can also be checked explicitly from the formula of Proposition 2.28 as explained in [EFK3], Remark 3.28. ∎

Define the **full Hecke operator** on $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1 = \mathcal{H}(\boldsymbol{\lambda}) \oplus \mathcal{H}(\boldsymbol{\lambda}')$ by the formula

$$H_{x,\mathrm{full}} = \begin{pmatrix} 0 & H_{x-} \\ H_{x+} & 0 \end{pmatrix}.$$

It follows that the operators $H_{x,\mathrm{full}}$ are self-adjoint and pairwise commuting.

2.16.6. Asymptotics of Hecke operators and the spectral decomposition Let us now discuss the asymptotics of Hecke operators as $x \to \infty$. Set $c := \lambda_{m+1} + 1$.

Proposition 2.31. *(i) If $c \neq 0$ then*

$$\|x\|^{-\frac{1}{2}}\,H_{x\pm} = \Gamma^F(\pm c)\|x\|^{\pm\frac{c}{2}}Q_\pm + \Gamma^F(\mp c)\|x\|^{\mp\frac{c}{2}}R_\pm + o(1),\ \ x \to \infty.$$

Thus

$$\|x\|^{-\frac{1}{2}} H_{x,\mathrm{full}} = \Gamma^F(c)\|x\|^{\frac{c}{2}} D + \Gamma^F(-c)\|x\|^{-\frac{c}{2}} D^\dagger + o(1),\ x \to \infty,$$

where

$$D := \begin{pmatrix} 0 & R_- \\ Q_+ & 0 \end{pmatrix}.$$

(ii) If $c = 0$ then one has

$$\|x\|^{-\frac{1}{2}} H_{x,\pm} = \log\|x\| + M + o(1),\ x \to \infty,$$

where

$$(M\varphi)(y_1,\dots,y_m) := \int_F \left(\varphi(y_1 - t_1 s,\dots,y_m - t_m s) + \frac{\varphi(\frac{y_1}{1-y_1 s^{-1}},\dots,\frac{y_m}{1-y_m s^{-1}})}{\prod_{j=1}^m \|1-y_j s^{-1}\|^{-\lambda_j}} - \varphi(y_1,\dots,y_m)\right)\left\|\frac{ds}{s}\right\|.$$

Note that Proposition 2.31(ii) generalizes [EFK3], Propositions 3.15(i) and 3.21.

Proof. (i) We follow the proof of [EFK3], Proposition 3.21. By (2.12) we have

$$\|x\|^{-\frac{1}{2}} (H_{x+}\varphi)(y_1,\dots,y_m) = \|x\|^{-\frac{c}{2}} \int_F \varphi\left(\frac{t_1 s x^{-1} - y_1}{1 - y_1 s^{-1}},\dots,\frac{t_m s x^{-1} - y_m}{1 - y_m s^{-1}}\right) \prod_{j=1}^m \left\|1 - y_j s^{-1}\right\|^{\lambda_j} \|s\|^c \left\|\frac{ds}{s}\right\|. \tag{2.14}$$

Now, as explained in the proof of [EFK3], Proposition 3.21, in the limit $x \to \infty$ the curve $Z_{x,\mathbf{y}}$ with parametrization $s \mapsto (\frac{t_1 s x^{-1} - y_1}{1-y_1 s^{-1}},\dots,\frac{t_m s x^{-1} - y_m}{1-y_m s^{-1}})$ along which we are integrating in (2.14) falls apart into two components corresponding to the regimes when $s = s(x)$ has a finite limit as $x \to \infty$ and when sx^{-1} has a finite limit when $x \to \infty$, respectively. As a result, similarly to the proof of [EFK3], Proposition 3.21, the integral (2.14) is asymptotic to the sum of two integrals over these components. Namely, we have

$$\|x\|^{-\frac{1}{2}} (H_{x+}\varphi)(y_1,\dots,y_m) = \|x\|^{-\frac{c}{2}} \int_F \varphi\left(\frac{y_1}{1 - y_1 s^{-1}},\dots,\frac{y_m}{1 - y_m s^{-1}}\right) \prod_{j=1}^m \left\|1 - y_j s^{-1}\right\|^{\lambda_j} \|s\|^c \left\|\frac{ds}{s}\right\| +$$

$$\|x\|^{\frac{c}{2}}\int_F \varphi(y_1-t_1s,\dots,y_m-t_ms)\,\|s\|^c\left\|\frac{ds}{s}\right\|+o(1),\ x\to\infty.$$

Now, the first integral is the operator $\Gamma(-c)R_+$ (formula (2.10)), while the second integral is the operator $\Gamma(c)Q_+$ (formula (2.11)), which implies the claimed asymptotics for H_{x+}. The asymptotics for H_{x-} is obtained by replacing c by $-c$.

(ii) follows from (i) by taking the limit $c\to 0$. Namely, write (i) in the form

$$\|x\|^{-\frac{1}{2}}(H_{x+}\varphi)(y_1,\dots,y_m)=$$

$$(\|x\|^{-\frac{c}{2}}-1)\int_F\varphi\left(\frac{y_1}{1-y_1s^{-1}},\dots,\frac{y_m}{1-y_ms^{-1}}\right)\prod_{j=1}^m\left\|1-y_js^{-1}\right\|^{\lambda_j}\|s\|^c\left\|\frac{ds}{s}\right\|+$$

$$(\|x\|^{\frac{c}{2}}-1)\int_F\varphi(y_1-t_1s,\dots,y_m-t_ms)\,\|s\|^c\left\|\frac{ds}{s}\right\|+$$

$$\int_F\left(\varphi(y_1-t_1s,\dots,y_m-t_ms)+\frac{\varphi(\frac{y_1}{1-y_1s^{-1}},\dots,\frac{y_m}{1-y_ms^{-1}})}{\prod_{j=1}^m\|1-y_js^{-1}\|^{-\lambda_j}}-\varphi(y_1,\dots,y_m)\right)\times$$

$$\|s\|^c\left\|\frac{ds}{s}\right\|+o(1),\ x\to\infty.$$

Now, each of the first two summands tends to $\frac{1}{2}\log\|x\|$ as $c\to 0$, while the third summand tends to M, as desired.[19] $\square$

Corollary 2.32. *(i) We have* $\cap_x \operatorname{Ker} H_{x\pm}=0$, $\cap_x\operatorname{Ker} H_{x,\mathrm{full}}=0$.

(ii) We have a spectral decomposition

$$\mathcal{H}=\oplus_k\mathcal{H}_k^{\pm},$$

where $\mathcal{H}_k^{\pm}$ *are finite dimensional joint eigenspaces of* $H_{x,\mathrm{full}}$ *with eigenvalues* $\pm\beta_k(x)$.

Proof. (i) It suffices to show that $\cap_x\operatorname{Ker} H_{x,\mathrm{full}}=0$. But this follows from Proposition 2.31 and the fact that $R_\pm, Q_\pm$ are unitary operators, hence so is D.

(ii) immediately follows from (i) and the compactness of $H_{x,\mathrm{full}}$. $\square$

Let $\pm\delta_k\in\mathbb{C}$ be the eigenvalue of D on $\mathcal{H}_k^{\pm}$, so $|\delta_k|=1$. We choose the signs so that $\arg\delta_k\in(-\frac{\pi}{2},\frac{\pi}{2}]$. Then Proposition 2.31 implies the following asymptotics for $\beta_k(x)$.

[19]Here it needs to be checked that the $o(1)$ term remains $o(1)$ as $c\to 0$. We leave this argument to the reader.

Corollary 2.33. *We have*

$$\beta_k(x) = \|x\|^{\frac{1}{2}} (2\,\mathrm{Re}(\delta_k \Gamma^F(c)\|x\|^{\frac{c}{2}}) + o(1)),\ x \to \infty.$$

So setting $\delta_k^* = e^{2\pi i\theta_k} := \frac{c\Gamma^F(c)}{|c\Gamma^F(c)|}\delta_k$, $-\pi < \theta_k \le \pi$, we get

$$\|x\|^{-\frac{1}{2}}\,\beta_k(x) = \frac{|c\Gamma^F(c)|}{c}(e^{2\pi i\theta_k}\|x\|^{\frac{c}{2}} - e^{-2\pi i\theta_k}\|x\|^{-\frac{c}{2}}) + o(1) =$$
$$\frac{|c\Gamma^F(c)|}{c}\Big(\cos\theta_k \cdot (\|x\|^{\frac{c}{2}} - \|x\|^{-\frac{c}{2}}) + i\sin\theta_k \cdot (\|x\|^{\frac{c}{2}} + \|x\|^{-\frac{c}{2}})\Big) + o(1),\ x \to \infty.$$

Thus, Proposition 2.31(ii) implies that

$$\theta_k = \frac{c}{2i}\mu^{(k)} + o(c),\ c \to 0,$$

where $\mu^{(k)} \in \mathbb{R}$ are the eigenvalues of M.

Thus we see that we have orthonormal bases $\{\mathbf{e}_k^0\}$ of $\mathcal{H}^0$ and $\{\mathbf{e}_k^1\}$ of $\mathcal{H}^1$ such that

$$Q_+\mathbf{e}_k^0 = \delta_k\mathbf{e}_k^1,\ Q_-\mathbf{e}_k^1 = \delta_k^{-1}\mathbf{e}_k^0,\ R_+\mathbf{e}_k^0 = \delta_k^{-1}\mathbf{e}_k^1,\ R_-\mathbf{e}_k^1 = \delta_k\mathbf{e}_k^0,$$

and the eigenvectors of $H_{x,\mathrm{full}}$ with eigenvalues $\pm\beta_k(x)$ are $\mathbf{e}_k^{\pm} = \mathbf{e}_k^0 \pm \mathbf{e}_k^1$. Thus the vectors $\mathbf{e}_k^0$ satisfy the equation

$$R_-H_{x+}\mathbf{e}_k^0 = \delta_k\beta_k(x)\mathbf{e}_k^0.$$

So the spectral problem for the operator $H_{x,\mathrm{full}}$ on $\mathcal{H}$ is equivalent to the spectral problem for R_-H_{x+} on $\mathcal{H}^0$, and the eigenvalues of R_-H_{x+} are

$$\widehat{\beta}_k(x) := \delta_k\beta_k(x) = \|x\|^{\frac{1}{2}}\,(\delta_k^2\Gamma(c)\,\|x\|^{\frac{c}{2}} + \Gamma(-c)\,\|x\|^{-\frac{c}{2}} + o(1)),\ x \to \infty.$$

Note that $\widehat{\beta}_k(x)$ do not depend on the above choice of signs for δ_k.

Example 2.34. Let $m = 1$ (3 points), then $H(\boldsymbol{\lambda}) \cong \mathbb{C}$ by sending φ to $\varphi(1)$. Let

$$\lambda_0 = -1 + a,\ \lambda_1 = -1 + b,\ t_1 = y_1 = 1.$$

Hence

$$Q_+ = \frac{1}{\Gamma^F(c)}\int_F \|1-s\|^{\frac{a+b-c-1}{2}}\,\|s\|^{c-1}\,\|ds\| = \frac{\Gamma^F(\frac{a+b-c+1}{2})}{\Gamma^F(\frac{a+b+c+1}{2})},$$

$$R_- = \frac{1}{\Gamma^F(c)} \int_F \|1-s\|^{\frac{-a+b-c-1}{2}} \|s\|^{\frac{a-b-c-1}{2}} \|ds\| = \frac{\Gamma^F(\frac{a-b-c+1}{2})}{\Gamma^F(\frac{a-b+c+1}{2})}.$$

Thus

$$\delta_k = \sqrt{\frac{\Gamma^F(\frac{a+b-c+1}{2})\Gamma^F(\frac{a-b-c+1}{2})}{\Gamma^F(\frac{a+b+c+1}{2})\Gamma^F(\frac{a-b+c+1}{2})}}.$$

Also by (2.13),

$$H_{x+} = \int_F \|s\|^{\frac{a-b+c-1}{2}} \|s-1\|^{\frac{-a+b+c-1}{2}} \|s-x\|^{\frac{a+b-c-1}{2}} \|ds\|,$$

so Proposition 2.31(i) takes the form

$$\int_F \|s\|^{\frac{a-b+c-1}{2}} \|s-1\|^{\frac{-a+b+c-1}{2}} \|s-x\|^{\frac{a+b-c-1}{2}} \|ds\| =$$
$$B^F(\tfrac{-a+b+c+1}{2}, \tfrac{a-b+c+1}{2})\|x\|^{\frac{a+b-c-1}{2}} + B^F(\tfrac{a+b-c+1}{2}, \tfrac{-a-b-c+1}{2})\|x\|^{\frac{a+b+c-1}{2}} +$$
$$o(\|x\|^{-\frac{1}{2}}),\ x \to \infty.$$

This asymptotic formula is also a special case of (2.25), when $\operatorname{Re}\beta = \operatorname{Re}\gamma = \frac{1}{2}$.

Remark 2.35. Analogously to [EFK3], Proposition 3.15(i), a similar asymptotic formula for H_x to Proposition 2.31 holds when $x \to t_j$ $0 \le j \le m$, with an additional factor S_j: if $\lambda_j \neq -1$ then

$$\|x-t_j\|^{-\frac{1}{2}} H_x = \Gamma^F(\lambda_j+1)\, \|x-t_j\|^{-\frac{\lambda_j+1}{2}} R^{(j)} S_j +$$
$$\Gamma^F(-\lambda_j-1)\, \|x-t_j\|^{\frac{\lambda_j+1}{2}} S_j R^{(j)} + o(1),\ x \to t_j,$$

where $R^{(j)} := R_{1,\dots,1,-1,1,\dots,1}$ with -1 in the j-th position. The proof and the computation of the limit $\lambda_j \to -1$ are parallel to the case $x \to \infty$.

Remark 2.36. This analysis may be extended to the **complementary series**, i.e., when some λ_j, instead of being in $-1+i\mathbb{R}$, are allowed to lie in the interval $(-2,0)$. For simplicity assume that $\lambda_{m+1} \in -1+i\mathbb{R}$, so that $M_{\lambda_{m+1}}$ is tempered and we can define a reasonable multiplicity space $\mathrm{Mult}_{PGL_2(F)}(M^*_{\lambda_{m+1}}, M_{\lambda_0} \otimes \cdots \otimes M_{\lambda_m})$. If $\lambda \in (-2,0)$, we still have $M_\lambda = L^2(\mathbb{P}^1(F), \|K\|^{-\frac{\lambda}{2}})$, but now with inner product

$$(f,g) = \frac{1}{\Gamma^F(-\lambda-1)} \int_{\mathbb{P}^1(F)^2} f(y)\overline{g(z)} \|y-z\|^{-\lambda-2} \|dydz\|^{\frac{\lambda+2}{2}}$$

(more precisely, the integral converges for $\lambda \in (-2,-1)$ but analytically continues to $\lambda \in (-2,0)$ as a positive definite inner product). Thus the inner product in $\mathcal{H}(\boldsymbol{\lambda})$ (translation invariant homogeneous functions of degree $\frac{1}{2}(\sum_{j=0}^{m} \lambda_j - \lambda_{m+1})$) also has to be modified accordingly and will become more complicated, but the formula for Hecke operators remains the same.

Remark 2.37. At least for $\ell = 1$, one should be able to extend this theory to the case of **admissible** representations V_i (not necessarily tempered, or even unitarizable) by working in the Schwartz space context instead of L^2 space (for example, using the approach of [BK2]). For instance, in this context the extension to complementary series from Remark 2.36 should be much more straightforward – we don't need to worry about positive inner products and can just do analytic continuation with respect to the Casimir eigenvalues $\frac{1}{2}(\lambda_j + 1)^2$ (which are no longer required to be real).

Remark 2.38. The material of Subsection 2.16 generalizes in a straightforward way when λ_j are taken to be arbitrary multiplicative characters of F of the form $\lambda_j(y) = \|y\|^{-1} \lambda_j^0(y)$, where λ_j^0 are unitary characters. The above setting is the special case when λ_j^0 are imaginary powers of the norm.

3. Analytic Langlands correspondence over $\mathbb{C}$

In this section we discuss the analytic Langlands correspondence over $\mathbb{C}$, including various twists. We begin by recalling the basic setup introduced in [EFK2].

3.1. The general setting of the analytic Langlands correspondence over $\mathbb{C}$

When one talks of Langlands correspondence for a group G, one usually means not just a formulation of a spectral problem for Hecke operators, but also a parametrization of their spectrum by data related to the Langlands dual group $G^\vee$. Such a description is essentially available for the **arithmetic** Langlands correspondence for a curve X over a finite field. In this case the Langlands conjecture describes the spectrum of Hecke operators in terms of étale $G^\vee$-local systems on X. On the other hand, for the **analytic** Langlands correspondence dealing with curves over a general non-archimedean field local field F, we cannot yet formulate even a conjectural description of the spectrum. But for archimedean fields we can use quantum Hitchin Hamiltonians commuting with Hecke operators to describe the spectrum (see Subsection 2.12). The most complete conjectural picture exists for $F = \mathbb{C}$

([EFK1, EFK2, EFK3]); we discuss it in this section. The case of curves over $F = \mathbb{R}$ is discussed in Section 4.

Consider first the unramified case. Let $B^\vee$ be a Borel subgroup of $G^\vee$ with maximal torus $T^\vee$, $Z^\vee$ the center of $G^\vee$, $\mathfrak{g}^\vee := \mathrm{Lie}\, G^\vee$, $\mathfrak{b}^\vee := \mathrm{Lie}\, B^\vee$, $\mathfrak{t}^\vee := \mathrm{Lie}\, T^\vee$. Let $Q^\vee \subset \Lambda$ be the root lattice of $G^\vee$, then $\mathrm{Hom}(\Lambda/Q^\vee, \mathbb{C}^\times) = Z^\vee$. Let $d_i, i = 1, \ldots, \mathrm{rank}\, G$ be the degrees of the basic invariants for G and $G^\vee$.

Definition 3.1 ([BD1, BD2])**.** A $G^\vee$**-oper** on X is a triple $(\mathcal{E}, \mathcal{E}_{B^\vee}, \nabla)$, where $\mathcal{E}$ is a $G^\vee$-bundle on X, $\mathcal{E}_{B^\vee}$ is its $B^\vee \cap [G^\vee, G^\vee]$-reduction, and ∇ is a connection on $\mathcal{E}$ which has the form

$$\nabla = d + (f + b(z))dz,\ \ b \in \mathfrak{b}^\vee[[z]]$$

for any trivialization of $\mathcal{E}_{B^\vee}$ (and hence $\mathcal{E}$) on the formal neighborhood of any point $x_0 \in X$, where f is the lower nilpotent element of a principal $\mathfrak{sl}_2$-triple $\{e, h, f\} \subset \mathfrak{g}^\vee$ such that $h \in \mathfrak{t}^\vee$ and $e \in \mathfrak{b}^\vee$ and z is a formal coordinate at x_0.[20]

The above $\mathfrak{sl}_2$ triple defines a **principal homomorphism** $\phi : SL_2 \to G^\vee$.

Since any two Borel subgroups of $G^\vee$ are conjugate to each other, any two maximal tori in a given Borel subgroup $B^\vee$ of $G^\vee$ are conjugate to each other by an element of $B^\vee$, and any two $\mathfrak{sl}_2$-triples of the kind considered in the above definition are conjugate by an element of the torus $T^\vee$, we obtain the following result.

Lemma 3.2. *Let $B'^\vee$ be another Borel subgroup of $G^\vee$. The spaces of $G^\vee$-opers corresponding to $B^\vee$ and $B'^\vee$ are canonically isomorphic.*

Given a flat $G^\vee$-bundle $(\mathcal{E}, \nabla)$, we may speak of an **oper structure** on it, which is a reduction $\mathcal{E}_{B^\vee}$ of $\mathcal{E}$ to $B^\vee \cap [G^\vee, G^\vee]$ satisfying the above condition.

Lemma 3.3 ([BD1, BD2])**.** *A flat $G^\vee$-bundle can have at most one oper structure.*

Thus it makes sense to say that a given flat $G^\vee$-bundle is or is not an oper.

Example 3.4. If $T^\vee$ is a torus then a $T^\vee$-oper on X is any connection ∇ on the trivial $T^\vee$-bundle on X. Thus $\nabla = d + \omega$ where $\omega \in H^0(X, K_X \otimes \mathrm{Lie}\, T^\vee)$.

[20] In fact, this definition is more restrictive than the one in [BD1, BD2], where $\mathcal{E}_{B^\vee}$ is assumed to be a $B^\vee$-bundle. But the two definitions coincide when $G^\vee$ is semisimple.

As explained in [BD1, BD2], $G^\vee_{\mathrm{ad}}$-opers on X are parametrized by a certain affine space $\mathrm{Op}_{G^\vee_{\mathrm{ad}}}(X)$ of dimension $(\mathrm{g}-1)\dim G$ – a torsor over the Hitchin base

$$\mathbf{Hitch} := \oplus_i H^0(X, K_X^{\otimes d_i}). \tag{3.1}$$

By Example 3.4, this is also true for a torus, hence for a product of a torus with an adjoint group, i.e., for any $G^\vee$ such that $[G^\vee, G^\vee]$ is adjoint. In other words, denoting by $Z^\vee_{\mathrm{der}}$ the intersection $Z^\vee \cap [G^\vee, G^\vee]$ of $Z^\vee$ with the derived group $[G^\vee, G^\vee]$, we see that this description is always valid for $G^\vee/Z^\vee_{\mathrm{der}}$-opers.

More generally, for arbitrary $G^\vee$ the variety $\mathrm{Op}_{G^\vee}(X)$ of $G^\vee$-opers on X is a torsor over the affine space $\mathrm{Op}_{G^\vee/Z^\vee_{\mathrm{der}}}(X)$ with fiber $H^1(X, Z^\vee_{\mathrm{der}})$. Moreover, any choice of a spin structure $K_X^{\frac{1}{2}}$ on X gives rise to a splitting of this torsor, i.e., fixes a canonical component $\mathrm{Op}^0_{G^\vee}(X) \cong \mathrm{Op}_{G^\vee/Z^\vee_{\mathrm{der}}}(X)$. Indeed, consider the unique up to isomorphism (by the Riemann-Roch theorem) nontrivial extension

$$0 \to K_X^{\frac{1}{2}} \to \mathcal{E}_{SL_2} \to K_X^{-\frac{1}{2}} \to 0 \tag{3.2}$$

defining an SL_2-bundle $\mathcal{E}_{SL_2}$ on X, and define $\mathcal{E}_{G^\vee} := \phi(\mathcal{E}_{SL_2})$. If the genus $\mathrm{g} > 1$, then according to [BD1], any connection on $\mathcal{E}_{G^\vee}$ is an oper. Moreover, there is a canonical isomorphism $H^0(X, K_X \otimes \mathrm{ad}\, \mathcal{E}_{G^\vee}) \cong \mathbf{Hitch}$ identifying the translation actions of $H^0(X, K_X \otimes \mathrm{ad}\, \mathcal{E}_{G^\vee})$ on connections and of $\mathbf{Hitch}$ on opers. So such opers form a component of $\mathrm{Op}_{G^\vee}(X)$, which we will denote by $\mathrm{Op}^0_{G^\vee}(X)$.[21] In other words, an oper from this component is just a connection on a certain fixed $G^\vee$-bundle $\mathcal{E}_{G^\vee}$.[22]

Given a $G^\vee$-oper $\chi = (\mathcal{E}, \mathcal{E}_{B^\vee}, \nabla)$, we have the underlying flat $G^\vee$-bundle $(\mathcal{E}, \nabla)$ and the corresponding $G^\vee$-local system on X. Recall that any flat $G^\vee$-bundle has at most one oper structure. Also, according to [BD2], §1.3,

[21] If $\mathrm{g} = 0$, the bundle underlying SL_2-opers is still the non-trivial extension $\mathcal{E}_{SL_2}$ given by (3.2) but it is isomorphic to the trivial SL_2-bundle in this case. The bundle underlying $G^\vee$-opers is $\mathcal{E}_{G^\vee} = \phi(\mathcal{E}_{SL_2})$, so it is isomorphic to the trivial $G^\vee$-bundle, and there is a unique $G^\vee$-oper that corresponds to the trivial connection on $\mathcal{E}_{G^\vee}$. If $\mathrm{g} = 1$, we should take instead the trivial extension (3.2) and set $\mathcal{E}_{G^\vee} = \phi(\mathcal{E}_{SL_2})$. In this case $K_X \simeq \mathcal{O}_X$ and so $K_X^{\pm\frac{1}{2}}$ is a square root of $\mathcal{O}_X$. With a choice of such a square root, we obtain a component in the space of $G^\vee$-opers which is isomorphic to $\mathbf{Hitch}$.

[22] Note that the associated PGL_2-bundle to $\mathcal{E}_{SL_2}$ is independent on the choice of $K_X^{\frac{1}{2}}$, Thus if ϕ factors through PGL_2 then the component $\mathrm{Op}^0_{G^\vee}(X)$ does not depend on the choice of $K_X^{\frac{1}{2}}$.

the automorphism group of every flat $G^\vee$-bundle underlying a $G^\vee$-oper is the center $Z^\vee$. Therefore, the space $\mathrm{Op}_{G^\vee}(X)$ can be realized as a certain half-dimensional (in fact, Lagrangian in the Atiyah-Bott holomorphic symplectic structure) complex analytic submanifold of the complex manifold $\mathrm{LocSys}^\circ_{G^\vee}(X)$ of $G^\vee$-local systems on X with the smallest possible group of automorphisms; namely, $Z^\vee$. The group $H^1(X, Z^\vee_{\mathrm{der}})$ naturally acts on $\mathrm{Op}_{G^\vee}(X)$.

If we choose a base point $x_0 \in X$, then the $G^\vee$-local system on X corresponding to χ gives rise to a monodromy representation $\rho_\chi : \pi_1(X, x_0) \to G^\vee(\mathbb{C})$ (it is well-defined up to conjugation by an element of $G^\vee(\mathbb{C})$). In this section, to simplify our notation when we discuss monodromy representations, we will write $G^\vee$ instead of $G^\vee(\mathbb{C})$.

Remark 3.5. Let $G^\vee = SL_2$ and χ any SL_2-oper on X. Denote by M the Zariski closure of the image of the monodromy representation $\rho_\chi : \pi_1(X, p) \to SL_2$. We claim that for $\mathrm{g} > 1$, $M = SL_2$. Indeed, by passing to a cover of X if needed, we can assume without loss of generality that M is connected. Hence it is either contained in a Borel subgroup of SL_2 or is SL_2 itself. If it's the former, then the vector bundle $\mathcal{E}_{SL_2}$ would contain a line subbundle $\mathcal{L}$ preserved by the oper connection. Then $\deg(\mathcal{L}) = 0$, and since $\deg(K_X^{-\frac{1}{2}}) < 0$ for $\mathrm{g} > 1$, it follows that the map $\mathcal{L} \to K_X^{-\frac{1}{2}}$ defined by the extension (3.2) is 0. But then $\mathcal{L}$ must be isomorphic to $K_X^{\frac{1}{2}}$ which is impossible since $\deg(K_X^{\frac{1}{2}}) > 0$. This also implies an analogous statement for $G^\vee = PGL_2$: if $\mathrm{g} > 1$, then for any PGL_2-oper χ the Zariski closure of the image of the corresponding monodromy representation $\rho_\chi : \pi_1(X, p) \to PGL_2(C)$ is equal to PGL_2.

Given a $G^\vee$-local system ρ on X and an algebraic representation $\varphi : G^\vee \to GL_N(\mathbb{C})$, we have a $GL_N(\mathbb{C})$-local system $\varphi(\rho)$ on X. There exists a unique $G^\vee$-local system $\overline{\rho}$ such that for every φ, $\varphi(\overline{\rho}) \cong \overline{\varphi(\rho)}$.

Definition 3.6. We say that a $G^\vee$-local system ρ on X is **real** if $\rho \cong \overline{\rho}$.

Thus the space $\mathrm{LocSys}^\circ_{G^\vee}(X)_\mathbb{R}$ of real local systems is a half-dimensional real submanifold of $\mathrm{LocSys}^\circ_{G^\vee}(X)$ (in fact, Lagrangian under the real part of the holomorphic symplectic form). As explained in [EFK2], Remark 1.9, ρ is real iff its monodromy group can be conjugated into an inner real form $G^\vee_\mathbb{R}$ of $G^\vee$.

Definition 3.7. A **real $G^\vee$-oper** is a $G^\vee$-oper such that the corresponding $G^\vee$-local system ρ is real.

In other words, a real oper is an intersection point of the above two half-dimensional submanifolds. It is expected (and known for $G^\vee = SL_2$, see [Fa]) that these manifolds intersect transversally, so the set of real opers is discrete. Moreover, it is conjectured in [EFK2] that for real opers the inner form $G^\vee_{\mathbb{R}}$ is, in fact, split, and this is known for $G^\vee = SL_2$ ([GKM]).[23]

Note that we may also consider the complex conjugate submanifold

$$\overline{\mathrm{Op}}_{G^\vee}(X) \subset \mathrm{LocSys}^\circ_{G^\vee}(X).$$

The points of this submanifold are local systems that are realized by an antiholomorphic $G^\vee$-oper (which we call an **anti-oper** for short). This is a third Lagrangian submanifold of $\mathrm{LocSys}^\circ_{G^\vee}(X)$ with respect to the real part of the holomorphic symplectic form, which intersects the other two submanifolds exactly at the same points where they intersect each other (i.e., at real opers). In other words, a real oper is the same thing as a real anti-oper and also the same as a local system that's both an oper and an anti-oper.

Now, the main conjecture of [EFK2] is as follows (we formulate it for semisimple G, as for abelian G it is not difficult and proved in [F4], see also [EFK2]). Recall that the manifold $\mathrm{Bun}^\circ_G(X)$ is the union of connected components $\mathrm{Bun}^\circ_{G,\alpha}(X)$ labeled by the first Chern class $c \in H^2(X, \pi_1(G)) = \pi_1(G)$ of a G-bundle on X, and that $\pi_1(G) = Z^{\vee *} = \Lambda/Q^\vee$.

Conjecture 3.8. (i) The Hilbert space $\mathcal{H} = L^2(\mathrm{Bun}^\circ_G(X))$ can be written as an orthogonal direct sum of 1-dimensional spaces

$$\mathcal{H} = \bigoplus_{\rho,\beta} \mathcal{H}_{\rho,\beta}$$

invariant under Hecke operators, where ρ runs over real $G^\vee$-opers in $\mathrm{Op}^0_{G^\vee}(X)$, and β runs over eigenvalues of Hecke operators corresponding to ρ. The quantum Hitchin Hamiltonians act on $\mathcal{H}_{\rho,\beta}$ via the character ρ.

(ii) The eigenvalue $\beta_\lambda(x, \overline{x})$ for the Hecke operator $H_{x,\lambda}$ in $\mathcal{H}_{\rho,\beta}$ is given up to scaling by the formula of [EFK2], Conjecture 5.1 (see Conjecture 3.38 below).

(iii) The set of such eigenvalues corresponding to a given ρ is a torsor over the group $Z^\vee = \mathrm{Hom}(\Lambda/Q^\vee, \mathbb{C}^\times)$ where the action of this group on eigenvalues is by multiplication, i.e.

$$(\xi \circ \beta)_\lambda = \xi(\lambda)\beta_\lambda.$$

[23] This is also easy to see for any $G^\vee$ in the tamely ramified case, see [EFK2], Remark 1.9.

(iv) The decomposition $\mathcal{H} = \oplus_{c \in Z^{\vee *}} L^2(\mathrm{Bun}^\circ_{G,c}(X))$ is invariant under quantum Hitchin Hamiltonians, and on each summand $L^2(\mathrm{Bun}^\circ_{G,c}(X))$ they have simple spectrum labeled by real $G^\vee$-opers ρ in $\mathrm{Op}^0_{G^\vee}(X)$. The Hecke operators $H_{x,\lambda}$ act between these summands, acting on labels c by $c \mapsto c + \lambda$, which gives rise to the action in (iii).

In in Corollary 3.18 below, we will recall the formula for the Hecke eigenvalues $\beta_\lambda(x, \overline{x})$ obtained in [EFK2], Corollary 1.19 in the case $G = PGL_n$ and $\lambda = \omega_1$. For G of types B_ℓ, C_ℓ, or G_2 and $\lambda = \omega_1$, we conjecture an analogous formula in Conjecture 3.28. In the general case, the formula for the Hecke eigenvalues is given in Conjecture 3.38 (it coincides with Conjecture 5.1 of [EFK2]).

Note that we have a free action of the finite group $H^1(X, Z)$ on $\mathrm{Bun}^\circ_G(X)$, where Z is the center of G, and this action commutes with Hecke and quantum Hitchin operators. Hence this group acts by a character $\chi_\rho \in H^1(X, Z)^* \cong H^1(X, Z^*)$ on each (1-dimensional) joint eigenspace of these operators corresponding to a real $G^\vee$-local system ρ and some choice of eigenvalue β. Let us explain how to compute χ_ρ.

