Pure and Applied Mathematics Quarterly
Volume 20, Number 1, 471–523, 2024

Classification of multiplicity free quasi-Hamiltonian manifolds

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Abstract: A quasi-Hamiltonian manifold is called multiplicity free if all of its symplectic reductions are 0-dimensional. In this paper, we classify compact, multiplicity free, twisted quasi-Hamiltonian manifolds for simply connected, compact Lie groups. Thereby, we recover old and find new examples of these structures.

Keywords: Multiplicity free, quasi-Hamiltonian manifolds.

Received June 8, 2022.

Introduction

Consider a compact, connected Lie group K. Within the class of Hamiltonian K-manifolds, the most basic ones are known as *multiplicity-free* manifolds. These objects can be characterized in several ways, such as the fact that their symplectic reductions are zero-dimensional or that their generic orbits are coisotropic. In the context of classical mechanics, multiplicity-free manifolds are intimately connected with completely integrable systems.

In the 1980s, Delzant launched a program to classify compact, multiplicity-free Hamiltonian manifolds. When K is a torus that acts effectively, he proved in [Del88] that such a manifold M is uniquely determined by its momentum image $\mathcal{P}_M$. Furthermore, Delzant was able to provide a characterization of the sets that take the form of $\mathcal{P}_M$: they are precisely the *simple polytopes*, i.e., polytopes which satisfy a technical integrality condition. Delzant extended his investigation to non-abelian groups of rank two in [Del90], which led him to conjecture that, in general, M is uniquely determined by its momentum image $\mathcal{P}_M$, i.e., the image of the invariant momentum map (see Section 2.3) and a certain lattice Λ_M that encodes the principal isotropy group.

Delzant's conjecture was subsequently confirmed in [Kno11] with an important step due to Losev [Los09b]. Furthermore, the pairs $(\mathcal{P}_M, \Lambda_M)$ arising this way were characterized in terms of smooth affine, spherical varieties. This completed Delzant's program.

Meanwhile, Alekseev, Malkin, and Meinrenken sought a way to describe Hamiltonian actions for the *loop group* of K, leading to the development of the notion of a *quasi-Hamiltonian manifold* [AMM98]. These finite-dimensional K-manifolds are equipped with a momentum map that takes values in K instead of the dual Lie algebra $\mathfrak{k}^*$.

In this paper, we extend the results in [Kno11] to the quasi-Hamiltonian case, by classifying compact, multiplicity-free quasi-Hamiltonian manifolds under the condition that K is simply connected. We also consider the case where the momentum map is twisted by an automorphism of K. Our result is that a compact multiplicity free quasi-Hamiltonian manifold M is classified by a pair $(\mathcal{P}, \Lambda)$, which is compact and *spherical* with respect to an affine root system (see Definition 2.5.2).

To prove this, we follow the approach in [Kno11] and first study quasi-Hamiltonian manifolds locally over $\mathcal{P}_M$. More precisely, let $(\mathcal{P}, \Lambda)$ be a fixed spherical pair. Then we show that the category of multiplicity free quasi-Hamiltonian manifolds M with $\mathcal{P}_M \subseteq \mathcal{P}$ open and $\Lambda_M = \Lambda$ forms a *gerbe*.

Crucial use is made of a reduction procedure to ordinary Hamiltonian manifolds due to Alekseev-Malkin-Meinrenken [AMM98] and Meinrenken [Mei17].

Since all automorphism groups are abelian, they form a sheaf of abelian groups over $\mathcal{P}$, called a *band*. To identify the band, we use the computations in the Hamiltonian case [Kno11].

A central point is to show that the higher cohomology of this band vanishes whenever $\mathcal{P}$ is convex. This cohomology computation is more involved than in the Hamiltonian case due to complications with affine root systems. General properties of gerbes then imply that there exists precisely one M with $(\mathcal{P}_M, \Lambda_M) = (\mathcal{P}, \Lambda)$.

We end the paper by constructing examples of compact multiplicity free quasi-Hamiltonian manifolds using our classification. We recover some old examples, such as the *double of a group* by Alekseev-Malkin-Meinrenken [AMM98], the *spinning* 4*-sphere* by Alekseev-Meinrenken-Woodward [AMW02], its generalization, the *spinning* $2n$*-sphere* by Hurtubise-Jeffreys-Sjamaar [HJS06], and the *quaternionic projective space* due to Eshmatov [Esh09]. We end the paper, by constructing some multiplicity free quasi-Hamiltonian manifolds which have not yet appeared in the literature. For example we list all multiplicity free quasi-Hamiltonian SU(2)-manifolds (twisted and untwisted) and Eshmatov's example is extended to all quaternionic Grassmannians. Most remarkable are multiplicity free quasi-Hamiltonian manifolds for which the momentum map is surjective. These exist for example for $K = \mathrm{SU}(n)$ and $K = \mathrm{Sp}(2n)$.

This paper replaces the preprint [Kno16] from 2016, which is now obsolete. Our classification theorem has already been successfully used by Paulus in his thesis [Pau18] to classify other interesting subclasses of multiplicity free quasi-Hamiltonian manifolds like those of with one-dimensional momentum polytope, see [Pau17, KP19]. Also the list of manifolds with surjective momentum map has been completed, see [PVS22].

Notation *a)* In the entire paper, K will be a compact connected Lie group. Whenever quasi-Hamiltonian manifolds are involved, K will additionally be assumed to be simply connected and equipped with an automorphism $k \mapsto {}^{\tau}k$ of K (a "twist") and a scalar product on its Lie algebra $\mathfrak{k}$ which is invariant for both K and τ.

b) We adopt the following conventions: a *polytope* is the convex hull of a finite subset of a finite dimensional real affine space while a *polyhedron* is cut out by finitely many affine linear inequalities $\alpha_1 \geq 0, \ldots, \alpha_n \geq 0$. It is well known, that polytopes are precisely the bounded polyhedra. A *polyhedral* *cone* is a subset of a real vector space which is cut out by linear inequalities.

c) As usual, we define the *fiber product* $X \times_Z Y$ with respect to two maps $\varphi : X \to Z$ and $\psi : Y \to Z$ as the set of $(x, y) \in X \times Y$ with $\varphi(x) = \psi(y)$. The fiber product is easily confused with the notation $K \times^L Y$ for an *associated fiber bundle*. The latter denotes by definition the orbit space $(K \times Y)/L$ where K is a group, L is a subgroup, Y is a set with an L-action, and L acts on $K \times Y$ via $l \cdot (k, y) := (kl^{-1}, ly)$.

Part 1
Local root systems and cohomology

The principal purpose of this part is to state and prove the Vanishing Theorem 1.4.1 which forms the central technical step in the proof of the Classification Theorem 2.6.1. In fact, the material of Sections 1.2, 1.3, and 1.4 is only used in the proof of Theorem 2.6.1.

1.1. Affine root systems

In this section we set up notation for (affine) root systems. In Section 2.2 they will be used to describe the geometry of the (twisted) conjugation action of a compact connected Lie group. They also control the automorphisms of multiplicity free quasi-Hamiltonian manifolds (see Theorem 2.6.4). In the latter case, the root systems are more complicated in that they don't have to be irreducible and that both finite and affine root systems may occur as irreducible components. Note also that our root systems carry a fixed metric as part of the structure. Thus the approach is very similar to (and inspired by) the treatises [Mac72, Mac03] of Macdonald.

Let $\overline{\mathfrak{a}}$ be an Euclidean vector space, i.e., a finite dimensional $\mathbb{R}$-vector space equipped with a positive definite scalar product $\langle \cdot, \cdot \rangle$ and let $\mathfrak{a}$ be an affine space for $\overline{\mathfrak{a}}$, i.e., a non-empty set equipped with a free and transitive $\overline{\mathfrak{a}}$-action

$$\mathfrak{a} \times \overline{\mathfrak{a}} \to \mathfrak{a} : (x, t) \mapsto x + t. \tag{1.1.1}$$

The set of affine linear functions on $\mathfrak{a}$ is denoted by $\mathrm{L}(\mathfrak{a})$. Since $\overline{\mathfrak{a}}$ carries a metric every $\alpha \in \mathrm{L}(\mathfrak{a})$ has a gradient $\overline{\alpha} := \nabla \alpha \in \overline{\mathfrak{a}}$. It is characterized by the equation

$$\alpha(x + t) = \alpha(x) + \langle \overline{\alpha}, t \rangle, \quad x \in \mathfrak{a}, t \in \overline{\mathfrak{a}}. \tag{1.1.2}$$

This way $\mathrm{L}(\mathfrak{a})$ is an extension

$$0 \to \mathbb{R} \to \mathrm{L}(\mathfrak{a}) \xrightarrow{\nabla} \overline{\mathfrak{a}} \to 0. \tag{1.1.3}$$

Every non-constant $\alpha \in \mathrm{L}(\mathfrak{a})$ defines an affine hyperplane $H_\alpha := \{\alpha = 0\}$ with normal vector $\overline{\alpha}$.

Similarly, let $M(\mathfrak{a})$ be the group of isometries of $\mathfrak{a}$. For every $w \in M(\mathfrak{a})$ let $\overline{w} \in \mathrm{O}(\overline{\mathfrak{a}})$ be its linear part. It is characterized by the equation

$$w(x+t) = w(x) + \overline{w}(t), \quad x \in \mathfrak{a}, t \in \overline{\mathfrak{a}}. \tag{1.1.4}$$

This way, we obtain a short exact sequence

$$0 \to \overline{\mathfrak{a}} \to M(\mathfrak{a}) \to \mathrm{O}(\overline{\mathfrak{a}}) \to 1. \tag{1.1.5}$$

For a subgroup W of $M(\mathfrak{a})$ let $\overline{W}$ be its image in $\mathrm{O}(\overline{\mathfrak{a}})$.

A *reflection* is an isometry $s \in M(\mathfrak{a})$ whose fixed point set $\mathfrak{a}^s$ is an affine hyperplane. Conversely, if $\alpha \in \mathrm{L}(\mathfrak{a})$ is non-constant then there is a unique reflection s_α with $\mathfrak{a}^{s_\alpha} = H_\alpha$ given by

$$s_\alpha(x) = x - \alpha(x)\,\overline{\alpha}^\vee, \quad x \in \mathfrak{a} \quad \text{with} \quad \overline{\alpha}^\vee := \frac{2}{\|\overline{\alpha}\|^2}\,\overline{\alpha} \in \overline{\mathfrak{a}}. \tag{1.1.6}$$

The induced action on $\mathrm{L}(\mathfrak{a})$ is given by

$$s_\alpha(\beta) = \beta - \langle \overline{\beta}, \overline{\alpha}^\vee \rangle \alpha, \quad \beta \in \mathrm{L}(\mathfrak{a}). \tag{1.1.7}$$

After these preliminaries, affine root systems are defined as follows:

1.1.1 Definition. A set $\Phi \subset \mathrm{L}(\mathfrak{a}) \setminus \mathbb{R}$ of non-constant affine linear functions is an *affine root system* if it has the following properties:

a) $\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$.
b) $\langle \overline{\alpha}, \overline{\beta}^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.
c) $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.
d) $\overline{\Phi} := \{\overline{\alpha} \in \overline{\mathfrak{a}} \mid \alpha \in \Phi\}$ is finite.

The *Weyl group of* Φ is the subgroup $W_\Phi \subseteq M(\mathfrak{a})$ generated by all reflections s_α, $\alpha \in \Phi$.

This definition differs slightly from Macdonald's in two respects: First, condition *a)* says that we consider only reduced root systems. Secondly, we do not assume $\mathrm{L}(\mathfrak{a})$ to be spanned by Φ or that Φ is infinite. In fact, $\Phi = \varnothing$ is also a root system.

If Φ is an affine root system then for any $x \in \mathfrak{a}$

$$\Phi_x := \{\alpha \in \Phi \mid \alpha(x) = 0\} \tag{1.1.8}$$

is a finite root system. Its Weyl group is the isotropy group of x in W_Φ (see e.g. [Bou68, V, §3, Prop. 1 (vii)]).

It is well known that every root system has a unique orthogonal decomposition

$$\mathfrak{a} = \mathfrak{a}_0 \times \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_n \quad \text{and} \quad \Phi = \Phi_1 \cup \cdots \cup \Phi_n, \tag{1.1.9}$$

such that the Weyl group $W_\Phi = W_{\Phi_1} \times \cdots \times W_{\Phi_n}$ acts trivially on $\mathfrak{a}_0$ and $\Phi_i \subset \mathrm{L}(\mathfrak{a}_\nu)$ is irreducible for $\nu \geq 1$. Each pair $(\mathfrak{a}_\nu, \Phi_\nu)$ with $\nu \geq 1$ corresponds either to a finite, to an affine, or to a twisted affine Dynkin diagram (see, e.g., [Kac90, Ch. 4]).

A *chamber of* Φ *(or* W_Φ*)* is a connected component of $\mathfrak{a} \setminus \bigcup_{\alpha\in\Phi} H_\alpha$. Its closure is called an *alcove*. It is known, that W_Φ acts simply transitively on the set of alcoves and that each alcove $\mathcal{A}$ is a fundamental domain for W_Φ. The reflections about the walls of $\mathcal{A}$ are called *simple* (with respect to $\mathcal{A}$). They generate W_Φ as a group.

If the factor Φ_ν, $\nu \geq 1$, is finite then its alcoves are simplicial cones. Otherwise, they are simplices. So in general, an alcove is a product of an affine space, a simplicial cone and a finite number of simplices.

The set $\overline{\Phi} \subseteq \overline{\mathfrak{a}}$ of gradients of affine roots is a finite, but possibly non-reduced root system (e.g., if the root system is of type $\mathsf{A}_{2n}^{(2)}$). Its Weyl group $W_{\overline{\Phi}}$ is the image $\overline{W}_\Phi$ of W_Φ in $\mathrm{O}(\overline{\mathfrak{a}})$.

If $\Lambda \subseteq \overline{\mathfrak{a}}$ is a lattice we will denote its dual lattice $\{\chi \in \overline{\mathfrak{a}} \mid \langle \Lambda, \chi\rangle \subseteq \mathbb{Z}\}$ by $\Lambda^\vee$.

1.1.2 Definition. Let $\Phi \subset \mathrm{L}(\mathfrak{a})$ be an affine root system. A *weight lattice for* Φ is a lattice $\Lambda \subseteq \overline{\mathfrak{a}}$ with

$$\overline{\Phi} \subset \Lambda \text{ and } \overline{\Phi}^\vee \subset \Lambda^\vee. \tag{1.1.10}$$

The pair (Φ, Λ) is called an *integral affine root system.*

Let (Φ, Λ) be an integral affine root system on $\mathfrak{a}$. Then its Weyl group will also act on the compact torus $A := \overline{\mathfrak{a}}/\Lambda^\vee$. The character group $\Xi(A)$ can be identified with Λ. More specifically, to $\chi \in \Lambda$ we attach the character

$$\widetilde{\chi}(a + \Lambda^\vee) := e^{2\pi i\,\langle\chi,a\rangle}. \tag{1.1.11}$$

For ease of notation, we are going to set $\widetilde{\alpha} := \widetilde{\overline{\alpha}}$ for $\alpha \in \Phi$. Dually, every $\eta \in \Lambda^\vee$ defines the cocharacter

$$\widetilde{\eta} : U(1) \to A : e^{2\pi i t} \mapsto t\eta + \Lambda^\vee. \tag{1.1.12}$$

Again, for $\alpha \in \Phi$ we write $\widetilde{\alpha}^\vee := \widetilde{\overline{\alpha}^\vee}$. Then we have the formula

$$\widetilde{\chi}(\widetilde{\alpha}^\vee(u)) = u^{\langle \chi, \overline{\alpha}^\vee \rangle} \text{ for all } \chi \in \Lambda, \alpha \in \Phi, u \in U(1). \tag{1.1.13}$$

Note that this implies in particular

$$\widetilde{\alpha}(\widetilde{\alpha}^\vee(u)) = u^2 \quad \text{for all } \alpha \in \Phi, u \in U(1). \tag{1.1.14}$$

The action of a reflection $s_\alpha \in W_\Phi$ is given by the formula

$$s_\alpha(a) = a \cdot \widetilde{\alpha}^\vee(\widetilde{\alpha}(a))^{-1} \quad \text{for all } a \in A. \tag{1.1.15}$$

1.2. Local root systems

In this section we introduce a localized version of a root system.

1.2.1 Definition. Let $\mathcal{P}$ be a subset of the affine space $\mathfrak{a}$. A *local root system on* $\mathcal{P}$ is a family $\Phi(*) = (\Phi(x))_{x \in \mathcal{P}}$ and a lattice $\Lambda \subseteq \overline{\mathfrak{a}}$ with:

a) For each $x \in \mathcal{P}$, the pair $(\Phi(x), \Lambda)$ is an integral affine root system on $\mathfrak{a}$.
b) Every $x \in \mathcal{P}$ has a neighborhood $U \subseteq \mathcal{P}$ such that $\Phi(y) = \Phi(x)_y$ for all $y \in U$.
c) Every $\alpha \in \Phi(x)$, $x \in \mathcal{P}$, has $\alpha|_\mathcal{P} \geq 0$ or $\alpha|_\mathcal{P} \leq 0$.

Observe that *b)* applied to $x = y$ means $\alpha(x) = 0$ for all $\alpha \in \Phi(x)$. Hence each of the root systems $\Phi(x)$ is finite.

If (Φ, Λ) is an integral affine root system and $\mathcal{P}$ is a subset of an alcove then the pair $((\Phi_x)_{x \in \mathcal{P}}, \Lambda)$ is a local root system. Systems of this type are called *trivial*.

1.2.2 Definition. A non-empty subset $\mathcal{P} \subseteq \mathfrak{a}$ is called *solid* if its *interior* $\mathcal{P}^0$ (i.e., the largest open subset contained in $\mathcal{P}$) is dense in $\mathcal{P}$. Observe that if $\mathcal{P}$ is convex then $\mathcal{P}$ is solid if and only if $\dim \mathcal{P} = \dim \mathfrak{a}$ if and only if $\mathcal{P}$ spans $\mathfrak{a}$ as an affine space.

The goal of this section is to prove the following triviality criterion.

1.2.3 Proposition. *Let* $(\Phi(*), \Lambda)$ *be a local root system on* $\mathcal{P} \subseteq \mathfrak{a}$ *and let* $W \subseteq M(\mathfrak{a})$ *be the subgroup generated by all local Weyl groups* $W(x)$ *of* $\Phi(x)$ *with* $x \in \mathcal{P}$. *Assume:*

a) The subset $\mathcal{P}$ is convex and solid.
b) Every W-orbit meets $\mathcal{P}$ in at most one point.

Then the local root system is trivial.

1.2.4 Remark. In the application later, $\mathfrak{a}$ will be a subspace of an affine space $\mathfrak{t}$ which carries an affine reflection group $\widetilde{W}$ such that each element of $W(x)$ is induced by an element of $\widetilde{W}$ and such that $\mathcal{P}$ lies in an alcove of $\widetilde{W}$. In this setting, condition *b)* in Proposition 1.2.3 holds since it already holds for $\widetilde{W}$-orbits in $\mathfrak{a}$.

The proof will proceed in two steps. First we consider the corresponding system of Weyl groups $(W(x))_{x\in\mathcal{P}}$ and prove that it is trivial under similar assumptions. From that we deduce that the local root system itself is trivial.

The reflection group analogue for Definition 1.2.1 is:

1.2.5 Definition. A *local reflection group on a subset $\mathcal{P} \subseteq \mathfrak{a}$* is a family $W(*) = (W(x))_{x\in\mathcal{P}}$ with the following properties:

a) $W(x)$ is a reflection group on $\mathfrak{a}$ for each $x \in \mathcal{P}$.
b) Every $x \in \mathcal{P}$ has a neighborhood $U \subseteq \mathcal{P}$ such that $W(y) = W(x)_y$ for all $y \in U$ where $W(x)_y$ is the isotropy group of y inside $W(x)$.
c) For every reflection $s \in W(x)$, $x \in \mathcal{P}$, the set $\mathcal{P}$ lies entirely in one of the two closed halfspaces determined by the reflection hyperplane $\mathfrak{a}^s$.

Again condition *b)* implies that x is a fixed point of $W(x)$ and therefore that $W(x)$ is a finite reflection group.

If $\Phi(*)$ is a local root system then its system of Weyl groups $(W(x))_{x\in\mathcal{P}}$ forms a local reflection group. Moreover, if W is an affine reflection group on $\mathfrak{a}$ and $\mathcal{P}$ a subset of an alcove then the family of isotropy groups $(W_x)_{x\in\mathcal{P}}$ is a local reflection group on $\mathcal{P}$. These local reflection groups will be called *trivial*.

The main tool for showing triviality is the following classical criterion for a given set of reflections to be the set of simple reflections of an affine reflection group.

