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Congruences for Hasse–Witt matrices and solutions of p-adic KZ equations

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To Corrado De Concini

Abstract: We prove general Dwork-type congruences for Hasse–Witt matrices attached to tuples of Laurent polynomials. We apply this result to establishing arithmetic and p-adic analytic properties of functions originating from polynomial solutions modulo p^s of Knizhnik–Zamolodchikov (KZ) equations, the solutions which come as coefficients of master polynomials and whose coefficients are integers. As an application we show that the p-adic KZ connection associated with the family of hyperelliptic curves $y^2 = (t - z_1)\dots(t-z_{2g+1})$ has an invariant subbundle of rank g. Notice that the corresponding complex KZ connection has no nontrivial subbundles due to the irreducibility of its monodromy representation.

Keywords: KZ equations, Dwork congruences, master polynomials, Hasse-Witt matrices.

1. Introduction

It is classical that the periods of the Legendre family $y^2 = t(t-1)(t-x)$ of elliptic curves viewed as functions of x satisfy the hypergeometric differential equation

$$x(1-x)I'' + (1-2x)I' - \frac{1}{4}I = 0; \tag{1.1}$$

the hypergeometric series

$$F(x) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \tag{1.2}$$

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$$= \frac{1}{\pi}\int_1^\infty t^{-1/2}(t-1)^{-1/2}(t-x)^{-1/2}dt = \sum_{k=0}^{\infty}\binom{-1/2}{k}^2 x^k$$

represents the analytic solution of (1.1) at the origin. In order to investigate the local zeta function of the x-fiber in the family, Dwork [4] studied the differential equation (1.1) p-adically using truncations of the infinite sum in (1.2),

$$F_s(x) = \sum_{k=0}^{p^s-1}\binom{-1/2}{k}^2 x^k \qquad \text{for} \quad s = 1, 2, \dots,$$

as p-adic approximations to its analytic solution (1.2). Clearly, the series $(F_s(x))$ converges to $F(x)$ as $s \to \infty$ in the disk $D_{0,1} = \{x \mid |x|_p < 1\}$. Dwork showed that the uniform limit $F_{s+1}(x)/F_s(x^p)$ as $s \to \infty$ exists in a larger domain $\mathfrak{D}_{\mathrm{Dw}}$, thus giving the p-adic analytic continuation of the function $F(x)/F(x^p)$, originally defined in $D_{0,1}$, to that larger domain. This limit, called the "unit root", defines a root of the local zeta function.

The second part of Dwork's investigation [4] concerned with the p-adic analytic continuation of the function $F(x)$ as a solution of (1.1). Dwork considered differential equation (1.1) as a system of first order linear differential equations for the vector $(F(x), F'(x))$ and approximated the direction-vector $(1, F'(x)/F(x))$ by rational functions $(1, F_s'(x)/F_s(x))$. He showed that the uniform limit as $s \to \infty$ of these rational functions does exist in the same larger domain $\mathfrak{D}_{\mathrm{Dw}}$, thus giving the p-adic analytic continuation to the domain $\mathfrak{D}_{\mathrm{Dw}}$ of the direction-vector $(1, F'(x)/F(x))$ (but not of the solution $(F(x), F'(x))$).

This fact indicates a clear difference of structure between solutions of complex analytic linear differential equations and their p-adic versions. A local solution of a complex linear differential equation can be analytically continued to a multi-valued analytic function on the domain of the definition of the differential equation, while in the p-adic setting only certain subspaces of the space of all local solutions can be analytically continued as subspaces to larger domains. In Dwork's situation only the one-dimensional subspace generated by $F(x)$ in the two-dimensional space of all local solutions at $x = 0$ can be p-adic analytically continued to the larger domain $\mathfrak{D}_{\mathrm{Dw}}$.

Dwork's work initiated significant research in p-adic differential equations and their applications to arithmetic of algebraic varieties. There is hardly a way to list all of them here, so we limit ourselves to mentioning some very recent contributions on the theme [1, 2]. The principal direction of that research is generalization of the first part of Dwork's work — on the relation between the unit root $F(x)/F(x^p)$ and the zeros of the local zeta function of the x-fiber in the Legendre family $y^2 = t(t-1)(t-x)$. In such a generalization the function $F(x)/F(x^p)$ becomes a square matrix with roots of its characteristic polynomial related to zeros of the local zeta function of the fibers of the corresponding family of algebraic varieties.

Our present paper is related to the second part of Dwork's investigation on the p-adic analytic continuation of the direction-vector $(1, F'(x)/F(x))$. We study this phenomenon for a system of Knizhnik–Zamolodchikov (KZ) differential equations. The KZ equations over $\mathbb{C}$ are objects of conformal field theory, representation theory, enumerative geometry, see for example [8, 5, 9]. In [14] the KZ equations over $\mathbb{C}$ were identified with the differential equations for flat sections of a suitable Gauss–Manin connection and solutions of the KZ differential equations were constructed in the form of multidimensional hypergeometric integrals. In that sense the KZ differential equations are distant relatives of the hypergeometric differential equation (1.1). It is known that the KZ equations and their solutions have remarkable properties, see for example [5, 17]. This motivates the study of KZ differential equations and their solutions over p-adic fields.

We consider the differential KZ equations over $\mathbb{C}$ in the special case, when the hypergeometric solutions are given by hyperelliptic integrals of genus g. In this case the space of solutions of the differential KZ equations is a $2g$-dimensional complex vector space. We also consider the p-adic version of the same differential equations. We show that the $2g$-dimensional space of local solutions of these p-adic differential KZ equations has a remarkable g-dimensional subspace of solutions which can be p-adic analytically continued as a subspace to a large domain $\mathfrak{D}^{(m),o}_{\mathrm{KZ}}$ in the space where the KZ equations are defined, see Theorems 6.7 and 6.9 for precise statements. This g-dimensional global subspace of solutions is defined as the uniform p-adic limit of the g-dimensional space of polynomial solutions of these KZ equations modulo p^s, the polynomial solutions constructed in [20].

In [15] general KZ differential equations were considered over the field $\mathbb{F}_p$ and their polynomial solutions were constructed. In [20] this construction was modified and polynomial solutions modulo p^s of the KZ equations associated with the hyperelliptic integrals were constructed. The polynomial solutions are vectors of polynomials with integer coefficients. They are "p^s-approximations" of the corresponding hyperelliptic integral solutions of the same differential KZ equations over $\mathbb{C}$. In [20] the constructed polynomial solutions are called the p^s-hypergeometric solutions. While the complex hyperelliptic integrals give the $2g$-dimensional space of all solutions of the complex KZ equations, the p^s-hypergeometric solutions span only a g-dimensional subspace. More general p^s-approximation constructions are discussed in [15, 12, 13].

In order to prove his two results stated above, Dwork developed in [4] two types of congruences,

$$\begin{aligned} F_{s+1}(x)/F_s(x^p) &\equiv F_s(x)/F_{s-1}(x^p) \pmod{p^s}, \\ F'_{s+1}(x)/F_{s+1}(x) &\equiv F'_s(x)/F_s(x) \pmod{p^s}, \end{aligned}$$

which are now called Dwork congruences. A suitable matrix form of Dwork congruences is used in most papers on p-adic differential equations. The closest versions of Dwork congruences related to our needs were developed in papers [10, 11, 22] by Mellit and Vlasenko. Motivated by our KZ equation considerations we give a generalization of Dwork congruences from [10, 11, 22] in Section 2, see Theorems 2.6 and 2.8. The proofs of these theorems are modifications of the proofs in [22].

In Sections 5 and 6 we apply Dwork congruences from Section 2 to the matrices composed of coordinates of the p^s-hypergeometric solutions and their antiderivatives. This application allows us to define the g-dimensional global subspace of solutions on a large domain $\mathfrak{D}_{\mathrm{KZ}}^{(m),o}$.

Notice that the main tool in [4] to prove the properties of the function $F(x)$ are polynomials $F_s(x)$ which are truncations of the power series expansion of the function $F(x)$. In our KZ equation case we do not have distinguished solutions whose power series expansions may be truncated and whose ratios could be p-adic analytically continued. Instead, we have a collection of p^s-hypergeometric solutions defined independently of any Dwork congruences, but which surprisingly satisfy appropriate Dwork congruences and give us a global subspace of solutions in the $s \to \infty$ limit.

This paper may be viewed as a continuation of our work [21] where the case $g = 1$ is developed. The Hasse–Witt matrices in [21] are of size 1×1 only, while in this paper the matrices are of arbitrary size.

Notice that in [20] the first authors considered the same KZ differential equations and p^s-hypergeometric solutions as in this paper. He showed that the space of p^s-hypergeometric solutions p-adically converges to the g-dimensional space of analytic solutions of the KZ differential equations in a suitable asymptotic zone. This statement is analogous to Dwork's observation in which his globally defined direction-vector $(1, F'(x)/F(x))$ represents the direction-vector of the unique analytic solution $(F(x), F'(x))$ of his differential equation at $x = 0$.

2. On ghosts

In this paper p is an odd prime. We denote by $\mathbb{Z}_p[w^{\pm 1}]$ the ring of Laurent polynomials in variables w with coefficients in $\mathbb{Z}_p$. A congruence $F(w) \equiv G(w) \pmod{p^s}$ for two Laurent polynomials from the ring is understood as the divisibility by p^s of all coefficients of $F(w) - G(w)$.

Throughout this section $x = (t, z)$, where $t = (t_1, \dots, t_r)$ and $z = (z_1, \dots, z_n)$ are two groups of variables.

2.1. Definition of ghosts

Let $\Lambda = (\Lambda_0(x), \Lambda_1(x), \dots, \Lambda_l(x))$ be a tuple of Laurent polynomials in $\mathbb{Z}_p[x^{\pm 1}]$. For every $0 \leqslant j \leqslant s \leqslant l$, we define the Laurent polynomials

$$W_s(x) = W_s(\Lambda_0, \dots, \Lambda_s)(x) := \Lambda_0(x)\Lambda_1(x)^p \dots \Lambda_s(x)^{p^s}$$

and

$$\begin{aligned} W_s^{(j)}(x) &= W_s^{(j)}(\Lambda_0, \dots, \Lambda_s)(x) \\ &:= W_{s-j}(\Lambda_j, \dots, \Lambda_s)(x) = \Lambda_j(x)\Lambda_{j+1}(x)^p \dots \Lambda_s(x)^{p^{s-j}}. \end{aligned}$$

Furthermore, we introduce the tuple $V_s = V_s(x) = V_s(\Lambda_0, \dots, \Lambda_s)(x)$, $s = 0, 1, \dots, l$, of Laurent polynomials in $\mathbb{Z}_p[x^{\pm 1}]$ by the recursive formula

$$V_s(x) = W_s(x) - \sum_{j=1}^{s} V_{j-1}(x) W_s^{(j)}(x^{p^j}). \tag{2.1}$$

The Laurent polynomials $V_s(x)$ are called *ghosts* associated with the tuple Λ. They are useful for studying the congruences related to the tuple, but they do not enter the final results. The ghosts $V_s(x)$ are essentially offered in Vlasenko's work [22], though stated there for a very particular situation. The ghosts $V_s(x)$ are quite different from the ghosts we use in our previous work [21] — those are rooted in Mellit's preprint [10].