Let $G^\vee_{\mathrm{sc}}$ be the simply connected cover of $G^\vee$. Recall that we have an exact sequence

$$1 \to H^1(X, \pi_1(G^\vee)) \to H^1(X, G^\vee_{\mathrm{sc}}) \to H^1(X, G^\vee) \to H^2(X, \pi_1(G^\vee)),$$

and that $\pi_1(G^\vee) = Z^*$. Thus every $G^\vee$-local system $\rho : \pi_1(X) \to G^\vee$ has a first Chern class $c_\rho \in H^2(X, Z^*)$. However, as explained above, if ρ is an oper then as a holomorphic bundle it reduces to the principal SL_2, so $c_\rho = 1$ (as $SL_2(\mathbb{C})$ is simply connected). Moreover, in this case there is a unique lift of ρ to a $G^\vee_{sc}$-oper ρ' in the canonical component $\mathrm{Op}^0_{G^\vee_{\mathrm{sc}}}(X)$. Now, the reality of ρ means that $\rho \cong \overline{\rho}$, where $\overline{\rho}$ is the complex conjugate of ρ, but then ρ' is not necessarily real: we have $\rho' \cong \eta\overline{\rho'}$ for a unique $\eta \in H^1(X, Z^*)$. We expect that $\chi_\rho = \eta$.[24]

3.2. Analytic Langlands correspondence twisted by Z-gerbes on X

The setting of the previous subsection has a twisted generalization where we take G simply connected, but instead of ordinary principal G-bundles take

[24] Note that the isomorphism $H^1(X, Z)^* \cong H^1(X, Z^*)$ comes from the cup product on $H^1(X, \mathbb{Z})$, so it is defined uniquely up to inversion and changes to inverse when we change the complex structure on X (hence the orientation) to the opposite one. So replacing this isomorphism by its inverse results just in replacing η by η^{-1}.

bundles twisted by Z-gerbes on X defined by $c \in H^2(X,Z) = Z$ (this is mentioned in [GW], Subsection 9.2). Such twisted bundles are defined on an open cover $\{U_i\}$ of X by holomorphic transition functions $g_{ij} : U_i \cap U_j \to G$ such that $g_{ij}g_{jk}g_{ki} = \widetilde{c}_{ijk} \in Z$ on $U_i \cap U_j \cap U_k$, where $\widetilde{c}$ is a Čech 2-cocycle representing c. Let $\mathrm{Bun}_G^\circ(X)_c$ be the variety of regularly stable twisted bundles with class c, and $\mathrm{Bun}_G^\circ(X)_{\mathrm{tw}}$ be the disjoint union of $\mathrm{Bun}_G^\circ(X)_c$ over all $c \in H^2(X,Z)$. We have a principal $H^1(X,Z)$-bundle

$$\mathrm{Bun}_G^\circ(X)_{\mathrm{tw}} \to \mathrm{Bun}_{G_{\mathrm{ad}}}^\circ(X).$$

Thus the Hilbert space $\mathcal{H} = L^2(\mathrm{Bun}_G^\circ(X)_{\mathrm{tw}})$ carries commuting actions of quantum Hitchin Hamiltonians and Hecke operators.

Conjecture 3.9. Conjecture 3.8 and the formula for χ_ρ holds in this twisted setting with the group $Z^\vee$ (trivial in our case) replaced by $\pi_1(G^\vee)$, the center of $G^\vee_{\mathrm{sc}}$.

3.3. Differential equations for the Hecke operators for $G = PGL_2$ and $X = \mathbb{P}^1$

In this subsection we consider the case of $G = PGL_2$ and $X = \mathbb{P}^1$ with parabolic structures at finitely many points. We will consider the case of $G = PGL_n$ and a smooth projective curve X of genus $\mathrm{g} > 1$ in in the next subsection.

We generalize the results of [EFK3], Subsection 4.2 to the twisted setting of Subsection 2.16. Namely, we show that for $F = \mathbb{R}$ the Hecke operators satisfy a second-order differential equation (the oper equation), while for $F = \mathbb{C}$ they satisfy a system of two such equations – holomorphic (the oper equation) and anti-holomorphic (the anti-oper equation, conjugate to the oper equation), which can be used to describe their spectrum. Let us now derive the oper equation.

We return to the setting of Subsection 2.16 for $F = \mathbb{R}$ or $F = \mathbb{C}$. Consider the Gaudin operators

$$G_i := \sum_{j \neq i} \frac{\Omega_{ij}}{t_i - t_j}, \ 0 \le i \le m,$$

where $\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h$ and

$$e = \partial_y, \ h = -2y\partial_y + \lambda, \ f = -y^2\partial_y + \lambda y \tag{3.3}$$

(this differs from [EFK3], (7.7) by the Chevalley involution). Thus, setting $\partial_i := \frac{\partial}{\partial y_i}$, we have ([EFK1], (7.8)):

$$G_i = \sum_{j\neq i} \frac{1}{t_i - t_j}\Big(-(y_i - y_j)^2\partial_i\partial_j + (y_i - y_j)(\lambda_i\partial_j - \lambda_j\partial_i) + \frac{\lambda_i\lambda_j}{2}\Big),$$

$$\widehat{G}_i := G_i - \sum_{j\neq i} \frac{\lambda_i\lambda_j}{2(t_i - t_j)} = \sum_{j\neq i} \frac{1}{t_i - t_j}\Big(-(y_i - y_j)^2\partial_i\partial_j + (y_i - y_j)(\lambda_i\partial_j - \lambda_j\partial_i)\Big).$$

Note that on translation invariant functions of $y_0, \dots, y_m$ we have

$$\sum_{i=0}^m G_i = \sum_{i=0}^m \widehat{G}_i = 0,\ \sum_{i=0}^m t_i G_i = E(E - \lambda - 1) + \frac{\lambda^2 - \sum_i \lambda_i^2}{4},$$
$$\sum_{i=0}^m t_i \widehat{G}_i = E(E - \lambda - 1),$$

where $E := \sum_{i=0}^m y_i\partial_i$ is the Euler vector field and $\lambda := \sum_i \lambda_i$ (see e.g. [EFK1], Section 7).

The following proposition is a complete analog of [EFK3], Proposition 4.3.

Proposition 3.10 (Universal oper equations)**.** *(i) We have*

$$\Big(\partial_x^2 - \sum_{i\geq 0} \frac{\lambda_i}{x - t_i}\partial_x\Big)\mathbb{H}_x - \mathbb{H}_x \sum_{i\geq 0} \frac{\widehat{G}_i}{x - t_i} = 0.$$

(ii) We have

$$\Big(\partial_x^2 - \sum_{i\geq 0} \frac{\lambda_i(\lambda_i + 2)}{4(x - t_i)^2}\Big) H_x - H_x \sum_{i\geq 0} \frac{G_i}{x - t_i} = 0.$$

Here the differential equations hold in the same sense as in [EFK3], Proposition 4.3.

Proof. (ii) easily follows from (i), so let us prove (i). The proof is parallel to the proof of [EFK3], Proposition 4.3. We only redo the algebraic part of the proof, as the analytic details are exactly the same. Let $u_i = u_i(s) := y_i - s$ and ψ, ψ_i, ψ_{ij} be the zeroth, first and second derivatives of ψ evaluated at the point $\mathbf{z}$ with coordinates $z_i := \frac{t_i - x}{y_i - s} = \frac{t_i - x}{u_i}$. Thus

$$\partial_i\psi_j = \frac{(x - t_i)\psi_{ij}}{u_i^2},\ \sum_{i\geq 0} \partial_i\psi_j = -\partial_s\psi_j \tag{3.4}$$

Also let

$$d\mu(s) := \prod_{i=0}^{m} \|s-y_i\|^{\lambda_i} \|ds\| . \tag{3.5}$$

We have

$$\partial_x(\mathbb{H}_x\psi) = \int_F \sum_{i\geq 0} \frac{\psi_i}{u_i} d\mu(s),\ \ \partial_x^2(\mathbb{H}_x\psi) = \int_F \sum_{i,j\geq 0} \frac{\psi_{ij}}{u_i u_j} d\mu(s).$$

Also

$$\sum_{i\geq 0} \frac{\mathbb{H}_x \widehat{G}_i \psi}{x-t_i} = -\int_F \sum_{i\neq j} \frac{(\frac{t_i-x}{u_i} - \frac{t_j-x}{u_j})^2 \psi_{ij} + (\frac{t_i-x}{u_i} - \frac{t_j-x}{u_j})(\lambda_j\psi_i - \lambda_i\psi_j)}{(x-t_i)(t_i-t_j)} d\mu(s).$$

Subtracting, we get

$$\partial_x^2(\mathbb{H}_x\psi) - \mathbb{H}_x \sum_{i\geq 0} \frac{\widehat{G}_i\psi}{x-t_i} =$$

$$\int_F \left(\sum_i \frac{\psi_{ii}}{u_i^2} + \sum_{i\neq j} \frac{(\frac{(t_i-x)^2}{u_i^2} + \frac{(t_j-x)^2}{u_j^2})\psi_{ij} + (\frac{t_i-x}{u_i} - \frac{t_j-x}{u_j})(\lambda_j\psi_i - \lambda_i\psi_j)}{(x-t_i)(t_i-t_j)} \right) d\mu(s) =$$

$$\int_F \left(\sum_{i,j\geq 0} \frac{(t_i-x)\psi_{ij}}{(t_j-x)u_i^2} + \sum_{i\neq j} \frac{(\frac{t_i-x}{u_i} - \frac{t_j-x}{u_j})(\lambda_j\psi_i - \lambda_i\psi_j)}{(x-t_i)(t_i-t_j)} \right) d\mu(s).$$

Now, using integration by parts, (3.4) and (3.5), we have

$$\int_F \sum_{i,j\geq 0} \frac{(t_i-x)\psi_{ij}}{(t_j-x)u_i^2} d\mu(s) = -\int_F \sum_{j\geq 0} \frac{1}{x-t_j} \sum_{i\geq 0} \partial_i \psi_j d\mu(s) =$$

$$\int_F \sum_{j\geq 0} \frac{1}{x-t_j} \partial_s \psi_j d\mu(s) = -\int_F \sum_{j\geq 0} \frac{\psi_j}{x-t_j} \sum_{i\geq 0} \frac{\lambda_i}{u_i} d\mu(s).$$

Thus

$$\partial_x^2(\mathbb{H}_x\psi) - \mathbb{H}_x \sum_{i\geq 0} \frac{\widehat{G}_i\psi}{x-t_i} =$$

$$-\int_F \left(\sum_{j\geq 0} \frac{\psi_j}{x-t_j} \sum_{i\geq 0} \frac{\lambda_i}{u_i} + \frac{1}{2} \sum_{i\neq j} \frac{(\frac{t_i-x}{u_i} - \frac{t_j-x}{u_j})(\lambda_j\psi_i - \lambda_i\psi_j)}{(t_i-x)(t_j-x)} \right) d\mu(s) =$$

$$-\int_F \left(\sum_{i\geq 0}\frac{\lambda_i}{(t_i-x)u_i}\psi_i+\sum_{i\neq j}\frac{\lambda_j}{(t_j-x)u_i}\psi_i\right)d\mu(s)=$$
$$-\int_F \left(\sum_{j\geq 0}\frac{\lambda_j}{t_j-x}\sum_{i\geq 0}\frac{\psi_i}{u_i}\right)d\mu(s)=\sum_{j\geq 0}\frac{\lambda_j}{x-t_j}\partial_x(\mathbb{H}_x\psi). \qquad \square$$

Also, similarly to [EFK3], Proposition 4.11, we have

Proposition 3.11.

$$[H_x, G_j]=0.$$

As shown in [EFK2], if $F=\mathbb{C}$ then the Hecke operators also satisfy an anti-holomorphic second-order differential equation (the anti-oper equation) which is the complex conjugate of the oper equation of Proposition 3.10(ii). Thus, introducing the operator-valued oper $\partial_x^2-S(x)$, where

$$S(x):=\sum_{i=0}^m\frac{\lambda_i(\lambda_i+2)}{4(x-t_i)^2}+\sum_{i=0}^m\frac{G_i}{x-t_i}, \tag{3.6}$$

for $F=\mathbb{C}$ we obtain

$$(\partial_x^2-S(x))H_x=0, \qquad (\overline{\partial}_x^2-\overline{S(x)})H_x=0 \tag{3.7}$$

(for $F=\mathbb{R}$ we only have the first equation).

We also obtain the following equation for the eigenvalues $\beta_k(x)$ of $H_{x,\mathrm{full}}$, which is a generalization of Corollary 4.14 of [EFK3].

Corollary 3.12. *The function $\beta_k(x)$ satisfies the differential equation*

$$L(\boldsymbol{\mu}_k)\beta_k(x)=0, \tag{3.8}$$

where

$$L(\boldsymbol{\mu}_k):=\partial_x^2-\sum_{i\geq 0}\frac{\lambda_i(\lambda_i+2)}{4(x-t_i)^2}-\sum_{i\geq 0}\frac{\mu_{i,k}}{x-t_i}$$

is an SL_2-oper, with

$$\sum_{i=0}^m\mu_{i,k}=0,\ \sum_{i=0}^m t_i\mu_{i,k}=\frac{\lambda_{m+1}(\lambda_{m+1}+2)}{4}-\sum_{i=0}^m\frac{\lambda_i(\lambda_i+2)}{4}. \tag{3.9}$$

Moreover, if $F=\mathbb{C}$, then β also satisfies the complex conjugate equation $\overline{L(\boldsymbol{\mu}_k)}\beta_k(x)=0$.

Note that equation (3.8) is Fuchsian at the points t_j with characteristic exponents $\frac{1}{2} \pm \frac{\lambda_j+1}{2}$, and by (3.9) it is also Fuchsian at ∞ with characteristic exponents $-\frac{1}{2} \mp \frac{\lambda_{m+1}+1}{2}$. In other words, basic solutions behave near t_j as $(x-t_j)^{\frac{1}{2}}$ and $(x-t_j)^{\frac{1}{2}}\log(x-t_j)$ if $\lambda_j = -1$ and as $(x-t_j)^{\frac{1}{2}\pm\frac{\lambda_j+1}{2}}$ else, while at ∞ they behave as $x^{\frac{1}{2}}, x^{\frac{1}{2}}\log x$ if $\lambda_{m+1} = -1$ and $x^{\frac{1}{2}\pm\frac{\lambda_{m+1}+1}{2}}$ else.[25] Thus the monodromy operators of (3.8) at t_j, $0 \le j \le m+1$ are conjugate to $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ if $\lambda_j = -1$ and to $\begin{pmatrix} e^{\pi i \lambda_j} & 0 \\ 0 & e^{-\pi i \lambda_j} \end{pmatrix}$ else.

Thus for $F = \mathbb{C}$ the spectral opers have a property that the system $L\beta = 0$, $\overline{L}\beta = 0$ has a single-valued solution. So the monodromy of such an oper must preserve a nondegenerate hermitian form. Moreover, as the above matrices cannot be conjugated to $SU(2)$, this form must be of signature (1,1). Thus spectral opers must have monodromy in $SU(1,1) \cong SL(2,\mathbb{R})$, i.e. they belong to the (discrete) set $\mathcal{B} = \mathcal{B}(\lambda_0, \dots, \lambda_{m+1})$ of real opers of the form of Corollary 3.12. Furthermore, the joint eigenspaces of $H_{x,\mathrm{full}}$ are 1-dimensional. In other words, we have

Theorem 3.13. *Theorem 4.15 of [EFK3] extends mutatis mutandis to the ramified setting with any weights $\lambda_j \in -1 + i\mathbb{R}$.*

Moreover, similarly to [EFK3], the set of spectral opers conjecturally coincides with $\mathcal{B}$, and this is definitely true at least for 4 and 5 points. The proofs of these facts are analogous to the proofs in the untwisted case given in [EFK3].

3.4. Differential equations for the Hecke operators for $G = PGL_n$

Let $G = PGL_n$ and X a smooth projective curve of genus $\mathrm{g} > 1$. Analogues of the universal oper equations of Proposition 3.10 were obtained in Subsection 1.4 and Section 4 of [EFK2]. In this subsection we summarize these results.

Recall the component $\mathrm{Op}^0_{SL_n}(X)$ of the space of SL_n-opers on X introduced in Subsection 3.1. For even n it depends on the choice of an equivalence class of the square root $K_X^{\frac{1}{2}}$ (a spin structure), which we will denote by γ.[26] It is an affine space which is a torsor over the vector space **Hitch** defined by formula (3.1).

[25] We remind that since the oper $L(\boldsymbol{\mu})$ is a map from $K^{-\frac{1}{2}}$ to $K^{\frac{3}{2}}$, solutions of the equation $L(\boldsymbol{\mu})\beta = 0$ are sections of $K^{-\frac{1}{2}}$, but we view them as functions by multiplying by $(dx)^{-\frac{1}{2}}$.

[26] In [EFK2], we denoted this component by $\mathrm{Op}^{\gamma}_{SL_n}(X)$.

Given $\chi \in \mathrm{Op}^0_{SL_n}(X)$, denote by $(\mathcal{V}_{\omega_1}, \nabla_\chi)$ the corresponding holomorphic flat rank n vector bundle on X whose determinant is identified with the trivial flat line bundle. The oper Borel reduction gives rise to an embedding

$$\kappa_{\omega_1} : K_X^{\frac{n-1}{2}} \hookrightarrow \mathcal{V}_{\omega_1},$$

and therefore an embedding

$$\widetilde{\kappa}_{\omega_1} : \mathcal{O}_X \hookrightarrow \mathcal{V}_{\omega_1} \otimes K_X^{-\frac{n-1}{2}}.$$

Hence we obtain a section

$$s_{\omega_1} := \widetilde{\kappa}_{\omega_1}(1) \in \Gamma(X, \mathcal{V}_{\omega_1} \otimes K_X^{-\frac{n-1}{2}}).$$

Likewise, we obtain a section

$$s_{\omega_{n-1}} \in \Gamma(X, \mathcal{V}_{\omega_{n-1}} \otimes K_X^{-\frac{n-1}{2}}) = \Gamma(X, \mathcal{V}^*_{\omega_1} \otimes K_X^{-\frac{n-1}{2}}).$$

Let $D^\gamma_n(X)$ be the affine space of nth order differential operators $P : K_X^{-\frac{n-1}{2}} \to K_X^{\frac{n+1}{2}}$, where for even n we use our chosen square root $K_X^{\frac{1}{2}}$, such that

1. $\mathrm{symb}(P) \in H^0(X, \mathcal{O}_X)$ equals 1;
2. The operator $P - (-1)^n P^*$, where $P^* : K_X^{-\frac{n-1}{2}} \to K_X^{\frac{n+1}{2}}$ is the algebraic adjoint operator (see [BB], Sect. 2.4), has order $n-2$.

The following statement is proved in [BD2], §2.8.

Lemma 3.14. *The assignment*

$$\chi \in \mathrm{Op}^0_{SL_n}(X) \quad \mapsto \quad P_\chi \in D^\gamma_n(X)$$

defines a bijection $\mathrm{Op}^0_{SL_n}(X) \simeq D^\gamma_n(X)$ *such that the sections* $s_{\omega_1} \in \Gamma(X, \mathcal{V}_{\omega_1} \otimes K_X^{-\frac{n-1}{2}})$ *and* $s_{\omega_{n-1}} \in \Gamma(X, \mathcal{V}_{\omega_{n-1}} \otimes K_X^{-\frac{n-1}{2}})$ *satisfy*

$$P_\chi \cdot s_{\omega_1} = 0, \qquad P^*_\chi \cdot s_{\omega_{n-1}} = 0,$$

where P^*_χ *is the algebraic adjoint of* P_χ*.*

Let $\mathcal{V}^{\mathrm{univ}}_{\omega_1}$ be the universal vector bundle over $\mathrm{Op}^0_{SL_n}(X) \times X$ equipped with a partial connection ∇^{univ} (along X) defined by the property

$$(\mathcal{V}^{\mathrm{univ}}_{\omega_1}, \nabla^{\mathrm{univ}})|_{\chi \times X} = (\mathcal{V}_{\omega_1}, \nabla_\chi).$$

Set $\mathcal{V}^{\mathrm{univ}}_{\omega_1,X} := \pi_*(\mathcal{V}^{\mathrm{univ}}_{\omega_1})$, where $\pi : \mathrm{Op}^0_{SL_n}(X) \times X \to X$ is the natural projection and π_* is the $\mathcal{O}$-module direct image. Using the connection ∇^{univ}, we obtain a left $\mathcal{D}_X$-module $\mathcal{V}^{\mathrm{univ}}_{\omega_1,X}$.

Now let $D_{PGL_n,\alpha}$ be the algebra of global holomorphic differential operators acting on the component $\mathrm{Bun}_{PGL_n,\alpha}$ of Bun_{PGL_n}. According to the results of [BD1], these algebras are isomorphic to each other and

$$D_{PGL_n,\alpha} \simeq \mathrm{Fun}\,\mathrm{Op}^0_{SL_n}(X).$$

From now on, we will use the notation D_{PGL_n} for $D_{PGL_n,\alpha}$.

Thus, D_{PGL_n} naturally acts on $\mathcal{V}^{\mathrm{univ}}_{\lambda,X}$, and this action commutes with the action of $\mathcal{D}_X$. We obtain the following result.

Lemma 3.15. *There is a unique n-th order differential operator*

$$\sigma : K_X^{-\frac{n-1}{2}} \to D_{PGL_n} \otimes K_X^{\frac{n+1}{2}} \tag{3.10}$$

satisfying the following property: for any $\chi \in \mathrm{Op}^0_{SL_n}(X) = \mathrm{Spec}\, D_{PGL_n}$, applying the corresponding homomorphism $D_{PGL_n} \to \mathbb{C}$ we obtain P_χ.

Let $\mathrm{Op}^0_{SL_n}(X)_{\mathbb{R}}$ be the set of real SL_n-opers in $\mathrm{Op}^0_{SL_n}(X)$. If χ is in $\mathrm{Op}^0_{SL_n}(X)_{\mathbb{R}}$, then the monodromy representations associated to χ and $\overline{\chi}$ are isomorphic. Therefore, $(\mathcal{V}_{\omega_1}, \nabla_\chi)$ and $(\overline{\mathcal{V}}_{\omega_1}, \overline{\nabla}_\chi)$ are isomorphic as C^∞ flat vector bundles on X. Hence we obtain a non-degenerate pairing

$$h_{\chi,\omega_1}(\cdot,\cdot) : (\mathcal{V}_{\omega_1}, \nabla_\chi) \otimes (\overline{\mathcal{V}}_{\omega_{n-1}}, \overline{\nabla}_\chi) \to (\mathcal{C}^\infty_X, d)$$

of C^∞ flat vector bundles on X. The flat vector bundle $(\mathcal{V}_{\omega_1}, \nabla_\chi)$ is known to be irreducible if $\mathrm{g} > 1$ (see [BD1], Sect. 3.1.5(iii)); therefore this pairing is unique up to a scalar.

The following results were proved in [EFK2], Theorem 1.18 and Corollary 1.16.

Theorem 3.16. *The Hecke operator H_{ω_1}, viewed as an operator-valued section of $\Omega_X^{-\frac{n-1}{2}} = K_X^{-\frac{n-1}{2}} \otimes \overline{K}_X^{-\frac{n-1}{2}}$, satisfies the system of differential equations*

$$\sigma \cdot H_{\omega_1} = 0, \qquad \overline{\sigma} \cdot H_{\omega_1} = 0. \tag{3.11}$$

Proposition 3.17. *$h_{\chi,\omega_1}(s_{\omega_1}, \overline{s_{\omega_{n-1}}})$ is a unique, up to a scalar, section Φ of $\Omega_X^{-\frac{n-1}{2}}$ which is a solution of the system of differential equations*

$$P_\chi \cdot \Phi = 0, \qquad \overline{P^*_\chi} \cdot \Phi = 0 \tag{3.12}$$

These two results have the following corollary describing the eigenvalues of the Hecke operator H_{ω_1} obtained in [EFK2], Corollary 1.19.

Corollary 3.18. *Each of the eigenvalues $\beta_{\omega_1}(x, \overline{x})$ of the Hecke operator H_{ω_1} on $\mathcal{H}$ corresponding to a real oper $\chi \in \mathrm{Op}^0_{SL_n}(X)_{\mathbb{R}}$ (see Conjecture 3.8) is equal to a scalar multiple of $h_{\chi,\omega_1}(s_{\omega_1}, \overline{s_{\omega_{n-1}}})$.*

3.5. General case

In [EFK2], Section 5, we described a conjectural analogue of this picture for general G, X, and $\lambda \in \Lambda^+$. In the general case, the analogues of the differential equations (3.11) satisfied by the Hecke operators are more complicated (see Subsection 3.8 below). However, there are several cases in which these equations can be presented in a simple form similar to equations (3.11). Those cases will be discussed in Subsection 3.6.

We start by introducing some notation following [EFK2], Section 5. For $\chi \in \mathrm{Op}^0_{G^\vee}(X)$ and $\lambda \in \Lambda^+$, we have the flat holomorphic vector bundle $(\mathcal{V}_\lambda, \nabla_{\chi,\lambda})$ on X associated to the irreducible representation V_λ of $G^\vee$ with highest weight λ (according to [BD2], §3, the corresponding vector bundles are isomorphic to each other for all $\chi \in \mathrm{Op}^0_{G^\vee}(X)$, which justifies the notation $\mathcal{V}_\lambda$; see Theorem 3.19 below). The oper Borel reduction gives rise to an embedding

$$\kappa_\lambda : K_X^{\langle\lambda,\rho\rangle} \hookrightarrow \mathcal{V}_\lambda \tag{3.13}$$

(if n is a half-integer, we take the power of $K_X^{\frac{1}{2}}$ in our chosen isomorphism class γ).

For $n \in \frac{1}{2}\mathbb{Z}$, denote by $\mathcal{D}_{X,n}$ the sheaf of differential operators acting on the line bundle K_X^n on X. We have

$$\mathcal{D}_{X,n} \simeq K_X^n \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} K_X^{-n}.$$

For $\lambda \in \Lambda_+$, set

$$d(\lambda) := 2\langle\lambda, \rho\rangle. \tag{3.14}$$

The $\mathcal{D}_X$-module structure on $\mathcal{V}_\lambda$ defined by the oper connection $\nabla_{\chi,\lambda}$ gives rise to a $\mathcal{D}_{X,-\frac{d(\lambda)}{2}}$-module structure on the $\mathcal{O}_X$-module

$$\mathcal{V}_\lambda^K := K_X^{-\frac{d(\lambda)}{2}} \underset{\mathcal{O}_X}{\otimes} \mathcal{V}_\lambda.$$

We denote this $\mathcal{D}_{X,-\frac{d(\lambda)}{2}}$-module by $\mathcal{V}^K_{\chi,\lambda}$.

The map (3.13) gives rise to a map

$$\widetilde{\kappa}_\lambda : \mathcal{O}_X \hookrightarrow K_X^{-\frac{d(\lambda)}{2}} \otimes \mathcal{V}_\lambda$$

and a non-zero section

$$s_\lambda := \widetilde{\kappa}_\lambda(1) \in \Gamma(X, K_X^{-\frac{d(\lambda)}{2}} \otimes \mathcal{V}_\lambda). \tag{3.15}$$

Following [EFK2], Section 5.2, we denote by $I_{\lambda,\chi}$ the left annihilating ideal of s_λ in the sheaf $\mathcal{D}_{X,-\frac{d(\lambda)}{2}}$.

Now suppose that $\chi \in \mathrm{Op}^0_{G^\vee}(X)_\mathbb{R}$. Then we have an isomorphism of C^∞ flat bundles

$$(\mathcal{V}_\lambda, \nabla_{\chi,\lambda}) \simeq (\overline{\mathcal{V}}_\lambda, \overline{\nabla}_{\chi,\lambda})$$

and hence a pairing

$$h_{\chi,\lambda}(\cdot,\cdot) : (\mathcal{V}_\lambda, \nabla_{\chi,\lambda}) \otimes (\overline{\mathcal{V}}_{-w_0(\lambda)}, \overline{\nabla}_{\chi,-w_0(\lambda)}) \to (\mathcal{C}^\infty_X, d)$$

as $V_\lambda^* \simeq V_{-w_0(\lambda)}$. Since $\langle -w_0(\lambda), \rho\rangle = \langle \lambda, \rho\rangle = \frac{d(\lambda)}{2}$, we have

$$\overline{s_{-w_0(\lambda)}} \in \Gamma(X, \overline{K}_X^{-\frac{d(\lambda)}{2}} \otimes \overline{\mathcal{V}}_{-w_0(\lambda)}).$$

We recall the results of [BD2], §3, on the structure of $(\mathcal{V}_\lambda, \nabla_{\chi,\lambda})$ and κ_λ. Fix a principal $\mathfrak{sl}_2$ subalgebra

$$\mathfrak{sl}_2 = \mathrm{span}\{e,h,f\} \subset \mathfrak{g}^\vee$$

such that $\mathrm{span}\{e,h\}$ is in the Borel subalgebra $\mathfrak{b}^\vee \subset \mathfrak{g}^\vee$ used in the definition of $G^\vee$-opers.

Denote by V_m the irreducible $(m+1)$-dimensional representation of $\mathfrak{sl}_2$. The irreducible representation V_λ of $G^\vee$ decomposes into a direct sum of irreducible representations of the principal $\mathfrak{sl}_2$ subalgebra:

$$V_\lambda \simeq V_{d(\lambda)} \oplus \left(\bigoplus_{m<d(\lambda)} V_m^{\oplus c_{m,\lambda}} \right), \qquad c_{m,\lambda} \in \mathbb{Z}_{\geq 0}, \tag{3.16}$$

where $d(\lambda)$ is given by formula (3.14).

Recall the rank two vector bundle $\mathcal{E}_{SL_2}$ which is a non-trivial extension (3.2) (as before, here we take $K_X^{\frac{1}{2}}$ in the isomorphism class γ that we have chosen). Define the rank $(m+1)$ vector bundle $\mathcal{E}_m := \mathrm{Sym}^m(\mathcal{E}_{SL_2})$ on X. By construction, $\mathcal{E}_m$ is equipped with a filtration $\{\mathcal{E}_m^{\leq i}\}_{i=0,\dots,m}$ such that

$$\mathcal{E}_m^{\leq i}/\mathcal{E}_m^{\leq(i-1)} \simeq K_X^{\frac{m}{2}-i}.$$

Fix isomorphisms

$$\jmath_m^i : \mathcal{E}_m^{\leq i}/\mathcal{E}_m^{\leq(i-1)} \to (\mathcal{E}_m^{\leq(i+1)}/\mathcal{E}_m^{\leq i}) \otimes K_X, \qquad i = 0, \dots, m.$$

Let $B_m \subset \mathrm{End}(\mathcal{E}_m)$ be the subbundle of endomorphisms preserving this filtration, and p the canonical section of $(\mathrm{End}(\mathcal{E}_m)/B_m) \otimes K_X$ such that $p(\mathcal{E}_m^{\leq i}) \subset \mathcal{E}_m^{\leq(i+1)} \otimes K_X$ and it induces the isomorphisms $\jmath_m^i$ on successive quotients.

The following results are due to Beilinson and Drinfeld [BD1, BD2].

Theorem 3.19. *(1) For any $\chi \in \mathrm{Op}^0_{SL_{m+1}}(X)$, we have $\mathcal{V}_{\chi,\omega_1} \simeq \mathcal{E}_m$, the oper Borel reduction corresponds to B_m, and $\nabla_{\chi,\omega_1} = d + p$ mod B_m.*

(2) For any $\chi \in \mathrm{Op}^0_{G^\vee}(X)$,

$$\mathcal{V}_\lambda \simeq \mathcal{E}_{d(\lambda)} \oplus \left(\bigoplus_{m<d(\lambda)} \mathcal{E}_m^{\oplus c_{m,\lambda}} \right), \tag{3.17}$$

where the numbers $c_{m,\lambda}$ are defined by (3.16), and

$$\nabla_{\chi,\lambda} = d + p \mod B \tag{3.18}$$

where B is the direct sum of $B_{d(\lambda)}$ and all B_m's corresponding to the summands of (3.17).

3.6. Differential equations for Hecke operators corresponding to principal weights

In the Subsection 3.4 we used the interpretation of SL_n-opers as scalar differential operators of order n (see Lemma 3.14). As we show in this subsection, an analogous interpretation is possible if G is a connected simple algebraic group such that $G^\vee$ has an irreducible representation V_λ with highest weight λ that remains irreducible under a principal $\mathfrak{sl}_2$ subalgebra of $\mathfrak{g}^\vee$. We will call such weights *principal*. According to the results of [SS], which go back to E. Dynkin in characteristic 0 (see Theorem 3.34 below), principal weights are $\lambda = \omega_1$ and ω_ℓ for $\mathfrak{g}^\vee$ of type A_ℓ, and $\lambda = \omega_1$ for $\mathfrak{g}^\vee$ of types B_ℓ, C_ℓ and

G_2. For A_ℓ, B_ℓ, and C_ℓ, the corresponding scalar differential operators were described in [DrS, BD2] (see also [FG] and [LM], where such operators were discussed in special cases).

