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The number of multiplicity-free primitive ideals associated with the rigid nilpotent orbits

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To Corrado De Concini with admiration

Abstract: Let G be a simple algebraic group defined over $\mathbb{C}$ and let e be a rigid nilpotent element in $\mathfrak{g} = \mathrm{Lie}(G)$. In this paper we prove that the finite W-algebra $U(\mathfrak{g}, e)$ admits either one or two 1-dimensional representations. Thanks to the results obtained earlier this boils down to showing that the finite W-algebras associated with the rigid nilpotent orbits of dimension 202 in the Lie algebras of type E_8 admit exactly two 1-dimensional representations. As a corollary, we complete the description of the multiplicity-free primitive ideals of $U(\mathfrak{g})$ associated with the rigid nilpotent G-orbits of $\mathfrak{g}$. At the end of the paper, we apply our results to enumerate the small irreducible representations of the related reduced enveloping algebras.

1. Introduction

Denote by G a simple algebraic group of adjoint type over $\mathbb{C}$ with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and let $\mathcal{X}$ be the set of all primitive ideals of the universal enveloping algebra $U(\mathfrak{g})$. We shall identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of an $(\mathrm{Ad}\, G)$-invariant non-degenerate symmetric bilinear form $(\,\cdot\,,\,\cdot\,)$ of $\mathfrak{g}$. Given $x \in \mathfrak{g}$ we write G_x the centraliser of x in G and write $\mathfrak{g}_x := \mathrm{Lie}(G_x)$.

It is well known that for any finitely generated $S(\mathfrak{g}^*)$-module M there exist prime ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ containing $\mathrm{Ann}_{S(\mathfrak{g}^*)}\, M$ and a chain $0 = R_0 \subset R_1 \subset \cdots \subset R_n = R$ of $S(\mathfrak{g}^*)$-modules such that $R_i/R_{i-1} \cong S(\mathfrak{g}^*)/\mathfrak{q}_i$ for $1 \le i \le n$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$ be the minimal elements in the set $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$. The zero sets $\mathcal{V}(\mathfrak{p}_i)$ of the $\mathfrak{p}_i$'s in $\mathfrak{g}$ are the irreducible components of the support $\mathrm{Supp}(M)$ of M. If $\mathfrak{p}$ is one of the $\mathfrak{p}_i$'s then we define $m(\mathfrak{p}) := \{1 \le i \le n \mid \mathfrak{q}_i = \mathfrak{p}\}$ and we call $m(\mathfrak{p})$ the *multiplicity* of $\mathcal{V}(\mathfrak{p})$ in $\mathrm{Supp}(M)$. The formal linear combination $\sum_{i=1}^{l} m(\mathfrak{p}_i)[\mathfrak{p}_i]$ is often referred to as the *associated cycle* of M and denoted $\mathrm{AC}(M)$.

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Given $I \in \mathcal{X}$ we can apply the above construction to the $S(\mathfrak{g}^*)$-module $S(\mathfrak{g}^*)/\mathrm{gr}(I)$ where $\mathrm{gr}(I)$ is the corresponding graded ideal in $\mathrm{gr}(U(\mathfrak{g})) = S(\mathfrak{g}) \cong S(\mathfrak{g}^*)$. The support of $S(\mathfrak{g}^*)/\operatorname{gr}(I)$ in $\mathfrak{g}$ is called the *associated variety* of I and denoted $\mathrm{V}(I)$. By Joseph's theorem, $\mathrm{V}(I)$ is the closure of a single nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ and, in particular, it is always irreducible. Hence in our situation the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\}$ is the singleton containing $J := \sqrt{\mathrm{gr}(I)}$ and we have that $\mathrm{AC}(S(\mathfrak{g}^*)/\operatorname{gr}(I)) = m(J)[J]$. The positive integer $m(J)$ is referred to the *multiplicity* of $\mathcal{O}$ in $U(\mathfrak{g})/I$ and denoted $\mathrm{mult}_{\mathcal{O}}(U(\mathfrak{g})/I)$.

For a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ we denote by $\mathcal{X}_{\mathcal{O}}$ the set of all $I \in \mathcal{X}$ with $\mathrm{V}(I) = \overline{\mathcal{O}}$. Following [25] we call $I \in \mathcal{X}_{\mathcal{O}}$ *multiplicity-free* if $\mathrm{mult}_{\mathcal{O}}(U(\mathfrak{g})/I) = 1$ and we say that a 2-sided ideal J of $U(\mathfrak{g})$ is *completely prime* if $U(\mathfrak{g})/J$ is a domain.

Classification of completely prime primitive ideals of $U(\mathfrak{g})$ is a classical problem of Lie Theory which finds applications in the theory of unitary representations of complex simple Lie groups. The subject has a very long history and many partial results can be found in the literature. In particular, it is known that any multiplicity-free primitive ideal is completely prime and that the converse fails outside type A for simple Lie algebras of rank ≥ 3; see [24] and [16] for more detail. A description of multiplicity-free primitive ideals in Lie algebras of types B, C and D was first obtained in [27]; that paper also solved the problem fo the majority of induced nilpotent orbits in exceptional Lie algebras.

Fix a nonzero nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ and let $\{e, h, f\}$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ with $e \in \mathcal{O}$. Let Q be the generalised Gelfand–Graev module associated with $\{e, h, f\}$; see [28] for more detail. Let $U(\mathfrak{g}, e) := (\mathrm{End}_{\mathfrak{g}}\, Q)^{\mathrm{op}}$, the finite W-algebra associated with $(\mathfrak{g}, e)$. If V is a finite dimensional irreducible $U(\mathfrak{g}, e)$-module, then Skryabin's theorem [19, Appendix] in conjunction with [21, Theorem 3.1(ii)] implies $Q \otimes_{U(\mathfrak{g}, e)} V$ is an irreducible $\mathfrak{g}$-module and its annihilator I_V in $U(\mathfrak{g})$ lies in $\mathcal{X}_{\mathcal{O}}$. Conversely, any primitive ideal in $\mathcal{X}_{\mathcal{O}}$ has this form for some finite dimensional irreducible $U(\mathfrak{g}, e)$-module V. This result was conjectured in [21, 3.4] and proved in [22, Theorem 1.1] for the primitive ideals admitting rational central characters. In full generality, the conjecture was first proved by Losev; see [11, Theorem 1.2.2(viii)]. A bit later, alternative proofs were found by Ginzburg in [7, 4.5] and by the first-named author in [23, Sect. 4]. The ideal I_V depends only on the image of V in the set $\mathrm{Irr}\, U(\mathfrak{g}, e)$ of all isoclasses of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules. We write $[V]$ for the class of V in $\mathrm{Irr}\, U(\mathfrak{g}, e)$.

It is well-known that group $C(e) := G_e \cap G_f$ is reductive and its finite quotient $\Gamma := C(e)/C(e)^\circ$ identifies with the component group of the centraliser G_e. From the Gan–Ginzburg realization of the finite W-algebra

$U(\mathfrak{g}, e)$ it follows that $C(e)$ acts on $U(\mathfrak{g}, e)$ by algebra automorphisms; see [5, Theorem 4.1]. By [21, Lemma 2.4], the connected component $C(e)^\circ$ preserves any 2-sided ideal of $U(\mathfrak{g}, e)$. As a result, we have a natural action of Γ on $\operatorname{Irr} U(\mathfrak{g}, e)$. For V as above, we let Γ_V denote the stabiliser of $[V]$ in Γ. In [15, 4.2], Losev proved that $I_{V'} = I_V$ if an only if $[V'] = [V]^\gamma$ for some $\gamma \in \Gamma$. In particular, $\dim V = \dim V'$. In conjunction with [15, Theorem 1.3.1(2)], this result of Losev also implies that

$$\operatorname{mult}_{\mathcal{O}}(U(\mathfrak{g})/I_V) = [\Gamma : \Gamma_V] \cdot (\dim V)^2.$$

As a consequence, a primitive ideal I_V is multiplicity-free if and only if $\dim V = 1$ and $\Gamma_V = \Gamma$. This brings our attention to the set $\mathcal{E}$ of all one-dimensional representations of $U(\mathfrak{g}, e)$ and its subset $\mathcal{E}^\Gamma$ consisting of all $C(e)$-stable such representations. Since $\mathcal{E}$ identifies with the maximal spectrum of the largest commutative quotient $U(\mathfrak{g}, e)^{\mathrm{ab}}$ of $U(\mathfrak{g}, e)$, it follows that $\mathcal{E}$ is an affine variety and $\mathcal{E}^\Gamma$ is a Zariski closed subset of $\mathcal{E}$.

If $\mathfrak{g}$ is a classical Lie algebra then it is proved in [27, Theorem 1] the variety $\mathcal{E}^\Gamma$ is isomorphic to the affine space $\mathbb{A}^{c_\Gamma(e)}$ where $c_\Gamma(e) = \dim(\mathfrak{g}_e/[\mathfrak{g}_e, \mathfrak{g}_e])^\Gamma$ (one should keep in mind here that the connected component of G_e acts trivially on $\mathfrak{g}_e/[\mathfrak{g}_e, \mathfrak{g}_e]$). This result continues to hold for $\mathfrak{g}$ exceptional provided that the orbit $\mathcal{O}$ is induced (in the sense of Lusztig–Spaltenstein) and not listed in [27, Table 0]. That table contains seven induced orbits (one in types F_4, E_6, E_7 and four in type E_8).

It is also known that $\mathcal{E} \neq \varnothing$ for all nilpotent orbits $\mathcal{O}$ in the finite dimensional simple Lie algebras $\mathfrak{g}$ and $\mathcal{E}$ is a finite set if and only if the orbit $\mathcal{O} \subset \mathfrak{g}$ is *rigid*, that is cannot be induced from a proper Levi subalgebra of $\mathfrak{g}$ in the sense of Lusztig–Spaltenstein. This was first conjectured in [21, Conjecture 3.1]. Several mathematicians contributed to the proof of this conjecture and we refer to [25, Introduction] for more detail on the history of the subject.

Furthermore, it is known that $\mathcal{E}^\Gamma \neq \varnothing$ in all cases. If e is rigid and $\mathfrak{g}$ is classical then $\mathfrak{g}_e = [\mathfrak{g}_e, \mathfrak{g}_e]$ by [30], whilst if $\mathfrak{g}$ is exceptional then either $\mathfrak{g}_e = [\mathfrak{g}_e, \mathfrak{g}_e]$ or $\mathfrak{g}_e = \mathbb{C}e \oplus [\mathfrak{g}_e, \mathfrak{g}_e]$ and the second case occurs for one rigid orbit in types G_2, F_4, E_7 and for three rigid orbits in type E_8; see [3, 26]. The Bala–Carter labels of these orbits are listed in Table 1.

Table 1: Rigid nilpotent elements with imperfect centralisers

Type of Φ	G_2	F_4	E_7	E_8	E_8	E_8
Label of e	$\widetilde{\mathrm{A}}_1$	$\widetilde{\mathrm{A}}_2 + \mathrm{A}_1$	$(\mathrm{A}_3 + \mathrm{A}_1)'$	$\mathrm{A}_3 + \mathrm{A}_1$	$\mathrm{A}_5 + \mathrm{A}_1$	$\mathrm{D}_5(\mathrm{a}_1) + \mathrm{A}_2$

Since $\mathcal{E}^\Gamma \neq \varnothing$, it follows from [27, Proposition 11] that for any simple Lie algebra $\mathfrak{g}$ the equality $\mathfrak{g}_e = [\mathfrak{g}_e, \mathfrak{g}_e]$ implies that $\mathcal{E}$ is a singleton. In view of the above we see that for any rigid nilpotent element in a classical Lie algebra the set $\mathcal{E} = \mathcal{E}^\Gamma$ contains one element, whilst for $\mathfrak{g}$ exceptional and e rigid the inequality $|\mathcal{E}| \geq 2$ may occur only for the six orbits listed in Table 1.