Lemma 2.1. *For $s = 0, 1, \dots, l$, we have $V_s(x) \equiv 0 \pmod{p^s}$.*

Proof. For $s = 0$ we have $V_0(x) = \Lambda_0(x)$ and no requirements on divisibility. For $s = 1$, we have modulo p:

$$V_1(x) = \Lambda_0(x)\Lambda_1(x)^p - V_0(x)\Lambda_1(x^p) = \Lambda_0(x)\big(\Lambda_1(x)^p - \Lambda_1(x^p)\big) \equiv 0.$$

More generally, for $s \geqslant 1$ applying $V_{j-1}(x) \equiv 0 \pmod{p^{j-1}}$ for $0 < j \leqslant s$ and

$$\begin{aligned}\Lambda_s(x)^{p^s} &\equiv \Lambda_s\big(x^p\big)^{p^{s-1}} \pmod{p^s} \equiv \Lambda_s\big(x^{p^2}\big)^{p^{s-2}} \pmod{p^{s-1}}\\ &\equiv \dots \equiv \Lambda_s\big(x^{p^j}\big)^{p^{s-j}} \pmod{p^{s-j+1}}\end{aligned}$$

(which follows from iterative use of $F(x^p)^{p^{i-1}} \equiv F(x)^{p^i} \pmod{p^i}$ valid for $i > 0$) we deduce modulo p^s:

$$\begin{aligned}V_s(x) &= W_{s-1}(x)\Lambda_s(x)^{p^s} - \sum_{j=1}^{s-1} V_{j-1}(x)W_{s-1}^{(j)}(x^{p^j})\Lambda_s\big(x^{p^j}\big)^{p^{s-j}} - V_{s-1}(x)\Lambda_s(x^{p^s})\\ &\equiv \Bigg(W_{s-1}(x) - \sum_{j=1}^{s-1} V_{j-1}(x)W_{s-1}^{(j)}(x^{p^j}) - V_{s-1}(x)\Bigg)\Lambda_s(x)^{p^s} = 0,\end{aligned}$$

giving the required statement. □

For a Laurent polynomial $F(t, z)$ in t, z, let $N(F) \subset \mathbb{R}^r$ be the Newton polytope of $F(t, z)$ with respect to the t *variables only.*

Lemma 2.2. *For $s = 0, 1, \dots, l$, we have*

$$N(V_s) \subset N(\Lambda_0) + pN(\Lambda_1) + \dots + p^s N(\Lambda_s).$$

Proof. This follows from (2.1) by induction on s. □

2.2. Admissible tuples

Let $\Delta \subset \mathbb{Z}^r$ be a finite subset.

Definition 2.3. *A tuple $(N_0, N_1, \dots, N_l)$ of convex polytopes in $\mathbb{R}^r$ is called* Δ-admissible *if for any $0 \leqslant i \leqslant j < l$ we have*

$$(\Delta + N_i + pN_{i+1} + \dots + p^{j-i}N_j) \cap p^{j-i+1}\mathbb{Z}^r \subset p^{j-i+1}\Delta.$$

Notice that subtuples $(N_i, N_{i+1}, \dots, N_j)$ of a Δ-admissible tuple are also Δ-admissible.

Example. Let $r = 1$, $\Delta = \{0, 1\} \subset \mathbb{Z}$ and $N = [-(p-1)/2, 3(p-1)/2] \subset \mathbb{R}$. Then the tuple $(N, N, \dots, N)$ is Δ-admissible.

Definition 2.4. *A tuple $(\Lambda_0(t, z), \Lambda_1(t, z), \dots, \Lambda_l(t, z))$ of Laurent polynomials is called* Δ-admissible *if the tuple $(N(\Lambda_0), N(\Lambda_1), \dots, N(\Lambda_l))$ is Δ-admissible.*

2.3. Hasse–Witt matrix

For $v \in \mathbb{Z}^r$ denote by $\operatorname{Coeff}_v F(t,z)$ the coefficient of t^v in the Laurent polynomial $F(t,z)$; clearly, this is a Laurent polynomial in z.

Given $m \geqslant 1$ and a finite subset $\Delta \subset \mathbb{Z}^r$ with $g = \#\Delta$, define the $g \times g$ *Hasse–Witt matrix* of level m of the Laurent polynomial $F(t,z)$ by the formula

$$A\big(m, F(t,z)\big) := \big(\operatorname{Coeff}_{p^m v - u} F(t,z)\big)_{u\in\Delta, v\in\Delta}. \tag{2.2}$$

The entries of this matrix are Laurent polynomials in z.

Furthermore, for a Laurent polynomial $G(z)$ define $\sigma(G(z)) = G(z^p)$.

Lemma 2.5. *Let $(\Lambda_0(t,z), \Lambda_1(t,z), \dots, \Lambda_l(t,z))$ be a Δ-admissible tuple of Laurent polynomials in $\mathbb{Z}_p[x^{\pm 1}] = \mathbb{Z}_p[t^{\pm 1}, z^{\pm 1}]$. Then for $0 \leqslant s \leqslant l$ we have*

(i) $A(s+1, V_s) \equiv 0 \pmod{p^s}$;

(ii)
$$\begin{aligned} A(s+1, W_s) = A(1, V_0) \cdot \sigma\big(A(s, W_s^{(1)})\big) \\ + A(2, V_1) \cdot \sigma^2\big(A(s-1, W_s^{(2)})\big) + \cdots \\ + A(s, V_{s-1}) \cdot \sigma^s\big(A(1, W_s^{(s)})\big) + A(s+1, V_s). \end{aligned}$$

Proof. Part (i) follows from Lemma 2.1. To prove (ii) consider the identity

$$\begin{aligned} &\Lambda_0(t,z)\Lambda_1(t,z)^p \dots \Lambda_s(t,z)^{p^s} \\ &\quad = \sum_{j=1}^{s} V_{j-1}(t,z)\Lambda_j\big(t^{p^j}, z^{p^j}\big)\Lambda_{j+1}\big(t^{p^j}, z^{p^j}\big)^p \dots \Lambda_s\big(t^{p^j}, z^{p^j}\big)^{p^{s-j}} + V_s(t,z), \end{aligned}$$

which is nothing else but (2.1). Let $u, v \in \Delta$. In order to calculate the coefficient of $t^{p^{s+1}v-u}$ in the term $V_{j-1}(t,z)\Lambda_j(t^{p^j}, z^{p^j}) \dots \Lambda_s(t^{p^j}, z^{p^j})^{p^{s-j}}$, we look for all pairs of vectors $w \in N(V_{j-1})$ and $y \in N(\Lambda_j(t,z) \dots \Lambda_s(t,z)^{p^{s-j}})$ such that

$$w + p^j y = p^{s+1}v - u,$$

hence $u + w \in p^j\mathbb{Z}^r$. On the other hand, it follows from Lemma 2.2 that $w \in N(\Lambda_0) + pN(\Lambda_1) + \cdots + p^{j-1}N(\Lambda_{j-1})$, so that

$$u + w \in \Delta + N(\Lambda_0) + pN(\Lambda_1) + \cdots + p^{j-1}N(\Lambda_{j-1}).$$

From the Δ-admissibility we deduce that $u+w=p^j\delta$ for some $\delta\in\Delta$, thus $w=p^j\delta-u$, $y=p^{s+1-j}v-\delta$ and

$$\begin{aligned}&\operatorname{Coeff}_{p^{s+1}v-u}\big(V_{j-1}(t,z)\Lambda_j\big(t^{p^j},z^{p^j}\big)\Lambda_{j+1}\big(t^{p^j},z^{p^j}\big)^p\dots\Lambda_s\big(t^{p^j},z^{p^j}\big)^{p^{s-j}}\big)\\&\quad=\sum_{\delta\in\Delta}\operatorname{Coeff}_{p^j\delta-u}\big(V_{j-1}(t,z)\big)\\&\qquad\times\sigma^j\big(\operatorname{Coeff}_{p^{s+1-j}v-\delta}\big(\Lambda_j(t,z)\Lambda_{j+1}(t,z)^p\dots\Lambda_s(t,z)^{p^{s-j}}\big)\big).\end{aligned}$$

This proves (ii). $\square$

Our next results discuss congruences of the type

$$F_1(z)F_2(z)^{-1}\equiv G_1(z)G_2(z)^{-1}\pmod{p^s},$$

where F_1,F_2,G_1,G_2 are $g\times g$ matrices whose entries are Laurent polynomials in z. We consider such congruences when the determinants $\det F_2(z)$ and $\det G_2(z)$ are Laurent polynomials both nonzero modulo p. Using Cramer's rule we write the entries of the inverse matrix $F_2(z)^{-1}$ in the form $f_{ij}(z)/\det F_2(z)$ for $f_{ij}(z)\in\mathbb{Z}_p[z^{\pm1}]$ and do a similar computation for $G_2(z)$. This presents the congruence $F_1(z)F_2(z)^{-1}\equiv G_1(z)G_2(z)^{-1}\pmod{p^s}$ in the form $F(z)\cdot1/\det F_2(z)\equiv G(z)\cdot1/\det G_2(z)\pmod{p^s}$ for some $g\times g$ matrices F,G with entries in $\mathbb{Z}_p[z^{\pm1}]$, while the latter is nothing else but the congruence $F(z)\cdot\det G_2(z)\equiv G(z)\cdot\det F_2(z)\pmod{p^s}$.

Theorem 2.6. *Let $(\Lambda_0(t,z),\Lambda_1(t,z),\dots,\Lambda_l(t,z))$ be a Δ-admissible tuple of Laurent polynomials in $\mathbb{Z}_p[x^{\pm1}]=\mathbb{Z}_p[t^{\pm1},z^{\pm1}]$.*

(i) *For $0\leqslant s\leqslant l$ we have*

$$\begin{aligned}&A\big(s+1,\Lambda_0(x)\Lambda_1(x)^p\dots\Lambda_s(x)^{p^s}\big)\\&\quad\equiv A\big(1,\Lambda_0(x)\big)\,\sigma\big(A\big(1,\Lambda_1(x)\big)\big)\dots\sigma^s\big(A\big(1,\Lambda_s(x)\big)\big)\pmod p.\end{aligned}$$

(ii) *Assume that the determinants of the matrices $A(1,\Lambda_i(t,z))$, $i=0,1,\dots,l$, are Laurent polynomials all nonzero modulo p. Then for $1\leqslant s\leqslant l$ the determinant of the matrix $A(s,\Lambda_1(x)\Lambda_2(x)^p\dots\Lambda_s(x)^{p^{s-1}})$ is a Laurent polynomial nonzero modulo p and we have modulo p^s:*

$$\begin{aligned}&A\big(s+1,\Lambda_0(x)\Lambda_1(x)^p\dots\Lambda_s(x)^{p^s}\big)\\&\qquad\times\sigma\big(A\big(s,\Lambda_1(x)\Lambda_2(x)^p\dots\Lambda_s(x)^{p^{s-1}}\big)\big)^{-1}\\&\quad\equiv A\big(s,\Lambda_0(x)\Lambda_1(x)^p\dots\Lambda_{s-1}(x)^{p^{s-1}}\big)\end{aligned}$$

$$\times \sigma\big(A\big(s-1, \Lambda_1(x)\Lambda_2(x)^p \dots \Lambda_{s-1}(x)^{p^{s-2}}\big)\big)^{-1},$$

where for $s = 1$ we understand the second factor on the right as the $g \times g$ identity matrix.