1.2.6 Lemma. *Let $\alpha_1, \ldots, \alpha_n \in \mathrm{L}(\mathfrak{a})$ be non-constant affine linear functions with:*

a) For any $i \neq j$, the angle between $\overline{\alpha}_i$ and $\overline{\alpha}_j$ equals $\pi - \frac{\pi}{\ell}$ with $\ell \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$.
b) There is a point $x \in \mathfrak{a}$ with $\alpha_i(x) > 0$ for all $i = 1, \ldots, n$.

Let $W \subseteq M(\mathfrak{a})$ be the group generated by the reflections $s_{\alpha_1}, \ldots, s_{\alpha_n}$. Then W is an affine reflection group,

$$\mathcal{A} := \{x \in \mathfrak{a} \mid \alpha_1(x) \geq 0, \ldots, \alpha_n(x) \geq 0\} \tag{1.2.1}$$

is an alcove for W, and the reflections $s_{\alpha_1}, \ldots, s_{\alpha_n}$ are the simple reflection of W with respect to $\mathcal{A}$.

Proof. Condition *b)* implies that $\mathcal{A}$ is a solid polyhedron. Let, after renumbering, $\alpha_1, \ldots, \alpha_m$ be the non-redundant functions defining $\mathcal{A}$, i.e., those whose intersection $\{\alpha_i = 0\} \cap \mathcal{A}$ is of codimension 1 in $\mathcal{A}$. Then a classical theorem (see e.g. Vinberg [Vin71, Thm. 1] for a much more general statement) asserts that, under condition *a)*, $s_{\alpha_1}, \ldots, s_{\alpha_m}$ are the simple reflections for an affine reflection group W and that $\mathcal{A}$ is a fundamental domain. So, it remains to show that $m = n$. Suppose not. Then α_{m+1} would be redundant. This implies that there are real numbers $c_1, \ldots, c_m \geq 0$ such that $\alpha_{m+1} = \sum_{i=1}^m c_i\alpha_i$. From *a)* we get that

$$\langle \overline{\alpha}_i, \overline{\alpha}_{m+1} \rangle = \|\overline{\alpha}_i\| \|\overline{\alpha}_{m+1}\| \cdot \cos\left(\pi - \frac{\pi}{\ell}\right) \leq 0 \tag{1.2.2}$$

for $i = 1, \ldots, m$ and therefore the contradiction $\langle \overline{\alpha}_{m+1}, \overline{\alpha}_{m+1} \rangle \leq 0$. $\square$

The triviality criterion for local reflection groups is:

1.2.7 Lemma. *Let $(W(x))_{x\in\mathcal{P}}$ be a local reflection group on $\mathcal{P} \subseteq \mathfrak{a}$. Let W be the group generated by all $W(x)$, $x \in \mathcal{P}$. Assume:*

a) $\mathcal{P}$ is convex and solid.
b) Every W-orbit in $\mathfrak{a}$ meets $\mathcal{P}$ in at most one point.

Then W is an affine reflection group with $W(x) = W_x$ for all $x \in \mathcal{P}$.

Proof. Let me first remark that it is important to keep in mind that the various reflection hyperplanes might not meet within $\mathcal{P}$. Typical is the situation of figure (2.7.15) where the shaded area is $\mathcal{P}$ and the local Weyl group at each vertex is generated by the reflection about the dashed lines through the vertex.

In a first step we claim that

$$W(x)_y = W(y)_x \quad \text{for all } x, y \in \mathcal{P}. \tag{1.2.3}$$

Indeed, let $l = [x, y] \subseteq \mathfrak{a}$ be the line segment joining x and y. Then $l \subseteq \mathcal{P}$ since $\mathcal{P}$ is convex. For any $z \in l$ let

$$W(z)_l := \{w \in W(z) \mid wu = u \text{ for all } u \in l\} \tag{1.2.4}$$

Then

$$W(u)_l = (W(z)_u)_l = W(z)_l \tag{1.2.5}$$

for all $u \in l$ which are sufficiently close to z. This means that the map $z \mapsto W(z)_l$ is locally constant, hence constant, on l. Thus

$$W(x)_y = W(x)_l = W(y)_l = W(y)_x. \tag{1.2.6}$$

Let $s = s_\alpha \in W$ be a reflection with fixed point set $H := \{\alpha = 0\}$, $i = 1, 2$. We claim that H does not meet $\mathcal{P}^0$, the interior of $\mathcal{P}$ inside $\mathfrak{a}$. Otherwise, there would be points $x, y \in \mathcal{P}^0$ with $\alpha(x) > 0$ and $\alpha(y) < 0$. The line segment joining x and y lies entirely in $\mathcal{P}^0$ and meets H in exactly one point z. Moreover there is an $\varepsilon > 0$ such that both points $z_\pm := z \pm \varepsilon\overline{\alpha}$ are in $\mathcal{P}^0$. But then z_+ and $z_- = s(z_+)$ would be two different points of $\mathcal{P}$ lying in the same W-orbit contradicting our assumption.

The claim implies that $\mathcal{P}^0$, being connected, lies entirely in one of the open halfspaces determined by H. Hence $\mathcal{P}$ lies entirely in one of the two closed halfspaces determined by H.

This reasoning applies, in particular, to all reflections contained in $W(x)$, where $x \in \mathcal{P}$. Thus, $\mathcal{P}$ is contained in a unique closed Weyl chamber $C(x) \subseteq \mathfrak{a}$ for $W(x)$. This chamber determines in turn a set $\Sigma(x) \subset W(x)$ of simple reflections. It is well-known that for every $y \in C(x)$ the set $\Sigma(x)_y := \{s \in \Sigma(x) \mid sy = y\}$ is a set of simple reflections for $W(x)_y$. Therefore equation (1.2.3) implies that

$$\Sigma(x)_y = \Sigma(y)_x \quad \text{for all } x, y \in \mathcal{P}. \tag{1.2.7}$$

Now let Σ be the union of all $\Sigma(x)$, $x \in \mathcal{P}$. Then

$$\Sigma(x) = \{s \in \Sigma \mid sx = x\} \tag{1.2.8}$$

for all $x \in \mathcal{P}$. Indeed, let $s \in \Sigma$ with $sx = x$. Then $s \in \Sigma(y)$ for some $y \in \mathcal{P}$. Thus, $s \in \Sigma(y)_x = \Sigma(x)_y \subseteq \Sigma(x)$.

For each $s \in \Sigma$ choose affine linear functions α_s with $s = s_{\alpha_s}$ and such that $\alpha_s \geq 0$ on $\mathcal{P}$. We are going to show that $\{\alpha_s \mid s \in \Sigma\}$ satisfies the assumptions of Lemma 1.2.6.

Let $s_1 \neq s_2 \in \Sigma$. Put $\alpha_i := \alpha_{s_i}$ and $H_i := \{\alpha_i = 0\}$. Assume first that H_1 and H_2 are parallel. Then $\overline{\alpha}_1 = c\overline{\alpha}_2$ with $c \neq 0$ and we have to show that $c < 0$. The functions α_i vanish, by construction, at some points $x_i \in \mathcal{P}$. Put $t := x_1 - x_2 \in \overline{\mathfrak{a}}$. Then $\langle \overline{\alpha}_1, t\rangle = -\alpha_1(x_2) < 0$ and $\langle \overline{\alpha}_2, t\rangle = \alpha_2(x_1) > 0$ which shows $c < 0$.

Now assume that H_1 and H_2 are not parallel. Then $E := H_1 \cap H_2$ is a subspace of codimension two. Let $W' \subseteq W$ be the dihedral group generated by s_1 and s_2 and let θ be the angle between $\overline{\alpha}_1$ and $\overline{\alpha}_2$. Then W' contains the rotation r around E with angle 2θ. If r had infinite order then the union of all $\langle r\rangle$-translates of, say, H_1 would be dense in $\mathfrak{a}$. Since $\mathcal{P}$ is solid that contradicts the assumption that every W-orbit meets $\mathcal{P}$ at most once. Therefore W' is a finite reflection group.

Now we claim that $\{s_1, s_2\}$ is a set of simple reflections for W'. If $E \cap \mathcal{P} \neq \varnothing$ this is clear since $s_1, s_2 \in \Sigma(x)$ for all $x \in E \cap \mathcal{P}$ (by eqn. (1.2.8)). So assume $E \cap \mathcal{P} = \varnothing$. Let C' be the unique Weyl chamber of W' which contains $\mathcal{P}$ and let $s_i' \in W'$, $i = 1, 2$, be the corresponding simple reflections. Choose functions α_i' with $s_i' = s_{\alpha_i'}$ such that $\alpha_i' \geq 0$ on $\mathcal{P}$. Observe that

$$E = \{\alpha_1 = \alpha_2 = 0\} = \{\alpha_1' = \alpha_2' = 0\} = \mathfrak{a}^{W'}. \tag{1.2.9}$$

Now fix $i \in \{1, 2\}$. Then $\alpha_i = c_1\alpha_1' + c_2\alpha_2'$ for some real numbers $c_1, c_2 \geq 0$. Suppose $c_1, c_2 > 0$, i.e., s_i is not simple. By construction $s_i \in W(x)$ for some $x \in \mathcal{P}$. Then

$$0 = \alpha_i(x) = c_1\alpha_1'(x) + c_2\alpha_2'(x) \tag{1.2.10}$$

implies $\alpha_1'(x) = \alpha_2'(x) = 0$ and therefore $x \in \mathcal{P} \cap E$ which is excluded.

The fact that s_1 and s_2 are simple reflections of W' implies that the angle between $\overline{\alpha}_1$ and $\overline{\alpha}_2$ is of the form $\pi - \frac{\pi}{\ell}$ with $\ell \in \mathbb{Z}_{\geq 2}$. Since condition *b)* of Lemma 1.2.6 is obvious from $\mathcal{P}^0 \subseteq \mathcal{A}$ and the fact that $\mathcal{P}$ is solid we can apply Lemma 1.2.6 and infer that W is an affine reflection group with alcove $\mathcal{A}$ containing $\mathcal{P}$ and that Σ is a set of simple reflections of W. Finally, (1.2.8) implies

$$W_x = \langle s \in \Sigma \mid sx = x\rangle = \langle \Sigma(x)\rangle = W(x) \tag{1.2.11}$$

for all $x \in \mathcal{P}$. $\square$

For the second step of the proof of Proposition 1.2.3 we analyze to what extent a root system Φ is determined by its Weyl group W and a weight lattice Λ.

Choose an alcove $\mathcal{A}$ of W and let $\Sigma \subseteq W$ be the set of simple reflections with respect to $\mathcal{A}$. For every $s \in \Sigma$ there is a unique affine linear function $\pi_s \in \mathrm{L}(\mathfrak{a})$ such that $\{\pi_s = 0\} = \mathfrak{a}^s$, $\pi_s|_{\mathcal{A}} \geq 0$, and $\overline{\pi}_s \in \Lambda$ is primitive.

Let (Φ, Λ) be an integral affine root system with Weyl group W and let $\alpha_s \in \Phi$ be the simple root corresponding to s. Then $\alpha_s = n\pi_s$ with $n \in \mathbb{Z}_{>0}$. Since $\overline{\alpha}_s^\vee = \frac{1}{n}\overline{\pi}_s^\vee \in \Lambda^\vee$ we have $\langle \chi, \overline{\alpha}_s^\vee \rangle = \frac{1}{n}\langle \chi, \overline{\pi}_s^\vee \rangle \in \mathbb{Z}$. Applied to $\chi = \pi_s$ one gets $n = 1$ or $n = 2$. Moreover, if $n = 2$ then $\langle \Lambda, \overline{\pi}_s^\vee \rangle = 2\mathbb{Z}$. Therefore, we define the set of *ambiguous reflections* as

$$\Sigma^a := \Sigma^a(\Lambda) = \{s \in \Sigma \mid \langle \Lambda, \overline{\pi}_s^\vee \rangle = 2\mathbb{Z}\} \tag{1.2.12}$$

and

$$\Sigma^a(\Phi) := \Sigma^a(\Phi, \Lambda) := \{s \in \Sigma \mid \alpha_s = 2\pi_s\}. \tag{1.2.13}$$

Then the discussion above implies that $\Sigma^a(\Phi) \subseteq \Sigma^a$ and that the set of simple roots of Φ and therefore Φ itself is determined by $\Sigma^a(\Phi)$. To see that every subset of Σ^a can be realized this way we need the following lemma

1.2.8 Lemma. *Let $s \in \Sigma^a$ and $t \in \Sigma$ be W-conjugate. Then $s = t$.*

Proof. Since $s, t \in \Sigma$ are conjugate there is a string of simple reflections $s = s_1, s_2, \ldots, s_n = t$ such that the order of $s_\nu s_{\nu+1}$ is odd for all $\nu = 1, \ldots, n-1$ (see e.g. [Bou68, IV, §1, Prop. 3]). For Weyl groups this happens only if π_{s_ν}, $\pi_{s_{\nu+1}}$ span a root system type A_2. Thus, if $s \neq t$ then $n \geq 2$ and $\langle \pi_{s_2}, \overline{\pi}_{s_1}^\vee \rangle = -1 \notin 2\mathbb{Z}$ in contradiction to s being ambiguous. $\square$

1.2.9 Lemma. *Fix an affine reflection group W on $\mathfrak{a}$, a W-invariant lattice $\Lambda \subseteq \overline{\mathfrak{a}}$, and an alcove $\mathcal{A}$ of W. Then the map $\Phi \mapsto \Sigma^a(\Phi)$ is a bijection between integral affine root systems (Φ, Λ) with $W_\Phi = W$ and subsets of Σ^a.*

Proof. It remains to prove that for every $I \subseteq \Sigma^a$ there is a root system Φ_I with $\Sigma^a(\Phi_I) = I$. If Φ_I exists at all then its set S_I of simple roots has to be $\{\alpha_s \mid s \in \Sigma\}$ with

$$\alpha_s := \begin{cases} 2\pi_s & \text{if } s \in I, \\ \pi_s & \text{if } s \in \Sigma \setminus I. \end{cases} \tag{1.2.14}$$

Then $\Phi_I = WS_I$ is a root system except that it might not be reduced. So suppose $\alpha, \beta \in \Phi_I$ are distinct and positively proportional. By applying an element of W and by the discussion above Lemma 1.2.8 we may assume without loss of generality that $\alpha = \pi_s$ and $\beta = 2\alpha_s$ for some $s \in \Sigma$. In particular $s \in \Sigma^a$. On the other hand there are $t \in \Sigma$ and $w \in W$ with $\beta = w\alpha_t$. Because of $s = s_\alpha = s_\beta = wtw^{-1}$ it follows from Lemma 1.2.8 that $s = t$ and therefore $\alpha_s = \alpha_t$. But that is impossible since α is primitive and β is not. $\square$

Now we can finish the

Proof of Proposition 1.2.3. The system $W(x)$ of Weyl groups forms a local reflection group on $\mathcal{P}$. Moreover, the assumptions on $\Phi(*)$ imply the assumptions of Lemma 1.2.7. Therefore, W is an affine reflection group with $W(x) = W_x$ for all $x \in \mathcal{P}$. In particular, $(\Phi(x), \Lambda)$ is an integral affine root system with Weyl group W_x. Let $\Sigma \subseteq W$ be the set of simple reflections. Then $\Sigma_x = \Sigma \cap W_x$ is a set of simple reflections for W_x. The integral affine root system $(\Phi(x), \Lambda)$ is therefore determined by a subset $\Sigma^a(x) := \Sigma^a(\Phi(x)) \subseteq \Sigma^a$. Now the same argument as for (1.2.3) also shows

$$\Phi(x)_y = \Phi(y)_x \text{ for all } x, y \in \mathcal{P}. \tag{1.2.15}$$

This implies that whenever $s \in \Sigma_x \cap \Sigma_y$ then $s \in \Sigma^a(x)$ if and only if $s \in \Sigma^a(y)$. Thus, the union $\Sigma^a(*) := \bigcup_{x \in \mathcal{P}} \Sigma^a(x)$ has the property that $\Sigma^a(*) \cap \Sigma_x = \Sigma^a(x)$ for all $x \in \mathcal{P}$. Let Φ be the root system with $\Sigma^a(\Phi) = \Sigma^a(*)$ whose existence is guaranteed by Lemma 1.2.9. Because of $\Sigma^a(\Phi_x) = \Sigma^a(\Phi) \cap \Sigma_x = \Sigma^a(x) = \Sigma^a(\Phi(x))$ we obtain $\Phi_x = \Phi(x)$, as required. $\square$

1.3. The automorphism sheaf

We keep the notation of Section 1.1: (Φ, Λ) is an integral affine root system on the affine space $\mathfrak{a}$ with Weyl group W and fundamental alcove $\mathcal{A}$. Recall the torus $A := \overline{\mathfrak{a}}/\Lambda^\vee$. Let, moreover, $\mathcal{P} \subseteq \mathcal{A}$ be a solid subset which is additionally assumed to be *locally polyhedral*, i.e., every $x \in \mathcal{P}$ has a neighborhood $U \subseteq \mathfrak{a}$ with $\mathcal{P} \cap U = \mathcal{Q} \cap U$ for some polyhedron $\mathcal{Q} \subseteq \mathfrak{a}$ depending on x.

In this section, we consider certain maps $\varphi : \mathcal{P} \to A$.

a) A map $\varphi : \mathcal{P} \to A$ is *smooth* if every point of $\mathcal{P}$ has an open neighborhood $U \subseteq \mathfrak{a}$ such that φ is the restriction of a smooth map $\widetilde{\varphi} : U \to A$ to $U \cap \mathcal{P}$. Let $\widehat{\mathcal{C}}_{\mathfrak{a},x}$ and $\widehat{\mathcal{C}}_{A,a}$ be the completions of the local ring of smooth functions (i.e., the rings of formal power series) in $x \in \mathfrak{a}$ and $a \in A$, respectively. Then a smooth map φ with $a = \varphi(x)$ induces an algebra homomorphism

$$\widehat{\varphi}_x : \widehat{\mathcal{C}}_{A,a} \to \widehat{\mathcal{C}}_{\mathfrak{a},x}. \tag{1.3.1}$$

In fact, any local extension $\widetilde{\varphi} : U \to A$ defines a homomorphism $\widehat{\widetilde{\varphi}}_x : \widehat{\mathcal{C}}_{A,a} \to \widehat{\mathcal{C}}_{\mathfrak{a},x}$ which, by continuity, is independent of the choice of $\widetilde{\varphi}$ since $\mathcal{P}^0$ is dense in $\mathcal{P}$. Thus $\widehat{\varphi}_x := \widehat{\widetilde{\varphi}}_x$ is well defined.

b) Since each tangent space of the product space $\mathfrak{a} \times A$ is canonically isomorphic to $\overline{\mathfrak{a}} \oplus \overline{\mathfrak{a}}$, the scalar product on $\overline{\mathfrak{a}}$ induces a symplectic structure on $\mathfrak{a} \times A$ by

$$(1.3.2)\quad \omega(\xi_1 + \eta_1, \xi_2 + \eta_2) = \langle \xi_1, \eta_2 \rangle - \langle \xi_2, \eta_1 \rangle, \quad \text{with } \xi_1, \xi_2, \eta_1, \eta_2 \in \overline{\mathfrak{a}}.$$

Using the identifications $\mathfrak{a} \cong \overline{\mathfrak{a}} \cong \overline{\mathfrak{a}}^*$ one can consider ω as the canonical symplectic form on the cotangent bundle $T_A^* = \overline{\mathfrak{a}}^* \times A$. A smooth map $\varphi : \mathcal{P} \to A$ is *closed* if the graph of $\varphi|_{\mathcal{P}^0}$ is a Lagrangian submanifold of $\mathfrak{a} \times A$.

c) A smooth map $\varphi : \mathcal{P} \to A$ is *W-equivariant* if for every $x \in \mathcal{P}$ the point $a = \varphi(x)$ is W_x-fixed (i.e., $w \in W, wx = x \Rightarrow wa = a$) and the induced homomorphism (1.3.1) is W_x-equivariant.

d) A smooth map $\varphi : \mathcal{P} \to A$ is *Φ-equivariant* if it is W-equivariant and

$$(1.3.3)\quad \widetilde{\alpha}(\varphi(x)) = 1 \text{ for all } x \in \mathcal{P} \text{ and all roots } \alpha \in \Phi \text{ with } \alpha(x) = 0,$$

where $\widetilde{\alpha}$ is as in (1.1.11).

1.3.1 Remarks. *a)* If $\mathcal{P}$ is convex, so in particular simply connected, the notion of closedness can be rephrased: Because $\exp : \overline{\mathfrak{a}} \to A$ is a covering the map φ can be lifted to a smooth map $\widetilde{\varphi} : \mathcal{P} \to \overline{\mathfrak{a}}$. Because of the identification $\overline{\mathfrak{a}} \cong \overline{\mathfrak{a}}^*$ one can interpret $\widetilde{\varphi}$ as a 1-form. Then it easy to see that φ is closed if and only if $\widetilde{\varphi}$ is closed as a 1-form (whence the name).

b) For $\alpha \in \Phi$ let $s_\alpha \in W$ be the corresponding reflection. Then $s_\alpha \in W_x$ if and only if $\alpha(x) = 0$. In this case, W-equivariance implies $s_\alpha(a) = a$ where $a = \varphi(x)$. This means

$$(1.3.4)\qquad \widetilde{\alpha}^\vee(\widetilde{\alpha}(a)) = 1 \in A$$

by equation (1.1.15). Applying $\widetilde{\alpha}$ to both sides, we see (equation (1.1.14)) that W-equivariance alone already implies

$$(1.3.5)\qquad \widetilde{\alpha}(a)^2 = 1 \in \mathbb{R}, \text{ i.e., } \widetilde{\alpha}(a) = \pm 1.$$

So Φ-equivariance just means that additionally $\widetilde{\alpha}(a)$ equals 1 instead of -1.