We have $V_\lambda \simeq V_{d(\lambda)}$ in the decomposition (3.16) (i.e. there are no lower terms) if and only if λ is a principal weight of $G^\vee$. In this case, we have the following corollary of Theorem 3.19.

Corollary 3.20. *Suppose that λ is a principal weight of $G^\vee$.*

(1) $(\mathcal{V}_\lambda, \nabla_{\chi,\lambda})$ together with its oper Borel reduction is an $SL_{d(\lambda)+1}$-oper.
(2) If $\mathrm{g} > 1$, then the flat vector bundle $(\mathcal{V}_\lambda, \nabla_{\chi,\lambda})$ is irreducible.

Proof. If λ is principal, then $\mathcal{V}_\lambda \simeq \mathcal{E}_{d(\lambda)}$. Part (1) readily follows from Theorem 3.19. By [BD1], Proposition 3.1.5(iii), if $\mathrm{g} > 1$, then the flat $G^\vee$-bundle underlying any $G^\vee$-oper does not admit a reduction to a nontrivial parabolic subgroup of $G^\vee$. This proves part (2). $\square$

Remark 3.21. If λ is not a principal weight, so V_λ is reducible as a representation of a principal $\mathfrak{sl}_2$ subalgebra of $\mathfrak{g}^\vee$ (see (3.16)), then there exists $\chi \in \mathrm{Op}^0_{G^\vee}(X)$ such that the flat vector bundle $(\mathcal{V}_\lambda, \nabla_{\chi,\lambda})$ is reducible. For example, this is so if the $G^\vee$-oper χ is in the image of the canonical embedding $\mathrm{Op}_{PGL_2}(X) \hookrightarrow \mathrm{Op}^0_{G^\vee}(X)$ constructed in [BD2], §3. Indeed, for such χ, the oper connection $\nabla_{\chi,\lambda}$ preserves the decomposition (3.17).

However, in the next subsection we will show that for generic $\chi \in \mathrm{Op}^0_{G^\vee}(X)$ the flat vector bundle $(\mathcal{V}_\lambda, \nabla_{\chi,\lambda})$ is irreducible for any $\lambda \in \Lambda^+$, if $\mathrm{g} > 1$ (see Corollary 3.32,(2)). $\square$

Recall from Subsection 3.5 that for any $\lambda \in \Lambda^+$ we have a canonical section $s_\lambda \in \Gamma(X, \mathcal{V}^K_\lambda)$ defined by the oper Borel reduction and the left annihilating ideal $I_{\lambda,\chi}$ of s_λ in the sheaf $\mathcal{D}_{X,-\frac{d(\lambda)}{2}}$. Let us specialize to the case of a principal weight λ. Corollary 3.20 then implies the following.

Lemma 3.22. *Suppose that λ is a principal weight of a group G. Then $\mathcal{V}^K_{\chi,\lambda}$ is an irreducible $\mathcal{D}_{X,-\frac{d(\lambda)}{2}}$-module, and we have an exact sequence of left $\mathcal{D}_{X,-\frac{d(\lambda)}{2}}$-modules*

$$0 \to I_{\lambda,\chi} \to \mathcal{D}_{X,-\frac{d(\lambda)}{2}} \to \mathcal{V}^K_{\chi,\lambda} \to 0$$

Recall from Subsection 3.5 that s_λ and $\overline{s}_{-w_0(\lambda)}$ are sections of $K^{-\frac{d(\lambda)}{2}}$ and $\overline{K}^{-\frac{d(\lambda)}{2}}$, respectively.

Proposition 3.23. *Let λ is a principal weight of a group G and $\chi \in \mathrm{Op}_{G^\vee}(X)_{\mathbb{R}}$. Then $h_{\chi,\lambda}(s_\lambda, \overline{s_{-w_0(\lambda)}})$ is a unique, up to a scalar, non-zero global C^∞ section of $\Omega_X^{-\frac{d(\lambda)}{2}}$ over X annihilated by the ideals $I_{\lambda,\chi}$ and $\overline{I_{-w_0(\lambda),\chi}}$.*

Proof. Clearly, $h_{\chi,\lambda}(s_\lambda, \overline{s_{-w_0(\lambda)}})$ satisfies the conditions of the lemma. Conversely, suppose that $\phi_\lambda(\chi)$ is a non-zero section of $\Omega_X^{-\frac{d(\lambda)}{2}}$ annihilated by the ideals $I_{\lambda,\chi}$ and $\overline{I_{-w_0(\lambda),\chi}}$. By the definition of these ideals, we then have a non-zero homomorphism of $\mathcal{D}_{X,-\frac{d(\lambda)}{2}} \otimes \overline{\mathcal{D}}_{X,-\frac{d(\lambda)}{2}}$-modules

$$\alpha_{\lambda,\chi} : \mathcal{V}^K_{\chi,\lambda} \otimes \overline{\mathcal{V}}^K_{\chi,-w_0(\lambda)} \to \Omega_X^{-\frac{d(\lambda)}{2}}$$

sending $s_\lambda \otimes \overline{s_{-w_0(\lambda)}}$ to $\phi_\lambda(\chi)$. Equivalently, we have a non-zero homomorphism of flat C^∞ vector bundles

$$(\mathcal{V}_\lambda, \nabla_{\chi,\lambda}) \otimes (\overline{\mathcal{V}}_{-w_0(\lambda)}, \overline{\nabla}_{\chi,-w_0(\lambda)}) \to (\mathcal{C}^\infty, d)$$

By Corollary 3.20, the flat vector bundle $(\mathcal{V}_\lambda, \nabla_{\chi,\lambda})$ is irreducible. Therefore the vector space of such homomorphisms is one-dimensional. Hence ϕ_λ is equal to a scalar multiple of $h_{\chi,\lambda}(s_\lambda, \overline{s_{-w_0(\lambda)}})$. □

For $G = PGL_n$, $\lambda = \omega_1$, let

$$I'_{\omega_1,\chi} := K_X^n \otimes I_{\omega_1,\chi}.$$

This is a left submodule of the $(\mathcal{D}_{X,\frac{n+1}{2}}, \mathcal{D}_{X,\frac{-n+1}{2}})$-bimodule of differential operators acting from $K_X^{\frac{-n+1}{2}}$ to $K_X^{\frac{n+1}{2}}$. The submodule $I'_{\omega_1,\chi}$ is generated by a globally defined nth order differential operator P_χ on X associated to χ by Lemma 3.14, that is

$$I'_{\omega_1,\chi} = \mathcal{D}_{X,\frac{n+1}{2}} \cdot P_\chi.$$

Therefore in this case a section annihilated by the ideal $I_{\lambda,\chi}$ is the same as a section satisfying the nth order differential equation (3.12).

A similar statement is true for a general principal weight λ of a group G. For $\chi \in \mathrm{Op}^0_{G^\vee}(X)$ let

$$I'_{\lambda,\chi} := K_X^{d(\lambda)+1} \otimes I_{\lambda,\chi},$$

which is a left submodule of the $(\mathcal{D}_{X,\frac{1+d(\lambda)}{2}}, \mathcal{D}_{X,-\frac{d(\lambda}{2}})$-bimodule of differential operators acting from $K_X^{-\frac{d(\lambda)}{2}}$ to $K_X^{\frac{1+d(\lambda)}{2}}$. As shown in the proof of Corollary 3.20, the flat vector bundle $(\mathcal{V}_\lambda, \nabla_{\chi,\lambda})$ has the structure of an $SL_{d(\lambda)+1}$-oper, which we will denote by $\widetilde{\chi}_\lambda$. This implies that the ideal $I'_{\lambda,\chi}$ is generated by the corresponding differential operator $P_{\widetilde{\chi_\lambda}}$ of order $d(\lambda)+1$ on X acting from $K_X^{-\frac{d(\lambda)}{2}}$ to $K_X^{1+\frac{d(\lambda)}{2}}$. Therefore, we again obtain that a section annihilated by the ideal $I_{\lambda,\chi}$ is the same as a section satisfying a differential equation of the form (3.12) of order $d(\lambda)+1$. Thus, Proposition 3.23 has the following equivalent reformulation which is an analogue of Corollary 3.17 for a general principal weight.

Corollary 3.24. *For a principal weight λ of a group G, $h_{\chi,\lambda}(s_\lambda, \overline{s_{-w_0(\lambda)}})$ is a unique, up to a scalar, non-zero section Φ of $\Omega_X^{-\frac{d(\lambda)}{2}}$ which is a solution of the system of differential equations*

$$P_{\widetilde{\chi_\lambda}} \cdot \Phi = 0, \qquad \overline{P^*_{\widetilde{\chi_\lambda}}} \cdot \Phi = 0. \tag{3.19}$$

Remark 3.25. (1) For $G^\vee$ of type B_ℓ (resp. C_ℓ) the map $\chi \mapsto P_{\widetilde{\chi_\lambda}}$ sets up a one-to-one correspondence between $\mathrm{Op}^0_{G^\vee}(X)$ and the space of self-adjoint (resp. anti-self adjoint) scalar differential operators, respectively, of order $d(\lambda)+1$ acting from $K_X^{-\frac{d(\lambda)}{2}}$ to $K_X^{1+\frac{d(\lambda)}{2}}$ and having symbol 1. This is proved in [BD2], §3 (following [DrS]) together with Lemma 3.14, which is the analogous statement for $G^\vee$ of type A_ℓ.

(2) If λ is not principal, then we do not expect that the ideal $I_{\lambda,\chi}$, or its twist such as $I'_{\lambda,\chi}$ is generated by a single globally defined differential operator. Hence in [EFK2], Section 5, we formulated everything in terms of the ideal $I_{\lambda,\chi}$ itself. See also Subsection 3.8 below.

Now we are going to formulate a conjectural analogue of Theorem 3.16 for principal weights (Conjecture 3.27).

For any $\lambda \in \Lambda^+$, let $\mathcal{V}^{\mathrm{univ}}_\lambda$ be the universal vector bundle over $\mathrm{Op}^0_{G^\vee}(X) \times X$ with a partial connection ∇^{univ} along X, such that

$$(\mathcal{V}^{\mathrm{univ}}_\lambda, \nabla^{\mathrm{univ}})|_{\chi \times X} = (\mathcal{V}_\lambda, \nabla_{\chi,\lambda}), \qquad \chi \in \mathrm{Op}^0_{G^\vee}(X).$$

Let $\pi : \mathrm{Op}^0_{G^\vee}(X) \times X \to X$ be the projection and set

$$\mathcal{V}^{\mathrm{univ}}_{X,\lambda} := \pi_*(\mathcal{V}^{\mathrm{univ}}_\lambda), \qquad \mathcal{V}^{K,\mathrm{univ}}_{X,\lambda} := K_X^{-\frac{d(\lambda)}{2}} \otimes \mathcal{V}^{\mathrm{univ}}_{X,\lambda}.$$

Then $\mathcal{V}^{K,\mathrm{univ}}_{X,\lambda}$ is naturally a $\mathcal{D}_{X,-\frac{d(\lambda)}{2}}$-module on X, equipped with a commuting action of $\mathrm{Fun}\,\mathrm{Op}^0_{G^\vee}(X) \simeq D_G$.

Moreover, the oper Borel reduction gives rise to an embedding

$$\kappa^{\mathrm{univ}}_\lambda : K_X^{-\frac{d(\lambda)}{2}} \hookrightarrow \mathcal{V}^{\mathrm{univ}}_{\lambda,X}$$

and hence a canonical section

$$s_\lambda^{\mathrm{univ}} \in \Gamma(X, \mathcal{V}_{X,\lambda}^{K,\mathrm{univ}}).$$

Consider the cyclic $D_G \otimes \mathcal{D}_{X,-\frac{d(\lambda)}{2}}$-module $(D_G \otimes \mathcal{D}_{X,-\frac{d(\lambda)}{2}}) \cdot s_\lambda^{\mathrm{univ}}$ generated by $s_\lambda^{\mathrm{univ}}$.

The next lemma, which follows from Corollary 3.20(2), is an analogue of Lemma 3.15 for a general principal weight.

Lemma 3.26. *For a principal weight λ of a group G, there is an isomorphism*

$$(D_G \otimes \mathcal{D}_{X,-\frac{d(\lambda)}{2}}) \cdot s_\lambda^{\mathrm{univ}} \simeq \mathcal{V}_{X,\lambda}^{K,\mathrm{univ}} \tag{3.20}$$

of $D_G \otimes \mathcal{D}_{X,-\frac{d(\lambda)}{2}}$-modules.

Recall that the Hecke operator H_λ is an operator-valued section of $\Omega_X^{-\frac{d(\lambda)}{2}}$. Hence we can apply to it the sheaf $\mathcal{D}_{X,-\frac{d(\lambda)}{2}}$ as well as the algebra D_G. The two actions commute, and they generate a $\mathcal{D}_{X,-\frac{d(\lambda)}{2}}$-module inside the sheaf of operator-valued C^∞ sections of $\Omega_X^{-\frac{d(\lambda)}{2}}$ on X. Let us denote this $\mathcal{D}_{X,-\frac{d(\lambda)}{2}}$-module by $\langle H_\lambda \rangle$. Likewise, we can apply to H_λ the sheaf $\overline{\mathcal{D}}_{X,-\frac{d(\lambda)}{2}}$ and the algebra $\overline{D}_G$. Denote the resulting $\overline{\mathcal{D}}_{X,-\frac{d(\lambda)}{2}}$-module by $\overline{\langle H_\lambda \rangle}$.

Recall that for any $\chi \in \mathrm{Op}^0_{G^\vee}$ we then have the corresponding differential operator $P_{\widetilde{\chi}_\lambda}$. These operators give rise to an analogue of the operator (3.15); namely,

$$\sigma : K_X^{-\frac{d(\lambda)}{2}} \to D_G \otimes K_X^{1+\frac{d(\lambda)}{2}} \tag{3.21}$$

satisfying the property that for any $\chi \in \mathrm{Op}^0_{G^\vee}(X) = \operatorname{Spec} D_G$, applying the corresponding homomorphism $D_G \to \mathbb{C}$ we obtain $P_{\widetilde{\chi}_\lambda}$.

The following statement is an analogue of Theorem 3.16 for a general principal weight.

Conjecture 3.27. For a principal weight λ of a group G, the Hecke operator H_λ satisfies the system of equations

$$\sigma \cdot H_\lambda = 0, \qquad \overline{\sigma} \cdot H_\lambda = 0. \tag{3.22}$$

Equivalently, there are isomorphisms

$$\langle H_\lambda \rangle \simeq \mathcal{V}_{X,\lambda}^{K,\mathrm{univ}}, \qquad \overline{\langle H_\lambda \rangle} \simeq \overline{\mathcal{V}_{X,\lambda}^{K,\mathrm{univ}}} \tag{3.23}$$

of $D_G \otimes \mathcal{D}_{X,-\frac{d(\lambda)}{2}}$-modules (resp. $\overline{D}_G \otimes \overline{\mathcal{D}}_{X,-\frac{d(\lambda)}{2}}$-modules).

The following conjecture follows from Conjecture 3.27 and Proposition 3.23 in the same way as Corollary 3.18 follows from Theorem 3.16 and Corollary 3.17 in the case $G = PGL_n, \lambda = \omega_1$.

Conjecture 3.28. Let λ be a principal weight of a group G. Each of the eigenvalues $\beta_\lambda(x, \overline{x})$ of the Hecke operator H_λ on $\mathcal{H}$ corresponding to a real oper $\chi \in \mathrm{Op}^0_{G^\vee}(X)_\mathbb{R}$ (see Conjecture 3.8) is equal to a scalar multiple of $h_\chi(s_\lambda, \overline{s_{-w_0(\lambda)}})$.

This is a special case (corresponding to the principal weights) of Conjecture 5.1 of [EFK2] which we mentioned in Conjecture 3.8(ii) above. In Subsection 3.8 we will discuss the general case (see Conjecture 3.38).

3.7. Monodromy of opers

Let G be a connected reductive algebraic group over $\mathbb{C}$ and X a smooth projective curve over $\mathbb{C}$ of genus $\mathrm{g} > 1$.

Let $\mathrm{LocSys}_{G^\vee}(X)$ be the stack of Betti $G^\vee$-local systems on X, and let $\mathrm{Conn}_{G^\vee}(X)$ be the stack of $G^\vee$-connections (i.e., de Rham $G^\vee$-local systems) on X. It contains the stack of $G^\vee$-opers which, according to [BD1, BD2], is the quotient of the variety of $G^\vee$-opers (which is a union of affine spaces which are torsors over **Hitch**) by the trivial action of the center $Z^\vee$ of $G^\vee$. Slightly abusing notation, in this section we will denote this stack by $\mathrm{Op}_{G^\vee}(X)$.

We have the analytic monodromy map $M : \mathrm{Conn}_{G^\vee}(X) \to \mathrm{LocSys}_{G^\vee}(X)$, which is an analytic isomorphism. Let $Z \subset \mathrm{Conn}_{G^\vee}(X)$ be the Zariski closed substack of connections whose differential Galois group is a proper subgroup of $G^\vee$.

Despite the map M not being algebraic, we have

Lemma 3.29. *$M(Z)$ is a Zariski closed substack of* $\mathrm{LocSys}_{G^\vee}(X)$.

Proof. $M(Z)$ can be defined algebraically in the Betti realization – it is the substack of local systems whose structure group is a proper subgroup of $G^\vee$. This implies the statement. □

Theorem 3.30. *Z does not contain $\mathrm{Op}_{G^\vee}(X)$. In other words, there exists a $G^\vee$-oper χ whose monodromy is Zariski dense in $G^\vee$ (equivalently, whose differential Galois group is the entire $G^\vee$).*

Proof. It is sufficient to prove the theorem in the case that $G^\vee$ is simple of adjoint type. In this case $\mathrm{Op}_{G^\vee}(X)$ is an affine space which we will view as a subvariety of $\mathrm{LocSys}_{G^\vee}(X)$. Since the automorphism group of the flat $G^\vee$-bundle underlying any $G^\vee$-oper is trivial if $G^\vee$ is of adjoint type (see

[BD2], §1.3), it follows that any $\chi \in \mathrm{Op}_{G^\vee}(X)$ has a Zariski open neighborhood $\mathrm{LocSys}_{G^\vee}(X)_\chi$ in $\mathrm{LocSys}_{G^\vee}(X)$ which is a smooth subvariety. The set $\mathrm{LocSys}_{G^\vee}(X)_\chi(\mathbb{C})$ of its $\mathbb{C}$-points is a smooth complex manifold.

Suppose that χ is a real $G^\vee$-oper. Then the set $\mathrm{LocSys}_{G^\vee}(X)_\chi(\mathbb{R})$ of $\mathbb{R}$-points of the variety $\mathrm{LocSys}_{G^\vee}(X)_\chi$ is a smooth real submanifold of $\mathrm{LocSys}_{G^\vee}(X)_\chi(\mathbb{C})$. Hence χ is a point of intersection of two smooth manifolds, $\mathrm{Op}_{G^\vee}(X)(\mathbb{C})$ and $\mathrm{LocSys}_{G^\vee}(X)_\chi(\mathbb{R})$, in $\mathrm{LocSys}_{G^\vee}(X)_\chi(\mathbb{C})$ (all viewed as real manifolds).

Following [Wa], call a real $G^\vee$-oper coming from a principal PGL_2 subgroup of $G^\vee$ *permissible*. According to Theorem A of [Wa], if χ is permissible, then the above two subvarieties are transversal at χ (note that this implies that permissible opers are discrete in $\mathrm{Op}_{G^\vee}(X)$). We will prove the Zariski density of the image of the monodromy representation of a generic $G^\vee$-oper by doing linear analysis around any given permissible $G^\vee$-oper (for instance, we can take the image of the real PGL_2-oper uniformizing X).

We start with an obvious lemma from linear algebra.

Lemma 3.31. *Let V be a finite dimensional real vector space of dimension $2d$ and U a complex subspace of $V_\mathbb{C}$ of dimension d transversal to V (as a real vector space), i.e., such that $U + V = V_\mathbb{C}$. Also, let W be a subspace of V. If U is contained in $W_\mathbb{C}$ then $W = V$.*

Proof. Since $U \oplus V = V_\mathbb{C}$, we have $W_\mathbb{C} \oplus V = V_\mathbb{C}$, hence $W = V$. □

According to Remark 3.5, for a permissible $G^\vee$-oper the Zariski closure of the corresponding monodromy representation is equal to a principal PGL_2 subgroup of $G^\vee$. Therefore, the Zariski closure of the monodromy group of a sufficiently generic $G^\vee$-oper χ has to be a subgroup K of $G^\vee$ containing its principal PGL_2 subgroup. Hence K is a semisimple group (it is the same for all sufficiently generic χ up to conjugacy). Fix a permissible $G^\vee$-oper ψ. Note that ψ is a smooth point of both $\mathrm{LocSys}_{G^\vee}(X)$ and $\mathrm{LocSys}_K(X)$, since the centralizer of the principal PGL_2 in $G^\vee$ is trivial (this also follows from the fact [BD2] we mentioned above that the group of automorphisms of any $G^\vee$-oper is trivial if $G^\vee$ is of adjoint type). Consider the tangent spaces $V := T_\psi \mathrm{LocSys}_{G^\vee}(X)(\mathbb{R})$, $U := T_\psi \mathrm{Op}_{G^\vee}(X)(\mathbb{C}) = \mathbf{Hitch}$, and $W := T_\psi \mathrm{LocSys}_K(X)(\mathbb{R})$. Then by assumption, U is contained in $W_\mathbb{C}$, and by Theorem A of [Wa], U and V (which have the same real dimension) are transversal in $V_\mathbb{C}$. Thus by Lemma 3.31 $W = V$, and hence $K = G^\vee$. This completes the proof of the theorem. □

Theorem 3.30 implies

Corollary 3.32. *(1) Opers $\chi \in \mathrm{Op}_{G^\vee}(X)$ whose differential Galois group is $G^\vee$ (equivalently, such that the Zariski closure of their monodromy is $G^\vee$) form a dense Zariski open subset $\mathcal{U} \subset \mathrm{Op}_{G^\vee}(X)$.*

(2) For a generic $\chi \in \mathrm{Op}_{G^\vee}(X)$ (namely, for $\chi \in \mathcal{U}$ from part (1)) the associated flat vector bundle $(\mathcal{V}_\lambda, \nabla_{\chi,\lambda})$ on X corresponding to an arbitrary dominant integral weight λ of $G^\vee$ is irreducible.

Suppose that $G^\vee$ is a simple algebraic group of adjoint type. Let χ be a $G^\vee$-oper and denote by M_χ the Zariski closure of the monodromy of χ. By Corollary 3.32, we have $M_\chi = G^\vee$ for generic χ. But this is not the case for some $G^\vee$-opers. For example, for the permissible $G^\vee$-opers χ discussed above (coming from a principal PGL_2 subgroup of $G^\vee$) we have $M_\chi = PGL_2$. The proof of the following result was communicated to us by D. Arinkin [Ari].

Theorem 3.33. *Suppose that the Zariski closure M_χ of the monodromy group of a $G^\vee$-oper χ on a curve of genus $\mathrm{g} > 1$ is a proper subgroup of $G^\vee$. Then $M_\chi \subset G'$, a proper simple subgroup of $G^\vee$ that contains a principal PGL_2 subgroup of $G^\vee$ and such that the flat $G^\vee$-bundle underlying χ is induced from a flat G'-bundle admitting a G'-oper structure.*

Proof. By going, if needed, to a finite cover of X, we can assume without loss of generality that M_χ is connected (the statement for this cover will imply the statement for X, since pullback of an oper to the cover is still an oper). By [BD1], Proposition 3.1.5(iii), M_χ is not contained in any non-trivial parabolic subgroup of $G^\vee$. Hence by Morozov's theorem (see e.g. [Bou], Ch. VIII, Section 10) it is contained in a proper connected reductive subgroup $G' \subset G^\vee$.

The flat $G^\vee$-bundle $E_\chi = (\mathcal{E}_{G^\vee}, \nabla_\chi)$ (where $\mathcal{E}_{G^\vee}$ is the $G^\vee$-bundle introduced after equation (3.2)) underlying χ can then be reduced to a flat G'-bundle $E_{G',\chi} = (\mathcal{E}_{G',\chi}, \nabla'_\chi)$, i.e. we have an isomorphism of flat $G^\vee$-bundles $E_{G',\chi} \times_{G'} G^\vee \cong E_\chi$.

Fix a point $x_0 \in X$. Conjugating inside $G^\vee$ if needed, we may assume that G' is the fiber of $\mathrm{Ad}(\mathcal{E}_{G',\chi})$ at x_0.

It is shown in [BH] that if H is a connected reductive group over $\mathbb{C}$, then any H-bundle $\mathcal{E}_H$ on X has a canonical Harder-Narasimhan reduction to a parabolic subgroup $P(\mathcal{E}_H)$ of H. These canonical Harder-Narasimhan reductions for $\mathcal{E}_{G',\chi}$ and $\mathcal{E}_{G^\vee}$ define a parabolic subgroup $P(\mathcal{E}_{G',\chi})$ of G' and the Borel subgroup $P(\mathcal{E}_{G^\vee}) = B^\vee \subset G^\vee$ (the one we have used in the definition of $G^\vee$-opers), respectively. These subgroups must be compatible. Therefore $P(\mathcal{E}_{G',\chi}) = B' := G' \cap B^\vee$ is a Borel subgroup of G', and $\mathcal{E}_{G',\chi} \cong \mathcal{E}_{B',\chi} \times_{B'} G'$, where $\mathcal{E}_{B',\chi}$ is a B'-bundle on X.

Let T' be a maximal torus in B' and $T^\vee$ a maximal torus of $B^\vee$, such that $T' \subset T^\vee$. Consider the $B'/[B',B'] \cong T'$-bundle $\mathcal{E}_{T',\chi}$ associated to $\mathcal{E}_{B',\chi}$. By construction, the induced $T^\vee$-bundle is the $T^\vee$-bundle $\mathcal{E}_{T^\vee}$ associated to the $B^\vee$-bundle $\mathcal{E}_{B^\vee}$ which is the oper $B^\vee$-reduction of the $G^\vee$-bundle $\mathcal{E}_{G^\vee}$. As explained at the beginning of Subsection 3.1, the $G^\vee$-bundle $\mathcal{E}_{G^\vee}$ is induced from the PGL_2-bundle corresponding to the vector bundle (3.2) under the principal embedding $PGL_2 \hookrightarrow G^\vee$. This implies that $\mathcal{E}_{T^\vee} \simeq K_X^\rho$ in the sense that for any character $\psi \in \mathbf{X}^*(T^\vee)$, the corresponding line bundle $\psi(\mathcal{E}_{T^\vee})$ is isomorphic to $K_X^{\langle\psi,\rho\rangle}$. This implies that the image of the cocharacter $\rho : \mathbb{G}_m \to T^\vee$ is contained in $T' \subset T^\vee$.

Let us trivialize the bundle $\mathcal{E}_{B',\chi}$ over the formal neighborhood of x_0, on which we pick a formal coordinate z. This trivializes $E_{G',\chi}$ as well as $\mathcal{E}_{B^\vee}$ and $\mathcal{E}_{G^\vee}$. We also obtain a trivialization of the corresponding adjoint vector bundles. The affine space of connections on $E_{G^\vee}$ is then identified with the space $d + \mathfrak{g}^\vee[[z]]dz$. The connections that come from connections on the G'-bundle $E_{G',\chi}$ belong to its subspace $d+\mathfrak{g}'[[z]]dz$. On the other hand, according to Definition 3.1, the oper connection ∇_χ has the form

$$\nabla_\chi = d + (f + b(z))dz, \tag{3.24}$$

where $f \in \mathfrak{g}^\vee$ is a principal nilpotent element satisfying $[\rho, f] = -f$ and $b(z) \in \mathfrak{b}^\vee[[z]]$.

Therefore $\mathfrak{g}'$ contains $f + b(0)$. Since $\mathfrak{g}'$ also contains ρ, we obtain that

$$\lim_{t\to 0} tAd(\rho(t))(f + b(0)) = f \in \mathfrak{g}'.$$

By the Jacobson-Morozov theorem, it now follows that G' contains a principal subgroup of $G^\vee$, as desired. Subtracting fdz from (3.24), we obtain that $b(z) \in \mathfrak{g}' \cap \mathfrak{b}^\vee[[z]] = \mathfrak{b}'[[z]]$. Therefore, $\mathcal{E}_{G',\chi}$ with its B'-reduction $\mathcal{E}_{B',\chi}$ and connection (3.24) is a G'-oper.

From the classical result of Dynkin (see [SS, EO]), which we recall in Theorem 3.34 below, it also follows that G' is simple. This completes the proof. $\square$

We recall the classification of pairs $G' \subset G^\vee$, where $G^\vee$ is a simple algebraic group over $\mathbb{C}$ and G' is its connected reductive subgroup containing a regular unipotent element of $G^\vee$ (and hence a principal PGL_2 subgroup of $G^\vee$). Following [EO], we will call such a pair (and the corresponding pair of Lie algebras) a *principal pair*. In the case when $G^\vee$ is of adjoint type that we are considering, it suffices to classify the principal pairs of Lie algebras. This classification is given by the following theorem, which is due to [SS] and in the characteristic 0 case goes back to the work of Dynkin (see [EO], Theorem 6.4).

Theorem 3.34. *The principal pairs of Lie algebras $\mathfrak{g}' \subset \mathfrak{g}^\vee$ (with a proper inclusion) are given by the following list:*

(1) $\mathfrak{sp}(2n) \subset \mathfrak{sl}(2n)$, $n \geq 2$;
(2) $\mathfrak{so}(2n+1) \subset \mathfrak{sl}(2n+1)$, $n \geq 2$;
(3) $\mathfrak{so}(2n+1) \subset \mathfrak{so}(2n+2)$, $n \geq 3$;
(4) $G_2 \subset \mathfrak{so}(7)$;
(5) $G_2 \subset \mathfrak{so}(8)$;
(6) $G_2 \subset \mathfrak{sl}(7)$;
(7) $F_4 \subset E_6$.
(8) $\mathfrak{sl}_2 \subset \mathfrak{g}^\vee$ *for any simple* $\mathfrak{g}^\vee$.

The Lie subalgebras $\mathfrak{g}'$ given in (1),(2),(3),(5),(7) are the invariant subalgebras of an automorphism of the Dynkin diagram $\mathfrak{g}^\vee$; (4) is obtained by composing (5) and (3); and (6) is obtained by composing (5) and (2).

Using Theorem 3.33 and Theorem 3.34, we can give a description of the possible Zariski closures M_χ of the monodromy groups of arbitrary $G^\vee$-opers χ. Namely, M_χ must be a simple subgroup of $G^\vee$ that contains a principal PGL_2 subgroup of $G^\vee$.

Lemma 3.35. *Suppose that a connected simple subgroup $H^\vee$ of a connected simple group $G^\vee$ contains a principal $(P)SL_2$ subgroup of $G^\vee$. Then we have a canonical map*

$$\alpha_{H^\vee,G^\vee} : \mathrm{Op}_{H^\vee}(X) \to \mathrm{Op}_{G^\vee}(X). \tag{3.25}$$

Proof. Under the condition of the lemma, the Borel subgroup $B_H^\vee$ of $H^\vee$ used in the definition of $H^\vee$-opers contains the Borel subgroup of a particular principal $(P)SL_2$ subgroup of $G^\vee$ with the Lie algebra spanned by the elements h and e of the corresponding principal $\mathfrak{sl}_2$ triple. Then $B_H^\vee$ is contained in a unique Borel subgroup $B^\vee$ of $G^\vee$. Definition 3.1 implies that for every $H^\vee$-oper η, the flat $G^\vee$-bundle induced from the flat $H^\vee$-bundle underlying η has a structure of $G^\vee$-oper with respect to $B^\vee$. This structure is unique by Lemma 3.3. Since the space of opers does not depend on the choice of a Borel subgroup by Lemma 3.2, we obtain a canonical map (3.25). □

Theorem 3.36. *Let M_χ be the Zariski closure of the monodromy group of a $G^\vee$-oper χ on a curve of genus $\mathrm{g} > 1$, where $G^\vee$ is a simple algebraic group of adjoint type. Then M_χ is a simple algebraic group of adjoint type containing a principal PGL_2 subgroup of $G^\vee$ and χ is in the image of the map $\alpha_{M_\chi,G^\vee}$.*

Proof. We argue by induction on $d = \dim(G^\vee)$, with the base being the case $d = 3$, which follows from Remark 3.5. Suppose that $M_\chi \neq G^\vee$. Then by Theorem 3.33 and Lemma 3.35, M_χ is contained in a simple subgroup $G' \subset G^\vee$ and χ is in the image of the map $\alpha_{G',G^\vee}$. But $\dim G' < \dim G^\vee$ and by passing to a finite cover of X, we can assume without loss of generality that G' is connected. Moreover, according to the list of Theorem 3.34, if $G^\vee$ is of adjoint type, then so is G'. Passing from $G^\vee$ to G' and using our inductive assumption, we obtain the result. ☐

Remark 3.37. Explicit examples of opers with proper subgroups $M_\chi \subset G^\vee$ (in genus 0 with ramification) can be found in [FG].