Proof. By Lemma 2.5 we have

$$\begin{aligned} &A\big(s+1, \Lambda_0(x)\Lambda_1(x)^p \dots \Lambda_s(x)^{p^s}\big) \\ &\quad \equiv A\big(1, \Lambda_0(x)\big) \cdot \sigma\big(A\big(s, \Lambda_1(x)\Lambda_2(x)^p \dots \Lambda_s(x)^{p^{s-1}}\big)\big) \pmod{p}. \end{aligned}$$

Iteration gives part (i) of the theorem.

If the determinants of the matrices $A(1, \Lambda_i(t,z))$, $i = 0, 1, \dots, l$, are Laurent polynomials all nonzero modulo p, then part (i) implies that the determinant

$$\begin{aligned} &\det A\big(s, \Lambda_1(x)\Lambda_2(x)^p \dots \Lambda_s(x)^{p^{s-1}}\big) \\ &\quad \equiv \prod_{j=1}^{s} \det \sigma^{j-1}\big(A\big(1, \Lambda_j(t,z)\big)\big) \pmod{p} \end{aligned}$$

is a Laurent polynomial nonzero modulo p. This proves the first statement of part (ii) of the theorem and allows us to consider the inverse matrices in the congruence of part (ii).

We prove part (ii) by induction on s. The case $s = 1$ follows from part (i). For $1 < s < l$ we substitute the expressions for $A(s+1, W_s(x))$ and $A(s, W_{s-1}(x))$ from part (ii) of Lemma 2.5 into the two sides of the desired congruence:

$$\begin{aligned} &A\big(s+1, \Lambda_0(x)\Lambda_1(x)^p \dots \Lambda_s(x)^{p^s}\big) \cdot \sigma\big(A\big(s, \Lambda_1(x)\Lambda_2(x)^p \dots \Lambda_s(x)^{p^{s-1}}\big)\big)^{-1} \\ &\quad = A(1, V_0) + \sum_{j=2}^{s} A(j, V_{j-1}) \cdot \sigma^j\big(A\big(s-j+1, W_s^{(j)}\big)\big) \cdot \sigma\big(A\big(s, W_s^{(1)}\big)\big)^{-1} \\ &\qquad + A(s+1, V_s) \cdot \sigma\big(A\big(s, W_s^{(1)}\big)\big)^{-1} \end{aligned}$$

and

$$\begin{aligned} &A\big(s, \Lambda_0(x)\Lambda_1(x)^p \dots \Lambda_{s-1}(x)^{p^{s-1}}\big) \\ &\qquad \times \sigma\big(A\big(s-1, \Lambda_1(x)\Lambda_2(x)^p \dots \Lambda_{s-1}(x)^{p^{s-2}}\big)\big)^{-1} \\ &\quad = A(1, V_0) + \sum_{j=2}^{s} A(j, V_{j-1}) \cdot \sigma^j\big(A\big(s-j, W_{s-1}^{(j)}\big)\big)\sigma\big(A\big(s-1, W_{s-1}^{(1)}\big)\big)^{-1}. \end{aligned}$$

Since we want to compare these two expressions modulo p^s, the last term in the upper sum containing $A(s+1, V_s) \equiv 0 \pmod{p^s}$ can be ignored.

Given $j = 2, \ldots, s$, we use the inductive hypothesis as follows:

$$\begin{aligned} &A(s-i+1, W_s^{(i)}) \cdot \sigma\big(A(s-i, W_s^{(i+1)})\big)^{-1} \\ &\quad \equiv A(s-i, W_{s-1}^{(i)}) \cdot \sigma\big(A(s-i-1, W_{s-1}^{(i+1)})\big)^{-1} \pmod{p^{s-i}} \end{aligned}$$

for $i = 1, \ldots, j-1$. Applying σ^{i-1} to the i-th congruence and multiplying them out lead to telescoping products on both sides:

$$\begin{aligned} &A(s, W_s^{(1)}) \cdot \sigma^{j-1}\big(A(s-j+1, W_s^{(j)})\big)^{-1} \\ &\quad \equiv A(s-1, W_{s-1}^{(1)}) \cdot \sigma^{j-1}\big(A(s-j, W_{s-1}^{(j)})\big)^{-1} \pmod{p^{s-j+1}}. \end{aligned}$$

By our assumptions these four matrices are invertible. Therefore, we can invert them to obtain the congruence

$$\begin{aligned} &\sigma^{j-1}\big(A(s-j+1, W_s^{(j)})\big) \cdot A(s, W_s^{(1)})^{-1} \\ &\quad \equiv \sigma^{j-1}\big(A(s-j, W_{s-1}^{(j)})\big) \cdot A(s-1, W_{s-1}^{(1)})^{-1} \pmod{p^{s-j+1}}. \end{aligned} \tag{2.3}$$

Since $A(j, V_{j-1}) \equiv 0 \pmod{p^{j-1}}$, application of σ to (2.3) and summation in j of the resulted congruences

$$\begin{aligned} &A(j, V_{j-1}(x)) \cdot \sigma^j\big(A(s-j+1, W_s^{(j)})\big) \cdot \sigma\big(A(s, W_s^{(1)})\big)^{-1} \\ &\quad \equiv A(j, V_{j-1}(x)) \cdot \sigma^j\big(A(s-j, W_{s-1}^{(j)})\big) \cdot \sigma\big(A(s-1, W_{s-1}^{(1)})\big)^{-1} \pmod{p^s} \end{aligned}$$

completes the proof of part (ii) of the theorem. ☐

Corollary 2.7. *Under the assumptions of part* (ii) *of Theorem* 2.6 *for* $1 \leqslant s \leqslant l$ *we have*:

$$\begin{aligned} &\det A\big(s+1, \Lambda_0(x)\Lambda_1(x)^p \ldots \Lambda_s(x)^{p^s}\big) \\ &\qquad \times \det \sigma\big(A\big(s-1, \Lambda_1(x)\Lambda_2(x)^p \ldots \Lambda_{s-1}(x)^{p^{s-2}}\big)\big) \\ &\quad \equiv \det A\big(s, \Lambda_0(x)\Lambda_1(x)^p \ldots \Lambda_{s-1}(x)^{p^{s-1}}\big) \\ &\qquad \times \det \sigma\big(A\big(s, \Lambda_1(x)\Lambda_2(x)^p \ldots \Lambda_s(x)^{p^{s-1}}\big)\big) \pmod{p^s}. \end{aligned}$$

2.4. Derivatives

Recall that $z = (z_1, \ldots, z_n)$. Denote

$$D_v = \frac{\partial}{\partial z_v}, \quad v = 1, \ldots, n.$$

Let $F_1(z), F_2(z), G_1(z), G_2(z) \in \mathbb{Z}_p[z^{\pm 1}]$ and $m \geqslant 1$. If

$$D_v\big(F_1(z)\big) \cdot F_2(z) \equiv D_v\big(G_1(z)\big) \cdot G_2(z) \pmod{p^s},$$

then

$$\begin{aligned} D_v\big(\sigma^m(F_1(z))\big) \cdot \sigma^m(F_2(z)) &- D_v\big(\sigma^m(G_1(z))\big) \cdot \sigma^m(G_2(z)) \\ &= D_v\big(F_1(z^{p^m})\big) \cdot F_2(z^{p^m}) - D_v\big(G_1(z^{p^m})\big) \cdot G_2(z^{p^m}) \\ &= p^m z_v^{p^m-1}\big(D_v(F_1(z)) \cdot F_2(z) - D_v(G_1(z)) \cdot G_2(z)\big)\big|_{z \to z^{p^m}} \\ &\equiv 0 \pmod{p^{s+m}}. \end{aligned} \tag{2.4}$$

Theorem 2.8. *Let $(\Lambda_0(t,z), \Lambda_1(t,z), \dots, \Lambda_l(t,z))$ be a Δ-admissible tuple of Laurent polynomials in $\mathbb{Z}_p[x^{\pm 1}] = \mathbb{Z}_p[t^{\pm 1}, z^{\pm 1}]$. Let $D = D_v$ for some $v = 1, \dots, n$. Then under assumptions of part* (ii) *of Theorem 2.6 we have*

$$\begin{aligned} D\big(\sigma^m(A(s+1, \Lambda_0\Lambda_1^p \dots \Lambda_s^{p^s}))\big) \,\cdot\, &\sigma^m\big(A(s+1, \Lambda_0\Lambda_1^p \dots \Lambda_s^{p^s})\big)^{-1} \\ &\equiv D\big(\sigma^m(A(s, \Lambda_0\Lambda_1^p \dots \Lambda_{s-1}^{p^{s-1}}))\big) \\ &\quad \times \sigma^m\big(A(s, \Lambda_0\Lambda_1^p \dots \Lambda_{s-1}^{p^{s-1}})\big)^{-1} \pmod{p^{s+m}} \end{aligned} \tag{2.5}$$

for $1 \leqslant s \leqslant l$ and $0 \leqslant m$.

Proof. First we notice that it is sufficient to establish the congruences (2.5) for $m = 0$, as the general m case follows from (2.4). So, we assume that $m = 0$ and proceed by induction on $s \geqslant 0$. For $s = 0$ the statement is trivially true.

Consider the case of general s. Using part (ii) of Lemma 2.5 we can write

$$\begin{aligned} &D\big(A(s+1, W_s)\big)\, A(s+1, W_s)^{-1} = \\ &\quad = \sum_{j=1}^{s+1} D\big(A(j, V_{j-1})\big)\, \sigma^j\big(A(s-j+1, W_s^{(j)})\big)\, A(s+1, W_s)^{-1} + \\ &\qquad + \sum_{j=1}^{s+1} A(j, V_{j-1})\, D\big(\sigma^j\big(A(s-j+1, W_s^{(j)})\big)\big)\, A(s+1, W_s)^{-1} \end{aligned}$$

and

$$\begin{aligned} &D\big(A(s, W_{s-1})\big)\, A(s, W_{s-1})^{-1} \\ &\quad = \sum_{j=1}^{s} D\big(A(j, V_{j-1})\big)\, \sigma^j\big(A(s-j, W_{s-1}^{(j)})\big)\, A(s, W_{s-1})^{-1} \end{aligned}$$

$$+\sum_{j=1}^{s} A(j,V_{j-1})\,D(\sigma^j(A(s-j,W_{s-1}^{(j)})))\,A(s,W_{s-1})^{-1}.$$

The summand corresponding to $j=s+1$ in the first expression vanishes modulo p^s, because $A(s+1,V_s)\equiv 0 \pmod{p^s}$, implying that $D(A(s+1,V_s))\equiv 0 \pmod{p^s}$. For the same reason $D(A(j,V_{j-1}))\equiv 0 \pmod{p^{j-1}}$ more generally; combining this with the congruence

$$\begin{aligned}\sigma^j(A(s-j+1,W_s^{(j)}))\,&A(s+1,W_s)^{-1}\\ &\equiv \sigma^j(A(s-j,W_{s-1}^{(j)}))\,A(s,W_{s-1})^{-1} \pmod{p^{s-j+1}}\end{aligned} \tag{2.6}$$

and summing over j we arrive at

$$\begin{aligned}\sum_{j=1}^{s+1} &D(A(j,V_{j-1}))\,\sigma^j(A(s-j+1,W_s^{(j)}))\,A(s+1,W_s)^{-1}\\ &\equiv \sum_{j=1}^{s} D(A(j,V_{j-1}))\,\sigma^j(A(s-j,W_{s-1}^{(j)}))\,A(s,W_{s-1})^{-1} \pmod{p^s}.\end{aligned}$$

Here (2.6) follows from (2.3), in which we take $j+1$ and $s+1$ for j and s and use $W_s=\Lambda_0\Lambda_1^p\dots\Lambda_s^{p^s}$ instead of $W_{s+1}^{(1)}=\Lambda_1\Lambda_2^p\dots\Lambda_{s+1}^{p^s}$.