Now we localize these definitions.

1.3.2 Definition. Let $((\Phi(x))_{x\in\mathcal{P}}, \Lambda)$ be a local system of roots on $\mathcal{P}$ and $U \subseteq \mathcal{P}$ open. Then $\mathfrak{L}^{\Phi(*)}_{\mathcal{P},\Lambda}(U)$ is the set of smooth and closed maps $U \to A$ which are $\Phi(x)$-equivariant for every $x \in U$.

Clearly $\mathfrak{L}^{\Phi(*)}_{\mathcal{P},\Lambda}$ is a sheaf of abelian groups on $\mathcal{P}$. If the local root system is trivial and comes from an integral affine root system (Φ, Λ) we simply write $\mathfrak{L}^{\Phi}_{\mathcal{P},\Lambda}$.

1.4. The vanishing theorem

In this section, we compute the cohomology of $\mathfrak{L}^{\Phi}_{\mathcal{P},\Lambda}$. This will be the central step in the proof of Theorem 2.6.1.

1.4.1 Theorem. *Let (Φ, Λ) be an integral affine root system on the affine space $\mathfrak{a}$, let $\mathcal{A} \subseteq \mathfrak{a}$ be an alcove and let $\mathcal{P} \subseteq \mathcal{A}$ be a convex, solid, locally polyhedral subset. Then $H^i(\mathcal{P}, \mathfrak{L}^{\Phi}_{\mathcal{P},\Lambda}) = 0$ for all $i \geq 1$.*

The proof will occupy the rest of this section. We start with a reduction step:

1.4.2 Lemma. *Let $\Lambda_1, \Lambda_2 \subseteq \overline{\mathfrak{a}}$ be two commensurable weight lattices for Φ (i.e. $\Lambda_1 \cap \Lambda_2$ is also a lattice). Then $H^i(\mathcal{P}, \mathfrak{L}^{\Phi}_{\mathcal{P},\Lambda_1}) = H^i(\mathcal{P}, \mathfrak{L}^{\Phi}_{\mathcal{P},\Lambda_2})$ for all $i \geq 1$*

Proof. By replacing Λ_1 with the intersection $\Lambda_1 \cap \Lambda_2$ we may assume $\Lambda_1 \subseteq \Lambda_2$. Then $A_1 := \overline{\mathfrak{a}}/\Lambda_1^\vee$ is a quotient of $A_2 := \overline{\mathfrak{a}}/\Lambda_2^\vee$ with kernel

$$E := \Lambda_1^\vee/\Lambda_2^\vee \subseteq A_2. \tag{1.4.1}$$

Let $U \subseteq \mathcal{P}$ be convex and open. Then any smooth map $\varphi_1 : U \to A_1$ can be lifted to a smooth map $\varphi_2 : U \to A_2$. This lifted map φ_2 is closed and Φ-equivariant if and only φ_1 is. Thus, we get a short exact sequence of sheaves

$$0 \longrightarrow E_{\mathcal{P}} \to \mathfrak{L}^{\Phi}_{\mathcal{P},\Lambda_2} \longrightarrow \mathfrak{L}^{\Phi}_{\mathcal{P},\Lambda_1} \longrightarrow 0 \tag{1.4.2}$$

where $E_{\mathcal{P}}$ denotes the constant sheaf on $\mathcal{P}$ with fiber E. Since $\mathcal{P}$ is convex, we have $H^i(\mathcal{P}, E_{\mathcal{P}}) = 0$ for $i \geq 1$ and the assertion follows. $\square$

A weight lattice will be called *of adjoint type* if

$$\Lambda = \mathbb{Z}\overline{\Phi} \oplus \Lambda^W \subseteq \mathbb{R}\overline{\Phi} \oplus \overline{\mathfrak{a}}^W = \overline{\mathfrak{a}}. \tag{1.4.3}$$

Since every weight lattice Λ is commensurable to $\mathbb{Z}\overline{\Phi} \oplus \Lambda^W$ Lemma 1.4.2 allows to assume that Λ is of adjoint type.

Next, we need a method to produce sections of $\mathfrak{L}^{\Phi}_{\mathcal{P},\Lambda}$. To this end, we define a function f on $\mathcal{P}$ to be *smooth* if it can be locally extended to a smooth function on an open subset of $\mathfrak{a}$. The differential df of a smooth function f can be considered as a smooth map $\mathcal{P} \to \overline{\mathfrak{a}}^*$. Since the form df is closed it follows from Remark 1.3.1 *a)* that $\varphi := \exp(\nabla f)$ defines a closed smooth map $\mathcal{P} \to A$. After localizing this construction, we get a homomorphism of sheaves

$$\varepsilon : \mathcal{C}_{\mathcal{P}} \to \mathfrak{L}^{\mathrm{closed}}_{\mathcal{P},\Lambda} \tag{1.4.4}$$

where $\mathcal{C}_{\mathcal{P}}$ is the sheaf of smooth functions on $\mathcal{P}$ and $\mathfrak{L}^{\mathrm{closed}}_{\mathcal{P},\Lambda}$ is the sheaf of closed maps from $\mathcal{P}$ to A.

1.4.3 Lemma. *The homomorphism ε is surjective, i.e., all closed maps φ from $\mathcal{P}$ to A are locally of the form*

$$\varphi(x) = \exp(\nabla f(x)) \tag{1.4.5}$$

where f is a smooth function on an open subset of $\mathcal{P}$.

Proof. Let $U \subseteq \mathfrak{a}$ be a convex open neighborhood of $a \in \mathcal{P}$ such that $\mathcal{P} \cap U$ is convex and φ is defined on $\mathcal{P} \cap U$. Then $\varphi|_{\mathcal{P}\cap U}$ lifts to a closed 1-form $\widetilde{\varphi} : \mathcal{P} \cap U \to \overline{\mathfrak{a}} \xrightarrow{\sim} \overline{\mathfrak{a}}^*$ which extends to a smooth 1-form ω on U. The derivative $d\omega$ of this form vanishes on $\mathcal{P} \cap U$. The convexity of U allows us to define the smooth function $f(x) := \int_{[a,x]} \omega$ in U where $[a, x]$ is the line segment from a to x. Because $\mathcal{P} \cap U$ is convex this line segment lies entirely in $\mathcal{P} \cap U$ when $x \in \mathcal{P} \cap U$. Since $d\omega$ vanishes on $\mathcal{P} \cap U$ (the proof of) Poincaré's Lemma shows $df|_{\mathcal{P}\cap U} = \widetilde{\varphi}$. $\square$

This construction produces closed maps to A. To get W-equivariant ones let f be a W*-invariant* smooth function in the sense that for each $x \in \mathcal{P}$ the Taylor series of f in x is W_x-invariant. Then $\varepsilon(f)$ is a W-equivariant closed map to A. We claim that $\varepsilon(f)$ is automatically Φ-equivariant. Indeed, consider the continuous family $\varphi_t = \varepsilon(tf)$, $t \in \mathbb{R}$ of closed maps and let $\alpha \in \Phi$ with $\alpha(x) = 0$. Since $\widetilde{\alpha}(\varphi_t(x)) \in \{\pm 1\}$ (see (1.3.5)) and $\varphi_0(x) = 1$ we get $\widetilde{\alpha}(\varphi_1(x)) = 1$ by continuity.

Thus, if we denote the sheaf of W-invariant smooth functions on $\mathcal{P}$ by $\mathcal{C}^W_{\mathcal{P}}$ we obtain a homomorphism of sheaves

$$\varepsilon^W : \mathcal{C}^W_{\mathcal{P}} \to \mathfrak{L}^{\Phi}_{\mathcal{P},\Lambda}. \tag{1.4.6}$$

The first step towards proving the vanishing theorem is:

1.4.4 Lemma. $H^i(\mathcal{P}, \mathcal{C}^W_{\mathcal{P}}) = 0$ *for* $i \geq 1$.

Proof. Since $\mathcal{P}$ is paracompact it suffices to show that $\mathcal{C}_{\mathcal{P}}^W$ is soft (see [Bre97, Thm. 9.11]). To this end we claim that there is a smooth closed embedding $\pi : \overline{\mathfrak{a}}/W \hookrightarrow \mathbb{R}^n$ with $n = \dim \mathfrak{a}$. It suffices to prove this claim for W irreducible. Then either W is finite in which case we can apply Chevalley's theorem or W is the affine Weyl group attached to a finite root system Φ_0 with Weyl group W_0 (observe that for every twisted affine root system there is an untwisted one with the same Weyl group). Then an embedding is provided by the smooth W-invariant functions $f_\omega(x) := \sum_{w\in W_0} \exp(2\pi i\omega(wx))$, where ω runs through the fundamental weights of Φ_0 (see, e.g., [Bou68, VI, §3.4, Thm. 1]).

Since $\pi(\mathcal{P})$ is homeomorphic to $\mathcal{P}$, it suffices to show that $\pi_*\mathcal{C}_{\mathcal{P}}$ is soft. Now observe that $\pi_*\mathcal{C}_{\mathcal{P}}$ is a $\mathcal{C}^\infty_{\mathbb{R}^n}$-module because the W_x-invariance of a Taylor series is preserved by multiplication with a W-invariant. Thus, $\pi_*\mathcal{C}_{\mathcal{P}}$ is a module for the soft sheaf of algebras $\mathcal{C}^\infty_{\mathbb{R}^n}$ and, therefore, itself soft (see [Bre97, Thms. 9.16 and 9.17]). $\square$

Now we investigate the cokernel of (1.4.6). To this end, consider the subgroup

$$A^\Phi := \{u \in A \mid \widetilde{\alpha}(u) = 1 \text{ for all } \alpha \in \Phi\} \tag{1.4.7}$$

of Φ-*fixed points* of A. By (1.1.15) it is contained in the subgroup A^W of W-fixed points.

Of particular interest will be the group A^{Φ_x} and its component group $\pi_0(A^{\Phi_x})$, where $x \in \mathcal{P}$. If y is close to x then $\Phi_y \subseteq \Phi_x$ and therefore

$$A^{\Phi_x} \subseteq A^{\Phi_y}. \tag{1.4.8}$$

This shows that there is a constructible sheaf $\mathfrak{C}_{\mathcal{P}}$ on $\mathcal{P}$ such that $\pi_0(A^{\Phi_x})$ is its stalk at x and the restriction maps $\pi_0(A^{\Phi_x}) \to \pi_0(A^{\Phi_y})$ are induced by (1.4.8). Its significance is given by

1.4.5 Lemma. *There is an exact sequence of sheaves of abelian groups*

$$\mathcal{C}_{\mathcal{P}}^W \xrightarrow{\varepsilon^W} \mathfrak{L}^\Phi_{\mathcal{P},\Lambda} \xrightarrow{\eta} \mathfrak{C}_{\mathcal{P}} \to 0. \tag{1.4.9}$$

Proof. Let $U \subseteq \mathcal{P}$ be a convex, open neighborhood of $x \in \mathcal{P}$ such that $\mathcal{P} \cap U$ is convex, as well. If $\varphi \in \mathfrak{L}^\Phi_{\mathcal{P},\Lambda}(U)$ then $\varphi(x) \in A^{\Phi_x}$, by Φ-equivariance. Thus we can define $\eta(\varphi)(x)$ to be the image of $\varphi(x)$ in $\pi_0(A^{\Phi_x})$.

Now let $u \in A^{\Phi_x}$ be a representative of some element $\overline{u} \in \pi_0(A^{\Phi_x})$. Then the constant map $\varphi : x \mapsto u$ is a section of $\mathfrak{L}^\Phi_{\mathcal{P},\Lambda}$ with $\eta(\varphi) = \overline{u}$, which shows that η is surjective.

On the other hand, for any section f of $\mathcal{C}_{\mathcal{P}}^W$, the image $\varepsilon^W(f)(x)$ lies in $\exp(\overline{\mathfrak{a}}^{W_x}) = (A^{\Phi_x})^0$, which shows $\operatorname{im}\varepsilon^W \subseteq \ker\eta$.

To show equality, let $\varphi : U \to A$ be a section of $\mathfrak{L}^{\Phi}_{\mathcal{P},\Lambda}$ with $\eta(\varphi) = 0$, i.e., $\varphi(x) \in (A^{\Phi_x})^0$ for all $x \in U$. Then there is a lift $\widetilde{\varphi} : U \to \overline{\mathfrak{a}}$ of φ with $\widetilde{\varphi}(x) \in \overline{\mathfrak{a}}^{W_x}$. Since $\widetilde{\varphi}$ is smooth and closed there is a smooth function f on U with $\nabla f = \widetilde{\varphi}$ (Lemma 1.4.3). Let $\widehat{f}$ be the Taylor series of f in x. Then the W_x-equivariance of $\widetilde{\varphi}$ implies $\nabla({}^w\widehat{f}) = \nabla\widehat{f}$ for all $w \in W_x$. Hence ${}^w\widehat{f} = \widehat{f} + c_w$ with a constant $c_w \in \mathbb{R}$. Evaluating this at x implies $c_w = 0$, i.e., f is in fact W_x-invariant. Therefore, $\varphi = \varepsilon^W(f)$ is indeed in the image of ε^W. □

To calculate the cohomology of $\mathfrak{C}_{\mathcal{P}}$ we need a more explicit description. The character group of A^{Φ} is given by

$$\Xi(A^{\Phi}) = \Lambda/\mathbb{Z}\overline{\Phi}. \tag{1.4.10}$$

In particular, $\pi_0(A^{\Phi}) = 0$ if and only if the root lattice $\mathbb{Z}\overline{\Phi}$ is a direct summand of Λ. More generally, we have

$$\Xi(\pi_0(A^{\Phi})) = \operatorname{Tors}(\Lambda/\mathbb{Z}\overline{\Phi}) = \frac{\Lambda \cap \mathbb{R}\overline{\Phi}}{\mathbb{Z}\overline{\Phi}}. \tag{1.4.11}$$

Dualizing, this is equivalent to

$$\pi_0(A^{\Phi}) = \frac{(\mathbb{Z}\overline{\Phi})^\vee}{\Lambda^\vee + (\mathbb{R}\overline{\Phi})^\vee}, \tag{1.4.12}$$

where $(\mathbb{Z}\overline{\Phi})^\vee$ is the coweight lattice and $(\mathbb{R}\overline{\Phi})^\vee$ is the orthogonal complement of $\mathbb{R}\overline{\Phi}$ in $\overline{\mathfrak{a}}$.

We compute $\mathfrak{C}_{\mathcal{P}}$ in two stages, the first treating the case of finite root systems.

1.4.6 Lemma. *Assume Φ is finite and $\Lambda = \mathbb{Z}\overline{\Phi}$. Then $\mathfrak{C}_{\mathcal{P}} = 0$.*

Proof. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots of $\overline{\Phi}$. Since these form a basis of $\overline{\mathfrak{a}}$ we get an isomorphism

$$\alpha_* : \overline{\mathfrak{a}} \to \mathbb{R}^n : x \mapsto (\langle\alpha_1, x\rangle, \ldots, \langle\alpha_n, x\rangle) \tag{1.4.13}$$

For any subset $I \subseteq \{1, \ldots, n\}$ let I' be its complement. For $k \in \{\mathbb{R}, \mathbb{Z}\}$ we set

$$k^I := \{(x_i) \in k^n \mid x_i = 0 \text{ for } i \in I'\} \cong k^{|I|} \tag{1.4.14}$$

For a fixed $x \in \mathcal{P}$, let $I := \{i \mid \alpha_i(x) = 0\}$. Then α_* maps $(\mathbb{Z}\overline{\Phi}_x)^\vee$, $(\mathbb{R}\overline{\Phi}_x)^\perp$, and $\Lambda^\vee$ to $\mathbb{Z}^I \oplus \mathbb{R}^{I'}$, $\mathbb{R}^{I'}$, and $\mathbb{Z}^n$, respectively, and the claim follows from (1.4.12). □

Now assume that Φ is an infinite irreducible root system with simple roots $S = \{\alpha_1, \dots, \alpha_n\}$. The *labels* of S are defined as the components of the unique primitive vector $\delta := (a_1, \dots, a_n) \in \mathbb{Z}^n_{>0}$ such that

$$a_1\overline{\alpha}_1 + \cdots + a_n\overline{\alpha}_n = 0. \tag{1.4.15}$$

For $I \subsetneq \{1, \dots, n\}$ let again $I' \neq \varnothing$ be its complement and

$$d_I := \gcd\{a_j \mid j \in I'\}. \tag{1.4.16}$$

1.4.7 Lemma. *Assume Φ is an infinite irreducible root system and $\Lambda = \mathbb{Z}\overline{\Phi}$. For any fixed $x \in \mathcal{A}$ let $\mathfrak{C}_x$ be the stalk of $\mathfrak{C}_\mathcal{P}$ in x and $I = I_x := \{i \mid \alpha_i(x) = 0\}$. Then there is a canonical isomorphism*

$$\mathfrak{C}_x = \pi_0(A^{\Phi_x}) \to \mathbb{Z}/d_I\mathbb{Z}. \tag{1.4.17}$$

Moreover, this isomorphism is compatible with the restriction homomorphisms of $\mathfrak{C}_\mathcal{P}$.

Proof. We keep the notation of the proof of Lemma 1.4.6. The map α_* of (1.4.13) with α_i replaced by $\overline{\alpha}_i$ identifies $\overline{\mathfrak{a}}$ with the hyperplane H of $\mathbb{R}^n$ which is perpendicular to δ. Thus (1.4.12) becomes

$$\pi_0(A^{\Phi_x}) = \frac{(\mathbb{Z}^I \oplus \mathbb{R}^{I'}) \cap H}{(\mathbb{R}^{I'} \cap H) + (\mathbb{Z}^n \cap H)} \tag{1.4.18}$$

Now consider the homomorphism

$$p_I : (\mathbb{Z}^I \oplus \mathbb{R}^{I'}) \cap H \to \mathbb{Z}/d_I\mathbb{Z} : (x_i) \mapsto \sum_{i \in I} a_i x_i + d_I\mathbb{Z}. \tag{1.4.19}$$

Since $d_{I'}$ and d_I are coprime there are $a', a \in \mathbb{Z}$ with $a'd_{I'} + ad_I = 1$. Because $I' \neq \varnothing$, there is $(x_i) \in (\mathbb{Z}^I \oplus \mathbb{R}^{I'}) \cap H$ with $\sum_{i \in I} a_i x_i = a'd_{I'}$. Then $p_I(x_i) = 1$, i.e., p_I is onto.

Next we claim that the kernel of p_I is precisely $E := (\mathbb{R}^{I'} \cap H) + (\mathbb{Z}^n \cap H)$. Clearly $\mathbb{R}^{I'} \cap H \subseteq \ker p_I$. Let $(x_i) \in \mathbb{Z}^n \cap H$. Then

$$\sum_{i \in I} a_i x_i = -\sum_{j \in I'} a_j x_j \in d_I\mathbb{Z} \tag{1.4.20}$$

shows that $E \subseteq \ker p_I$. To show the converse, let $(x_i) \in \ker p_I$. Then, by definition, $\sum_{i\in I} a_i x_i \in d_I\mathbb{Z}$. Hence there is $(y_i) \in \mathbb{Z}^{I'}$ with

$$\sum_{i\in I} a_i x_i = -\sum_{j\in I'} a_j y_j. \tag{1.4.21}$$

Now define

$$\overline{x}_i := \begin{cases} x_i & \text{if } i \in I \\ y_i & \text{if } i \in I' \end{cases} \tag{1.4.22}$$

Then $(\overline{x}_i) \in \mathbb{Z}^n \cap H$ with $(x_i) - (\overline{x}_i) \in \mathbb{R}^{I'} \cap H$, proving the claim. Thus p_I induces an isomorphism between $\pi_0(A^{\Phi_x})$ and $\mathbb{Z}/d_I\mathbb{Z}$.