3.8. General case, continued

Let G be a connected simple algebraic group and $\lambda \in \Lambda^+$. We will use the notation introduced in Subsection 3.6. In Conjecture 5.1 of [EFK2] we gave a conjectural formula for the eigenvalues $\beta_\lambda(x, \overline{x})$ of the Hecke operator H_λ. It generalizes Corollary 3.18 in the case $G = PGL_n$ and Conjecture 3.28 in the case of principal weights. Actually, for each $\chi \in \mathrm{Op}^0_{G^\vee}(X)_\mathbb{R}$ there are (conjecturally) finitely many eigenvalues which differ from each other by a root of unity (see [EFK2], Remark 5.1). Hence the statement Conjecture 5.1 of [EFK2] is made in the form "up to a non-zero scalar." Here it is in terms of the notation of the present paper.

Conjecture 3.38. The eigenvalues $\beta_\lambda(x, \overline{x})$ of the Hecke operator H_λ corresponding to $\chi \in \mathrm{Op}^\gamma_{G^\vee}(X)_\mathbb{R}$ are equal to $h_{\chi,\lambda}(s_\lambda, \overline{s_{-w_0(\lambda)}})$ up to a non-zero scalar.

In [EFK2], Section 5, we gave the following strategy for proving this conjecture:

(1) Prove that eigenvalues $\beta_\lambda(x, \overline{x})$ satisfy a system or differential equations

$$D\beta_\lambda(x, \overline{x}) = 0, \qquad \overline{D'}\beta_\lambda(x, \overline{x}) = 0, \qquad D \in I_{\lambda,\chi}, D' \in I_{-w_0(\lambda),\chi}. \tag{3.26}$$

This is essentially the statement of [EFK2], Conjecture 5.5.

(2) Show that every solution of the system (3.26) is equal to $h_{\chi,\lambda}(s_\lambda, \overline{s_{-w_0(\lambda)}})$ up to a non-zero scalar. In the case when the monodromy representation of the flat vector bundle $(\mathcal{V}_\lambda, \nabla_{\chi,\lambda})$ is irreducible, this is the statement of [EFK2], Corollary 5.4.

We now consider this statement in the case when the monodromy representation of the flat vector bundle $(\mathcal{V}_\lambda, \nabla_{\chi,\lambda})$ is reducible, using the results of the previous subsection. For simplicity, we will assume that $G^\vee$ is of adjoint type.

Proposition 3.39. *For every $\chi \in \mathrm{Op}^0_{G^\vee}(X)_{\mathbb{R}}$, $h_{\chi,\lambda}(s_\lambda, \overline{s_{-w_0(\lambda)}})$ is a unique, up to a scalar, non-zero section of $\Omega_X^{-\frac{d(\lambda)}{2}}$ annihilated by the ideals $I_{\lambda,\chi}$ and $\overline{I_{-w_0(\lambda),\chi}}$.*

Proof. In the case when the monodromy representation of the flat vector bundle $(\mathcal{V}_\lambda, \nabla_{\chi,\lambda})$ is irreducible, this was proved in [EFK2], Corollary 5.4.

Suppose now that $(\mathcal{V}_\lambda, \nabla_{\chi,\lambda})$ is reducible. This means that the Zariksi closure $M = M_\chi$ of the monodromy of χ is a proper subgroup of $G^\vee$. By Theorem 3.36, M is a simple subgroup of $G^\vee$ (also of adjoint type) that contains a principal PGL_2 subgroup of $G^\vee$, and χ is the image of an M-oper χ^M under the map $\alpha_{M,G^\vee} : \mathrm{Op}_M(X) \to \mathrm{Op}_{G^\vee}(X)$.

Under the action of M, the representation V_λ of $G^\vee$ decomposes into a direct sum of irreducible representations of M. Denote by V^M_λ its component containing the highest weight subspace of V_λ with respect to the Borel subgroup $B^\vee \cap M$ of M (where $B^\vee$ is the Borel subgroup of $G^\vee$ we have used to define $G^\vee$-opers). Since $\mathfrak{m} := \mathrm{Lie}(M)$ contains a principal $\mathfrak{sl}_2$ subalgebra, V^M_λ contains the highest component $V_{d(\lambda)}$ in the decomposition (3.16) of V_λ under the principal $\mathfrak{sl}_2$. This implies that the dual representation $(V^M_\lambda)^*$ is isomorphic to $V^M_{-w_0(\lambda)}$, the component containing the highest weight subspace of $V_{-w_0(\lambda)} = V^*_\lambda$.

The decomposition of V_λ into a direct sum of irreducible representations of M gives rise to a direct sum decomposition of the flat vector bundle $(\mathcal{V}_\lambda, \nabla_{\chi,\lambda})$ into irreducible flat vector bundles each having real monodromy. In particular, the flat subbundle $(\mathcal{V}^M_\lambda, \nabla_{\chi,\lambda})$ of $(\mathcal{V}_\lambda, \nabla_{\chi,\lambda})$ corresponding to the highest weight component $V^M_\lambda \subset V_\lambda$ is irreducible and has real monodromy. Therefore, there is a unique up to a scalar non-zero pairing

$$h^M_{\chi,\lambda}(\cdot,\cdot) : (\mathcal{V}^M_\lambda, \nabla_{\chi,\lambda}) \otimes (\overline{\mathcal{V}}^M_{-w_0(\lambda)}, \overline{\nabla}_{\chi,-w_0(\lambda)}) \to (\mathcal{C}^\infty_X, d),$$

where $\mathcal{V}^M_{-w_0(\lambda)}$ is the subbundle of $\mathcal{V}_{-w_0(\lambda)}$ corresponding to $V^M_{-w_0(\lambda)} \simeq (V^M_\lambda)^*$. From the above direct sum decomposition it is clear that $h^M_{\chi,\lambda}(\cdot,\cdot)$ is equal to the restriction of $h_{\chi,\lambda}(\cdot,\cdot)$ to $\mathcal{V}^M_\lambda \otimes \overline{\mathcal{V}}^M_{-w_0(\lambda)}$ up to a non-zero scalar.

Moreover, the canonical section $s_\lambda \in \Gamma(X, K_X^{-\frac{d(\lambda)}{2}} \otimes \mathcal{V}_\lambda)$ corresponding to the Borel reduction of the $G^\vee$-oper χ (see formula (3.15)) is equal to the image of the canonical section $s^M_\lambda \in \Gamma(X, K_X^{-\frac{d(\lambda)}{2}} \otimes \mathcal{V}^M_\lambda)$ for the M-oper χ^M under the embedding $\mathcal{V}^M_\lambda \hookrightarrow \mathcal{V}_\lambda$. And likewise, for the canonical sections $s_{-w_0(\lambda)}$ and $s^M_{-w_0(\lambda)}$.

Using irreducibility of the flat bundle $(\mathcal{V}^M_\lambda, \nabla_{\chi,\lambda})$ in the same way as in the proof of Corollary 3.23, we obtain that $h^M_{\chi,\lambda}(s^M_\lambda, \overline{s^M_{-w_0(\lambda)}})$ is a unique, up to a scalar, non-zero section of $\Omega_X^{-\frac{d(\lambda)}{2}}$ annihilated by the ideals $I_{\lambda,\chi}$ and $\overline{I_{-w_0(\lambda),\chi}}$. But according to the above discussion, $h^M_{\chi,\lambda}(s^M_\lambda, \overline{s^M_{-w_0(\lambda)}})$ is equal to $h_{\chi,\lambda}(s_\lambda, \overline{s_{-w_0(\lambda)}})$ up to a non-zero scalar. This implies the statement of the proposition. □

This proves part (2) of the above argument. Hence in order to prove Conjecture 5.1 of [EFK2] it remains to prove Conjecture 5.5 of [EFK2] (see [EFK2], Section 5.3 for an outline of how to prove it using the results of [BD1]).

3.9. Functoriality in the analytic Langlands correspondence

Consider the framework of the Langlands Program for a smooth projective curve X over a finite field $\mathbb{F}_q$. Let G and H be two split reductive algebraic groups over $\mathbb{F}_q$, and $G^\vee$ and $H^\vee$ their Langlands dual groups. The *Langlands functoriality principle* (see [Art] for a survey) is the statement that for any homomorphism

$$a : H^\vee \to G^\vee \tag{3.27}$$

between them there should be a map (sometimes called *transfer*) from the set of L-packets of tempered automorphic representations of $H(\mathbb{A}_F)$ to the set of L-packets of tempered automorphic representations of $G(\mathbb{A}_F)$ (where $\mathbb{A}_F$ is the ring of adeles of $F = \mathbb{F}_q(X)$, the function field of X).

The existence of such a map is quite surprising: even though we have a homomorphism (3.27) of dual groups $a : H^\vee \to G^\vee$, there is *a priori* no connection between the groups G and H. The explanation is found on the dual side of the Langlands correspondence, which gives a parameterization of L-packets of tempered automorphic representations of $G(\mathbb{A})$ in terms of homomorphisms $W(F) \to G^\vee$, where $W(F)$ is the Weil group of F. Given a homomorphism (3.27), every Langlands parameter $\sigma : W(F) \to H^\vee$ for $H(\mathbb{A})$ gives rise to a Langlands parameter $a \circ \sigma : W(F) \to G^\vee$ for $G(\mathbb{A}_F)$.

This interpretation also makes it clear that functoriality should satisfy the following *transitivity* property: if K is another reductive group and we have a chain of homomorphisms of dual groups:

$$K^\vee \to H^\vee \to G^\vee, \tag{3.28}$$

then the composition of the transfers from $K(\mathbb{A}_F)$ to $H(\mathbb{A}_F)$ and from $H(\mathbb{A}_F)$ to $G(\mathbb{A}_F)$ should coincide with the transfer obtained directly from the composition $K^\vee \to G^\vee$.

Another important property is that the Hecke eigenvalues of the automorphic representations (at the unramified places) should match under the transfer in a natural way.

What should be the analogues of the Langlands functoriality and the transfers in the analytic Langlands correspondence for a curve X over $\mathbb{C}$?

According to our main Conjecture 3.8, the role of the Langlands parameters for a reductive group X is now played by real $G^\vee$-opers. From Theorem 3.36, it is clear that the homomorphisms (3.27) that we should consider are the ones that map a principal SL_2 (or PGL_2) subgroup of $H^\vee$ to a principal SL_2 (or PGL_2) subgroup of $G^\vee$. We will call such homomorphisms *principal*. For simple $G^\vee$, at the level of Lie algebras, the list of principal homomorphisms is given in Theorem 3.34 following [SS] and [EO] (note also that we have discussed principal embeddings in the case when $\mathfrak{g}^\vee = \mathfrak{sl}_n$ in Subsection 3.6.)

Suppose for simplicity that $G^\vee$ is of adjoint type. Then it follows from Theorem 3.34 that $H^\vee$ is also of adjoint type. Given a principal homomorphism (3.27), by Lemma 3.35 we obtain a canonical map

$$\alpha_{H^\vee,G^\vee} : \mathrm{Op}_{H^\vee}(X) \to \mathrm{Op}_{G^\vee}(X) \tag{3.29}$$

which is an embedding of affine spaces in the case when $H^\vee$ and $G^\vee$ are of adjoint type. The following result follows immediately from Theorem 3.36.

Proposition 3.40. *A $G^\vee$-oper in the image of $\alpha_{H^\vee,G^\vee}$ which is not in the image of $\alpha_{K^\vee,G^\vee}$ for any $K^\vee \subset H^\vee$ consists of the $G^\vee$-opers on X such that the Zariski closure of their monodromy is equal to $H^\vee$.*

Thus, we obtain a stratification of the affine space $\mathrm{Op}_{G^\vee}(X)$ by affine subspaces given by the images of the embeddings $\alpha_{H^\vee,G^\vee}$ corresponding to all principal homomorphisms (3.27). It gives rise to the corresponding family of embeddings of the sets of real opers, which are the Langlands parameters of the analytic Langlands correspondence:

$$\alpha^{\mathbb{R}}_{H^\vee,G^\vee} : \mathrm{Op}_{H^\vee}(X)_{\mathbb{R}} \hookrightarrow \mathrm{Op}_{G^\vee}(X)_{\mathbb{R}}. \tag{3.30}$$

Since these maps are transitive for a pair of embeddings (3.28), Conjecture 3.8 implies that an analogue of the transitivity property holds in the analytic Langlands correspondence.

Next, by analogy with the Langlands functoriality discussed above, we expect that the eigenvalues of the Hecke operators for the groups G and H related by a principal homomorphism (3.27) should match. Let us verify that this is compatible with our Conjecture 3.38 (which is Conjecture 5.1 of [EFK2]) that gives an explicit formula for these eigenvalues up to a scalar).

Given $\lambda \in \Lambda^+_G$, let $a(\lambda) \in \Lambda^+_H$ be the dominant integral weight of $H^\vee$ obtained via the homomorphism (3.27). We have the corresponding Hecke operators H_λ and $H_{a(\lambda)}$ for the groups G and H, respectively, depending on a point of X. According to Conjecture 3.38, the eigenvalues H_λ and $H_{a(\lambda)}$ are parametrized by real opers in $\mathrm{Op}_{G^\vee}(X)_\mathbb{R}$ and $\mathrm{Op}_{H^\vee}(X)_\mathbb{R}$, respectively (more precisely, for each real oper, we expect finitely many Hecke eigenvalues differing by a root of unity; see [EFK2], Remark 5.1 for more details). Given $\chi \in \mathrm{Op}_{H^\vee}(X)_\mathbb{R}$, denote by $a(\chi)$ the image of χ under the embedding (3.30). Let $\beta^\chi_{a(\lambda)}(x,\overline{x})$ be the eigenvalue of $H_{a(\lambda)}$, and $\beta^{a(\chi)}_\lambda(x,\overline{x})$ the corresponding eigenvalue of H_λ. Matching of these eigenvalues means that they are equal up to an overall non-zero scalar (independent of $x, \overline{x}$).

Our conjectural formula for these eigenvalues in Conjecture 3.38 says that $\beta^\chi_{a(\lambda)}(x,\overline{x}) = h^{H^\vee}_{\chi,\lambda}(s_{a(\lambda)}, \overline{s_{a(-w_0(\lambda))}})$ and $\beta^{a(\chi)}_\lambda(x,\overline{x}) = h^{G^\vee}_{a(\chi),\lambda}(s_\lambda, \overline{s_{-w_0(\lambda)}}))$ up to non-zero scalars. But we have shown in the proof of Proposition 3.39 that the two expressions are proportional to each other. Hence we find that our conjectural formulas for the Hecke eigenvalues are indeed compatible with the analytic Langlands version of functoriality.

Remark 3.41. An important example of functoriality in the case of a curve over a finite field comes from the embedding of a maximal torus of the group $G^\vee$, $T^\vee \hookrightarrow G^\vee$. In this case, the corresponding automorphic functions for the group G over the adeles are known as the *Eisenstein series*. The above discussion explains why we do not expect analogues of Eisenstein series in the analytic Langlands correspondence for a simple algebraic group G and a curve over $\mathbb{C}$ (as we can see from Conjecture 3.8): such an embedding is not principal and therefore should not lead to functoriality.

3.10. Analytic Langlands correspondence twisted by an Aut G-torsor on X

Analytic Langlands correspondence can be naturally generalized to the case when the connected reductive group G is replaced by a flat group scheme $\mathcal{G}$ over X with fibers isomorphic to G. For simplicity let us discuss this theory in the case of complex curves ($F = \mathbb{C}$).

Example 3.42. Suppose $G = \mathbb{G}_m^n$ is an n-dimensional torus. Then the possible groups $\mathcal{G}$ are parametrized by homomorphisms $\phi : \pi_1(X) \to GL_n(\mathbb{Z})$ with finite image Γ. Let $\widetilde{X}$ be the cover of X corresponding to the kernel of ϕ. Then Γ acts on $\widetilde{X}$ and $\mathcal{G}$ trivializes over $\widetilde{X}$, so the corresponding moduli stack $\mathrm{Bun}_{\mathcal{G}}(X)$ is the stack of Γ-equivariant G-bundles on $\widetilde{X}$. Set-theoretically, this is the subgroup of $\mathrm{Pic}(\widetilde{X})^n$ consisting of bundles E with a consistent family of isomorphisms $\gamma_* E \cong \phi(\gamma)E$, $\gamma \in \Gamma$. In this case it is easy to see (see [EFK2]) that the spectrum of Hecke operators is parametrized by Γ-equivariant real $G^\vee = \mathbb{G}_m^n$-opers on $\widetilde{X}$, i.e., lifts $\rho : \pi_1(X) \to \Gamma \ltimes \mathbb{G}_m^n(\mathbb{R})$ of ϕ such that the local system $\rho : \mathrm{Ker}\,\phi = \pi_1(\widetilde{X}) \to \mathbb{G}_m^n(\mathbb{R})$ is an oper (hence also an anti-oper), similarly to Subsection 3.1.

Example 3.43. Suppose that G is adjoint. In this case, the possible groups $\mathcal{G}$ are classified by $H^1(X, \mathrm{Aut}\,\Delta_G \ltimes \mathcal{O}_{X,G})$, where $\mathcal{O}_{X,G}$ is the sheaf of regular functions on X with values in G. Two elements $\theta_1, \theta_2 \in H^1(X, \mathrm{Aut}\,\Delta_G \ltimes \mathcal{O}_{X,G})$ which map to the same element in $H^1(X, \mathrm{Aut}\,\Delta_G) = \mathrm{Hom}(\pi_1(X), \mathrm{Aut}\,\Delta_G)$ define Morita equivalent groups $\mathcal{G}$, so the corresponding moduli spaces are the same ([Br], Subsection 1.6, Proposition 1.2). In other words, similarly to Subsection 2.4, the theory depends only on the inner class of $\mathcal{G}$ (see [Br], Remark 1.2). This is, in fact, a general feature which extends beyond $F = \mathbb{C}$.

Thus we may restrict ourselves to groups $\mathcal{G}$ obtained from maps $\phi : \pi_1(X) \to \mathrm{Aut}\,\Delta_G$. So, similarly to Example 3.42 we may define $\Gamma := \mathrm{Im}\,\phi$ and realize the corresponding stack $\mathrm{Bun}_{\mathcal{G}}(X)$ as the stack of Γ-equivariant principal G-bundles on the Γ-cover $\widetilde{X}$ of X. As in Example 3.42, we expect that the spectrum of Hecke operators is parametrized by Γ-equivariant real $G^\vee$-opers on $\widetilde{X}$, i.e. lifts $\rho : \pi_1(X) \to \Gamma \ltimes G^\vee(\mathbb{R})$ of ϕ whose restriction to $\mathrm{Ker}\,\phi$ is an oper (hence also an anti-oper).

Remark 3.44. More generally, suppose a finite group Γ acts simultaneously on G by root datum automorphisms and on a curve $\widetilde{X}$. Then we can consider harmonic analysis on the space $\mathrm{Bun}_G^\circ(\widetilde{X})^\Gamma$ of Γ-equivariant regularly stable G-bundles on $\widetilde{X}$. If Γ acts on $\widetilde{X}$ freely, this reduces to the above setting with $X = \widetilde{X}/\Gamma$, but the theory extends naturally to the case when the action is not necessarily free. We note that such moduli spaces of twisted bundles have been recently studied in connection with twisted conformal blocks and twisted Verlinde formula, see [DM, HK]. In the framework of the usual Langlands correspondence over function fields, such twisted setting is considered in [L], Section 12.

4. Analytic Langlands correspondence over $\mathbb{R}$

4.1. The general setup

In this section we will focus on the case $F = \mathbb{R}$ and propose a conjectural description of the spectrum of Hecke operators in terms of $G^\vee$-opers satisfying suitable reality conditions, generalizing the results of [EFK3], Subsection 4.7. Much of our analysis is based on [GW], Section 6.

We first specialize the setting of Subsection 2.2 to the case $F = \mathbb{R}$, so $F_{\mathrm{sep}} = \mathbb{C}$ and $\Gamma_F = \mathbb{Z}/2$. We will only consider either real or complex points of algebraic groups G, and will write G for $G(\mathbb{C})$ when no confusion is possible. Let G a split connected reductive group defined over $\mathbb{Q}$ and $G^\vee$ its Langlands dual group. Let $Z, Z^\vee$ be the centers of $G, G^\vee$. These groups are equipped with a natural operation of complex conjugation, $g \mapsto \overline{g}$. A **real structure** on G is a holomorphic automorphism $\theta : G \to G$ satisfying the 1-cocycle condition $\theta \circ \theta^* = \mathrm{Id}$, where $\theta^*(g) := \overline{\theta(\overline{g})}$. Two such 1-cocycles differ by a coboundary iff the corresponding real structures are isomorphic. In fact, one can (and usually does) choose a representative θ of its cohomology class so that θ commutes with complex conjugation, i.e., $\theta^* = \theta$ and $\theta^2 = \mathrm{Id}$, which gives rise to the **Satake diagram** of the corresponding real form.[27] Such θ gives rise to an **antiholomorphic involution** $\sigma(g) := \theta(\overline{g})$. The corresponding **group of real points** $G^\sigma = G^\sigma(\mathbb{R})$ (which may be disconnected) is the subgroup of $g \in G$ stable under σ, i.e., satisfying $\theta(g) = \overline{g}$. The inner class of σ gives rise to a root datum involution $s = s_\sigma$ for G which is also one for $G^\vee$.

Recall [ABV] that to G, s we may attach the **Langlands L-group** ${}^LG = {}^LG_s$, the semidirect product of $\mathbb{Z}/2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by $G^\vee$, with the action of $\mathbb{Z}/2$ defined by $\omega \circ s$, where ω is the **Chevalley involution** defining the compact form of G.

Let $X = X(\mathbb{C})$ be a compact complex Riemann surface of genus $\mathrm{g} \geq 2$. Let $\tau : X \to X$ be an antiholomorphic involution. We specialize the setting of Subsections 2.4, 2.5, 2.8 to the case $F = \mathbb{R}$. Given a holomorphic principal G-bundle P on X, we can define the antiholomorphic bundle $\tau(P)$, hence a holomorphic bundle $(\sigma, \tau)(P)$. A **pseudo-real structure** on P is an isomorphism $A : (\sigma, \tau)(P) \to P$. Such a structure defines a class α_P in

$$H^2(\mathbb{Z}/2, Z^s(\mathbb{C})) = \mathrm{Ker}(1 - s|_Z)/\,\mathrm{Im}(1 + s|_Z)$$

[27] Another possibility is to choose θ to commute with the complex conjugation of the compact form of G, which gives rise to the **Vogan diagram** of the real form.

which depends only on P and not on A. A pseudo-real structure A on P is a **real structure** if

$$A \circ (\sigma, \tau)(A) = 1 \tag{4.1}$$

(in particular, this means that the cocycle a_A and hence the class α_P equals 1). For example, if G is adjoint then the isomorphism A is unique if exists and (4.1) is automatic if $\mathrm{Aut}(P) = 1$, which happens for regularly stable bundles. Note that if $g \in G$ and $g\sigma(g) = 1$ then

$$Ag^{-1} : (\mathrm{Ad}(g)\sigma, \tau)(P) \to P$$

satisfies (4.1) with σ replaced by $\sigma' = \mathrm{Ad}(g)\sigma$. Thus the space of such regularly stable bundles depends only on the inner class s of σ, in agreement with Subsection 2.4 (see also [BGH], Proposition 3.8). Following Subsection 2.4, we denote this space by $\mathrm{Bun}^\circ_{G,s}(X, \tau)$.

The space $\mathrm{Bun}^\circ_{G,s}(X, \tau)$ is a real analytic manifold, which is a disjoint union of open submanifolds $\mathrm{Bun}^\circ_{G,s,\alpha}(X, \tau)$, $\alpha \in H^2(\mathbb{Z}/2, Z^s(\mathbb{C}))$. Moreover, for every character χ of

$$H^1(\mathbb{Z}/2, Z^s(\mathbb{C})) = \mathrm{Ker}(1 + s|_Z)/\,\mathrm{Im}(1 - s|_Z)$$

we have a Hermitian line bundle $\mathcal{L}_\chi$ on each $\mathrm{Bun}^\circ_{G,s,\alpha}(X, \tau)$ defined in Subsection 2.8.

Let

$$\mathcal{H}(s, \tau, \alpha, \chi) := L^2(\mathrm{Bun}^\circ_{G,s,\alpha}(X, \tau), \mathcal{L}_\chi)$$

be the Hilbert space of L^2 half-densities on $\mathrm{Bun}^\circ_{G,s,\alpha}(X, \tau)$ valued in $\mathcal{L}_\chi$. Let

$$\mathcal{H}(s, \tau, \alpha) = \oplus_\chi \mathcal{H}(s, \tau, \alpha, \chi),\ \ \mathcal{H}(s, \tau) = \oplus_\alpha \mathcal{H}(s, \tau, \alpha).$$

We have (conjecturally) a spectral decomposition of $\mathcal{H}(s, \tau)$ under the action Hecke operators compatible with the (α, χ)-grading.

Remark 4.1. To be more precise, the definition of the Langlands L-group in [ABV] uses s instead of $\omega \circ s$. This is in fact a major difference between the classical Langlands correspondence for real groups and the analytic Langlands correspondence for curves over $\mathbb{R}$. An explanation of this phenomenon is provided by [EFK1], Proposition 3.6, which says that taking the formal adjoint of quantum Hitchin Hamiltonians corresponds to applying the Chevalley involution on opers. See also Remark 4.3 below.

4.2. The case when τ has no fixed points

We first consider the easier case when τ has no fixed points. Let ρ be a local system on the non-orientable surface X/τ with structure group ${}^LG$. If we choose a base point $p \in X/\tau$ then such a local system corresponds to a homomorphism $\pi_1(X/\tau, p) \to {}^LG$ which is unique up to conjugation. We will say that ρ is an **L-system** if it attaches to every orientation-reversing path in X/τ a conjugacy class in ${}^LG$ that maps to the nontrivial element in $\mathbb{Z}/2$. The following conjecture is equivalent to the conjecture made in [GW], Section 6.2 on the basis of insights from 4-dimensional supersymmetric gauge theory (as well as the duality proposal from [BS]).

Conjecture 4.2. (i) There is an orthogonal decomposition

$$\mathcal{H}(s,\tau,1) = \bigoplus_{\rho} \mathcal{H}(s,\tau,1)_\rho,$$

where ρ runs over L-systems on X/τ with values in ${}^LG = {}^LG_s$ whose pullback to X have the structure of a $G^\vee$-oper.

(ii) For $\lambda \in \Lambda_+$ the Hecke operator $H_{\lambda x + s(\lambda)\overline{x}}$ acts on $\mathcal{H}(s,\tau,1)_\rho$ by the eigenvalue $\boldsymbol{\beta}_{\rho,\lambda}(x,\overline{x})$ defined by the formula in [EFK2], Conjecture 5.1. In particular, if $G = PGL_n$, $\lambda = \omega_1$ and $L_\rho = \partial^n + a_2\partial^{n-2} + \cdots + a_n$ is the SL_n-oper (i.e., holomorphic differential operator $K_X^{\frac{1-n}{2}} \to K_X^{\frac{1+n}{2}}$) corresponding to ρ then $\boldsymbol{\beta}_{\lambda,\rho}(x,\overline{x})$ is the (unique up to scaling) single-valued section of $|K_X|^{1-n}$ satisfying the system of oper equations $L_\rho\boldsymbol{\beta} = 0, \overline{L_\rho^*}\boldsymbol{\beta} = 0$.

Remark 4.3. More precisely, as was explained to us by E. Witten, what comes from ordinary gauge theory is this picture for the **compact** inner class s. To obtain other inner classes, one needs to consider **twisted gauge theory** where the twisting is by a root datum automorphism of G. Namely, gauge fields in this theory are invariant under complex conjugation τ composed with this automorphism. This may be seen as the physical explanation of the appearance of the Chevalley involution in the definition of ${}^LG$ in analytic Langlands correspondence, which does not happen in the usual Langlands correspondence for real groups.

Example 4.4. Let $s = \omega$ (the compact inner class). Then ${}^LG = \mathbb{Z}/2 \times G^\vee$, so an L-system is the same thing as a $G^\vee$-local system on X/τ. So in this case according to Conjecture 4.2, the spectral local systems are ρ which are isomorphic to ρ^τ and such that ρ is an oper (hence also an anti-oper), so ρ is a real oper "with real coefficients". But among these we should only choose those local systems that descend to X/τ and the eigenspaces are labeled by these extensions.

More precisely, recall that opers for adjoint groups have no nontrivial automorphisms ([BD2], §1.3). So for any connected reductive G we get an obstruction for such ρ to descend to X/τ which lies in $Z^\vee/(Z^\vee)^2 = H^2(\mathbb{Z}/2, Z^\vee)$. Moreover, if this obstruction vanishes then the freedom for choosing the extension is in a torsor over $H^1(\mathbb{Z}/2, Z^\vee) = Z_2^\vee$, the 2-torsion subgroup in $Z^\vee$.

Indeed, $\pi_1(X/\tau)$ is generated by $\pi_1(X)$ and an element t such that $tbt^{-1} = \gamma(b)$ for some automorphism γ of $\pi_1(X)$, and $t^2 = c \in \pi_1(X)$, so that $\gamma^2(b) = cbc^{-1}$. So given a representation $\rho : \pi_1(X) \to G^\vee$, an L-system would be given by an assignment $\rho(t) = T \in G^\vee$ such that (1) $T^2 = \rho(c)$ and (2) $T\rho(a)T^{-1} = \rho(\gamma(a))$. If $\rho \cong \rho \circ \gamma$ then T satisfying (2) is unique up to multiplying by $u \in Z^\vee$, and $T^2 = \rho(c)z$, $z \in Z^\vee$. Moreover, if T is replaced by Tu then z is replaced by zu^2, hence the obstruction to satisfying (1) lies in $Z^\vee/(Z^\vee)^2$. And if this obstruction vanishes, then the choices of T form a torsor over $Z_2^\vee$ acting by $T \mapsto Tz$.

Remark 4.5. As pointed out in [GW], Section 6, this reality condition on the $G^\vee$-oper on X is equivalent to the condition that ρ extends as a topological local system to the 3-manifold

$$U_\tau := (X \times [-1, 1])/(\tau, -\mathrm{Id})$$

whose boundary is X, introduced in [GW], and this extension is a part of the data. This follows from the fact that the inclusion $X/\tau \hookrightarrow U_\tau$ is a homotopy equivalence.

Remark 4.6. We have the inflation-restriction exact sequence

$$H^1(\pi_1(X), Z^s(\mathbb{C}))^{\mathbb{Z}/2} \to H^2(\mathbb{Z}/2, Z^s(\mathbb{C})) \to H^2(\pi_1(X/\tau), Z^s(\mathbb{C})).$$

Let $\overline{\alpha}$ be the image in $H^2(\pi_1(X/\tau), Z^s(\mathbb{C}))$ of $\alpha \in H^2(\mathbb{Z}/2, Z^s(\mathbb{C}))$. So $\overline{\alpha} = 1$ iff α is the image of $\eta \in H^1(\pi_1(X), Z^s(\mathbb{C}))^{\mathbb{Z}/2} = H^1(X, Z^s(\mathbb{C}))^{\mathbb{Z}/2}$, which corresponds to a pseudo-real Z^s-bundle on X. Multiplication by η acts on the space of pseudo-real bundles commuting with Hecke operators, changing α_P to $\alpha_P + \alpha$. This implies that if $\overline{\alpha} = 1$ then Conjecture 4.2 generalizes in a straightforward way to give the spectral decomposition of $\mathcal{H}(s, \tau, \alpha)$: namely, the spectrum of the Hecke operators is the same as in $\mathcal{H}(s, \tau, 1)$. More generally, this shows that the spectrum of Hecke operators on $\mathcal{H}(s, \tau, \alpha)$ for general α depends only on $\overline{\alpha}$.

It remains to describe the spectrum in the case when $\overline{\alpha} \neq 1$. As was explained to us by D. Gaiotto, in this case Conjecture 4.2 can be generalized by considering gauge theory on the 3-manifold U_τ (homotopy equivalent to X/τ) twisted by the Z^s-gerbe corresponding to $\overline{\alpha}$. Mathematically this corresponds to the setting of Subsection 3.2 extended to the case $F = \mathbb{R}$. We omit the details.