Let T be a maximal torus of G and $\mathfrak{t} = \mathrm{Lie}(T)$. Let Φ be the root system of $\mathfrak{g}$ with respect to T and let Π be a basis of simple roots in Φ. By Duflo's theorem [4], any primitive ideal $I \in \mathcal{X}$ has the form $I = I(\lambda) := \mathrm{Ann}_{U(\mathfrak{g})}\, L(\lambda)$ for some irreducible highest weight $\mathfrak{g}$-modules $L(\lambda)$ with $\lambda \in \mathfrak{t}^*$, and all multiplicity-free primitive ideals I constructed in [25] are given in their Duflo realisations. It is known that if $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\alpha \in \Pi$ then $\mathrm{V}(I)$ is the closure of a *special* (in the sense of Lusztig) nilpotent orbit in $\mathfrak{g}$. One also knows that to any $\mathfrak{sl}_2$-triple $\{e, h, f\}$ in $\mathfrak{g}$ with e special there corresponds an $\mathfrak{sl}_2$-triple $\{e^\vee, h^\vee, f^\vee\}$ in the Langlands dual Lie algebra $\mathfrak{g}^\vee$ with $h^\vee \in \mathfrak{t}^*$. As Barbasch–Vogan observed in [1, Proposition 5.10], for e special and rigid there is a unique choice of $h^\vee$ such that $\langle \frac{1}{2}h^\vee, \alpha^\vee \rangle \in \{0, 1\}$ for all $\alpha \in \Pi$. Furthermore, in this case we have that $I(\frac{1}{2}h^\vee - \rho) \in \mathcal{X}_\mathcal{O}$ (here ρ is the half-sum of the positive roots of Φ with respect to Π and $\mathcal{O}$ is the nilpotent orbit containing e).

If $\mathfrak{g}$ is classical and e is special rigid, then it follows from [17] that one of the Duflo realisations of the multiplicity-free primitive ideal in $\mathcal{X}_\mathcal{O}$ is obtained by using the Arthur–Barbasch–Vogan recipe described above. By [25, Theorem A], this result continues to hold for the special rigid nilpotent orbits in exceptional Lie algebras. (It is worth mentioning here that all nilpotent elements listed in Table 1 are non-special.) It was also proved in [25] that for any orbit $\mathcal{O}$ listed in Table 1 the set $\mathcal{X}_\mathcal{O}$ contains (at least) two multiplicity-free primitive ideals and their Duflo realisations $I(\Lambda)$ and $I(\Lambda')$ were found in all cases by using a method described by Losev in [14, 5.3].

It should be stressed at this point that in the case of rigid nilpotent orbits in exceptional Lie algebras the set $\mathcal{E}$ was first investigated by Goodwin–Röhrle–Ubly [8] and Ubly [29] who relied on some custom GAP code. In particular, it was checked in [8] that $|\mathcal{E}| = 2$ for all orbits in types G_2, F_4 and E_7 listed in Table 1. After [8] was submitted Ubly has improved the GAP code and was able to check that $|\mathcal{E}| = 2$ for the nilpotent orbit in type with Bala–Carter label $\mathrm{A}_3 + \mathrm{A}_1$ in type E_8; see [29]. This left open the two largest rigid nilpotent orbits (of dimension 202) in Lie algebras of type E_8.

The main result of this paper is the following:

Theorem A. *If e lies in a nilpotent orbit $\mathcal{O}$ listed in Table 1 then $|\mathcal{E}| = |\mathcal{E}^\Gamma| = 2$. Consequently, the set $\mathcal{X}_\mathcal{O}$ contains two multiplicity-free primitive ideals.*

Combined with the main results of [25], Theorem A provides a full list of all multiplicity-free primitive ideals of $U(\mathfrak{g})$ associated with rigid nilpotent orbits. Since $\Gamma = \{1\}$ for all nilpotent elements listed in Table 1, in order to prove the theorem we just need to show that $|\mathcal{E}| = 2$ for the nilpotent elements in Lie algebras of type E_8 labelled $\mathrm{A}_5 + \mathrm{A}_1$ and $\mathrm{D}_5(\mathrm{a}_1) + \mathrm{A}_2$. By the proof of Proposition 2.1 in [25] and by [28, Proposition 5.4], the largest commutative quotient $U(\mathfrak{g}, e)^{\mathrm{ab}}$ of $U(\mathfrak{g}, e)$ is generated by the image of a Casimir element of $U(\mathfrak{g})$ in $U(\mathfrak{g}, e)^{\mathrm{ab}}$; we call it c. Looking very closely at the commutators of certain PBW generators of Kazhdan degree 5 in $U(\mathfrak{g}, e)$ we are able to show that $\lambda c^2 + \eta c + \xi = 0$ for some $\lambda \in \mathbb{C}^\times$ and $\eta, \xi \in \mathbb{C}$. This quadratic equation results from investigating certain elements of Kazhdan degree 8 in the graded Poisson algebra $\mathcal{P}(\mathfrak{g}, e)$ associated with the Kazhdan filtration of $U(\mathfrak{g}, e)$.

Let $R = \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}]$. In [28, 4.1], a natural R-form, Q_R, of the Gelfand–Graev module Q was introduced, and it was proved for e rigid that the ring $U(\mathfrak{g}_R, e) := \mathrm{End}_{\mathfrak{g}}(Q_R)^{\mathrm{op}}$ has a nice PBW basis over R. In the present paper, we use these results to carry out all our computations over the ring R. In particular, we show that $\lambda \in R^\times$ and $\eta, \xi \in R$. The explicit form of Λ and Λ' in [25, 3.16, 3.17] in conjunction with [28, Theorem 1.2] and [19, Theorem 2.3] then enables us to obtain the following:

Theorem B. *Let* $\mathfrak{g}_{\Bbbk} := \mathrm{Lie}(G_{\Bbbk})$ *be a Lie algebra of type* E_8 *over an algebraically closed field* $\Bbbk$ *of characteristic* $p > 5$ *and let* e *be a nilpotent element of* $\mathfrak{g}_{\Bbbk}$ *with Bala–Carter label* $\mathrm{A}_5 + \mathrm{A}_1$ *or* $\mathrm{D}_5(\mathrm{A}_1) + \mathrm{A}_2$. *Let* $\chi \in \mathfrak{g}_{\Bbbk}^*$ *be such that* $\chi(x) = \kappa(e, x)$ *for all* $x \in \mathfrak{g}_{\Bbbk}^*$ *where* κ *is the Killing form of* $\mathfrak{g}_{\Bbbk}$. *Then the reduced enveloping algebra* $U_\chi(\mathfrak{g}_{\Bbbk})$ *has two simple modules of dimension* $p^{d(\chi)}$ *where* $d(\chi) = 101$ *is half the dimension of the coadjoint* $G_{\Bbbk}$*-orbit of* χ.

We recall that $U_\chi(\mathfrak{g}_{\Bbbk}) = U(\mathfrak{g}_{\Bbbk})/I_\chi$ where I_χ is the 2-sided ideal of $U(\mathfrak{g}_{\Bbbk})$ generated by all elements $x^p - x^{[p]} - \chi(x)^p$ with $x \in \mathfrak{g}_{\Bbbk}$ (here $x \mapsto x^{[p]}$ is the $[p]$-th power map of the restricted Lie algebra $\mathfrak{g}_{\Bbbk}$). By the Kac–Weisfeiler conjecture (proved in [18]) any finite-dimensional $U_\chi(\mathfrak{g}_{\Bbbk})$-module has dimension divisible by $p^{d(\chi)}$. It would be interesting to prove an analogue of Theorem B for the first four orbits in Table 1 and to reestablish the remaining results of [8] and [29] by the methods of the present paper.

2. Notation and preliminaries

Let $G_{\mathbb{Z}}$ be a Chevalley group scheme of type E_8 and $\mathfrak{g}_{\mathbb{Z}} = \mathrm{Lie}(G_{\mathbb{Z}})$. Let $R = \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}]$ (recall that 2, 3 and 5 are bad primes for $G_{\mathbb{Z}}$). We set $\mathfrak{g}_R := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$, and $\mathfrak{g} := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. Let Φ be the root system of $G_{\mathbb{Z}}$ with respect to a maximal split torus $T_{\mathbb{Z}}$ of $G_{\mathbb{Z}}$. Let $\Pi = \{\alpha_1, \ldots, \alpha_8\}$ be a set of simple roots in Φ and write Φ_+ for the set of positive roots of Φ with respect to Φ. We always use Bourbaki's numbering of simple roots; see [2, Planche VII].

We choose a Chevalley system $\bigcup_{\alpha\in\Phi_+}\{h_\alpha, e_\alpha, f_\alpha\}$ of $\mathfrak{g}_\mathbb{Z}$ so that that the signs of the structure constants $N_{\alpha,\beta}\in\{-1,0,1\}$ with $\alpha,\beta\in\Phi$ follow the conventions of [9] and [13]. Recall that $h_\alpha = [e_\alpha, f_\alpha]$ for all $\alpha\in\Phi_+$. We set $e_i := e_{\alpha_i}$, $f_i := f_{\alpha_i}$ and $h_i := h_{\alpha_i}$ for all $\alpha_i\in\Pi$ and denote by $(\,\cdot\,,\,\cdot\,)$ the $\mathbb{Z}$-valued invariant symmetric bilinear form on $\mathfrak{g}_\mathbb{Z}$ such that $(e_\alpha, f_\alpha) = 1$ for all $\alpha\in\Phi_+$.

Given $x\in\mathfrak{g}$ we denote by $\mathfrak{g}_x$ the centraliser of x in $\mathfrak{g}$. Of course, our main concern is with the nilpotent elements $e\in\mathfrak{g}_\mathbb{Z}$ labelled $\mathrm{A}_5+\mathrm{A}_1$ and $\mathrm{D}_5(\mathrm{a}_1)+\mathrm{A}_2$. A lot of useful information on the structure of $\mathfrak{g}_e$ can be found in [12, pp. 149, 150]. We note that the cocharacter $\tau\in X_*(T_\mathbb{Z})$ introduced in *op. cit.* is optimal for e in the sense of the Kempf–Rousseau theory; see [20] for detail. The adjoint action of $\tau(\mathbb{C}^*)$ on $\mathfrak{g}$ gives rise to a $\mathbb{Z}$-grading $\mathfrak{g}_e = \bigoplus_{i\in\mathbb{Z}_{\geq 0}}\mathfrak{g}_e(e)$ of $\mathfrak{g}_e$. As explained in [28, 3.4], this grading is defined over R, that is $\mathfrak{g}_{R,e} := \mathfrak{g}_e\cap\mathfrak{g}_R = \bigoplus_{i\in\mathbb{Z}_{\geq 0}}\mathfrak{g}_{R,e}(i)$ where $\mathfrak{g}_{R,e}(i) = \mathfrak{g}_R\cap\mathfrak{g}_e(i)$. Also, $\mathfrak{g}_{R,e}$ is a direct summand of the Lie ring $\mathfrak{g}_R$.

In what follows we adopt the notation introduced in [19] and [28]. Let Q be the generalised Gelfand–Graev module associated with e and write Q_R and $U(\mathfrak{g}_R, e)$ for the R-forms of Q and $U(\mathfrak{g}, e)$ defined in [28, 4.1, 5.1]. We write $\mathcal{F}_i(Q)$ and $\mathcal{F}_i(Q_R)$ for the i-th components of the Kazhdan filtration of Q and Q_R, respectively, and regard $U(\mathfrak{g}, e)$ as a subspace of Q. By [26, 4.5] and [28, Sect. 5], the associative algebra $U(\mathfrak{g}, e)$ is generated by elements Θ_y with $y\in\bigcup_{i\leq 5}\mathfrak{g}_e(i)$ and every such element is defined over R, i.e. has the property

$$\Theta(y) = y + \sum_{|(\mathbf{i},\mathbf{j})|_e\,\leq n_k+2,\ |\mathbf{i}|+|\mathbf{j}|\geq 2} \lambda_{\mathbf{i},\mathbf{j}}(y)x^{\mathbf{i}}z^{\mathbf{j}} \tag{2.1}$$

for some $\lambda_{\mathbf{i},\mathbf{j}}(y)\in R$; see [28, 4.2]. The monomials $x^{\mathbf{i}}z^{\mathbf{j}}$ involved in (2.1) will be described in more detail in Subsection 3.3.