For the final assertion, we denote I by I_x. Let $y \in \mathcal{P}$ be close enough to x such that $I_y \subseteq I_x$. Then $d_{I_y}|d_{I_x}$. Thus, we have to show that the diagram

$$\begin{tikzcd} (\mathbb{Z}^{I_x} \oplus \mathbb{R}^{I'_x}) \cap H \arrow[r, "p_{I_x}"] \arrow[d, hook] & \mathbb{Z}/d_{I_x}\mathbb{Z} \arrow[d, two heads, "{[1]\mapsto[1]}"] \\ (\mathbb{Z}^{I_y} \oplus \mathbb{R}^{I'_y}) \cap H \arrow[r, "p_{I_y}"] & \mathbb{Z}/d_{I_y}\mathbb{Z} \end{tikzcd} \tag{1.4.23}$$

commutes. But this follows from $d_{I_y} | a_i$ for all $i \in I_x \setminus I_y$. □

From this we deduce:

1.4.8 Lemma. *Assume Λ is of adjoint type. Then $\mathfrak{C}_\mathcal{P}$ has a finite filtration such that each factor is a constant sheaf supported on a face of $\mathcal{P}$.*

Proof. Let $\mathfrak{a} = \mathfrak{a}_0 \times \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_m$ and $\Phi = \Phi_1 \cup \cdots \cup \Phi_m$ be the unique decomposition of $(\mathfrak{a}, \Phi)$ into a trivial part $\mathfrak{a}_0$ and irreducible parts $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$. The alcove $\mathcal{A}$ of Φ will split accordingly as $\mathcal{A} = \mathfrak{a}_0 \times \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$. Then $\mathfrak{C}_P = \mathfrak{C}^{(1)} \oplus \ldots \oplus \mathfrak{C}^{(m)}$ where $\mathfrak{C}^{(i)}$ is the pull-back of $\mathfrak{C}_{\mathcal{A}_i}$ to $\mathcal{P}$. Thus it suffices to show the assertion for $\mathfrak{C} := \mathfrak{C}^{(i)}$ for any i. By Lemma 1.4.6 we may also assume that Φ_i is infinite. Let $\alpha_1, \ldots, \alpha_n \in \Phi_i$ be the simple roots.

For any prime power p^e let $\mathfrak{C}[p^e] \subseteq \mathfrak{C}_\mathcal{P}$ be the kernel of multiplication by p^e. The union $\mathfrak{C}[p^\infty]$ over all e is the p-primary component of $\mathfrak{C}_\mathcal{P}$. Since $\mathfrak{C}_\mathcal{P}$ is the direct sum of its primary components it suffices to show the assertion for $\mathfrak{C}[p^\infty]$. Now it follows from Lemma 1.4.7 that $\mathfrak{C}[p^e]/\mathfrak{C}[p^{e-1}]$ is a constant sheaf with stalks $\mathbb{Z}/p\mathbb{Z}$ which is supported in the face

$$\{x \in \mathcal{P} \mid \alpha_i(x) = 0 \text{ for all } i \text{ with } p^e \nmid a_i\} \tag{1.4.24}$$

□

Since constant sheaves on contractible spaces have trivial cohomology, we get:

1.4.9 Corollary. *Assume Λ is of adjoint type and that $\mathcal{P}$ is convex. Then $H^i(\mathcal{P}, \mathfrak{C}_{\mathcal{P}}) = 0$ for all $i \geq 1$.*

Next we study the kernel of ε^W (cf. (1.4.6)). Observe that the constant sheaf $\mathbb{R}_{\mathcal{P}}$ is contained in the kernel. From this we get a homomorphism

$$\bar{\varepsilon}^W : \mathcal{C}^W_{\mathcal{P}}/\mathbb{R}_{\mathcal{P}} \to \mathfrak{L}^{\Phi}_{\mathcal{P},\Lambda} \tag{1.4.25}$$

1.4.10 Lemma. *Let $\mathfrak{K}_{\mathcal{P}}$ be the kernel of $\bar{\varepsilon}^W$. Then its stalk $\mathfrak{K}_x$ at $x \in \mathcal{P}$ is equal to $\Lambda^\vee \cap (\mathbb{R}\overline{\Phi}_x)^\vee$.*

Proof. Let $x \in \mathcal{P}$ and let $U \subseteq \mathcal{P}$ be a small convex open neighborhood. A smooth function f on U is in the kernel of ε if and only if its gradient is in $\Lambda^\vee$. Continuity implies that ∇f must be in fact constant. This implies that f is an affine linear function with $\overline{f} = \nabla f \in \Lambda^\vee$. Moreover, f is a section of $\mathcal{C}^W_{\mathcal{P}}$ if and only if $\overline{f}$ is W_x-invariant. This means $\overline{f}$ should be orthogonal to all $\overline{\alpha} \in \overline{\Phi}_x$. ☐

In the following lemma let $\Lambda^\vee_{\mathcal{P}}$ be the constant sheaf with stalks $\Lambda^\vee$ on $\mathcal{P}$. Similarly, $\mathbb{Z}_{H_i\cap\mathcal{P}}$ will be the constant sheaf with stalks $\mathbb{Z}$ on $H_i \cap \mathcal{P}$ which is then extended by zero to $\mathcal{P}$.

1.4.11 Lemma. *Let Λ be of adjoint type and let $\alpha_1, \ldots, \alpha_n$ be the simple roots of Φ. Let H_i be the hyperplane $\{\alpha_i = 0\}$. Then the sheaf $\mathfrak{K}_{\mathcal{P}}$ fits into an exact sequence*

$$0 \to \mathfrak{K}_{\mathcal{P}} \to \Lambda^\vee_{\mathcal{P}} \xrightarrow{\varrho} \bigoplus_{i=1}^{n} \mathbb{Z}_{H_i\cap\mathcal{P}} \xrightarrow{\psi} \mathfrak{C}_{\mathcal{P}} \to 0. \tag{1.4.26}$$

If Φ is an infinite irreducible root system then ψ maps the generator of the stalk $\mathbb{Z}_{H_i\cap\mathcal{P},x}$ in $x \in \mathcal{P}$ to the class $a_i + d_I\mathbb{Z}$ (notation as in Lemma 1.4.7).

Proof. All sheaves are restrictions of the corresponding sheaves on $\mathcal{A}$ to $\mathcal{P}$. Thus we may assume that $\mathcal{P} = \mathcal{A}$. Therefore we may treat every factor of the root system Φ separately. Thus, we may assume that Φ is either finite or irreducible and infinite.

For $x \in \mathcal{P}$ we have to show that the stalk $\mathfrak{K}_x = \Lambda^\vee \cap (\mathbb{R}\overline{\Phi}_x)^\vee$ fits into an exact sequence

$$0 \to \mathfrak{K}_x \to \Lambda^\vee \xrightarrow{\varrho_x} \mathbb{Z}^{I_x} \xrightarrow{\psi_x} \mathfrak{C}_x \to 0. \tag{1.4.27}$$

First, we define ϱ_x as $\varrho_x(v) := (\langle\overline{\alpha}_i, v\rangle)_{i\in I_x} \in \mathbb{Z}^{I_x}$. Then $\mathfrak{K}_x$ is the kernel of ϱ_x by Lemma 1.4.10.

If Φ is finite then the set of all $\overline{\alpha}_i$ with $i \in I_x$ is part of a dual basis of $\Lambda^\vee$. Thus, ϱ_x is surjective and (1.4.27) is exact since $\mathfrak{C}_x = 0$ in this case by Lemma 1.4.6.

Now assume that Φ is irreducible and infinite. Then $\mathfrak{C}_x = \mathbb{Z}/d_{I_x}\mathbb{Z}$ by Lemma 1.4.7 and we define ψ_x as $\psi_x(y_i) := \sum_{i\in I_x} a_i y_i + d_{I_x}\mathbb{Z}$. Identifying $\overline{\mathfrak{a}}$ with the hyperplane H and $\Lambda^\vee$ with $\mathbb{Z}^n$ as in the proof of Lemma 1.4.7 we have to show that

$$\mathbb{Z}^n \cap H \overset{\varrho_x}{\to} \mathbb{Z}^{I_x} \overset{\psi_x}{\to} \mathbb{Z}/d_I\mathbb{Z} \to 0 \tag{1.4.28}$$

is exact where ϱ_x is the projection $(x_i) \mapsto (x_i)_{i\in I_x}$. Surjectivity of ψ_x follows again from $\gcd(d_I, d_{I'}) = 1$. Moreover, the kernel of ψ_x consists of all $(y_i)_{i\in I_x}$ which can be extended to an n-tuple $(y_i)_{i=1}^n \in \mathbb{Z}^n$ with $\sum_{i=1}^n a_i y_i = 0$, since this is equivalent to $\sum_{i\in I_x} a_i y_i$ being divisible by d_I (see proof of Lemma 1.4.7). $\square$

1.4.12 Lemma. *Assume Λ is of adjoint type and that $\mathcal{P}$ is convex. Then the homomorphism*

$$H^0(\psi) : H^0(\mathcal{P}, \bigoplus_i \mathbb{Z}_{H_i\cap\mathcal{P}}) \to H^0(\mathcal{P}, \mathfrak{C}_\mathcal{P}) \tag{1.4.29}$$

is surjective.

Proof. Both sides decompose as direct sums according to the decomposition of Φ into factors. Thus we may assume that Φ is irreducible. Then there is nothing to prove if Φ is finite since then $\mathfrak{C}_\mathcal{P} = 0$. So assume that Φ is infinite.

Let p be a prime and let $\mathfrak{C}[p]$ be the p-primary component of $\mathfrak{C}_\mathcal{P}$. Then it suffices to show that the composition of $H^0(\psi)$ with the projection onto $H^0(\mathfrak{C}_\mathcal{P})[p] = H^0(\mathfrak{C}[p])$ is surjective.

To this end define the faces $\mathcal{F}_j \subseteq \mathcal{A}$, $j \geq 0$, by the equations $\alpha_i = 0$ with $p^j \nmid a_i$. Then $\mathcal{A} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \cdots$ is a descending chain of faces of $\mathcal{A}$. Let e be maximal with $\mathcal{P}_e := \mathcal{F}_e \cap \mathcal{P} \neq \varnothing$. Then

$$\mathcal{P} = \mathcal{P}_0 \supseteq \mathcal{P}_1 \supset \cdots \supseteq \mathcal{P}_e \supset \mathcal{P}_{e+1} = \varnothing \tag{1.4.30}$$

is a chain of of closed subsets of $\mathcal{P}$. Let $\mathcal{P}'_j := \mathcal{P}_j \setminus \mathcal{P}_{j+1} = \mathcal{P} \cap (\mathcal{F}_j \setminus \mathcal{F}_{j+1})$. Then the convexity of $\mathcal{P}$ implies that also $\mathcal{P}'_j$ is convex, hence contractible. Moreover, the explicit description of $\mathfrak{C}_\mathcal{P}$ of Lemma 1.4.7 implies that the restriction $\mathfrak{C}'_j$ of $\mathfrak{C}[p]$ to $\mathcal{P}'_j$ is locally constant with fiber $\mathbb{Z}/p^j\mathbb{Z}$. It follows that either $\mathcal{P}'_j$ is empty or $H^0(\mathcal{P}'_j, \mathfrak{C}[p]) = \mathbb{Z}/p^j\mathbb{Z}$. Therefore, a global section of $\mathfrak{C}[p]$ is given by a system of elements of $H^0(\mathcal{P}'_j, \mathfrak{C}[p])$ which is compatible with the canonical restriction maps. This immediately implies $H^0(\mathcal{P}, \mathfrak{C}[p]) = H^0(\mathcal{P}_e, \mathfrak{C}[p]) = \mathbb{Z}/p^e\mathbb{Z}$.

Finally, since the labels a_* of Φ are coprime there is at least one label a_i which is not divisible by p. Then $\mathcal{P}_j \subseteq H_i \cap \mathcal{P}$ for all $j \geq 1$ and ψ maps the generator of $\mathbb{Z}_{H_i \cap \mathcal{P}}$ to $a_i + p^e\mathbb{Z} \in H^0(\mathcal{P}, \mathfrak{C}[p])$ which yields the assertion. $\square$

1.4.13 Lemma. *Assume Λ is of adjoint type. Then $H^i(\mathcal{P}, \mathfrak{K}_\mathcal{P}) = 0$ for all $i \geq 2$.*

Proof. Let $\mathfrak{T}$ be the kernel of ψ, yielding a short exact sequence

$$0 \to \mathfrak{T} \to \bigoplus_i \mathbb{Z}_{H_i \cap \mathcal{P}} \xrightarrow{\psi} \mathfrak{C}_\mathcal{P} \to 0. \tag{1.4.31}$$

Since $H_i \cap \mathcal{P}$ is convex, hence contractible, the higher cohomology of $\bigoplus_i \mathbb{Z}_{H_i \cap \mathcal{P}}$ vanishes. We already proved that $H^i(\mathcal{P}, \mathfrak{C}_\mathcal{P}) = 0$ for all $i \geq 1$ in Corollary 1.4.9. Combined with the surjectivity of $H^0(\psi)$ this implies that $H^i(\mathcal{P}, \mathfrak{T}) = 0$ for all $i \geq 1$. Since also $H^i(\mathcal{P}, \Lambda^\vee) = 0$ for all $i \geq 1$, the short exact sequence

$$0 \to \mathfrak{K}_\mathcal{P} \to \Lambda^\vee_\mathcal{P} \to \mathfrak{T} \to 0 \tag{1.4.32}$$

induced by (1.4.26) implies $H^i(\mathcal{P}, \mathfrak{K}_\mathcal{P}) = 0$ for all $i \geq 2$. $\square$

Proof of Theorem 1.4.1. By Lemma 1.4.2 we may assume that Λ is of adjoint type. Consider the short exact sequence

$$0 \to \mathbb{R}_\mathcal{P} \to \mathcal{C}^W_\mathcal{P} \to \mathcal{C}^W_\mathcal{P}/\mathbb{R}_\mathcal{P} \to 0. \tag{1.4.33}$$

The higher cohomology of the two sheaves on the left vanishes (see Lemma 1.4.4 for the second one) and thus so does that of the right hand sheaf. Let $\mathfrak{S} \subseteq \mathfrak{L}^\Phi_{\mathcal{P},\Lambda}$ be the image of ε^W. Then we get a short exact sequence

$$0 \to \mathfrak{K}_\mathcal{P} \to \mathcal{C}^W_\mathcal{P}/\mathbb{R}_\mathcal{P} \to \mathfrak{S} \to 0 \tag{1.4.34}$$

from (1.4.25). Lemma 1.4.13 implies $H^i(\mathcal{P}, \mathfrak{S}) = 0$ for all $i \geq 1$. Finally, Corollary 1.4.9 and the short exact sequence

$$0 \to \mathfrak{S} \to \mathfrak{L}^\Phi_{\mathcal{P},\Lambda} \to \mathfrak{C}_\mathcal{P} \to 0 \tag{1.4.35}$$

induced by (1.4.9) imply $H^i(\mathcal{P}, \mathfrak{L}^\Phi_{\mathcal{P},\Lambda}) = 0$ for all $i \geq 1$. $\square$

Part 2
Multiplicity free quasi-Hamiltonian manifolds

This part constitutes the main part of the paper. Except for Section 1.1 on affine root systems, the content of the preceding part will only be used in the proof of the classification Theorem 2.6.1.

2.1. Quasi-Hamiltonian manifolds

Recall the definition of Hamiltonian manifolds:

2.1.1 Definition. Let K be Lie group. A *Hamiltonian K-manifold* is a K-manifold M which is equipped with 2-form ω and a smooth map $m : M \to \mathfrak{k}^*$ (the momentum map) such that

a) m is K-equivariant,
b) the 2-form ω is K-invariant, closed and non-degenerate,
c) $\omega(\xi x, \eta) = \langle \xi, m_* \eta \rangle$ for all $\xi \in \mathfrak{k}$, $x \in M$, and $\eta \in T_x M$.

The concept of *quasi-Hamiltonian manifolds* is a multiplicative version of Hamiltonian manifolds. It was introduced in [AMM98, §8] (see also [GS05, §1.4] for a short survey). The main difference is that the momentum map has values in the Lie group instead of the dual of the Lie algebra.

To define quasi-Hamiltonian manifolds one needs the Lie algebra $\mathfrak{k}$ to be equipped with an $\mathrm{Ad}\, K$-invariant scalar product. This allows us to identify $\mathfrak{k}^*$ with $\mathfrak{k}$.

Not essential but natural is to consider Lie groups with a *twist*, i.e., a fixed automorphism $k \mapsto {}^\tau k$ of K. The target of the momentum map will then be K as a set but with K acting on it by *τ-twisted conjugation*:

$$k *_\tau g := k \cdot g \cdot {}^\tau k^{-1}. \tag{2.1.1}$$

The set K with this action will be denoted by $K\tau$. To distinguish elements of the group K from those of the space $K\tau$ we frequently denote the latter by $k\tau$ with $k \in K$.

Next, we need to introduce a couple of differential forms. First, let θ and $\overline{\theta}$ be the two canonical $\mathfrak{k}$-valued 1-forms on K induced by left and right translation:

$$\theta(k\xi) = \xi = \overline{\theta}(\xi k) \quad \text{with } \xi \in \mathfrak{k} \text{ and } k \in K. \tag{2.1.2}$$

These are combined to a $\mathfrak{k}$-valued 1-form on $K\tau$:

$$\Theta_\tau := \tfrac{1}{2}\big(\,\overline{\theta} + {}^{\tau^{-1}}\theta\,\big). \tag{2.1.3}$$

Explicitly

$$\Theta_\tau(k\xi) = \tfrac{1}{2}(\mathrm{Ad}(k)\xi + {}^{\tau^{-1}}\xi). \tag{2.1.4}$$

Using the scalar product on $\mathfrak{k}$ one defines the canonical biinvariant closed 3-form on K

$$\chi := \tfrac{1}{12}\langle\theta,[\theta,\theta]\rangle = \tfrac{1}{12}\langle\overline{\theta},[\overline{\theta},\overline{\theta}]\rangle. \tag{2.1.5}$$

2.1.2 Definition. Let K be a Lie group which is equipped with a K-invariant scalar product on $\mathfrak{k}$ and an automorphism τ. A *quasi-Hamiltonian $K\tau$-manifold* is a K-manifold M equipped with a 2-form ω and a smooth map $m : M \to K\tau$ (the *group valued momentum map*) such that:

a) The map $m : M \to K\tau$ is K-equivariant,
b) the form ω is K-invariant and satisfies $\mathrm{d}\omega = -m^*\chi$,
c) $\omega(\xi x, \eta) = \langle\xi, m^*\Theta_\tau(\eta)\rangle$ for all $\xi \in \mathfrak{k}$, $x \in M$, and $\eta \in T_xM$,
d) $\ker\omega_x = \{\xi x \in T_xM \mid \xi \in \mathfrak{k} \text{ with } \mathrm{Ad}\,m(x)({}^\tau\xi) + \xi = 0\}$.

2.1.3 Remark. This definition originates very naturally from studying (ordinary) Hamiltonian actions of the twisted loop group

$$\mathcal{L}_\tau K := \{\varphi : \mathbb{R} \to K \mid \varphi(t+1) = {}^\tau\varphi(t)\}. \tag{2.1.6}$$

See [AMM98] for the untwisted case and [Mei17] or the first version of this paper [Kno16] for the twisted version.

For the proof of Proposition 2.4.2 we will need the following observation:

2.1.4 Lemma. *Let M be a quasi-Hamiltonian manifold, $x \in M$, and $E_x :=$ $\ker(1 + \mathrm{Ad}\,m(x) \circ \tau) \subseteq \mathfrak{k}$. Then*

$$(\mathfrak{k}x)^\perp = \ker D_x m + E_x x. \tag{2.1.7}$$

Proof. The inclusion "$\supseteq$" follows from part *c)* of Definition 2.1.2 and $E_x x = \ker\omega_x$ (part *d)*). For the opposite inclusion let $\eta \in (\mathfrak{k}x)^\perp$. Put $a := m(x)$ and $\sigma := \frac{1}{2}m_*(\eta)a^{-1} \in \mathfrak{k}$. Then part *c)* implies $\sigma \in E_x$. Moreover, the equivariance of m (part *a)*) implies

$$m_*(\sigma x) = \sigma *_\tau a = \sigma a - a\,{}^\tau\sigma = 2\sigma a = m_*(\eta). \tag{2.1.8}$$

Thus if $\varrho := \eta - \sigma x$ then $\eta = \varrho + \sigma x$ with $\varrho \in \ker D_x m$ and $\sigma x \in E_x x$. $\square$

2.2. Twisted conjugacy classes

In this section we recall the geometry of $K\tau$ as a K-manifold. As sources we use mostly the papers [Wen01], [MW04], and [Mei17]. From now on K is assumed to be a *simply connected* and *compact* Lie group. For our take on affine root systems see Section 1.1.

Let $T \subseteq K$ be a maximal torus, $\mathfrak{t} = \operatorname{Lie} T$ its Lie algebra, and $\Xi(T)$ its character group. For any $\chi \in \Xi(T)$ there is a unique $a_\chi \in \mathfrak{t}$ with

$$\chi(\exp \xi) = e^{2\pi i \langle a_\chi, \xi \rangle} \quad \text{for all } \xi \in \mathfrak{t}. \tag{2.2.1}$$

Then $\chi \mapsto a_\chi$ identifies $\Xi(T)$ with a lattice in $\mathfrak{t}$. In particular, the root system $\overline{\Phi}(\mathfrak{k}, \mathfrak{t}) \subseteq \Xi(T)$ of $\mathfrak{k}$ can be considered as a subset of $\mathfrak{t}$.