Example 4.7. Let $G = K \times K$ for some complex group K, and s be the permutation of components (the only real form in this inner class is K regarded as a real group). In this case $\mathrm{Bun}^\circ_{G,s}(X,\tau) = \mathrm{Bun}^\circ_G(X)$, the usual moduli space for the complex field. Also ${}^LG_s = {}^LG_{\omega\circ s} = \mathbb{Z}/2 \ltimes (K^\vee \times K^\vee)$, where $\mathbb{Z}/2$ acts by permutation. So an L-system is a $K^\vee \times K^\vee$ local system on X of the form (ρ,ρ^τ). Thus the spectrum is parametrized by ρ such that both ρ and ρ^τ are opers, i.e., ρ is both an oper and an anti-oper, i.e. a real oper, which agrees with the main conjecture from [EFK2]. (Note that in this case $H^i(\mathbb{Z}/2, Z^\vee) = 1$ so there is no obstructions or freedom for extensions.)

Example 4.8.[28] Let us verify Conjecture 4.2 for $G = GL_1$. In this case the possible s are 1 and -1, each being its entire inner class. So consider two cases:

1. Compact case: $s = -1$. Then the spectrum is parametrized by characters of $\pi_1(X/\tau)$, i.e., elements of $H^1(X/\tau, \mathbb{C}^\times) = (\mathbb{C}^\times)^g \times \mathbb{Z}/2$, which come from GL_1-opers.

2. Split case: $s = 1$. Then the spectrum is parametrized by $H^1(X/\tau, \mathbb{C}^\times_\tau)$, where $\mathbb{C}^\times_\tau$ is the local system where τ acts by inversion. We have $H^1(X/\tau, \mathbb{C}^\times_\tau) = (\mathbb{C}^\times)^g$, and the spectrum is parametrized by such local systems that come from GL_1-opers.

In both cases the resulting "spectral" opers form a lattice $\mathbb{Z}^g$. They are of the form $d + \phi$ where (roughly speaking) ϕ in the first case has integral periods on τ-antiinvariant cycles and in the second case integral periods on τ-invariant cycles.

Example 4.9. Consider the simplest instance of the previous example, with genus 1 curve $X = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ and coordinate $z = x + iy$, with $\tau(z) = \overline{z} + \frac{1}{2}$. Then X/τ is the Klein bottle with $\pi_1(X/\tau)$ generated by t and b with $tbt^{-1} = b^{-1}$.

1. In the compact case $s = -1$ we need to consider characters of this group, which send b to ± 1 and t to any nonzero number. So the corresponding opers are $L = d+\phi$ where ϕ has half-integral period in the imaginary direction, i.e. $\phi = \pi n$, $n \in \mathbb{Z}$. Thus the Hecke eigenvalue is expected to be proportional to $e^{2\pi i n y_0} = e^{\pi n(z_0 - \overline{z}_0)}$.

[28] This is based on the letter [W] in which E. Witten kindly explained to us the predictions of [GW] in the abelian case.

And indeed, this is what we see if we compute the eigenvalues of Hecke operators. Namely, the moduli of bundles of degree 0 admitting a real structure consist of two circles $x = 0$ and $x = \frac{1}{2}$, call them S_0 and S_1, with coordinate $y \in [0, 1)$, swapped by τ. However, the space of real bundles of degree 0 is the union of their **double covers** $\widetilde{S}_0$ and $\widetilde{S}_1$. The reason is that a real bundle is a bundle admitting a real structure with a choice of an isomorphism $A : (s, \tau)(E) \to E$ such that $A \circ (s, \tau)(A) = 1$, which is defined up to sign, and there is no canonical choice of this sign (one can check that it changes as we go around the circle). So the eigenbasis of Hecke operators is $\psi_n^+ = (e^{\pi i n y}, e^{\pi i n y})$ and $\psi_n^- = (e^{\pi i n y}, -e^{\pi i n y})$, $n \in \mathbb{Z}$, with eigenvalues $e^{2\pi i n y_0}$ and $-e^{2\pi i n y_0}$. This also shows that we have a 2-dimensional space corresponding to each oper, which agrees with the fact that we have two extensions for each oper to a local system on X/τ (as $Z_2^\vee = \mathbb{Z}/2$).

2. In the split case $s = 1$ we need to consider homomorphisms $\pi_1(X/\tau) \to \mathbb{Z}/2 \ltimes \mathbb{C}^\times$ that send t to $\{-1, 1\}$, so b goes to any nonzero number, while t^2 maps to 1. So the corresponding opers are $L = d + \phi$ where ϕ has integral period in the real direction, i.e. $\phi = 2\pi i n$, $n \in \mathbb{Z}$. So the Hecke eigenvalue is expected to be proportional to $e^{4\pi i n x_0} = e^{2\pi i n (z_0 + \overline{z}_0)}$.

And indeed, this is what we see. Namely, in this case bundles admitting a real structure form the circle S defined by the equation $y = 0$, with coordinate x. The circle $y = \frac{1}{2}$ consists of pseudo-real bundles, i.e., those for which $A \circ (s, \tau)(A) < 0$ for any isomorphism $A : (s, \tau)E \to E$, so it does not contribute. Moreover, in this case the choice of A such that $A \circ (s, \tau)(A) = 1$ is unique up to isomorphism if exists. So the set of real bundles is S (i.e., we don't get double covers) and the basis of eigenfunctions is $\psi_n = e^{2\pi i n x}$, $n \in \mathbb{Z}$, with Hecke eigenvalue $e^{4\pi i n x_0}$. Also extension of opers is unique and the space corresponding to each oper is 1-dimensional.

Remark 4.10. We can derive that every eigenvalue $\boldsymbol{\beta}(x, \overline{x})$ of the Hecke operator $H_{\lambda x + s(\lambda)\overline{x}}$ is indeed of the form $\boldsymbol{\beta}_{\rho,\lambda}(x, \overline{x})$ for some oper ρ satisfying the reality condition of Conjecture 4.2 (for semisimple G) from Conjecture 5.5 of [EFK2] (which is proved in Theorem 1.18 of *loc. cit.* for $G = PGL_n$ and $\lambda = \omega_1$). Namely, this statement implies that each eigenvalue of the Hecke operators is a unique (up to a scalar) solution of the system of linear differential equations $L\boldsymbol{\beta} = 0, \overline{L}^* \boldsymbol{\beta} = 0$, where L runs over the holomorphic differential operators from the annihilating ideal $I_{\lambda,\rho}$ introduced in Section 5 of [EFK2]. Therefore, we obtain that this system has a single-valued solution on X invariant under τ. The topological condition on the oper ρ given in Conjecture 4.2 should follow from this similarly to the argument of [EFK1], Corollary 1.19.

4.3. The case when τ has fixed points: genus 0 with $m+2$ real ramification points

4.3.1. The untwisted case We will now consider the case when τ has fixed points, which is more complicated. We restrict ourselves to $G = PGL_2$. We start with the genus zero case with ramification points considered in [EFK3].

Let $t_0 < t_1 < \cdots < t_m, t_{m+1} = \infty \in \mathbb{RP}^1$ be the ramification points. Let $a, b \in \mathbb{R}$, $x = a + ib \in \mathbb{C}$, and let $H_{x,\overline{x}}$ be the Hecke operator from [EFK3], Example 3.30, obtained by averaging over Hecke modifications at $(x, \overline{x})$ using the lines $(s, \overline{s})$. The following lemma is straightforward.

Lemma 4.11. *The eigenvalues $\boldsymbol{\beta}_k(x, \overline{x})$ of $H_{x,\overline{x}}$ satisfy the equality*

$$\boldsymbol{\beta}_k(x, \overline{x})|_{x=a} = \beta_k(a)^2$$

for $a \in \mathbb{R}$, where β_k are the eigenvalues of H_a.

Let us now study the function $\boldsymbol{\beta}_k(x, \overline{x})$ for $x \notin \mathbb{R}$. To do so, note that $\boldsymbol{\beta} = \boldsymbol{\beta}_k$ satisfies the oper equations

$$L\boldsymbol{\beta} = 0, \ \overline{L}\boldsymbol{\beta} = 0,$$

where $L = L(\boldsymbol{\mu}_k)$ (using the notation of [EFK3], Subsection 4.4). Recall also that the points t_j divide $\mathbb{RP}^1$ into intervals $I_j = (t_j, t_{j+1})$, and that in [EFK3], Subsection 4.7 we defined the functions f_j, g_j on I_j. Recall that on I_j we have $\beta(x) = f_j(x)$. Thus by Lemma 4.11 along the interval I_j we have

$$\boldsymbol{\beta}(x, \overline{x}) = |f_j(x)|^2 + \gamma_j \operatorname{Im}(\overline{f_j(x)} g_j(x)), \ \gamma_j \in \mathbb{R},$$

for x on and above I_j.

Let the function $g_j^* : I_j \to \mathbb{R}$ be defined by

$$g_j = b_j f_j - a_j g_j^*,$$

where $a_j, b_j \in \mathbb{R}$ are as in [EFK3], Subsection 4.7. Then

$$\boldsymbol{\beta}(x, \overline{x}) = |f_j(x)|^2 - a_j \gamma_j \operatorname{Im}(\overline{f_j(x)} g_j^*(x)).$$

For an analytic function h on I_j let h^{a} be its analytic continuation from I_j to I_{j-1} along a path passing above t_j. Recall from [EFK3], Subsection 4.7, that

$$f_{j-1} = i f_j^{\mathrm{a}} + g_j^{\mathrm{a}}, \ g_{j-1}^* = i g_j^{\mathrm{a}}. \tag{4.2}$$

This yields

$$\boldsymbol{\beta} = |if_j^{\mathrm{a}} + g_j^{\mathrm{a}}|^2 - a_{j-1}\gamma_{j-1}\operatorname{Im}((\overline{f_j^{\mathrm{a}}} + i\overline{g_j^{\mathrm{a}}})g_j^{\mathrm{a}}) =$$
$$|f_j^{\mathrm{a}}|^2 + (2 - a_{j-1}\gamma_{j-1})\operatorname{Im}(\overline{f_j^{\mathrm{a}}}g_j^{\mathrm{a}}) + |g_j^{\mathrm{a}}|^2 - a_{j-1}\gamma_{j-1}|g_j^{\mathrm{a}}|^2,$$

on I_{j-1}. The last two terms must cancel, so $a_{j-1}\gamma_{j-1} = 1$. Thus we get

$$\boldsymbol{\beta} = |f_j^{\mathrm{a}}|^2 + \operatorname{Im}(\overline{f_j^{\mathrm{a}}}g_j^{\mathrm{a}}),$$

on I_{j-1}. But we also have

$$\boldsymbol{\beta} = |f_j^{\mathrm{a}}|^2 + \gamma_j \operatorname{Im}(\overline{f_j^{\mathrm{a}}}g_j^{\mathrm{a}}).$$

Thus for all j we have $\gamma_j = 1$, hence $a_j = 1$ (i.e. the local system is **balanced**, in agreement with [EFK3], Subsection 4.7). Thus we obtain

Proposition 4.12. *We have*

$$\boldsymbol{\beta}(x, \overline{x}) = |f_j(x)|^2 + \operatorname{Im}(\overline{f_j(x)}g_j(x))$$

on and above I_j.

In particular, for 3 points this gives an explicit formula for the function $\boldsymbol{\beta}$ in terms of classical elliptic integrals (see [EFK3], Example 4.5).

Corollary 4.13. *The one-sided normal derivative of $\boldsymbol{\beta}(x, \overline{x})$ at the real line (with x approaching from above) equals π. Thus*

$$\boldsymbol{\beta}(a + ib, a - ib) = \beta(a)^2 + \pi|b| + o(|b|),\ \ b \to 0.$$

In particular, $\boldsymbol{\beta}(x, \overline{x})$ is continuous, but only one-sided differentiable on the real locus (excluding ramification points).

Proof. It is easy to check that the normal derivative equals the Wronskian $W(f_j, g_j)$, which equals π, as explained in [EFK3] (Proof of Proposition 4.25). □

Remark 4.14. Note that the statement of Corollary 4.13 makes sense on any real curve near its real point (indeed f_j, g_j are $-1/2$-forms, so their Wronskian is a function, and it makes sense to say that it equals π). Moreover, it holds for any real curve since it is a local statement and it holds in genus zero by Corollary 4.13. It can also be checked by direct computation of the normal derivative.

4.3.2. The twisted case Consider now the twisted case with arbitrary twisting parameters $\lambda_j = -1 + c_j, c_j \in i\mathbb{R}$. In this case the story is similar to the previous subsection and [EFK3], Subsection 4.7. Namely, fix k and consider the function $\beta_k(x)$. On the interval I_j, $0 \le j \le m$, we have the following solutions of the oper equation $L(\boldsymbol{\mu}_k)\beta = 0$: first of all,

$$f_j(x) = \beta_k(x)|_{I_j} \sim \delta_{kj}\Gamma^{\mathbb{R}}(c_j)(x-t_j)^{\frac{1-c_j}{2}} + \delta_{kj}^{-1}\Gamma^{\mathbb{R}}(-c_j)(x-t_j)^{\frac{1+c_j}{2}},\ x \to t_j+,$$

and also

$$\widehat{g}_j(x) \sim \frac{\pi\delta_{kj}^{-1}}{c_j\Gamma^{\mathbb{R}}(c_j)}(x-t_j)^{\frac{1+c_j}{2}},\ x \to t_j+.$$

The function $\widehat{g}_j$ is not real-valued on the real axis, however, so let us look for a real-valued solution of the form

$$g_j = \widehat{g}_j + i\xi_j f_j,\ \xi_j \in \mathbb{R}.$$

A short calculation using that

$$\Gamma^{\mathbb{R}}(c) = \Gamma(c)\cos\frac{\pi c}{2}$$

yields

$$\xi_j = \frac{\Lambda_j - \Lambda_j^{-1}}{\Lambda_j + \Lambda_j^{-1}}, \quad \Lambda_j := e^{\frac{\pi i c_j}{2}},$$

and the Wronskian $W(f_j, g_j) = W(f_j, \widehat{g}_j)$ equals π.

Similarly, let $\widehat{g}_{j-1}^*(x)$ be the solution of the oper equation on I_{j-1} of the form

$$\widehat{g}_{j-1}^*(x) = \frac{\pi\delta_{kj}^{-1}}{c_j\Gamma^{\mathbb{R}}(c_j)}(t_j - x)^{\frac{1+c_j}{2}},\ x \to t_j-.$$

and $g_{j-1}^* := \widehat{g}_{j-1}^* + i\xi_{j-1}f_{j-1}$ be the corresponding real solution. Then

$$g_j = b_j f_j - a_j g_j^*,\ a_j, b_j \in \mathbb{R}, a_j \neq 0.$$

Also instead of (4.2) we get

$$\begin{pmatrix} f_{j-1} \\ g_{j-1}^* \end{pmatrix} = J_j \begin{pmatrix} f_j^{\mathrm{a}} \\ g_j^{\mathrm{a}} \end{pmatrix}, \tag{4.3}$$

where

$$J_j := \frac{\Lambda_j + \Lambda_j^{-1}}{2} \begin{pmatrix} i & -\frac{(\Lambda_j - \Lambda_j^{-1})^2}{(\Lambda_j + \Lambda_j^{-1})^2} \\ 1 & i \end{pmatrix}.$$

It follows that the monodromy of our oper in appropriate bases looks as follows:

$$M(t_{j+1}- \to t_j+) = B_j := \begin{pmatrix} 1 & b_j \\ 0 & -a_j \end{pmatrix}, \; M^{\pm}(t_j+ \to t_j-) = J_j^{\pm 1}, \tag{4.4}$$

where $M^{\pm}$ denotes the monodromy above and below the real axis, respectively. Also $W(f_{j-1}, g_{j-1}^*) = -\pi$, hence $a_j = 1$.

When $j = m+1$, the formulas are the same, except that $x - t_j$ is replaced by $-1/x$ and $J_j^{\pm 1}$ is replaced by $-J_j^{\pm 1}$.

We thus obtain a deformation of the theory of balanced local systems and opers described in [EFK3], Subsection 4.7. Namely, similarly to [EFK3], given a sufficiently generic 2-dimensional local system ∇ on X with ramifications at t_j and regular local monodromies with eigenvalues $-\Lambda_j^{\pm 2}$, it can be written (generically in two different ways) in the form (4.4), where $a_j, b_j \in \mathbb{C}$ and B_j must satisfy the equations

$$\prod_{j=0}^{m+1} J_j B_j = -1, \; \prod_{j=0}^{m+1} J_j^{-1} B_j = -1, \tag{4.5}$$

which are deformations of equations (4.7) of [EFK3] (the total monodromy around the circle above and below the real axis is trivial).

Define a **$\mathbf{\Lambda}$-balancing** of ∇ to be an isomorphism (considered up to scaling) of ∇ with a local system (4.4) such that $a_j = 1$ for all j. In this case, as in [EFK3], the two equations in (4.5) are, in fact, equivalent, since $SJ_j^{-1}B_jS^{-1} = J_jB_j$, where $S := \left(\begin{smallmatrix} 1 & 2i \\ 0 & 1 \end{smallmatrix}\right)$. We call such a local system ∇ **$\mathbf{\Lambda}$-balanced** if it is equipped with a balancing. Generically a local system admits at most one balancing, as in [EFK3].

We obtain the following analog of [EFK3], Proposition 4.25 and Theorem 4.29. Denote by $\mathcal{B}_{\mathbf{\Lambda}}$ the set of $\mathbf{\Lambda}$-balanced opers.

Theorem 4.15. *The spectral opers for Hecke operators $H_{x,\mathrm{full}}$ are $\mathbf{\Lambda}$-balanced with balancing defined by the eigenvalue $\beta_k(x)$, and $b_i \in \mathbb{R}$, as in [EFK3], Proposition 4.7. Thus the spectrum $\Sigma_{\mathbf{\Lambda}}$ of the Hecke operators is a subset of $\mathcal{B}_{\mathbf{\Lambda}}$.*

Moreover, as in [EFK3], we expect that these sets are, in fact, equal, and can show this for 4 and 5 points.

4.4. The case when τ has fixed points: real and quaternionic ovals

Now suppose X is an arbitrary real curve and C is a connected component (oval) of $X(\mathbb{R})$. Given a real PGL_2-bundle P on X (with respect to the split form of the group), recall that its fiber P_x is a $PGL_2(\mathbb{C})$-torsor. So for a point $x \in C$, the real structure on P defines a map $A : P_x \to P_x$ such that $Ag = \overline{g}A$ for $g \in PGL_2(\mathbb{C})$, and $A^2 = 1$. Pick $p \in P_x$, then $A(p) = bp$ for a unique $b \in PGL_2(\mathbb{C})$, so $A(bp) = \overline{b}A(p) = \overline{b}bp$, thus $\overline{b}b = 1$. Replacing p with $q := gp$, we get

$$A(q) = A(gp) = \overline{g}A(p) = \overline{g}bp = \overline{g}bg^{-1}q.$$

Thus b is well defined up to $b \mapsto \overline{g}bg^{-1}$. It is easy to show that such b fall into two orbits of this action – that of $b = 1$ (which we call **real**) and that of $b = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ (which we call **quaternionic**). Namely, if b_* is a lift of b to $SL_2(\mathbb{C})$ then $\overline{b_*}b_* = 1$ or -1, and this is what determines the type of b (real for 1, quaternionic for -1). In the language of Subsections 2.6, 2.7, P is real at x if the associated real form G^σ of G is split and quaternionic if G^σ is compact.

It is clear that the type of P at x is independent on x as it varies along C. Thus given P, on every oval C_i of $X(\mathbb{R})$, $i = 1, \dots, r$, P is either real or quaternionic. So the manifold $\mathrm{Bun}^\circ_{G,s}(X, \tau)$ splits into 2^r disconnected parts according to the type of P at each C_i (some of which can be empty). In fact, how many of them (and which ones) are non-empty is specified on p.18 of [BGH] and references therein.

Remark 4.16. 1. The analog of this for general groups is as follows (see Subsections 2.6, 2.7). By Subsection 2.6, every real G-bundle P on X and every $x \in X(\mathbb{R})$ defines a real form of G in the inner class $C(s)$ which is continuous, hence locally constant, with respect to x (see also [GW], Section 6). So each component C_i of $X(\mathbb{R})$ carries a real form G^{σ_i} of G in $C(s)$ – the type of P at C_i. For example, as explained above, if $G = SL_2$ then C_i is real if the form of G attached to C_i is $SL_2(\mathbb{R})$ and quaternionic if it is SU_2.

2. On components containing tame ramification points, in the untwisted setting this form has to be quasi-split, as we need a real Borel subalgebra to define parabolic structures. So in particular for $G = PGL_2$ all contours containing ramification points must be real. More generally, if we consider parabolic structures for an arbitrary parabolic subgroup P of G, the corresponding ramification points can occur only on components for which the corresponding real group contains a form of P.

3. More generally, following Subsections 2.14, 2.15, at ramification points p one can place unitary representations π_i of the complex group G if $\tau(p) \neq p$

and of the real form G_i if $p \in C_i$ (the twisted setting). In this case we no longer have the restriction that G_i should be quasi-split if $p \in C_i$ (e.g. see Subsection 5.6). The untwisted setting of (2) then corresponds to taking π_i to be the spherical principal series representations with central character of $-\rho$, consisting of L^2-half-densities on the flag manifold, which requires the groups G_i to be quasi-split.

4.5. Real ovals, separating real locus

Now consider the case of PGL_2-bundles on an arbitrary real curve X without ramification points, and let us characterize the part of the spectrum coming from bundles for which all ovals are real. Let the set $X(\mathbb{R})$ of fixed points of τ be the union of ovals $C_1, \dots, C_n \subset X$. Assume first that these ovals cut X into two pieces X_+, X_- swapped by τ. Then the behavior of the Hecke eigenvalue $\boldsymbol{\beta}(x, \overline{x})$ gives us conditions which allow us to pinpoint opers L (with real coefficients) that can occur in the spectrum of Hecke operators.

Namely, first of all, the eigenvalue $\beta(x), x \in X(\mathbb{R})$ of the Hecke operator H_x is a solution of the oper equation $L\beta = 0$ periodic along C_j. So (assuming this eigenvalue is not identically zero), the local system ρ_L must satisfy

Condition 1. The monodromies of ρ_L around C_j are unipotent.

Indeed, this is necessary for the existence of the periodic solution $\beta(x)$, since the monodromy lies in SL_2.

Also, since the eigenvalue $\boldsymbol{\beta}(x, \overline{x})$ of $H_{x,\overline{x}}$ is single-valued on X_+ and satisfies the oper equations $L\boldsymbol{\beta} = \overline{L}\boldsymbol{\beta} = 0$, we see that ρ_L must also satisfy

Condition 2. The monodromy representation $\rho_L : \pi_1(X_+) \to SL_2(\mathbb{C})$ lands in $SL_2(\mathbb{R}) \cong SU(1,1)$, up to conjugation.

To write this condition more explicitly, fix a base point $x_0 \in X_+$ and paths p_j from x_0 to $c_j \in C_j$. This defines elements $\delta_j := p_j^{-1} C_j p_j \in \pi_1(X, x_0)$, where we agree that C_j begins and ends at c_j and is oriented so that when we travel around it, X_+ remains on the left. Let g_+ be the genus of X_+ and $A_k, B_k, 1 \le k \le \mathrm{g}_+$ be the A-cycles and B-cycles of X_+. Then δ_j, A_k, B_k generate $\pi_1(X_+, x_0)$ with defining relation

$$\prod_{k=1}^{\mathrm{g}_+} [A_k, B_k] \prod_{j=1}^{n} \delta_j = 1$$

(for a suitable choice of the paths p_j). Then Condition 2 is equivalent to the condition that there exists a basis in which the matrices $\rho_L(\delta_j), \rho_L(A_k), \rho_L(B_k)$ are real for all j, k.

Definition 4.17. We will say that a local system ρ on X_+ (not necessarily an oper) satisfying Conditions 1, 2 is **balanced**.

Let us now reformulate Condition 2 on the local system ρ_L attached to the oper L in a more analytic language, assuming that Condition 1 holds for ρ_L. By Condition 1, for each j we have a nonzero real periodic solution of the equation $Lf = 0$ on C_j, call it $f_j(x)$. We also have the other real solution $g_j(x)$ which changes by a multiple of f_j when we go around C_j such that the Wronskian $W(f_j, g_j) = \pi$.[29] We will use the notation f_j, g_j also for the analytic continuations of f_i, g_j to a neighborhood of I_j. Motivated by Proposition 4.12, introduce

Condition 2a. There is a single-valued solution $\boldsymbol{\beta}$ of the system

$$L\boldsymbol{\beta} = 0,\ \overline{L}\boldsymbol{\beta} = 0$$

on $X \setminus \cup_j C_j$ such that near each C_j, we have

$$\boldsymbol{\beta}(x, \overline{x}) = \varepsilon_{j1}|f_j(x)|^2 \pm \varepsilon_{j2}\,\mathrm{Im}(\overline{f_j(x)}g_j(x))$$

for a suitable choice of f_j, g_j with $W(f_j, g_j) = \pi$ (unique up to sign) and $\varepsilon_{j1}, \varepsilon_{j2} = \pm 1$, where the sign in front of the second summand is $+$ if x is above C_j and $-$ if x is below C_j.[30]

Proposition 4.18. *If Condition 1 holds and $\rho_L(C_j) \neq 1$ for all j then Condition 2a is equivalent to Condition 2.*

Proof. Since $\boldsymbol{\beta}$ is a single-valued solution of the oper equations $L\boldsymbol{\beta} = 0$, $\overline{L}\boldsymbol{\beta} = 0$, Condition 2a implies Condition 2. To prove the converse, note that Condition 1 implies that there exist bases $\{f_j, g_j\}$ of the fibers of the local system ρ_L at the points c_j in which $\rho_L(C_j) = \left(\begin{smallmatrix} 1 & \lambda_j \\ 0 & 1 \end{smallmatrix}\right)$ for $\lambda_j \in \mathbb{R}$, and Condition 2 implies that on these fibers there are nondegenerate Hermitian forms invariant under $\rho_L(C_j)$ and compatible with the operators $\rho_L(p_k p_j^{-1})$, and $\det \rho_L(p_k p_j^{-1}) = 1$.

Now, nondegenerate Hermitian forms in two variables X, Y invariant under the matrix $\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ for nonzero $\lambda \in \mathbb{R}$ are of the form $p|X|^2 + q\,\mathrm{Im}(\overline{X}Y)$, where $p, q \in \mathbb{R}, q \neq 0$. This implies that the Hermitian form at c_j in the basis

[29] Note that if $\rho(C_j) \neq 1$ then f_j is uniquely defined up to scaling, and once it is chosen, g_j is uniquely defined up to adding a real multiple of f_j.

[30] Note that the function $\mathrm{Im}(\overline{f_j(x)}g_j(x))$ does not depend on the choice of g_j, is single-valued in the neighborhood of C_j, and vanishes on C_j. Hence $\boldsymbol{\beta}(x, \overline{x})$ is continuous but only one-sided differentiable on C_j.

f_j, g_j has the form $p_j|X|^2 + q_j \operatorname{Im}(\overline{X}Y)$, where $p_j, q_j \in \mathbb{R}$, $q_j \neq 0$. We can now renormalize f_j, g_j by reciprocal positive constants to make sure that $p_j = \pm 1$.

It remains to show that all q_j are the same up to sign, then we can renormalize $\boldsymbol{\beta}$ to make sure that $q_j = \pm 1$, and construct the solution $\boldsymbol{\beta}$ satisfying Condition 2a from the invariant Hermitian form on the fibers of ρ_L. But this follows from the equality $\det \rho_L(p_k p_j^{-1}) = 1$. □

Remark 4.19. 1. The continuation of g_j around C_j gives $g_j + \lambda_j f_j$, and the real numbers λ_j attached to the components C_j do not depend on any choices and are invariants of a balanced oper.

2. If the representation ρ_L of $\pi_1(X_+)$ is irreducible then the solution $\boldsymbol{\beta}$ satisfying Conditions 1 and 2a is unique up to sign if exists.

The above discussion implies

Proposition 4.20. *Every oper appearing in the spectrum of Hecke operators is balanced.*

There is, however, another condition satisfied by spectral opers. Namely, let us say that a balanced oper L is **positive** if there is a solution $\boldsymbol{\beta}$ satisfying Condition 2a with $\varepsilon_{j1} = \varepsilon_{j2} = 1$ for all j.

Proposition 4.21. *Every oper appearing in the spectrum of Hecke operators is positive.*

Proof. Since $\boldsymbol{\beta}_k(x, x) = \beta_k(x)^2$ for $x \in X(\mathbb{R})$, ε_{j1} are all the same, so can be assumed to be 1. Then it follows from Remark 4.14 that we also have $\varepsilon_{j2} = 1$ for all j. □

Conjecture 4.22. The spectrum of Hecke operators is labeled by positive balanced opers, possibly with finitely many eigenvalues corresponding to the same oper.

Example 4.23. Let X be of genus 2 with $X(\mathbb{R})$ having 3 components, C_1, C_2, C_3 that cut X into two trinions X_+, X_-. Thus $\mathrm{g}_+ = 0$, so $\pi_1(X_+)$ is generated by δ_j, $j = 1, 2, 3$, with defining relation $\delta_1\delta_2\delta_3 = 1$. So Condition 1, saying that all $A_i := \rho(\delta_i)$ are unipotent, implies that they all commute, as $A_1A_2A_3 = 1$. In spite of having three equations ($\operatorname{Tr}\rho(\delta_i) = 2$, $i = 1, 2, 3$), one can show that the space of such (real) local systems on X is of codimension 2 (i.e., 4-dimensional over $\mathbb{R}$) so it is **not** a complete intersection. However, Condition 2 provides one more equation to get a 3-dimensional real manifold. Therefore if appropriate transversality holds, then Conditions 1, 2 and the oper condition imply the discreteness of the spectrum. Namely, isomorphism classes of (nontrivial) unipotent representations of the group $\pi_1(X_+, x_0) = F_2$ are labeled by one complex parameter κ, which lives in $\mathbb{CP}^1$ (namely, $\rho(\delta_2) = \rho(\delta_1)^\kappa$). So Condition 2 just tells us that $\kappa \in \mathbb{RP}^1$.

Let us write down a formula for $\boldsymbol{\beta}(x, \overline{x})$ in this case, for $\rho = \rho_L$. If Condition 1 holds then we have a global holomorphic solution $f(x)$ of the equation $Lf = 0$ on X_+, generically unique up to scaling. We can normalize it to be real on C_1. We also have another holomorphic solution $g(x)$ on X_+ which is changed by a multiple of $f(x)$ when we go around cycles, which we can normalize so that $W(f, g) = \pi$. Then g is uniquely determined up to adding a multiple of f. We can make sure that g is real on C_1, then the remaining freedom is adding to g a real multiple of f. Now consider the function

$$\boldsymbol{\beta}(x, \overline{x}) := |f(x)|^2 + \mathrm{Im}(\overline{f(x)}g(x)).$$

Note that this function is independent on the choice of g, and is single-valued since κ is real. This function is the eigenvalue of $H_{x,\overline{x}}$ when f is normalized so that $f(x)|_{C_1} = \beta(x)$ (then this will also hold on C_2 and C_3, up to sign). Thus we see that in this case every balanced oper is automatically positive.

Example 4.24. Suppose X has genus $\mathrm{g} \geq 3$ with $X(\mathbb{R})$ having $\mathrm{g}+1$ components $C_1, \ldots, C_{\mathrm{g}+1}$; thus $\mathrm{g}_+ = 0$ so $\pi_1(X_+)$ is generated by δ_j with $\prod_{j=1}^{\mathrm{g}+1} \delta_j = 1$. Then Condition 1 imposes $\mathrm{g}+1$ constraints on the local system: we have that $A_j := \rho(\delta_j)$ are unipotent for $j = 1, \ldots, \mathrm{g}+1$. Moreover, now this actually defines a complete intersection (unlike the previous example, which is a degenerate case). Once this condition is imposed, our representation ρ of $\pi_1(X_+, x_0) = F_{\mathrm{g}}$ is a point of the **unipotent SL_2-character variety $\mathcal{M}^{\mathrm{unip}}_{0,\mathrm{g}+1}$** for the sphere with $\mathrm{g}+1$ holes, which has (complex) dimension $2(\mathrm{g}-2)$. Thus the condition that this point is real is $2(\mathrm{g}-2)$ real equations. So altogether we get $\mathrm{g}+1+2(\mathrm{g}-2) = 3\mathrm{g}-3$ real equations, i.e. if appropriate transversality holds then we should get a real submanifold of middle dimension $3\mathrm{g}-3$ in the $6\mathrm{g}-6$-dimensional manifold of local systems, as needed for discrete spectrum.