3. Dealing with the orbit $\mathbf{A_5A_1}$

3.1. A relation in $\mathfrak{g}_\mathbf{e}(6)$ involving four elements of weight 3

Following [12, p. 149] we choose $e = e_1+e_2+e_4+e_5+e_6+e_7$. Then

$$f = f_1+5f_2+8f_4+9f_5+8f_6+5f_7$$

and $h = h_1 + 2h_2 - 9h_3 + 2h_4 + 2h_5 + 2h_6 + 2h_7 - 9h_8$. The Lie algebra $\mathfrak{g}_e(0)$ consists of two commuting $\mathfrak{sl}_2$-triples generated by $e_{\tilde\alpha}, f_{\tilde\alpha}$ and $e' := e_{\substack{1232100\\2}} + e_{\substack{1232110\\1}} - e_{\substack{1222210\\1}}$, $f' := f_{\substack{1232100\\2}} + f_{\substack{1232110\\1}} - f_{\substack{1222210\\1}}$. The 4-dimensional graded component $\mathfrak{g}_e(3)$ is a direct sum of two $\mathfrak{g}_e(0)$-modules of highest weights $(1,0)$ and $(0,1)$. As in *loc. cit.* we choose

$$\begin{aligned} v &:= e_{\substack{1243211\\2}} - e_{\substack{1233221\\2}} + e_{\substack{1233321\\1}}, \\ v' &:= e_{\substack{1121100\\1}} - e_{\substack{0122100\\1}} - e_{\substack{0121110\\1}} + 2e_{\substack{1111110\\1}} \end{aligned}$$

as corresponding highest weight vectors. Setting $v := -[f_{\tilde\alpha}, u]$ and $v' := -[f', u']$ and using the structure constants $N_{\alpha,\beta}$ tabulated in [13, Appendix] we then check directly that

$$\begin{aligned} u &:= f_{\substack{1232111\\2}} - f_{\substack{1232211\\1}} + f_{\substack{1222221\\1}}, \\ u' &:= f_{\substack{1110000\\1}} + f_{\substack{1111000\\0}} + f_{\substack{0111000\\1}} + 2f_{\substack{0111100\\0}}. \end{aligned}$$

One has to keep in mind here that

$$\begin{gathered} N_{\substack{1222221\\1},\substack{1243211\\2}} = N_{\substack{1232221\\1},\substack{1233211\\2}} = N_{\substack{1232111\\2},\substack{1233321\\1}} = 1, \\ N_{\substack{0111100\\0},\substack{1111110\\1}} = N_{\substack{1111000\\0},\substack{0121110\\1}} = N_{\substack{1110000\\1},\substack{0122100\\1}} = -N_{\substack{0111000\\1},\substack{1121100\\1}} = 1, \end{gathered}$$

and $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ for all $\alpha, \beta \in \Phi_+$; see [9, p. 409] and [13, Appendix]. Let

$$w := e_{\substack{0011000\\1}} + e_{\substack{0011100\\0}} + e_{\substack{0001110\\0}}.$$

Since both $[u,v]$ and $[u',v']$ lie in $\mathfrak{g}_e(6)$ and have weight $(0,0)$ with respect to $\mathfrak{g}_e(0)$ it follows from [12, p. 149] that $[u,v] = aw$ and $[u',v'] = bw$ for some $a, b \in \mathbb{C}$. Applying $\operatorname{ad} e_4$ to both sides of the equation $[u,v] = aw$ gives $[[e_4,u],v] + [u,[e_4,v]] = a[e_4,w]$ implying that

$$-[[e_4, f_{\substack{1232211\\1}}], v] - [u, [e_4, e_{\substack{1233221\\2}}]] = a[e_4, e_{\substack{0001110\\0}}].$$

It follows from [13, Appendix] that $[e_4, e_{\substack{0001110\\0}}] = e_{\substack{0011110\\0}}$ and $[e_4, e_{\substack{1233221\\2}}] = e_{\substack{1243221\\2}}$. Also, $[e_{\substack{1232211\\1}}, f_4] = \varepsilon e_{\substack{1222211\\1}}$ for some $\varepsilon \in \{\pm 1\}$. As $N_{\alpha_4, -\substack{1232211\\1}} = N_{\substack{1232211\\1}, -\alpha_4}$ by [9, p. 409], applying $\operatorname{ad} e_4$ to both sides of the last equation gives $[e_{\substack{1232211\\1}}, h_4] = \varepsilon [e_4, e_{\substack{1222211\\1}}]$. In view of [13, Appendix] this yields $-e_{\substack{1232211\\1}} = \varepsilon e_{\substack{1232211\\1}}$ forcing $\varepsilon = -1$. As a result,

$$[f_{\substack{1222211\\1}}, e_{\substack{1233321\\1}}] - [f_{\substack{1232111\\2}}, e_{\substack{1243221\\2}}] = a e_{\substack{0011110\\0}}.$$

Using [13, Appendix] we see $[e_{\substack{0011110\\0}}, e_{\substack{1222211\\1}}] = e_{\substack{1233321\\1}}$ and $[e_{\substack{0011110\\0}}, e_{\substack{1232111\\2}}] = -e_{\substack{1243221\\2}}$. Therefore,

$$[f_{\substack{1222211\\1}}, [e_{\substack{0011110\\0}}, e_{\substack{1222211\\1}}]] + [f_{\substack{1232111\\2}}, [e_{\substack{0011110\\0}}, e_{\substack{1232111\\2}}]] = ae_{\substack{0011110\\0}}.$$

Equivalently, $-[e_{\substack{0011110\\0}}, h_{\substack{1222211\\1}}] - [e_{\substack{0011110\\0}}, h_{\substack{1232111\\2}}] = ae_{\substack{0011110\\0}}$. Thus $a = -2$ so that

$$[u, v] = -2w.$$

Since $[e_4, u'] = 0$, applying $\operatorname{ad} e_4$ to both sides of the equation $[u', v'] = bw$ we get

$$[u', [e_4, 2e_{\substack{1111110\\1}}]] = 2[u', e_{\substack{1121110\\1}}] = b[e_4, e_{\substack{0001110\\0}}] = be_{\substack{0011110\\0}}$$

(we use the fact that $N_{\alpha_4, \substack{1111110\\1}} = 1$ which follows from the conventions in [9]). Our formula for u' implies that $[u', e_{\substack{1121110\\1}}] = [f_{\substack{1110000\\1}}, e_{\substack{1121110\\1}}]$. As $[e_{\substack{0011110\\0}}, e_{\substack{1110000\\1}}] = -e_{\substack{1121110\\1}}$ by [13, Appendix], we now obtain

$$-2[f_{\substack{1110000\\1}}, [e_{\substack{0011110\\0}}, e_{\substack{1110000\\1}}]] = 2[e_{\substack{0011110\\0}}, h_{\substack{1110000\\1}}] = be_{\substack{0011110\\0}}.$$

Hence $b = 2$ so that $[u', v'] = 2w$. In view of the above the following relation holds in $\mathfrak{g}_e(6)$:

$$[u, v] + [u', v'] = 0. \tag{3.1}$$

3.2. Searching for a quadratic relation in $\mathbf{U(\mathfrak{g}, e)^{ab}}$

Our hope is that despite (3.1) the element $[\Theta_u, \Theta_v] + [\Theta_{u'}, \Theta_{v'}] \in U(\mathfrak{g}, e)$ is nonzero; moreover, that lies in $\mathcal{F}_8(Q) \setminus \mathcal{F}_7(Q)$. Let

$$\mathcal{P}(\mathfrak{g}, e) = \big(\operatorname{gr}_{\mathcal{F}}(U(\mathfrak{g}, e)), \{\,\cdot\,,\,\cdot\,\}\big)$$

denote the Poisson algebra associate with Kazhdan-filtered algebra $U(\mathfrak{g}, e)$. It is well-known that $\mathcal{P}(\mathfrak{g}, e)$ identifies with the algebra of regular functions on the Slodowy slice $e + \mathfrak{g}_f$ to adjoint G-orbit e; see [19, 5]. We identify $\mathcal{P}(\mathfrak{g}, e)$ with the symmetric algebra $S(\mathfrak{g}_e)$ by using the isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ induced by the G-invariant symmetric bilinear form $(\,\cdot\,,\cdot\,)$ on $\mathfrak{g}$. We write $\mathcal{I}$ for the ideal of $\mathcal{P}(\mathfrak{g}, e)$ generated by $\bigcup_{i\neq 2} \mathfrak{g}_e(i)$ and put $\bar{\mathcal{P}} := \mathcal{P}(\mathfrak{g}, e)/\mathcal{I}$. Obviously, $\bar{\mathcal{P}} \cong S(\mathfrak{g}_2(2))$ as $\mathbb{C}$-algebras.

Given $y \in \mathfrak{g}_e(i)$ we write θ_y for the $\mathcal{F}$-symbol of Θ_y in $\mathcal{P}_{i+2}(\mathfrak{g}, e)$. We put $\varphi := \{\theta_u, \theta_v\} + \{\theta_{u'}, \theta_{v'}\}$, an element of $\mathcal{P}_8$ (possibly zero), and denote by $\bar{\varphi}$

the image of φ in $\bar{\mathcal{P}}$. By [12, p. 149], the graded component $\mathfrak{g}_e(2) = \mathfrak{g}_e(2)^{\mathfrak{g}_e(0)}$ is spanned by e and $e_1 = e_{\alpha_1}$. In view of (3.1) and [19, Theorem 4.6(iv)] the linear part of φ is zero and there exist scalars λ, μ, ν such that

$$\bar{\varphi} = \lambda e^2 + \mu e e_1 + \nu e_1^2.$$

In fact, the main results of [28, Theorem 1.2] imply that $\lambda, \mu, \nu \in R$. Since it follows from [25, Prop. 2.1] and [28, 5.2] that the commutative quotient $U(\mathfrak{g}, e)^{\mathrm{ab}}$ is generated by the image of Θ_e, we wish to take a closer look at the image of $\{\theta_u, \theta_v\} + \{\theta_{u'}, \theta_{v'}\}$ in $\bar{\mathcal{P}}$.

By [12, p. 149], the graded component $\mathfrak{g}_e(1)$ is an irreducible $\mathfrak{g}_e(0)$-module generated by $e_{\substack{2343210\\2}}$, a highest weight vector of weight $(0,3)$ for $\mathfrak{g}_e(0)$. Hence $[\mathfrak{g}_e(1), \mathfrak{g}_e(1)] \subseteq \mathfrak{g}_e(2) = \mathfrak{g}_e(2)^{\mathfrak{g}_e(0)}$ has dimension ≤ 1. On the other hand, a rough calculation relying on the above expression of f' shows that $(\operatorname{ad} f')^3(e_{\substack{2343210\\2}}) \in Re_1$. Since in the present case $\mathfrak{g}_e = \mathbb{C}e \oplus [\mathfrak{g}_e, \mathfrak{g}_e]$, we see that

$$[\mathfrak{g}_e, \mathfrak{g}_e](2) = [\mathfrak{g}_e(1), \mathfrak{g}_e(1)] + [\mathfrak{g}_e(0), \mathfrak{g}_e(2)^{\mathfrak{g}_e(0)}] = [\mathfrak{g}_e(1), \mathfrak{g}_e(1)]^{\mathfrak{g}_e(0)}$$

has codimension 1 in $\mathfrak{g}_e(2)$. The preceding remark now entails that $e_1 \in [\mathfrak{g}_e(1), \mathfrak{g}_e(1)]$.

Since it is immediate from [25, Prop. 2.1] and [28, 5.2] that the largest commutative quotient of $U(\mathfrak{g}, e)$ is generated by the image of Θ_e we would find a desired quadratic relation in $U(\mathfrak{g}, e)^{\mathrm{ab}}$ if we managed to prove that the coefficient λ of $\bar{\varphi}$ is nonzero. Indeed, let I_c denote the 2-sided ideal of $U(\mathfrak{g}, e)$ generated by all commutators. If it happens that $\lambda \in R^\times$ then the element $[\Theta_u, \Theta_v] + [\Theta_{u'}, \Theta_{v'}] \in I_c \cap Q_R$ has Kazhdan degree 8 and is congruent to $\lambda\Theta_e^2$ modulo $I_c \cap U(\mathfrak{g}_R, e) + \mathcal{F}_7(Q_R)$. As [28, Prop. 5.4] yields

$$U(\mathfrak{g}, e) \cap \mathcal{F}_7(Q_R) \subset R1 + R\Theta_e + I_c \cap U(\mathfrak{g}_R, e)$$

the latter would imply that $\lambda\Theta_e^2 + \eta\Theta_e + \xi 1 \in I_c$ for some $\lambda \in R^\times$ and $\eta, \xi \in R$.