2.2.1 Theorem. *Let K be a simply connected compact Lie group K and τ an automorphism of K. Then there is a τ-stable maximal torus $T \subseteq K$ and an integral affine root system $(\Phi_\tau, \Lambda_\tau)$ on $\mathfrak{a} = \mathfrak{t}^\tau$, the τ-fixed part of $\mathfrak{t} = \operatorname{Lie} T$, with the following properties:*

a) (Comparison) Let $\operatorname{pr}_\mathfrak{a} : \mathfrak{t} \to \mathfrak{a}$ be the orthogonal projection. Then $\overline{\Phi}_\tau = \operatorname{pr}_\mathfrak{a} \overline{\Phi}(\mathfrak{k}, \mathfrak{t})$ and $\Lambda_\tau = \operatorname{pr}_\mathfrak{a} \Xi(T)$. Moreover, $\Lambda_\tau = \langle \overline{\Phi}_\tau^\vee \rangle^\vee$ (the weight lattice of *Φ_τ).*

b) (Orbit space) Let $\mathcal{A} \subseteq \mathfrak{a}$ be an alcove of Φ_τ. Then the composition

$$c : \mathcal{A} \subseteq \mathfrak{a} \xrightarrow{\exp} K \to K\tau/K \tag{2.2.2}$$

is a homeomorphism.

c) (Orbits) For $a \in \mathcal{A}$ let $u = \exp(a)$,

$$K_{a\tau} := \{k \in K \mid k *_\tau u = u\}, \tag{2.2.3}$$

and

$$\Phi_\tau(a) := \{\alpha \in \Phi_\tau \mid \alpha(a) = 0\}. \tag{2.2.4}$$

Then $K_{a\tau}$ is a connected subgroup of K with maximal torus $\exp \mathfrak{a}$ and root datum $(\overline{\Phi_\tau(a)}, \Lambda_\tau)$.

Proof. Morally, all assertions are contained in the papers [Wen01], [MW04], and [Mei17] but it is a bit unclear to which degree of generality they are proven. Unproblematic is the case when K is simple and τ is a diagram automorphism, i.e., is induced by an automorphism of the Dynkin diagram:

Parts *a)* and *b)* follow from the discussion in [MW04, §3] while *c)* is [MW04, §4].

The general case follows easily.

Step 1: Reduction to "K simple". If K decomposes into a nontrivial product $K_1 \times K_2$ with τ-invariant factors then all assertions reduce to the factors K_i. So assume that this is not the case. Then τ permutes the simple factors of K cyclically. Thus we may assume that $K = K_0 \times \cdots \times K_0$ (with m simple factors) and that τ acts as

$$ {}^{\tau}(k_1, k_2, \ldots, k_m) = (k_2, \ldots, k_m, {}^{\tau_0}k_1) \tag{2.2.5} $$

where $\tau_0 \in \operatorname{Aut} K_0$. The τ-twisted action then has the form

$$ (k_i)\ (g_i)\ {}^{\tau}(k_i)^{-1} = (k_1 g_1 k_2^{-1}, \ldots, k_{m-1} g_{m-1} k_m^{-1}, k_m g_m {}^{\tau_0}k_1^{-1}). \tag{2.2.6} $$

It follows that the twisted action of $1 \times K_0^{m-1} \subseteq K$ is free with quotient map

$$ K\tau \to K_0\tau_0 : (g_1, \ldots, g_m) \mapsto g_1 \ldots g_m \tag{2.2.7} $$

which is equivariant with respect to the first copy of K_0. This implies easily that $T = T_0^m$, $\mathfrak{a} = \mathfrak{a}_0 := \mathfrak{t}_0^{\tau_0}$ diagonally embedded in $\mathfrak{a}_0^m \subseteq \mathfrak{t}$, $\Lambda_\tau = \Lambda_{\tau_0}$, and $\Phi_\tau = m^*\Phi_{\tau_0}$ have all required properties, where $(m^*\alpha)(a) := \frac{1}{m}\alpha(ma)$.

Step 2: Reduction to "τ diagram automorphism". Fix a maximal torus $T_0 \subseteq K$ and consider all diagram automorphism with respect to T_0 (and a choice of positive roots). Since these represent all classes of $\operatorname{Out} K$ there is $h \in K$ and a diagram automorphism τ_0 with $\tau = \operatorname{Ad}(h) \circ \tau_0$. Let $\mathcal{A}_0$ be an alcove for Φ_{τ_0}. Then, by *b)*, there are $a \in \mathcal{A}_0$ and $t \in K$ with $h = t *_{\tau_0} u$ where $u := \exp a$. Then a short calculation yields

$$ \tau = \operatorname{Ad}(t)\ \operatorname{Ad}(u)\tau_0\ \operatorname{Ad}(t)^{-1}. \tag{2.2.8} $$

The automorphism $\tau_1 := \operatorname{Ad}(u)\tau_0 = \tau_0 \operatorname{Ad}(u)$ is intertwined with τ_0 via right translation by u. So, Φ_{τ_1} is the root system Φ_{τ_0} translated by $-a$ living on the same torus $T_1 = T_0$. Finally, put $T = \operatorname{Ad}(t)T_0$ and define Φ_τ on T by "transport of structure". □

2.2.2 Remark. The quadruple $(T, \mathfrak{a}, \Phi_\tau, \Lambda_\tau)$ is actually uniquely determined by K and τ up to conjugation by $K_\tau = K^\tau$. To see this, observe that $\mathfrak{a}$ is, by *c)*, a Cartan subspace of $\mathfrak{k}^\tau$, hence unique up to conjugation by K^τ. Then there is only one choice for T namely the centralizer of $\mathfrak{a}$ in K. For this one has to prove that $\mathfrak{a}$ is regular, i.e., no root of K vanishes on $\mathfrak{a}$. Since $a\tau$ with

$a \in \exp \mathfrak{a}$ and τ yield the same subspace $\mathfrak{a}$ one may assume that τ is a diagram automorphism. But then $\mathfrak{a}$ is defined by the equalities $\alpha = {}^\tau\alpha$ where α runs through all simple roots of K. Thus $\mathfrak{a}$ contains the sum of the fundamental coweights which clearly has a nonzero scalar product with every root. Finally assertion *b)* determines an alcove $\mathcal{A}$. Then Φ_τ and Λ_τ are again determined by property *c)*.

Next we describe the local structure of $K\tau$. For $a \in \mathcal{A}$ let $\mathcal{C}_a := \mathbb{R}_{\geq 0}(\mathcal{A}-a)$ be the tangent cone of $\mathcal{A}$ in a. It is a Weyl chamber of $\operatorname{Lie} K_{a\tau}$.

2.2.3 Corollary. *Keep the notation from Theorem 2.2.1 and let $a \in \mathcal{A}$. There exists an open neighborhood U of a in $\mathcal{A}$ such that $U - a$ is an open neighborhood of 0 in $\mathcal{C}_a$ and such that*

$$K \times^L \mathfrak{l}_{U-a} \xrightarrow{\sim} (K\tau)_U : [k,\xi] \mapsto k *_\tau (\exp(\xi)\exp(a)) \tag{2.2.9}$$

is a K-diffeomorphism where $L = K_{a\tau}$, $\mathfrak{l}_{U-a} := \mathfrak{l} \times_{\mathcal{C}_a} (U-a)$, and $(K\tau)_U := K\tau \times_{\mathcal{A}} U$.

Proof. Consider the twisted orbit $Y := K *_\tau u$ of $u := \exp(a)$ in $K\tau$. The isotropy group of u is L. Because of $\tau(u) = u$ we have $u *_\tau u = u$ and therefore $u \in L$. Then it has been shown in [Mei17, Prop. 2.5] that $L\tau \subseteq K\tau$ is a slice of Y in u. This means that $L\tau$ is L-stable and that it is transversal to Y in u. The slice theorem implies that $K \times^L L\tau \to K\tau : [k, l\tau] \mapsto (k *_\tau l)\tau$ is a diffeomorphism on a K-invariant neighborhood of $K \times^L \{u\}$.

Now observe that $l \in L$ means $lu\,{}^\tau l^{-1} = l *_\tau u = u$ and therefore ${}^\tau l = u^{-1}lu$. Hence the τ-twisted action of L on itself is conjugation twisted by $\operatorname{Ad} u^{-1}$. Thus, the map $L\tau \to L : l\tau \mapsto lu^{-1}$ intertwines the twisted conjugation on L with the usual conjugation action. Moreover, it sends u to $e \in L$. This implies that the map $K \times^L L \to K\tau : [k,l] \mapsto k *_\tau (lu)$ is a diffeomorphism near $K \times^L \{e\}$ where L acts on itself by conjugation. The assertion follows now from the fact that $\exp : \mathfrak{l} \to L$ is a diffeomorphism in a small open neighborhood $\mathfrak{l}_{U-a}$ of 0. $\square$

2.2.4 Remarks. *a)* The type of the root system Φ_τ depends only on the image $\overline{\tau}$ of τ in $\operatorname{Out}(K)$. If K is simple let X_n with $\mathsf{X} \in \{\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D},\mathsf{E},\mathsf{F},\mathsf{G}\}$ be the type of the Dynkin diagram of K. Then Φ_τ is the affine root system of type $X_n^{(r)}$ in the notation of [Kac90] where $r \in \{1,2,3\}$ is the order of $\overline{\tau}$. The general case is reduced to this one by the first reduction in the proof of Theorem 2.2.1.

b) As seen in the proof of Theorem 2.2.1, multiplying τ with $\operatorname{Ad} u$, where $u = \exp(a)$ with $a \in \mathcal{A}$ results in a translation of the root system Φ_τ by $-a$.

Accordingly the alcove $\mathcal{A}$ will also be translated which means that 0 may no longer be a vertex. Take, e.g., $K = \mathrm{SU}(2)$ and let ${}^{\tau}k = \overline{k}$ (complex conjugation). Then τ is also conjugation by $j := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus we can take $T = \mathrm{SO}(2)$ and have $\mathfrak{a} = \mathfrak{t} = \mathbb{R}j$. Since $\mathcal{A}_{\mathrm{id}_K} = [0, \pi]j$ and $j = \exp(\frac{\pi}{2}j)$ we have $\mathcal{A}_\tau = \mathcal{A}_{\mathrm{id}_K} - \frac{\pi}{2}j = [-\frac{\pi}{2}, \frac{\pi}{2}]j$. Thus $\exp \mathcal{A}_\tau$ is a fundamental domain for the twisted action $k *_\tau g = kg\overline{k}^{-1} = kgk^t$.

2.3. The local structure of quasi-Hamiltonian manifolds

Now we describe how to transfer the results on the local structure of $K\tau$ to quasi-Hamiltonian manifolds. Thereby we follow mostly [AMM98] and [Mei17].

Let $m : M \to K\tau$ be the momentum map of a quasi-Hamiltonian manifold. Then the inverse of the homeomorphism $c : \mathcal{A} \to K\tau/K$ of (2.2.2) yields a map $m_+ = c^{-1} \circ \pi \circ m$, the *invariant momentum map*, which fits into the commutative diagram

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "m_+"] & K\tau \arrow[d, "\pi"] \\ \mathcal{A} \arrow[r, "c"] & K\tau/K \end{tikzcd} \tag{2.3.1}$$

The image $m_+(M) \subseteq \mathcal{A}$ will be called the *momentum image* of M and will be denoted by $\mathcal{P}_M$. Our first goal is to describe M locally over $\mathcal{P}_M$.

Let a, L and U be as in Corollary 2.2.3 and assume $m_+(M) \subseteq U$. Put

$$\log_L M := M \times_{K\tau} \mathfrak{l} = M \times_{(K\tau)_U} \mathfrak{l}_{U-a} \tag{2.3.2}$$

(with respect to the map $\mathfrak{l} \to K\tau : \xi \mapsto \exp(\xi)\exp(a)$). Then (2.2.9) shows that

$$K \times^L \log_L M \to M : [k, (m, \xi)] \mapsto km \tag{2.3.3}$$

is a homeomorphism. Using the two projections $\iota : \log_L M \hookrightarrow M$ and $m_0 : \log_L M \to \mathfrak{l} \cong \mathfrak{l}^*$ one defines the 2-form $\omega_0 = \iota^*\omega - m_0^*\widetilde{\omega}$ on $\log_L M$ where $\widetilde{\omega}$ is the 2-form on $\mathfrak{l}$ which was defined in [AMM98, Lemma 3.3].

2.3.1 Remark. Let (M_0, m_0, ω_0) be a Hamiltonian K-manifold. Then completely analogously to m_+ one can define the invariant momentum map $(m_0)_+$ for M_0. It has values in a Weyl chamber $\mathfrak{t}^+$ inside the dual Cartan subalgebra $\mathfrak{t}^*$ of $\mathfrak{k}$. For details see [Kno11, Eqn. (2.1)].

2.3.2 Theorem. *Let a, L and U be as in Corollary 2.2.3.*

a) Let (M, m, ω) be a quasi-Hamiltonian $K\tau$-manifold with $m_+(M) \subset U$. Then the triple $(\log_L M, m_0, \omega_0)$ is a Hamiltonian L-manifold.

b) $\log_L$ is an equivalence of the category of quasi-Hamiltonian $K\tau$-manifolds M with $m_+(M) \subseteq U$ and the category of Hamiltonian L-manifolds M_0 with $(m_0)_+(M_0) \subseteq U - a$ where in both categories the morphisms are the isomorphisms. Moreover, as a manifold, we have $M = K \times^L \log_L M$.

c) Let $m_+(M) \subseteq U$. Then $\log_L$ preserves the momentum image in the sense that $m_+(M) = (m_0)_+(\log_L M) + a$.

Proof. a) The construction of $\log_L M$ can be performed in three steps. For the first observe that $L = K_{a\tau}$ is τ-stable. Hence the inclusion $L\tau \hookrightarrow K\tau$ is L-equivariant. By [Mei17, Prop. 4.1] the preimage $M_2 := M \times_{K\tau} L\tau$ has the structure of a quasi-Hamiltonian $L\tau$-manifold. For the second step put $M_1 := M_2$ with momentum map changed to $x \mapsto m(x)u^{-1}$. Recall from the proof of Corollary 2.2.3 that the map $L\tau \to L : l\tau \mapsto lu^{-1}$ intertwines the twisted conjugation on L with the usual conjugation. Then M_1 is an untwisted quasi-Hamiltonian L-manifold. Finally, $\log_L M = M_0$ is the pull-back of M_1 via the exponential map $\exp : \mathfrak{l} \to L$. Now the assertion follows from [AMM98, Remark 3.3].

b) For the inverse functor we invert each of the three steps above separately. We start with a Hamiltonian L-manifold (M_0, m_0, ω_0) satisfying $(m_0)_+(M_0) \subseteq U - a$. Then according to [AMM98, Prop. 3.4] the triple $(M_1, m_1, \omega_1) := (M_0, \exp \circ q \circ m_0, \omega_0 + m_0^* \widetilde{\omega})$ is an (untwisted) quasi-Hamiltonian L-manifold with $(m_1)_+(M_1) \subseteq U - a$. Here q is the identification $\mathfrak{l}^* \xrightarrow{\sim} \mathfrak{l}$. Observe that Prop. 3.4 of [AMM98] is applicable since the exponential function is locally invertible on $\mathfrak{l}_{U-a}$. Thus, the functor $M_0 \mapsto M_1$ inverts the functor $M_1 \mapsto M_0$.

In the second step we put $(M_2, m_2, \omega_2) := (M_1, m_1 \cdot u, \omega_1)$. Then M_2 is an $\mathrm{Ad}(u^{-1})$-twisted quasi-Hamiltonian L-manifold with $(m_2)_+(M_2) \subseteq U$ satisfying $m_+(M) \subseteq U$. This inverts the functor $M_2 \mapsto M_1$.

It remains to invert the functor $M \mapsto M_2 = m^{-1}(L)$. This means in particular that we need to provide $M := K \times^L M_2$ with a quasi-Hamiltonian $K\tau$-structure whenever M_2 is an $\mathrm{Ad}\, u^{-1}$-twisted quasi-Hamiltonian L-manifold with $(m_2)_+(M_2) \subseteq U$.

To this end recall the double $D(K)$ of a group K from [AMM98, §3.2]. As a set it is $D(K) = K \times K$ with $K \times K$-action $(u, v) * (a, b) = (uav^{-1}, vbu^{-1})$ and $K \times K$-valued momentum map $m(a, b) = (ab, a^{-1}b^{-1})$. Now we consider $K\tau$ as a connected component of the semidirect product $\widetilde{K} := K \rtimes \mathbb{Z}\tau$. Then inside $\widetilde{K}$ products like $b \cdot \tau$ and $a(b \cdot \tau)a^{-1} = ab(\tau a^{-1} \tau^{-1})\tau = (a *_\tau b) \cdot \tau$ make sense. The latter formula shows that τ-twisted conjugation becomes ordinary conjugation on $K\tau$.

Now we define the *twisted double* $D_\tau(K)$ as the connected component $K\times K\tau$ inside $D(\widetilde{K})=\widetilde{K}\times\widetilde{K}$. Then $D_\tau(K)$ is $K\tau\times K\tau^{-1}$-quasi-Hamiltonian with $K\times K$-action

$$(u,v)*(a,b\cdot\tau)=(uav^{-1},vb\tau u^{-1})=(uav^{-1},vb\,{}^{\tau}u^{-1}\cdot\tau) \tag{2.3.4}$$

and momentum map

$$m_D(a,b\cdot\tau)=(ab\tau,a^{-1}\tau^{-1}b^{-1})=(ab\tau,a^{-1}\,{}^{\tau^{-1}}b^{-1}\cdot\tau^{-1}). \tag{2.3.5}$$

Let

$$Z:=m_D^{-1}(K\tau\times(L\tau)_U^{-1})=\{(b,c\tau)\in D_\tau(K)\mid c\,{}^{\tau}b\in(L\tau)_U\}. \tag{2.3.6}$$

where $(L\tau)_U$ is the open subset $L\tau\cap(K\tau)_U=L\tau\times_{\mathcal{A}}U$ of $L\tau$ and $(L\tau)_U^{-1}\subseteq K\tau^{-1}$ is the set of its inverses. Again by [Mei17, Prop. 4.1] this is a quasi-Hamiltonian $K\tau\times L\tau^{-1}$-manifold. One easily checks that the map

$$K\times(L\tau)_U\to Z:(a,l\tau)\mapsto(a,l\tau a^{-1}) \tag{2.3.7}$$

is invertible, yielding a quasi-Hamiltonian structure on $K\times(L\tau)_U$ with $K\tau\times L\tau^{-1}$-valued momentum map.

Now consider the fusion product $Z\otimes_L M_2$ with respect to the factor L (see [AMM98, §6]*.) which equals $Z\times M_2$ as a manifold and is $K\tau\times L$-quasi-Hamiltonian with momentum map

$$m_\otimes:(a,l\tau,x)\mapsto(al\tau a^{-1},(l\tau)^{-1}m(x))\in K\tau\times L \tag{2.3.8}$$

where we have identified Z with $K\times(L\tau)_U$ using (2.3.7). The q-Hamiltonian reduction

$$M':=(Z\otimes_L M_2)/\!\!/L:=m_\otimes^{-1}(K\tau\times\{1\})/L \tag{2.3.9}$$

defined in [AMM98, § 5]* with respect to L is a quasi-Hamiltonian $K\tau$-manifold. It is diffeomorphic to $M:=K\times^L M_2$ via the map

$$M\to M':[a,x]\mapsto(a,m(x),x). \tag{2.3.10}$$

*Strictly speaking, the paper [AMM98] deals only with the untwisted case but as explained above one may consider twisted $K\tau$-manifolds as open subsets of (untwisted) $\widetilde{K}$-manifolds to which [AMM98] applies.

This provides M with a $K\tau$-quasi-Hamiltonian structure. Since all constructions are functorial for isomorphisms we have defined a functor $M_2 \mapsto M$. It remains to show that it is quasiinverse to the functor $M \mapsto M_2$ defined above.

We start with the composition $M_2 \mapsto M \mapsto M_2$ which needs to be isomorphic to the identity functor. For this, let M_2 be as above. Then it suffices to show that the map

$$\varphi : M_2 \to (K \times^L M_2) \times_{K\tau} L\tau : x \mapsto ([1, x], m_2(x)) \tag{2.3.11}$$

is an isomorphism of quasi-Hamiltonian $L\tau$-manifolds. Tracing through all definitions one sees easily that φ is a diffeomorphism which is compatible with the L-actions and the momentum maps. It is more difficult to see that that the 2-forms match up, as well.