To match the other sums we recall the inductive hypothesis in the form

$$\begin{aligned}D(\sigma^j(A(s-j+1,W_s^{(j)})))\,&\sigma^j(A(s-j+1,W_s^{(j)}))^{-1}\\ &\equiv D(\sigma^j(A(s-j,W_{s-1}^{(j)})))\,\sigma^j(A(s-j,W_{s-1}^{(j)}))^{-1} \pmod{p^s}\end{aligned} \tag{2.7}$$

and notice that both sides in (2.7) are congruent to zero modulo p^j by formula (2.4). Therefore, multiplying congruences (2.7) and (2.6) (in this order!) we obtain

$$\begin{aligned}D(\sigma^j(A(s-j+1,W_s^{(j)})))\,&A(s+1,W_s)^{-1}\\ &\equiv D(\sigma^j(A(s-j,W_{s-1}^{(j)})))\,A(s,W_{s-1})^{-1} \pmod{p^s};\end{aligned}$$

then multiplying both sides of this congruence by $A(j,V_{j-1})$ from the left and summing over j we deduce

$$\sum_{j=1}^{s} A(j,V_{j-1})\,D(\sigma^j(A(s-j+1,W_s^{(j)})))\,A(s+1,W_s)^{-1}$$

$$\equiv \sum_{j=1}^{s} A(j, V_{j-1})\, D(\sigma^j(A(s-j, W^{(j)}_{s-1})))\, A(s, W_{s-1})^{-1} \pmod{p^s}.$$

This completes the proof of the theorem. □

There are similar congruences for higher order derivatives of the matrices $A(s+1, W_s)$. We restrict ourselves with the second order derivatives.

Theorem 2.9. *Let $(\Lambda_0(t,z), \Lambda_1(t,z), \dots, \Lambda_l(t,z))$ be a Δ-admissible tuple of Laurent polynomials in $\mathbb{Z}_p[x^{\pm 1}] = \mathbb{Z}_p[t^{\pm 1}, z^{\pm 1}]$. Then under assumptions of part* (ii) *of Theorem 2.6 we have*

$$\begin{aligned} &D_u(D_v(A(s+1, \Lambda_0\Lambda_1^p \dots \Lambda_s^{p^s}))) \cdot A(s+1, \Lambda_0\Lambda_1^p \dots \Lambda_s^{p^s})^{-1} \\ &\quad \equiv D_u(D_v(A(s, \Lambda_0\Lambda_1^p \dots \Lambda_{s-1}^{p^{s-1}}))) \\ &\quad\quad \times A(s, \Lambda_0\Lambda_1^p \dots \Lambda_{s-1}^{p^{s-1}})^{-1} \pmod{p^{s+2m}} \end{aligned} \tag{2.8}$$

for all $1 \leqslant u, v \leqslant n$ and $1 \leqslant s \leqslant l$.

Proof. Notice that, for an invertible matrix $F(z)$ and a derivation D, we have $D(F^{-1}) = -F^{-1}\, D(F)\, F^{-1}$.

We apply the derivation D_u to congruence (2.5) with $D = D_v$ and $m = 0$:

$$\begin{aligned} &D_u(D_v(A(s+1, \dots)))\, A(s+1, \dots)^{-1} \\ &\quad - D_v(A(s+1, \dots))\, A(s+1, \dots)^{-1}\, D_u(A(s+1, \dots))\, A(s+1, \dots)^{-1} \\ &\equiv D_u(D_v(A(s, \dots)))\, A(s, \dots)^{-1} - \\ &\quad - D_v(A(s, \dots))\, A(s, \dots)^{-1}\, D_u(A(s, \dots))\, A(s, \dots)^{-1} \pmod{p^s}, \end{aligned}$$

where we write $A(s+1, \dots)$ and $A(s, \dots)$ for

$$A(s+1, \Lambda_0\Lambda_1^p \dots \Lambda_s^{p^s}) \quad \text{and} \quad A(s, \Lambda_0\Lambda_1^p \dots \Lambda_{s-1}^{p^{s-1}}).$$

It remains to apply (2.5) with $D = D_u$ and $D = D_v$ and $m = 0$ to see that the second terms on both sides agree modulo p^s. After their cancellation we are left with the required congruences in (2.8). □

3. Convergences

3.1. Infinite tuples

Let $\Lambda = (\Lambda_0(x), \Lambda_1(x), \Lambda_2(x), \dots)$ be an infinite Δ-admissible tuple of Laurent polynomials in $\mathbb{Z}_p[x^{\pm 1}]$ with only finitely many distinct elements. Thus there is a finite set $\{F_1(x), \dots, F_f(x)\} \subset \mathbb{Z}_p[x^{\pm 1}] = \mathbb{Z}_p[t^{\pm 1}, z^{\pm 1}]$ of distinct Laurent polynomials such that for any j there is a unique $1 \leqslant i_j \leqslant f$ with $\Lambda_j(x) = F_{i_j}(x)$.

Definition 3.1. *The Δ-admissible tuple Λ is called* nondegenerate*, if for any $i = 1, \dots, f$, the Laurent polynomial* $\det A(1, F_i(x)) \in \mathbb{Z}_p[z^{\pm 1}]$ *is nonzero modulo p.*

Recall the notation:

$$W_s(x) = \Lambda_0(x)\Lambda_1(x)^p \dots \Lambda_s(x)^{p^s},$$
$$W_s^{(j)}(x) = \Lambda_j(x)\Lambda_{j+1}(x)^p \dots \Lambda_s(x)^{p^{s-j}}.$$

If a Δ-admissible tuple Λ is nondegenerate, then for any $0 \leqslant j \leqslant s$, the Laurent polynomials $\det A(s-j+1, W_s^{(j)}(x)) \in \mathbb{Z}_p[z^{\pm 1}]$ are not congruent to zero modulo p and we may consider congruences involving the inverse matrices $A(s-j+1, W_s^{(j)}(x))^{-1}$.

3.2. Domain of convergence

Recall that $z = (z_1, \dots, z_n)$. Denote

$$\mathfrak{D} = \big\{z \in \mathbb{Z}_p^n \ \big| \ |\det A\big(1, F_i(t,z)\big)|_p = 1, \ i = 1, \dots, f\big\}.$$

Lemma 3.2. *For any $0 \leqslant j \leqslant s$ and $a \in \mathfrak{D}$ we have*

$$|\det A\big(s-j+1, W_s^{(j)}(t,a)\big)|_p = 1. \qquad \square$$

Corollary 3.3. *All entries of $A(s-j+1, W_s^{(j)}(t,z))^{-1}$ are rational functions in z regular on $\mathfrak{D}$. For every $a \in \mathfrak{D}$ all entries of $A(s-j+1, W_s^{(j)}(t,a))$ and $A(s-j+1, W_s^{(j)}(t,a))^{-1}$ are elements of $\mathbb{Z}_p$.* $\square$

Theorem 3.4. *Let $\Lambda = (\Lambda_0(x), \Lambda_1(x), \Lambda_2(x), \dots)$ be an infinite nondegenerate Δ-admissible tuple. Consider the sequence of $g \times g$ matrices*

$$\Big(A\big(s+1, W_s(t,z)\big) \cdot \sigma\big(A\big(s, W_s^{(1)}(t,z)\big)\big)^{-1}\Big)_{s \geqslant 0} \tag{3.1}$$

whose entries are rational functions in z regular on the domain $\mathfrak{D}$. This sequence uniformly converges on $\mathfrak{D}$ as $s \to \infty$ to an analytic $g \times g$ matrix with values in $\mathbb{Z}_p$. Denote this matrix by $\mathcal{A}_\Lambda(z)$. For $a \in \mathfrak{D}$ we have

$$|\det \mathcal{A}_\Lambda(a)|_p = 1 \tag{3.2}$$

and the matrix $\mathcal{A}_\Lambda(a)$ is invertible.

Proof. By part (i) of Theorem 2.6 we have $|\det\sigma(A(s,W_s^{(1)}(t,a)))|_p = 1$ for $a\in\mathfrak{D}$. Hence $A(s+1,W_s(t,z))\cdot\sigma(A(s,W_s^{(1)}(t,z)))^{-1}$ is a matrix of rational functions in z regular on $\mathfrak{D}$. Moreover, if $a\in\mathfrak{D}$, then every entry of this matrix is an element of $\mathbb{Z}_p$. The uniform convergence on $\mathfrak{D}$ of the sequence (3.1) is a corollary of part (ii) of Theorem 2.6. Equation (3.2) follows from part (i) of Theorem 2.6. The theorem is proved. $\square$

Theorem 3.5. *Let $\Lambda = (\Lambda_0(x),\Lambda_1(x),\Lambda_2(x),\dots)$ be an infinite nondegenerate Δ-admissible tuple, and $D = D_v$, $v=1,\dots,n$. Given $m\geqslant 0$ consider the sequence of $g\times g$ matrices*

$$\big(D\big(\sigma^m(A(s+1,W_s(x)))\big)\cdot\sigma^m\big(A(s+1,W_{s+1}(x))\big)^{-1}\big)_{s\geqslant 0}$$

whose entries are rational functions in z regular on the domain $\mathfrak{D}$. This sequence uniformly converges on $\mathfrak{D}$ as $s\to\infty$ to an analytic $g\times g$ matrix with values in $\mathbb{Z}_p$. Denote this matrix by $\mathcal{A}_{\Lambda,D\sigma^m}(z)$.

Proof. The theorem is a corollary of Theorem 2.8. $\square$

Theorem 3.6. *Let $\Lambda = (\Lambda_0(x),\Lambda_1(x),\Lambda_2(x),\dots)$ be an infinite nondegenerate Δ-admissible tuple. Given $1\leqslant u,v\leqslant n$, consider the sequence of $g\times g$ matrices*

$$\big(D_u\big(D_v(A(s+1,W_s(x)))\big)\cdot A\big(s+1,W_{s+1}(x)\big)^{-1}\big)_{s\geqslant 0}$$

whose entries are rational functions in z regular on the domain $\mathfrak{D}$. This sequence uniformly converges on $\mathfrak{D}$ as $s\to\infty$ to an analytic $g\times g$ matrix with values in $\mathbb{Z}_p$. Denote this matrix by $\mathcal{A}_{\Lambda,D_uD_v}(z)$.