From the expression for f in Subsection 3.1 we get $(e, f) = 5 + 8 + 9 + 8 + 5 + 1 = 36$. As $(e, f_1) = (e_1, f_1) = 1$ we obtain $(e_1, f - f_1) = 0$ and $(e, f - f_0) = 35$. Since all elements of $\mathcal{I}$ vanish on $f - f_1$ this gives

$$\varphi(f - f_1) = \lambda(e, f - f_1)^2 = 5^2 7^2 \lambda. \tag{3.2}$$

This formula indicates that we might expect some complications in characteristic 7.

3.3. Computing $\boldsymbol{\lambda}$, part 1

In order to determine λ we need a more explicit formula for commutators $[\Theta_a,\Theta_b]$ with $a,b\in\mathfrak{g}_e(3)$. For that purpose, it is more convenient to use the construction of $U(\mathfrak{g},e)$ introduced by Gan–Ginzburg in [5]. Let $\chi\in\mathfrak{g}^*$ be such that $\chi(x)=(e,x)$ for all $x\in\mathfrak{g}$ and set $\mathfrak{n}':=\bigoplus_{i\le -2}\mathfrak{g}(i)$ and $\mathfrak{n}:=\bigoplus_{i\le 1}\mathfrak{g}(i)$. Let $\mathcal{J}_\chi$ denote the left ideal of $U(\mathfrak{g})$ generated by all $x-\chi(x)$ with $x\in\mathfrak{n}'$ and put $\widehat{Q}:=U(\mathfrak{g})/\mathcal{J}_\chi$. Since χ vanishes on $[\mathfrak{n},\mathfrak{n}']\subseteq\bigoplus_{i\le -3}\mathfrak{g}(i)$, the left ideal $\mathcal{J}_\chi$ is stable under the adjoint action of $\mathfrak{n}$. Therefore, $\mathfrak{n}$ acts on $\widehat{Q}$. Moreover, the fixed point space $\widehat{Q}^{\operatorname{ad}\mathfrak{n}}$ carries a natural algebra structure given by $(x+\mathcal{J}_\chi)(y+\mathcal{J}_\chi)=xy+\mathcal{J}_\chi$ for all $x+\mathcal{J}_\chi, y+\mathcal{J}_\chi\in\widehat{Q}_\chi$. By [5, Theorem 4.1], $U(\mathfrak{g},e)\cong\widehat{Q}^{\operatorname{ad}\mathfrak{n}}$ as algebras. The Kazhdan filtration $\mathcal{F}$ of $\widehat{Q}$ (induced by that of $U(\mathfrak{g})$) is nonnegative.

Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be the non-degenerate symplectic form on $\mathfrak{g}(-1)$ given by $\langle x,y\rangle=(e,[x,y])$ for all $x,y\in\mathfrak{g}(-1)$ and let $z_1,\dots,z_s,z_{s+1},\dots,z_{2s}$ be a basis of $\mathfrak{g}(-1)$ such that $\langle z_{i+s},z_j\rangle=\delta_{ij}$ and $\langle z_i,z_j\rangle=\langle z_{i+s},z_{j+s}\rangle=0$ for all $1\le i,j\le s$. Let $\mathfrak{p}=\bigoplus_{i\ge 0}\mathfrak{g}(i)$, the parabolic subalgebra associated with the cocharacter τ, and let $x_1,\dots,x_m$ be a homogeneous basis of $\mathfrak{p}$ such that $x_1,\dots,x_r$ is a basis of $\mathfrak{g}_e\subset\mathfrak{p}$ and $x_i\in\mathfrak{g}(n_i)$ for some $n_i\in\mathbb{Z}_{\ge 0}$ (and all $i\le m$). Given $(\mathbf{i},\mathbf{j})\in\mathbb{Z}^m_{\ge 0}\times Z^{2s}_{\ge 0}$ we set $x^{\mathbf{i}}z^{\mathbf{j}}:=x_1^{i_1}\cdots x_m^{i_m}z_1^{j_1}\cdots z_{2s}^{j_{2s}}$. Clearly, $\mathcal{F}_d(\widehat{Q})\subset U(\mathfrak{g})/\mathcal{J}_\chi$ has $\mathbb{C}$-basis consisting of all $x^{\mathbf{i}}z^{\mathbf{j}}$ with

$$|(\mathbf{i},\mathbf{j})|_e:=\sum_{k=1}^m i_k(n_k+2)+\sum_{k=1}^{2s} j_k=\mathrm{wt}_h(x^{\mathbf{i}}z^{\mathbf{j}})+2\deg(x^{\mathbf{i}}z^{\mathbf{j}})\le d.$$

As explained in [21, 2.1] the algebra $U(\mathfrak{g},e)$ has a PBW basis consisting of monomials $\Theta^{\mathbf{i}}:=\Theta_1^{i_1}\cdots\Theta_r^{x_r}$ with $\mathbf{i}\in\mathbb{Z}^r_{\ge 0}$, where

$$\Theta_k=x_k+\sum_{|(\mathbf{i},\mathbf{j})|_e\le n_k+2,\ |\mathbf{i}|+|\mathbf{j}|\ge 2}\lambda^k_{\mathbf{i},\mathbf{j}}x^{\mathbf{i}}z^{\mathbf{j}},\qquad 1\le k\le r,$$

where $\lambda^k_{\mathbf{i},\mathbf{j}}\in\mathbb{C}$ and $\lambda^k_{\mathbf{i},\mathbf{j}}=0$ whenever $\mathbf{j}=\mathbf{0}$ and $i_j=0$ for $j>r$. The elements $\{\Theta_k\mid 1\le k\le r\}$ are unique by [28, Lemma 2.4].

Given $a=\sum_i\xi_ix_i\in\mathfrak{g}_e$ we put $\Theta_a:=\sum_i\xi_i\Theta_i$. Following [21, 2.4] we denote by $\mathbf{A}_e$ the associative $\mathbb{C}$-algebra generated by $z_1,\dots,z_s,z_{s+1},\dots,z_{2s}$ subject to the relations $[z_{i+s},z_j]=\delta_{ij}$ and $[z_i,z_j]=[z_{i+s},z_{j+s}]=0$ for all $1\le i,j\le s$. Clearly, $\mathbf{A}_e\cong\mathbf{A}_s(\mathbb{C})$, the s-th Weyl algebra over $\mathbb{C}$. Let $i\mapsto i^*$ denote the involution of the index set $\{1,\dots,s,s+1,\dots,2s\}$ such that $i^*=i+s$ for $i\le s$ and $i^*=i-s$ for $i>s$, and put $z_i^*:=(-1)^{p(i)}z_{i^*}$

where $p(i) = 0$ if $i \le s$ and $p(i) = 1$ if $i > s$. Then $[z_i^*, z_j] \in \delta_{ij} + \mathcal{J}_\chi$ for all $i \le 2s$.

Let $a \in \mathfrak{g}_e(d)$ where $d \ge 1$. As $\mathfrak{g}(-1) \subset \mathfrak{n}$ and $\Theta_a \in \widehat{Q}^{\mathfrak{n}}$ it is straightforward to see that

$$\begin{aligned}\Theta_a \equiv a + \sum_{i=1}^{2s}[a, z_i^*]z_i + \sum_{|(\mathbf{i},\mathbf{0})|_e = d+2,\ |\mathbf{i}|=2} \lambda_{\mathbf{i},\mathbf{0}}(a)x^{\mathbf{i}} \\ + \sum_{|(\mathbf{i},\mathbf{j})|_e = d+2,\ |\mathbf{i}|+|\mathbf{j}|\ge 3} \lambda_{\mathbf{i},\mathbf{j}}(a)x^{\mathbf{i}}z^{\mathbf{j}} \mod \mathcal{F}_{d+1}(\widehat{Q})\end{aligned}$$

where $\lambda_{\mathbf{i},\mathbf{j}}(a) \in \mathbb{C}$. By [21, Prop. 2.2], there exists an injective homomorphism of $\mathbb{C}$-algebras $\widetilde{\mu}\colon U(\mathfrak{g}, e) \hookrightarrow U(\mathfrak{p}) \otimes \mathbf{A}_e^{\mathrm{op}}$ such that

$$\tilde{\mu}(\Theta_k) = x_k \otimes 1 + \sum_{|(\mathbf{i},\mathbf{j})|_e \le n_k+2,\ |\mathbf{i}|+|\mathbf{j}|\ge 2} \lambda_{\mathbf{i},\mathbf{j}}^k x^{\mathbf{i}} \otimes z^{\mathbf{j}}, \qquad 1 \le k \le r.$$

If $u_1, u_2 \in U(\mathfrak{p})$ and $c_1, c_2 \in \mathbf{A}_e^{\mathrm{op}}$ then

$$[u_1 \otimes c_1, u_2 \otimes c_2] = u_1u_2 \otimes c_2c_1 - u_2u_1 \otimes c_1c_2 = u_1u_2 \otimes [c_1, c_2] + [u_1, u_2] \otimes c_1c_2.$$

Now let $a \in \mathfrak{g}_e(d_1)$ and $b \in \mathfrak{g}_e(d_2)$, where d_1, d_2 are positive integers. Combining the above expressions for Θ_a and Θ_b with the preceding remark and properties of $\tilde{\mu}$ one observes that

$$\begin{aligned}[\Theta_a, \Theta_b] \equiv [a, b] + \sum_{i=1}^{2s}[[a, b]z_i^*]z_i + \sum_{i=1}^{2s}[a, z_i^*][b, z_i] + q(a, b) \\ + \sum_{|(\mathbf{i},\mathbf{j})|_e = d_1+d_2+2,\ |\mathbf{i}|+|\mathbf{j}|\ge 3} \lambda_{\mathbf{i},\mathbf{j}}(a, b)x^{\mathbf{i}}z^{\mathbf{j}} \mod \mathcal{F}_{d_1+d_2+1}(\widehat{Q}),\end{aligned}$$

where $\lambda_{\mathbf{i},\mathbf{j}}(a, b) \in \mathbb{C}$ and $q(a, b)$ is a linear combination of $[a, x_i]x_j$ with $n_i + n_j = d_2 + 2$ and $[b, x_i]x_j$ with $n_i + n_j = d_1 + 2$. In view of (3.1) this implies that

$$\begin{aligned}\{\theta_u, \theta_v\} + \{\theta_{u'}, \theta_{v'}\} = \sum_{i=1}^{2s} \left([u, z_i^*][v, z_i] + [u', z_i^*][v', z_i]\right) \\ + q(u, v, u', v') + \text{terms of standard degree} \ge 3,\end{aligned}$$

where $q(u, v, u', v') = q(u, v) + q(u', v')$. All terms of standard degree ≥ 3 involved in $\{\theta_u, \theta_v\} + \{\theta_{u'}, \theta_{v'}\}$ have Kazhdan degree 8. Therefore, they must vanish at $f - f_1 \in \mathfrak{g}(-2)$. Since each quadratic monomial involved in $q(u, v, u', v')$ has a linear factor of standard degree ≥ 3 we also have that $q(u, v, u', v')(f - f_1) = 0$. As a consequence,

$$(\{\theta_u, \theta_v\} + \{\theta_{u'}, \theta_{v'}\})(f - f_1) = \sum_{i=1}^{2s}([u, z_i^*], f - f_1)([v, z_i], f - f_1) + \sum_{i=1}^{2s}([u', z_i^*], f - f_1)([v', z_i], f - f_1).$$

3.4. Computing $\boldsymbol{\lambda}$, part 2

Our deliberation in Subsection 3.3 show that in order to determine λ we need to evaluate two sums:

$$A := \sum_{i=1}^{2s}([u, z_i^*], f - f_1)([v, z_i], f - f_1),$$
$$\text{and} \quad B := \sum_{i=1}^{2s}([u', z_i^*], f - f_1)([v', z_i], f - f_1).$$