Example 4.25. More generally, suppose $X(\mathbb{R})$ is the union of $C_1, \ldots, C_n$ where $n \leq \mathrm{g}+1$. Then X_+ has genus $\mathrm{g}_+ = \frac{\mathrm{g}+1-n}{2}$ (so $\mathrm{g}+1-n$ must be even). So Condition 1 gives us n real equations, and then we end up in the unipotent character variety $\mathcal{M}^{\mathrm{unip}}_{\frac{\mathrm{g}+1-n}{2},n}$, which has (complex) dimension $d = 3(\mathrm{g}+1-n)-6+2n = 3\mathrm{g}-3-n$. So altogether we again get $3\mathrm{g}-3-n+n = 3\mathrm{g}-3$ real equations, as needed for discrete spectrum.

4.6. Real ovals, non-separating real locus

Now suppose that $X(\mathbb{R})$ still consists of n circles $C_1, \ldots, C_n$ of real type but now is non-separating. To handle this case, let Σ be a connected non-orientable surface with n holes of Euler characteristic χ, and let $\mathcal{M}^{\mathrm{unip},-}_{\chi,n}$ be the corresponding unipotent character variety. Note that Σ can be obtained by gluing $s = 1$ or 2 Möbius strips into an orientable surface Σ_+ with $n+s$ holes and the same Euler characteristic χ. We have $2-2\mathrm{g}(\Sigma_+)-n-s=\chi$, so $\mathrm{g}(\Sigma_+) = 1-\frac{n+s+\chi}{2}$, and s is such that $n+s+\chi$ is even. Thus $\mathcal{M}^{\mathrm{unip}}_{\mathrm{g}(\Sigma_+),n+s}$ has dimension $-3(n+s+\chi)+2(n+s) = -n-s+3\chi$. So $\dim \mathcal{M}^{\mathrm{unip},-}_{\chi,n} = -n-3\chi$ (since we no longer have unipotency condition at s of the $n+s$ holes, where we glue in a Möbius strip).

Now let $\Sigma = X/\tau$. This is a non-orientable surface of Euler characteristic $\chi = 1-\mathrm{g}$ and n holes. So $\dim \mathcal{M}^{\mathrm{unip},-}_{\chi,n} = -n-3\chi = 3\mathrm{g}-3-n$. Thus Condition 1 gives n equations, and the real locus in $\mathcal{M}^{\mathrm{unip},-}_{\chi,n}$ another $3\mathrm{g}-3-n$ equations, so altogether we get $3\mathrm{g}-3$ equations, again as needed.

4.7. Quaternionic ovals

For a quaternionic oval C_j the Hecke operator H_x for $x \in C_j$ is not defined, so the function $\beta(x)$ is not defined either. As a result, it is not hard to show that

$$\lim_{x\to C_j} H_{x,\overline{x}} = 0$$

and thus $\boldsymbol{\beta}(x,\overline{x}) = 0$ for $x \in C_j$. More specifically, the function $\boldsymbol{\beta}(x,\overline{x})$ near C_j has the form

$$\boldsymbol{\beta}(x,\overline{x}) = \pm\,\mathrm{Im}(\overline{f_j(x)}g_j(x)), \tag{4.6}$$

where as before f_j, g_j are real solutions of the oper on C_j such that $W(f_j,g_j) = \pi$, so we have

$$\boldsymbol{\beta}(x,\overline{x}) \sim \pi|b|$$

where b is the distance from x to the oval (this makes sense because $\boldsymbol{\beta}$ is not a function but actually a $-1/2$-density, i.e., $|b|$ is really $|db/b|^{-1}$). However, the normalization of f_j is now not fixed, so we are free to multiply f_j by a nonzero real scalar and divide g_j by the same scalar.

Also, we no longer have a condition that monodromy around C_j is unipotent. It only has to have real eigenvalues $\mu_j^{\pm 1}$, so it can preserve the indefinite Hermitian form defined by $\boldsymbol{\beta}$ (see (4.6)), and f_j, g_j are the corresponding eigenvectors.

4.8. Conditions with ramification points

In presence of ramification points on C_j (in which case in absence of twists C_j is necessarily of real type), the story should be the same, except that by monodromy along C_j we should mean monodromy “in the sense of principal value”, as explained in [EFK3], Subsection 4.7; i.e., when we continue through a ramification point, we take minus the half-sum of upper and lower analytic continuation. So for genus 0 we recover exactly the answer from [EFK3], Subsection 4.7. In this case the real locus divides $\mathbb{CP}^1$ into two disks, which are simply connected, so we don't have any conditions similar to Subsection 4.2, and the balancing conditions on ρ_L can be formulated solely in terms of the neighborhood of the real locus. This is exactly what happens in [EFK3], Subsection 4.7.

5. Analytic Langlands correspondence and Gaudin model

In this section we discuss the Gaudin model and its generalizations and relate these models to various settings of the analytic Langlands correspondence on $\mathbb{P}^1$ with ramification points over $\mathbb{R}$ and $\mathbb{C}$.

Initially, the Gaudin model associated to a simple Lie algebra $\mathfrak{g}$ was defined for the tensor products of finite-dimensional representations of $\mathfrak{g}$. But in fact the Gaudin Hamiltonians (which we have already encountered in Subsection 3.3 in the case $\mathfrak{g} = \mathfrak{sl}_2$) give rise to well-defined commuting operators on the tensor product of any representations of $\mathfrak{g}$. If this tensor product has a weight space decomposition with finite-dimensional weight spaces, then since these subspaces are preserved by the Gaudin Hamiltonians, the corresponding spectral problem is clearly well-defined. This is the case, for example, when all representations are of highest weight or lowest weight.

If this is not the case, the spectral problem may still be well-defined if there is a natural Hilbert space structure on a completion of this tensor product and the Gaudin Hamiltonians can be extended to self-adjoint strongly commuting operators on it. This conjecturally happens when the representations of $\mathfrak{g}$ come from tempered unitary representations of a connected real Lie group $G(\mathbb{R})$ whose complexified Lie algebra is $\mathfrak{g}$; for example, this happens for representations of the unitary principal series of $SL_2(\mathbb{R})$. The traditional methods of Bethe Ansatz can no longer be used in this case. But here we get into the setting of the analytic Langlands correspondence for $G = SL_2$, $X = \mathbb{P}^1$, and $F = \mathbb{R}$ with real ramification points discussed in Section 4 (with the Gaudin Hamiltonians being the Hitchin Hamiltonians). Hence we can use the results of Section 4 to describe the spectrum of the Gaudin Hamiltonians.

In fact, we will show that even the original case of the tensor product of finite-dimensional representations of $\mathfrak{g}$ can be interpreted in the framework of the analytic Langlands correspondence (namely, it appears in the quaternionic case discussed in Subsection 4.7). Applying our results, we obtain a new interpretation of the description of the spectrum of the Gaudin Hamiltonians in this case in terms of monodromy-free opers [F2, R]. This description is closely related to the Bethe Ansatz in the Gaudin model. We discuss all this in Subsections 5.1–5.3 and 5.5–5.7.

In Subsection 5.4 we consider the case of the tensor product of the infinite-dimensional contragredient Verma modules with arbitrary highest weights. In this case the weight subspaces of the tensor product are finite-dimensional and the spectral problem for the Gaudin Hamiltonians is well-defined. It is natural to expect that the spectrum is given by opers whose monodromy is contained in a Borel subgroup of $G^\vee$, the Lie group of adjoint type associated to the Lie algebra $\mathfrak{g}^\vee$. In a follow-up paper [EF] we intend to prove this result using the tools of the present paper.

In Subsection 5.8 we interpret this result, in the case when the highest weights satisfy a certain reality condition, as a description of the spectrum for the tensor product of the holomorphic discrete series representations of $G(\mathbb{R})$. In Subsection 5.9 we introduce chiral versions of the Hecke operators acting on the tensor product of contragredient Verma modules.

Next, in Subsection 5.10 we discuss the infinite-dimensional case. First, we describe the spectrum for the tensor product of unitary principal series representations in terms of balanced opers. This is essentially the statement of Theorem 4.15. This case is very interesting because we cannot use the ordinary Bethe Ansatz method (since these representations don't have highest weight vectors). We also comment on the case of a tensor product of discrete series representations involving both holomorphic and anti-holomorphic ones.

In Subsection 5.11 we interpret the analytic Langlands correspondence for $\mathbb{P}^1$ with parabolic structures over $\mathbb{C}$ (discussed in Subsection 3.3) as a "double" of the Gaudin model.

In all of these settings, our description of the spectrum relies on the existence of the Hecke operators, which commute with the Gaudin Hamiltonians and satisfy differential equations (the universal oper equations). These equations can be used to describe the analytic properties of the opers encoding the possible eigenvalues of the Gaudin Hamiltonians.

Interestingly, the role of the Hecke operators is played by the Gaudin model analogues of Baxter's Q-operators or closely related operators (see Subsection 5.7). This allows us to regard the generalized Bethe Ansatz method and Baxter's Q-operators as arising from special cases of the tamely ramified analytic Langlands correspondence in genus 0. We also discuss a q-deformation of this story, which has to do with the quantum integrable models of XXZ type associated to the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$, in Subsections 5.12 and 5.13.

5.1. The Gaudin model and monodromy-free opers

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and G the connected simply-connected algebraic group with the Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a triangular decomposition of $\mathfrak{g}$ with Weyl group $W \subset \mathrm{Aut}(\mathfrak{h})$. Let $\Lambda_G^+ \subset \mathfrak{h}^*$ be the set of dominant integral weights of $\mathfrak{g}$ (and G), and let V_λ be the finite-dimensional irreducible representation of G with highest weight $\lambda \in \Lambda_G^+$.

Let

$$\{t_i\} := \{t_0, t_1, \dots, t_{m+1} = \infty\}$$

be a collection of $(m+2)$ distinct points on $\mathbb{P}^1$, with the last point t_{m+1} identified with the point ∞ (with respect to a once and for all chosen global coordinate x on $\mathbb{P}^1$). Fix $\lambda_i \in \Lambda_G^+$, $0 \le i \le m+1$, and let

$$\mathcal{H} := (V_{\lambda_0} \otimes \cdots \otimes V_{\lambda_{m+1}})^{\mathfrak{g}} \simeq (V_{\lambda_0} \otimes \cdots \otimes V_{\lambda_m})^{\mathfrak{n}_+}[\lambda_{m+1}^*]. \tag{5.1}$$

Thus, $\mathcal{H}$ is the subspace of singular vectors of weight $\lambda_{m+1}^* = -w_0(\lambda_{m+1})$ in $V_{\lambda_0} \otimes \cdots \otimes V_{\lambda_m}$, where $w_0 \in W$ is the maximal element.

On the space $\mathcal{H}$ acts the commutative subalgebra $\mathcal{G} \subset (U(\mathfrak{g})^{\otimes m+1})^{\mathfrak{g}}$ of the **generalized Gaudin Hamiltonians** introduced in [FFR] (see also [F1, F2]).[31] The algebra $\mathcal{G}$ includes the original (quadratic) Gaudin Hamiltonians

$$G_i := \sum_{j \neq i} \sum_a \frac{J^{a(i)} J_a^{(j)}}{t_j - t_i}, \tag{5.2}$$

where $\{J^a\}$ and $\{J_a\}$ are two bases of the Lie algebra $\mathfrak{g}$ dual to each other with respect to the normalized non-degenerate invariant bilinear form. In the case $\mathfrak{g} = \mathfrak{sl}_2$, the algebra $\mathcal{G}$ is generated by the G_i's. For groups of rank greater than 1, there are also higher Gaudin Hamiltonians.

Joint eigenvectors and eigenvalues of the algebra $\mathcal{G}$ in $\mathcal{H}$ have been constructed explicitly in [FFR] generalizing the classical Bethe Ansatz method in the case $\mathfrak{g} = \mathfrak{sl}_2$ (in [RV] an alternative proof was given that these vectors are eigenvectors of the G_i's). It has been proved in [SV] (see also [F1]) that for $\mathfrak{g} = \mathfrak{sl}_2$ these eigenvectors form a basis in $\mathcal{H}$ if the collection $\{t_i\}$ is generic. But for other groups this is not always the case. The reasons for this are explained in [F2], Section 5.5. An explicit counterexample in the case $\mathfrak{g} = \mathfrak{sl}_3$ has been given in [MV].

However, an alternative description of the spectrum of the algebra $\mathcal{G}$ on the space $\mathcal{H}$, which does not rely on explicit formulas for the eigenvectors,

[31] This algebra is called the **Bethe algebra** or the **Gaudin algebra**.

was conjectured in [F2] (following [F1]) and proved in [R]. Namely, let $G^\vee$ be the Langlands dual group of G (thus, $G^\vee$ is the Lie group of adjoint type with associated with the Lie algebra $\mathfrak{g}^\vee$ which is Langlands dual to $\mathfrak{g}$). There is a bijection between the joint spectrum of the algebra $\mathcal{G}$ of generalized Gaudin Hamiltonians on $\mathcal{H}$ given by formula (5.1) (without multiplicities) and the set of **monodromy-free** $G^\vee$-opers on $\mathbb{P}^1$ with regular singularities at the points t_i, with **residues** $\varpi(-\lambda_i - \rho) \in \mathfrak{h}^*/W$, where ϖ is the projection $\mathfrak{h}^* \to \mathfrak{h}^*/W$.[32] (In the case of $\mathfrak{g} = \mathfrak{sl}_n$, a similar result for the spectrum of a certain deformation of $\mathcal{G}$ follows from [MTV1].) Moreover, it is shown in [R] that for a generic collection $\{t_i\}$ the algebra $\mathcal{G}$ is diagonalizable on $\mathcal{H}$ and its spectrum is simple. The precise statement is given in Theorem 5.5 below.

5.2. The Gaudin model for $\mathfrak{g} = \mathfrak{sl}_2$

Consider first the Gaudin model for $\mathfrak{g} = \mathfrak{sl}_2$. Note that $w_0(\lambda) = -\lambda$, so $\lambda^* = \lambda$ for all weights λ. We identify the dominant integral weights λ_i with non-negative integers and define n by the formula

$$2n = \sum_{i=0}^{m} \lambda_i - \lambda_{m+1}. \tag{5.3}$$

For the space

$$\mathcal{H} = \left(\bigotimes_{i=0}^{m} V_{\lambda_i} \right)^{\mathfrak{n}_+} [\lambda_{m+1}] \tag{5.4}$$

to be non-zero, n must be a non-negative integer.

The algebra $\mathcal{G}$ of Gaudin Hamiltonians is in this case generated by the G_i's given by formula (5.2). The original formulation of Bethe Ansatz for diagonalization of these operators is the following. Given a collection $\mathbf{w} = \{w_1, \ldots, w_n\}$ of distinct complex numbers such that $w_j \neq t_i$ for all i and j, define the **Bethe vector** by the formula

$$v_{\mathbf{w}} := f(w_1) \ldots f(w_m) v, \tag{5.5}$$

where

$$f(w) := \sum_{i=0}^{m} \frac{f_i}{w - t_i} \tag{5.6}$$

[32] The notion of residue was introduced in [BD1], Section 3.8.11; see also [F3], Subsection 9.1.

and v is the tensor product of the highest weight vectors of the representations V_{λ_i}.

The following is the system of **Bethe Ansatz equations** on the numbers $w_j, j = 1, \ldots, n$:

$$\sum_{i=0}^{m} \frac{\lambda_i}{w_j - t_i} = \sum_{s \neq j} \frac{2}{w_j - w_s}, \qquad j = 1, \ldots, n. \tag{5.7}$$

Theorem 5.1 ([SV]). *For a generic collection $\{t_i\}$, the vectors $v_{\mathbf{w}}$ with $\mathbf{w} = \{w_1, \ldots, w_n\}$ satisfying the system (5.7) form an eigenbasis of $\mathcal{H}$. The eigenvalue μ_i of G_i on $v_{\mathbf{w}}$ is given by the formula*

$$\mu_i = \lambda_i \left(\sum_{k \neq i} \frac{\lambda_k}{2(t_i - t_k)} - \sum_{j=1}^{n} \frac{1}{t_i - w_j} \right). \tag{5.8}$$

This result is referenced as **completeness** of Bethe Ansatz for $\mathfrak{g} = \mathfrak{sl}_2$.

Analogs of Bethe vectors have been constructed for an arbitrary Lie algebra $\mathfrak{g}$ [FFR, RV]. Unfortunately, they do not give an eigenbasis for a general $\mathfrak{g}$ even for a generic collection $\{t_i\}$, so Bethe Ansatz is incomplete; a counterexample has been found already for $\mathfrak{g} = \mathfrak{sl}_3$ [MV].

Luckily, there is an alternative approach to describing the joint spectrum of the Gaudin Hamiltonians on the space $\mathcal{H}$ given by (5.1). It uses a realization [FFR, F1, F2] of the algebra of Gaudin Hamiltonians as the quotient of the center of the completed enveloping algebra of the affine Kac-Moody algebra at the critical level and its isomorphism with the algebra of functions on the space of $G^\vee$-opers on the punctured disc [FF, F3]. We will now explain this approach in the case of $\mathfrak{g} = \mathfrak{sl}_2$ and connect it to the Bethe Ansatz discussed above.

For $\mathfrak{g} = \mathfrak{sl}_2$, we have $G = SL_2$ and $G^\vee = PGL_2$. The Bethe Ansatz equations can be interpreted in terms of monodromy-free PGL_2-opers on $\mathbb{P}^1$ as follows.

Recall that a PGL_2-oper is a second order-differential operator acting from $K^{-\frac{1}{2}}$ to $K^{\frac{3}{2}}$ of the form $\partial_x^2 - v(x)$. Such an oper is said to have a regular singularity at the point $x = t$ with residue $\varpi(\lambda+1) = \frac{1}{2}(\lambda+1)^2$ if its expansion near this point has the form

$$\partial_x^2 - \tfrac{\lambda(\lambda+2)}{4(x-t)^2} + O(\tfrac{1}{x-t}), \ x \to t$$

(where for $t = \infty$ we take the expansion in $1/x$).

Lemma 5.2. *If*

$$\partial_x^2 - v(x) = (\partial_x - u(x))(\partial_x + u(x)), \tag{5.9}$$

where

$$u(x) = \sum_{i=0}^{m} \frac{\lambda_i}{2(x-t_i)} - \sum_{j=1}^{n} \frac{1}{x-w_j} \tag{5.10}$$

for some distinct $w_j, 1 \le j \le n$, with $w_j \neq t_i$, satisfying the system (5.7), then the PGL_2-oper $\partial_x^2 - v(x)$ on $\mathbb{P}^1$ has trivial monodromy representation

$$\pi_1(\mathbb{P}^1 \backslash \{t_0, \dots, t_m, t_{m+1} = \infty\}) \to PGL_2 \tag{5.11}$$

and regular singularities at the points t_i, with residues $\varpi(\lambda_i + 1)$, $\lambda_i \in \mathbb{Z}_{\ge 0}$, $0 \le i \le m+1$.

Moreover, the converse is also true for a generic collection $\{t_0, \dots, t_m, t_{m+1} = \infty\}$.

Proof. If the PGL_2-oper $\partial_x^2 - v(x)$ has the form (5.9) with $u(x)$ given by formula (5.10), then the section

$$\Phi := \Phi(x) dx^{-\frac{1}{2}}, \ \Phi(x) := \prod_{i=0}^{m} (x-t_i)^{-\frac{\lambda_i}{2}} \prod_{j=1}^{n} (x-w_j) \tag{5.12}$$

of $K^{-\frac{1}{2}}$ is a solution of the equation

$$(\partial_x^2 - v(x))\Phi(x) = 0. \tag{5.13}$$

Moreover,

$$\Phi_*(x) := \Phi(x) \int \Phi^{-2}(x) dx$$

is then another, linearly independent local solution of the same equation. This solution is a single-valued global solution on $\mathbb{P}^1$ with singularities only at the points $\{t_i\}$ if and only if equations (5.7) are satisfied (in fact, the j-th equation in (5.7) is equivalent to it having no monodromy at $x = w_j$).

Conversely, suppose that $\partial_x^2 - v(x)$ is a PGL_2-oper on $\mathbb{P}^1$ which has regular singularities at t_i's with residues $\varpi(\lambda_i + 1)$, $\lambda_i \in \mathbb{Z}_{\ge 0}$ and trivial monodromy representation (5.11). Equation (5.13) then must have a solution of the form

$$\Phi := \Phi(x) dx^{-\frac{1}{2}}, \ \Phi(x) := \prod_{i=0}^{m} (x-t_i)^{-\frac{\lambda_i}{2}} Q(x), \tag{5.14}$$

where $Q(x)$ is a polynomial of degree n. According to Theorem 13 of [SV], the roots of this polynomial do not belong to the set $\{t_i\}$ for generic collections $\{t_i\}$. Writing

$$Q(x) = \prod_{j=1}^{n}(x - w_j), \tag{5.15}$$

we obtain formula (5.9). Moreover, formula (5.13) then implies that the w_j's are pairwise distinct. Using the argument of the preceding paragraph, we find that the w_j's must satisfy Bethe Ansatz equations (5.7). This completes the proof. □

Lemma 5.2 links Bethe Ansatz equations to monodromy-free PGL_2-opers. In fact, the j-th equation in (5.7) is equivalent to the property that oper (5.9) has no singularity at $x = w_j$.

Theorem 5.1 describes the case of generic parameters $\{t_i\}$. For special collections $\{t_i\}$ the Gaudin Hamiltonians may not be diagonalizable and/or the Bethe vectors may not give a basis of $\mathcal{H}$ (see the examples in Subsection 5.5). Nevertheless, it turns out that the joint spectrum of the Gaudin Hamiltonians is in bijection with the monodromy-free PGL_2-opers satisfying the conditions of Lemma 5.2 **for all** collections $\{t_i\}$, as stated in the following theorem. Its part (1) was proved in [F1]; part (2) was conjectured in [F2] and proved in [R] (without explicit construction of eigenvectors).

Theorem 5.3. *(1) The joint eigenvalues $\{\mu_i\}_{i=0,\dots,m}$ of the Gaudin Hamiltonians $\{G_i\}_{i=0,\dots,m}$ acting on the space (5.4) are such that the PGL_2-oper on $\mathbb{P}^1$*

$$L(\boldsymbol{\mu}) = \partial_x^2 - \sum_{i=0}^{m} \frac{\lambda_i(\lambda_i+2)}{4(x-t_i)^2} - \sum_{i=0}^{m} \frac{\mu_i}{x - t_i} \tag{5.16}$$

has regular singularity with residue $\varpi(\lambda_{m+1}+1)$ at $t_{m+1} = \infty$ and trivial monodromy representation (5.11).

(2) For any collection $\{t_0, t_1, \dots, t_{m+1} = \infty\}$ this defines a one-to-one correspondence between the set of joint eigenvalues (without multiplicity) of the Gaudin Hamiltonians and the set of all such PGL_2-opers.

Remark 5.4. 1. Under this correspondence, the eigenvalues μ_i of the Gaudin operators G_i are given by the formula

$$\mu_i = \mathrm{Res}_{t_i}\left(\frac{\Phi''(x)}{\Phi(x)}\right) = \lambda_i\left(\sum_{k\neq i} \frac{\lambda_k}{2(t_i - t_k)} - \sum_{j=1}^{n} \frac{1}{t_i - w_j}\right),$$

where $\Phi(x)dx^{-\frac{1}{2}}$ is the solution (5.14) (compare with formula (5.8)).

The conditions that $L(\boldsymbol{\mu})$ has a regular singularity and residue $\varpi(\lambda_{m+1}+1)$ at $t_{m+1}=\infty$ are equivalent to the conditions on the eigenvalues μ_i's given by the first and second equations in (3.9), respectively.

2. The equation $L(\boldsymbol{\mu})\beta = 0$ can we written in the form of connection with first order poles in two different ways. Namely, setting

$$\mathbf{b} := \begin{pmatrix} \partial_x\beta + (\sum_{j=0}^m \frac{\lambda_j}{2(x-t_j)})\beta \\ \beta \end{pmatrix},$$

for the first way we get

$$\partial_x \mathbf{b} = \begin{pmatrix} \sum_{j=0}^m \frac{\lambda_j}{2(x-t_j)} & \sum_{j=0}^m \frac{\widehat{\mu}_j}{x-t_j} \\ 1 & -\sum_{j=0}^m \frac{\lambda_j}{2(x-t_j)} \end{pmatrix} \mathbf{b}. \tag{5.17}$$

The second way is the same but replacing λ_j with $-\lambda_j - 2$. Note that near $x = t_j$ equation (5.17) in the variable $z := x - t_j$ looks like

$$\partial_z \mathbf{b} = \begin{pmatrix} \frac{\lambda_j}{2z} + a_j(z) & \frac{\widehat{\mu}_j}{z} + c_j(z) \\ 1 & -\frac{\lambda_j}{2z} - a_j(z) \end{pmatrix} \mathbf{b}$$

where a_j, c_j are regular at $z = 0$. So we can make the residue of the matrix on the right hand side independent of $\widehat{\mu}_j$ by setting $\widetilde{\mathbf{b}} := \mathrm{diag}(z^{\frac{1}{2}}, z^{-\frac{1}{2}})\mathbf{b}$. Then we get

$$\partial_z \widetilde{\mathbf{b}} = \begin{pmatrix} \frac{\lambda_j+1}{2z} + a_j(z) & \widehat{\mu}_j + zc_j(z) \\ \frac{1}{z} & -\frac{\lambda_j+1}{2z} - a_j(z) \end{pmatrix} \widetilde{\mathbf{b}}.$$

Now the residue is the regular element

$$\begin{pmatrix} \frac{\lambda_j+1}{2} & 0 \\ 1 & -\frac{\lambda_j+1}{2} \end{pmatrix}$$

which maps to $\varpi(\lambda_j + 1) = \frac{1}{2}(\lambda_j + 1)^2$ under the map $A \mapsto \mathrm{Tr}\, A^2$.

5.3. The case of a general Lie algebra $\mathfrak{g}$

Theorem 5.3 generalizes to an arbitrary simple Lie algebra $\mathfrak{g}$ as follows.

Theorem 5.5. *(1) For any collection $\{t_0, \ldots, t_{m+1} = \infty\}$, there is a one-to-one correspondence between the set of joint eigenvalues (without multiplicity) of the algebra $\mathcal{G}$ of generalized Gaudin Hamiltonians on the space $\mathcal{H}$*

given by (5.1) *and the set of all* $G^\vee$*-opers on* $\mathbb{P}^1$ *with regular singularities and residues* $\varpi(-\lambda_i - \rho)$ *at* t_i *and with trivial monodromy representation*

$$\pi_1(\mathbb{P}^1 \backslash \{t_0, \dots, t_m, t_{m+1} = \infty\}) \to G^\vee.$$

(2) For generic $\{t_0, \dots, t_m\}$*, the algebra* $\mathcal{G}$ *of generalized Gaudin Hamiltonians is diagonalizable and has simple spectrum on the space* $\mathcal{H}$ *given by* (5.1)*.*

In [F2], part (1) of this theorem was conjectured and a map in one direction (from the set of joint eigenvalues to the set of monodromy-free opers) was constructed. The bijectivity of this map (and hence the statement of part (1)) was proved in [R]. Part (2) was also proved in [R].

Analytic Langlands correspondence provides a novel conceptual framework for (and in many cases, solution of) the problem of diagonalization of the Gaudin Hamiltonians. In particular, Theorem 5.3 will be derived using this framework in Subsection 5.6. We will then extend this framework to infinite-dimensional representations.

5.4. The Gaudin model with complex weights

The theory of Subsections 5.1, 5.2, and 5.3 can be "analytically continued" to complex weights λ_j such that $\alpha := \sum_{i=0}^m \lambda_i - \lambda_{j+1}^* \in Q_+$ (nonnegative integer linear combination of simple roots). Namely, we may replace the finite dimensional modules V_{λ_j}, $0 \leq j \leq m$, by contragredient Verma modules $\nabla(\lambda_j)$ over $\mathfrak{g}$ with highest weights λ_j, and consider the action of Gaudin Hamiltonians in

$$\mathcal{H} := (\nabla(\lambda_0) \otimes \cdots \otimes \nabla(\lambda_m))^{\mathfrak{n}_+}[\lambda_{m+1}^*]. \tag{5.18}$$

First consider the case of $\mathfrak{sl}_2$, so $\lambda_{m+1}^* = \lambda_{m+1}$ and equation (5.3) is satisfied for a non-negative integer n. Analytically continuing the explicit formulas for the Bethe vectors $v_{\mathbf{w}}$ and using Theorem 5.1, we obtain the following result.

Theorem 5.6. *For generic collections* $\{t_0, t_1, \dots, t_{m+1} = \infty\}$ *and* $\{\lambda_0, \lambda_1, \dots, \lambda_{m+1}\}$ *satisfying formula* (5.3) *with a non-negative integer* n*:*

(1) The Bethe vectors $v_{\mathbf{w}}$ *given by formula* (5.5) *with* $\mathbf{w} = \{w_1, \dots, w_n\}$ *satisfying the system* (5.7) *form an eigenbasis of the space* $\mathcal{H}$ *given by* (5.18)*. The eigenvalue* μ_i *of* G_i *on* $v_{\mathbf{w}}$ *is given by the formula* (5.8)*.*

(2) The spectrum of the Gaudin Hamiltonians is simple and the set of their joint eigenvalues is in one-to-one correspondence with the set of PGL_2*-opers*

on $\mathbb{P}^1$ with regular singularities and residues $\varpi(\lambda_i+1)$ at $t_i, i=0,\dots,m+1$, and with **solvable monodromy**, *i.e. contained in a Borel subgroup $B^\vee \subset PGL_2$. Under this correspondence, the collection $\{\mu_i\}$ of joint eigenvalues of the Gaudin Hamiltonians $G_i, i=0,1,\dots,m$, maps to the PGL_2-oper given by formula (5.16).*

Proof. Suppose that λ_i and $\lambda_{m+1}^* = \lambda_{m+1}$ are dominant integral weights. Then we have a canonical inclusion (up to a scalar)

$$(5.19)\quad (V_{\lambda_0}\otimes\cdots\otimes V_{\lambda_m})^{\mathfrak{n}_+}[\lambda_{m+1}] \hookrightarrow (\nabla(\lambda_0)\otimes\cdots\otimes\nabla(\lambda_m))^{\mathfrak{n}_+}[\lambda_{m+1}] = \mathcal{H},$$

which commutes with the Gaudin Hamiltonians.

For any fixed n given by formula (5.3), this map is an isomorphism for sufficiently large $\{\lambda_i\}$. Suppose that this is the case. Theorem 5.1 then implies that for generic $\{t_i\}$ the Bethe vectors $v_{\mathbf{w}}$ given by formula (5.5), with $\mathbf{w}=\{w_1,\dots,w_n\}$ satisfying the system (5.7), form an eigenbasis of $\mathcal{H}$, and moreover, the spectrum of the Gaudin Hamiltonians on $\mathcal{H}$ is simple. Since this is an open condition, the same is true for generic $\{\lambda_i\}$ and $\{t_i\}$. Moreover, explicit calculation shows that the eigenvalues μ_i of the Gaudin Hamiltonians G_i are still given by formula (5.8). This proves part (1).

Consider the corresponding oper $L(\boldsymbol{\mu})$ on $\mathbb{P}^1$ given by formula (5.16). By construction, it has regular singularities and residues $\varpi(\lambda_i+1)$ at $t_i, i = 0,\dots,m+1$. Since equations (5.7) are satisfied, this oper can be written as the Miura transformation (5.9), where $u(z)$ is given by formula (5.10). Therefore Φ given by formula (5.14) is a solution of the equation $L(\boldsymbol{\mu})\Phi=0$. This implies that the monodromy of $L(\boldsymbol{\mu})$ is contained in a Borel subgroup of PGL_2.