Proof. The theorem is a corollary of Theorem 2.9. $\square$

Let $\Lambda = (\Lambda_0(x),\Lambda_1(x),\Lambda_2(x),\dots)$ be an infinite nondegenerate Δ-admissible tuple. Let $1\leqslant u,v\leqslant n$. Consider the $g\times g$ matrix valued functions $\mathcal{A}_{\Lambda,\frac{\partial}{\partial z_u}\sigma^0}(z)$, $\mathcal{A}_{\Lambda,\frac{\partial}{\partial z_v}\sigma^0}(z)$ in Theorem 3.5 and denote them by $\mathcal{A}_u(z)$, $\mathcal{A}_v(z)$, respectively. Consider the $g\times g$ matrix valued function $\mathcal{A}_{\Lambda,\frac{\partial}{\partial z_u}\frac{\partial}{\partial z_v}}(z)$ in Theorem 3.6 and denote it by $\mathcal{A}_{u,v}(z)$. All the three functions are analytic on $\mathfrak{D}$.

Lemma 3.7. *We have*

$$\frac{\partial}{\partial z_u}\mathcal{A}_v = \mathcal{A}_{u,v} - \mathcal{A}_v\mathcal{A}_u.$$

Proof. The lemma follows from the formula

$$\frac{\partial}{\partial z_u}\left(\frac{\partial A}{\partial z_v}\cdot A^{-1}\right) = \frac{\partial^2 A}{\partial z_u\partial z_v}\cdot A^{-1} - \frac{\partial A}{\partial z_v}\cdot A^{-1}\cdot\frac{\partial A}{\partial z_v}\cdot A^{-1}. \qquad\square$$

4. KZ equations

4.1. KZ equations

Let $\mathfrak{g}$ be a simple Lie algebra with an invariant scalar product. The Casimir element is $\Omega = \sum_i h_i \otimes h_i \in \mathfrak{g} \otimes \mathfrak{g}$, where $(h_i) \subset \mathfrak{g}$ is an orthonormal basis. Let $V = \otimes_{i=1}^n V_i$ be a tensor product of $\mathfrak{g}$-modules, $\kappa \in \mathbb{C}^\times$ a nonzero number. The *KZ equations* is the system of differential equations on a V-valued function $I(z_1, \dots, z_n)$,

$$\frac{\partial I}{\partial z_i} = \frac{1}{\kappa} \sum\nolimits_{j \neq i} \frac{\Omega_{i,j}}{z_i - z_j} I, \qquad i = 1, \dots, n,$$

where $\Omega_{i,j} : V \to V$ is the Casimir operator acting in the ith and jth tensor factors, see [8, 5].

This system is a system of Fuchsian first order linear differential equations. The equations are defined on the complement in $\mathbb{C}^n$ to the union of all diagonal hyperplanes.

The object of our discussion is the following particular case. Let $n = 2g+1$ be an odd positive integer. We consider the following system of differential and algebraic equations for a column n-vector $I = (I_1, \dots, I_n)$ depending on variables $z = (z_1, \dots, z_n)$:

$$\frac{\partial I}{\partial z_i} = \frac{1}{2} \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} I, \quad i = 1, \dots, n, \qquad I_1 + \dots + I_n = 0, \tag{4.1}$$

where $z = (z_1, \dots, z_n)$, the $n \times n$-matrices Ω_{ij} have the form

$$\Omega_{ij} = \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \end{array} \begin{pmatrix} & \overset{i}{\vdots} & & \overset{j}{\vdots} & \\ \cdots & -1 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & 1 & \cdots & -1 & \cdots \\ & \vdots & & \vdots & \end{pmatrix},$$

and all other entries are zero. This joint system of *differential and algebraic equations* will be called the *system of KZ equations* in this paper.

For $i = 1, \dots, n$ denote

$$H_i(z) = \tfrac{1}{2} \sum\nolimits_{j \neq i} \tfrac{\Omega_{ij}}{z_i - z_j}, \qquad \nabla_i^{\mathrm{KZ}} = \tfrac{\partial}{\partial z_i} - H_i(z), \qquad i = 1, \dots, n. \tag{4.2}$$

The linear operators $H_i(z)$ are called the Gaudin Hamiltonians. The KZ equations can be written as the system of equations,

$$\nabla_i^{\mathrm{KZ}} I = 0, \quad i = 1, \dots, n, \qquad I_1 + \dots + I_n = 0.$$

System (4.1) is the system of the differential KZ equations with parameter $\kappa = 2$ associated with the Lie algebra $\mathfrak{sl}_2$ and the subspace of singular vectors of weight $2g-1$ of the tensor power $(\mathbb{C}^2)^{\otimes(2g+1)}$ of two-dimensional irreducible $\mathfrak{sl}_2$-modules, up to a gauge transformation, see this example in [17, Section 1.1].

4.2. Solutions over $\mathbb{C}$

Define the *master function*

$$\Phi(t, z) = (t - z_1)^{-1/2} \dots (t - z_n)^{-1/2}$$

and the column n-vector

$$I^{(C)}(z) = (I_1, \dots, I_n) := \int_C \left(\frac{\Phi(t, z)}{t - z_1}, \dots, \frac{\Phi(t, z)}{t - z_n} \right) dt, \tag{4.3}$$

where $C \subset \mathbb{C} - \{z_1, \dots, z_n\}$ is a contour on which the integrand takes its initial value when t encircles C.

Theorem 4.1 (cf. [20])**.** *The function $I^{(C)}(z)$ is a solution of system* (4.1)*.*

This theorem is a very particular case of the results in [14].

Proof. The theorem follows from Stokes' theorem and the two identities:

$$-\frac{1}{2} \left(\frac{\Phi(t, z)}{t - z_1} + \dots + \frac{\Phi(t, z)}{t - z_n} \right) = \frac{\partial \Phi}{\partial t}(t, z), \tag{4.4}$$

$$\left(\frac{\partial}{\partial z_i} - \frac{1}{2} \sum_{j \neq i} \frac{\Omega_{i,j}}{z_i - z_j} \right) \left(\frac{\Phi(t, z)}{t - z_1}, \dots, \frac{\Phi(t, z)}{t - z_n} \right) = \frac{\partial \Psi^i}{\partial t}(t, z), \tag{4.5}$$

where $\Psi^i(t, z)$ is the column n-vector $(0, \dots, 0, -\frac{\Phi(t,z)}{t-z_i}, 0, \dots, 0)$ with the nonzero element at the i-th place. $\square$

Theorem 4.2 (cf. [16, Formula (1.3)])**.** *All solutions of system* (4.1) *have this form. Namely, the complex vector space of solutions of the form* (4.3) *is* $(n-1)$*-dimensional.*

4.3. Solutions as vectors of first derivatives

Consider the hyperelliptic integral

$$T(z) = T^{(C)}(z) = \int_C \Phi(t,z)\,dt.$$

Then

$$I^{(C)}(z) = 2\Big(\frac{\partial T^{(C)}}{\partial z_1}, \dots, \frac{\partial T^{(C)}}{\partial z_n}\Big).$$

Denote $\nabla T = (\frac{\partial T}{\partial z_1}, \dots, \frac{\partial T}{\partial z_n})$. Then the column gradient vector of the function $T(z)$ satisfies the following system of (KZ) equations

$$\nabla_i^{\mathrm{KZ}} \nabla T = 0, \quad i = 1, \dots, n, \qquad \frac{\partial T}{\partial z_1} + \dots + \frac{\partial T}{\partial z_n} = 0.$$

This is a system of second order linear differential equations on the function $T(z)$.

4.4. Solutions modulo p^s

For an integer $s \geqslant 1$ define the *master polynomial*

$$\Phi_s(t,z) = \big((t-z_1)\dots(t-z_n)\big)^{(p^s-1)/2}.$$

Recall that $n = 2g+1$. For $\ell = 1, \dots, g$ define the column n-vector

$$I_{s,\ell}(z) = (I_{s,\ell,1}, \dots, I_{s,\ell,n})$$

as the coefficient of $t^{\ell p^s - 1}$ in the column n-vector of polynomials

$$\Big(\frac{\Phi_s(t,z)}{t-z_1}, \dots, \frac{\Phi_s(t,z)}{t-z_n}\Big).$$

Notice that $\deg_t \frac{\Phi_s(t,z)}{t-z_i} = (2g+1)\frac{p^s-1}{2} - 1$. If $\ell \notin \{1, \dots, g\}$, then $\frac{\Phi_s(t,z)}{t-z_i}$ does not have the monomial $t^{\ell p^s - 1}$.

Theorem 4.3 ([20]). *The column n-vector $I_{s,\ell}(z)$ of polynomials in z is a solution of system* (4.1) *modulo p^s.*

The column n-vectors $I_{s,\ell}(z)$, $\ell = 1, \dots, g$, were called in [20] the p^s-*hypergeometric solutions* of the KZ equations (4.1).

Proof. We have the following modifications of identities (4.4), (4.5):

$$\frac{p^s-1}{2}\left(\frac{\Phi_s(t,z)}{t-z_1}+\dots+\frac{\Phi_s(t,z)}{t-z_n}\right) = \frac{\partial \Phi_s}{\partial t}(t,z),$$
$$\left(\frac{\partial}{\partial z_i}+\frac{p^s-1}{2}\sum_{j\neq i}\frac{\Omega_{i,j}}{z_i-z_j}\right)\left(\frac{\Phi_s(t,z)}{t-z_1},\dots,\frac{\Phi_s(t,z)}{t-z_n}\right) = \frac{\partial \Psi_s^i}{\partial t}(t,z),$$

where $\Psi_s^i(t,z)$ is the column n-vector $(0,\dots,0,-\frac{\Phi_s(t,z)}{t-z_i},0,\dots,0)$ with the nonzero element at the i-th place. Theorem 4.3 follows from these identities. □

Consider the $n \times g$ matrix

$$I_s(z) = (I_{s,1},\dots,I_{s,g}) = (I_{s,\ell,i})_{\ell=1,\dots,g}^{i=1,\dots,n},$$

where $I_{s,\ell,i}$ stays at the ℓ-th column and i-th row. The matrix $I_s(z)$ satisfies the KZ equations,

$$\begin{aligned} &\nabla_i^{\mathrm{KZ}} I_s(z) = 0, \qquad i = 1,\dots,n, \\ &I_{s,\ell,1}(z)+\dots+I_{s,\ell,n}(z) = 0, \qquad \ell = 1,\dots,g, \end{aligned}$$

modulo p^s.