For this observe that $x \in M_2$ is mapped by φ to the L-orbit of

$$(1, m_2(x), x) \in K \times L \times M_2 \subseteq D_\tau(K) \times M_2. \tag{2.3.12}$$

Recall the explicit formula of [AMM98, Thm. 6.1] for the 2-form on the fusion product of two quasi-Hamiltonian manifolds (M', ω', m') and (M'', ω'', m''):

$$\pi_1^* \omega' + \pi_2^* \omega'' + \tfrac{1}{2} \langle m'^* \theta, m''^* \overline{\theta} \rangle. \tag{2.3.13}$$

We apply this formula to $M' = Z$ and $M'' = M_2$. Let $\iota : L \to L : h \mapsto h^{-1}$ be the inversion. Because of $\iota^* \theta = -\overline{\theta}$ the pull-back of the third term in (2.3.13) to M_2 vanishes. To see that also the first summand vanishes on M_2 we look at the explicit form of ω_D on $D(\widetilde{K})$ (see [AMM98, Prop. 3.2]):

$$\omega_D = \tfrac{1}{2} \langle p_1^* \theta, p_2^* \overline{\theta} \rangle + \tfrac{1}{2} \langle p_1^* \overline{\theta}, p_2^* \theta \rangle \tag{2.3.14}$$

where p_1, p_2 are the two projections of $D(\widetilde{K})$ to $\widetilde{K}$. The map from M_2 to $Z \subseteq D_\tau(K)$ is $x \mapsto (1, m_2(x))$. Hence p_1 is constant on M_2 implying that the pull-backs of $p_1^* \theta$ and $p_1^* \overline{\theta}$, hence of ω_D to M_2 vanish. This finishes the proof that φ is an isomorphism of quasi-Hamiltonian manifolds.

Now let (M, m, ω) be given with $m_*(M) \subseteq U$ and $M_2 := M \times_{K\tau} L\tau$ as in part *a)*. Then we show that

$$\psi : K \times^L M_2 \to M : [k, x] \mapsto kx \tag{2.3.15}$$

is an isomorphism of quasi-Hamiltonian $K\tau$-manifolds. As above it is easy to check that ψ is a K-equivariant diffeomorphism which is compatible with the momentum maps. At stake are again the 2-forms.

The map ψ defines a second 2-form ω' on M such that (M, m, ω') is quasi-Hamiltonian. Moreover, by the previous discussion the pull-backs of ω and ω' to M_2 coincide. Thus, to show $\omega = \omega'$ it suffices to show that ω is uniquely determined by m and its pullback to M_2. Let $x \in M$. By K-invariance we may assume that $x \in M_2$. The momentum map property *c)* of Definition 2.1.2 allows to compute $\omega_x(\xi, \eta)$ where $\xi \in \mathfrak{k}x$ and $\eta \in T_xM$. Moreover, also $\omega_x(\xi, \eta)$ is known for $\xi, \eta \in T_xM_2$. Because of $\mathfrak{k}x + T_xM_2 = T_xM$ we proved our assertion.

c) is obvious. □

Theorem 2.3.2 can be used to analyze the local structure of an arbitrary M. To enforce the requirement $m_+(M) \subseteq U$ we apply the theorem to the open part

$$M_U := m_+^{-1}(U) = M \times_{\mathcal{A}} U \subseteq M. \tag{2.3.16}$$

There is a subtlety however in that M_U might not be connected despite M being so. This is a problem since some fundamental structure theorems like convexity of the momentum image $m_+(M)$ or connectedness of fibers of m depend crucially on connectedness of M. Therefore we impose an extra condition on M which is automatic in case M is compact.

2.3.3 Definition. A (quasi-)Hamiltonian manifold (M, m) is *locally convex* if all fibers of m_+ are connected and $m_+ : M \to m_+(M)$ is an open map. It is *convex* if in addition $\mathcal{P}_M = m_+(M)$ is convex.

2.3.4 Lemma. *Let M be a (quasi-)Hamiltonian manifold.*

a) If M is locally convex then M_U is locally convex for all $U \subseteq \mathcal{A}$ open. If additionally, $\mathcal{P}_M \cap U$ is connected then M_U is connected.

b) If M is locally convex then the momentum image $\mathcal{P}_M$ is locally polyhedral (hence locally convex, locally closed, and locally connected). In particular, every $a \in \mathcal{P}_M$ has a neighborhood U such that $\mathcal{P}_M \cap U$, and therefore M_U, is convex.

c) Assume that all fibers of m_+ are connected and that m_+ is proper onto $\mathcal{P}_M$. Then M is locally convex.

d) Assume M is compact. Then M is convex and $\mathcal{P}_M$ is a polytope.

Proof. *a)* is easy point set topology.

b) Using Theorem 2.3.2 and *a)* we may assume that M is Hamiltonian. Then [Kno02, Cor. 2.8, Thm. 5.1] implies that every $x \in M$ has a K-invariant open neighborhood $V \subseteq M$ such that $m_+(V)$ is open in C_x where C_x is a polyhedral cone with vertex $m_+(x)$. Since m_+ is open, $m_+(V)$ is also open in $m_+(M)$.

c) It suffices to show that every $x \in M$ has an open neighborhood $W \subseteq M$ such that $W \to m_+(M)$ is an open map. Again by Theorem 2.3.2 we may assume that M is Hamiltonian (possibly not connected). Choose a neighborhood V_x of x and let $a = m_+(x)$ and C_x be as above. By [Kno02, Cor. 6.1], the cone C_x is locally constant in $x \in m_+^{-1}(a)$. Since the fiber $F = m_+^{-1}(a)$ is connected this implies that $C = C_x$ is the same for all $x \in F$. The fiber F being compact, there are finitely many $x_1, \ldots, x_n \in F$ with $F \subseteq V := V_{x_1} \cup \cdots \cup V_{x_n}$. Because all restrictions $V_{x_i} \to C$ of m_+ are open, the same holds for V. Because of properness, the set $m_+(M \setminus V)$ is closed in $m_+(M)$. Hence the complement

$$U = m_+(M) \setminus m_+(M \setminus V) = \{b \in m_+(M) \mid m^{-1}(b) \subseteq V\} \tag{2.3.17}$$

is an open neighborhood of a in $m_+(M)$ with $U \subseteq m_+(V)$. Thus, $W := m_+^{-1}(U) \subseteq V$ has the required property.

d) This is [Mei17, Thm. 4.4]. Openness of m_+ follows from *c)*. □

Besides the momentum image $\mathcal{P}_M$, the most important invariant of a (quasi)-Hamiltonian is its principal isotropy group. We briefly recall its definition. Before that we need to make the following

2.3.5 Remark. Assume M is quasi-Hamiltonian with $m_+(M) \subseteq \mathcal{F}$ where $\mathcal{F}$ is a face of $\mathcal{A}$. Let $a \in m_+(M)$ and let $U \subseteq \mathcal{A}$ be as in Theorem 2.3.2. Then $m^{-1}((L\tau)_U)$ and $\log_L M$ actually only depend on the intersection of U with $\mathcal{F}$. Therefore, one may as well regard U as an open subset of $\mathcal{F}$ which can be extended to an open subset of $\mathcal{A}$ if needed.

2.3.6 Lemma. *For a connected, locally convex quasi-Hamiltonian $K\tau$-manifold let $\mathcal{F} \subseteq \mathcal{A}$ be the smallest face of $\mathcal{A}$ containing $\mathcal{P}_M$ and let $U \subseteq \mathcal{F}$ be its relative interior. Then $L_M := K_{a\tau}$ is independent of $a \in U$. The preimage $M_0 := m^{-1}(L\tau) \cap m_+^{-1}(U)$ is a connected quasi-Hamiltonian L_M-manifold such that $K \times^{L_M} M_0 \to M$ is a diffeomorphism onto an open dense subset of M. Let $L'_M \subseteq L_M$ be the kernel of $L_M \to \mathrm{Aut}(M_0)$. Then $A_M = L_M/L'_M$ is a torus acting freely on a dense open subset of M_0. In particular, L'_M is a generic isotropy group for the K-action on M.*

Proof. All assertions are local over $\mathcal{P}_M$. Theorem 2.3.2 implies therefore that we may assume that M is Hamiltonian. But in this context, all assertions have been proved in [LMTW98]. □

In practice, it is more convenient to encode L'_M by the character group $\Lambda_M := \Xi(A_M)$. Let $A \subseteq L_M$ be the subtorus with $\mathrm{Lie}\, A = \mathfrak{a} = \mathfrak{t}^\tau$. Then

$A \to A_M$ is surjective and we can consider Λ_M as a subgroup of $\Xi(A) \otimes \mathbb{R} = \mathfrak{a}^* = \mathfrak{a}$.

The two objects $\mathcal{P}_M$ and Λ_M are not unrelated: Let $\mathfrak{a}_M \subseteq \mathfrak{a}$ be the affine span of $\mathcal{P}_M$. Since $\mathfrak{a}_{M_0} = \mathfrak{a}_M - a = \overline{\mathfrak{a}}_M$ and $\Lambda_{M_0} = \Lambda_M$ we see that Λ_M is a lattice inside $\overline{\mathfrak{a}}_M$. Its dimension will be called the *rank* $\operatorname{rk} M$ of M.

2.4. Multiplicity free manifolds

We can use Lemma 2.3.6 to compute the dimension of M where M is connected, locally convex and quasi-Hamiltonian. Since m_+ is K-invariant we get an induced map $M/K \to \mathcal{P}_M$ which is surjective by definition. This implies the inequality

$$\dim M/K \geq \dim \mathcal{P}_M = \operatorname{rk} M. \tag{2.4.1}$$

On the other side, the generic isotropy group L'_M has dimension $\dim L_M - \dim A_M = \dim L_M - \operatorname{rk} M$. Thus (2.4.1) is equivalent to

$$\dim M \geq \dim K - \dim L_M + 2 \operatorname{rk} M. \tag{2.4.2}$$

This suggests the following

2.4.1 Definition. A quasi-Hamiltonian manifold M is *multiplicity free* if it is connected, locally convex, and the equivalent inequalities (2.4.1) and (2.4.2) are actually equalities.

Observe, that this is a verbatim generalization of the concept of multiplicity freeness of ordinary Hamiltonian manifolds (see, e.g., [Kno11, Def. 2.1]). As in that case, multiplicity freeness for quasi-Hamiltonian manifolds has numerous equivalent characterizations. The following statement mentions a couple of them. Thereby we adopt the convention that a submanifold $N \subseteq M$ is called *coisotropic* if $(T_xN)^\perp \subseteq T_xN$ for all $x \in N$ even in the case that the 2-form ω is degenerate.

2.4.2 Proposition. *For a connected, locally convex quasi-Hamiltonian manifold M the following are equivalent:*

a) M is multiplicity free.
b) The induced map $m_+/K : M/K \to \mathcal{P}_M$ is a homeomorphism.
c) $\mathfrak{k}_{m(x)}x = \ker D_xm$ for x in a non-empty open subset of M.
d) The orbit Kx is coisotropic for x in a non-empty open subset of M.
e) $\log_L M$ is a multiplicity free Hamiltonian manifold for all (or one) open subset(s) U as in Theorem 2.3.2 with $U \cap \mathcal{P}_M$ connected.

Proof. *a)*⇔*e)* Let $M_0 := \log_L M$ which is connected by Lemma 2.3.4*a)*. Since there are open embeddings $M_0/L \hookrightarrow M/K$ and $\mathcal{P}_{M_0} \hookrightarrow \mathcal{P}_M - a$ we get that $\dim M/K = \dim \mathcal{P}_M$ if and only if $\dim M_0/L = \dim \mathcal{P}_{M_0}$.

a)⇔*b)* The manifold M is by definition multiplicity free if and only if $\dim M/K = \dim \mathcal{P}_M$. Since the fibers of m_+ are connected, this is equivalent to m_+/K being bijective over a non-empty open subset U of $\mathcal{P}_M$. Now it suffices to prove that m_+/K is (globally) injective since m_+/K is already surjective and open. Using Theorem 2.3.2 we may assume that M is actually Hamiltonian. Let $V \subseteq \mathcal{P}_M$ the non-empty set of points in which m_+/K is locally invertible. It follows from the symplectic slice theorem (see e.g. [GS05, 2.3]) that m_+/K is locally analytic (even semialgebraic). It follows that $(m_+/K)^{-1}(V)$ is open and dense in M/K. Since M/K is connected it suffices to show that V is closed. So let $a \in \mathcal{P}_M$, $b \in F := (m_+/K)^{-1}(a)$ and $a_i \in V$ a sequence so that the preimages $b_i \in M/K$ converge to b. Consider the open subset $U := M/K \setminus \{b_i \mid i\} \cup \{b\}$. Then the image $(m_+/K)(U)$ is an open subset of $\mathcal{P}$ which does not contain the a_i. Thus it does not contain a either, i.e., m_+/K is injective over a.

a)⇔*c)*: It follows from Lemma 2.3.6 that the generic orbit in $m(M)$ is isomorphic to K/L_M and that $\dim m(M)/K = \dim \mathcal{P} = \operatorname{rk} M$. Thus

$$\dim m(M) = \dim K - \dim L_M + \operatorname{rk} M. \tag{2.4.3}$$

Thus, M is multiplicity free if and only if $\dim m(M) = \dim - \operatorname{rk} M$ or, equivalently, $\dim F_x = \operatorname{rk} M$ where $F_x = m^{-1}(m(x)) \subseteq M_0$ is a generic fiber of m. On the other side, the isotropy group $K_{m(x)}$ is conjugate to L_M which means that the generic $K_{m(x)}$-orbits in F_x have dimension $\operatorname{rk} M$. Thus M is multiplicity free if and only if $K_{m(x)}$ has open orbits in F_x which is precisely *c)*.

c)⇔*d)*: Condition *c)* is equivalent to $\ker D_x m \subseteq \mathfrak{k}x$. Hence the equivalence with *d)* follows from Lemma 2.1.4. □

2.4.3 Remark. Generally, the fiber of m_+/K over $a \in \mathcal{P}_M$ is called the *symplectic reduction of M in a*. So, another characterization of multiplicity freeness of a locally convex, connected M is that all its symplectic reductions are points.

2.5. Local models

In this section we make the first step towards the goal of classifying multiplicity free manifolds. More precisely, we characterize the pairs $(\mathcal{P}, \Lambda)$ which eventually will be shown to be those of the form $(\mathcal{P}_M, \Lambda_M)$ for M multiplicity free.

Let $G = K^{\mathbb{C}}$ be the complexification of K which is a connected complex reductive group. Let $B \subseteq G$ be a Borel subgroup containing a maximal torus $T \subseteq K$ and let $\Xi^{\mathbb{C}}(B) := \mathrm{Hom}(B, \mathbb{C}^*) \cong \mathbb{Z}^{\mathrm{rk}\, G}$ be its algebraic character group. Since $\Xi^{\mathbb{C}}(B)$ can be identified with $\Xi(T)$ the space $\Xi^{\mathbb{C}}(B) \otimes \mathbb{R}$ identifies with $\mathfrak{t}^*$ and therefore with $\mathfrak{t}$. The choice of B determines a Weyl chamber $\mathfrak{t}^+ \subseteq \mathfrak{t}$ and characters lying in $\mathfrak{t}^+$ are called dominant. Recall, that there is a $1:1$-correspondence $\chi \mapsto L_\chi$ between dominant characters and irreducible representations of G.

An algebraic G-variety X is called *spherical* if B has a Zariski dense orbit in X. In the following we are only interested in the case when X is affine. Then there is a purely representation theoretic criterion for sphericity due to Vinberg-Kimel'fel'd, [VK78]: Let $\mathbb{C}[X]$ be the ring of regular functions on X. Then X is spherical if and only if $\mathbb{C}[X]$ is multiplicity free, i.e., it is a direct sum of *distinct* irreducible representations of G. Thus, for a spherical variety there is a well defined set $\Lambda_X^+ \subseteq \mathfrak{t}^+$ of dominant weights such that

$$\mathbb{C}[X] \cong \bigoplus_{\chi \in \Lambda_X^+} L_\chi. \tag{2.5.1}$$

2.5.1 Definition. For an affine spherical G-variety X let $\mathcal{P}_X := \mathbb{R}_{\geq 0}\Lambda^+$ be the convex cone and let $\Lambda_X := \mathbb{Z}\Lambda_X^+$ be the subgroup spanned by Λ_X^+ in $\mathfrak{t}^*$. Then $(\mathcal{P}_X, \Lambda_X)$ will be called the *spherical pair* determined by X.

Recall the notation of Theorem 2.2.1 and in particular that every $a \in \mathcal{A}$ determines a subgroup $K_{a\tau} \subseteq K$. Because $\mathfrak{a} \subseteq \mathfrak{k}_{a\tau}$ is a Cartan subalgebra, the characters of the complexification $K_{a\tau}^{\mathbb{C}}$ can be considered to be elements of $\mathfrak{a}$. Thus, if X is a spherical $K_{a\tau}^{\mathbb{C}}$-variety then $\mathcal{P}_X$ and Λ_X^+ will be subsets of $\mathfrak{a}$.

2.5.2 Definition. Let $\mathcal{P} \subseteq \mathcal{A}$ be a subset and let $\Lambda \subseteq \mathfrak{a}$ be a subgroup.

a) The pair $(\mathcal{P}, \Lambda)$ is *spherical in* $a \in \mathcal{P}$ if there is a *smooth* affine spherical $K_{a\tau}^{\mathbb{C}}$-variety X and a neighborhood U of a in $\mathfrak{a}$ such that

$$((\mathcal{P}_X + a) \cap U, \Lambda_X) = (\mathcal{P} \cap U, \Lambda). \tag{2.5.2}$$

b) The pair $(\mathcal{P}, \Lambda)$ is *spherical* if $\mathcal{P}$ is connected and $(\mathcal{P}, \Lambda)$ is spherical in all $a \in \mathcal{P}$.

c) The pair $(\mathcal{P}, \Lambda)$ is *convex* if $\mathcal{P}$ is convex.

First we remark that sphericity is a necessary condition:

2.5.3 Lemma. *Let M be a (convex and) multiplicity free quasi-Hamiltonian $K\tau$-manifold. Then $(\mathcal{P}_M, \Lambda_M)$ is (convex and) spherical.*

Proof. Using the Local Structure Theorem 2.3.2, the assertion is reduced to the Hamiltonian case where it is follows from [Kno11, Thm. 11.2]. □

The following remarks are not necessary to understand the proof of the classification theorem but they are useful for recognizing and constructing spherical pairs.