Conversely, suppose that the numbers $\boldsymbol{\mu}=\{\mu_i\}$ are such that the PGL_2-oper $L(\boldsymbol{\mu})$ satisfies the conditions of the theorem. According to Lemma 5.7 below, if $\lambda_{m+1}\notin\{-2,-3,\dots,-n-1\}$, then $L(\boldsymbol{\mu})$ is equal to the Miura transformation (5.9) of $u(x)$ given by formula (5.10). The set of numbers $\mathbf{w}=\{w_j\}$ appearing in $u(x)$ then must satisfy equations (5.7). But then the corresponding Bethe vector $v_{\mathbf{w}}$ is an eigenvector of the G_i's with the eigenvalues μ_i's. Since we know that these vectors form an eigenbasis for generic $\{t_i\}$ and $\{\lambda_i\}$, we obtain the statement of part (2). □

Lemma 5.7. *Let $L(\boldsymbol{\mu})$ be a PGL_2-oper of the form (5.16) with $\lambda_i\in\mathbb{C}$ that has a regular singularity at ∞ with residue $\varpi(\lambda_{m+1}+1)$, where $\lambda_{m+1}\notin\{-2,-3,\dots,-n-1\}$ and satisfies equation (5.3) with a non-negative integer n. Then for any collection $\{\lambda_i\}$ and generic $\{t_i\}$ the equation $L(\boldsymbol{\mu})\Phi=0$ has* *a unique solution of the form*

$$\Phi := \Phi(x) dx^{-\frac{1}{2}}, \ \Phi(x) := \prod_{i=0}^{m} (x - t_i)^{-\frac{\lambda_i}{2}} Q(x), \tag{5.20}$$

where $Q(x)$ is a polynomial of degree n with n distinct roots $w_1, \dots, w_n$ that are also distinct from $\{t_i\}$ and satisfy the Bethe Ansatz equations (5.7). Moreover, if $\lambda_i \notin \{0, 1, \dots, n-1\}$ for all $i = 0, \dots, m$, then this is so for all $\{t_i\}$.

Equivalently, under the above the above conditions, $L(\boldsymbol{\mu})$ is equal to the Miura transformation (5.9) of $u(x)$ given by formula (5.10).

Proof. Substituting (5.20) into the equation $L(\boldsymbol{\mu})\Phi = 0$, we obtain the equation

$$\left(\partial_x^2 - \sum_{i=0}^{m} \frac{\lambda_i}{x - t_i} \partial_x - \sum_{i=0}^{m} \frac{\widehat{\mu}_i}{x - t_i} \right) Q(x) = 0, \tag{5.21}$$

where

$$\widehat{\mu}_i := \mu_i - \sum_{j \neq i} \frac{\lambda_i \lambda_j}{2(t_i - t_j)}.$$

One of the characteristic exponents of this equation at ∞ is equal to n. Hence for any $\lambda_i \in \mathbb{C}, i = 0, \dots, m+1$, satisfying (5.3) there is a solution of this differential equation of the form $x^n + \sum_{j=1}^{\infty} Q_j x^{n-j}$, as long as the coefficients $Q_1, \dots, Q_n$ are uniquely determined by this condition. If this is the case, we set $Q_j = 0$ for $j > n$, and this will give us a solution since it does so when all λ_i's are positive integers. The standard theory of ODE implies that $Q_1, \dots, Q_n$ are indeed uniquely determined if and only if the second characteristic exponent of equation (5.21) is not in $\{0, 1, \dots, n-1\}$, which translates into the condition $\lambda_{m+1} \notin \{-2, -3, \dots, -n-1\}$.

Denote by $w_1, \dots, w_n$ the roots of $Q(x)$ counted with multiplicity. If $w_j \neq t_i$ for all $i = 0, \dots, m$, equation (5.21) implies that w_j is a simple root.

Suppose now that $w_j = t_i$ for a some i and $j \in J_i \subset \{1, \dots, n\}$. Then the expansion of the solution (5.20) of the equation $L(\boldsymbol{\mu})\Phi = 0$ near $x = t_i$ is equal to

$$(x - t_i)^{-\frac{\lambda_i}{2} + |J_i|}(1 + O(x - t_i))$$

up to a non-zero scalar factor. But since the leading term of $L(\boldsymbol{\mu})$ at $x = t_i$ is $\lambda_i(\lambda_i + 2)/4(x - t_i)$, this is only possible if $|J_i| = \lambda_i + 1$ which means that $\lambda_i \in \{0, 1, \dots, n-1\}$.

If $\lambda_i \notin \{0, 1, \dots, n-1\}$ for all $i = 0, \dots, m$, we find that $w_j \neq t_i$ for all i and any collection $\{t_i\}$. If $\lambda_i \in \{0, 1, \dots, n-1\}$ for some i, then we find that

$w_j \neq t_i$ for all i provided that the collection $\{t_i\}$ is sufficiently generic. That's because we know from [SV] that this is so when all λ_i's are non-negative integers (see the last paragraph of the proof of Lemma 5.2). □

Motivated by Theorem 5.5, it is natural to conjecture the following statements for a general Lie algebra $\mathfrak{g}$.

Conjecture 5.8. (1) For all $\{t_i\}$ and $\{\lambda_i\}$, there is a one-to-one correspondence between the set of joint eigenvalues of the algebra $\mathcal{G}$ of generalized Gaudin Hamiltonians on $\mathcal{H}$ given by formula (5.18) (without multiplicity) and the set of $G^\vee$-opers on $\mathbb{P}^1$ with regular singularities and residues $\varpi(-\lambda_i - \rho)$ at t_i, $i = 0, \ldots, m+1$, and solvable monodromy.

(2) Suppose that λ_i and λ^*_{m+1} are dominant integral weights. Then the monodromy of the corresponding $G^\vee$-opers is unipotent, and the inclusion (5.19) corresponds to the inclusion of the set of opers with trivial monodromy into the set of opers with unipotent monodromy.

(3) Let $I \subset \{1, \ldots, m\}$ and

$$\mathcal{H} := \left(\bigotimes_{i \in I} V_{\lambda_i} \otimes \bigotimes_{i \notin I} \nabla(\lambda_i) \right)^{\mathfrak{n}_+} [\lambda^*_{m+1}],$$

where λ_i are dominant integral for $i \in I$. Then the set of joint eigenvalues of the algebra $\mathcal{G}$ on $\mathcal{H}$ is in one-to-one correspondence with the set of $G^\vee$-opers as above with solvable monodromy and trivial local monodromy around the points t_i, $i \in I$.

(4) More generally, suppose that for $i = 0, \ldots, m$, $\mathfrak{p}_i$ are parabolic subalgebras of $\mathfrak{g}$ containing the positive Borel subalgebra $\mathfrak{b}_+ \subset \mathfrak{g}$, $\nabla(\lambda_i, \mathfrak{p}_i)$ are parabolic contragredient Verma modules for $\mathfrak{p}_i$, and

$$\mathcal{H} := \left(\bigotimes_{i=0}^{m} \nabla(\lambda_i, \mathfrak{p}_i) \right)^{\mathfrak{n}_+} [\lambda^*_{m+1}].$$

Then the set of joint eigenvalues of the algebra $\mathcal{G}$ on $\mathcal{H}$ is in one-to-one correspondence with the set of $G^\vee$-opers as above with solvable monodromy and local monodromy at t_i belonging to $Z(L_i^\vee)U_i^\vee \subset G^\vee$, where $L_i^\vee, U_i^\vee \subset P_i^\vee$ are the Levi factor and unipotent radical of the positive parabolic $P_i^\vee \subset G^\vee$ dual to $\mathfrak{p}_i$, and $Z(L_i^\vee)$ is the center of $L_i^\vee$.

Note that part (3) is a special case of (4) where $\mathfrak{p}_i = \mathfrak{g}$ if $i \in I$ and $\mathfrak{p}_i = \mathfrak{b}_+$ if $i \notin I$.

In a follow-up paper [EF], we plan to prove one direction of part (1); namely, construct an injective map from the set of joint eigenvalues of $\mathcal{G}$

to the corresponding set of $G^\vee$-opers with solvable monodromy. It follows from the results [MTV2] that if $\mathfrak{g} = \mathfrak{sl}_n$ and the set $\{\lambda_i\}$ is generic, then the cardinality of the subset of the set of PGL_n-opers from part (1) corresponding (via the Miura transformation) to solutions of the Bethe Ansatz equations is not greater than the dimension of $\mathcal{H}$ given by (5.18). If for genetic $\{\lambda_i\}$ all PGL_n-opers from part (1) have this form and the spectrum of $\mathcal{G}$ is simple, this would imply part (1) for $\mathfrak{g} = \mathfrak{sl}_n$ and generic $\{\lambda_i\}$.

Remark 5.9. Still more generally, one may consider the action of the algebra $\mathcal{G}$ on

$$(M_0 \otimes \cdots \otimes M_m)^{\mathfrak{n}_+}[\lambda_{m+1}^*],$$

where M_i are arbitrary objects from category $\mathcal{O}$. It would be very interesting to parametrize the spectrum of this action, but here we don't even have a conjectural picture yet.

5.5. Examples of Bethe vectors

In this subsection for readers convenience we give some (well known) examples of Bethe vectors in the space $\mathcal{H}$ defined by formula (5.18) and various phenomena related to them.

Example 5.10. Let $\mathfrak{g} = \mathfrak{sl}_2$ and assume that λ_j is generic for all $0 \le j \le m$, and let $n = 1$. Then $\dim \mathcal{H} = m$ and the Bethe Ansatz equation has the form

$$\sum_{i=0}^{m} \frac{\lambda_i}{w - t_i} = 0,$$

which reduces to a polynomial equation of degree m in w, so its solution set $S \subset \mathbb{C}$ has cardinality m. For every $w \in S$ the Bethe vector v_w has the form:

$$v_w = \left(\sum_{i=0}^{m} \frac{f_i}{w - t_i} \right) v, \tag{5.22}$$

where v is the tensor product of the highest weight vectors of $\nabla(\lambda_i)$ and f_i is $f \in \mathfrak{sl}_2$ acting in the i-th factor (this is a special case of formula (5.5)). These vectors form a basis in $\mathcal{H}$.

For special weights, however, the Gaudin Hamiltonians may fail to be semisimple and the Bethe Ansatz equations may have fewer solutions than $\dim \mathcal{H}$ (possibly none at all), or infinitely many solutions. Thus Bethe vectors may fail to form a basis of $\mathcal{H}$. Such things happen, for instance, when λ_j are dominant integral for $0 \le j \le m$ while λ^*_{m+1} is integral, but not dominant (see [F2]).

Example 5.11. Let $m = 2$, so we have four ramification points t_0, t_1, t_2, ∞. Let E_k be the space of $\mathfrak{n}_+$-invariant vectors in $\nabla(\lambda_0) \otimes \nabla(\lambda_1) \otimes \nabla(\lambda_2)$ of weight $\sum_{i=0}^2 \lambda_i - 2k$. Thus $\dim E_k = k + 1$.

In E_0 we have a unique up to scaling Bethe vector, which is merely the tensor product v of highest weight vectors of $\nabla(\lambda_i)$. The eigenvalues of G_i on v are $\mu_i^0 := \sum_{j \ne i} \frac{\lambda_i \lambda_j}{2(t_i - t_j)}$.

Now consider Bethe vectors in E_1. The Bethe Ansatz equation has the form

$$\sum_{i=0}^{2} \frac{1}{w - t_i} = 0,$$

i.e.,

$$P(\boldsymbol{\lambda}, \mathbf{t}, w) := \lambda_0 (w - t_1)(w - t_2) + \lambda_1 (w - t_0)(w - t_2) + \lambda_2 (w - t_0)(w - t_1) = 0, \tag{5.23}$$

so it is quadratic if $\sum_{i=0}^2 \lambda_i \ne 0$, and for generic t_j has two solutions $w_\pm$ giving rise to two Bethe vectors v_+, v_- which form a basis of E_1. The eigenvalues of G_i on the vectors $v_\pm$ are $\mu_i^\pm := \mu_i^0 - \frac{\lambda_i}{t_i - w_\pm}$.

Now consider the case $\sum_{i=0}^2 \lambda_i = 0$. In this case

$$E_k = \mathrm{Hom}(\Delta(-2k), \nabla(\lambda_0) \otimes \nabla(\lambda_1) \otimes \nabla(\lambda_2)),$$

so we have an injective map $R = \Delta_3(f) : E_0 \to E_1$ which is defined by restricting of a homomorphism $\Delta(0) \to \nabla(\lambda_1) \otimes \nabla(\lambda_2) \otimes \nabla(\lambda_3)$ to $\Delta(-2) \subset \Delta(0)$. Thus $Rv \in E_1$ is an eigenvector of G_i with eigenvalues μ_i^0 (as $[R, G_i] = 0$).

The vector Rv is **not**, however, a Bethe vector of the form (5.5). Namely, if $\sum_{i=0}^2 \lambda_i = 0$ then the quadratic term in the Bethe Ansatz equation (5.23) drops out and it becomes linear, so has only one finite solution

$$w_+ = -\frac{\lambda_0 t_1 t_2 + \lambda_1 t_0 t_2 + \lambda_2 t_0 t_1}{\lambda_0 t_0 + \lambda_1 t_1 + \lambda_2 t_2}$$

(provided that the denominator $\sum_{i=0}^2 \lambda_i t_i$ is nonzero). The second solution w_- escapes to ∞ as we approach the hyperplane $\sum_{i=0}^2 \lambda_i = 0$. Thus we obtain only one Bethe vector v_+; the second vector $v_- = Rv$ is a limit of Bethe vectors from generic λ_i, but is not itself a Bethe vector, as it does not correspond to a finite solution of the Bethe Ansatz equations. Nevertheless, the vectors v_+, v_- are still an eigenbasis in E_1 for the operators G_i, which for generic t_i still act regularly and semisimply on this space.

It is instructive to consider the resonance case when $\sum_{i=0}^2 \lambda_i = \sum_{i=0}^2 \lambda_i t_i = 0$ but λ_i don't vanish simultaneously. Then the Bethe Ansatz equations have no solutions, so there are no Bethe vectors at all. The operators G_i are not semisimple on E_1 in this case, so their only joint eigenvector in E_1 up to scaling is v_-.

Finally, in the most degenerate case $\lambda_0 = \lambda_1 = \lambda_2 = 0$, the Bethe Ansatz equation is vacuous, so any $w \in \mathbb{C}$ is a solution. In this case, G_i act by scalars on E_1 and there are infinitely many Bethe vectors given by (5.22).

A slightly more interesting example is $\sum_{i=0}^2 \lambda_i = 2$ (assuming that otherwise λ_i are generic). In this case we have the injective restriction map $R = \Delta_3(f) : E_1 \to E_2$. Generically we have a basis of Bethe vectors v_+, v_- of E_1 as above, so we have eigenvectors Rv_+, Rv_- of G_i with the same eigenvalues.

These vectors are not Bethe vectors, however. Indeed, the Bethe Ansatz equations for E_2 have the form

$$\sum_{i=0}^{2} \frac{\lambda_i}{w_1 - t_i} = \frac{2}{w_1 - w_2}, \quad \sum_{i=0}^{2} \frac{\lambda_i}{w_2 - t_i} = \frac{2}{w_2 - w_1},$$

which implies that

$$\begin{aligned} 2(w_1 - t_0)(w_1 - t_1)(w_1 - t_2) &= (w_1 - w_2)P(\boldsymbol{\lambda}, \mathbf{t}, w_1), \\ 2(w_2 - t_0)(w_2 - t_1)(w_2 - t_2) &= (w_2 - w_1)P(\boldsymbol{\lambda}, \mathbf{t}, w_2). \end{aligned}$$

This is a system of two cubic equations in w_1, w_2, so for generic λ_i by Bezout's theorem it has $3 \cdot 3 = 9$ solutions. However, three of these solutions are $w_1 = w_2 = t_i$, $i = 0, 1, 2$, which are not solutions of the Bethe Ansatz equations. This leaves us with 6 solutions, or 3 modulo the symmetry exchanging w_1 and w_2, each defining a Bethe vector. But on the hyperplane $\sum_{i=0}^2 \lambda_i = 2$ two of these solutions run away to infinity, tending to $(\infty, w_\pm)$. This leaves us with just one solution $\mathbf{w}$ which gives rise to a single Bethe vector $v_{\mathbf{w}}$. The vectors $v_{\mathbf{w}}, Rv_+, Rv_-$ form a basis of E_2.

5.6. Analytic Langlands correspondence over $\mathbb{R}$ for compact groups

In this subsection and the next, we incorporate the Gaudin model for finite-dimensional representations into the framework of the analytic Langlands correspondence.

Let $F=\mathbb{R}$ and $X=\mathbb{P}^1$ with the usual real structure and distinct real ramification points $t_0,\dots,t_{m+1}$ (in this cyclic order). As before, we will set $t_{m+1}=\infty$, so $t_0<\cdots<t_m$. Let the group G^{σ_i}, $0\le i\le m+1$, be the compact form G_c of G for all i. Place the finite-dimensional irreducible representation V_{λ_i} of G_c with highest weight λ_i at the point t_i. Then, as we explained in Subsection 2.15, the Hilbert space of the analytic Langlands correspondence is defined precisely by formula (5.1). Also the quantum Hitchin Hamiltonians are exactly the Gaudin Hamiltonians in this case.

It remains to explain why in our setting the monodromy-free condition is precisely the topological reality condition on spectral opers arising from the analytic Langlands correspondence. Let us do so for $G=SL_2$. We expect that this argument can be generalized to all simple Lie groups G.

In view of Remark 2.15, we can define the Hecke operator $H_{x,\overline{x}}$ for $x\in\mathbb{C}$ (with coweight 1 of $SL_2=(G/(\pm1))^\vee$ attached to both x and $\overline{x}$). Recall that since the real locus $X(\mathbb{R})$ is a quaternionic oval, the eigenvalue of the Hecke operator $H_{x,\overline{x}}$ is given by formula (4.6):

$$\boldsymbol{\beta}(x,\overline{x})=\mathrm{Im}(\overline{f_j^+(x)}g_j^+(x)),\ \mathrm{Im}(x)>0$$

near the interval (t_j,t_{j+1}), where f_j^+,g_j^+ are the basic local solutions of the oper equation $L\beta=0$ near t_j such that

$$g_j^+(t_j+u)=\pi u^{\frac{\lambda_j}{2}+1}(1+ug_j^0(u)),$$
$$f_j^+(t_j+u)=u^{-\frac{\lambda_j}{2}}(1+uf_j^0(u))+\gamma_j u^{\frac{\lambda_j}{2}+1}(1+ug_j^0(u))\log u,$$

where $f_j^0(u),g_j^0(u)\in\mathbb{R}[[u]]$, $\gamma_j\in\mathbb{R}$. Consider also the basic local solutions f_j^-,g_j^- of the oper equation near t_j such that for small $u>0$

$$g_j^-(t_j-u)=\pi u^{\frac{\lambda_j}{2}+1}(1-ug_j^0(-u)),$$
$$f_j^-(t_j-u)=u^{-\frac{\lambda_j}{2}}(1-uf_j^0(-u))-\gamma_j u^{\frac{\lambda_j}{2}+1}(1-ug_j^0(-u))\log u.$$

Then the half-monodromy matrix along the upper (respectively, lower) half-circle around t_j between the bases f_j^+,g_j^+ and f_j^-,g_j^- is

$$J_j^\pm=\begin{pmatrix} i^{\mp\lambda_j} & 0\\ i^{\mp(1+\lambda_j)}\gamma_j & -i^{\mp\lambda_j}\end{pmatrix}.$$

Thus the real analytic continuation along the upper half-circle transforms $\boldsymbol{\beta}=\mathrm{Im}(\overline{f_j^+}g_j^+)$ to $\pm\,\mathrm{Im}(\overline{f_j^-}g_j^-)\mp\gamma_j|g_j^-|^2$. So to preserve the vanishing condition for

$\boldsymbol{\beta}$ on the real locus, we must have $\gamma_j = 0$. This means that the monodromy of the oper L around t_j, which is given by the matrix $(J_j^-)^{-1}J_j^+ = (-1)^{\lambda_j}$ is trivial in PGL_2 for all j, i.e. the oper is monodromy-free.

5.7. The connection between Hecke operators and Baxter's $\boldsymbol{Q}$-operator

Let us go back to the case of the Gaudin model with the space of states $\mathcal{H}$ given by formula (5.18). The following result is proved in the same way as Lemma 5.7.

Theorem 5.12. *For arbitrary collections $\{t_i\}$ and $\{\lambda_i\}$ satisfying (5.3) with*

$$\lambda_{m+1} \notin \{-2, -3, \ldots, -n-1\},$$

there is a unique linear operator $\mathbf{Q}(x)$ acting on $\mathcal{H}$ given by formula (5.18) commuting with the Gaudin Hamiltonians G_i, which is a monic polynomial in x of degree n satisfying the universal oper equation (compare with Proposition 3.10(i)):

$$\left(\partial_x^2 - \sum_{i=0}^{m} \frac{\lambda_i}{x - t_i}\partial_x\right)\mathbf{Q}(x) - \mathbf{Q}(x)\sum_{i=0}^{m}\frac{\widehat{G}_i}{x - t_i} = 0, \tag{5.24}$$

where

$$\widehat{G}_i := G_i - \sum_{j \neq i} \frac{\lambda_i \lambda_j}{2(t_i - t_j)}.$$

In particular, if $v \in \mathcal{H}$ is an eigenvector of the G_i's with eigenvalues $\boldsymbol{\mu} = \{\mu_i\}$, we have

$$\mathbf{Q}(x)v = Q_v(x)v,$$

where $Q_v(x)$ is the polynomial appearing in the corresponding solution (5.20) of the equation $L(\boldsymbol{\mu})\Phi = 0$.

As we will see in Subsection 5.12, this operator can be obtained as the $q \to 1$ limit of the celebrated Baxter Q-operator introduced by R. Baxter in the study of integrable quantum spin chains. For this reason we call $\mathbf{Q}(x)$ the **Baxter Q-operator of the Gaudin model** (or Baxter's Q-operator for short).

On the other hand, as we explain presently, this operator $\mathbf{Q}(x)$ may be viewed as an algebraic version of the Hecke operators of the analytic Langlands correspondence.

Let us switch to the setting of the previous subsection (for $G = SL_2$). Thus, $\mathcal{H}$ is now given by formula (5.4). The proof of Theorem 5.12 (which follows the proof of Lemma 5.7) carries over without changes to this case. Thus, we obtain a Q-operator in this case as well, for which we will use the same notation. Moreover, the condition $\lambda_{m+1} \notin \{-2, -3, \ldots, -n-1\}$ becomes vacuous in this case because all λ_i's are assumed to be dominant integral.

It is clear that the resulting operator is the restriction of the Q-operator acting on the tensor product of contragredient Verma modules with dominant integral weights (which is described in Theorem 5.12) to the tensor product of the corresponding finite-dimensional representations under the embedding (5.19).

Consider now the finite-dimensional case with $\mathcal{H}$ given by formula (5.4). Let $L(\boldsymbol{\mu})$ be a monodromy-free PGL_2-oper corresponding via Theorem 5.3 to a set of joint eigenvalues $\boldsymbol{\mu} = \{\mu_i\}$ of the Gaudin Hamiltonians acting on $\mathcal{H}$. Formula (4.6) implies that the eigenvalue $\boldsymbol{\beta}(x, \overline{x})$ of the Hecke operator $H_{x,\overline{x}}$ corresponding to the PGL_2-oper $L(\boldsymbol{\mu})$ is, up to scaling by an x-independent constant, given by

$$\boldsymbol{\beta}(x, \overline{x}) \sim |\Phi(x)|^2 \operatorname{Im} \int_{x_0}^{x} \Phi^{-2}(z) dz, \ \operatorname{Im}(x) \geq 0,$$

where Φ is defined by (5.14), and the same expression with a minus sign if $\operatorname{Im}(x) < 0$. Here the lower limit x_0 of integration can be any point of $X(\mathbb{R})$; recall from Subsection 5.6 that if the oper is monodromy-free then the imaginary part of the integral is independent on the choice of this point.

Thus, again up to a constant,

$$H_{x,\overline{x}} \sim \prod_{i=0}^{m} |x - t_i|^{-\lambda_i} \mathbf{Q}(x)^\dagger \mathbf{Q}(x) \operatorname{Im} \int_{x_0}^{x} \mathbf{Q}^{-2}(z) \prod_{i=0}^{m} (z - t_i)^{\lambda_i} dz, \ \operatorname{Im}(x) > 0,$$

$$\mathbb{H}_{x,\overline{x}} \sim \mathbf{Q}(x)^\dagger \mathbf{Q}(x) \operatorname{Im} \int_{x_0}^{x} \mathbf{Q}^{-2}(z) \prod_{i=0}^{m} (z - t_i)^{\lambda_i} dz, \ \operatorname{Im}(x) > 0.$$

Thus we see that the Hecke operator can be expressed in a rather direct way in terms of Baxter's Q-operator of the Gaudin model.

5.8. Analytic Langlands correspondence for discrete series representations

Consider now another example of the analytic Langlands correspondence in the setting of the Gaudin model, which involves discrete series representations of $SL_2(\mathbb{R})$. As in Subsection 5.6, we take $F = \mathbb{R}$, $X = \mathbb{P}^1$ with the standard real structure and **real** ramification points $t_0, \ldots, t_{m+1} \in \mathbb{R}$, but now set G^σ to be the split real group $SL_2(\mathbb{R})$. So this setting is somewhat different from the one of Subsection 5.6, where we deal with compact groups; they are related by analytic continuation in highest weights.

Let $V_i = \widehat{\Delta}(-r_i)$ be the completion of the Verma module for $0 \le i \le m$, and $V_{m+1} = \widehat{\Delta}(-r_{m+1})^*$, where $r_i \in \mathbb{Z}_{\ge 1}$. So V_i is a holomorphic discrete series representation for all $0 \le i \le m$, while V_{m+1} is an antiholomorphic discrete series representation.[33] Then

$$\begin{aligned}\mathcal{H}_{V_0,\ldots,V_{m+1}} &= \mathrm{Hom}(\Delta(-r_{m+1}), \Delta(-r_0) \otimes \cdots \otimes \Delta(-r_m)) \\ &= (\Delta(-r_0) \otimes \cdots \otimes \Delta(-r_m))^{\mathfrak{n}_+}_{-r_{m+1}}.\end{aligned}$$

So the Hilbert space is finite dimensional in this case. It is non-zero if and only if

$$r_{m+1} - \sum_{i=0}^{m} r_i = 2n, \tag{5.25}$$

where n is a non-negative integer. It is easy to see that the Gaudin operators G_i are self-adjoint, and hence diagonalizable, in this case.

Theorem 5.13. *For all real $\{t_i\}$ and all $\{r_i \in \mathbb{Z}_{\ge 1}\}$ the Gaudin Hamiltonians are diagonalizable on $\mathcal{H}_{V_0,\ldots,V_{m+1}}$ with simple joint spectrum and there is a one-to-one correspondence between the set of their joint eigenvalues and the set of PGL_2-opers on $\mathbb{P}^1$ with regular singularities and residues $\varpi(-r_i + 1)$ at $t_i, i = 0, \ldots, m+1$, and solvable monodromy.*

Proof. According to Theorem 5.6, this statement holds for generic collections $\{t_i\}$ and $\{\lambda_i = -r_i\}$. However, Corollary 2.4.6 of [V] implies that in the case that all r_i's are positive (so that all λ_i's are negative) and all t_i's are real, the Bethe vectors $v_{\mathbf{w}}$ corresponding to the solutions $\mathbf{w} = \{w_1, \ldots, w_n\}$ (with n defined by formula (5.25)) of the Bethe Ansatz equations (5.7) form a basis of eigenvectors of the Gaudin Hamiltonians. As before, for each such solution $\mathbf{w}$, denote by $\boldsymbol{\mu} = \{\mu_i\}$ the corresponding set of joint eigenvalues of the Gaudin Hamiltonians $\{G_i\}$. Then the PGL_2-oper $L(\boldsymbol{\mu}) = \partial_x^2 - v(x)$ given by formula (5.16) (with $\lambda_i = -r_i$) is equal to the Miura transformation (5.9)

[33] More precisely, the representations $\widehat{\Delta}(-1)$, $\widehat{\Delta}(-1)^*$ are limit of discrete series representations.

of $u(x)$ given by formula (5.10). Equivalently, equation $L(\boldsymbol{\mu})\Phi = 0$ has a solution Φ given by formula (5.20) with

$$Q(x) = \prod_{j=1}^{n}(x - w_j). \tag{5.26}$$

This shows that all opers $L(\boldsymbol{\mu})$ corresponding to the joint eigenvalues $\boldsymbol{\mu}$ of the Gaudin Hamiltonians have solvable monodromy representation.

Suppose that the joint spectrum of the Gaudin Hamiltonians is not simple. Then there are two different solutions $\mathbf{w}_1$ and $\mathbf{w}_2$ of the Bethe Ansatz equations (5.7) such that the corresponding joint eigenvalues coincide; that is, $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ and so $L(\boldsymbol{\mu}_1) = L(\boldsymbol{\mu}_2)$. But then equation $L(\boldsymbol{\mu}_1)\Phi = 0$ has two different solutions Φ_1 and Φ_2 of the form (5.20) with the monic polynomials $Q_1(x)$ and $Q_2(x)$ of degree n of the form (5.26) corresponding to $\mathbf{w}_1$ and $\mathbf{w}_2$. Therefore, $\Phi = \Phi_1 - \Phi_2$ is also a solution of $L(\boldsymbol{\mu}_1)\Phi = 0$, such that the corresponding polynomial $Q(x) = Q_1(x) - Q_2(x)$ is non-zero and has degree $0 \le k \le n-1$.

It follows from our condition (5.25) that near $t_{m+1} = \infty$, any solution of the equation $L(\boldsymbol{\mu}_1)\Phi = 0$ has the expansion (up to a non-zero scalar) either $y^{r_m/2}(1 + O(y))$ or $y^{(-r_{m+1}+2)/2}(1 + O(y))$, where $y = x^{-1}$. Therefore, if $L(\boldsymbol{\mu}_1)\Phi = 0$ has a solution of the form (5.20) with $Q(x)$ of degree $0 \le k \le n-1$, then $r_{m+1} = n - k + 1$. But this is impossible because (5.25) also implies that $r_{m+1} > 2n$. This shows that the joint spectrum of the Gaudin Hamiltonians is simple.

Conversely, suppose that a PGL_2-oper $L(\boldsymbol{\mu}) = \partial_x^2 - v(x)$ has regular singularities with residues $\varpi(-r_i + 1)$ at $t_i, i = 0, \dots, m+1$, where $r_i \in \mathbb{Z}_{\ge 1}, i = 0, \dots, m+1$. Then the r_i's must satisfy equation (5.25), which implies that $r_{m+1} \notin \{2, 3, \dots, n+1\}$. Therefore, by Lemma 5.7, for all $\{t_i\}$, the PGL_2-oper $L(\boldsymbol{\mu})$ is equal to the Miura transformation (5.9) of $u(x)$ given by formula (5.10) with $\lambda_i = -r_i$, with all w_j's being distinct and different from the t_i's. But then the set of numbers $\mathbf{w} = \{w_j\}$ appearing in $u(x)$ must satisfy Bethe Ansatz equations (5.7), so the corresponding Bethe vector $v_{\mathbf{w}}$ is an eigenvector of the G_i's with the eigenvalues μ_i's. Since we know from above that these vectors form an eigenbasis, this completes the proof. □

Remark 5.14. 1. We hope that this statement can be proved within the framework of the analytic Langlands correspondence, using the results of Subsection 4.3.

2. The above argument showing the simplicity of the joint spectrum of the Gaudin Hamiltonians can also be used to prove the simplicity of the joint spectrum of the Gaudin Hamiltonians in the setting of Subsection 5.6.

3. The above proof shows that the statement of Theorem 5.13 remains true if the r_i's are arbitrary positive real numbers constrained by equation (5.25). If r_i is not an integer, one cannot define an action of $SL(2,\mathbb{R})$ on a completion of $\Delta(-r_i)$. But it can be interpreted as a unitary representation of the universal cover of $SL(2,\mathbb{R})$, and according to Remark 2.22, such representations can also included into the framework of the analytic Langlands correspondence.

5.9. Chiral analytic Langlands correspondence

In this subsection we consider another setting in which the analytic Langlands correspondence can be linked to a particular Gaudin model. Let $F=\mathbb{C}$, $X=\mathbb{P}^1$ with ramification points $t_0,\dots,t_m,t_{m+1}=\infty$, and suppose that the representations V_j of G are holomorphic. Then in the definition of the Hecke operator $H_{x,\lambda}$ we may replace the integral $\int_{\mathcal{Z}_{\lambda,x}(P)}\psi(Q)dQd\overline{Q}$ over the complex variety $\mathcal{Z}_{\lambda,x}(P)$ by the "contour" integral $\int_C\psi(Q)dQ$, where $C\subset\mathcal{Z}_{\lambda,x}(P)$ is a real cycle, and by Cauchy's theorem the result is stable under deformations of C. Since V_j are not Hermitian, the corresponding space $\mathcal{H}$ will not carry a Hermitian form, but we may still consider eigenvectors and eigenvalues of Hecke operators (in the spirit of Remark 2.37).

Admittedly, there are very few holomorphic irreducible representations of G (only the finite dimensional ones), but we may in fact take V_j to be certain representations of the Lie algebra $\mathfrak{g}=\operatorname{Lie}G$ which do not necessarily integrate to G. We call this setting the **chiral analytic Langlands correspondence**. Let us show how it works in an example with $G=SL_2$.