5. Congruences for solutions of KZ equations

5.1. Congruences for Hasse–Witt matrices of KZ equations

Let $p > 2g+1$,

$$\Delta = \{1,\dots,g\} \subset \mathbb{Z}, \qquad N = \big[0, gp+(p-1)/2-g\big] \subset \mathbb{R}. \tag{5.1}$$

Lemma 5.1. *The infinite tuple $(N, N, \dots)$ is Δ-admissible, see Definition 2.3.* □

Denote

$$F(t,z) := \Phi_1(t,z) = \big((t-z_1)\dots(t-z_n)\big)^{(p-1)/2}.$$

The Newton polytope of $F(t,z)$ with respect to variable t is the interval $N=[0,gp+(p-1)/2-g]$, see (5.1), and

$$\Phi_s(t,z)=F(t,z)\cdot F(t,z)^p\dots F(t,z)^{p^{s-1}}.$$

The infinite tuple $(F(t,z),F(t,z),\dots)$ is Δ-admissible, see Definition 2.4.

For $s\geqslant 1$ consider the Hasse–Witt $g\times g$ matrix

$$A(s,\Phi_s(t,z))=\big(\mathrm{Coeff}_{p^s v-u}(\Phi_s(t,z))\big)_{u,v=1,\dots,g},$$

see (2.2). The entries of this matrix are polynomials in z.

Theorem 5.2. *The polynomial* $\det A(1,F(t,z))$ *is nonzero modulo* p.

Proof. Consider the lexicographical ordering of monomials $z_1^{d_1}\dots z_{2g+1}^{d_{2g+1}}$. We have $z_1>\dots>z_{2g+1}$ and so on. For a nonzero Laurent polynomial $f(z)=\sum_{d_1,\dots,d_{2g+1}}a_{d_1,\dots,d_{2g+1}}z_1^{d_1}\dots z_{2g+1}^{d_{2g+1}}$ with coefficients in $\mathbb{Z}$, the nonzero summand $a_{d_1,\dots,d_{2g+1}}z_1^{d_1}\dots z_{2g+1}^{d_{2g+1}}$ with the largest monomial $z_1^{d_1}\dots z_{2g+1}^{d_{2g+1}}$ is called the leading term of $f(z)$.

If $f(z)$ and $g(z)$ are two nonzero Laurent polynomials, then the leading term of $f(z)g(z)$ equals the product of the leading terms of $f(z)$ and $g(z)$.

Denote $A(1,F(t,z))=:(A_{u,v}(z))_{u,v=1,\dots,g}$.

Lemma 5.3. *If* $p>2g+1$*, the leading term of* $A_{u,v}(z)$ *equals*

$$\pm\binom{(p-1)/2}{v-u}(z_1z_2\dots z_{2g+1-2v})^{(p-1)/2}/z_{2g+1-2v}^{v-u},\qquad \text{if } v\geqslant u,$$
$$\pm\binom{(p-1)/2}{u-v}(z_1z_2\dots z_{2g+1-2v})^{(p-1)/2}z_{2g+2-2v}^{u-v},\qquad \text{if } v\leqslant u.$$

For example, for $g=2$ *the matrix of leading terms is*

$$\begin{pmatrix}\pm(z_1z_2z_3)^{(p-1)/2} & \pm\binom{(p-1)/2}{1}z_1^{(p-1)/2}/z_1\\ \pm\binom{(p-1)/2}{1}(z_1z_2z_3)^{(p-1)/2}z_4 & \pm z_1^{(p-1)/2}\end{pmatrix}. \tag{5.2}$$

Proof. The proof is by inspection. $\square$

It is easy to see that the leading term of the determinant of the matrix of leading terms of $A_{u,v}(z)$ equals the product of diagonal elements,

$$\pm\prod_{v=1}^{g}(z_1\dots z_{2g+1-2v})^{(p-1)/2}. \tag{5.3}$$

This term is not congruent to zero modulo p. This proves Theorem 5.2. □

Corollary 5.4. *The Δ-admissible infinite tuple $(F(t,z), F(t,z), \dots)$ satisfies the assumptions of Theorem 2.6. Therefore,*

(i) *for $s \geqslant 1$ we have*

(5.4)
$$A\big(s, \Phi_s(t,z)\big) \equiv A\big(1, F(t,z)\big) \cdot \sigma\big(A\big(1, F(t,z)\big)\big) \dots \sigma^{s-1}\big(A\big(1, F(t,z)\big)\big) \pmod{p};$$

(ii) *for $s \geqslant 1$ the determinant of the matrix $A(s, \Phi_s(t,z))$ is a polynomial, which is nonzero modulo p, and we have modulo p^s:*

$$A\big(s+1, \Phi_{s+1}(t,z)\big) \cdot \sigma\big(A\big(s, \Phi_s(t,z)\big)\big)^{-1} \equiv A\big(s, \Phi_s(t,z)\big) \cdot \sigma\big(A\big(s-1, \Phi_{s-1}(t,z)\big)\big)^{-1},$$

where for $s = 1$ we understand the second factor on the right as the $g \times g$ identity matrix.

Proof. The corollary follows from Theorems 5.2 and 2.6. □

5.2. Congruences for frames of solutions of KZ equations

Theorem 5.5. *We have the following congruences of $n \times g$ matrices.*

(i) *For $s \geqslant 1$,*

$$I_{s+1}(z) \cdot A\big(s+1, \Phi_{s+1}(t,z)\big)^{-1} \equiv I_s(z) \cdot A\big(s, \Phi_s(t,z)\big)^{-1} \pmod{p^s}.$$

(ii) *For $s \geqslant 1$ and $j = 1, \dots, n$,*

$$\frac{\partial I_{s+1}}{\partial z_j}(z) \cdot A\big(s+1, \Phi_{s+1}(t,z)\big)^{-1} \equiv \frac{\partial I_s}{\partial z_j}(z) \cdot A\big(s, \Phi_s(t,z)\big)^{-1} \pmod{p^s}.$$

Proof. Consider the first row of the Hasse–Witt matrix $A(s, \Phi_s(t,z))$,

$$\big(A_{1,1}\big(s, \Phi_s(t,z)\big), \dots, A_{1,g}\big(s, \Phi_s(t,z)\big)\big),$$

$$\text{where} \quad A_{1,\ell}\big(s,\Phi_s(t,z)\big) = \operatorname{Coeff}_{\ell p^s-1}\big(\Phi_s(t,z)\big).$$

For each $A_{1,\ell}(s,\Phi_s(t,z))$ we view the gradient

$$\nabla A_{1,\ell}\big(s,\Phi_s(t,z)\big) = \left(\frac{\partial A_{1,\ell}(s)}{\partial z_1},\dots,\frac{\partial A_{1,\ell}(s)}{\partial z_n}\right)$$

as a column n-vector. The resulting $n \times g$ matrix of gradients

$$\nabla A(s,z) := \big(\nabla A_{1,1}(s,\Phi_s(t,z)),\dots,\nabla A_{1,g}(s,\Phi_s(t,z))\big)$$

is proportion to the matrix $I_s(z)$, $\nabla A(s,z) = \frac{1-p^s}{2} I_s(z)$. By Theorems 2.8 and 2.9 we have modulo p^s,

$$\begin{aligned}
\nabla A(s+1,z)\cdot A\big(s+1,\Phi_{s+1}(t,z)\big)^{-1} &\equiv \nabla A(s,z)\cdot A\big(s,\Phi_s(t,z)\big)^{-1},\\
\frac{\partial}{\partial z_j}\big(\nabla A(s+1,z)\big)\cdot A\big(s+1,\Phi_{s+1}(t,z)\big)^{-1}& \\
\equiv \frac{\partial}{\partial z_j}\big(\nabla A(s,z)\big)\cdot A\big(s,\Phi_s(t,z)\big)^{-1}&.
\end{aligned}$$

These congruences imply the theorem. □

Corollary 5.6. *For $s \geqslant 1$ we have*

$$I_s(z)\cdot A\big(s,\Phi_s(t,z)\big)^{-1} \equiv I_1(z)\cdot A\big(1,\Phi_1(t,z)\big)^{-1} \pmod p.$$

6. Convergence of solutions of KZ equations

6.1. Nonzero polynomials

Lemma 6.1. *Let $z=(z_1,\dots,z_n)$. Let $F(z)\in\mathbb{F}_p[z]$ be a nonzero polynomial, $\deg F(z)\leqslant d$ for some d. Let $p^m>d$. Then there are at least $\frac{p^{mn}-1}{p^m-1}(p^m-1-d)+1$ points of $(\mathbb{F}_{p^m})^n$ where $F(z)$ is nonzero.*

Proof. First we show that there exists $a\in(\mathbb{F}_{p^m})^n$ such that $F(a)\neq 0$. The proof is by induction on n. If $n=1$, then the nonzero polynomial $F(z_1)$ cannot have more than d zeros. Hence there exists $a\in\mathbb{F}_{p^m}$ such that $F(a)\neq 0$.

Assume that the existence of a is proved for all nonzero polynomials with less than n variables. Write $F(z)=\sum_i c_i(z_2,\dots,z_n)z_1^i$. By the induction assumption, there exists $a_2,\dots,a_n\in\mathbb{F}_{p^m}$ such that $c_i(a_2,\dots,a_n)\neq 0$ for at least one i. Hence $F(z_1,a_2\dots,a_n)$ is a nonzero polynomial of degree $\leqslant d$ which defines a nonzero function of z_1. The existence of a is proved.

Let $a \in (\mathbb{F}_{p^m})^n$ be such that $F(a) \neq 0$. In $(\mathbb{F}_{p^m})^n$ there are $\frac{p^{mn}-1}{p^m-1}$ distinct lines through a. On each of the lines there are at least $p^m - 1 - d$ points different from a where $F(z)$ is nonzero. Hence the total number of points where $F(z) \neq 0$ is at least $\frac{p^{mn}-1}{p^m-1}(p^m - 1 - d) + 1$. $\square$

6.2. Unramified extensions of $\mathbb{Q}_p$

We fix an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. For every m, there is a unique unramified extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}_p}$ of degree m, denoted by $\mathbb{Q}_p^{(m)}$. This can be obtained by attaching to $\mathbb{Q}_p$ a primitive root of 1 of order $p^m - 1$. The norm $|\cdot|_p$ on $\mathbb{Q}_p$ extends to a norm $|\cdot|_p$ on $\mathbb{Q}_p^{(m)}$. Let

$$\mathbb{Z}_p^{(m)} = \{a \in \mathbb{Q}_p^{(m)} \mid |a|_p \leqslant 1\}$$

denote the ring of integers in $\mathbb{Q}_p^{(m)}$. The ring $\mathbb{Z}_p^{(m)}$ has the unique maximal ideal

$$\mathbb{M}_p^{(m)} = \{a \in \mathbb{Q}_p^{(m)} \mid |a|_p < 1\},$$

such that $\mathbb{Z}_p^{(m)}/\mathbb{M}_p^{(m)}$ is isomorphic to the finite field $\mathbb{F}_{p^m}$.

For every $u \in \mathbb{F}_{p^m}$ there is a unique $\tilde{u} \in \mathbb{Z}_p^{(m)}$ that is a lift of u and such that $\tilde{u}^{p^m} = \tilde{u}$. The element $\tilde{u}$ is called the Teichmuller lift of u.