To simplify notation we put $E := \operatorname{ad} e$, $H := \operatorname{ad} H$, $F = \operatorname{ad} f$ and $H_1 := \operatorname{ad} h_1 = \operatorname{ad} h_{\alpha_1}$. Since $u, u', v, v' \in \mathfrak{g}_e(3)$ there exist $u_- \in \mathbb{C}F^3(u)$, $v_- \in \mathbb{C}F^3(v)$, $u'_- \in \mathbb{C}F^3(u')$ and $v'_- \in \mathbb{C}F^3(v')$ such that $u = E^3(u_-)$, $v = E^3(v_-)$, $u' = E^3(u'_-)$ and $v' = E^3(v'_-)$. As $\mathfrak{g}_e \subset \mathfrak{p}$ the $\mathfrak{sl}_2$-theory shows that the elements u_-, v_-, u'_-, v'_- lie in $\mathfrak{g}_f(-3)$. Using the $\mathfrak{g}$-invariance of $(\,\cdot\,,\,\cdot\,)$ and the fact that $E^3(f - f_1) = 0$ we get

$$\begin{aligned}
A &= \sum_{i=1}^{2s}([E^3(u_-), z_i^*], f - f_1)([E^3(v_-), z_i], f - f_1) \\
&= \sum_{i=1}^{2s}(z_i^*, [E^3(u_-), f - f_1])(z_i, [E^3(v_-), f - f_1]) \\
&= \sum_{i=1}^{2s}(z_i^*, E^3([u_-, f - f_1]) \\
&\quad - 3E([E(u_-), h - h_1]))(z_i, E^3([v_-, f - f_1]) - 3E([E(v_-), h - h_1])) \\
&= \sum_{i=1}^{2s}(e, [E^2([u_-, f - f_1]) \\
&\quad - 3[E(u_-), h - h_1], z_i^*])(e, [E^2([v_-, f - f_1]) - 3[E(v_-), h - h_1], z_i]).
\end{aligned}$$

Our choice of the z_i^*'s implies that $\langle x,y\rangle = \sum_{i=1}^{2s}\langle z_i^*,x\rangle\langle z_i,y\rangle$ for all $x,y\in\mathfrak{g}(-1)$. The definition of $\langle\,\cdot\,,\cdot\,\rangle$ then yields

$$\begin{aligned}
A &= (e, [[E^2([u_-,f-f_1]) - 3[E(u_-),h-h_1],[E^2([v_-,f-f_1])\\
&\quad - 3[E(v_-),h-h_1]])\\
&= \big([E^3(u_-),f-f_1],E^2([v_-,f-f_1]) - 3[E(v_-),h-h_1]\big)\\
&= \big([u,f-f_1],E^2([v_-,f-f_1])\big) - 3\big([u,f-f_1],[E(v_-),h-h_1]\big)\\
&= 2([u,e_1],[v_-,f-f_1]) - 3(u,[f-f_1,[[e,v_-],h-h_1]])\\
&= 2([[u,e_1],f_1],v_-) - 3(u,[[-h+h_1,v_-],h-h_1])\\
&\quad + 3(u,[[e,[f_1,v_-]],h-h_1]) - 6(u,[[e,v_-],f-f_1])\\
&= 2([[u,e_1],f_1],v_-) - 3\big(u,(H-H_1)^2(v_-)\big)\\
&\quad + 3([e,[u,[f_1,v_-]],h-h_1) - 6([e,[u,v_-]],f-f_1)\\
&= 2([[u,e_1],f_1],v_-) - 3\big(u,(H-H_1)^2(v_-)\big)\\
&\quad - 6([f_1,v_-],[u,e-e_1]) + 6([u,v_-],h-h_1)\\
&= 2([[u,e_1],f_1],v_-) - 3\big(u,(H-H_1)^2(v_-)\big)\\
&\quad + 6(u,[e_1,[f_1,v_-]]) - 6(u,[h-h_1,v_-])\\
&= 8([[u,e_1],f_1],v_-) - 3\big((H-H_1)(H-H_1-2)(u),v_-\big).
\end{aligned}$$

Absolutely similarly we obtain that

$$B = 8([[u',e_1],f_1],v'_-) - 3\big((H-H_1)(H-H_1-2)(u'),v'_-\big).$$

The expression for v in Subsection 3.1 yields $[e_1,u]=[h_1,u]=0$. Since $[h,u]=3u$ this implies that $A=-9(u,v_-)$. Also, $u'=u'_1+u'_2$ where

$$u'_1 = f_{\substack{1110000\\1}} + f_{\substack{1111000\\0}} \quad\text{and}\quad u'_2 = f_{\substack{0111000\\1}} + 2f_{\substack{0111100\\0}}.$$

As $[e_1,u'_2]=[f_1,u'_1]=0$ and $[h_1,u'_1]=-u'_1$ we have

$$[[u',e_1],f_1] = [[u'_1,e_1],f_1] = [u'_1,h_1] = u'_1.$$

As $[h_1,u'_2]=u'_2$ and $u'_1,u'_2\in\mathfrak{g}(3)$ we have

$$(H-H_1)(H-H_1-2)(u') = (H_1-3)(H_1-1)(u') = -2(H_1-3)(u'_1) = 8u'_1.$$

From this it is immediate that $B = 8(u'_1,v'_-) - 24(u'_1,v'_-) = -16(u'_1,v'_-)$ and

$$A+B = -9(u,v_-) - 16(u'_1,v'_-).$$

Recall that $v = E^3(v_-)$ and $v' = E^3(v'_-)$. Since both v and v' have weight 3 it is straightforward to check that $v_- = \frac{1}{36}F^3(v)$ and $v'_- - = \frac{1}{36}F^3(v')$. As a result,

$$36(A+B) = 9(F^3(u), v) + 16(F^3(u'_1), v').$$

Our next step is to compute $F^3(u_1) = (\operatorname{ad} f)^3(f_{\substack{1110000\\1}} + f_{\substack{1111000\\0}})$. The formula for f in Subsection 3.1 shows that

$$[f, u'_1] = 9[f_5, f_{\substack{1110000\\1}}] + 5[f_2, f_{\substack{1111000\\0}}] + 8[f_6, f_{\substack{1111000\\0}}].$$

Using the structure constants and conventions of [9] we get

$$[f, u'_1] = 4(f_{\substack{1111000\\1}} + 2f_{\substack{1111100\\0}}).$$

Then

$$\begin{aligned}
[f,[f,u'_1]] &= 4\big(10[f_1, f_{\substack{1111100\\0}}] + 8[f_4, f_{\substack{1111000\\1}}] + 8[f_6, f_{\substack{1111000\\1}}] + 8[f_7, f_{\substack{1111100\\0}}]\big)\\
&= 4(-10f_{\substack{1111100\\1}} - 8f_{\substack{1121000\\1}} + 8f_{\substack{1111100\\1}} + 10f_{\substack{1111110\\0}})\\
&= 8(-f_{\substack{1111100\\1}} - 4f_{\substack{1121000\\1}} + 5f_{\substack{1111110\\0}}).
\end{aligned}$$

Finally,

$$\begin{aligned}
F^3(u'_1) &= 8\big(-8[f_4, f_{\substack{1111100\\1}}] - 5[f_7, f_{\substack{1111100\\1}}] - 32[f_6, f_{\substack{1121000\\1}}] + 25[f_2, f_{\substack{1111110\\0}}]\big)\\
&= 8(8f_{\substack{1121100\\1}} - 5f_{\substack{1111110\\1}} - 32f_{\substack{1121100\\1}} - 25f_{\substack{1111110\\1}})\\
&= -48(4f_{\substack{1121100\\1}} + 5f_{\substack{1111110\\1}}).
\end{aligned}$$

Therefore,

$$(F^3(u'_1), v') = -48(4f_{\substack{1121100\\1}} + 5f_{\substack{1111110\\1}}, e_{\substack{1121100\\1}} + 2e_{\substack{1111110\\1}}) = -2^5\cdot 3\cdot 7.$$

Next we determine $F^3(u) = (\operatorname{ad} f)^3(f_{\substack{1232111\\2}} - f_{\substack{1232211\\1}} + f_{\substack{1222221\\1}})$. Here we use conventions of [9] and the structure constants from [13, Appendix]. We have

$$\begin{aligned}
[f,u] &= 8[f_6, f_{\substack{1232111\\2}}] - 5[f_2, f_{\substack{1232211\\1}}] - 5[f_7, f_{\substack{1232211\\1}}] - 9[f_5, f_{\substack{1232211\\1}}]\\
&\quad + 8[f_4, f_{\substack{1222221\\1}}]\\
&= -8f_{\substack{1232211\\2}} + 5f_{\substack{1232211\\2}} + 5f_{\substack{1232221\\1}} + 9f_{\substack{1233211\\1}} - 8f_{\substack{1232221\\1}}\\
&= -3(f_{\substack{1232211\\2}} - 3f_{\substack{1233211\\1}} + f_{\substack{1232221\\1}}).
\end{aligned}$$

Then

$$\begin{aligned}[f,[f,u]] &= -3\big(9[f_5, f_{\substack{1232211\\2}}] + 5[f_7, f_{\substack{1232211\\2}}] - 15[f_2, f_{\substack{1233211\\1}}] - 15[f_7, f_{\substack{1233211\\1}}] \\ &\quad + 5[f_2, f_{\substack{1232221\\1}}] + 9[f_5, f_{\substack{1232221\\1}}]\big) = -3(-9f_{\substack{1233211\\2}} - 5f_{\substack{1232221\\2}} \\ &\quad - 5f_{\substack{1232221\\2}} - 9f_{\substack{1233221\\1}} + 15f_{\substack{1233211\\2}} + 15f_{\substack{1233221\\1}}) \\ &= 6(5f_{\substack{1232221\\2}} - 3f_{\substack{1233221\\1}} - 3f_{\substack{1233211\\2}}).\end{aligned}$$

Finally,

$$\begin{aligned}F^3(u) &= 6\big(45[f_5, f_{\substack{1232221\\2}}] - 15[f_2, f_{\substack{1233221\\1}}] - 24[f_6, f_{\substack{1233221\\1}}] - 24[f_4, f_{\substack{1233211\\2}}] \\ &\quad - 15[f_7, f_{\substack{1233211\\2}}]\big) \\ &= 6(-45f_{\substack{1233221\\2}} + 15f_{\substack{1233221\\2}} + 24f_{\substack{1233321\\1}} + 24f_{\substack{1243211\\2}} \\ &\quad + 15f_{\substack{1233221\\2}}) = 18(-5f_{\substack{1233221\\2}} + 8f_{\substack{1233321\\1}} + 8f_{\substack{1243211\\2}}).\end{aligned}$$

Therefore,

$$\begin{aligned}(F^3(u), v) &= 18(-5f_{\substack{1233221\\2}} + 8f_{\substack{1233321\\1}} + 8f_{\substack{1243211\\2}}, e_{\substack{1243211\\2}} - e_{\substack{1233221\\2}} + e_{\substack{1233321\\1}}) \\ &= 18(5+8+8) = 2\cdot 3^3\cdot 7.\end{aligned}$$

As a result, $36(A+B) = -16\cdot 2^5\cdot 3\cdot 7 + 9\cdot 2\cdot 3^3\cdot 7 = 6\cdot 7\cdot(9^2-16^2) = -6\cdot 5^2\cdot 7^2$. In view of (3.2) we now deduce that $5^2\cdot 7^2\lambda = A+B = -\frac{1}{6}\cdot 5^2\cdot 7^2$ forcing $\lambda = -\frac{1}{6}$. This enables us to conclude that in the present case $\dim U(\mathfrak{g}, e)^{\mathrm{ab}} = 2$. It is quite remarkable that 7^2 gets cancelled and we obtain $\lambda \in R^\times$ at the end!

Remark 3.1. For safety, we have used GAP [6] to double-check our computations and obtained the same result; i.e. $36(A+B) = -6\cdot 5^2\cdot 7^2$.[1]

4. Dealing with the orbit $\mathbf{D_5(a_1)A_2}$

4.1. A relation in $\mathfrak{g}_e(6)$ involving two elements of weight 3

Following [12, p. 150] we choose $e = e_1+e_2+e_3+e_5+e_7+e_8+e_{\alpha_2+\alpha_4}+e_{\alpha_4+\alpha_5}$ where $e_{\alpha_2+\alpha_4} = [e_2, e_4]$ and $e_{\alpha_4+\alpha_5} = [e_4, e_5]$. Then $h = 6h_1 + 7h_2 + 10h_3 + 12h_4 + 7h_5 + 2h_7 + 2h_8$. As $f_{\alpha_2+\alpha_4} = -[f_2, f_4]$ and $f_{\alpha_4+\alpha_5} = -[f_4, f_5]$ by the conventions of [9] a direct verification shows that

$$f = 6f_1 + f_2 + 10f_3 + f_5 + 2f_7 + 2f_8 - 6[f_2, f_4] - 6[f_4, f_5].$$

[1]The relevant code is available at https://github.com/davistem/the_number_of_multiplicity-free_primitive_ideals/

Therefore, $(e, f) = 6 + 1 + 10 + 1 + 2 + 2 + 6 + 6 = 34$.