2.5.4 Remarks. Let $(\mathcal{P}, \Lambda)$ be a spherical pair.

a) Let $\mathfrak{a}_\mathcal{P} \subseteq \mathfrak{a}$ be the affine subspace spanned by $\mathcal{P}$. Then the group Λ is a lattice in the group of translations $\overline{\mathfrak{a}}_\mathcal{P}$ which in turn follows from (2.5.2) and the fact that $\mathcal{P}_X$ and Λ_X have the same $\mathbb{R}$-span.

b) The subset $\mathcal{P}$ is locally polyhedral and therefore locally convex, locally closed and solid inside $\mathfrak{a}_\mathcal{P}$. It follows, in particular, that $\mathcal{P}$ has a well defined dimension, namely $\dim \mathcal{P} := \dim \mathfrak{a}_\mathcal{P} = \operatorname{rk} \Lambda$. This follows from (2.5.2), the fact that $\mathcal{P}_X$ is a finitely generated cone and the connectedness of $\mathcal{P}$.

c) The *tangent cone* $C_a\mathcal{P}$ of $\mathcal{P}$ in a is defined as the convex cone generated in $\mathfrak{a}$ by $(\mathcal{P} \cap U) - a$ where $\mathcal{P} \cap U$ is a convex neighborhood of a in $\mathcal{P}$. It is easy to see that $C_a\mathcal{P}$ is independent of the choice of U. From Definition 2.5.2 follows

$$(\mathcal{P}_X, \Lambda_X) = (C_a\mathcal{P}, \Lambda). \tag{2.5.3}$$

which means that $(\mathcal{P}_X, \Lambda_X)$ is uniquely determined by $\mathcal{P}$, Λ and a.

d) For any affine spherical variety X the set Λ_X^+ is a monoid, i.e., is additively closed. The reason for this is that $\mathbb{C}[X]$ is a domain. If X is normal (e.g., smooth) then $\Lambda_X^+ = \mathcal{P}_X \cap \Lambda_X$. This means that Λ_X^+ and $(\mathcal{P}_X, \Lambda_X)$ determine each other.

e) It is a non-trivial theorem of Losev [Los09a, Thm. 1.3] that a smooth affine spherical G-variety is uniquely determined by its weight monoid Λ_X^+. With *c)* and *d)* this implies that the $K_{a\tau}^\mathbb{C}$-variety X from Definition 2.5.2 is uniquely determined by $\mathcal{P}, \Lambda$ and a. We call X the *local model* of $(\mathcal{P}, \Lambda)$ in a.

f) Assume $(\mathcal{P}, \Lambda) = (\mathcal{P}_M, \Lambda_M)$ for some multiplicity free quasi-Hamiltonian manifold M and let X be the local model in $a \in \mathcal{P}$. We sketch how a neighborhood of $m_+^{-1}(a)$ in M is described by X. Since X is affine, it can be embedded equivariantly into a $K_{a\tau}^\mathbb{C}$-vector space Z as a closed $K_{a\tau}^\mathbb{C}$-subvariety. The choice of a $K_{a\tau}$-invariant Hermitian scalar product on Z induces a Kähler-hence symplectic structure on X. With $\widetilde{m}(x) := (\xi \mapsto \frac{1}{2}\langle \xi x, x\rangle) \in \mathfrak{k}_{a\tau}^*$, the variety X becomes a Hamiltonian $K_{a\tau}$-manifold and $(\mathcal{P}_X, \Lambda_X)$ is also the pair attached to X when considered as a Hamiltonian manifold (see [Sja98, Thm. 4.9] or [Los09a, Prop. 8.6(3)]). Moreover, there is an open neighborhood U of a in $\mathfrak{a}$ such that

$$\begin{tikzcd} K \times^{K_{a\tau}} \widetilde{m}_+^{-1}(\mathcal{P}_X \cap (U-a)) \arrow[r, hook, "\text{open}"] \arrow[d, "\sim"'] & K \times^{K_{a\tau}} X \\ m_+^{-1}(\mathcal{P} \cap U) \arrow[r, hook, "\text{open}"] & M \end{tikzcd} \tag{2.5.4}$$

g) Affine spherical varieties have been classified by Knop-Van Steirteghem in [KVS06]. Pezzini-Van Steirteghem, [PVS19], developed an algorithm for deciding the sphericity of a pair $(\mathcal{P}, \Lambda)$ in any given point $a \in \mathcal{P}$.

h) For deciding sphericity in all points of $\mathcal{P}$ the following observations help:

2.5.5 Lemma. *The set of points $a \in \mathcal{P}$ where $(\mathcal{P}, \Lambda)$ is spherical is open in $\mathcal{P}$.*

Proof. Let a, U and X as in Definition 2.5.2*a)*. Recall from Remark 2.5.4 *f)* that X carries a structure as multiplicity free Hamiltonian K-manifold with pair $(\mathcal{P}_X, \Lambda_X)$. Now it follows from [Kno11, Thm. 11.2] applied to X that every point of $\mathcal{P} \cap U = (\mathcal{P}_X + a) \cap U$ is spherical. ☐

The tangent cone is constant on open intervals:

2.5.6 Lemma. *Let $\mathcal{P} \subseteq \mathfrak{a}$ be a locally convex subset and let $I \subseteq \mathcal{P}$ be an open interval. Then the tangent cone $C_a\mathcal{P}$ is the same for all $a \in I$.*

Proof. It suffices to show that $a \mapsto C_a\mathcal{P}$ is locally constant on I. Fix $a \in I$. Since the tangent cone depends only on a neighborhood of a we can replace $\mathcal{P}$ by a convex neighborhood $\mathcal{P} \cap U$ and therefore assume that $\mathcal{P}$ is convex. Now let $b \in I$ be so close to a that the two points $b' := b + (b-a)$ and $a' := a + (a-b)$ still lie in I. Then

$$\begin{aligned} C_a\mathcal{P} &= \mathbb{R}_{\geq 0}(\mathcal{P} - a) = \mathbb{R}_{\geq 0}(\mathcal{P} - b + (b' - b)) \\ &\subseteq \mathbb{R}_{\geq 0}(\mathcal{P} - b) + \mathbb{R}_{\geq 0}(b' - b) \subseteq C_b\mathcal{P} \end{aligned} \tag{2.5.5}$$

and similarly $C_b\mathcal{P} \subseteq C_a\mathcal{P}$. ☐

Now for compact sets $\mathcal{P}$ we have the following criterion:

2.5.7 Proposition. *Let $\mathcal{P} \subseteq \mathcal{A}$ be compact. Then a pair $(\mathcal{P}, \Lambda)$ is spherical if and only if $\mathcal{P}$ is a polytope and the pair $(\mathcal{P}, \Lambda)$ is spherical in every vertex of $\mathcal{P}$.*

Proof. Assume first that $(\mathcal{P}, \Lambda)$ is spherical. Then $\mathcal{P}$ is a polytope by Lemma 2.5.8 below.

Now assume conversely that $\mathcal{P}$ is a polytope such that $(\mathcal{P}, \Lambda)$ is spherical in every extremal point. Then it follows from the preceding two lemmas that the set of spherical points is convex. Since it contains all extremal points it is all of $\mathcal{P}$. □

2.5.8 Lemma. *Let $\mathfrak{a}$ be an affine space and let $\mathcal{P} \subseteq \mathfrak{a}$ be locally polyhedral, compact, and connected. Then $\mathcal{P}$ is a polytope.*

Proof. Since $\mathcal{P}$ is closed, connected, and locally convex it is convex (Tietze [Tie28, Satz 1]). It follows that $\mathcal{P}$ is the convex hull of its extremal points (Krein–Milman). But there are only finitely many extremal points since $\mathcal{P}$ is locally polyhedral. So $\mathcal{P}$ is a polytope. □

2.6. Classification of multiplicity free quasi-Hamiltonian manifolds

Now we can tie all strings together and prove our main theorem.

2.6.1 Theorem. *Let K be a simply connected compact Lie group with twist τ and let $\mathcal{A} \subseteq \mathfrak{a}$ be as in Theorem 2.2.1. Let $\mathcal{P} \subseteq \mathcal{A}$ be a subset and $\Lambda \subseteq \mathfrak{a}$ a subgroup. Then there is a convex multiplicity free quasi-Hamiltonian $K\tau$-manifold M with $(\mathcal{P}_M, \Lambda_M) = (\mathcal{P}, \Lambda)$ if and only if $(\mathcal{P}, \Lambda)$ is convex and spherical. Moreover, this M is unique up to isomorphism.*

For compact multiplicity free quasi-Hamiltonian manifolds this means:

2.6.2 Corollary. *Compact multiplicity free quasi-Hamiltonian $K\tau$-manifolds are classified by pairs $(\mathcal{P}, \Lambda)$ for which $\mathcal{P} \subseteq \mathcal{A}$ is a polytope and $(\mathcal{P}, \Lambda)$ is spherical in every vertex of $\mathcal{P}$.*

Proof. Follows from Theorem 2.6.1 using Lemma 2.3.4*d)* and Proposition 2.5.7. □

We reduce the proof of the main Theorem 2.6.1 to a statement about automorphisms. For this we use the language of *gerbes*.

Let $(\mathcal{P}, \Lambda)$ be a spherical pair. For any open subset $U \subseteq \mathcal{P}$ let $\mathsf{M}_{\mathcal{P},\Lambda}(U)$ be the category whose objects are multiplicity free quasi-Hamiltonian $K\tau$-manifolds M with $(\mathcal{P}_M, \Lambda_M) = (U, \Lambda)$. The morphisms are K-equivariant diffeomorphisms $\varphi : M \to M'$ such that $\omega = \varphi^*\omega'$ and that $m = m' \circ \varphi$. Since all morphisms are invertible, $\mathsf{M}_{\mathcal{P},\Lambda}(U)$ is a *groupoid*.

For any pair of open subsets $U \subseteq V \subseteq \mathcal{P}$ the groupoids are linked by restriction functors $\operatorname{res}^V_U : \mathsf{M}_{\mathcal{P},\Lambda}(V) \to \mathsf{M}_{\mathcal{P},\Lambda}(U) : M \mapsto m^{-1}(U)$ which satisfy $\operatorname{res}^V_U \circ \operatorname{res}^W_V = \operatorname{res}^W_U$ whenever $U \subseteq V \subseteq W$.

This means that $\mathsf{M}_{\mathcal{P},\Lambda}$ is a *presheaf of groupoids* over $\mathcal{P}$. Since all morphisms and objects can be glued along any gluing data, the system of categories $\mathsf{M}_{\mathcal{P},\Lambda}$ is even a *sheaf of groupoids*, also known as *stack* (see e.g. [Bry93, Def. 5.2.1] for a precise definition). A very particular kind of stacks are *gerbes* which means that they have the following two additional properties:

a) M is locally non-empty, i.e. every point $a \in \mathcal{P}$ has an open neighborhood $U \subseteq \mathcal{P}$ such $\mathsf{M}(U) \neq \varnothing$ and

b) any two objects $M, M' \in \mathsf{M}(V)$ with $V \subseteq \mathcal{P}$ open are locally isomorphic, i.e., every $a \in V$ has an open neighborhood $U \subseteq V$ such that $\operatorname{res}^V_U M \cong \operatorname{res}^V_U M'$.

See e.g. [Bry93, Def. 5.2.4] (where the definition of a gerbe is combined with that of a band) or [SP23, Def. 8.11.2]. In the next two proofs we extensively use the local structure theorem 2.3.2 and the obvious fact that if it holds for an open set U then it will also hold for all smaller open subsets V regardless of whether V contains the base point a or not.

2.6.3 Theorem. *Let $(\mathcal{P}, \Lambda)$ be spherical. Then $\mathsf{M}_{\mathcal{P},\Lambda}$ is a gerbe over $\mathcal{P}$.*

Proof. We have to prove that *a)* and *b)* hold for $\mathsf{M}_{\mathcal{P},\Lambda}$.

Hereby, *a)* is basically the definition of a spherical pair: let X be a smooth affine spherical variety with (2.5.2). Then X has a K-Hamiltonian structure such that $(\mathcal{P}_X, \Lambda_X) = (C_a\mathcal{P}, \Lambda)$ (cf. remark 2.5.4 *f)*). Choosing U small enough as in Theorem 2.3.2, there is a quasi-Hamiltonian manifold M with $m_+(M) = (a + C_aP) \cap U$ and $\log_L M = X_{U-a}$. Then M has the required properties.

The second assertion *b)* follows using Theorem 2.3.2 from [Los09b, Thm. 1.3] to the effect that the local model X is uniquely determined by Λ^+_X (cf. 2.5.4 *e)* and [Kno11, Thm. 2.4]). □

A particular nice type of gerbes are those for which the automorphism group of every object is abelian. In this case, the automorphism groups combine to a sheaf of abelian groups $\mathfrak{L}_{\mathsf{M}}$ on $\mathcal{P}$, the so-called *band* of M. More precisely, let $M, M' \in \mathsf{M}(U)$ be two objects over U and $\varphi : M \xrightarrow{\sim} M'$ an isomorphisms. Then φ induces an isomorphism $\widetilde{\varphi} : \operatorname{Aut}(M) \xrightarrow{\sim} \operatorname{Aut}(M')$ by $\widetilde{\varphi}(f) = \varphi f \varphi^{-1}$. If $\psi : M \xrightarrow{\sim} M'$ is another isomorphism then one checks easily that $\widetilde{\psi}^{-1}\widetilde{\varphi} \in \operatorname{Aut}(\operatorname{Aut}(M))$ is conjugation by $\psi^{-1}\varphi \in \operatorname{Aut}(M)$. Thus, if $\operatorname{Aut}(M)$ is abelian, then $\operatorname{Aut}(M)$ depends only on the isomorphism class of M. This means, that $\mathfrak{L}^{\#}_{\mathsf{M}}(U) := \mathrm{Aut}(M)$ with $M \in \mathsf{M}(U)$ is a well-defined presheaf of abelian groups on $\mathcal{P}$. The band $\mathfrak{L}_{\mathsf{M}}$ is by definition the sheafification of $\mathfrak{L}^{\#}_{\mathsf{M}}$. See [SP23, Lemma 8.11.8] for details.

In the remainder of this section we are going to finally make use of the results of first part of this paper. Recall, in particular, the sheaf of abelian groups $\mathfrak{L}^{\Phi(*)}_{\mathcal{P},\Lambda}$ from Definition 1.3.2.

2.6.4 Theorem. *Let $(\mathcal{P},\Lambda)$ be spherical. Then the gerbe $\mathsf{M} := \mathsf{M}_{\mathcal{P},\Lambda}$ has abelian automorphism groups. Its band $\mathfrak{L}_{\mathsf{M}}$ is canonically isomorphic to $\mathfrak{L}^{\Phi(*)}_{\mathcal{P},\Lambda}$ for a unique local root system on $\mathcal{P} \subseteq \mathfrak{a}_{\mathcal{P}}$.*

Proof. For every $a \in \mathcal{P}$ choose a open neighborhood V as in Corollary 2.2.3 and let $\overline{\mathsf{M}}$ be the gerbe of Hamiltonian manifolds over $U := \mathcal{P} \cap V$. Then the Local Structure Theorem 2.3.2 yields an isomorphism of gerbes $\overline{\mathsf{M}} \xrightarrow{\sim} \mathsf{M}|_U$. The automorphism groups of the objects of $\overline{\mathsf{M}}$ have been determined in [Kno11, Thm. 9.2]. Translated into the language of gerbes the result is that there is unique root system $\Phi(a)$ with $\alpha(a) = 0$ for all $\alpha \in \Phi(a)$ such that the band of $\overline{\mathsf{M}}$ is isomorphic to $\mathfrak{L}^{\Phi(a)}_{U,\Lambda}$. Now it follows from [Kno11, eq. (9.4)] that the system $((\Phi(a))_{a\in\mathcal{P}}, \Lambda)$ forms a local root system on $\mathcal{P}$. In other words, the band $\mathfrak{L}_{\mathsf{M}}$ is locally isomorphic to $\mathfrak{L} := \mathfrak{L}^{\Phi(*)}_{\mathcal{P},\Lambda}$.

We claim that that the local isomorphisms $\Phi_U : \mathfrak{L}|_U \to \mathfrak{L}_{\mathsf{M}}|_U$ glue to a global isomorphism $\Phi : \mathfrak{L} \xrightarrow{\sim} \mathfrak{L}_{\mathsf{M}}$. This is not completely obvious since the local model of $(\mathcal{P},\Lambda)$ at $a \in \mathcal{P}$ is a Hamiltonian manifold for the group $K_{a\tau}$ which therefore does not depend continuously on a.

To bypass this problem we restrict the isomorphisms Φ_U to $\mathcal{P}^0$, the interior of $\mathcal{P}$ inside $\mathfrak{a}_{\mathcal{P}}$. Since $\mathcal{P}^0$ is dense in $\mathcal{P}$ and since the sections of both $\mathfrak{L}$ and $\mathfrak{L}_{\mathsf{M}}$ are continuous it suffices to prove compatibility on $\mathcal{P}^0$.

To this end, let $a \in \mathcal{P}$ and let $U \subseteq \mathcal{P}$ be a convex neighborhood of a in $\mathcal{P}$. Then $U^0 := U \cap \mathcal{P}^0$ is open in $\mathcal{P}$ and convex, as well. Choose U small enough such that an object $M \in \mathsf{M}(U)$ exists. Let $L := K_{a\tau}$. Then we can choose U such that also $\overline{M} := \log_L M$ exists. With $M^0 := M_{U^0} = m_+^{-1}(U^0)$ and $\overline{M}^0 := \overline{M}_{U^0} = \overline{m}_+^{-1}(U^0)$ we obtain the following diagram:

$$\begin{tikzcd} \mathfrak{L}(U^0) \arrow[d, "\sim"'] & C^\infty(U^0) \arrow[l, two heads, "\varepsilon"'] \arrow[dl, "\overline{h}"'] \arrow[d, "h"] \\ \mathrm{Aut}(\overline{M}^0) \arrow[r, "\sim"] & \mathrm{Aut}(M^0). \end{tikzcd} \tag{2.6.1}$$

Here ε is the map $f \mapsto \exp(\nabla f)$ from (1.4.4) which is surjective by Lemma 1.4.3. The maps marked with h and $\overline{h}$ map f to the Hamiltonian flow

$\exp(H_f)$ on M^0 and $\overline{M}^0$, respectively (see [AMM98, Prop. 4.6] for Hamiltonian flows on quasi-Hamiltonian manifolds). The upper left triangle is commutative by [Kno11, Thm. 9.1]. One can easily check that the Hamiltonian flow on M restricts to the corresponding Hamiltonian flow on $\overline{M}$. So the bottom right triangle commutes as well. It follows that the upper right triangle commutes. Because of $\mathrm{Aut}(M^0) = \mathfrak{L}_{\mathsf{M}}(U^0)$ we obtain the commutative triangle

$$\begin{tikzcd} \mathfrak{L}(U^0) \arrow[dr, "\Phi_U|_{U_0}"'] & C^\infty(U^0) \arrow[l, two heads, "\eta"'] \arrow[d, "h"] \\ & \mathfrak{L}_{\mathsf{M}}(U^0). \end{tikzcd} \tag{2.6.2}$$

Since both η and h depend only on U^0 (instead of U), the same holds for the restriction $\Phi_U|_{U_0}$. This shows that all isomorphisms Φ_U coincide on the intersection of their domains. Hence they glue to a global isomorphism Φ. □

If $\mathcal{P}$ is convex we can say more:

2.6.5 Corollary. *Let $(\mathcal{P}, \Lambda)$ be a spherical pair with $\mathcal{P}$ convex. Then the higher cohomology of the band of $\mathsf{M}_{\mathcal{P},\Lambda}$ vanishes.*

Proof. It follows from Proposition 1.2.3 that the local root system $\Phi(*)$ is trivial. Indeed, every convex set is solid in its affine span. Moreover, it follows from [Kno11, Thm. 4.1] that every local Weyl group $W(a)$ is a subquotient of the Weyl group of $K_{a\tau}$. Thus every element $w \in W(a)$ can be lifted to an element $\widetilde{w}$ of W_{Φ_τ} with $\widetilde{w}\mathfrak{a}_{\mathcal{P}} = \mathfrak{a}_{\mathcal{P}}$. Thus, the second condition follows from $\mathcal{P} \subseteq \mathcal{A}$ and the fact that every W_{Φ_τ}-orbit meets $\mathcal{A}$ in exactly one point.

Now the assertion follows from Theorem 1.4.1. □

Proof of Theorem 2.6.1. Let $\mathsf{M} := \mathsf{M}_{\mathcal{P},\Lambda}$ and let $M_1, M_2 \in \mathrm{Obj}\,\mathsf{M}(\mathcal{P})$ be two global objects. Then the sheaf $\mathfrak{I} := \mathrm{Isom}_{\mathcal{P}}(M_1, M_2)$ is a torsor for $\mathfrak{L}_{\mathsf{M}}$. The vanishing of $H^1(\mathcal{P}, \mathfrak{L}_{\mathsf{M}})$ implies that $\mathfrak{I}$ has a global section, i.e., M_1 and M_2 are isomorphic (see [Bry93, 5.2.5(1)]). This shows uniqueness of M.

For the existence, observe that there are arbitrary fine open coverings $\mathcal{P} = \bigcup_\mu U_\mu$ which are good in the sense that H^1 of $\mathfrak{L}_{\mathsf{M}}$ vanishes on all U_μ and $U_\mu \cap U_\nu$. Indeed one can take for the U_μ intersections of $\mathcal{P}$ with a small open balls. Then all U_μ and $U_\mu \cap U_\nu$ are convex, hence have vanishing H^1. Under these circumstances the vanishing of $H^2(\mathcal{P}, \mathfrak{L})$ implies that M is the only gerbe with band $\mathfrak{L}_{\mathsf{M}}$ ([Bry93, 5.2.8], see also [SP23, Lemma 21.11.1]). Thus M is isomorphic to the category of all $\mathfrak{L}_{\mathsf{M}}$-torsors which has a global object, namely the trivial torsor. So $\mathsf{M}(\mathcal{P}) \neq \varnothing$. □

2.7. Examples

We conclude this paper with a series of examples. It should be mentioned that Paulus has obtained many more in his thesis [Pau18].

Doubles A particularly important quasi-Hamiltonian manifold is the *double* $D(K)$ of a Lie group K. It was defined in [AMM98] and since it was used for the proof of Theorem 2.3.2. Hence, our construction is just an a posteriori reason for the existence of $D(K)$. In case K is compact and simply connected the double has a nice description in terms of a spherical pair: Recall from the proof of Theorem 2.3.2 that the acting group is $\overline{K} = K \times K$. As a manifold $D(K)$ equals $K \times K$ with $\overline{K}$ acting on $D(K)$ as

$$(x, y) * (a, b) = (xay^{-1}, xby^{-1}). \tag{2.7.1}$$

The momentum map is

$$m(a, b) = (ab^{-1}, a^{-1}b). \tag{2.7.2}$$

(this differs from [AMM98, §3.2] by the coordinate change $(a, b) \mapsto (a, b^{-1})$ on $D(K)$). Let $\mathcal{A}$ and Λ be the alcove and the weight lattice of K. Then $\overline{\mathcal{A}} = \mathcal{A} \times \mathcal{A}$ and $\overline{\Lambda} = \Lambda \oplus \Lambda$ are alcove and weight lattice of $\overline{K}$. Let w_0 the longest element of the Weyl group W of K and $\delta = \mathrm{id} \times (-w_0) : \mathfrak{t} \to \mathfrak{t} \oplus \mathfrak{t}$. Then

$$(\mathcal{P}_{D(K)}, \Lambda_{D(K)}) = (\delta(\mathcal{A}), \delta(\Lambda)) \subseteq (\overline{\mathcal{A}}, \overline{\Lambda}). \tag{2.7.3}$$

Indeed, let $T \subseteq K$ be a maximal torus and $(a, b) \in \overline{T} := T \times T$. Then (2.7.2) shows that $\mathcal{P}_{D(K)}$ is the set of $(a_1, a_2) \in \overline{\mathcal{A}}$ such that $\exp(a_1)$ is the w_0-conjugate of $\exp(-a_2)$. This shows $\mathcal{P}_{D(K)} = \delta(\mathcal{A})$. Furthermore, for generic (a, b) the stabilizer of $m(a, b)$ is $\overline{T}$ which shows $L_{D(K)} = \overline{T}$. The stabilizer of (a, b) in $L_{D(K)}$ is the diagonal torus ΔT. Thus $A_{D(K)} = \overline{T}/\Delta T \cong T$ which implies $\Lambda_{D(K)} = \delta(\Lambda)$.