6.3. Domain $\mathfrak{D}_B$

For $u \in \mathbb{F}_{p^m}$ and $r > 0$ denote

$$D_{u,r} = \{a \in \mathbb{Z}_p^{(m)} \mid |a - \tilde{u}|_p < r\}.$$

We have the partition

$$\mathbb{Z}_p^{(m)} = \bigcup_{u \in \mathbb{F}_{p^m}} D_{u,1}.$$

Recall $z = (z_1, \dots, z_n)$. For $B(z) \in \mathbb{Z}[z]$, define

$$\mathfrak{D}_B = \{a \in (\mathbb{Z}_p^{(m)})^n \;\mid\; |B(a)|_p = 1\}.$$

Let $\bar{B}(z)$ be the projection of $B(z)$ to $\mathbb{F}_p[z] \subset \mathbb{F}_{p^m}[z]$. Then $\mathfrak{D}_B$ is the union of unit polydiscs,

$$\mathfrak{D}_B = \bigcup_{\substack{u_1,\dots,u_n \in \mathbb{F}_{p^m} \\ \bar{B}(u_1,\dots,u_n) \neq 0}} D_{u_1,1} \times \dots \times D_{u_n,1}.$$

For any k we have

$$\begin{aligned} \{a \in (\mathbb{Z}_p^{(m)})^n \mid |B(a^{p^k})|_p = 1\} &= \bigcup_{\substack{u_1,\dots,u_n \in \mathbb{F}_{p^m} \\ \sigma^k(\bar{B}(u_1,\dots,u_n)) \neq 0}} D_{u_1,1} \times \dots \times D_{u_n,1} \\ &= \bigcup_{\substack{u_1,\dots,u_n \in F_{p^m} \\ \bar{B}(u_1,\dots,u_n) \neq 0}} D_{u_1,1} \times \dots \times D_{u_n,1} = \mathfrak{D}_B. \end{aligned}$$

6.4. Uniqueness theorem

Let $\mathfrak{D} \subset (\mathbb{Z}_p^{(m)})^n$ be the union of some of the unit polydiscs $D_{u_1,1} \times \dots \times D_{u_n,1}$, where $u_1, \dots, u_n \in \mathbb{F}_{p^m}$.

Let $(F_i(z))_{i=1}^\infty$ and $(G_i(z))_{i=1}^\infty$ be two sequences of rational functions on $(\mathbb{Z}_p^{(m)})^n$. Assume that each of the rational functions has the form $P(z)/Q(z)$, where $P(z), Q(z) \in \mathbb{Z}[z]$, and for any polydisc $D_{u_1,1} \times \dots \times D_{u_n,1} \subset \mathfrak{D}$, we have

$$|Q(\tilde{u}_1, \dots, \tilde{u}_n)|_p = 1,$$

which implies that

$$|Q(a_1, \dots, a_n)|_p = 1, \qquad \forall\ (a_1, \dots, a_n) \in \mathfrak{D}.$$

Assume that the sequences $(F_i(z))_{i=1}^\infty$ and $(G_i(z))_{i=1}^\infty$ uniformly converge on $\mathfrak{D}$ to analytic functions, which we denote by $F(z)$ and $G(z)$, respectively.

Theorem 6.2. *Under these assumptions, if $F(z) = G(z)$ on an open non-empty subset of $\mathfrak{D}$. Then $F(z) = G(z)$ on $\mathfrak{D}$.*

The following proof was communicated to us by Vladimir Berkovich.

Proof. The domain $\mathfrak{D}$ is a disjoint union of open unit polydiscs, and so it gives rise to a similar domain $\mathfrak{D}'$ over the algebraic closure of $\mathbb{Q}_p^{(m)}$. Each rational function of our sequence has no poles in $\mathfrak{D}'$. This property implies that the restriction of the function to each of open unit polydisc contained in $\mathfrak{D}$ is a formal power series convergent at all points of the polydisc.

First, recall the definition and some properties of the affine space $\mathbb{A}^n$ over a non-Archimedean field $\mathbb{K}$ (as $\mathbb{Q}_p^{(m)}$). As a space it is the set of all multiplicative seminorms $|\cdot|_x \colon \mathbb{K}[T_1, \ldots, T_n] \to \mathbb{R}_+$ that extend the (multiplicative) valuation $|\cdot| \colon \mathbb{K} \to \mathbb{R}_+$, and it is provided with the weakest topology with respect to which all functions $\mathbb{A}^n \to \mathbb{R}_+ \colon x \mapsto |f|_x$ for polynomials f are continuous. We need only one point g, called the Gauss point and defined as follows: $|\sum_\mu a_\mu T^\mu|_g = \max_\mu |a_\mu|$. One can show that

(1) the point g lies in the closure of each open polydisc of radius one with center at a point $t \in \mathbb{K}^n$ with $|t_i| \leqslant 1$, and
(2) for each bounded convergent power series f on such an open polydisc, the real valued function $x \mapsto |f|_x$ extends by continuity to the point g, and one has $|f|_x \leqslant |f|_g$ for all points of the polydisc.

Uniqueness Since F and G are uniform limits of rational functions regular on $\mathfrak{D}$, their restrictions to each open polydisc in $\mathfrak{D}$ are bounded convergent power series and, in particular, the number $|F - G|_g$ is well defined and one has $|F - G|_x \leqslant |F - G|_g$ for all points $x \in \mathfrak{D}$. If $F(x) = G(x)$ for points from a nonempty open subset of an open unit polydisc, then $F(x) = G(x)$ for all points of the polydisc (it is uniqueness property for convergent power series) and, therefore, $|F - G|_g = 0$. This implies that $F = G$ on $\mathfrak{D}$. $\square$

6.5. Domain of convergence

By Theorem 5.2 the polynomial $\det A(1, F(t, z)) \in \mathbb{Z}[z]$ is nonzero modulo p. The polynomial $\det A(1, F(t, z))$ is a homogeneous polynomial in z of degree

$$d = \frac{p-1}{2} g^2, \tag{6.1}$$

cf. (5.3). Define

$$\mathfrak{D}_{\mathrm{KZ}}^{(m)} = \{a \in (\mathbb{Z}_p^{(m)})^n \mid |\det A(1, F(t, a))|_p = 1\}.$$

By Lemma 6.1 the domain $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$ is nonempty if $p^m > d$. In what follows we assume that $p^m > d$.

Remark. The space $(\mathbb{Z}_{p^m})^n$ is the disjoint union of p^{mn} unit polydiscs $D_{u_1,1} \times \cdots \times D_{u_n,1}$. By Lemma 6.1 at least $\frac{p^{mn}-1}{p^m-1}(p^m - 1 - d) + 1 > p^{mn}(1 - \frac{d}{p^m-1})$ of them belong to $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$. So, as m grows almost all polydiscs belong to $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$.

We have $|\det A(s,\Phi_s(t,a))|_p = 1$ for $a \in \mathfrak{D}_{\mathrm{KZ}}^{(m)}$. All entries of $A(s,\Phi_s(t,z))^{-1}$ are rational functions in z regular on $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$. For every $a \in \mathfrak{D}_{\mathrm{KZ}}^{(m)}$ all entries of $A(s,\Phi_s(t,a))$ and $A(s,\Phi_s(t,a))^{-1}$ are elements of $\mathbb{Z}_p^{(m)}$.

Theorem 6.3. *The sequence of $g \times g$ matrices*

$$\big(A\big(s+1,\Phi_s(t,z)\big)\cdot\sigma\big(A\big(s,\Phi_{s-1}(t,z)\big)\big)^{-1}\big)_{s\geqslant 1},$$

whose entries are rational functions in z regular on $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$, uniformly converges on $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$ as $s \to \infty$ to an analytic $g \times g$ matrix which will be denoted by $\mathcal{A}(z)$. For $a \in \mathfrak{D}_{\mathrm{KZ}}^{(m)}$ we have

$$|\det\mathcal{A}(a)|_p = 1$$

and the matrix $\mathcal{A}(a)$ is invertible.

Proof. The theorem follows from Theorem 3.4. □

Theorem 6.4. *For $i = 1,\dots,n$ the sequence of $g \times g$ matrices*

$$\bigg(\bigg(\frac{\partial}{\partial z_i}A\big(s,\Phi_s(t,z)\big)\bigg)\cdot A\big(s,\Phi_s(t,z)\big)^{-1}\bigg)_{s\geqslant 1},$$

whose entries are rational functions in z regular on $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$, uniformly converges on $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$ as $s \to \infty$ to an analytic $g \times g$ matrix, which will be denoted by $\mathcal{A}^{(i)}(z)$.

The sequence of $n \times g$ matrices

$$\big(I_s(z)\cdot A\big(s,\Phi_s(t,z)\big)^{-1}\big)_{s\geqslant 1},$$

whose entries are rational functions in z regular on $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$, uniformly converges on $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$ as $s \to \infty$ to an analytic $n \times g$ matrix which will be denoted by $\mathcal{I}(z)$.

For $i = 1,\dots,n$ the sequence of $n \times g$ matrices

$$\bigg(\frac{\partial I_s}{\partial z_i}(z)\cdot A\big(s,\Phi_s(t,z)\big)^{-1}\bigg)_{s\geqslant 1},$$

whose entries are rational functions in z regular on $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$, uniformly converges on $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$ as $s \to \infty$ to an analytic $n \times g$ matrix which will be denoted by $\mathcal{I}^{(i)}(z)$.

We have

$$\frac{\partial \mathcal{I}}{\partial z_i} = \mathcal{I}^{(i)} - \mathcal{I} \cdot \mathcal{A}^{(i)}.$$

Proof. The theorem follows from Theorems 3.5, 3.6, and Lemma 3.7. □

Theorem 6.5. *We have the following system of equations on* $\mathfrak{D}^{(m)}_{\mathrm{KZ}}$*:*

$$\mathcal{I}^{(i)} = H_i \cdot \mathcal{I}, \qquad i = 1, \dots, n,$$

where H_i *are the Gaudin Hamiltonians defined in* (4.2)*.*

Proof. The theorem is a corollary of Theorem 4.3. □

Corollary 6.6. *For* $a \in \mathfrak{D}^{(m)}_{\mathrm{KZ}}$ *we have*

$$\mathcal{I}(a) \equiv I_1(a) \cdot A\big(1, \Phi_1(t,a)\big)^{-1} \pmod{p}.$$

Proof. The corollary follows from Corollary 5.6 and Theorem 6.4. □

6.6. Vector bundle $\mathcal{L} \to \mathfrak{D}^{(m),o}_{\mathrm{KZ}}$

Denote

$$W = \big\{(I_1, \dots, I_n) \in \big(\mathbb{Q}^{(m)}_p\big)^n \mid I_1 + \dots + I_n = 0\big\}.$$

We consider vectors $(I_1, \dots, I_n)$ as column vectors. The differential operators ∇^{KZ}_i, $i = 1, \dots, n$, define a connection on the trivial bundle $W \times \mathfrak{D}^{(m)}_{\mathrm{KZ}} \to \mathfrak{D}^{(m)}_{\mathrm{KZ}}$, called the KZ connection. The connection has singularities at the diagonal hyperplanes in $(\mathbb{Z}^{(m)}_p)^n$ and is well-defined over

$$\mathfrak{D}^{(m),o}_{\mathrm{KZ}} = \big\{a = (a_1, \dots, a_n) \in \big(\mathbb{Z}^{(m)}_p\big)^n \mid |\det A\big(1, F(t,a)\big)|_p = 1,\ a_i \neq a_j \ \ \forall i,j\big\}.$$

The KZ connection is flat,

$$\big[\nabla^{\mathrm{KZ}}_i, \nabla^{\mathrm{KZ}}_j\big] = 0 \qquad \forall\, i, j,$$

see [5]. The flat sections of the KZ connection are solutions of system (4.1) of KZ equations.