The Lie algebra $\mathfrak{g}_e(0) \cong \mathfrak{sl}(2)$ is spanned by

$$\begin{aligned} e' &:= e_{\substack{1232221\\1}} - 2e_{\substack{1233210\\2}} - e_{\substack{1232211\\2}} - e_{\substack{1233211\\1}}, \\ f' &:= 2f_{\substack{1232221\\1}} - f_{\substack{1233210\\2}} - f_{\substack{1232211\\2}} - f_{\substack{1233211\\1}} \\ \text{and } h' &:= 2\varpi_6^\vee, \quad \text{where} \quad \varpi_6^\vee(e_i) = \delta_{i,6}\, e_i \quad \text{for } 1 \le i \le 8. \end{aligned}$$

The 4-dimensional graded component $\mathfrak{g}_e(3)$ is a direct sum of two $\mathfrak{g}_e(0)$-modules of highest weights 1. As in *loc. cit.* we choose

$$u := e_{\substack{1221110\\1}} + e_{\substack{1121111\\1}} - e_{\substack{1222100\\1}} - 2e_{\substack{1122110\\1}} + 3e_{\substack{1111111\\1}} + e_{\substack{1232100\\1}}$$

as a highest weight vector of one of these modules and set $v := [f', u]$. By standard properties of the root system Φ,

$$\begin{aligned} v = {} & 2[f_{\substack{1232221\\1}}, e_{\substack{1221110\\1}}] + 2[f_{\substack{1232221\\1}}, e_{\substack{1121111\\1}}] + [f_{\substack{1233210\\2}}, e_{\substack{1222100\\1}}] \\ & + 2[f_{\substack{1233210\\2}}, e_{\substack{1122110\\1}}] - [f_{\substack{1232211\\2}}, e_{\substack{1121111\\1}}] - 3[f_{\substack{1232211\\2}}, e_{\substack{111111\\1}}] \\ & + [f_{\substack{1233211\\1}}, e_{\substack{1222100\\1}}] - [f_{\substack{1233211\\1}}, e_{\substack{1232100\\1}}]. \end{aligned}$$

From [13, Appendix] we get

$$\begin{gathered} N_{\substack{0011111\\0}, \substack{1221110\\1}} = N_{\substack{0111110\\0}, \substack{1121111\\1}} = N_{\substack{0121100\\1}, \substack{1111111\\1}} = N_{\substack{0001111\\0}, \substack{1232100\\1}} = -1, \\ N_{\substack{0011110\\1}, \substack{1222100\\1}} = N_{\substack{0111100\\1}, \substack{1121111\\1}} = N_{\substack{0011111\\0}, \substack{1222100\\1}} = N_{\substack{0111100\\1}, \substack{1122110\\1}} = 1. \end{gathered}$$

Since $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ by [9], a straightforward computation shows that

$$\begin{aligned} v &= 2f_{\substack{0011111\\0}} + 2f_{\substack{0111110\\0}} - f_{\substack{0011100\\1}} - 2f_{\substack{0111100\\1}} \\ &\quad + f_{\substack{0111100\\1}} - 3f_{\substack{012110\\1}} - f_{\substack{0011111\\0}} - f_{\substack{0001111\\0}} \\ &= -f_{\substack{0001111\\0}} - f_{\substack{0011100\\1}} - f_{\substack{0111100\\1}} + 2f_{\substack{0111110\\0}} - 3f_{\substack{012110\\1}} + f_{\substack{0011111\\0}}. \end{aligned}$$

It is worth mentioning that v also appears in the extended (unpublished) version of [12] as a linear combination of vectors v_{12} and v_{14}.

Since u is a highest weight vector of weight 1 for $\mathfrak{g}_e(0)$ it must be that $[u, v] \in \mathfrak{g}_e(6)^{\mathfrak{g}_e(0)}$. By [12, p. 150], the latter subspace is spanned by $e_{\substack{1110000\\1}} + e_{\substack{1111000\\0}} + 2e_{\substack{0121000\\1}}$. On the other hand, a rough calculation (ignoring the signs of structure constants) shows that $[u, v]$ is a linear combination of $e_{\substack{1110000\\1}}$ and

$e_{\substack{1111000\\0}}$. This implies that

$$[u, v] = 0. \tag{4.1}$$

4.2. Searching for a quadratic relation in $\mathbf{U(\mathfrak{g}, e)^{ab}}$

Similar to our discussion in (3.2) we hope (with fingers crossed) that the element $[\Theta_u, \Theta_v]$ lies in $\mathcal{F}_8(Q) \setminus \mathcal{F}_7(Q)$. For that purpose we have to look closely at the element $\varphi := \{\theta_u, \theta_v\} \in \mathcal{P}_8(\mathfrak{g}, e)$ Here, as before, θ_y denotes the $\mathcal{F}$-symbol of Θ_y in the Poisson algebra $\mathcal{P}(\mathfrak{g}, e) = \mathrm{gr}_{\mathcal{F}}(U(\mathfrak{g}, e))$.

As in (3.2) we identify $\mathcal{P}(\mathfrak{g}, e)$ with the symmetric algebra $S(\mathfrak{g}_e)$ and write $\mathcal{J}$ for the ideal of $\mathcal{P}(\mathfrak{g}, e)$ generated by the graded subspace $[\mathfrak{g}_e(0), \mathfrak{g}_e(2)] \oplus \sum_{i \neq 2} \mathfrak{g}_e(i)$. We know from [12, p. 150] that $[\mathfrak{g}_e(0), \mathfrak{g}_e(2)]$ is an irreducible $\mathfrak{g}_e(0)$-module of highest weight 4 and $\mathfrak{g}_e(2) = [\mathfrak{g}_e(0), \mathfrak{g}_e(2)] \oplus \mathfrak{g}_e(2)^{\mathfrak{g}_e(0)}$. Furthermore, $\mathfrak{g}_e(2)^{\mathfrak{g}_e(0)}$ is a 2-dimensional subspace spanned by e and $e_0 := e_2 + e_5 + e_7 + e_8$. It follows that the factor-algebra $\bar{\mathcal{P}}(\mathfrak{g}, e) := \mathcal{P}(\mathfrak{g}, e)/\mathcal{J}$ is isomorphic to a polynomial algebra in e and e_0. We let $\bar{\varphi}$ denote the image of φ in $\bar{\mathcal{P}}(\mathfrak{g}, e)$. Then

$$\bar{\varphi} = \lambda e^2 + \mu e e_0 + \nu e_0^2,$$

and the main results of [28, Theorem 1.2] imply that the scalars λ, μ, ν lie in the ring R. Since is immediate from [25, Prop. 2.1] and [28, 5.2] that the largest commutative quotient of $U(\mathfrak{g}, e)$ is generated by the image of Θ_e we would find a desired quadratic relation in $U(\mathfrak{g}, e)^{\mathrm{ab}}$ if we managed to prove that the coefficient λ of $\bar{\varphi}$ is nonzero. Indeed, let I_c denote the 2-sided ideal of $U(\mathfrak{g}, e)$ generated by all commutators. If $\lambda \in R^\times$ then the element $[\Theta_u, \Theta_v] \in I_c \cap Q_R$ has Kazhdan degree 8 and is congruent to $\lambda \Theta_e^2$ modulo $I_c \cap U(\mathfrak{g}_R, e) + \mathcal{F}_7(Q_R)$. As it follows from [28, Prop. 5.4] that

$$U(\mathfrak{g}, e) \cap \mathcal{F}_7(Q_R) \subset R1 + R\Theta_e + I_c \cap U(\mathfrak{g}_R, e)$$

the latter would imply that $\lambda \Theta_e^2 + \eta \Theta_e + \xi 1 \in I_c$ for some $\eta, \xi \in R$.

Lemma 4.1. *We have that* $[\mathfrak{g}_e(1), \mathfrak{g}_e(1)]^{\mathfrak{g}_e(0)} = \mathbb{C} e_0$.

Proof. It follows from [26, 4.4] that $[\mathfrak{g}_e, \mathfrak{g}_e](2) = [\mathfrak{g}_e(0), \mathfrak{g}_e(2)] + [\mathfrak{g}_e(1), \mathfrak{g}_e(1)]$ has codimension 1 in $\mathfrak{g}_e(2)$. Hence $[\mathfrak{g}_e, \mathfrak{g}_e](2)^{\mathfrak{g}_e(0)} \neq \{0\}$. On the other hand, [12, p. 150] shows that $[\mathfrak{g}_e(0), \mathfrak{g}_e(2)] \cong L(4)$ and $[\mathfrak{g}_e(1), \mathfrak{g}_e(1)]$ is a homomorphic image of $\wedge^2 L(3)$, where $L(r)$ stands for the irreducible $\mathfrak{sl}(2)$-module of highest weight r. This implies that the subspace $[\mathfrak{g}_e(1), \mathfrak{g}_e(1)]^{\mathfrak{g}_e(0)}$ is 1-dimensional.

By [12, p. 150], the $\mathfrak{g}_e(0)$-module $\mathfrak{g}_e(1)$ is generated by the highest weight vector

$$w := e_{\substack{2344321\\2}} - e_{\substack{1354321\\3}}.$$

Given a root $\gamma \in \Phi$ we write $\nu_3(\gamma)$ for the coefficient of α_3 in the expression of γ as a linear combination of the simple roots $\alpha_i \in \Pi$, and we denote by t_3 the derivation of $\mathfrak{g}$ such that $t_3(e_\gamma) = \nu_3(\gamma)e_\gamma$ for all $\gamma \in \Phi$. Then $t_3(w) = 3w$ and $t_3(f') = -2f'$. Our preceding remarks show that the subspace $[\mathfrak{g}_e(1), \mathfrak{g}_e(1)]^{\mathfrak{g}_e(0)}$ is spanned by a nonzero vector of the form

$$a[(\mathrm{ad}\ f')^3(w), w] + b[(\mathrm{ad}\ f')^2(w), (\mathrm{ad}\ f')(w)]$$

with $a, b \in \mathbb{C}$. Since such a vector is a linear combination of e and e_0 and lies in the kernel of t_3 we now deduce that $[\mathfrak{g}_e(1), \mathfrak{g}_e(1)]^{\mathfrak{g}_e(0)} = \mathbb{C}e_0$ as stated (one should keep in mind here that $t_3(e_0) = 0$ and $t_3(e) = e_3 \neq 0$). $\square$

Let $h_0 = [e_0, f] = h_2 + h_5 + 2h_7 + 2h_8$. Since $[e, e_0] = 0$ we have that $[h_0, e] = [[e_0, f], e] = [h, e_0] = 2e_0$. Since $h_0 \in \mathfrak{t}$, each e_i is an eigenvector for $\mathrm{ad}\, h_0$ this forces $[h_0, e_0] = 2e_0$. Next we set $f_0 := \frac{1}{2}[f, [f, e_0]] = \frac{1}{2}[h_0, f]$ and observe that

$$[e, f_0] = \frac{1}{2}\big([h, [f, e_0]] + [f, [h, e_0]]\big) = [f, e_0] = -h_0.$$

Since $[f, e_0] = -h_0$ we get $[f_0, e_0] = \frac{1}{2}[[h_0, f], e_0] = [e_0, f] = -[e, f_0]$ which yields

$$[f_0, [f_0, e]] = -[f_0, [f, e_0]] = -[f, [f_0, e_0]] = [f, [e, f_0]] = -[h, f_0] = 2f_0.$$

As both f and f_0 lie in $\mathfrak{g}_f(-2)^{\mathfrak{g}_e(0)}$, they are orthogonal to $[\mathfrak{g}_e(0), \mathfrak{g}_e(2)]$ with respect to our symmetric bilinear form $(\,\cdot\,,\,\cdot\,)$. Since

$$(e_0, f_0) = (e_0, \frac{1}{2}[h_0, f]) = \frac{1}{2}([e_0, h_0], f) = -(e_0, f) = -(1 + 1 + 2 + 2) = -6$$

we have that $(e_0, f + f_0) = 0$. As $(e, f_0) = \frac{1}{2}(e, [f, [f, e_0]]) = \frac{1}{2}(h, [f, e_0]]) = -(f, e_0) = -6$ we get $(e, f + f_0) = (e, f) - (e_0, f_0) = 34 - 6 = 28$. Since the ideal $\mathcal{J}$ vanishes on $\mathfrak{g}_f(-2)^{\mathfrak{g}_e(0)}$ it follows that

$$\varphi(f + f_0) = \bar{\varphi}(f + f_0) = \lambda(e, f + f_0)^2 = 2^4 7^2 \lambda. \tag{4.2}$$

As in (3.2) this indicates that we might expect some complications in characteristic 7.