2.7.1 Remark. The case of doubles shows that the classification of quasi-Hamiltonian manifolds does depend on the choice of an invariant scalar product on $\mathfrak{k}$. To see this observe that the alcove $\overline{\mathcal{A}}$ for $\overline{K} = \mathrm{SU}(2) \times \mathrm{SU}(2)$ is a rectangle whose side lengths depend on the chosen metric. The double $D(\mathrm{SU}(2))$ corresponds to the case when $\mathcal{P}$ is the diagonal of $\mathcal{A}$. In order for $(\mathcal{P}, \Lambda)$ to be spherical, $\mathcal{P}$ has to be parallel to the sum $\alpha + \alpha'$ of the simple roots of $\overline{K}$. This holds if and only if $\overline{\mathcal{A}}$ is a square, i.e., when the metrics on both factors of $\overline{K}$ are the same. This phenomenon does not occur when K is simple or for Hamiltonian manifolds.

Groups of rank 1 Let $K = \mathrm{SU}(2)$. Then $\mathcal{A}$ is an interval and $\mathcal{P}_M \subseteq \mathcal{A}$ is a subinterval. If $\mathcal{P}_M \neq \mathcal{A}$ then M is of the form $K \times^L M_0$ (see Theorem 2.3.2) for some Hamiltonian L-manifold M_0 with $L \subseteq K$. Quasi-Hamiltonian manifolds which are not of this form will be called *genuine*. Since genuine multiplicity free quasi-Hamiltonian SU(2)-manifolds have necessarily $\mathcal{P}_M = \mathcal{A}$ we just have to check which lattice Λ_M can occur. Because the possible local models in the end points are the $\mathrm{SL}(2,\mathbb{C})$-varieties $\mathbb{C}^2$, $\mathrm{SL}(2,\mathbb{C})/\mathbb{C}^*$ and $\mathrm{SL}(2,\mathbb{C})/N(\mathbb{C}^*)$ we get 3 different genuine multiplicity free quasi-Hamiltonian SU(2)-manifolds:

- $\Lambda_M = P \cong \mathbb{Z}\omega$, the weight lattice of SU(2). Here M is obtained by equivariantly gluing two copies of the closed unit disk D in $\mathbb{C}^2$ along their boundary S^3. One can check that the result of such a glueing is always diffeomorphic to the 4-sphere S^4. This example has been found by Alekseev-Meinrenken-Woodward [AMW02] under the name "spinning 4-sphere".
- $\Lambda_M = 2P$. In this case one can show that $M \cong \mathbf{P}^1(\mathbb{C}) \times \mathbf{P}^1(\mathbb{C})$.
- $\Lambda_M = 4P$. Here, M is the quotient of the previous case by the switching involution. Hence $M \cong \mathbf{P}^2(\mathbb{C})$.

There is another affine root system of rank 1, namely $\mathsf{A}_2^{(2)}$. It is the root system of $K = \mathrm{SU}(3)$ with the twist being an outer automorphism τ of K, e.g., complex conjugation. The alcove $\mathcal{A}$ is an interval and the two simple roots α_0, α_1 satisfy $\overline{\alpha}_0 = -2\overline{\alpha}_1$. The weight lattice of the affine root system is $P = \mathbb{Z}\overline{\alpha}_1$. The centralizers corresponding to the end points are SU(2) and SO(3), respectively. Let $\mathcal{P}_M = \mathcal{A}$. Then a discussion as above yields two cases

- $\Lambda_M = P$: In this case, the local models are $\mathbb{C}^2$ and $\mathrm{SO}(3,\mathbb{C})/\,\mathrm{SO}(2,\mathbb{C})$.
- $\Lambda_M = 2P$. In this case, the local models are $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SO}(2,\mathbb{C})$ and $\mathrm{SO}(3,\mathbb{C})/\,\mathrm{O}(2,\mathbb{C})$.

Note that $\Lambda_M = 4P$ does not work since $\Lambda_X^+ = \mathbb{Z}_{\geq 0}(4\overline{\alpha}_1)$ is not the weight monoid of any *smooth* affine spherical $\mathrm{SO}(3,\mathbb{C})$-variety.

Manifolds of rank 1 The spinning 4-sphere has been generalized by Hurtubise-Jeffrey-Sjamaar in [HJS06] to that of a spinning $2n$-sphere. In our terms it can be constructed as follows: let $K = \mathrm{SU}(n)$. Then the alcove $\mathcal{A}$ has n vertices, namely $x_0 = 0$ and the fundamental weights $x_i = \omega_i$, $i = 1, \ldots, n-1$. Let $\mathcal{P}$ be the edge joining x_0 and x_1. Let $\Lambda = \mathbb{Z}\omega_1$. Then $(\mathcal{P}, \Lambda)$ with $\Lambda = \mathbb{Z}\omega_1$ is a spherical pair. Indeed, the smooth affine spherical $\mathrm{SL}(n,\mathbb{C})$-variety $X = \mathbb{C}^n$ has weight monoid $\mathbb{Z}_{\geq 0}\omega_1$. This shows that it is a local model at the vertex x_0. The situation in x_1 is similar: the centralizer is still $K = \mathrm{SU}(n)$

but the simple root system is different, namely $\alpha_2, \alpha_3, \ldots, \alpha_{n-1}, \alpha_n = \alpha_0$. The last fundamental weight with respect to this system is $-\omega_1$. Therefore the monoid $C_{x_1}\mathcal{P} \cap \Lambda = \mathbb{Z}_{\geq 0}(-\omega_1)$ has a model, as well, namely again $\mathbb{C}^n$. Glued together this yields the spinning $2n$-sphere.

Eshmatov, [Esh09], has found an analogue of the spinning $2n$-sphere for the symplectic group. More precisely, he showed that the quaternionic projective space $\mathbf{P}^n(\mathbb{H})$ carries a structure of a multiplicity free quasi-Hamiltonian $\mathrm{Sp}(2n)$-manifold. Using our theory, this example can be obtained as follows. Let $K = \mathrm{Sp}(2n)$ and let $\varepsilon_1, \ldots, \varepsilon_n$ be the standard basis of the Cartan subalgebra $\mathfrak{t}$. Let $\mathcal{P}$ be the line segment joining the origin $x_0 = 0$ with $x_1 = \frac{1}{2}\varepsilon_1$. This is an edge of the fundamental alcove $\mathcal{A}$. Put $\Lambda := \mathbb{Z}\varepsilon_1$. Then the smooth affine spherical $\mathrm{Sp}(2n, \mathbb{C})$-variety $\mathbb{C}^{2n}$ is a local model in x_0. The other endpoint x_1 behaves differently, though. In this case the simple roots of the centralizer K_{x_1} are $\alpha_0, \alpha_2, \alpha_3, \ldots, \alpha_n$ which yields $K_{x_1} = \mathrm{Sp}(2) \times \mathrm{Sp}(2n-2)$. Moreover $-\omega_1$ is now the fundamental weight of the first factor of K_{x_1}. The local model with weight monoid $\mathbb{Z}_{\geq 0}(-\omega_1)$ is $\mathbb{C}^2$ with the second factor of K_{x_1} acting trivially. This shows that M is obtained by gluing the open pieces $U_1 = \mathbb{C}^n$ and

$$U_2 = \mathrm{Sp}(2n) \overset{\mathrm{Sp}(2)\times\mathrm{Sp}(2n-2)}{\times} \mathbb{C}^2. \tag{2.7.4}$$

This example has been further generalized by Knop-Paulus in [KP19]. We keep $K = \mathrm{Sp}(2n)$. Then the vertices of $\mathcal{A}$ are $x_k := \frac{1}{2}\sum_{i=1}^{k} \varepsilon_k$ for $k = 0, \ldots, n$. Fix k with $k > 0$ and let $\mathcal{P}_k$ be the line segment joining x_{k-1} and x_k. Let moreover $\Lambda_k := \mathbb{Z}\varepsilon_k$. Then one shows as above that $(\mathcal{P}_k, \Lambda_k)$ is spherical and it is even possible to identify the corresponding manifold:

2.7.2 Theorem. *Let n, k be integers with $1 \leq k \leq n$. Then there is a multiplicity free quasi-Hamiltonian $\mathrm{Sp}(2n)$-manifold structure on the quaternionic Grassmannian $M = \mathrm{Gr}_k(\mathbb{H}^{n+1})$ with $(\mathcal{P}_M, \Lambda_M) = (\mathcal{P}_k, \Lambda_k)$.*

Proof. The open pieces at x_{k-1} and x_k, respectively, are the spaces

$$X_1 := \mathrm{Sp}(2n) \times^{H_{k-1}} \mathbb{C}^{2n-2k+2} \text{ and } X_2 := \mathrm{Sp}(2n) \times^{H_k} \mathbb{C}^{2k} \tag{2.7.5}$$

where $H_k := \mathrm{Sp}(2k) \times \mathrm{Sp}(2n-2k) \subseteq \mathrm{Sp}(2n)$ and they glue to a multiplicity free quasi-Hamiltonian manifold M. Now recall that $\mathrm{Sp}(2n)$ can also be interpreted as the unitary group of $\mathbb{H}^n$. Then H_k is the isotropy group of $\mathbb{H}^k \subseteq \mathbb{H}^n$. Therefore X_2 can be identified with the universal bundle $\widetilde{\mathrm{Gr}}_k(\mathbb{H}^n)$ over the quaternionic Grassmannian $\mathrm{Gr}_k(\mathbb{H}^n)$. Similarly, X_1 is isomorphic to $\widetilde{\mathrm{Gr}}_{n-k+1}(\mathbb{H}^n)$. Now consider the space $\mathbb{H}^{n+1} = \mathbb{H}^n \oplus \mathbb{H}$ where K acts on the first factor. Let $e := (0,1)$ be the fixed point. Each element of $\widetilde{\mathrm{Gr}}_k(\mathbb{H}^n)$ can be interpreted as a pair (L, v) with $L \in \mathrm{Gr}_k(\mathbb{H}^n)$ and $v \in L$. Let $\Gamma_{L,v} \subseteq \mathbb{H}^n \oplus \mathbb{H}$ be the graph of the map $L \to \mathbb{H} : u \mapsto \langle u, v\rangle$. Then the map $(L, v) \mapsto \Gamma_{L,v}$ identifies $X_2 = \widetilde{\mathrm{Gr}}_k(\mathbb{H}^n)$ with the open subset of all $\widetilde{L} \in \mathrm{Gr}_k(\mathbb{H}^{n+1})$ with $e \notin \widetilde{L}$. Similarly, X_1 be identified with the set of all $\widetilde{L} \in \mathrm{Gr}_k(\mathbb{H}^{n+1})$ with $e \notin \widetilde{L}^\perp$. So $\mathrm{Gr}_k(\mathbb{H}^{n+1})$ is also obtained by gluing X_1 and X_2. One can check using, e.g., [AA92] or [AA93, Thm. 7.1] that all such gluings give diffeomorphic results. So $M \cong \mathrm{Gr}_k(\mathbb{H}^{n+1})$. □

Surjective momentum maps It is interesting to look at multiplicity free quasi-Hamiltonian manifolds M which are in a sense as big as possible. For us this means that $\mathcal{P}_M$ is the entire alcove $\mathcal{A}$ and Λ_M is the weight lattice P of $\overline{\Phi}$. In geometric terms, these are the multiplicity free quasi-Hamiltonian manifolds where the momentum map is surjective and where the principal isotropy group is trivial.

2.7.3 Proposition. *Let (K, τ) be one of the following three cases:*

$$(\mathrm{SU}(n), \mathrm{id}), \quad (\mathrm{Sp}(2n), \mathrm{id}), \quad (\mathrm{SU}(2n+1), k \mapsto \overline{k}) \tag{2.7.6}$$

(the last τ is complex conjugation). Then $(\mathcal{A}, P)$ is spherical, i.e., there is a unique multiplicity free quasi-Hamiltonian $K\tau$-manifold M whose momentum map is surjective and such that K acts freely on M.

Proof. It suffices to find a local model in each of the vertices a of $\mathcal{A}$. For that, each case will be treated separately.

$(K, \tau) = (\mathrm{SU}(n), \mathrm{id})$: We start with $a = 0 \in \mathcal{P} = \mathcal{A}$. Then $K_a = K$ and $C_a\mathcal{A}$ is the dominant Weyl chamber. Therefore, we have to show that there is a smooth affine $\mathrm{SL}(n, \mathbb{C})$-variety X_n such that $\mathbb{C}[X] = \bigoplus_\chi L(\chi)$ where χ runs through all dominant weights. Such a variety does in general not exist for an arbitrary reductive group but it does for $\mathrm{SL}(n, \mathbb{C})$, namely

$$X_n := \begin{cases} \mathrm{SL}(n, \mathbb{C}) \overset{\mathrm{Sp}(n,\mathbb{C})}{\times} \mathbb{C}^n & \text{if } n \text{ is even,} \\ \mathrm{SL}(n, \mathbb{C}) / \mathrm{Sp}(n-1, \mathbb{C}) & \text{if } n \text{ is odd.} \end{cases} \tag{2.7.7}$$

Thus $(\mathcal{A}, \Lambda)$ is spherical in $a = 0$. But then it is also spherical in all other vertices of $\mathcal{A}$ since they differ only in a translation by an element of the center.

$(K, \tau) = (\mathrm{Sp}(2n), \mathrm{id})$: A local model in $a = 0$ is

$$Y_n := \begin{cases} \mathrm{Sp}(2n, \mathbb{C}) \overset{\mathrm{Sp}(n,\mathbb{C})\times \mathrm{Sp}(n,\mathbb{C})}{\times} \mathbb{C}^n & \text{if } n \text{ is even,} \\ \mathrm{Sp}(2n, \mathbb{C}) \overset{\mathrm{Sp}(n-1,\mathbb{C})\times \mathrm{Sp}(n+1,\mathbb{C})}{\times} \mathbb{C}^{n+1} & \text{if } n \text{ is odd.} \end{cases} \tag{2.7.8}$$

In general, $\mathcal{A}$ has $n+1$ vertices $x_0 = 0, x_1, \ldots, x_n$ which are enumerated in such a way that $\alpha_i(x_i) \neq 0$ where $\alpha_0, \ldots, \alpha_n$ are the simple roots. Then the centralizer of x_i in K is $L = \mathrm{Sp}(2i) \times \mathrm{Sp}(2n-2i)$. Since $\Lambda = \mathbb{Z}^n = \mathbb{Z}^i \oplus \mathbb{Z}^{n-i}$ splits accordingly, the manifolds

$$Y_{i,n-i} := \mathrm{Sp}(2n) \overset{\mathrm{Sp}(2i)\times\mathrm{Sp}(2n-2i)}{\times} (Y_i \times Y_{n-i}) \tag{2.7.9}$$

are the open pieces in x_i.

$(K, \tau) = (\mathrm{SU}(2n+1), \tau)$ with τ an outer automorphism: Here, the Dynkin diagram of (K, τ) is of type $\mathsf{A}_{2n}^{(2)}$. In this case, $\mathcal{A}$ has $n+1$ vertices $x_0, \ldots, x_n$ such that the centralizer of x_i is $L = \mathrm{Sp}(2i) \times \mathrm{SO}(2n+1-2i)$. It is well-known that the coordinate ring of

$$Z_n := \mathrm{SO}(2n+1, \mathbb{C})/GL(n, \mathbb{C}) \tag{2.7.10}$$

contains all irreducible $\mathrm{SO}(2n+1, \mathbb{C})$-modules exactly once. So

$$Z_{i,n-i} := \mathrm{SU}(2n+1) \overset{\mathrm{Sp}(2i)\times\mathrm{SO}(2n+1-2i)}{\times} (Y_i \times Z_{n-i}) \tag{2.7.11}$$

is an open piece in x_i. □

2.7.4 Remark. Paulus, [Pau18], has determined all multiplicity free quasi-Hamiltonian manifolds with surjective momentum map. Thereby he showed that the manifolds above are the only ones where K is simple and the generic isotropy is trivial.

It is also interesting to determine the (global) root system Φ_M generated by the local root system from Theorem 2.6.4 using Proposition 1.2.3. For that it suffices to calculate its simple roots, the so called *spherical roots* of M. To do this we use that the spherical roots of the local models are known.

We only treat the case $(K, \tau) = (\mathrm{SU}(n), \mathrm{id})$ in Proposition 2.7.3. The simple affine roots of K are

$$\alpha_0 = 1 + x_n - x_1, \alpha_1 = x_1 - x_2, \ldots, \alpha_{n-1} = x_{n-1} - x_n. \tag{2.7.12}$$

The spherical roots of X_n are $\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \ldots, \alpha_{n-2} + \alpha_{n-1}$. For n odd, see [BP15] while the even case is handled in [Lun07]. It follows from the comparison results of [Kno11], in particular Thms. 3.3 and 9.1, that the spherical roots of X_n are the simple roots of the local root systems of M. Therefore, the simple roots of Φ_M are

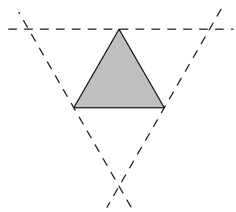
$$1 + x_n - x_2, x_1 - x_3, x_2 - x_4, \ldots, x_{n-2} - x_n, 1 + x_{n-1} - x_1. \tag{2.7.13}$$

Hence

$$\Phi_M \cong \begin{cases} \mathsf{A}^{(1)}_{\frac{n}{2}-1} \times \mathsf{A}^{(1)}_{\frac{n}{2}-1} & n \text{ even,} \\ \mathsf{A}^{(1)}_{n-1} & n \text{ odd.} \end{cases} \tag{2.7.14}$$

Observe that in the odd case the root systems of K and M are isomorphic but they are not the same. For example, for $n = 3$, i.e., $K = \mathrm{SU}(3)$, one gets the picture

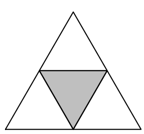
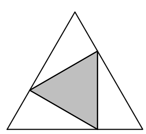
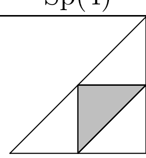
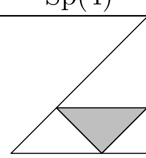
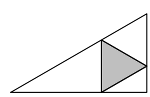
(2.7.15)



where the gray triangle denotes $\mathcal{P} = \mathcal{A}$ and the axes of the simple reflections of Φ_M are marked by dashed lines. There is also something to be observed in the even case: here all roots of Φ_M are perpendicular to the vector $\delta = (1, -1, \ldots, 1, -1) \in \overline{\mathfrak{a}}$. Let f be an affine linear function of $\mathfrak{a}$ with $\nabla f = \delta$. Then f is W_M-invariant and $t \mapsto \varepsilon(tf)$ from (1.4.6) defines a 1-parameter subgroup of automorphisms of M. Since δ lies in the weight lattice, this action factors through an action of an 1-dimensional torus. Thus, the $\mathrm{SU}(n)$-action on M extends to an $U(1) \times \mathrm{SU}(n)$-action.

Inscribed triangles For the last example, we toyed with triangles inscribed in a triangular alcove. Here are some examples of spherical pairs $(\mathcal{P}, \Lambda)$:

(2.7.16)

K	$\mathrm{SU}(3)$	$\mathrm{SU}(3)$	$\mathrm{Sp}(4)$	$\mathrm{Sp}(4)$	G_2
$\mathcal{P} \subseteq \mathcal{A}$					
Λ	P or R	R	R	R	R

We make no claim of completeness. In particular, we considered only untwisted groups. The letters P and R denote the weight and the root lattice of K, respectively. At each vertex, the complexified centralizer L is isogenous to $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^*$. Then one can show that the local models are either of the form $X = \mathrm{SL}(2,\mathbb{C})/\mu_n$ in case $\mathcal{P}$ touches $\mathcal{A}$ in form of a reflection and $X = \mathrm{SL}(2,\mathbb{C}) \times^{\mathbb{C}^*} \mathbb{C}$ otherwise.

2.7.5 Remark. As communicated to me by Eckhart Meinrenken, the first triangle has also been found by Chris Woodward (unpublished).

Acknowledgements

I would like to thank Chris Woodward who, a long time ago, suggested the topic of this paper to me. Thanks are also due to Kay Paulus and Bart Van Steirteghem for numerous discussions about this paper. Finally, I am indebted to the unknown referee for the many comments which highly improved the paper.

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