For $\lambda\in\mathbb{C}$ let $\Delta(\lambda),\nabla(\lambda)$ be the Verma and contragredient Verma modules over $\mathfrak{sl}_2$ with highest weight λ. For $0\le i\le m$ take $V_i:=\nabla(\lambda_i)$, and let $V_{m+1}=\Delta(\lambda_{m+1})^*$ be the graded dual of $\Delta(\lambda_{m+1})$, i.e., the contragredient Verma module with lowest weight $-\lambda_{m+1}$. Assume that $\sum_{i=0}^m\lambda_i-\lambda_{m+1}=2n$ and that $\lambda_{m+1}=r-1$, where $n,r\in\mathbb{Z}_{\ge0}$. Then the space

$$\mathcal{H}:=\operatorname{Hom}(V_{m+1}^*,V_0\otimes\dots\otimes V_m)=\operatorname{Hom}(\Delta(r-1),\nabla(\lambda_0)\otimes\dots\otimes\nabla(\lambda_m))$$

has dimension $\binom{n+m}{m}$; it is the space of singular vectors in $V_0\otimes\dots\otimes V_m$ of weight $r-1$. We realize $\nabla(\lambda)$ as $\mathbb{C}[y]$ with the action of $\mathfrak{g}$ given by (3.3). Then the space $\mathcal{H}$ is realized as the space of homogeneous polynomials of degree n in $y_0,\dots,y_m$ which are invariant under simultaneous translations. Consider also the space

$$\mathcal{H}':=\operatorname{Hom}(\Delta(-r-1),\nabla(\lambda_0)\otimes\dots\otimes\nabla(\lambda_m))$$

of dimension $\binom{n+r+m}{m}$; it is the space of homogeneous polynomials of $y_0, \dots, y_m$ of degree $n+r$. Then we can define the (modified) **chiral Hecke operator** $\mathbb{H}_x^{\rm chir} : \mathcal{H} \to \mathcal{H}'$ by analogy with (2.6), replacing integration over $\mathbb{C}$ by contour integration:

$$(5.27)\qquad (\mathbb{H}_x^{\rm chir}\psi)(y_0, \dots, y_m) = \oint \psi\left(\frac{t_0 - x}{s - y_0}, \dots, \frac{t_m - x}{s - y_m}\right) \prod_{j=0}^{m} (s - y_j)^{\lambda_j} ds,$$

where $\oint f(s)ds$ denotes the residue of $f(s)ds$ at ∞. Note that this residue is well defined since $\sum_{i=0}^{m} \lambda_i = n+r-1 \in \mathbb{Z}$, hence the integrand is single-valued near ∞. So this formula makes sense even though the $\mathfrak{g}$-modules V_i do not integrate to G.

Note that we have an inclusion $\iota : \Delta(-r-1) \hookrightarrow \Delta(r-1)$, so we have the restriction operator $R : \mathcal{H} \to \mathcal{H}'$.

Lemma 5.15. *Under suitable normalization of ι, we have*

$$(R\psi)(y_0, \dots, y_m) = \oint \psi\left(\frac{1}{s - y_0}, \dots, \frac{1}{s - y_m}\right) \prod_{j=0}^{m} (s - y_j)^{\lambda_j} ds.$$

Proof. The proof is similar to the proof of Lemma 2.26. We have

$$R\psi = \Delta_{m+1}(\tfrac{f^r}{r!})\psi = \oint \Delta_{m+1}(\exp(s^{-1}f)\psi) s^{r-1} ds.$$

The 1-parameter group generated by $f \in \mathfrak{sl}_2$ consists of fractional linear transformations $z \mapsto \frac{z}{tz+1}$. Therefore

$$R\psi = \oint \psi\left(\frac{y_0 s}{s - y_0}, \dots, \frac{y_m s}{s - y_m}\right) s^{r-1-\sum_j \lambda_j} \prod_{j=0}^{m} (s - y_j)^{\lambda_j} ds =$$
$$\oint \psi\left(\frac{1}{s - y_0}, \dots, \frac{1}{s - y_m}\right) \prod_{j=0}^{m} (s - y_j)^{\lambda_j} ds. \qquad \square$$

By Lemma 5.15, $\mathbb{H}_x^{\rm chir} \sim x^n R$, $x \to \infty$. Also it is easy to show that $\mathbb{H}_x^{\rm chir}$ satisfies the universal oper equation of Proposition 3.10(i) (the proof is the same as for $\mathbb{H}_x$), hence it commutes with Gaudin Hamiltonians. So the discussion of Section 5.2 implies

Proposition 5.16. *The chiral Hecke operator is the composition of the restriction operator and the Baxter Q-operator:*

$$\mathbb{H}_x^{\rm chir} = R\mathbf{Q}(x).$$

5.10. Infinite dimensional generalizations, including principal series

The above discussion shows that tamely ramified analytic Langlands correspondence in genus 0 may be viewed as a generalization of the Gaudin model in which we can consider the action of Gaudin Hamiltonians on tensor products of any admissible representations of a real reductive group G, so that the space of states can be infinite dimensional. For instance, in the case $G = SL_2$ and $F = \mathbb{R}$ (Subsection 5.2), we may replace finite dimensional representations V_i of $SU(2)$ by principal series representations $\mathcal{V}_i$ of $SL_2(\mathbb{R})$ (taking the real locus $X(\mathbb{R}) = \mathbb{RP}^1 \subset X(\mathbb{C}) = \mathbb{CP}^1$ to be real rather than quaternionic). Then the Gaudin operators $\{G_i\}$ act as self-adjoint unbounded strongly commuting operators on the Hilbert space

$$\mathcal{H} := \mathrm{Mult}_{SL_2(\mathbb{R})}(\mathcal{V}^*_{m+1}, \mathcal{V}_0 \otimes \cdots \otimes \mathcal{V}_m).$$

So the spectral problem for $\{G_i\}$ becomes analytic and can no longer be solved by algebraic Bethe Ansatz. Nevertheless, the description of the spectrum of the operators $\{G_i\}$ in terms of monodromy-free opers (see Theorem 5.5) can be generalized: the spectrum is parametrized (at least conjecturally) by the set of **balanced opers** with prescribed residues, as explained in Subsection 4.3.2. Namely, we have the following result.

Theorem 5.17. *The set of joint joint eigenvalues μ_i of the Gaudin operators G_i on the Hilbert space $\mathrm{Mult}_{SL_2(\mathbb{R})}(\mathcal{V}^*_{m+1}, \mathcal{V}_0 \otimes \cdots \otimes \mathcal{V}_m)$ is in one-to-one correspondence with the set of $\boldsymbol{\Lambda}$-balanced opers on $\mathbb{P}^1$ of the form*

$$L(\boldsymbol{\mu}) = \partial_x^2 - \sum_{i=0}^{m} \frac{\lambda_i(\lambda_i+2)}{4(x-t_i)^2} - \sum_{i=0}^{m} \frac{\mu_i}{x-t_i},$$

subject to the equations

$$\sum_{i=0}^{m} \mu_i = 0, \ \sum_{i=0}^{m} t_i\mu_i = \frac{\lambda_{m+1}(\lambda_{m+1}+2)}{4} - \sum_{i=0}^{m} \frac{\lambda_i(\lambda_i+2)}{4}, \tag{5.28}$$

where λ_j are the parameters of the principal series representations $\mathcal{V}_j := L^2(\mathbb{RP}^1, |K|^{-\frac{\lambda_j}{2}})$ and $\boldsymbol{\Lambda} = (\Lambda_0, \ldots, \Lambda_{m+1})$, $\Lambda_j = -ie^{\frac{\pi i \lambda_j}{2}}$.

Indeed, this follows from the fact that the Gaudin operators $\{G_i\}$ strongly commute with Hecke operators, hence have the same spectral decomposition, and the latter is described in Subsection 4.3.2.

Similarly, one can generalize to the infinite dimensional case the setting of Subsection 5.8. Namely, we may consider the situation when $V_i = \widehat{\Delta}(-r_i)$, $0 \le i \le p$, and $V_i = \widehat{\Delta}(-r_i)^*$ for $p+1 \le i \le m+1$, i.e., we have p holomorphic discrete series representations and $m-p+1$ antiholomorphic ones. Then the Hilbert space $\mathcal{H}$ is infinite dimensional if $1 \le p \le m-1$, so the problem is again no longer algebraic. Since discrete series representations are composition factors of (non-unitary) principal series representations, one can presumably generalize the analysis of Subsection 4.3.2 to describe the spectrum in this case, but we will not discuss this here.

5.11. Double Gaudin model

The discussion of the analytic Langlands correspondence for a group G on $\mathbb{P}^1$ with parabolic points over $\mathbb{C}$ in Subsection 3.3 can be framed as a "double" of the Gaudin model for the Lie algebra $\mathfrak{g}$ in which we combine holomorphic and anti-holomorphic degrees of freedom.

For concreteness, consider the case of SL_2. Then we associate to the points $t_i, i = 0, 1, \dots, m+1$, with $t_{m+1} = \infty$ as before (these are the parabolic points) spherical unitary principal series representations of $SL_2(\mathbb{C})$, which are completions of **spherical Harish-Chandra bimodules**, i.e., spaces of finite type linear maps between Verma modules for $\mathfrak{sl}_2$. We have two commuting actions of the Lie algebra $\mathfrak{sl}_2$ on such representations (by left and right multiplications), which we can view as holomorphic and anti-holomorphic. Therefore, on such representations act not only the above Gaudin Hamiltonians G_i but also their complex conjugates $\overline{G}_i$. Hence it is natural to form the generating series $S(x)$ given by formula (3.6) and its complex conjugate $\overline{S}(\overline{x})$.

We have proved in [EFK2, EFK3] that the Hecke operator H_x (which is a section of $K^{-1/2} \otimes \overline{K}^{-1/2}$ on $\mathbb{P}^1$) satisfies a system of two second-order differential equations (3.7), see Corollary 3.12.

The corresponding eigenvalues of the Hecke operators are single-valued C^∞ solutions of this system with $S(x)$ and $\overline{S}(\overline{x})$ replaced by the corresponding eigenvalues $v(x)$ and $\overline{v}(x)$. As explained in Subsection 3.1, such single-valued solutions exist if and only if the oper $\partial_x^2 - v(x)$, where $v(z)$ is given by formula (5.16), has real monodromy, and such opers are called **real opers**. Theorem 3.13 implies the following result.

Theorem 5.18. *If $\{\mu_i\}$ is the set of joint eigenvalues of the Gaudin operators $\{G_i\}$ on the Hilbert space (2.9) then the PGL_2-oper*

$$L(\boldsymbol{\mu}) = \partial_x^2 - \sum_{i=0}^{m} \frac{\lambda_i(\lambda_i+2)}{4(x-t_i)^2} - \sum_{i=0}^{m} \frac{\mu_i}{x-t_i}$$

on $\mathbb{P}^1$ satisfies equations (5.28) and has real monodromy. The corresponding eigenvalues of $\overline{G}_i$ are equal to $\overline{\mu}_i$.

Moreover, we conjecture that there is a one-to-one correspondence between the spectrum of Gaudin operators $\{G_i\}$ on this Hilbert space and the set of such opers. This is an extension of a conjecture from [EFK3] from the case when all $\lambda_i = -1$ to the case when $\lambda_i \in -1 + \mathbb{R}\sqrt{-1}$. In the case when $m = 3$ or 4 it is proved in the same way as in [EFK3].

5.12. q-deformation of the Gaudin model

The q-deformation of this Gaudin model is known as the **XXZ model**. The role of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_2$ is now played by the corresponding quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$.[34] In particular, the role of the representations V_{λ_k} attached to the points t_k is now played by the corresponding evaluation representations of $U_q(\widehat{\mathfrak{sl}}_2)$, and the role of the space $\mathcal{H}$ given by (5.4) is played by the space $\mathcal{H}_q$ in which we replace each V_{λ_k} by the corresponding evaluation representation (see [FH] for more details).

The role of the generating function $S(x)$ of the Gaudin Hamiltonians given by formula (3.6) is now played by the **transfer-matrix** $\mathbf{T}(z)$ associated to the two-dimensional evaluation representation of $U_q(\widehat{\mathfrak{sl}}_2)$ corresponding to z (see [FH]).

The q-deformation of the Hecke operator H_z is now closely related to the transfer-matrix $\mathbf{H}(z)$ corresponding to an infinite-dimensional representation of a Borel subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$ called prefundamental, see [FH]. Just as in the $q = 1$ case, it is equal to the product of a scalar factor that depends only on the values of $\{t_k\}$ and $\{\lambda_k\}$ and an operator $\mathbf{Q}(z)$ acting on $\mathcal{H}_q$ whose eigenvalues are polynomials in z. The latter is is known as **Baxter's Q-operator**.

Moreover, there is an analogue of the second-order differential equation (3.7). It is a second-order **difference** equation

$$(D^2 - D\mathbf{T}(z) + 1)\mathbf{H}(z) = 0, \tag{5.29}$$

where D is a shift operator, $(D \cdot f)(z) := f(zq^2)$, i.e., $D := q^{2z\partial}$. If we write $\mathbf{T}(z) = 2 + h^2 z^2 S(z) + \cdots$, where $q = e^{\hbar}$, expand equation (5.29) in a power

[34] There is also an intermediate deformation called the XXX model, which preserves the classical $\mathfrak{sl}_2$ symmetry; its full symmetry algebra is the (doubled) Yangian $Y(\mathfrak{sl}_2)$, which is a degeneration of $U_q(\widehat{\mathfrak{sl}}_2)$. The material of this subsection applies mutatis mutandis to the XXX model.

series in $\hbar$, divide by $\hbar^2$ and set $\hbar = 0$ (i.e. $q = 1$), we obtain equation (3.7). In this sense, equation (5.29) is indeed a q-deformation of equation (3.7).

The second-order difference operator in the RHS of (5.29) is known as a q-**oper** for the group SL_2. The notion of a q-oper has been defined for an arbitrary simple Lie group in [FKSZ].

Relation (5.29) can be interpreted as a relation in the Grothendieck ring of a certain category of representations of a Borel subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$, see e.g. [FH]. It is often written in the form

$$\mathbf{T}(z) = \frac{\mathbf{H}(zq^2)}{\mathbf{H}(z)} + \frac{\mathbf{H}(zq^{-2})}{\mathbf{H}(z)}, \tag{5.30}$$

and it can be further rewritten as a q-difference equation on $\mathbf{Q}(z)$ known as the **Baxter TQ-relation**:

$$\mathbf{T}(z) = A(z)\frac{\mathbf{Q}(zq^2)}{\mathbf{Q}(z)} + D(z)\frac{\mathbf{Q}(zq^{-2})}{\mathbf{Q}(z)} \tag{5.31}$$

(here $A(z)$ and $D(z)$ are functions that only depend on the parameters $\{t_k\}$ and $\{\lambda_k\}$ of $\mathcal{H}_q$, see e.g. [FH], formula (1.1)). It has been conjectured by Baxter and proved in [FH] that the eigenvalues of $\mathbf{Q}(z)$ on $\mathcal{H}_q$ are polynomials whose roots satisfy a q-deformation of the Bethe Ansatz equations (5.7) (see [FH], Section 5.6).

At the level of the eigenvalues, we obtain that the Baxter TQ-relation (5.31) expressing the eigenvalues of $\mathbf{T}(z)$ on $\mathcal{H}_q$ is a q-deformation of the Miura transformation (5.9) expressing the eigenvalues of $S(z)$ on $\mathcal{H}$. This was one of the key observations of [FR] (Section 1.3 and Remark 6), see also [F1] (Section 6.8). (Moreover, both maps preserve natural Poisson structures [FR].)

An insight of the present paper is that Baxter's Q-operator $\mathbf{Q}(z)$ may be viewed as a q-deformation of an operator closely related to the Hecke operator H_z of the tamely ramified analytic Langlands correspondence for SL_2 on $\mathbb{P}^1$. There is a similar statement for an arbitrary simple algebraic group G. This opens the door to the investigation of q-deformation of the analytic Langlands correspondence (which we expect to exist in the ramified setting in genus 0 and 1).

5.13. *q*-deformation of the double Gaudin model

In conclusion, let us speculate what this q-deformation of the analytic Langlands correspondence should look like for $F = \mathbb{C}$ and $G = SL_2$ in the case of $X = \mathbb{P}^1$ with parabolic structures. As we explained in Subsection 5.11, for $q = 1$ this is the double Gaudin model.

In the undeformed situation (with twists) we place at the parabolic points the spherical unitary principal series representations of $SL_2(\mathbb{C})$, which are completions of spherical Harish-Chandra bimodules, on which we have an action of the Gaudin Hamiltonians G_i and their complex conjugates $\overline{G}_i$. Hence it is natural to form the generating series $S(z)$ given by formula (3.6) and its complex conjugate $\overline{S}(\overline{z})$.

As shown in Subsection 3.3 (following [EFK2, EFK3]), the Hecke operator H_z satisfies the following system of two second-order differential equations:

$$(\partial_z^2 - S(z))H_z = 0, \qquad (\overline{\partial}_z^2 - \overline{S}(\overline{z}))H_z = 0. \tag{5.32}$$

Such a solution exists if and only if the corresponding oper has real monodromy, and such opers are called real opers. Locally, the Hecke eigenvalue can be written in the form

$$\Psi_0(z)\overline{\Psi}_1(\overline{z}) + \Psi_1(z)\overline{\Psi}_0(\overline{z}), \tag{5.33}$$

where $\Psi_0(z)$ and $\Psi_1(\overline{z})$ are two linearly independent local solutions of equation (5.13) (see [EFK3]).

In light of the discussion of the previous subsection, we expect that the q-deformation of this picture for $0 < q < 1$ should look as follows. We also expect that a similar picture exists for $F = \mathbb{R}$, and also for elliptic curves.

1. The spherical unitary principal series representations of $SL_2(\mathbb{C})$ should be replaced by their q-analogs defined in [Pu]; they are completions of spherical Harish-Chandra $U_q(\mathfrak{sl}_2)$-bimodules, which are spaces of finite maps between Verma modules for $U_q(\mathfrak{sl}_2)$.

2. We should view these as bimodules over $U_q(\widehat{\mathfrak{sl}}_2)$ via the evaluation homomorphism. Then we can define the transfer-matrices $\mathbf{T}(z)$ and $\overline{\mathbf{T}}(\overline{z})$ which are the q-analogues of $S(z)$ and $\overline{S}(\overline{z})$.

3. The Hecke operator $\mathbf{H}_z$ should be given by a suitable q-deformation of the formulas of Subsection 2.16.4 satisfying the following universal q-oper equations:

$$(D^2 - D\mathbf{T}(z) + 1)\mathbf{H}_z = 0, \qquad (\overline{D}^2 - \overline{D}\overline{\mathbf{T}}(\overline{z}) + 1)\mathbf{H}_z = 0, \tag{5.34}$$

where $D := q^{2z\partial}$, $\overline{D} := \overline{q}^{2\overline{z\partial}}$.

The challenge is to make sense of the system (5.34) which at the outset is only well-defined if the eigenvalues of $\mathbf{H}_z$ extend to analytic functions in

z and $\overline{z}$ (viewed as independent variables). We may also consider the setting where $q = e^\hbar$ and $\hbar$ is a formal variable, in which this analytic subtlety disappears. Finally, we need to identify the property of the solutions of (5.34) that is a q-analogue of the existence of a single-valued C^∞ solution of the oper equations.

Answering this question should lead us to the notion of a **real q-oper**. We leave this interesting topic for a future paper.

Acknowledgements

It is our pleasure to dedicate this paper to Corrado De Concini, who has been an inspiration to us mathematically and personally for many decades.

The material of Section 4 arose from our discussions with Davide Gaiotto and Edward Witten in the Spring of 2021. We are very grateful to them for these discussions, which motivated us to write this section. We thank Dima Arinkin for providing a proof of Theorem 3.33. We also thank Nikita Nekrasov for valuable discussions, and Ilya Dumanskii for communication regarding the material of Subsection 2.16.

P. E.'s work was partially supported by the NSF grant DMS-2001318. The project has received funding from ERC under grant agreement No. 669655.

References

[ABV] J. Adams, D. Barbasch, and D. Vogan, The Langlands Classification and Irreducible Characters for Real Reductive Groups, Progress in Math. 104, Birkhäuser Boston, 1992. MR1162533

[Ari] D. Arinkin, private communication.

[Art] J. Arthur, The principle of functoriality, Bull. Am. Math. Soc. **40** (2002) 39–53. MR1943132

[BS] D. Baraglia and L. Schaposnik, Higgs bundles and (A, B, A)-branes, Commun. Math. Phys. **331** (2014) 1271–1300, arXiv:1305.4638. MR3248065

[BB] A. Beilinson and J. Bernstein, A proof of Jantzen conjectures, in: I.M. Gelfand Seminar, S. Gelfand, S. Gindikin (eds.), Adv. in Soviet Math. 16, Part 1, pp. 1–50, AMS, Providence, 1993. MR1237825

[BD1] A. Beilinson and V. Drinfeld, Quantization of Hitchin's integrable system and Hecke eigensheaves, https://math.uchicago.edu/drinfeld/langlands/QuantizationHitchin.pdf

[BD2] A. Beilinson and V. Drinfeld, Opers, arXiv:math/0501398.

[BGH] I. Biswas, O. Garcia-Prada, and J. Hurtubise, Pseudo-real principal G-bundles over a real curve, J. Lond. Math. Soc. **93** (2016) 47–64, arXiv:1502.00563. MR3455781

[BH] I. Biswas and Y. I. Holla, Harder-Narasimhan reduction of a principal bundle, Nagoya Math. J. **174** (2004) 201–223. MR2066109

[BK1] A. Braverman and D. Kazhdan, Some examples of Hecke algebras for two-dimensional local fields, Nagoya Math. J. **184** (2006) 57–84, arXiv:math/0510538. MR2285231

[BK2] A. Braverman and D. Kazhdan, Automorphic functions on moduli spaces of bundles on curves over local fields: a survey, to appear in Proceedings of ICM-2022, arXiv:2112.08139. MR4680270

[BKP1] A. Braverman, D. Kazhdan, and A. Polishchuk, Automorphic functions for nilpotent extensions of curves over finite fields, arXiv:2303.16259.

[BKP2] A. Braverman, D. Kazhdan, and A. Polishchuk, Hecke operators for curves over non-archimedean local fields and related finite rings, arXiv:2305.09595.

[BT] F. Bruhat and J. Tits, Groupes algébriques sur un corps local. Chapitre III. Compléments et applications á la cohomologie galoisienne, J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math. **34** (1987) 671–698. MR0927605

[Br] L. Breen, Notes on 1- and 2-gerbes, in: Towards Higher Categories, John C. Baez, J. Peter May (eds.), The IMA Volumes in Mathematics and its Applications, pp. 193–237, Springer, New York, 2010, arXiv:math/0611317. MR2664168

[Bou] N. Bourbaki, Groupes et Algébres de Lie, VII, VIII, Hermann, Paris, 1975. MR0453824

[Ca] W. Casselman, Introduction to the theory of induced representations of p-adic reductive groups, https://personal.math.ubc.ca/cass/research/pdf/p-adic-book.pdf

[D] J. Dixmier, C^*-algebras, North Holland, 2011. MR0458185

[DM] T. Deshpande and S. Mukhopadhyay, Crossed modular categories and the Verlinde formula for twisted conformal blocks, Camb. J. Math. **11** (2023) 159–297, arXiv:1909.10799. MR4600543

[DrS] V.G. Drinfeld and V.V. Sokolov, Equations of Korteweg-deVries type and simple Lie algebras, Sov. Math. Dokl. **23** (1981) 457–462. MR0615463

[EF] P. Etingof and E. Frenkel, in preparation.

[EFK1] P. Etingof, E. Frenkel, and D. Kazhdan, An analytic version of the Langlands correspondence for complex curves, in: Integrability, Quantization, and Geometry, Vol. II. Dedicated to the Memory of Boris Dubrovin, S. Novikov, et al. (eds.), Proc. Sympos. Pure Math. 103.2, pp. 137–202, Amer. Math. Soc., Providence, RI, 2021, arXiv:1908.09677. MR4285697

[EFK2] P. Etingof, E. Frenkel, and D. Kazhdan, Hecke operators and analytic Langlands correspondence for curves over local fields, Duke Math. J. **172** (2023) 2015–2071, arXiv:2103.01509. MR4627247

[EFK3] P. Etingof, E. Frenkel, and D. Kazhdan, Analytic Langlands correspondence for PGL_2 on $\mathbb{P}^1$ with parabolic structures over local fields, Geom. Funct. Anal. **32** (2022) 725–831, arXiv:2106.05243. MR4472583

[EO] P. Etingof and V. Ostrik, On semisimplification of tensor categories, in: Representation Theory and Algebraic Geometry, V. Baranovsky, et al. (eds.), Trends in Mathematics, Birkhäuser, 2022, arXiv:1801.04409. MR4486913

[Fa] G. Faltings, Real projective structures on Riemann surfaces, Compos. Math. **48** (1983) 223–269. MR0700005

[FF] B. Feigin and E. Frenkel, Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras, Int. J. Mod. Phys. A **7**, Suppl. 1A (1992) 197–215. MR1187549

[FFR] B. Feigin, E. Frenkel, and N. Reshetikhin, Gaudin model, Bethe Ansatz and critical level, Commun. Math. Phys. **166** (1994) 27–62, arXiv:hep-th/9402022. MR1309540

[F1] E. Frenkel, Affine algebras, Langlands duality and Bethe Ansatz, in: Proceedings of the International Congress of Mathematical Physics, Paris, 1994, D. Iagolnitzer (ed.), pp. 606–642, International Press, Cambridge, 1995, arXiv:q-alg/9506003. MR1370720

[F2] E. Frenkel, Gaudin model and opers, in: Infinite Dimensional Algebras and Quantum Integrable Systems, P. Kulish, et al.

(eds.), Progress in Math. 237, pp. 1–58, Birkhäuser, Basel, 2005, arXiv:math/0407524. MR2160841

[F3] E. Frenkel, Langlands Correspondence for Loop Groups, Cambridge University Press, 2007. MR2332156

[F4] E. Frenkel, Is there an analytic theory of automorphic functions for complex algebraic curves? SIGMA **16** (2020) 042, arXiv:1812.08160. MR4098670

[FG] E. Frenkel and B. Gross, A rigid irregular connection on the projective line, Ann. Math. **170** (2009) 1469–1512, arXiv:0901.2163. MR2600880

[FH] E. Frenkel and D. Hernandez, Baxter's relations and spectra of quantum integrable models, Duke Math. J. **164** (2015) 2407–2460, arXiv:1308.3444. MR3397389

[FKSZ] E. Frenkel, P. Koroteev, D.S. Sage, and A.M. Zeitlin, q-opers, QQ-systems, and Bethe Ansatz, J. Eur. Math. Soc. (2023), arXiv:2002.07344. MR4542420

[FR] E. Frenkel and N. Reshetikhin, Quantum affine algebras and deformations of the Virasoro and $\mathcal{W}$-algebras, Commun. Math. Phys. **178** (1996) 237–264, arXiv:q-alg/9505025. MR1387950

[GW] D. Gaiotto and E. Witten, Gauge theory and the analytic form of the geometric Langlands correspondence, Ann. Henri Poincaré (2022), arXiv:2107.01732. MR4699908

[GKM] D. Gallo, M. Kapovich, and A. Marden, The monodromy groups of Schwarzian equations on closed Riemann surfaces, Ann. Math. **151** (2000) 625–704. MR1765706

[HK] J. Hong, S. Kumar, Twisted conformal blocks and their dimension, arXiv:2207.09578.

[JLN] Saebyeok Jeong, Norton Lee, and Nikita Nekrasov, Parallel surface defects, Hecke operators, and quantum Hitchin system, arXiv:2304.04656.

[K] M. Kneser, Lectures on Galois Cohomology of Classical Groups, Tata Inst. of Fund. Res., 1969. MR0340440

[Ko] M. Kontsevich, Notes on motives in finite characteristic, in: Algebra, Arithmetic, and Geometry, Vol. II, In Honor of Yu.I. Manin, Y.

Tschinkel, Y. Zarhin (eds.), Progress in Math. 270, pp. 213–247, Birkhäuser Boston, 2009, arXiv:math/0702206.

[L] V. Lafforgue, Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale, J. Am. Math. Soc. **31** (2018) 719–891. MR3787407

[La] R.P. Langlands, On analytic form of geometric theory of automorphic forms (in Russian), Preprint http://publications.ias.edu/rpl/paper/2678

[LM] K. Lu and E. Mukhin, On the Gaudin model of type G_2, Commun. Contemp. Math. **21** (2019) 1850012. MR3947061

[Ma] R. P. Martin, Tensor products for $SL_2(\mathcal{K})$, Trans. Am. Math. Soc. **239** (1978) 197–211. MR0487045

[MTV1] E. Mukhin, V. Tarasov, and A. Varchenko, Spaces of quasi-exponentials and representations of $\mathfrak{gl}_N$, J. Phys. A **41** (2008) 194017. MR2452191

[MTV2] E. Mukhin, V. Tarasov, and A. Varchenko, Higher Lamé equations and critical points of master functions, Mosc. Math. J. **7** (2007) 533–542. MR2343147

[MV] E. Mukhin and A. Varchenko, Multiple orthogonal polynomials and a counterexample to the Gaudin Bethe Ansatz conjecture, Trans. Am. Math. Soc. **359** (2007) 5383–5418. MR2327035

[Na] M. A. Naimark, Decomposition of a tensorial product of irreducible representations of the proper Lorentz group into irreducible representations, Tr. Mosk. Mat. Obŝ. **8** (1959) 121–153. MR0114139

[NRS] N. Nekrasov, A. Rosly, and S. Shatashvili, Darboux coordinates, Yang-Yang functional, and gauge theory, Nucl. Phys. B **216** (2011) 69–93, arXiv:1103.3919. MR2851597

[P] L. Pukánszky, On the Kronecker products of irreducible representations of the 2×2 real unimodular group. I, Trans. Am. Math. Soc. **100** (1961) 116–152. MR0172962

[Pu] W. Pusz, Irreducible unitary representations of quantum Lorentz group, Commun. Math. Phys. **152** (1993) 591–626. MR1213303

[Re] J. Repka, Tensor products of unitary representations of $SL_2(\mathbb{R})$, Am. J. Math. **100** (1978) 747–774. MR0509073

[RV] N. Reshetikhin and A. Varchenko, Quasiclassical asymptotics of solutions to the KZ equations, in: Geometry, Topology, and Physics, pp. 293–322, International Press, Cambridge, MA, 1995. MR1358621

[R] L. Rybnikov, A proof of the Gaudin Bethe Ansatz conjecture, Int. Math. Res. Not., **22** (2020) 8766–8785, arXiv:1608.04625. MR4216703

[SS] J. Saxl and G. Seitz, Subgroups of algebraic groups containing regular unipotent elements, J. Lond. Math. Soc. **55** (1997) 370–386. MR1438641

[SV] I.B. Scherbak and A.N. Varchenko, Critical points of functions, sl_2 representations, and Fuchsian differential equations with only univalued solutions, Mosc. Math. J. **3** (2003) 621–645. MR2025276

[S] J.-P. Serre, Galois Cohomology, Springer Monographs in Mathematics, Springer Berlin, Heidelberg, 2001. (Translation of Cohomologie Galoisienne, Lecture Notes in Math. 5, Springer Berlin, Heidelberg, 1964.) MR0180551

[Te] J. Teschner, Quantisation conditions of the quantum Hitchin system and the real geometric Langlands correspondence, in: Geometry and Physics, Vol. I. In Honour of Nigel Hitchin, A. Dancer, et al. (eds.), pp. 347–375, Oxford University Press, 2018, arXiv:1707.07873. MR3932266

[T] N.Q. Thắng, On Galois cohomology of semisimple groups over local and global fields of positive characteristic, I, Math. Z. **259** (2008) 457–470. MR2390091

[V] A. Varchenko, Critical points of the product of powers of linear functions and families of bases of singular vectors, Compos. Math. **97** (1995) 385–401. MR1353281

[Wa] Y. Wakabayashi, Opers with real monodromy and the Eichler-Shimura isomorphism, arXiv:2309.12203.

[Wi] F. Williams, Tensor Products of Principal Series Representations, Lecture Notes in Math. 358, Springer Berlin, Heidelberg, 1973. MR0349903

[W] E. Witten, Letter to the authors, March 2021.

[Z] H. Zelaci, On very stability of principal G-bundles, Geom. Dedic. **204** (2020) 165–173. MR4056697

Pavel Etingof
Department of Mathematics
MIT
Cambridge, MA 02139
USA
E-mail: etingof@math.mit.edu

Edward Frenkel
Department of Mathematics
University of California
Berkeley, CA 94720
USA
E-mail: frenkel@math.berkeley.edu

David Kazhdan
Einstein Institute of Mathematics, Edmond J. Safra Campus, Givaat Ram
The Hebrew University of Jerusalem
Jerusalem, 91904
Israel
E-mail: kazhdan@math.huji.ac.il