For any $a \in \mathfrak{D}_{\mathrm{KZ}}^{(m)}$ let $\mathcal{L}_a \subset W$ be the vector subspace generated by columns of the $n \times g$ matrix $\mathcal{I}(a)$. Then

$$\mathcal{L} := \bigcup\nolimits_{a \in \mathfrak{D}_{\mathrm{KZ}}^{(m)}} \mathcal{L}_a \to \mathfrak{D}_{\mathrm{KZ}}^{(m)}$$

is an analytic distribution of vector subspaces in the fibers of the trivial bundle $W \times \mathfrak{D}_{\mathrm{KZ}}^{(m)} \to \mathfrak{D}_{\mathrm{KZ}}^{(m)}$.

Theorem 6.7. *The distribution $\mathcal{L} \to \mathfrak{D}_{\mathrm{KZ}}^{(m)}$ is invariant with respect to the KZ connection. In other words, if $s(z)$ is a local section of $\mathcal{L} \to \mathfrak{D}_{\mathrm{KZ}}^{(m)}$, then the sections $\nabla_i^{\mathrm{KZ}} s(z)$, $i = 1, \dots, n$, also are sections of $\mathcal{L} \to \mathfrak{D}_{\mathrm{KZ}}^{(m)}$.*

Proof. Let $\mathcal{I}(z) = (\mathcal{I}_1(z), \dots, \mathcal{I}_g(z))$ be columns of the $n \times g$ matrix $\mathcal{I}(z)$. Let $a \in \mathfrak{D}_{\mathrm{KZ}}^{(m)}$. Let $c(z) = (c_1(z), \dots, c_g(z))$ be a column vector of analytic functions at a. Consider a local section of the distribution $\mathcal{L} \to \mathfrak{D}_{\mathrm{KZ}}^{(m)}$,

$$s(z) = \sum_{j=1}^{g} c_j(z) \mathcal{I}_j(z) =: \mathcal{I} \cdot c.$$

Then

$$\begin{aligned}
\nabla_i^{\mathrm{KZ}} s(z) &= -H_i \cdot \mathcal{I} \cdot c + \frac{\partial \mathcal{I}}{\partial z_i} \cdot c + \mathcal{I} \cdot \frac{\partial c}{\partial z_i} \\
&= -H_i \cdot \mathcal{I} \cdot c + \big(\mathcal{I}^{(i)} - \mathcal{I} \cdot \mathcal{A}^{(i)}\big) \cdot c + \mathcal{I} \cdot \frac{\partial c}{\partial z_i} \\
&= -H_i \cdot \mathcal{I} \cdot c + \big(H_i \cdot \mathcal{I} - \mathcal{I} \cdot \mathcal{A}^{(i)}\big) \cdot c + \mathcal{I} \cdot \frac{\partial c}{\partial z_i} \\
&= -\mathcal{I} \cdot \mathcal{A}^{(i)} \cdot c + \mathcal{I} \cdot \frac{\partial c}{\partial z_i}.
\end{aligned}$$

Clearly, the last expression is a local section of $\mathcal{L} \to \mathfrak{D}_{\mathrm{KZ}}^{(m)}$. $\square$

Theorem 6.8. *The function $a \mapsto \dim_{\mathbb{Q}_p^{(m)}} \mathcal{L}_a$ is constant on $\mathfrak{D}_{\mathrm{KZ}}^{(m),o}$, in other words, $\mathcal{L} \to \mathfrak{D}_{\mathrm{KZ}}^{(m)}$ is a vector bundle over $\mathfrak{D}_{\mathrm{KZ}}^{(m),o} \subset \mathfrak{D}_{\mathrm{KZ}}^{(m)}$.*

Proof. First, we prove that the function $a \mapsto \dim_{\mathbb{Q}_p^{(m)}} \mathcal{L}_a$ is locally constant on $\mathfrak{D}_{\mathrm{KZ}}^{(m),o}$. This holds true in the following more general setting. Let k be a positive integer, $a \in (\mathbb{Z}_p^{(m)})^n$. Let $B_i(z)$, $i = 1, \dots, n$, be $k \times k$ matrices defined and analytic in a neighborhood of a. The differential operators $\mathcal{B}_i = \frac{\partial}{\partial z_i} - B_i(z)$,

$i = 1, \dots, n$, act on $(\mathbb{Q}_p^{(m)})^k$-valued functions defined and analytic in a neighborhood of a. The operators $(\mathcal{B}_i)$ define a connection $\nabla^{\mathcal{B}}$ on the restriction of the trivial bundle $(\mathbb{Q}_p^{(m)})^k \times (\mathbb{Q}_p^{(m)})^n \to (\mathbb{Q}_p^{(m)})^n$ to a neighborhood of a. Assume that the connection is flat, $[\mathcal{B}_i, \mathcal{B}_j] = 0$ for all i, j. Then, for a sufficiently small neighborhood D of a, the space $\mathcal{S}$ of solutions of the system $\mathcal{B}_i y = 0$, $i = 1, \dots, n$, on D is a k-dimensional $\mathbb{Q}_p^{(m)}$-vector space. For any $b \in D$ the values of solutions at b span $(\mathbb{Q}_p^{(m)})^k$. Under these assumptions, the $\nabla^{\mathcal{B}}$-invariant subspace distributions in fibers of $(\mathbb{Q}_p^{(m)})^k \times (\mathbb{Z}_p^{(m)})^n \to (\mathbb{Z}_p^{(m)})^n$ over D are labeled by $\mathbb{Q}_p$-vector subspaces $Y \subset \mathcal{S}$. The corresponding distribution assigns to $b \in D$ the subspace $\{y(b) \mid y \in Y\} \subset (\mathbb{Q}_p^{(m)})^k$. Such distributions have constant rank. Hence the function $a \mapsto \dim_{\mathbb{Q}_p^{(m)}} \mathcal{L}_a$ is locally constant on $\mathfrak{D}_{\mathrm{KZ}}^{(m),o}$.

By Theorem 6.2 the locally constant function $a \mapsto \dim_{\mathbb{Q}_p^{(m)}} \mathcal{L}_a$ cannot take more than one value on $\mathfrak{D}_{\mathrm{KZ}}^{(m),o}$ since the dimension $\dim_{\mathbb{Q}_p^{(m)}} \mathcal{L}_a$ may drop from its maximal value only on a proper analytic subset of $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$. The theorem is proved. $\square$

Recall that d is the degree of the polynomial $\det A(1, F(t, z))$, see (6.1).

Theorem 6.9. *If $p^m > 2d$, then the analytic vector bundle $\mathcal{L} \to \mathfrak{D}_{\mathrm{KZ}}^{(m),o}$ is of rank g.*

Proof. First we show that there is a $g \times g$ minor of the $n \times g$ matrix valued function $\mathcal{I}(z)$ which is nonzero on $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$. By Corollary 6.6 this fact holds true if there is a $g \times g$ minor of the $n \times g$ matrix $I_1(z) \cdot A(1, \Phi_1(t, z))^{-1}$, which defines a function on $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$ nonzero modulo p. Since $|\det A(1, \Phi_1(t, a))|_p = 1$ for any $a \in \mathfrak{D}_{\mathrm{KZ}}^{(m)}$, it is enough to prove that there is a nonzero $g \times g$ minor of the $n \times g$ polynomial matrix $I_1(z)$. But this fact follows from [20, Lemma 7.2], also see [19, Lemma 6.1]. More precisely, [20, Lemma 7.2] implies that the leading term of the $g \times g$ minor in rows $1, 3, \dots, 2g-1$ equals

$$\pm \prod_{l=1}^{g} \binom{(p-1)/2}{l} z_1^{(p-1)/2} \dots z_{2g-2l}^{(p-1)/2} z_{2g-2l+1}^{(p-1)/2-l}.$$

The degree of this minor equals

$$g(2g+1)\frac{p-1}{2} - p(1 + \dots + g) = g^2 \frac{p-1}{2} - \frac{g(g+1)}{2} \ < \ g^2 \frac{p-1}{2} = d.$$

Thus, we have two polynomials of degree $\leqslant d$: this minor and $\det A(1, \Phi_1(t, z))$. Both polynomials are nonzero modulo p. By Lemma 6.1 if $p^m > 2d$, then this minor is nonzero on $\mathfrak{D}_{\mathrm{KZ}}^{(m),o}$. $\square$

6.7. Remarks

6.7.1. It was shown in [20, Section 10.4] that the span of columns of the $n \times g$ matrix $I_s(z)$ has a p-adic limit as $s \to \infty$ when z belongs to one of the asymptotic zones of the KZ equations. The limit is a g-dimensional space of power series solutions of the KZ equations with respect to the coordinates attached to that asymptotic zone. It is not clear yet if that asymptotic zone belongs to $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$.

6.7.2. One may expect that the subbundle $\mathcal{L} \to \mathfrak{D}_{\mathrm{KZ}}^{(m),o}$ can be extended to a rank g subbundle over $\mathfrak{D}_{\mathrm{KZ}}^{(m)} - \mathfrak{D}_{\mathrm{KZ}}^{(m),o}$, the union of the diagonal hyperplanes in $\mathfrak{D}_{\mathrm{KZ}}^{(m)}$.

6.7.3. Following Dwork we may expect that locally at any point $a \in D_{\mathrm{KZ}}^{(m),o}$, the solutions of the KZ equations with values in $\mathcal{L} \to \mathfrak{D}_{\mathrm{KZ}}^{(m),o}$ are given at a by power series in $z_i - a_i$, $i = 1, \dots, n$, bounded in their polydiscs of convergence, while any other local solution at a is given by a power series unbounded in its polydisc of convergence, cf. [4] and [20, Theorem A.4].

6.7.4. The KZ connection ∇_i^{KZ}, $i = 1, \dots, n$, over $\mathbb{C}$ is identified with the Gauss–Manin connection of the family of hyperelliptic curves $y^2 = (t - z_1) \dots (t - z_n)$. The monodromy representation of that Gauss–Manin connection is described in [3, Appendix]. The image of the monodromy representation is so big that the connection does not have proper invariant subbundles. The monodromy representation is irreducible, see [7, Lemma 6]. Thus the existence of the invariant subbundle $\mathcal{L} \to \mathfrak{D}_{\mathrm{KZ}}^{(m),o}$ is a p-adic feature.

6.7.5. The invariant subbundles of the KZ connection over $\mathbb{C}$ usually are related to some additional conformal block constructions, for example, see [6, 15, 18]. Apparently our subbundle $\mathcal{L} \to \mathfrak{D}_{\mathrm{KZ}}^{(m),o}$ is of a different p-adic nature.

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