4.3. Computing $\boldsymbol{\lambda}$

In order to determine λ we use the method described in Subsections 3.3 and 3.4. We adopt the notation introduced there and put $E := \operatorname{ad} e$, $E_0 := \operatorname{ad} e_0$, $H := \operatorname{ad} h$, $H_0 := \operatorname{ad} h_0$, $F = \operatorname{ad} f$ and $F_0 := \operatorname{ad} f_0$. Since u and v are in $\mathfrak{g}_e(3)$ there exist $u_- \in \mathbb{C}F^3(u)$ and $v_- \in \mathbb{C}F^3(v)$ such that $u = E^3(u_-)$ and $v = E^3(v_-)$. As $\mathfrak{g}_e \cap \mathfrak{g}(-5) = \{0\}$ it follows from the $\mathfrak{sl}_2$-theory that the elements u_- and v_- lie in $\mathfrak{g}_f(-3)$. Arguing as in Subsection 3.3 we observe that

$$\{\theta_u, \theta_v\} = \sum_{i=1}^{2s} [u, z_i^*][v, z_i] + q(u, v) + \text{terms of standard degree} \geq 3.$$

Since all terms of standard degree ≥ 3 involved in $\{\theta_u, \theta_v\}$ have Kazhdan degree 8 they must vanish at $f + f_0 \in \mathfrak{g}(-2)$. Since each quadratic monomial involved in $q(u, v)$ has a linear factor of standard degree ≥ 3 we also have that $q(u, v)(f + f_0) = 0$. Using the $\mathfrak{g}$-invariance of $(\,\cdot\,,\,\cdot\,)$ and the fact that $E^3(f + f_0) = 0$ we get $\{\theta_u, \theta_v\}(f + f_0) =$

$$\begin{aligned}
&= \sum_{i=1}^{2s} ([u, z_i^*], f + f_0)([v, z_i], f + f_0) \\
&= \sum_{i=1}^{2s} ([E^3(u_-), z_i^*], f + f_0)([E^3(v_-), z_i], f + f_0) \\
&= \sum_{i=1}^{2s} (z_i^*, [E^3(u_-), f + f_0])(z_i, [E^3(v_-), f + f_0]) \\
&= \sum_{i=1}^{2s} (z_i^*, E^3([u_-, f + f_0]) - 3E([E(u_-), h - h_0]))(z_i, E^3([v_-, f + f_0]) \\
&\quad - 3E([E(v_-), h - h_0])) \\
&= \sum_{i=1}^{2s} (e, [E^2([u_-, f + f_0]) - 3[E(u_-), h - h_0], z_i^*])(e, [E^2([v_-, f - f_1]) \\
&\quad - 3[E(v_-), h - h_0], z_i]).
\end{aligned}$$

Here we used the fact that $E(f + f_0) = h + [e, f_0] = h - [e_0, f] = h - h_0$. As before, our choice of the z_i^*'s implies that $\langle x, y\rangle = \sum_{i=1}^{2s} \langle z_i^*, x\rangle \langle z_i, y\rangle$ for all $x, y \in \mathfrak{g}(-1)$. The definition of $\langle\,\cdot\,,\,\cdot\,\rangle$ then yields: $\{\theta_u, \theta_v\}(f + f_0)$

$$= (e, [[E^2([u_-, f + f_0]) - 3[E(u_-), h - h_0],$$

$$\begin{aligned}&[E^2([v_-, f+f_0]) - 3[E(v_-), h-h_0]]\big)\\ &= \big([E^3(u_-), f+f_0], E^2([v_-, f+f_0]) - 3[E(v_-), h-h_0]\big).\end{aligned}$$

One should keep in mind here that $E^3([u_-, f+f_0]) - 3[e, [E(u_-), E(f-f_0)]] = [E^3(u_-), f+f_0]$ which holds since $E^3(f+f_0) = 0$. As $E^4(u_-) = 0$ the latter equals to

$$([u, E^2(f_0)], [v_-, f_0]) - 3(u, [f+f_0, [E(v_-), h-h_0]]) \tag{4.3}$$

thanks to the $\mathfrak{g}$-invariance of $(\,\cdot\,,\,\cdot\,)$. Recall that $-h_0 = [e, f_0] = [f, e_0]$ and $[h, f_0] = -2f_0$. Also, $[f, v_-] = 0$ and $[f, f_0] = 0$. By the Jacobi identity, (4.3) equals to

$$\begin{aligned}&([u, E^2(f_0)], [v_-, f_0]) - 3(u, [[[f+f_0, e], v_-], h-h_0])\\ &\qquad - 3(u, [[e, [f+f_0, v_-]], h-h_0])\\ &\qquad - 3(u, [[e, v_-], 2f + [f, [e, f_0]] + 2f_0 + [f_0, [e, f_0]]]])\\ &\quad = ([u, [e, [f, e_0]]], [v_-, f_0]) - 3(u, (H + [E, F_0])^2(v_-))\\ &\qquad - 3([u, [e, [f_0, v_-]], h + [e, f_0])\\ &\qquad - 3(u, [E(v_-), 2f + 2f_0 + 2f_0 - {F_0}^2(e)]).\end{aligned}$$

Since h commutes with $[u, [e, [f_0, v_-]]$ the last expression equals

$$\begin{aligned}&2([u, e_0], [v_-, f_0]) - 3\big(u, (H + [E, F_0])^2(v_-)\big) + 3([u, [e, f_0]], [e, [f_0, v_-]])\\ &\qquad - 3(u, [E(v_-), 2f + 2f_0 + 2f_0 - 2f_0]) = -2([[u, e_0], f_0], v_-)\\ &\qquad - 3\big(u, (H + [E, F_0])^2(v_-)\big) - 3([u, E^2(f_0)], [f_0, v_-])\\ &\qquad + 6([u, v_-], h + [e, f_0])\\ &\quad = -2([[u, e_0], f_0], v_-) - 3\big((H - [E_0, F])^2(u), v_-)\big) - 6([u, e_0], [f_0, v_-])\\ &\qquad + 6((H - [E_0, F])(u), v_-)\\ &\quad = -8([f_0, [e_0, u]], v_-) - 3\big((3 - [E_0, F])^2(u), v_-\big) + 6\big((3 - [E_0, F])(u), v_-\big)\\ &\quad = -3\big(([E_0, F]^2 - 4[E_0, F] + 3)(u), v_-\big) - 8\big([f_0, [e_0, u]], v_-\big).\end{aligned}$$

Finally, a GAP computation[2] reveals that

$$-3\big(([E_0, F]^2 - 4[E_0, F] + 3)(u), v_-\big) - 8\big([f_0, [e_0, u]], v_-\big) = 1176 = 2^3 \cdot 3 \cdot 7^2.$$

[2]Again, see https://github.com/davistem/the_number_of_multiplicity-free_primitive_ideals/ for the code.

In view of (4.2) the factor 7^2 gets cancelled and we obtain $\lambda = \frac{3}{2} \in R^\times$. Arguing as in Subsection 3.2 we now deduce that $U(\mathfrak{g}, e)^{\mathrm{ab}}$ has dimension 2.

Remark 4.2. For safety, we have also used GAP to compute the expressions (4.3) and (4.4), and the number 1176 was the output in both cases.

4.4. The modular case

In this subsection we prove Theorem B. First suppose that e has Bala–Carter label $\mathrm{A}_5 + \mathrm{A}_1$. By [25, 3.16], we then have

$$\begin{aligned}
\Lambda + \rho &= \frac{1}{3}\varpi_1 + \frac{1}{3}\varpi_2 + \frac{1}{6}\varpi_3 + \frac{1}{6}\varpi_4 + \frac{1}{6}\varpi_5 + \frac{1}{6}\varpi_6 + \frac{1}{6}\varpi_7 + \frac{1}{6}\varpi_8,\\
\Lambda' + \rho &= \frac{1}{3}\varpi_1 + \frac{1}{3}\varpi_2 + \frac{1}{6}\varpi_3 + \frac{7}{6}\varpi_4 - \frac{11}{6}\varpi_5 + \frac{7}{6}\varpi_6 + \frac{1}{6}\varpi_7 + \frac{1}{6}\varpi_8.
\end{aligned}$$

In view of [2, Planche VII], we get

$$\begin{aligned}
\Lambda + \rho &= \frac{1}{6}(\varpi_1 + \varpi_2) + \frac{1}{6}\rho\\
&= \frac{1}{3}\varepsilon_8 + \frac{1}{12}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + 5\varepsilon_8)\\
&\quad + \frac{1}{6}(\varepsilon_2 + 2\varepsilon_3 + 3\varepsilon_4 + 4\varepsilon_5 + 5\varepsilon_6 + 6\varepsilon_7 + 23\varepsilon_8).
\end{aligned}$$

Using the standard coordinates of $\mathbb{R}^8$ we obtain

$$\Lambda + \rho = \frac{1}{12}(1, 3, 5, 7, 9, 11, 13, 55).$$

Next we observe that $\Lambda' + \rho = \Lambda + \rho + \varpi_4 - 2\varpi_5 + \varpi_6$. Since

$$\begin{aligned}
&\varpi_4 - 2\varpi_5 + \varpi_6\\
&\quad = (0, 0, 1, 1, 1, 1, 1, 5) - 2(0, 0, 0, 1, 1, 1, 1, 4) + (0, 0, 0, 0, 1, 1, 1, 3)\\
&\quad = (0, 0, 1, -1, 0, 0, 0)
\end{aligned}$$

by [2, Planche VII], we have $\Lambda' + \rho = \frac{1}{12}(1, 3, 17, -5, 9, 11, 13, 55)$. It follows that

$$\begin{aligned}
(\Lambda + \rho \,|\, \Lambda + \rho) - (\Lambda' + \rho \,|\, \Lambda' + \rho) &= \frac{1}{144}((5^2 - 17^2) + (7^2 - 5^2))\\
&= \frac{1}{144}(7^2 - 17^2)) = -\frac{5}{3}.
\end{aligned} \tag{4.4}$$

Now suppose that e has Bala–Carter label $\mathrm{D}_5(\mathrm{a}_1)+\mathrm{A}_2$. By [25, 3.17],

$$\begin{aligned}\Lambda+\rho&=-\frac{1}{4}\varpi_1-\frac{1}{4}\varpi_2-\frac{1}{4}\varpi_3+\varpi_4-\frac{1}{4}\varpi_5+\varpi_6+-\frac{1}{4}\varpi_7-\frac{1}{4}\varpi_8,\\ \Lambda'+\rho&=-\frac{1}{4}\varpi_1-\frac{1}{4}\varpi_2-\frac{1}{4}\varpi_3+2\varpi_4-\frac{9}{4}\varpi_5+2\varpi_6-\frac{1}{4}\varpi_7-\frac{1}{4}\varpi_8.\end{aligned}$$

Hence $\Lambda+\rho=-\frac{1}{4}\rho+\frac{5}{4}(\varpi_4+\varpi_6)=$

$$=-\frac{1}{4}(0,1,2,3,4,5,6,23)+\frac{5}{4}(0,0,1,1,2,2,2,8)=\frac{1}{4}(0,-1,3,2,6,5,4,17).$$

Similarly,

$$\begin{aligned}\Lambda'+\rho&=-\frac{1}{4}\rho+\frac{9}{4}(\varpi_4+\varpi_6)-2\varpi_5\\ &=-\frac{1}{4}(0,1,2,3,4,5,6,23)+\frac{9}{4}(0,0,1,1,2,2,2,8)-(0,0,0,0,2,2,2,8)\\ &=\frac{1}{4}(0,-1,7,6,-3,-4,-5,17).\end{aligned}$$

Therefore,

$$(\Lambda+\rho\,|\,\Lambda+\rho)-(\Lambda'+\rho\,|\,\Lambda'+\rho)=\frac{1}{16}(2^2-7^2)=-\frac{45}{16}.$$

This shows that in both cases the element $(\Lambda+\rho\,|\,\Lambda+\rho)-(\Lambda'+\rho\,|\,\Lambda'+\rho)$ is invertible in R. We set $r:=(\Lambda'+\rho\,|\,\Lambda'+\rho)-(\rho\,|\,\rho)$ and $r':=(\Lambda'+\rho\,|\,\Lambda'+\rho)-(\rho\,|\,\rho)$. Clearly, $r,r'\in R$.

Since the ideals $I(\Lambda)$ and $I(\Lambda')$ are multiplicity-free, our discussion in the introduction shows that $I(\Lambda)=I_V$ and $I(\Lambda')=I_{V'}$ for some 1-dimensional $U(\mathfrak{g},e)$-modules V and V'. There exist 2-sided ideals I and I' of codimension 1 in $U(\mathfrak{g},e)$ such that $V=U(\mathfrak{g},e)/I$ and $V'=U(\mathfrak{g},e)/I'$. As $L(\Lambda)$ and $L(\Lambda')$ are highest weight modules, we can find a Casimir element $C\in U(\mathfrak{g}_R)$ which acts on $L(\Lambda)$ and $L(\Lambda')$ as $r\mathrm{Id}$ and $r'\mathrm{Id}$, respectively.

Obviously, $C-r\in I$, $C-r'\in I'$, and the ideals I and I' contain all commutators in $U(\mathfrak{g},e)$. Put $I_R:=I\cap U(\mathfrak{g}_R,e)$, $I'_R:=I'\cap U(\mathfrak{g}_R,e)$ and $V_R:=U(\mathfrak{g}_R,e)/I_R$, $V'_R:=U(\mathfrak{g}_R,e)/I'_R$. It follows from [28, Proposition 5.4] that $U(\mathfrak{g}_R,e)=R\,1\oplus I_R$ and $U(\mathfrak{g}_R,e)=R\,1\oplus I'_R$.

To ease notation we identify e with its image in $\mathfrak{g}_{\Bbbk}=\mathfrak{g}_R\otimes_R\Bbbk$ (this will cause no confusion). Following [10] we let $U(\mathfrak{g}_{\Bbbk},e)$ denote the modular finite W-algebras associated with the pair $(\mathfrak{g}_{\Bbbk},e)$. By [28, Theorem 1.2(1)], we have that $U(\mathfrak{g}_{\Bbbk},e)\cong U(\mathfrak{g}_R,e)\otimes_R\Bbbk$ as $\Bbbk$-algebras. Our computations in

Subsections 3.3 and 3.4 imply that the image of C in the largest commutative quotient of $U(\mathfrak{g}_{\Bbbk}, e)$ satisfies a non-trivial quadratic equation. As a consequence, $U(\mathfrak{g}_{\Bbbk}, e)$ cannot have more than two 1-dimensional representations. On the other hand, the formulae for $r - r'$ obtained earlier yield that in each case the image of $r - r'$ in $R/pR \subset \Bbbk$ is nonzero for any good prime p of $G_{\mathbb{Z}}$. This entails that $V_{\Bbbk} := V_R \otimes_R \Bbbk$ and $V'_{\Bbbk} := V'_R \otimes_R \Bbbk$ are the only non-equivalent 1-dimensional representations of $U(\mathfrak{g}_{\Bbbk}, e)$.

Given $\xi \in \mathfrak{g}_{\Bbbk}^*$ we let $\mathfrak{g}_{\Bbbk}^{\xi}$ denote the coadjoint stabiliser of ξ in $\mathfrak{g}_{\Bbbk}$. As explained in [10, 8.1] the modular finite W-algebra $U(\mathfrak{g}_{\Bbbk}, e)$ contains a large central subalgebra $Z_p(\mathfrak{g}_{\Bbbk}, e)$ isomorphic to a polynomial algebra in $\dim \mathfrak{g}_{\Bbbk}^{\chi}$ variables. The algebra $U(\mathfrak{g}_{\Bbbk}, e)$ is free $Z_p(\mathfrak{g}_{\Bbbk}, e)$-module of rank $p^{\dim \mathfrak{g}_{\Bbbk}^{\chi}}$ and the maximal spectrum of $Z_p(\mathfrak{g}_{\Bbbk}, e)$ identifies with a Frobernius twist of a *good transverse slice* $\mathbb{S}_{\chi} = \chi + \tilde{\kappa}(\mathfrak{o})$ to the coadjoint orbit of χ. Here $\tilde{\kappa}\colon\ \mathfrak{g}_{\Bbbk} \to \mathfrak{g}_{\Bbbk}^*$ is the $G_{\Bbbk}$-module isomorphism induced by the Killing form κ and $\mathfrak{o}$ is a graded subspace of $\bigoplus_{i\le 0}\, \mathfrak{g}_{\Bbbk}(i)$ complementary to the tangent space $T_e((\mathrm{Ad}\, G_{\Bbbk})\, e) = [e, \mathfrak{g}_{\Bbbk}]$.

Every $\xi \in \mathbb{S}_{\chi}$ gives rise to a maximal ideal J_{ξ} of $Z_p(\mathfrak{g}_{\Bbbk}, e)$ which leads to a *p-central reduction*

$$U_{\xi}(\mathfrak{g}_{\Bbbk}, e) := U(\mathfrak{g}_{\Bbbk}, e)/J_{\xi}\, U(\mathfrak{g}_{\Bbbk}, e) \cong U(\mathfrak{g}_{\Bbbk}, e) \otimes_{Z_p(\mathfrak{g}_{\Bbbk}, e)} \Bbbk_{\xi}.$$

By [23, Lemma 2.2(iii)] and [10, Sections 8 and 9], for every $\xi \in \mathbb{S}_{\chi}$ we have an algebra isomorphism

$$U_{\xi}(\mathfrak{g}_{\Bbbk}) \cong \mathrm{Mat}_{p^{d(\chi)}}(U_{\xi}(\mathfrak{g}_{\Bbbk}, e)). \tag{4.5}$$

The 1-dimensional $U(\mathfrak{g}_{\Bbbk}, e)$-modules $V_{\Bbbk}$ and $V'_{\Bbbk}$ are annihilated by some maximal ideals J_{η} and $J_{\eta'}$ of $Z_p(\mathfrak{g}_{\Bbbk}, e)$. Therefore, $V_{\Bbbk}$ and $V'_{\Bbbk}$ are 1-dimensional modules over the p-central reductions $U_{\eta}(\mathfrak{g}_k, e)$ and $U_{\eta'}(\mathfrak{g}_{\Bbbk}, e)$, respectively. By (4.5), the reduced enveloping algebras $U_{\eta}(\mathfrak{g}_{\Bbbk})$ and $U_{\eta'}(\mathfrak{g}_{\Bbbk})$ with $\eta, \eta' \in \mathbb{S}_{\chi}$ afford simple modules of dimension $p^{d(\chi)}$; we call them $\widetilde{V}_{\Bbbk}$ and $\widetilde{V}'_{\Bbbk}$. As explained in [23, Lemma 2.2(iii)] and [10, Sections 8 and 9] we may assume further that the $U(\mathfrak{g}_{\Bbbk})$-modules $\widetilde{V}_{\Bbbk}$ and $\widetilde{V}'_{\Bbbk}$ are generated by their 1-dimensional subspaces $V_{\Bbbk}$ and $V'_{\Bbbk}$, respectively.

At this point we invoke a contracting $\Bbbk^{\times}$-action on $\mathbb{S}_{\chi}$ given by $\mu(t) \cdot \xi = t^{-2}(\mathrm{Ad}^*\, \tau(t))\, \xi$ for all $t \in \Bbbk^{\times}$ and $\xi \in \mathbb{S}_{\chi}$. It shows, in particular, that $\dim(\mathrm{Ad}\, G_{\Bbbk})\, \xi \ge \dim(\mathrm{Ad}\, G_{\Bbbk})\, \chi$ for every $\xi \in \mathbb{S}_{\chi}$. In conjunction with the main result of [18] this entails that $\dim(\mathrm{Ad}\, G_{\Bbbk})\, \eta = \dim(\mathrm{Ad}\, G_{\Bbbk})\, \eta' = \dim(\mathrm{Ad}\, G_{\Bbbk})\, \chi$. By [26, Theorem 3.8], the $G_{\Bbbk}$-orbit of e is rigid in $\mathfrak{g}_{\Bbbk}$. Therefore, χ lies in a single sheet of $\mathfrak{g}_{\Bbbk}^*$ which coincides with the coadjoint orbit of χ. Since the contracting action of $\mu(\Bbbk^\times)$ on $\mathbb{S}_\chi$ now shows that both η and η' lie in the only sheet of $\mathfrak{g}_\Bbbk^*$ containing χ, we deduce that $\chi = (\mathrm{Ad}^*\, g)\,\eta$ and $\chi = (\mathrm{Ad}^*\, g')\,\eta'$ for some $g, g' \in G_\Bbbk$.

Given $\xi \in \mathfrak{g}_\Bbbk^*$ we denote by I_ξ the 2-sided ideal of $U(\mathfrak{g}_\Bbbk)$ generated by all elements $x^p - x^{[p]} - \xi(x)^p$ with $x \in \mathfrak{g}_\Bbbk$. It is well-known (and easy to check) that for any $y \in G_\Bbbk$ the automorphism $\mathrm{Ad}\, y$ of $U(\mathfrak{g}_\Bbbk)$ sends I_ξ onto $I_{(\mathrm{Ad}^*\, y)\,\xi}$ and thus gives rise to an algebra isomorphism between the respective reduced enveloping algebras. The image $C_\Bbbk$ of our Casimir element C in $U(\mathfrak{g}_\Bbbk) = U(\mathfrak{g}_R) \otimes_R \Bbbk$ lies in the Harish-Chandra centre of $U(\mathfrak{g}_k)$. Hence $(\mathrm{Ad}\, y)(C_\Bbbk - a) = C_\Bbbk - a$ for all $y \in G_\Bbbk$ and $a \in \Bbbk$.

Let $\tilde{I}$ and $\tilde{I}'$ denote the annihilators of $\widetilde{V}_\Bbbk$ and $\widetilde{V}_\Bbbk'$ in $U(\mathfrak{g}_\Bbbk)$, and write $\bar{r}$ and $\bar{r}'$ for the images of r and r' in $\Bbbk$. The above discussion shows that $\tilde{I}$ contains I_η and $C_\Bbbk - \bar{r}$ whereas $\tilde{I}'$ contains $I_{\eta'}$ and $C_\Bbbk - \bar{r}'$. By construction, $\tilde{I}/I_\eta$ and $\tilde{I}'/I_{\eta'}$ have codimension $p^{2d(\chi)}$ in $U_\eta(\mathfrak{g}_\Bbbk)$ and $U_{\eta'}(\mathfrak{g}_\Bbbk)$, respectively. Hence the 2-sided ideals $(\mathrm{Ad}\, g)(\tilde{I})/(\mathrm{Ad}\, g)(I_\eta) = (\mathrm{Ad}\, g)(\tilde{I})/I_\chi$ and $(\mathrm{Ad}\, g')(\tilde{I}')/(\mathrm{Ad}\, g)(I_{\eta'}) = (\mathrm{Ad}\, g)(\tilde{I}')/I_\chi$ have codimension $p^{2d(\chi)}$ in $U_\chi(\mathfrak{g}_\Bbbk) = U(\mathfrak{g}_\Bbbk)/I_\chi$. These ideals are distinct since $(\mathrm{Ad}\, g)(C_\Bbbk) = (\mathrm{Ad}\, g')(C_\Bbbk) = C_\Bbbk$ and $\bar{r} \neq \bar{r}'$. Thanks to the main result of [18] this yields that $U_\chi(\mathfrak{g}_\Bbbk)$ has at least two simple modules of dimension $p^{d(\chi)}$. On the other hand, being a homomorphic image of $U(\mathfrak{g}_\Bbbk, e)$ the algebra $U_\chi(\mathfrak{g}_\Bbbk, e)$ cannot have more than two 1-dimensional representations. Applying (4.5) with $\xi = \chi$ we finally deduce that $U_\chi(\mathfrak{g}_\Bbbk)$ has exactly two simple modules of dimension $p^{d(\chi)}$. This completes the proof of Theorem B.

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